HOW THE ACTIVITY OF PROVING IS CONSTITUTED IN A CYPRIOT CLASSROOM FOR 12 YEAR OLD STUDENTS

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DECLARATION

I hereby declare that, except where explicit attribution is made, the work presented in this thesis is entirely my own.

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ABSTRACT

The aim of this study is to identify how the activity of proving is constituted in a Cypriot primary classroom for 12 year old students. Through Cultural-Historical Activity Theory (CHAT), the influence of research literature, curriculum prescriptions, the students and critically the teacher is documented. The evolution of objects, in particular the aims of the teacher, and other components in the activity systems is traced.

Within the qualitative enquiry, this study employs CHAT alongside a collaborative design approach to explore the way the teacher is working with the students to foreground mathematical argumentation. The research is situated in a Cypriot primary school classroom with the researcher having the role of teacher researcher. The usual class teacher and researcher co-developed Dynamic Geometry Environment (DGE)-based tasks to be used with the children. As a result it was possible to track how the nature of the teacher's objects changed and how contradictions emerged. Evidence from the curriculum documentation and from classroom observations was used to develop the activity systems of exploring and explaining.

One important finding lies in how exploring and explaining were key sub-systems within the central activity system of proving as they provided a key pathway, which often included defining. Processes of explaining, defining and exploring appeared to create a fertile ground for the development of proving. I refer to these developments as pre-proving. However, it turns out that there are inherent contradictions within explaining and exploring that hinder the constitution of proving in the classroom.

An emerging primary contradiction was apparent in the multifaceted nature of the object of both exploring and explaining to both facilitate mathematical argumentation and address a prescribed curriculum. Due to the tension between these objects, the teacher was often faced with dilemmas such as whether to open up playful activity or close it down to focus on the curriculum specifics. These led to a constant struggle in the teacher's everyday practice. I report also on how primary contradictions led inevitably to higher-level contradictions between other components of the activity systems.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

TABLE OF CONTENTS OF APPENDICES

CHAPTER I

THE RESEARCH BACKGROUND

1.1. Introduction and Background of the author

Today's complex and continuously changing society places responsibilities upon educators to understand more about children and therefore develop teaching and the quality of learning. Indeed, research is considered to be a fundamental tool for expanding knowledge regarding these important educational issues, as well as for improving classroom practices (Robert-Holmes, 2005). Personally, my experience as a primary school teacher has led me to investigate how pupils explain and justify geometric conjectures experienced in carefully designed activities within a widely used (or well known) dynamic environment.

A key question is to ask how students might be supported in developing mathematical reasoning and concepts of proof at all levels of schooling (Küchemann and Hoyles, 2006). In the case of geometry this is a very challenging task, as shown by a brief look at the history of geometry in schools.

In the twentieth century, an increasing emphasis was given to the experimental and practical aspects of geometry (French, 2004). For much of that century reasoning and proof were seen in relation to an abstract logical system and geometrical tasks were seen merely as problems that involved a procedural application of a theorem, as they were beyond the ability of most students to solve through reasoning. According to French (2004) this approach prevented students from developing skills in making conjectures, explaining the steps followed in order to tackle a certain mathematical activity and justifying the procedures adopted.

Where should emphasis be placed in a geometry curriculum? French (2004) has argued that both induction and deduction should be taught at schools, in order for students to appreciate at first the need for proof in geometry, to understand the nature of proof, and finally gain competence in constructing proofs. Furthermore, he claims that naming, describing, classifying and making links to measurement, position and movement is the first step to providing an intuitive feel for the properties of geometric concepts, and the relationships among them. Waring (2000) emphasizes making opportunities for analyzing statements, drawing conclusions from them, or for recognizing that some claims just do not follow. In this way, students would be given the opportunity to learn early on to recognize the form of informal mathematical statements, the applicable methods of mathematical reasoning, and how the truth of the components ensures truth of complex statements.

When I was at primary, lower secondary and secondary school I never had any problems in mathematics. I always had good marks. Being a secondary school student that chose mathematics as a specialised subject, I was always enthusiastic in solving problems related with Euclidean geometry. Proving in geometry was an important aspect of the Cypriot mathematics curriculum. Even though it was important for us, the students, to be able to solve exercises so as to get a good mark, our teachers wanted us to also know what we were doing and why. Being given a task in the form 'prove that…' was customary in our school mathematics practice. However, even though proving was a challenging task, I was fascinated by mathematical proofs and I would find it difficult to comprehend why other people would say that they hated geometry and that they could not 'see' things so as to be able to prove them. Fear or hatred was an obstructing factor in their relationship with geometry. Many of my classmates perceived themselves as failures in this subject. The day the positions for universities were announced and I found out that I would study in the department of Primary Education at the Aristotle University of Thessaloniki, I promised myself that I would do a PhD related with geometry so as to find ways to help people 'see' things in geometry the way I did; originally a naïve idea that gradually developed into a research proposal. During my first degree and my MSc in Mathematics Education, I was fascinated by the opportunities offered by Dynamic Geometry Environments in supporting someone's geometric thinking. All these experiences led to me applying for a PhD. As a teacher, I experienced powerful personal motivation to help students to an appreciation and understanding of mathematics. As a result, my prior and ongoing teaching experience have not only motivated this research study, but also shaped the way it has transformed.

As a researcher, I was initially interested in creating occasions for pupils to make explicit their reasoning and exploring the ways they can be reinforced in developing their reasoning skills in geometry. To be more precise, I was interested in identifying the explanations and justifications upper primary school students make in geometry when interacting with a dynamic geometry environment, to identify features of tasks that may favour the process of formulating a conjecture and proving it, and to investigate the interactions that take place within this process.

When I carried out a first exploratory study in Cyprus, I worked with pairs of children outside the classroom; as a teacher, I kept asking myself how different the results might have been if these tasks had been employed instead in a classroom.

A second exploratory study was undertaken in two classrooms, from different primary schools in Cyprus. The results of this experiment were very interesting. In both classrooms, pairs of students were exploring tasks in a DGE. The attitude of the students towards the researcher, their behaviour while working on the computer, the questions asked regarding the exploration of the tasks and their comments at the end of the pilot led to me seeking more information about the teacher of the classroom as well as information regarding the organisation of each school. The information gathered led to yet more questions. Even though the main purpose was initially to investigate the way upper primary school students explain and justify geometric conjectures while interacting with a dynamic geometry environment, the fact that the socio-cultural aspects of this system had an impact on the students' activity could not be ignored. At that point, I began to consider how Cultural-Historical Activity Theory (CHAT) might enable me to study the effect of the tasks at different levels. Implementing this framework seemed most appropriate in seeking answers that I had both as a teacher as well as a researcher.

Reflecting on the exploratory studies led to the decision to investigate proving activity in a classroom in a primary school, with both the teacher and the students participating in this study.

1.2. Synopsis of the thesis

Following the previous section which was an initial step in situating the background of this research and its general goals, the remainder of this first chapter sets the scene so as for the reader to have an overview of what was involved in undertaking this research. The remainder of this thesis provides the theoretical background of this study, the methodological framework and presents the findings of the study conducted.

Chapter 2 is concerned with the epistemological, psychological and pedagogical aspects of mathematics. The main themes that are being utilised in exploring this area of study are critically illustrated and elaborated on. That is, Chapter II explores the meaning of proving, the students' conceptions of proving as well as the roots of proving. The argument developed throughout the review of the literature concludes by stating the more detailed articulation of the purpose of this study. The overriding research question of this thesis is explicated.

Chapter 3 introduces the two exploratory studies and discusses the way they informed the research design of the main study. In this chapter, the study setting, my role as a researcher, the method for data collection, the emergent findings as well as the analysis of the findings of the preliminary studies presented highlight their influence to the general direction to subsequent work.

Chapter 4 is concerned with the theoretical considerations of Cultural-Historical Activity Theory. Furthermore, it provides a justification on why this approach is suitable for exploring how proving is constituted in the mathematics classroom. In the light of this discussion, the research question is restated in three main research aspirations.

Chapter 5 continues by elaborating on the methodological approach that alongside Cultural-Historical Activity Theory will enable this study to explore its main objectives. This is followed by a more detailed presentation of the research plan. This chapter progresses by providing and justifying the theoretical assumptions underlying the methodological approaches which I will use in my own research design by considering both their strengths and weaknesses. The chapter then proceeds with a description of the data analysis process implemented in this study. It also exemplifies how the ethical issues that needed to be taken into consideration for this study to maintain theoretical sophistication and methodological rigour.

Chapters 6, 7 and 8 are devoted to the presentation of the three phases of the study. Each chapter explicates the method for conducting each phase of data collection, presents and discusses the findings emerging from the data gathered. Chapters 7 and 8 also provide a detailed description of my role as a researcher.

Chapter 9 constitutes the final phase of the analysis. The aim of this chapter is to place the study's findings in a broader theoretical context by conducting a retrospective analysis on the entire data set generated from the three phases of this study. This is being achieved by employing the main aspirations of Cultural-Historical Activity Theory, alongside the available research literature that informs this study.

Chapter 10 draws together the elements of the thesis and offers a discussion of the results in a form of answering the main themes of the research questions, which through the analysis of the findings are further developed. It then communicates the contribution to the research field within the domain of the study. This is realised by connecting the research findings with the basic studies delineated within the review of literature. In addition, the limitations of the study are acknowledged and discussed and possible directions for further research are identified. The chapter also lists this thesis' contributions and outlines its implications on teaching, teacher education and curriculum development. With the concluding remarks this research study is reflected upon as a whole.

CHAPTER II

PROVING IN THE PRIMARY CLASSROOM

2.1. Introduction

This chapter provides a systematic and critical review of the available research literature, in order to compose a careful and analytical argument from which this research drew.

Existing research on mathematical proof addresses a number of different aspects of proof and proving. The perspectives investigated are related with the historical-epistemological, cognitive and didactical characteristics of proof. The main issues the studies are attempting to tackle are the definition of the roles and function of proof in diverse historical and institutional settings, the function which proof should have in mathematics and school mathematics, the analysis of arguments produced by students, proof in the context of dynamic software, the relationship between different activities involved in the process of proving, students' difficulties with proof and the ways students can be supported when understanding or constructing proofs (instructional and curricular issues).

To begin with, the literature review of the current research on proof is explored so as to provide a formal definition of mathematical proof, the various functions of proof as well as the relationship between argumentation and proof. Then the students' conceptions of proving are elaborated on. The psychological as well as the sociocultural factors that play a role on the students' emergence of the meaning of proof are identified and analysed. That is, the epistemological aspects that characterise geometry both as a formal mathematical discipline and a part of the physical world, and the nature of geometrical concept, as well as the theories that illustrate the development of geometrical concepts and the systems of categories of individual's argumentative behaviour are presented. What is more, theories that argue about the construction of meanings in mathematics are introduced.

A discussion concerning the roots of proving will follow. The notions of explanation, justification, definition and argumentation are explored and a construct of 'pre-proving' will be introduced.

After illustrating key themes, and giving a systematic and critical review of the available research literature, a careful and analytical argument that supports this research is composed and the main purpose of this study is presented.

2.2. What is proving?

In this section, the review of the literature is concerned with identifying the formal aspects of proof and proving. That is, by providing a formal definition of proof, the differing functions of proof will be illustrated. A discussion regarding the interplay of argumentation and proof will follow.

2.2.1 Defining proof and proving

Proof and proving are notions that are employed in the mathematics education literature with differing significations (Reid, 2001; 2005). In fact, no explicit general definition of a proof is shared by the entire mathematics community. This is also evident in Balacheff (2002; 2008), who concludes that this apparent lack of agreement on what is meant by proof in the mathematics education research can be, to some extent, justified due to the lack of transparency in articulating the perspective each study follows regarding what counts as proof. Thus, to foster the communication of ideas, it is imperative that one clarifies the terminology.

A proof can be thought of as an argument accepted by a community at a given time. Mathematical proof is a *'proof accepted by mathematicians. As a discourse, mathematical proofs have now-a-days a specific structure and follow well defined rules that have been formalised by logicians'* (Balacheff, 1988a, p.2). Formal mathematical proof as a clear robust idea can be defined as *'a finite sequence of formulas in a given system, where each formula of the sequence is either an axiom of the system or is derived from preceding formulas by rules of inference of the system'* (Hanna and Barbeau, 2010, p.98). Formal mathematical proof can also be defined as *'any justification which satisfies the requirements of abstraction, rigor, language demanded by professional mathematicians to accept a mathematical statement as valid within an axiomatic system'* (Marrades and Gutiérrez, 2000, p.89). The mental act of proving constitutes *'the process of removing or instilling doubts about an assertion'* (Harel, 2007, p.65), the outcome of which is proof as an end product.

Before deciding the role of proof in the mathematics classroom, one must not only perceive proof as the ultimate method of verification, but to consider the whole range of meanings which proof has in mathematics practice (Hanna, 2000). De Villiers (1999b), expanding Bell's (1976) initial categorisation between the functions of verification, illumination and systematisation presents a range of functions which proof demonstrates in mathematical practice.

Verification. The first role of proof is concerned with establishing the truth of a proposition. Hoyles (1997) argues that although the notion of proof incorporates a variety of meanings in differing contexts, it has a distinctive role within the mathematics community:

> It has traditionally separated mathematics from the empirical sciences as an indubitable method of testing knowledge which contrasts natural induction with empirical pursuits. Deductive mathematical proof offers human beings the purest form of how to distinguish right from wrong (p.7).

Explanation. Explaining why a theorem or a property is true should be provided when verification does not appear enough for providing insight into why the conjecture is true (De Villiers, 1999a, 1999b). Hanna (1990) argues that some proofs have a more explanatory nature than others and talks about proofs that explain and proofs that prove. Proofs that prove show that a theorem is true, whereas proofs that explain also provide reasons that are based on the mathematical ideas involved. Following this, Hersh (1993) and Hanna (1995) state that, while in mathematics practice and research proof is just a convincing argument, the main function in mathematics education is that of explanation. It is not enough only to convince students that a statement is true. The truth of a proposition functions as a satisfactory proof for students only when it involves interpretation, understanding, reasoning and sense-making.

Systematization. The main objective of systematising the various results into a deductive system of axioms, major concepts and theories is *'not to check whether certain statements are really true, but to organise logically unrelated individual statements that are already known to be true into a coherent unified whole'* (De Villiers, 1999b, p.10).

Discovery. Proof can function as the discovery or invention of new results in two ways. New mathematical results can be discovered by means of intuition and/or quasi-empirical methods and then proved deductively. New results can also be realised in a purely deductive way (De Villiers, 1999b).

Communication. According to De Villiers (1999b) proof constitutes a means for communicating mathematical results between mathematicians, between teachers and students and among students themselves. Balacheff (1991) stresses the importance of proof as a tool for both establishing the validity of a statement, as well as a tool for communication. Proof may well be considered as any factual evidence that helps to establish the truth of something. However, it is only through communication within the community of mathematicians or the mathematics classroom that its importance becomes apparent; only when it involves interpretation, understanding, reasoning and sense-making proof functions as a satisfactory proof for an individual.

Proof can also function as the construction of empirical theory, the incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective, and the exploration of definition and of the consequences of assumptions (Hanna, 2000).

In recent years, it is being argued that proofs have an additional value; proofs are bearers of mathematical knowledge (Rav, 1999). Hanna and Barbeau (2010) share the view that a proof may display *'fresh methods, tools, strategies, and concepts that are of wider applicability in mathematics and open up new mathematical directions'*, (p.86), and therefore bring to mathematics new knowledge. Regarding the mathematics classroom, they support their argument with examples of proofs that are common to secondary school mathematics curricula to illustrate that the exposure to unfamiliar methods, tools, strategies and concepts *'can convey to students a much* *richer understanding of mathematics'* and a broader picture on the nature of mathematics (p.97).

Having in mind the broad range of meanings and functions of proof, it appears that different functions of proof are related to mathematics and school mathematics. In mathematics practice the main function of proof is justification and verification, while in the educational domain its main function is that of explaining why, discovering and communicating (De Villiers, 1998).

Having explored the various meanings the notion of proof encompasses, the relationship between argumentation and proof must also be considered, as this undoubtedly influences the discussion on proof and proving.

2.2.2 Argumentation versus proof

The debate related with the relationship between argumentation and proof is recent. Different studies employ differing tools and attempt to compare the process of proving and arguing from a cognitive and epistemological point of view. These perspectives are based on Duval's view on argumentation and proof and either build on this approach or criticise it.

According to Duval (2007) both a cognitive and logical gap can be located between argumentation and proof. In formal proof the inferences of each step constitute the entrance condition of the inference rule of the following step. In other words, formal proof progresses step-by-step such that the conclusion from one step is affirmed and enables the consideration of the next step. In argumentation the inferences of preceding phases are reinterpreted from different points of view. Duval (2007) uses the organization and the possible shift of the meaning of a proposition (statement which has a value for itself and a status in relationship to one another) in a certain discourse in order to explain a move from standard argumentation to proof. He talks about the epistemic (the degree of certainty or conviction associated to a proposition) and true value of a proposition and takes into account these features of proof and proving in order to understand deductive reasoning. In deductive reasoning the epistemic value of a proved proposition is that of being true and necessary whereas in argumentative reasoning the propositions have not the same value and the epistemic value depends on the content.

The position that distance exists between argumentation and proof is reinforced by Balacheff (1991) who perceives argumentation as one of the limitations of social interaction regarding mathematical proof. Balacheff stresses the social side of argumentation to which students are exposed in their everyday life. The primary aim of this discourse is not necessarily related to establishing the truth of a certain statement. On the contrary, in mathematical proofs one has to *'fit the requirement for the use of some knowledge taken from a common body of knowledge on which people (mathematicians) agree'*, (p.189). Thus students are involved in argumentation the aim of which is not to demonstrate the truth of an assertion, nor to show the logical validity of reasoning, but rather to obtain the agreement of a partner in the interaction. Balacheff (1999) concludes that social argumentation constitutes an epistemological obstacle to the learning of mathematical proof, and more generally of proof in mathematics. That is, when students transfer social processes and behaviours that are closely related with social argumentation, the construction of a mathematical proof might appear problematic. Consequently mathematical proof should be learned 'against' argumentation, so that students become aware of the specificity of proof. Nevertheless, the fact that pupils appear able to argue due to social interaction can be exploited as a resource for learning by establishing the distinction between argumentation and proving and supporting the development of proving.

While argumentative and proving discourses have undoubted differences (Mogetta, 2001), a selection of researchers have also attempted to explore instances where unity may exist between argumentation and proof that leads to the formulation of conjectures in mathematical contexts and the proving process.

From a cognitive point of view, Garuti et al (1996) show experimental evidence of unity between the production of conjectures and the construction of a proof. They propose the *'cognitive unity'* of theorems as a tool appropriate for predicting the level of difficulties students meet when proving a given statement. This theoretical construct relies on the continuity that exists between the production of a conjecture and the possible construction of its proof. The following quotation describes this phenomenon:

> During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices: during the subsequent proving stage, the student links up with this process in a coherent way, organising some of the justifications produced during the construction of the statement according to a logical chain (p.113).

They argue that this unity can be achieved through the construction of a cycle of dynamic exploration which facilitates the connection between conjecturing and proving; exploring, producing a conjecture, coming back to the exploration, reorganising it into a proof (Garuti et al, 1998, p.347). They also define the gap between the exploration of the statement and the proving process as *'the distance between the arguments for the plausibility of the conjecture, produced during the exploration of the statement and the arguments which can be exploited during the proving process'*, (p.348).

Pedemonte (2001) argues that the comparison of the content (language, mathematical theory, the drawing, the heuristic) between argumentation and proof is not sufficient for an in depth analysis and understanding of the possible continuity or distance between these processes. Even if students are exposed in situations or environments that facilitate the construction of conjectures, this is not enough to cover the phenomena where they fail to construct the proof. She considers continuity between the argumentation process that produces a conjecture and proof related to function and structure (Pedemonte, 2007). Both argumentation and proof in mathematics may be thought as rational justifications that are developed in order to convince oneself or the 'universal audience' (the mathematical community, the classroom, the teacher) about the truth of a statement in a certain theoretical field (Pedemonte, 2007). Regarding the structural continuity between these processes, Pedemonte (2003) argues that this unity may not have as immediate consequence the successful construction of proof. Sometimes the structural distance needs to be covered for students to construct correct proofs.

Knipping (2003) and Vincent et al (2005) also explore the overall structure of the argumentation developed by students when engaged in proving discourses in ordinary classroom situations. Their observations indicate that proving situations appear to have complex and distinct argumentation structures. Thus, the elements that are involved in a proving discourse need to be counted so as to form a holistic picture of what is taking place during the argumentation and proof construction.

A consideration of the aforementioned indicates that approaching mathematical proof and proving is a complicated activity. Thus, approaching proof and proving in the mathematics classroom encompasses many challenges. Explanation, justification and communication are the functions proof should have for school mathematics. A deeper examination on the discussion regarding argumentation and proof also supports that. Keeping in mind the above, it can be argued that these functions of proof should be incorporated in the introduction of proof and proving to students. In order for this to be successfully achieved, the students' conceptions of proving need to be identified, and the psychological and sociocultural factors that influence the students' emergence for proving need to be explored. These issues are discussed in the section that follows.

2.3. Students' conceptions of proving

Traditionally, proof had a restricted place in the mathematics curriculum; it was mainly viewed as a high school geometry topic. Nowadays the teaching of proof as a fundamental part of school mathematics is widely embraced (Ball et al, 2002, Hanna, 1995; 2000, Stylianides, 2007a, Yackel and Hanna, 2003). Still, students have difficulties when approaching the construction and understanding of proofs.

The empirical studies that have addressed the difficulties students face when attempting to either read or write proofs, have been mainly situated within secondary and undergraduate settings. These difficulties may be attributed to several factors; the curriculum, the textbooks, instruction, the teacher's background, the students themselves (i.e. student interest in mathematics). Identifying areas that are problematic in the teaching and learning of proof strengthens the need to incorporate aspects of proof across all levels of schooling. However, even though there is a gap in the existing research related with elementary school students understanding proof (Stylianou et al, 2009), recent studies demonstrate that elementary school students can engage in building logical arguments in order to establish certainty. These studies are being presented subsequently.

One of the main difficulties that has been highlighted in the available mathematics education research literature is that students do not understand the fundamental distinction between empirical and deductive arguments (Balacheff, 1988b). When they are presented with mathematical problems they consider as proof the mathematical argument for the specific case or diagram and tend to use empirical arguments. Hoyles, (1997) provides a rationalisation for why this happens, the fact that *'proof requires the co-ordination of a range of competencies - identifying assumptions, organising logical arguments-each of which, individually, is by no means trivial'*, (p.7). When a student is able to co-ordinate these competencies then he/she can start to master both empirical and theoretical arguments.

This argument is also supported by Hanna (1995). She argues that even if students have the appropriate level of experience and understand the mathematical language involved in the activity, if they have not been prepared through their mathematics schooling to follow the reasoning of an argument, they will not be able or confident in evaluating and constructing a proof. Thus, in order for students to engage in reasoning interactively, whether in small groups or in the whole class setting, they must have been taught to follow a mathematical argument. In addition to this, Dreyfus (1999) makes use of examples from his research to demonstrate how beginning undergraduates, even if they appeared to have a satisfactory understanding of the question and the answer, were unable to give an appropriate explanation or argumentation.

The above research findings support the results of the survey carried out with Year 10 students of England and Wales within the project 'Justifying and Proving in School Mathematics' (Healy and Hoyles, 1998). They show that students are capable of conjecturing and arguing using everyday language and most of them recognise that an empirical justification is not enough. However, they do not know how to give a formal argument.

Going further, Küchemann and Hoyles (2006; 2009) in reporting some findings of the 'Longitudinal Proof Project', which studied the development of high-attaining students' mathematical reasoning over 3 years, investigated patterns in the use of empirical and structural reasoning. The findings indicated that structural reasoning increased over the years but still the use of empirical arguments was widespread.

Initially, this was interpreted as further evidence of students' lack of appreciation of the power of structural reasoning (Küchemann and Hoyles, 2009, p.188). However, Küchemann and Hoyles (2009) now offer a different perspective *'that distinguishes a more advanced use of empirical reasoning, namely to check the validity of a structural argument'*, (p.189). Thus, they suggest that *'although recourse to empirical data may in many cases indicate a naïve understanding of proof, it need not do so'*, (p.189).

Despite the difficulties students encounter when learning proof, studies also point to even young elementary students engaging in logical arguments. Stylianides and Stylianides (2008) reviewed psychological research on the development of students' ability for deductive reasoning in the context of proof. This review indicates that the students' ability for deductive reasoning follows a developmental trajectory and that forms of deductive reasoning begin to emerge in the early elementary grades (p.111). Students being able to use deductive reasoning so as to construct arguments and proofs is also evident in the mathematics education research literature.

For instance, Maher and Martino (1996), in studying one student's development of the idea of mathematical proof over a 5 year period, reveal that deductive reasoning can emerge naturally when challenged to justify one's work. Maher (2009) argues that *'in the process of convincing oneself and others of the validity of a solution, arguments are presented by young children that take the form of mathematical proof'*, (p.130).

In a similar way, Schifter (2009) argues that students can engage in the process of proof when they notice a regularity in the number system and are challenged to argue for the truth of a claim about an infinite class. By exploring episodes from Year 1 to Year 4 classes she demonstrates that reasoning from visual representations to justify general claims is possible.

Correspondingly, Mueller and Maher (2009) provide evidence of Year 6 students naturally using different types of arguments in justifying their solutions to tasks: direct reasoning, reasoning by contradiction, upper and lower bounds, and case-based reasoning (p.29).

Going further, Reid (2002) draws from a large study, the aim of which is to develop descriptions of mathematical reasoning, and describes one pattern of reasoning from the mathematical activity of grade 5 students. Reid identifies features of a pattern of reasoning that should be considered so as to argue whether a pattern of reasoning is mathematical or not: conjecturing a general rule, testing the conjecture and using that rule.

Other studies also show evidence of students at the elementary level engaging in mathematical reasoning. What is not really evident though in the primary level through these studies is proof and proving as it has been defined previously in Section 2.1.1. What may be seen is an activity that relates to proving.

This has been recognised by Stylianides (2007a; 2007b). Stylianides acknowledges the need for proof and proving to become part of the students' experiences throughout schooling. As a response to the students' difficulties with proving and to the formal definition of proving being inappropriate for primary school, he proposes a conceptualisation of the meaning of proof in school mathematics. He defines proof as a *'mathematical argument, a connected sequence of assertions intended to verify or refute a mathematical claim'* (Stylianides, 2007a, p.291). This argument has three components; the set of accepted statements, the modes of argumentation and modes of argument representation. To be more precise, an argument should use statements that are being accepted in the classroom community without further justification, be formulated and be communicated using forms of reasoning and expression that are *'known to or within the conceptual reach of the classroom community'*, (p.291).

In employing this definition of proof to analyse three classroom episodes from third grade, Stylianides identifies the base argument, the starting point as well as the ensuing argument, the argument resulting from the teacher's instruction and checks whether this argument meets the definition of proof.

Stylianides suggests that this conceptualisation is appropriate to utilise in school mathematics as it honours mathematics as a discipline and students as mathematics learners (the intellectual-honesty principle) and promotes a consistent meaning of proof throughout the grades (the continuum principle). Adopting this conceptualisation also prevents empirical arguments from being considered as proof and supports studying the role of the teacher in supporting students' deductive reasoning.

Lo and McCrory (2009) build on Stylianides's work and propose a fourth element of a mathematical argument: it is relative to objectives within the context (context dependence) which determine what needs to be proved.

This conceptualisation of the meaning of proof for school mathematics constitutes a useful aid in cultivating proof in the classroom, as it is broader than a strict definition of mathematical proof. However, this conceptualisation may prove restrictive if the focus is to understand how proof and proving is shaped by the practices in the mathematic classroom. To elaborate more, if one's aim is to understand how proving is constituted in the classroom, a wider network of ideas is required that moves beyond this relatively narrow definition to areas such as explaining and justifying as these ideas no doubt have an impact on how proof in the narrow sense is constituted. Thus, the focus should not only be on proof as the culminating stage of mathematical activity, but also on the proving process and how this is shaped by the classroom environment. This is in accordance with Herbst and Balacheff (2009) who argue that research on proof in classrooms *'needs to go beyond the description of the customary notion of proof embedded in a class's mathematical work'*, (p.49).

Furthermore, considering the different roles and meanings of proof, *'it is reasonable to question how proof can occur as a connecting theme across the grades as students' understanding of proof emerges'* (Stylianou et al, 2009) p.8). Nonetheless, the continuity in how the notion of proof is conceptualised across the grades may be realised by promoting an appreciation for *'building reasonable, logical arguments while using mathematical tools'* that are within the reach of the students (Stylianou et al, 2009, p.8). That is, by developing an appreciation for and understanding of mathematical ways of knowing and thinking as well as understanding what constitutes sufficient evidence, the students will gain the experience with proof necessary to move towards deductive reasoning.

To conclude, even though students in the elementary level may not engage very often or very deeply in proving in the formal sense (if at all), they engage in activity that is connected to proving. As this study is interested in the constitution of proving in the primary classroom, it needs to go beyond Stylianides' definition. In order to study how proving emerges during elementary level, I need to review a range of psychological and sociocultural factors which lie at the root of students' conceptions of proving.

2.3.1 The emergence of students' meanings for proof

It has been illustrated in Section 2.2. that students view empirical evidence as proof and mathematical proof simply as evidence. This can be attributed to several factors; identifying types of proofs and proof schemes, as well as the institutional and instructional constraints that impose particular demands on the existence of proof in classrooms contributes to a more complete understanding of the student's meanings for proof.

2.3.1.1 Psychological Factors

From a psychological perspective, research examining students' conceptions of proof has led to the identification of compatible classifications of students' understanding of proof related to their reasoning ability and the construction of formal proof.

Balacheff (1988b) examined how students become convinced about the validity of a proposed solution and provided a hierarchy of justifications *'which hold a privileged position in the cognitive development of proof'*, (p.218). He identified two types of proofs. Pragmatic proofs are proofs based on real actions or representations of mathematical objects. Conceptual proofs are proofs that are detached from actions and *'rest on formulations of the properties in question and relations between them'* (p.217). The movement from empirical to deductive arguments involves a shift in the language used and a differentiation between the objects and relations involved. He distinguishes different levels of proofs that constitute the foundation for the development of a formal mathematical proof. The first type is naïve empiricism. It consists of asserting the truth of a statement by observing only a small number of cases. The second type is the crucial experiment. The problem of the generality of a supported conjecture is explicitly posed. However, the assertion is recognised as proof by its producers. Validation is established through the examination of a particular case which is considered the least particular. The generic example concerns making explicit reasons for the validity of a statement by means of operations or transformations on an object which is presented not as a particular case but as a characteristic representative of a class of objects (p.219). The thought experiment *'invokes action by internalising it and detaching itself from a particular representation'*, (p.219).

In a similar way, Harel and Sowder (1998) and Sowder and Harel (1998), by giving priority to the function of proof as a convincing argument, reported three broad categories of students' proof schemes, a classification that is not based on a priori determination, but constitutes *'the individual's scheme of doubts, truths and convictions in a given social context'* (Harel and Sowder, 1998, p.245). These categories include the external conviction proof schemes where what convinces the student and what the student offers to persuade others relies on: an outside source; the empirical based proof schemes which consists of justifications made with the examples on the basis of physical evidence or experience; and the analytical/deductive proof schemes where the validation of a conjecture is attained by means of logical deductions (Harel, 2007).

Going further, as the secondary mathematical focus of this study is geometry, the epistemological perspective of geometry as well as the psychological characteristics that underpin the learning and teaching of geometry also need to be taken into consideration, in order to establish the psychological factors that impact on the emergence of students' meaning of proof.

Geometry is characterized by a duality; it is *'a theoretical domain and an area of practical experience'* (Fujita and Jones, 2003, p.47). As an activity it encompasses formal features focusing on logical relations, definitions and proofs of geometrical concepts with no direct connection with spatial experience (Laborde et al, 2006) and at the same time the properties and relationships between points, lines, planes and figures that are taught or even memorized at school are part of someone's everyday experience. That is, beyond its abstraction, geometry can help people understand *'the space around them, the structures of the buildings, the bridges, the cranes, furniture and the thousands of machines and mechanisms that influence their daily lives'* (The Royal Society, 2001, p.28).

Laborde (1995) in a similar way, considering the graphical representations of geometrical objects, draws a distinction between their 'theoretical referent', that is the 'figure' and the 'material entity', the 'drawing' (p.37). Pratt and Ainley (1997) provide certain examples to clarify the duality figure-drawing,

> A drawing incorporates many relations which are to be disregarded when considering the corresponding figure. For example, the drawing of a line contains thickness; the drawing of a tangent to a circle intersects the circle in a line segment. In contrast, the line as a figure is an ideal, which cannot be represented in reality as it has no thickness; the figure for a tangent to a circle meets the circle at a point, which has position but no dimension. Furthermore, a drawing is fixed as a single case, whereas the figure is often intended to represent an infinite set of cases (p.296)

The fact that the nature of geometry embodies a theoretical/empirical duality has an effect on how geometrical objects are represented. Fischbein (1993) reflecting on this issue considers geometrical concepts as "*a mixture of two independent, defined entities that is abstract ideas (concepts), on one hand, and sensory representations reflecting some concrete operations, on the other*", (p.140). According to the theory of figural concepts, geometrical figures are "*mental entities which possess simultaneously conceptual and figural properties*", (p.160). For instance, a square is a figure that reveals logical relationships that can be derived by its definition, and is an image too. Consequently, the figural and conceptual aspects of mental objects interact with each other during geometrical reasoning. He also argues that figural concepts do not develop naturally and that "*the process of building figural concepts in the students' mind should not be considered a spontaneous effect of usual geometry courses*", (p.156). That is, deliberate teaching is needed for the fusion between the visual image and the properties of the figural concept.

Employing Fischbein's theory of figural concepts, Fujita et al (2004) argue that for successful reasoning in geometry, students must be given the opportunity to exercise skills in "*creating and manipulating geometrical figures in the mind, perceiving geometrical properties, relating images to concepts and theorems in geometry and deciding where and how to start when solving problems in geometry*", (p.5). For this to be achieved, students must have the chance to manipulate and get engaged with many different representations of several geometrical figures. By moving gradually through different stages of representation they can also learn the interrelationships among ideas and link these to their own informal knowledge and strategies, and they develop and extend their reasoning and manipulation skills. Fujita et al (2004) call the skill "*to imagine, create and manipulate geometrical figures in the mind when solving problems in geometry*", geometrical intuition.

Additionally, Mariotti (1994) also used Fischbein's theory and investigated the dialectic interplay between the two components of figural concepts. She concluded that, while geometrical reasoning results in the interplay between "*observing the object as it appears*" and "*relating to the properties which characterise the geometrical figure*", (p.234) in practice, there is a conflict between the figural and the conceptual. For example, even though students are aware of the definition of a parallelogram, they may nevertheless find it difficult to recognise the various shapes that are compatible with that definition. An oblique parallelogram, a square and a rectangle are figurally different and make the similarities within the classification disappear (Pratt and Davison, 2003, p.32). Thus, difficulties and errors in geometric reasoning can be interpreted in terms of a 'rupture between the two components, whereas successful reasoning can be interpreted as a good harmony between them' (Mariotti, 1997).

2.3.1.1. Sociocultural factors

From a sociocultural perspective, the emergence of student's meanings for proof is also influenced by elements of the practices in the mathematics classroom. Thus, research on proof in the mathematics classroom should also be directed towards understanding the role of culture in shaping the proof and proving practices of the classroom.

Yackel and Cobb (1996), in developing a framework that enables the understanding of the social beliefs and practices, describe the expectations and obligations that are constituted in the classroom as classroom social norms. These normative aspects of the classroom may be applied to any subject area. Concerning normative understandings or interpretations of mathematical discussions that are specific to mathematics and students' mathematical activity, they identify sociomathematical norms. They clarify this distinction between social and sociomathematical norms by providing the following example:

> The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm. Likewise, the understanding that when discussing a problem students should offer solutions different from those already contributed is a social norm, whereas the understanding of what constitutes mathematical difference is a sociomathematical norm (Yackel and Cobb, 1996, p.461).

Examples of what constitutes a sociomathematical norm are the normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient and mathematically elegant in the classroom, as well as what counts as an acceptable explanation and justification. Regarding the norms related to justification and explanation, '*these include that students explain and justify their thinking, that they listen to and attempt to make sense of the explanations of others, and that explanations describe actions on objects that are experientially real for them*' (Yackel, 2001, p.17). The concepts of the social and sociomathematical norms have been utilised in a comprehensive way in studies focusing on classroom interactions, collaborative student pairs, from the kindergarten to the university level.

Kazemi and Stipek (2001) examined the upper-elementary mathematics classroom practices that focus on the deeper understanding of mathematical ideas, relations and concepts. Through their analysis they identified the following sociomathemtical norms: an explanation consists of a mathematical argument, not simply a procedural description; mathematical thinking involves understanding relations among multiple strategies; errors provide opportunities to reconceptualize a problem, explore contradictions in solutions, and pursue alternative strategies; and that collaborative work involves individual accountability and reaching consensus through mathematical argumentation.

Stylianou and Blanton (2002) analysed teaching episodes in order to document how the sociomathematical norms of explanation and justification are constituted in undergraduate mathematics. Their findings indicate that students' interactions developed in accordance to this normative understanding. The passive acceptance of the instructor's authority developed towards the expectation that students become active contributors to the class and all share common understandings. What is more, their arguments gradually shifted from empirical and procedural to deductive and conceptual.

Concerning the sociomathematical norm of what counts as an acceptable explanation and justification, the desirable scenario is for all students to share the necessary rules of discourse that would lead them to agree that an argument is a proof. However, there might be students who use different and sometimes conflicting rules of discourse about deriving new truth from existing ones. Once again, the role of the teacher appears important.

Despite the importance of establishing mathematical norms in the classroom, Küchemann and Hoyles (2006) argue that these norms might have a negative impact in motivating proving. For instance, students who learn the sociomathematical norms expected of written explanations using geometrical knowledge they have recently been taught may reject their intuitive spatial reasoning when manipulating shape and space. Having in mind this, they might try to find 'deeper reasons' instead, for example, of using a counter example (p.604).

The way the sociomathematical norms may influence the students' ability to construct proofs in geometry has also been explored by Martin and McCrone (2003). By focusing on the teachers' pedagogical choices (for example demonstrating and describing the model of the needed proof strategies before setting students to work) and the norms established due to these choices (for example the students are not responsible for producing proofs other than those that fit the format described by the teacher), they illustrate how norms may or may not have an educational value on the learning of mathematical proofs (p.30).

Consideration of all the above implies the need for providing students with opportunities to experience, make explicit and develop their geometrical reasoning and understanding. Such occasions can be created when a student is called to convince a classmate of a guess or conjecture during a collaborative phase, when a student asks for help, when a teacher wants to gain an insight about students' thinking in order to help them, assess their progress or even when he/she attempts to move them from a descriptive to justificative mode of thinking about what they are doing. Martin et al (2005) argue that "*within such an environment, actively* *participating students and teachers can contribute to the negotiation of classroom mathematical practices as well as to the development of individual ability to construct formal proof*', (p.121).

2.4. The roots of proving

As stated in the preceding paragraphs, it is the objective of this study to investigate proving as constituted in the primary classroom. That is, this study will attempt to understand the nature and role of proof in the mathematics classroom, by focusing on proof as constituted through the class' performances. To achieve this, it has been argued that Stylianides' (2007a; 2007b) conceptualisation of the meaning of proof is not broad enough so as to capture what might not look exactly like proving at a given moment in time, but might be something important for the emergence of proving at a later stage. In order to achieve this, I now refer to pre-proving, that aspect of mathematical reasoning that might nurture proving.

A metaphor might help to explain the notion of pre-proving. Consider pre-algebra, which is well established in the mathematics education literature. Students typically engage with pre-algebra in upper primary and lower secondary school, a preparation course before the beginning of formal algebra. Linchevski (1995) states that *'the role of pre-algebra is to develop the more primitive, concrete preconcepts that are necessary for the development of the higher, more abstract concepts'* (p.114). Pre-algebra is considered a stage of transition from the environment of arithmetic to that of formal algebra. Considering the fact that pre-algebra is mostly considered as generalising, it can be argued that what generalising is for algebra, mathematical reasoning is for proving. Not all generalising is of an algebraic nature; those aspects of generalising that are in fact algebraic in nature are often referred to as pre-algebra (Amit and Neria, 2008). Similarly, though not all mathematical reasoning will lead to or be directed towards proving, those aspects which make up pre-proving, have a particular role for being the roots or potential roots of proving.

Keeping in mind the above metaphor, this section proceeds with the identification of the roots of proving; the aspects of mathematical reasoning that may facilitate the transition from inductive/empirical reasoning toward deductive reasoning and toward a greater level of generality.

2.4.1 Explanation, justification, argumentation and definition

Mathematical reasoning is used in a number of different ways in mathematics education research. Thompson (1996) defines mathematical reasoning as *'purposeful inference, deduction, induction, and association in the areas of quantity and structure',* (p.267). Keeping this in mind, Steen (1999) argues that this phrase may either denote *'the distinctively mathematical methodology of axiomatic reasoning, logical deduction and formal inference'* or signal a *'much broader quantitative and geometric craft that blends analysis and intuition with reasoning and inference, both rigorous and suggestive',* (p.270). Ball and Bass, (2003) define mathematical reasoning as a set of practices and norms that are collective, not merely individual or idiosyncratic, and that are rooted in the discipline. According to them mathematical reasoning entails two processes: (i) the reasoning of inquiry where mathematical reasoning can serve as an instrument of inquiry in discovering and exploring new ideas, and (ii) the reasoning of justification where mathematical reasoning functions centrally in justifying or proving mathematical claims. Aspects of mathematical reasoning, as suggested in the literature include justifying and generalizing which are considered key practices involved in mathematical reasoning, with symbolizing, representing and communicating the key practices that support them (Ball and Bass, 2003).

Yackel and Hana (2003) extend the definition of mathematical reasoning by recognizing its social aspects and argue that two aspects of reasoning that at a later stage result in a formal mathematical proof are the notions of justification and argumentation. They claim that argumentation, explanation and justification provide a foundation for further work on developing deductive reasoning and the transition to a more formal mathematical study in which proof and proving are central.

To integrate aspects of proof across all levels of schooling, the notions of explanation, justification and argumentation must be defined. To be more accurate, while we accept these notions as important aspects of proof, we must also be able to explain how the features of argumentation, explanation and justification can be judged as valid and that they constitute proof.

For the purpose of this research study I consider mathematical explanation an act of communication, the purpose of which is to clarify aspects of one's mathematical thinking that might not be apparent to others (Yackel and Cobb, 1996). Hanna (1995) gives examples of the different forms of this explanatory function of proof; it can either be presented as a calculation, a visual demonstration, a guided discussion observing proper rules of argumentation, a preformal proof, an informal proof, or a proof that conforms to strict norms of rigour.

Justification is *'the discourse of an individual who aims to establish for somebody else the validity of a statement'* (Balacheff, 1988a, p.2). The validity of this justification is originally connected with the person that expresses it. Justification can be considered as *'explaining, arguing, corroborating, verifying a particular statement'* (Mariotti, 2007, p.288).

Bell (1976) identified two categories of justifications students use in problems related with proof. Each category constitutes a variety of answers, which according to Bell correspond to a hierarchy of completeness. Empirical justification relies on the use of examples as the elements that contribute to the drawing of the conclusion. Deductive justification includes responses with a deductive element in order to connect data with conclusions.

The perspective on the relationship between explanation, justification, argumentation and proof, taken on board by this research, by taking into account both their meanings as perceived by the author, as well as illustrated in the research literature is as follows: mathematical argumentation is a discursive activity (written or oral) based on arguments, reasoning that supports or disproves something. It includes the argumentative and proving process; the exploration process, the formulation of hypotheses and conjectures, explaining and justifying the steps followed towards the outcome and the proof of the statement. As stated before, either distance or unity might exist between the argumentative and proving discourse. However, as argumentation and proof are part of a continuum, proof is at the core of mathematical argumentation, both a justification and explanation and a valid argument.

Given that proof is both a justification and an explanation, it can be argued that emphasis should be placed in these two aspects of mathematical reasoning. A consideration of the functions of proof that are considered important for school mathematics, the notion of sociomathematical norms as well as the above discussion regarding explanation and justification, strengthens the assertion that explanation, justification and communication constitute these elements of mathematical reasoning and argumentation that might foster the development of the students' ability for deductive reasoning. That is, describing, conveying and exchanging ideas through the act of communication, explaining and justifying statements influences the appearance of proof and the transition from unsophisticated empirical arguments to the level of sophistication that might be expected at the tertiary level.

In addition to the above, Ball and Bass (2003) also stress the crucial role of definition and terminology in mathematical reasoning. They argue that definitions *'originate in and emerge from new ideas and concepts and develop through active investigation and reflection'*, as well as facilitate reasoning about *'those new ideas by naming and specification',* (p.33). Can it be argued that mathematical definitions and defining constitute possible roots of proving? The subsequent part of this subsection focuses on the key role definitions play in mathematics. Emphasis is given on the interplay between defining and proving. It is acknowledged that the following review of the literature regarding mathematical definitions is not exhaustive. Despite this, it provides the reader with a comprehensive understanding of the issues pertaining to the successful use of definitions in the classroom setting.

In discussing definitions in mathematics, mathematicians and mathematics educators address, initially, the distinction between ordinary or dictionary definitions (descriptive, extracted or synthetic definitions) and mathematical definitions (stipulated or analytic definitions). De Villiers (1998), drawing on Freudenthal (1973), elaborated the activity of defining by making a distinction between descriptive defining which *'outlines a known object by singling out a few characteristic properties' and constructive defining which 'models new objects out of familiar ones'* (Freudenthal, 1973, p.458). In a similar way, Edwards and Ward (2004; 2008) explore the work of philosophers and lexicographers and adopt the terms 'extracted' and 'stipulated' definitions. Extracted definitions are definitions extracted from a body of evidence (Edwards and Ward, 2004, p.412). They are based on examples of actual usage whereas a stipulated definition is *'the explicit and self-conscious setting up of the meaning-relation between some word and some object, the act of assigning an object to a name (or a name to an object)'*, (Robinson, 1962, p.59).

In describing the way students use definitions in mathematics, Tall and Vinner (1981) introduced the terms concept image and concept definition. The concept image is a nonverbal representation of an individual's understanding of a concept. It includes the visual representations, the mental pictures, the impressions and the experiences associated with the concept name. The term concept definition refers to a form of words used to specify a concept and can be personal or formal (Harel et al, 2006). The concept definition can be the stipulated definition assigned to a given concept.

In addition to distinguishing types of defining in actively engaging students in defining mathematical concepts, the requirements for mathematical definitions valued by mathematicians are also being described in the mathematics education research literature. That is, a mathematical definition should be unambiguous, non-contradictory, hierarchical, as well as invariant under changes of representation. Also, mathematical definitions should have precision in terminology and be easily comprehended by students (Morgan, 2005; 2006, Harel et al, 2006).

In discussing these characteristics of mathematical definitions, Borasi (1992) identifies two functions that definitions must fulfil. According to her, a definition of a given mathematical concept should (i) allow us to discriminate between instances and noninstances of the concept with certainty, consistency, and efficiency (by simply checking whether a potential candidate satisfies all the properties stated in the definition) and (ii) 'capture' and synthesize the mathematical essence of the concept (all the properties belonging to the concept should be logically derivable from those included in its definition), (p.17-18).

By considering the aforementioned, Morgan (2005) concludes that the use of definitions in mathematics includes the following characteristics:

> There exists a possibility of conflict with intuitive images of the concept being defined, especially with images formed by generalising from examples;

> definitions form a generative basis for logical deduction, not only of known properties of the concept but of new properties; definitions may be created deliberately in particular forms in order to facilitate the construction of theorems and proofs; a single object may be defined in several logically but not conceptually equivalent ways and such alternative definitions facilitate the generation of different types of theorems, proofs and solution methods (p.108).

Keeping in mind the characteristics and functions of definitions, it is concluded that mathematical definitions may function in several ways for learners of mathematics. Morgan (2006) highlights the idea of choice and purposeful formulation of definitions. She argues that this active role in relation to definition should not only be a characteristic of the ways mathematicians use definitions but also stressed at the school level. These remarks point towards using the term 'defining' as a mathematical activity rather than definitions to stress the role of defining in student progress from informal to more formal ways of reasoning (Mariotti and Fischbein, 1997, Zandieh and Rasmussen, 2010). Therefore, defining activity includes formulating a definition, negotiating what one wants a definition to be (and why), and refining or revising a definition that can occur as students are proving a statement, generating conjectures, creating examples, and trying out or 'proving' a definition (Zandieh and Rasmussen, 2010, p.59). Thus, the construction of definitions as a mathematical activity should be considered as important a process as the aforementioned mathematical activities and be part of the mathematics teaching (De Villiers, 1998).

A consideration of the above provides an indication of the dialectical interplay between concept formation, definition construction, and proof. This is further explored in the mathematics research literature (Harel et al, 2006). Ouvrier-Buffet (2004) in engaging university students in definition construction considered zero-definitions, tentative or 'working definitions' emerging at the start of an investigation and proof-generated definitions, directly linked to problem situations and attempts at proof. Maher (2002) also provides evidence of students' mathematical activities leading to both descriptive and constructive definitions, in situations where justification and proof were expected. Larsen and Zandieh (2005), in developing a framework for making sense of the role of proving in students' defining activity, made a categorization of the ways proving activity contributes to defining activity: proof as motivation for defining; proof as a guide for defining; and proof as a way to assess defining.

Nevertheless, studies also show that students do not necessarily understand the content of relevant definitions or how to use these definitions in proof (Moore, 1994; Healy and Hoyles, 2000). That is, even though students undoubtedly encounter definitions throughout their mathematical practices, the difficulties identified at the secondary and tertiary level imply that their earlier experiences may not provide a basis for using definitions in ways that go beyond the development of concepts. To elaborate more, a mathematically incorrect way to apply a definition may be due to an incomplete or faulty understanding of the concept that is to be defined or due to a mathematically incorrect understanding of the role of mathematical definitions in general (Edwards and Ward, 2004, p.414). Much of students' difficulty in using definitions in proofs lies in the fact that the students' concept images are not well connected to the concept definition (Tall and Vinner, 1981). A consideration of the above indicates that definitions need to become operable for an individual (Selden, 2012). According to Bills and Tall (1998) a definition is formally operable for a student if the student *'is able to use it in creating or (meaningfully) reproducing a formal argument [proof]'*, (p.104).

By considering the way mathematical definitions were used by the undergraduate students in their studies, Edwards and Ward (2004) further discuss the implications of these findings for teaching. To begin with, they argue that the special nature of mathematical definitions should be treated as a concept in its own right. Furthermore, they argue that introducing the concept image/concept definition terminology in a course may provide both the instructor and the students with a framework that helps in better understanding the struggles faced by the required shift to the concept definition-based logical reasoning. The above should also be taken into account in the mathematical preparation of future teachers (Edwards and Ward, 2004; Stylianides et al, 2013). Regarding the professional development of mathematics teachers, Leikin and Winicki (2001) further argue that engaging teachers in discussing 'what is a definition' and 'how to define' allows focusing on logical relationships between mathematical statements, didactical sequences of learning, mathematical connections, and mathematical communications (p.63). While the aforementioned studies focused on secondary and university students, Keiser (2000) also points to activities that engage elementary school students in forming and critically evaluating their own definitions. A consideration of the above discussion highlights the role of the teacher. Nevertheless, the role of the teacher will be further discussed in Section 5.6.

The above review of the literature indicates that defining may also be considered as a potential root for proving. That is, engaging in defining activity may have the qualities of proving. To be more elaborative, formulating a definition entails a process that focuses on logical relationships between mathematical statements, explaining and justifying. These elements have been identified as potential roots of proving. At the same time, it has been stated previously that proving activity may lead to defining activity. Thus, proving and defining activity are interrelated. Additionally, an activity involving concept image and concept definition can be related with proving. That is, engaging students into an activity that relates to the functions and characteristics of proving may lead to a fusion of the concept image and concept definition related to the notion of mathematical proof.

Though, it is through exploration and investigation that all these aforementioned interrelated elements surface and develop in the process of proving. Thus, when discussing the roots of proving, exploration, which activates intuition and encourages thinking, constitutes another notion that should be taken into consideration. Considerable research studies have considered the importance of integrating exploration in proving. However, it is also being acknowledged that these studies have not explicitly defined 'exploration'. For the purposes of this study, I adapt the unification of the two different positions of interpreting exploration as proposed by Hsieh et al (2012). That is, exploration as a mental process and, exploration as *'an activity that involves manipulation of and interaction with external environments'*, (p.282).

2.4.2 The notion of pre-proving

In the light of the above discussion, I now turn to the roots of proving. Exploration, experimentation, conjecturing and hypothesising are important aspects of mathematical reasoning. However, mathematical reasoning is not always directed to proving. Yet, it is through explanation, justification and communication that particular ways of acting may be considered proofs or potential proofs. Thus, pre-proving refers to those elements that direct mathematical reasoning towards the ultimate goal of formal proving; that is exploration, explanation, justification, definition and communication. Rather than a definition as such, pre-proving as stated here is more of an orientation towards student activity, that will later steer my analysis, focusing on those aspects of reasoning that appear to have the qualities of proving, even though they may not be proving in themselves. I expect that the nature of pre-proving will become more evident during the analysis and discussion so that a clearer definition should emerge from my study.

At this point, another remark needs to be made. While a metaphor has been provided to clarify the notion of pre-proving, in order for the reader to gain a better understanding of the way the notion of pre-proving is utilised in this study, a distinction needs to be made between this notion and the terminology employed in the mathematics education research literature that relates proof and proving with the level of formality of proof. That is, pre-proving should not be considered as a pre-formal proof which refers to *'a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises'* (Blum and Kirsch, 1991, p.187). According to Reid (2001) pre-formal proofs, as a possible step to semi-formal proofs, appear in working notes and conversations, and may encompass hidden assumptions, analogies, and informal language and notations. Going further, the notion of pre-proving should not be mistaken with the territory before proof, a metaphor introduced by Edwards (1997) to describe the space of potential intellectual precursors to proof (p.189). While Edwards refers to noticing, constructing and describing patterns, conjecturing, inductive and deductive reasoning as the elements of reasoning activities that can be located 'before' formal proof, in this study, the notion of pre-proving does not aim at interpreting students' behaviour when working with proof. While acknowledging that the aforementioned elements support the goal of establishing certainty, the notion of pre-proving activity also refers to all other aspects of mathematical activity.

2.5. Summary and purpose of this study

It is now acknowledged that proof and proving should become part of students' experiences throughout their schooling. However, secondary school students as well as undergraduate students face difficulties when giving formal mathematical arguments. At the same time, research that addresses proof at different grade levels and shows how upper primary school students approach and construct proofs is still limited. It is also argued that argumentation, explanation and justification provide a foundation for further work on developing deductive reasoning and the transition to a more formal mathematical study in which proof and proving are central. Stylianides (2007a) proposes a conceptualisation of the meaning of proof for school mathematics. However, in order to explore proof in the context of the day-to-day practice of a mathematics class, where proof might not even be the main object of the teaching, this conceptualisation needs to be expanded. It has been argued that to understand how proving is constituted in the classroom, one needs not only to capture instances of generation of general statements that request proofs of their truths, but also instances of the generation of ideas that have the potential at some point to inform the customary notion of proof. This is what, for the purposes of this study, is described as pre-proving. Thus, in exploring how proving is constituted in the primary classroom, the role of exploring, explaining, defining, justifying and communicating in being the roots or potential roots of proving also need to be established.

In the social environment of the classroom, the tools and tasks used, the rules of the classroom, the way the students work together, the way the teacher negotiates meanings and other external factors that have an impact on what is going on in the classroom, all interact, interrelate and influence each other.

It is the aim of this research to explore how all the aforementioned components of the classroom environment are developed and transformed so as to understand the way proving is constituted in the mathematics classroom. By doing so, the study will be able to unfold and negotiate the elements that influence pre-proving and proving activity.

Thus, it is the purpose of this research to explore pre-proving and proving in the elementary mathematics classroom and the way the structural resources of the classroom's surrounding setting shape this process.

The main research question of this study is:

How is the activity of proving being constituted in the classroom for upper primary school students?

Keeping in mind the main objective of this study, a theoretical approach that facilitates investigating how the various aspects and culture of the classroom might promote and constitute proving is required. The following chapter focuses on the two preliminary studies undertaken prior to the main study, in order explore the aforementioned ideas and focus my thinking about the research. How reflection on these studies directed the research in considering Cultural-Historical Activity Theory as the most appropriate tool that offers the means for handling this complexity in coming to understand how proving might be constituted in the classroom will follow.

CHAPTER III

A REVIEW OF THE PRELIMINARY STUDIES

3.1. Introduction

This chapter introduces and reviews the two exploratory studies conducted prior to the main study. It also discusses their impact on the final research design. To be more precise, each section begins with a description of the study setting. This is followed by a description of the data collection, analysis and the study findings. How reflecting on the exploratory studies informed the research design of the main study will follow.

3.2. Exploratory Study I

As discussed in Chapter I, the initial objective of this research study was to explore students' thinking in change. To be more accurate, I was interested in investigating upper primary school students' mathematical argumentation. Keeping in mind the affordances provided by Dynamic Geometry Environments, the study aimed at exploring how students explain and justify geometric conjectures experienced through a DGE, as well as how the features of the task shape the process of formulating a conjecture and proving it. The research methodology considered most appropriate to follow was a design research approach (Edelson, 2002). This research methodology is placed within the paradigm of experimental design (Cobb et al, 2003). This approach is characterised by research the aim of which is to 'test and refine educational designs based on theoretical principles derived from prior research' (Collins et al, 2004). It also provides the opportunity of developing new and deeper questions, leading this way into deeper understanding. This meant that the research would be comprised of cycles of research. This exploratory study, a first cycle of research, was used as a starting point, the background on which the subsequent cycles would draw upon. Thus, the exploratory study aimed at locating the pupils in the literature as well as understanding what they already know and what is difficult for them. What is more, it would identify what mathematics is for students in order to explore at a later stage whether this changes and how. A consideration of the aforementioned led to a decision to use group work (pairs of students) as a means for students to articulate ideas, due to the dynamic feedback inherent in the discussion (Hoyles, 1985).

3.2.1 The study setting

The study was conducted in a public primary school in Nicosia, the capital of Cyprus. The participants were 4 pairs of Year 6 students (12 years old) from the same classroom. The students participating in this study were the students whose parents agreed and signed the consent and authorisation form. According to the teacher, the students were of mixed abilities. Furthermore, the students had never used a Dynamic Geometry Environment before. The students were observed outside the classroom setting - the computer lab. At this point it should be elucidated that although the teacher instructed the students to go with me, I clarified to students that at any point they were free to withdraw and return to their regular lesson.

The data were collected through video recordings and field notes. To be more elaborative, the participants were video recorded while exploring two DGE-based tasks. The video recorder was positioned in such a place that allowed the recording of the computer screen and, therefore, the students' gestures. Furthermore, I used each session (exploration of the DGE-based tasks) to gain students' feedback regarding their experience. It was considered more effective to combine this 'interviewing' with the exploration of the tasks. That is, in a 'chatting' mode, I had informal discussions with each pair of students after the exploration of each task.

3.2.2 My role as a researcher

Keeping in mind the purpose of this exploratory study, as exemplified in Section 3.2., and given the fact that I would observe and interact with pairs of students in a familiar setting yet outside the classroom, my role as a researcher pointed to one of acting as a teacher. As researcher/teacher, my role was to facilitate rather than to instruct. Several challenges and dilemmas can be encountered by being a researcher/teacher (Ainley, 1999; Wong, 1995). In order not to compromise the research design and address this perspective, decisions regarding certain strategies that would be followed had to be made.

A characteristic of the researcher acting as a teacher is the ongoing struggle in developing a context in which *'the social meanings of the involved language and actions are negotiated by the participants'* (Cobb and Steffe, 1983, p.84). Initially, it was my intention to diminish the impact of the power differential between researcher/teacher and student. As a consequence, I took small steps that would nurture a more collaborative, less obviously hierarchical milieu. I decided to place myself beside or at right-angles to students. I also used informal speech, for a moderately natural conversational tone.

Furthermore, in order to elucidate participant responses and actions, emphasis was put in initiating and asking probes. This was imperative as I decided to use no pre-printed materials (see Section 3.2.3. below). At the outset, the tasks and the questions were posed using the minimum amount of language. When considered necessary, additional information was provided to the students in order to clarify aspects of the task and/or to encourage them. I also decided to ask extended probing questions to assist students in the event of encountering difficulties. However, in the instances where the need for input was required, I would quickly withdraw after having made the intervention. As a point of clarification, it should also be noted that I deliberately tried to avoid using formal mathematical language. Nevertheless, the probing questions relevant to the DGE-based tasks used are provided in Section 3.2.3. below, where the precise details of the tasks employed are presented and elaborated on.

3.2.3 The DGE-based tasks

Keeping in mind what is stated in the available research literature concerning task design (see Section 5.5.), the DGE-based tasks employed in this exploratory study consisted of tasks that can only be solved in a dynamic geometry environment. By considering the purpose of the exploratory study, the tasks were designed in such a way so as to allow a confrontation between what is predicted and what is observed. That is, in the event where the students' hypothesis turned out to be wrong, that would be a good opportunity for asking 'Why is it so?' and calling for an explanation and justification. Additionally, the tasks introduced would also expose pupils' perceptions of the distinctions between a drawing and a geometrical figure, as well as robust constructions versus those that could be messed up.

Furthermore, the tasks designed were related to 'circle'. As the mathematics curriculum in Cyprus includes an introduction to 'circle' in Year 5 (see Section 6.2.2.2,), it was expected that the pupils would be, to an extent, familiar with this area of mathematics. It should be made explicit that I did not draw on the mathematics curriculum in designing the DGE-based tasks. However, the information provided by the official documentation was taken into consideration in attempting to anticipate students' responses to the tasks. Doing this proved supportive when asking probing questions and valuable in maintaining focus. The proposed tasks were piloted with two Year 6 primary school teachers.

The DGE-based tasks presented and described below derive originally from my own teaching experience and a variety of sources in the available research and pedagogical literature. The above remark also applies for the DGE-based tasks employed in Exploratory Study II (see Section 3.3.3.).

DGE-based Task 1: Angles in the same segment

This task is concerned with a circle theorem: the angle at the center is twice the angle at the circumference for angles which are subtended from the same arc. Since it was expected that students would be familiar with circle, it was decided to design tasks that would give them the opportunity to explore circle theorems. The rationale for designing this task was for the students to have the opportunity to investigate the circle theorem by determining all possible configurations of A, B, C and D such that the condition <ADC =2 <ABC holds. Figure 3.1a. below shows a screenshot of the DGE-based task. In Figure 3.1a. the circle is hidden in the task and it is expected to emerge as a result of the students finding points that satisfy the condition.

To begin with, the students could freely drag the points, and make inferences concerning the relationship between the segments. The students were then asked to investigate the theorem. The students could make inferences regarding what shape is emerging and check their predictions. That is, they could use a marker to mark point B on the screen (a transparent slide). Figure 3.1b. below shows a screenshot of the

DGE-based task, where the dashed circle emerges during the students' investigation. The students were then asked to describe the curve. Even though it is acknowledged that asking students to hypothesise and thus give an explanation regarding why they got this shape is difficult, an attempt was made. Following this, the students were asked to change the length of the distance between the points A and C.

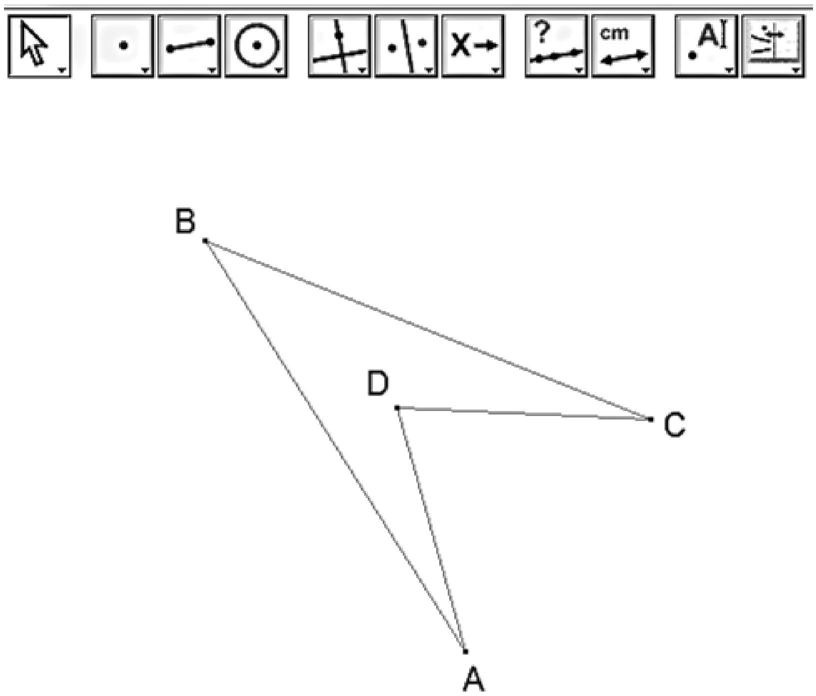


Figure 3.1a.: Angles in the same segment

As pointed out in Section 3.2.2., my role as a researcher/teacher was to ask prompt questions and facilitate the students while exploring the DGE-based tasks rather than to instruct. The prompt questions used while the pairs were interacting with the Dynamic Geometry Environment and the tasks were: What do you observe? What if you drag the other point? Use the 'Trace' command. Which curve do you think is being constructed? Can you identify some properties of the constructed curve? Why do you get this shape?

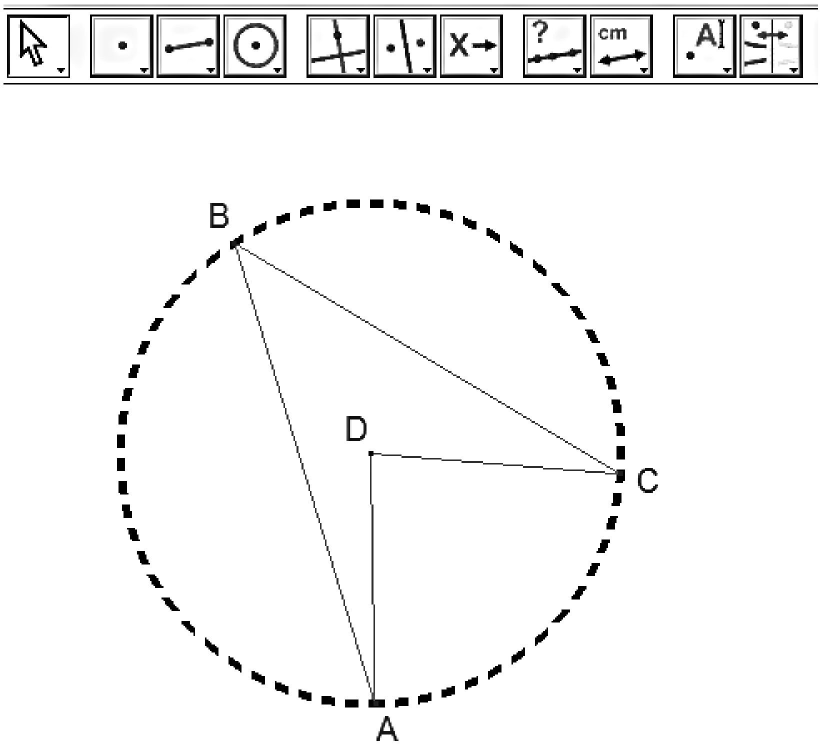


Figure 3.1b.: Angles in the same segment

DGE-based Task 2: Ellipse

The rationale for designing this task was for the students to have the opportunity to explore one method for constructing an ellipse. Through the construction, students would have the opportunity to discover that the sum of the distances from a point on an ellipse to its foci is always constant (one definition of an ellipse is as the set of points in a plane such that the sum of the distances from the two fixed points (foci) in that plane is constant). Thus, this task could initiate the formation of hypotheses and mathematical argumentation.

To begin with, the students could freely move point A (see Figure 3.2.), and make inferences concerning the relationship between the segment and the diagram. The students were then asked by keeping the sum of the two segments stable to drag point A and make inferences regarding what is being shaped. Following this, the students had the opportunity to check their predictions. That is, they could use a marker to mark point A on the screen (a transparent slide). The students were then asked to describe the curve and explain why they got this shape. Following this, the students were asked to change the length of the distance between the points B and G and follow the same procedure as before.

Regarding naming the curve, it was expected that if the students did not know the name of the curve (for example the ellipse), that they would say what it reminded them (again for the ellipse it might remind them of a closed curve resembling a flattened circle, the shadow of a circle tilted towards the light).

Figure 3.2.below shows a screenshot of the DGE-bases Task 2.

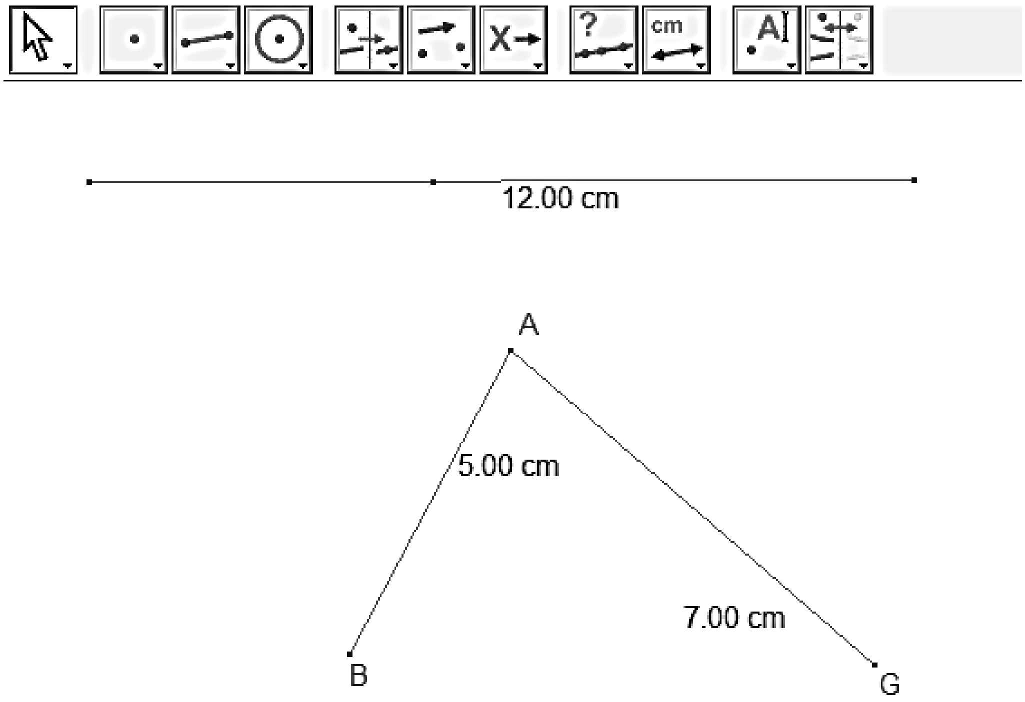


Figure 3.2.: Ellipse

As with DGE-based Task 1, my role as a researcher/teacher was to prompt and facilitate rather than to instruct. The questions posed while the pairs were interacting with the Dynamic Geometry Environment and the task were of the following nature: What do you observe? What if you drag the other point? Use the 'Trace' command. Which curve do you think is being constructed? Can you identify some properties of the constructed curve? Can you explain why you get this shape?

3.2.4 Data analysis

Initially, before proceeding with the data analysis, it should be clarified that it has been decided not to give a detailed analysis as the impact of these exploratory studies was to influence my general direction to subsequent work rather than to seek to elaborate specific research questions. (In fact I do not refer to these studies as 'pilots' because of their preliminary nature, directed towards orienting myself to the issues, rather than testing out a proposed method). Hence, only a summary is provided below so that the reader can understand the general essence of the development of the thesis at the outset.

To begin with, it was noticed that during the initial exploration of the DGE-based Task 1, students' exploration could be characterized as conservative. That is, the students were reluctant to experiment. In fact, three pairs were waiting for each other to take the initiative to start exploring the environment. This can be partially justified as the students were not familiar with dynamic geometry environments. This observation where the students' work at the beginning may have not been that productive may also be comprehended by taking into account that proposing students to investigate some construction in a very open way, without an explicit goal, may generate some insecurity (Ponte, 2007). Nevertheless, the students gradually felt more comfortable in exploring the environment, and therefore the task.

For DGE-based Task 1, the exploration of the task began by measuring the angles and finding the numerical values that satisfied the condition. Following this, the students would either drag points A and C, or B. In the first case, they would get random points that did not generate a pattern. When point B was dragged, and the points were marked on the screen, a recognizable pattern would emerge. All pairs of students concluded that when moving point B and the condition holds, a circle is constructed. In this circle, point D is the center of the circle. All participants also concluded that changing the distance between points A or C and repeating their focused dragging of point B, the size of the circle would also change.

The following episode shows a pair of students exploring the circle theorem, after concluding that in a circle the angle at the center is twice the angle at the circumference for angles which stand on the same arc.

An illustrative episode concerning DGE-based Task 1

S1 moves point A so that AB is the diameter.

S1: Now we have the diameter.

S2: Yes.

R: Yes. *(S2 moves point B so that the condition holds).*

S2: The angle is half.

S1: And a right-angle triangle is constructed.

S2: But not always.

R: How many degrees is the angle?

S2: 90°.

S1: It will always be a right-angle triangle.

S2: The angle must … we need to draw a diameter, find the angle of the diameter…

S1: It will be 180°. The diameter is like a line. It will always be 180°.

S2: No, not always, it depends on how big the circle is.

R: You can check your hypothesis.

In this episode students' dragging led them to a special case of the inscribed angle theorem. That is, the students are exploring Thale's theorem: an angle inscribed in a semicircle is always a right angle. This episode is an indication of the students engaging in a process of explanation and justification. That is, S1 was responsive to the theorem where the diameter subtends a right angle to any point on a circle's circumference and attempts to explain her thinking to her classmate and justify this assertion. S2 engages in this discussion and as he is not convinced; he tries to articulate why he believes this is not the case. He seems to be responsive to his classmate's justification but does not fully accept the generalizability of this assertion. Both students agreed to further test the hypothesis made by S2. In the end, both students decided that this statement holds. What seems interesting in this episode is the fact the students did not feel threatened by the presence of the researcher but felt comfortable in exchanging ideas even though their suggestions may prove to be inadequate.

Concerning the second task, the students' initial 'predictions' were related to what they had observed in the first task. That is, all pairs participating in this exploratory study said that the shape would be a circle. However, marking the points on the slide revealed a differing curve. The following episode illustrates a pair of students attempting to describe the shape that was being constructed.

An illustrative episode concerning DGE-based Task 2

S3: It will be oval.
S4: It's like a circle.
S3: A rectangular circle.
S4: A potato.
S3: It's a circle…but…not quite circle.
S4: A circle.
S3: Like a ball of American football … like eggplants.

The discussion that followed describing the shape was related with exploring what happens when the distance between points B and C changed. Two pairs hypothesized that if the length of AB and BG was kept the same then the shape constructed would be a circle, as according to them AB and BG were the radii of the circle. The result of this exploration was not the expected one. Nevertheless, through their exploration these pairs reached a special case where the curve was a circle. The discussion between the pairs led them to conclude that perhaps points B and G were, for the 'oval' shape two centers. However, this was a hypothesis by them that remained untested. All pairs concluded that by keeping the sum of the distance of AB and AC the same and changing the distance between B and C, one could either get rather big 'oval' circles or more 'squeezed' circles.

3.2.5 Reflecting on the study

Reflections on this exploratory study indicated at the beginning that the students did not have a conceptual understanding of the geometrical notions explored. Instead, pupils seemed to rely mainly on perception and failed to distance themselves from the geometrical 'drawing'. That is, they relied on their perception of what a circle looks like rather than on its properties such as all points being equidistant from a fixed point. Perhaps the task was set up in such a way that it did not offer opportunities for the use of conceptual knowledge. As a result, the prototypical image of circle was not the one expected ('extended prototype' of circle). To elaborate more, the prototypical image of a circle is a flat image where the observer looks on from a point on the circle's perpendicular axis. Here the oval might have been seen as a non-prototypical view of a circle (for example from an oblique angle). In practice it was identified as something other than a circle. However, throughout the exploration of the DGE-based Task 2, a different way of thinking about circle was emerging indicating thinking in change. That is, the students related initially their observations with their intuitive knowledge. However, as this proved not to be enough, they turned to the properties of the shape, thinking about centres and radii, and concluding that maybe this oval shape has two 'centres'.

Concerning student's experiences regarding the utilisation of the above tasks, the students commented that the tasks put them in a process of thinking and that they felt like they were playing. The fact that all students referred to the word 'play' when describing their interaction with this environment cannot be neglected. For these students 'play' seemed to mean to explore, wonder about, enjoy. Considering this outcome, it can be argued that the students were motivated in engaging with the tasks.

In regards with the utilization of the tasks, it was acknowledged that the task design had its limitations. For instance, if point B was moved below points A and C, the condition would not stand. Thus, the shape constructed would be an arc. Furthermore, depending on the size of the circle, as the points were marked on the transparent slide, moving the screen would mess up students' work. Furthermore, opportunities for explaining and justifying were mostly (though not exclusively) limited to arguments around what they perceived, rather than drawing on geometric conceptual knowledge.

Nevertheless, the question concerning the utilisation of these tasks in the formal classroom setting remained. How would the social setting surrounding the classroom influence students' argumentation? If the episode regarding a right-angle triangle inscribed in a circle emerged in the classroom, how would the teacher engage in the argumentation process? Would the students consider these tasks as playful activities if they were part of the mathematics lesson and how would the teacher respond to this?

This issue led to considering that for the empirical work in this thesis, the study should perhaps investigate argumentation within the classroom setting. What is more, even though this initial study followed a design research approach, it was conducted without clear theoretical foundations. Thus, while designing and implementing the second exploratory study, it was also my goal to identify the theoretical framework that would allow me to explore these issues more deeply.

3.3. Exploratory Study II

Keeping in mind the issues raised from the initial exploratory study, it was decided to conduct another exploratory study to further explore the aforementioned ideas.

3.3.1 The study setting

The second exploratory study was conducted in two Year 6 classrooms in two public primary schools in Nicosia, the capital of Cyprus. Both schools implemented the mathematics curriculum proposed by the Cyprus Ministry of Education and Culture. Furthermore, both schools had a good reputation concerning their academic results. The Principal of the second school was explicitly encouraging teachers to utilise ICT in their teaching. The teacher of class A had more years of experience compared to the teacher in class B. The participants were the students. Class A consisted of 22 students (10 girls, 12 boys) of mixed abilities. The teacher of this class was the teacher of the students that participated in the first exploratory study. Class B had 24 students (11 girls, 13 boys). These students were also of mixed abilities. The students in both classrooms did not have previous experience with DGE in their mathematics lessons.

By taking into consideration the reflections on the first exploratory study it was decided that prior to the implementation of this exploratory study the students would have an introductory lesson to the DGE employed in the study, in order for the students to get familiar with most of the DGE tools as well as the principle of robustness of constructions in DGEs.

Even though the teachers were not participating in the study, they were present in the classroom and were left free to intervene. The participants were observed after their parents agreed and signed the consent and authorisation form. Before proceeding, it should be made explicit that in classroom A the teacher would circulate among the students and engage students in a process of explaining, whereas in the second classroom the teacher was just an observer.

The exploratory study data were collected through classroom observations, field notes, and the students' written work (worksheet). To be more accurate, the sessions were video recorded. Two video recorders were positioned in such places that allowed the recording of the whole classroom.

3.3.2 My role as a researcher

Conducting this exploratory study within the classroom setting inevitably meant that I had to redefine my role of a researcher acting as teacher. As a researcher my aim was to explore students' thinking and understanding and what they can do. In order to maintain my research focus and facilitate students' argumentation, rather than to instruct, I developed and followed specific strategies.

Initially, I decided to use pre-printed material. A more detailed rationale for providing students with pre-printed material is presented in Section 3.3.3. During the exploration of the DGE-based tasks, my interaction with the students was related firstly with answering questions related with the tools the software provided. In asking probing questions, I employed the same steps as in Exploratory Study I (see Section 3.2.2.).

During whole classroom discussion, I attempted to engage students in a discourse where they accurately convey information, as well as assist them in creating meaning. However, I was cautious to keep my talk to a minimum, in order for the students to have more opportunities to share, listen and respond to each other's ideas.

Nevertheless, while acknowledging the methodological complexities that exist when one person teaches and researches at the same time, being a singular researcher/teacher provided a great advantage. That is, I had the opportunity to immediately seize opportunities to pursue and draw attention to events that proved promising in terms of capturing interesting details and therefore collecting relevant data. This is in accordance with Stake (1995) who affirms it is *'essential to have the interpretive powers of the research team in immediate touch with developing events and ongoing revelations, partly to redirect observations and to pursue emerging issues'* (p.41–2).

3.3.3 The DGE-based tasks

The pairs were provided with a worksheet that consisted of two DGE-based tasks to be attempted. The rational for providing students with a worksheet was twofold; (i) as a video recorder would be placed in the middle of the classroom, a worksheet would give me access to the work of each pair and (ii) the students would have the opportunity to work at their own pace.

DGE-based Task 1: Circle as a geometrical locus

This is a locus problem illustrating the situation in which the solver can follow the moving of point x on a circle with point O as the centre and AO and OB as radii (see Figure 3.3.).

To be more descriptive, let the parallelogram AOCD with the A and O points as fixed ones and C and D points as mobile ones, so that the line DC becomes parallel with AO. The geometrical locus point X - the projection of the A point on the line limited by points D and M (the middle point of BD) - is represented by the circle which has the centre in the O point and AB as the diameter.

The students could initially investigate what happens when you move the points in the figure. The students were expected to conclude that some points are static and cannot be moved, whereas points C and D could be dragged. The students were also expected to conclude that point X would move only when points C and D were dragged. Following this, the students would investigate what happens when you move point C so that the length of AO is kept the same. For instance, the students could make observations concerning changes to the parallelogram. The students were also expected to observe that point X is moving along BD in such a way that a curve is being shaped. The students could formulate a hypothesis concerning the curve that is being shaped. Subsequently, the students would have the opportunity to check their predictions. That is, the students would have the opportunity to mark on a transparency several points of point X moving on BD as the length of AO is kept the same. Adding to the above, by using the 'Trace' tool and the drag-mode, they could obtain a circle with point O as the centre.

The students were expected to give a written response explaining how and why a circle was being shaped. Adding to this, they were asked to identify the properties of the circle on the diagram. The students were also asked to explore what happens when the length of AO changes. Even though the notion of locus is presented in secondary school teaching, DGE-based Task 1 provided the opportunity for students to explore circle as an example of loci. Thus, it could initiate the formation of hypotheses and mathematical argumentation.

Figure 3.3. below shows a graphical representation of the construction related to the DGE-based Task 1.

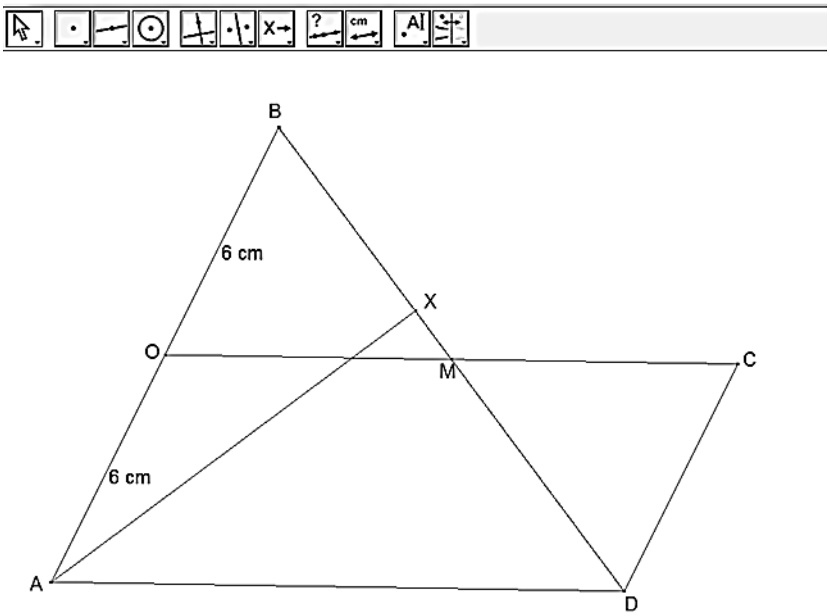


Figure 3.3.: Circle as a geometrical locus

DGE-based Task 2: Triangles in a circle

This task is concerned with the triangles that can be constructed in a 6-point-circle. The students could do some constructions, investigate properties of triangles drawn in circles, investigate circle theorems, investigate angles in similar shapes as well as engage with explanation and justification. Figure 3.4. below shows a graphical representation of the construction related to the DGE-based Task 2.

Initially the students were encouraged to construct triangles in the 6-point circle. The condition was not to use the radius as a side of a triangle. The worksheet consisted of the following questions: What sorts of triangle can be found? What are the properties of these triangles? How many different triangles are there? What are their angles? What might be inferred about triangles with right-angles in them? What is the largest angle that can be found in such triangles? What is the smallest angle that can be found in such triangles? What difference would it make if the centre of the circle were allowed as a vertex? What can be said about all such triangles? Students could also investigate what happens when the radius of the circle changes. What is more, they could explore what happens with 8 or 12 point circles.

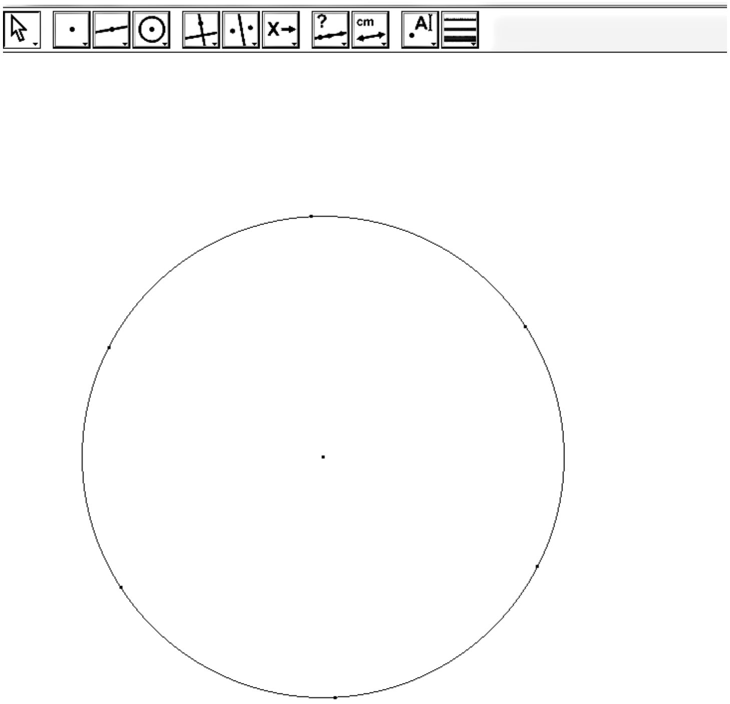


Figure 3.4.: Triangles in a circle

As pupils investigate what happens with circles with different numbers of points, more general questions would be posed. This would happen through classroom discussion. The following questions would be posed: Under what conditions are your triangles obtuse-angled? Under what conditions are your triangles acute-angled? When are they right-angled? How many different triangles can you get on an n-point circle? What is the largest angle that can be found in a triangle drawn in an n-point circle? Can you prove why your triangles yield the angels they do?

3.3.4 Data analysis

Analysis of the observational data highlighted the fact that the norms of argumentation can differ profoundly from one classroom to another. Consequently, this revealed the influence of both the classroom level as well as the institutional and educational level in student's work. That is, in the first classroom (class A), the students, even though they fully explored the tasks, only completed 1-3 questions of the worksheet. While exploring the tasks, the classroom of students was quite noisy, and the students were seeking confirmation by the teacher before writing something down. Despite this, they shared their ideas among their classmates.

In the second classroom (class B), it was observed that the students were accustomed to sharing a worksheet when working in pairs. The pairs were collaborating in a relatively quiet way, often working independently from the teacher. At the end, they were able to write their observations and ideas related to the exploration of the tasks.

The initial exploration of DGE-based Task 1 involved students randomly dragging the points on the figure. This 'messing up' led students to conclude that some points could only be moved when specific points were dragged. Following this the students explored the task. All students predicted that the shape would be a circle. The students, in explaining why point M was the centre of the circle, detached themselves from the drawing and gave an explanation by using the definition of circle. That is, they stated that it was the centre of the circle as it was equidistant from the circumference. The students, in answering why a circle would be constructed, did not relate their response to circle as geometrical locus.

Concerning DGE-based Task 2, most pairs found that 8 inscribed triangles could be constructed in a 6-point circle. These triangles were right-angle triangles and isosceles triangles. The angles in these triangles were either 90° or 45°. Five pairs of students in class B used the radius of the circle as a side of the triangle. For this reason, they concluded that 20 triangles could be drawn. These triangles could either be right-angle triangles, equilateral or isosceles triangles. The students' observation concerning the size of the angles is that the biggest angle the constructed triangle could have is 120° and the smallest 30°.

Concerning the triangles that had a right-angle, half of the pairs inferred that they are called right-angle triangles. The other pairs concluded that one side of these triangles is the diameter of the circle. In answering which conditions should stand so that a triangle has a right angle, one pair stated that this happens when a rectangle is divided diagonally. The other pairs' response was related to their previous inference. For instance, some pairs (class B) wrote:

"If one point of the circle is the centre and opposite there is another point, then the triangles will have a right angle".

"One side of the triangle will be the diameter and the other two sides will meet at a point on the circumference of the circle".

"The triangle will have a right angle if the diameter of the circle is one of its sides".

In discussing the general questions through classroom discussion, the students were elaborating their ideas but the class did not reach a conclusion.

3.3.5 Reflecting on the study

In this second cycle of research, the analysis of the data revealed differences among the two classrooms. Even though this is something that is generally expected, in exploring students' explanations and justifications where the goal goes beyond categorising students' reasoning all elements that influence students' argumentation need to be accounted for. How does the classroom and its surrounding influence students' proving practices?

It was observed that some students did not seek further explanations regarding their observations. Furthermore, it can be concluded that the second investigation task gave the opportunity to students of different levels of mathematics ability to pursue it with different levels of depth. The above observations raise a question concerning the teacher's role. How would the teacher orchestrate the classroom argumentation? One could speculate, for instance, that given the fact the teacher in class A would circulate among the students and facilitate their engagement with the tasks, the students would complete the worksheet.

Thus, the question concerning the utilisation of these tasks in the formal classroom setting remained after this exploratory study as well: How are explanations communicated for differing audiences?

It has already been mentioned that these initial studies were conducted without clear theoretical foundations. At this point, I began to consider the framework that for the purposes of this study would provide an insight into the features that drive and influence students' argumentation in the classroom. Reflecting on the second exploratory study led to considering that Cultural-Historical Activity Theory could prove a powerful tool for investigating proving in the classroom. That is, Cultural-Historical Activity Theory would help map out and take account of the socio-cultural aspects of the classroom as a system and their impact on the students' activity. Keeping in mind the aforementioned, it was concluded that the main research would be undertaken in the classroom, with both the teacher and students participating in the study.

3.4. Conclusion

This chapter has described the exploratory studies conducted that had an impact on the general direction of the main study. The purpose of these studies was not to seek answers to specific research questions, but to explore how students' explanations and justifications could be investigated and to begin to identify what might be important issues.

The exploratory studies implemented add to the research literature indicating that upper primary school students engage in pre-proving activity (see Section 2.3.). However, in investigating pre-proving activity, the elements constituting the social dimension of the classroom ought to be made explicit. The exploratory studies directed the research towards considering Cultural-Historical Activity Theory as the most appropriate tool that offers the means for handling this complexity in coming to understand how proving might be constituted in the classroom.

Undoubtedly, investing time in exploratory studies enhanced the quality of the main research study. For instance, even though a worksheet can provide the researcher with fruitful data, the discussion among the pairs appeared to offer richer data around students' argumentative process. In addition to the above, task design would also benefit from these undertaken studies. That is, the tasks employed, even though appropriate for an introduction to DGEs and exploration, they were not as appropriate for supporting students in conjecturing and justifying. This observation would be taken into consideration when designing tasks that support students' argumentation for the main study.

What is more, a consideration of how Cultural-Historical Activity Theory could account for the differences that emerged between the two classrooms made clear which information should be collected so as to portray the educational and institutional level. A decision would of course need to be made about whether to work with two classrooms or whether, given the limitations in scope of the project, more could be learned by an intensive study of one classroom.

Keeping in mind the aforementioned, the following chapter explores and elaborates the theoretical considerations of Cultural-Historical Activity Theory. Furthermore, it provides a justification on why this approach is suitable for exploring the way proving is constituted in the mathematics classroom. Following this argument, the research questions are presented.

CHAPTER IV

USING CULTURAL-HISTORICAL ACTIVITY THEORY TO EXPLORE PROVING IN THE PRIMARY CLASSROOM

4.1. Introduction

Cultural-Historical Activity Theory (CHAT) provides me with a theoretical framework for conceptualizing the constitution of proving in the classroom. It provides me with a theoretical basis to identify forces that interact to shape pre-proving activity in a complex environment that includes not only what the children do but also, to name but a few, the received curriculum, the 'wisdom' from generations of mathematics education researchers, the school's inspirations and constraints and the teacher's agenda. Cultural-Historical Activity Theory offers a means for handling this complexity in coming to understand how proving might be constituted in the classroom.

In this chapter the theoretical background of Cultural-Historical Activity Theory is presented with a particular discussion on the main concepts that are relevant to this study, along with the ways it has been utilized across disciplines. A specific emphasis is given to mathematics education. Following this, a justification will be provided on why this approach is suitable for this study by exploring how these concepts can be used to investigate proving as constituted in the mathematics classroom. Drawing on this justification, the initial research question presented in Chapter II will be redefined with an initial description of the path that this research will follow.

4.2. Cultural-Historical Activity Theory

Cultural-Historical Activity Theory (CHAT), or Activity Theory as it is also known, is a psychological and multidisciplinary theory, which has its roots in the socio-cultural perspectives of Vygotsky (1978) and has subsequently been developed by Leont'ev (1978, 1981) and particularly by Engeström.

Activity Theory has its threefold historical origins in classical German philosophy (from Kant and Hegel), in the writings of Marx and Engels and in the Soviet Russian cultural-historical pedagogy of Vygotsky, Leont'ev, and Luria (Engeström, 1999, p.19-20).

Kuutti (1996) defines Cultural-Historical Activity Theory as *'a philosophical and cross-disciplinary framework for studying different forms of human practices a development processes, both individual and social levels interlinked at the same time'*, (p.25).

The development and growth of the theory spread across three generations of CHAT each of which builds upon the previous one. Vygotsky, who laid its basic foundations, highlighted the cultural mediation of actions (see Figure 4.1. below). That is, the relationship between the subject and the object is not direct but mediated through the use of a tool which can be both physical and intellectual. However, in his work the unit of analysis remained individually focused (Engeström, 2001).

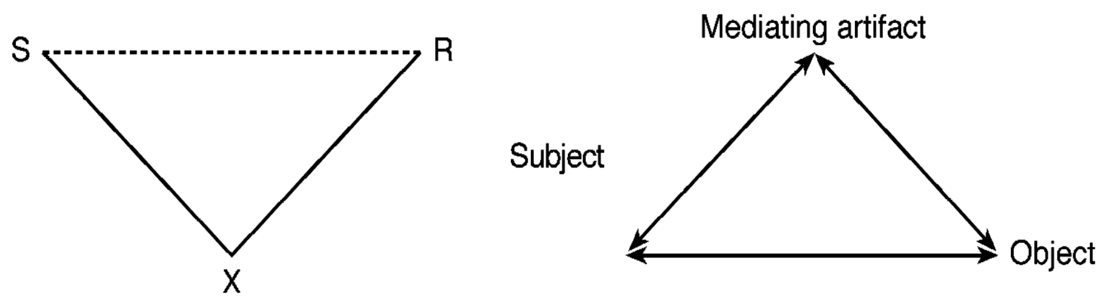


Figure 4.1.: Vygotsky's model of mediated act and its common reformulation

This limitation was overcome by the second generation. Leont'ev (1978, 1981), by building on the idea of mediation, instigated the use of the word activity. He conceptualized activity as composed of three different units of analysis. According to Leont'ev (1981), an activity can also be analysed in three hierarchical levels which contribute to a unique understanding of human production (see Figure 4.2. below). By considering this, a distinction is made between the activity and the actions and operations that comprise it. Actions are *'associated with individual knowledge and skills'* (Barab et al, 2004, p.202), subordinated to individual needs and facilitated by tools. Actions are directed towards the attainment of certain goals which can be distinguished from the motive of the overall activity (Cole, 1985, p.152). Actions are chains of operations which are *'habitual routines associated with an action and are influenced by current conditions of the overall activity'* (Barab et al, 2004, p.202). Thus, operations are the most basic level of activity, are routinized and unconscious components of actions and they do not have their own goals. Roth and Lee (2007), describe the relationship between action (goal) and activity (motive) as *'dialectical, for actions constitute activities, but activities motivate particular action sequences'*, (p.201). In this tiered explanation of activity (see Figure 4.2. below), the arrows indicate the two-way relationships involved.

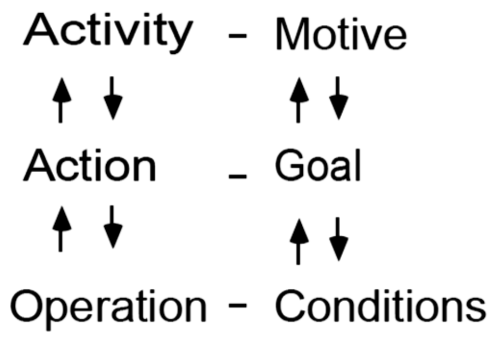


Figure 4.2.: The hierarchical levels of an activity as developed by Leont'ev (1981)

Leont'ev (1981) tried to theorise the activity that is performed collectively versus the activity that is performed individually. Engeström acknowledged the contextualized nature of human activity and further developed the theory by regarding activity as an expansive social system under constant transformation. Engeström expanded Vygotsky's activity triangle of tool mediation and introduced the activity system (see Figure 4.3. below). This reconceptualised pictorial representation of a generic activity system is a general model of human activity that reflects its collaborative nature. Engeström's extended and holistic view of human activity embodies the idea that both individual and social levels interlink at the same time.

In CHAT, the unit of analysis is an activity. The term activity is employed to describe a *'coherent, stable, relatively long term endeavor directed to an articulated or identifiable goal or object'* (Rochelle, 1998, p.84). Activities can also be portrayed as *'longer-term formations of chains and networks composed of individual and cooperative actions',* (p.84).

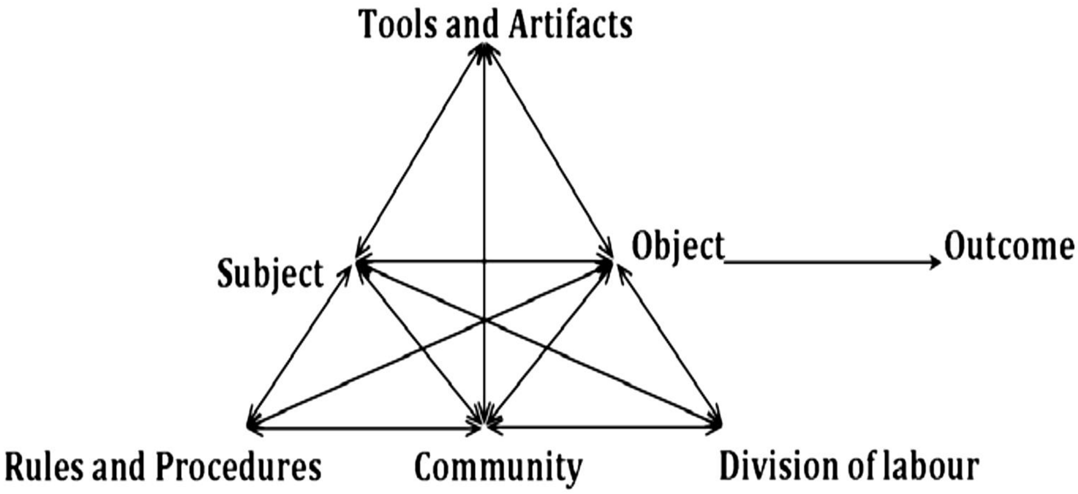


Figure 4.3.: The basic schematic of an Activity System as developed by Engeström (1987)

In CHAT the subject is an individual or a group engaged in the activity. The object or objective of the activity is the focus of the activity. It can be raw materials, conceptual understandings or even problem spaces. The tool or instrument is the means with which the subjects are performing the activity. Regarding mathematics education the means that mediate the activity can be either mathematical concepts, strategies, procedures, language, gestures, descriptions, explanations, group work or computing technologies. The motive that drives the activity is the transformation of the object into an outcome. The outcome is defined as the intended or not results, final products of the defined objectives of the activity.

Engeström argues that, while artefacts are either internal or external, an emphasis should be given to the different ways the artefacts are being used (Engeström, 1999). He distinguishes four types of artefacts:

> The first type is what artifacts, used to identify and describe objects. The second type is how artifacts, used to guide and direct processes and procedures on, within, or between objects. The third type is why artifacts, used to diagnose

> and explain the properties and behavior of objects. Finally the fourth type is where to artifacts, used to envision the future state or potential development of objects, including institutions and social systems (Engeström, 1999, p.381).

Regarding the nature of the object of the activity it is emphasized that it is always challenging to define it due to the polyphony and dynamics of the activity system (Engeström, 1999, Kaptelinin, 2005). The object is multidimensional, ambiguous and contradictory and open to change. The object is *'interlinked with the subject to the extent that its construction and transformation depends on the subjects' will and motivation',* (p.196) even though the subjects are not necessarily aware of it, or have multiple interpretations of it (p.34). That is, the object orientates the subject in a direction of action and at the same time it is *'shaped by the multiple domains of mediation constituting the activity'* (Kanes, 2002, p.8).

Correspondingly, it is dynamic as the activity is the ongoing construction and reconstruction of the object and it might not be fully reached or conquered (Engeström, 1999). It should also be noted that the object might not be shared among the people involved in the activity system. But even if a joint object exists, it is not always the case that a commonly shared language or tools are available in order for the activity to be directed towards this joint object (Kallio, cited in Engeström et al, 1999). A new joint system of activity might then be necessary so as to be able to manage the joint activity. Despite this, the question whether it is possible or desirable to have a completely shared object in an activity still remains. Nevertheless, uncovering and *'defining the object demands object-specific historical analysis. By understanding the history of its origin, one can understand the transformation of the object of an activity as well as the contradictions within the activity system'* (Kallio, cited in Engeström et al, 1999, p.34).

Even though it is not the scope of this section to extensively elaborate on all the issues existing in CHAT, it is considered important to provide the reader further details concerning the misunderstandings that may appear regarding the 'object of the activity'. In unraveling the complexity of the object as 'the sense-maker' of the activity, Kaptelinin (2005) turns primarily to the translations of the concepts of the activity system from Russian to English and secondly to the meaning given to the object from the perspective of the second and third generation of activity theory. That is, a linguistic gap appears when reconstructing the meaning of this concept from the language in which it was developed to other languages (p.8). Furthermore, as the work of Leont'ev was individually focused, whereas Engeström considered the activity as a collective phenomenon, a conceptual gap also exists. In order to shed light on the existing confusion regarding the object Kaptelinin (2005) argues that one has to make a distinction between the object and the motive. Kaptelinin concludes that this is not to be considered as the Achilles' heel of activity theory but as complementary versions of activity theory dealing *'successfully with practical and research issues in their respective domains, that is, psychology and organizational change',* (p.11). This is also why when considering CHAT the literature refers to the three generations of Activity Theory.

The bottom part of the triangle in Figure 4.3. illustrates the context in which the activities occur. The social components of an activity system which define and influence the activity are the division of labour, the community and rules. Division of labour shows the group dynamics, the division of tasks, power and status among different actors of the system. This component helps to differentiate what is accomplished collectively or individually. The community is the environment in which the activity is carried out. Individuals or groups, who are all concerned with the same object directly or indirectly, constitute the community of the activity system. Rules and regulations refer to principles of regulation of action and interaction and conventions of behaviour (Engeström, 1999).

In this collective activity system in which individual and group actions are embedded, the components are interdependent and interconnected. An activity system functions as a unit that is transformed over time through transactions inside and outside the system. Thus, its tensions and potentials can be understood against its own history. According to Engeström (2001), *'history itself needs to be studied as local history of the activity and its objects, and as history of the theoretical ideas and tools that have shaped the activity',* (p.136).

The third generation of CHAT is building on and expanding upon the first two generations, by developing conceptual tools to understand dialogue, multiple perspectives and voices and networks. Engeström (2001) models these perspectives by expanding the single activity system to include as the minimum unit of analysis two interacting activity systems (see Figure 4.4.). In this model, activity systems interact and overlap with other activity systems, implying that the elements of an activity system are always produced by some other activity. This model also highlights the emerging shared object between the minimum unit of analysis of two activity systems (there may be more).

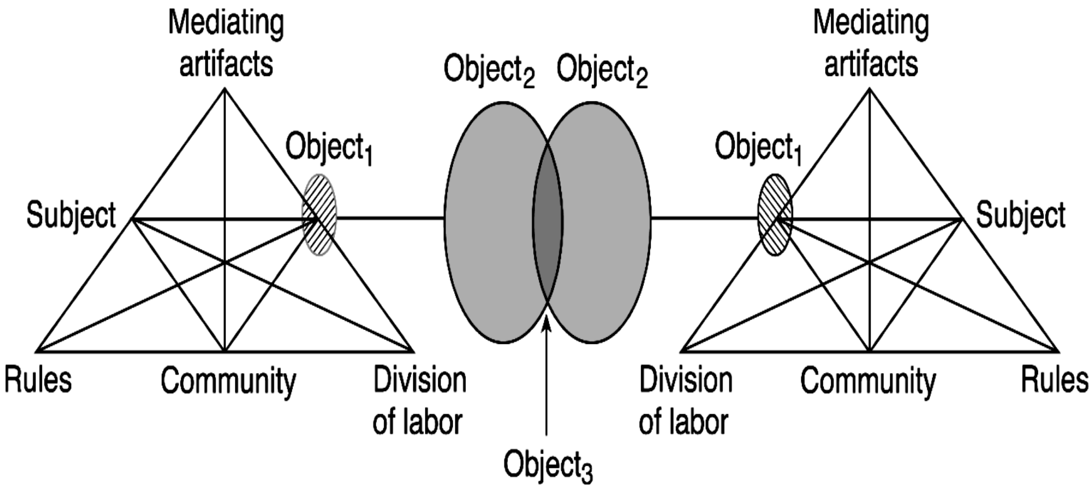


Figure 4.4: Two interacting activity systems as a minimal model for the third generation of CHAT. (Engeström, 2001, p.136)

Among the basic principles of Cultural-Historical Activity Theory is the notion of contradictions. Contradictions are imbalances, ruptures and problems that can occur within and between components of the activity system as well as across entire activity systems. This conceptualization of contradictions, as manifested in CHAT, should be differentiated from mere problems or disorienting dilemmas from the subject-only perspective as they are more deeply rooted in a sociohistorical context. (Engeström, 2001). According to Virkkunen and Kuutti (2000) *'contradictions are fundamental tensions and misalignments in the structure that typically manifest themselves as problems, ruptures and breakdowns in the functioning of the activity system'*, (p.302).

In the process of transformation, Engeström identifies the systemic tensions into four levels (illustrated in Figure 4.5). Primary contradictions occur within each component of the activity system and manifest themselves in secondary contradictions which take the form of tensions between components. According to Engeström the primary contradiction is in essence economic in nature. That is, it is originated on the opposition between the use value of the product or service (meet specific needs) and its exchange value (for example its commercial potential).

Tertiary contradictions appear between the object/motive and the culturally advanced form of the central activity. These contradictions are originated when the object/motive of the new activity confronts the object/motive of the dominant activity. Finally, quaternary contradictions are tensions between the central activity system and adjacent activities.

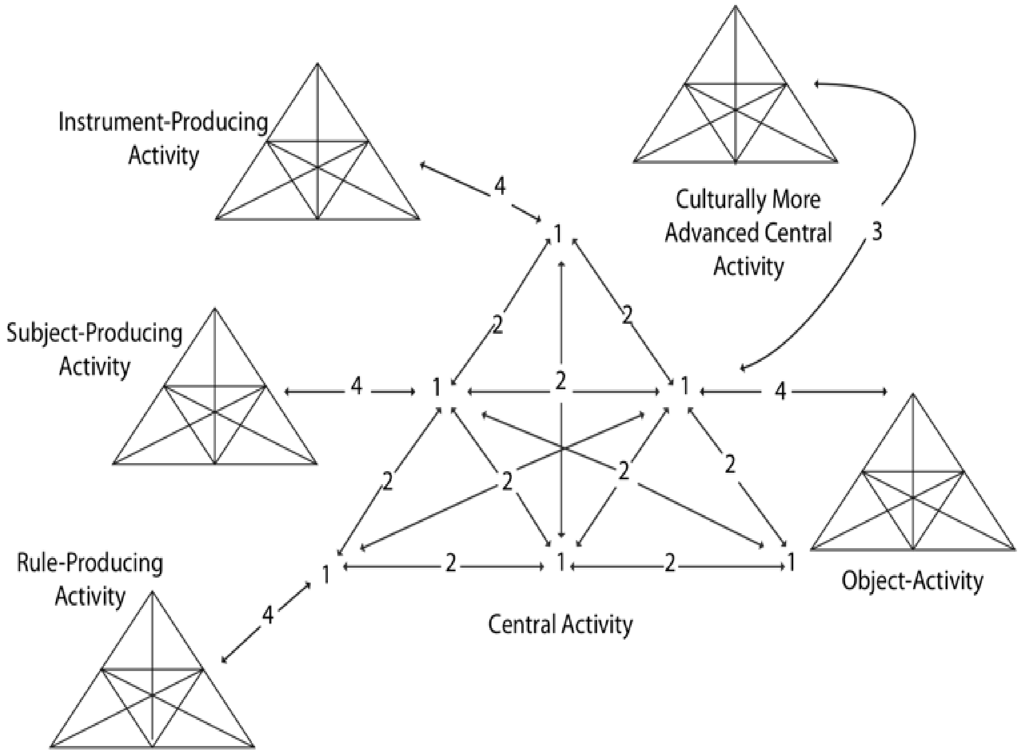


Figure 4.5.: Contradictions within human activity.
(Adopted from Engeström, 1987)

In order to further explain these levels of contradictions, I will use the classroom as an illustrative example. From the perspective of CHAT, a primary contradiction can be identified by focusing on one of the elements of the activity system, for example the teacher. The teacher might wish to promote exploration and investigation in the classroom, but at the same time to perceive the students' work as playing. The secondary contradiction might be expressed by the teacher (the subject) as experiencing conflict with time and the interactive whiteboards (tool) management. If the teacher is faced with situations where he/she has to employ an advanced method in order to achieve an objective (introduction of new technology), then a tertiary contradiction can arise. If the object of the classroom, which might be to provide students with higher order skills in ICT, differentiates from the object of the school, which can be to increase school ranking by improving examination results, then a quaternary contradiction is identified (Lim and Hang, 2003).

Engeström and Sannino (2011) acknowledge that the notion of contradictions may be used in a vague and ambiguous way and attempt to overcome this by introducing four types of discursive manifestations of contradictions: a dilemma is an expression or exchange of incompatible evaluations, either between people or within the discourse of a single person; conflicts take the form of resistance, disagreement, argument and criticism; critical conflicts are situations in which *'people face inner doubts that paralyze then in front of contradictory motives unsolvable the subject alone'*, (p.374); and double binds are processes in which *'actors repeatedly dance pressing and equally unacceptable alternatives in their activity systems'*, (p.374).

These generated disturbances can inform a number of possible interventions and therefore become sources of change and development (Engeström, 2001). What is more, by acknowledging these conflicts, both the subject and the community engaged in the activity are pushed to reflect and collaborate towards either a modification of procedures, or a revision of the theory or both (Williams et al, 2001). That is, contradictions may lead to transformations and expansions of the system and thus become tools for supporting motivation and learning. As Barab et al (2002) utter, *'when systemic tensions are brought into a healthy balance they can facilitate a meaningful interplay that enriches and adds dynamism to the learning process'*, (p.104).

Oliver and Pelletier (2006) analyse secondary contradictions of a student's game in an attempt to draw conclusions about learning from games. They state that *'resolved contradictions between the subject and the tool indicate examples of skill development (understood as proficient use of a tool) and that resolved contradictions between the subject and rules illustrate examples of learning socially accepted concepts or practice'*, (p.75). What is more, they assert that these contradictions are most likely to indicate individual learning.

By considering the aforementioned, CHAT may be summarized with the help of five principles (Engeström, 2001). According to the first principle, the prime unit of analysis is a collective, artefact-mediated and object-oriented activity system. The second principle is the multi-voicedness of activity systems. The participants of an activity system *'carry their own diverse histories'* and the activity system itself *'carries multiple layers and strands of history engraved in its artefacts, rules and conventions'* (Engeström, 2001, p. 136). The multi-voicedness constitutes the ground for tensions and innovation. The second principle leads to the third principle of CHAT which refers to historicity. According to Engeström (2001) by considering the history of activity systems one can understand their problems as well as their potentials as *'parts of older phases of activities stay often embedded in them as they develop'* (Kuutti, 1996, p.26). The fourth principle is the central role of contradictions as source of change and development. As discussed previously in this section, the ruptures may drive the system to change. The fifth principle proclaims the possibility of expansive transformations of activity systems through the reconceptualization of the object and the motive of activity.

4.3. How is Cultural-Historical Activity Theory being utilized across disciplines?

Cultural-Historical Activity Theory is an interdisciplinary theoretical and practical movement that has been concretized and applied to different problems and domains (Lompscher, 2006, p.49). For instance, the first and second generation Cultural-Historical Activity Theory was utilized in developmental and educational psychology, engineering psychology and ergonomics. With the model of Engeström, the third generation was employed in *'different branches of societal activity'* (Lompscher, p.48) such as technology, health care, science and agriculture. Cultural-Historical Activity Theory can be utilized either within an interventionist framework or as an analytical tool for understanding complex human situations that can be observed in natural setting (Yamagata-Lynch, 2010).

This section draws from the education literature, with a specific focus on mathematics to illustrate how studies adopted, referred to or were influenced by Cultural-Historical Activity Theory. Roth et al (2009) in reflecting on several contributions from the education domain, illustrate that CHAT can be a fruitful framework for understanding tool mediation in teaching and learning, making visible normally invisible structures, processes, relations, and configurations, investigating issues concerning a larger system or across systems, rethinking and empowering learning and creating structures and collaborations to facilitate change (p.145).

In a similar way, Nussbaumer (2012) provides an overview of the various applications of CHAT in classroom research during the last decade. This review concentrates on the studies that make use of and clarify CHAT constructs, unit of analysis, mediation, as well as internalization/externalization of learning. With this review they make explicit the complexity of the framework in application of CHAT. Nussbaumer argues that, if CHAT is only used for validation of research methods without applying and adapting its principles and constructs, it indicates a researcher's lack of understanding and comprehending of its underpinning principles (p.46). Despite this, this review highlights the significance of CHAT *'in contributing to the understanding of certain complex situational teaching and learning activities'*, (p.46).

Karakus (2014) adds to the aforementioned statement, arguing that in understanding these complex activities, an Activity Theory framework '*requires extensive, in-depth, qualitative analysis of the context'*, (p.159).

Cowan and Butler (2013) employed Activity Theory in an action research study in order to analyse the role of the teacher during mobile learning. Through their findings they propose an enhanced AT model assuming a three-dimensional representation that encapsulates the teacher at the heart of the activity system. They also recognize the limitations of this enhanced model.

Concerning mathematics education, Nunez (2009a, 2009b, 2012) provides examples of how the activity systems and the components of the activity system have been operationalised in the mathematics educational research community. It has been used to describe mathematics departments (Beswick et al, 2007), to assist understanding the position of the student both in college mathematics and workplace practice (Williams et al, 2001, Goodchild and Jaworski, 2005), to construct a picture of computer classrooms using teachers' descriptions (Hardam, 2005a, 2005b), the mediating role of tools in learning mathematical concepts (Groves and Dale, 2005), whereas Kanes (2001) operationalizes the notion of numeracy as a cultural historical activity system whose object is numerical knowledge and Jurdak (2006) regards mathematical problem solving in the school context as an activity system.

In identifying contradictions, the mathematics educational community either introduced a new instrument, a new object or transferred to a new context or situation. Studies that introduced a new instrument or object into the central activity system illustrate disruptions of the dynamics among components generating secondary contradictions (Hardam, 2005a; Goodchild and Jaworski, 2005) and between micro and institutional level generating tertiary contradictions (Lim and Hang, 2003). Regarding transfer to a new context or situation, Nunez (2012) states that *'when a student transfers from applying mathematics in college to applying mathematics in the workplace, the move can generate a primary or inner contradiction in cases where there is a need to re-learn practical aspects of college mathematics at work, even if a student is theoretically proficient'*, (p.92).

Jonassen and Ronrer-Murphy (1999) illustrate a process for utilizing Activity Theory as a framework for designing and describing constructivist learning environments. They identify 6 steps necessary to portray the way Activity Theory as a framework determines the components of activity systems for designing learning activities. Firstly, the purpose of the activity system must be clarified, as well as the subject and the relevant context in which the activities occur (p.70). Then, the components of the activity system must be defined and the activity structure must be analysed. Furthermore, the tools should be analysed, focusing on those that provide direct or indirect communication among subject, community and object (p.74). The following step includes the analysis of the internal, subject-driven contextual bounds and the external, community-driven contextual bounds. The final step is to analyse the system dynamics.

Jurita and Nussbaum (2007) employ the aforementioned steps along with a framework for designing collaborative learning in order to study mobile computer supported collaborative learning (MCSCL). In their approach AT is adopted as a theoretical framework and not as a methodology. To be more precise, they elaborate on the social (face-to-face communication between the members) and technological (communication between the members and the handhelds or the handhelds themselves) aspects of the classroom situations, and describe each component of the activity system according to these components. What is more, in order to better analyse student's behaviour, their observation guidelines included communication, interaction, coordination, discussion, negotiation and technology appropriation. In evaluating this framework for teaching basic mathematics skills, they argue that this framework facilitated the study of students' social interactions. Keeping in mind that AT is primarily a descriptive tool, they conclude that AT *'embodies a qualitative approach that offers different lens for analyzing a learning process and its outcome, focusing on the activities people are engaged in'*, (p.214). However, it should be noted that even though their framework was based on AT, only one principle of AT was taken into consideration.

Hardam (2005a) investigated the potential shift in a lecture's pedagogical practice by introducing a computer-based learning environment. In achieving this, the face-to-face lesson and the computer laboratory lesson were analysed using the triangle system. An outcome of this case study was the recognition of the difficulty that exists in trying to identify the object in a non traditional classroom setting. What is more, the utilization of activity theory appeared a useful tool in identifying contradictions and forcing changes in and between systems.

Hardman (2007a; 2007b) further explores the prospect of developing a framework which helps unfolding the object of an activity. In order to investigate pedagogical practices in primary school mathematics classroom, she uses evaluative episodes which she describes as the pedagogical moments the teacher exploits to restate something he/she has already covered in order to identify the object the teacher acts on in his/her lesson. By categorizing the form of the questions asked and the statements used by the teacher, as well as the actions taken during this episode, the researcher temporarily freezes the activity system and unfolds its components. This study shows that analysis of observational data may lead to the identification of the object of an activity.

Ho (2007) also employed Activity Theory to investigate teachers' classroom practices. In this study, Grounded Theory was employed and categories of actions the teachers perform during the lesson along with the time spent for each action were coded in order to explore the extent to which the teacher gives emphasis to mathematical problem solving. During classroom teaching, the categories of actions are heuristics-instruction, teaching concepts/skills, going over assigned work, students' activities and other events that are not specific actions but helped towards accurately accounting class time. With this research study it is argued that the elaborated scheme can help address the question of how much emphasis is given to problem solving in the classroom. While the actions of the subject can reveal the object of the activity system, this study did not really track the object of mathematics teaching oriented to problem solving using the actions identified and categorized. Thus the object of the activity was not fully explored through its transformation over a period of time.

Groves and Dale, (2005), in exploring the way in which the calculator acted as a tool for learning in the development of number knowledge of six students during their first two years of primary schooling, decided to employ Activity Theory as an analytical tool in an attempt to look holistically at the individual children's learning in the social environment of the classroom. Even though their main focus was the mediation of this particular tool, it became obvious that the high level performance of the students could not be attributed solely to the use of the calculator; attention should be drawn to the broader context of the classroom community, the teacher's beliefs and intentions, the classroom norms and the division of labour. This was achieved by exploring the relationships between the child, the calculator, the teacher and the classroom environment. This was achieved by employing the activity system and reporting findings related with the learning environment, the role of the calculators, the learning outcomes for the students as well as the teachers' perceptions on the students' learning.

In reflecting on the way CHAT was appropriated as a way to report findings, Groves and Dale recognise that more emphasis was given to specific components of the activity system and thus neglecting other components of the activity triangle. They also acknowledge that other principles of CHAT were not taken into consideration as for example the activity system being the prime unit of analysis and the identification and exploration of potential contradictions. Keeping in mind the insights AT provided in exploring the mediation role of calculators, they argue that the activity system also provides a framework appropriate for framing the research. They conclude that *'activity theory can play a significant role in the planning of* *future collaborative classroom research aimed at envisioning and implementing new practices that take into account the constraints and affordance inherent in the activity system as a whole'*, (p.10).

By considering the concluding remarks of Groves and Dale, (2005), as well as the way in which Cultural-Historical Activity Theory has been adopted in the other studies reported in this chapter, it can be argued that in employing activity theory in designing and analyzing practices in context it is important to consider all components of the activity system. If researchers only use the upper triangle of the activity system and ignore the other components, they get an incomplete view of the way the interrelationships of the components give guidance to the system towards transformations and expansions. Although a limited view may be the focus of some research, depending upon their research questions, there may be certain queries that these studies cannot answer due to their focus on only a part of the activity triangle. What is more, the identification of contradictions that emerge in all four levels and their potential resolution enriches understanding concerning systemic dynamics, motivation and learning. As the notion of contradiction is a key idea in modern Cultural-Historical Activity Theory (third generation), then the potential for observing contradictions between elements of the triangle is limited if only a part of the triangle is researched. Thus, the extended, holistic view of activity systems allows for the contribution of multiple perspectives when investigating learning as a socio-cultural activity.

A consideration of the aforementioned studies that were developed and carried out using a Cultural-Historical Activity Theory methodology leads towards the emergence of several themes. Initially, it is noticed that Cultural-Historical Activity Theory is always employed alongside another research methodology. What is more, it becomes evident that in order to track the object of an activity system, observation seems an appropriate research technique to achieve this. Even though these methodological issues will be further explored and discussed in the methodology chapter, what should be stressed at this point is that describing and analyzing classrooms through the lens of activity theory is a challenging task. As Cultural-Historical Activity Theory is not *'a fixed and finished body of strictly defined statements'* (Engeström, 1993, p.64), replicating the way Cultural-Historical Activity

Theory has been used for research should be avoided due to the diversity in the research objectives and contexts and thus the diversity in the interpretation of CHAT concepts directed at these objectives. Still, research on how CHAT has been utilised in the classroom offers insights into how objectives may be investigated. Nevertheless, how the constructs from the aforementioned studies may shed light on the data collection and analysis process of this particular study will be explored subsequently. Going further, evidence regarding research in mathematics education that takes into consideration all key ideas of Cultural-Historical Activity Theory is still limited.

Having this in mind, the following subsection elaborates on Cultural-Historical Activity Theory as a lens for exploring the way the activity of proving develops and is transformed in the classroom and analyzing students' proving.

4.4. Using Cultural-Historical Activity Theory as a lens for analyzing students' proving

In Cultural-Historical Activity Theory, knowing and learning are studied in context. As Stevenson (2004) illustrates, *'knowing is not isolated from the world of activity, it is imminent in it and occurs through the various elements of a human activity system'*, (p.192). Thus, knowing and learning can be understood more holistically if considered in the context where it occurs. Additionally, it is argued that CHAT provides a framework that brings together both the constructivist and socio-cultural perspectives in mathematics learning (Cobb, 1994, Roth and Radford, 2011).

It is the main objective of this study to investigate the way proving and pre-proving is constituted in the mathematics classroom and the way the structural resources of the classroom's surrounding setting shape this process. To achieve this, Cultural-Historical Activity Theory offers a means to address this complexity. That is, in this research study, CHAT is used both as a framework for conceptualising the research and formulating the research design. Whereas other studies only employ CHAT as an analytical tool, this study utilizes the CHAT constructs so as to set up the research questions and design the research, a process that naturally leads to different types of data as well as the contradictions that may lie between them. By gathering these different data, I can look into the way similar issues are presented in contradictory ways in the different components of the triangle.

Having in mind all the aforementioned, it can be argued that Cultural-Historical Activity Theory is an appropriate framework for capturing the way argumentation is developed in a mathematics classroom. That is, by using Cultural-Historical Activity Theory we can study not only the way by which students make conjectures and justify them, but also the way their surrounding influences in either a positive or negative way the process of argumentation. By perceiving an activity system as consisting of nests of activity triangles, we can also examine the effects of changes in components of the activity system in students' motivation in engaging in the activity and directing their actions towards the outcome. Furthermore, given that activity systems are characterized by historicity, students' thinking-in-change can be studied against the local history of the activity and the *'theoretical ideas and tools that have shaped the activity'* (Engeström, 2001, p.137).

In addition to the above, when each element of the activity system is portrayed and thoroughly analysed then an in depth understanding of the outcome of the activity might be possible as it would not be comprehended in isolation from the social context in which it emerges.

In light of the above, this study aims at exploring the way the activity system supports or hinders proving. Regarding tool mediation, it also seeks to research how dynamic geometric environments mediate argumentation in different settings. Keeping in mind the fact that Dynamic Geometry Environments may provide opportunities to the students to express mathematical ideas which is an essential element when approaching formal mathematical proofs, information from the appropriation of such a tool and its impact on the activity system will guide the elaboration regarding the study's objective. In this context, this study will also reflect on the way the tool *'expresses the rules of the activity, shapes the community, formulates the object, positions the subject and affords or constraints the actions of the subject working on the object'* (Boag-Munroe, 2004, p.169). In these classroom settings the use of the tool by the teacher will also be analysed, as well as the interplay between pre-specified teacher-student instruction versus student-directed learning in order to uncover the constraints that direct the process of proving.

Nevertheless, a more detailed justification regarding employing DGE in this research will be provided in Section 5.3.

By introducing the triangle model of human activity as a framework for capturing the way argumentation is developed, it is also intended to identify contradictions that might occur.

Keeping in mind that the activity system is a micro context within broader macro context levels (Jaworski and Potari, 2009), this study will also attempt to identify the way the activity of a mathematics classroom is influenced and dependent upon the structure and organization of the school and the ministry of education as wider educational contexts. This is also in accordance with Balacheff (2009) who argues that among the important pieces in trying to understand the nature and role of proof in a mathematics class is describing the general usage of the word proof in these contexts and the demands this usage imposes in the classroom (p.45).

Thus, the research questions of this research project are the following:

- What is the object of developing proving in the classroom?
- Are any types of contradictions identified?
- How does the subject engage with proving in the classroom?

Having identified the research questions within an activity theory approach, the following chapter introduces the methodology employed to conduct the research, and drawing upon the methodological issues, it presents the design of the main study conducted so as to systematically investigate the research objectives and elaborate the research questions.

CHAPTER V

METHODOLOGY

5.1. Introduction

In understanding how the activity of proving is constituted in a mathematics classroom, this research study explores, describes and analyzes the activity of proving as it changes and develops over time in the specific naturalistic social setting of the mathematical classroom. What is more, as this study will attempt to set up research opportunities that may lead to unforeseen areas of discovery, it will follow an inductive approach allowing for models to emerge from the data itself. Therefore, it seems appropriate to employ a qualitative research methodology as it is naturalistic, descriptive, inductive and concerned with process and meaning (Bogdan and Biklen, 2003).

While both qualitative and quantitative methodologies aim at understanding phenomena, the qualitative paradigm *'represents a broad view that to understand human affairs it is insufficient to rely on quantitative surveys and statistics, and necessary instead to delve deep into the subjective qualities that govern behavior'* (Holliday, 2002, p.7). Going further, qualitative research does not refer *'to the quantifying of qualitative data but rather to a non mathematical process of interpretation, carried out for the purpose of discovering concepts and relationships in raw data and then organizing these to a theoretical explanatory scheme'* (Strauss and Corbin, 1998, p.11).

Within the qualitative enquiry, several research perspectives attempt to explore, illuminate and interpret bits of reality (Luttrell, 2010). This is in accordance with Cultural-Historical Activity Theory, which, as has been illustrated in Chapter IV, constitutes the theoretical approach that informs this study. Tolman (1999) and

Yamagata-Lynch (2010) illustrate that qualitative research methods are most appropriate for a Cultural-Historical Activity Theory approach as they provide rich descriptions of the activity systems under investigation. Despite this, it is also acknowledged that Cultural-Historical Activity Theory does not provide any clear methodology and techniques to guide how activities are to be recognized, described and analysed. This is in accordance with Engeström (2008) who suggests that Cultural-Historical Activity Theory is *'an evolving framework which needs to develop further as it is applied in empirical studies'*, (p.382).

Cultural-Historical Activity Theory is a framework that allows the analysis and description in an activity system and is utilized in differing disciplines. In order to capture the meanings under investigation from a Cultural-Historical Activity Theory approach, a descriptive theoretical framework derived from CHAT and other research methodologies is required. Research studies across disciplines have used CHAT along with ethnographic methods, teaching experiments, action research, discourse analyses to improve practices, study emergent contradictions and the way these contradictions are confronted. It is the intention of this study to provide such a descriptive framework, even though it is acknowledged that it will also be open to negotiation as the research evolves, according to the needs of the study.

In light of the above, this chapter introduces the methodology employed to conduct the research, and, drawing upon the methodological issues, it explains how this research project developed. This chapter begins by relating Cultural-Historical Activity Theory to other research methodologies. That is, a conceptual framework is provided using ideas from design based research and action research, followed by an explanation why these ideas appear relevant for this study. It then provides the argument why collaborative design alongside CHAT enables this study to systematically investigate the study's objectives. This is followed by a more detailed presentation of the research plan. The theoretical issues related with tools and instruments, task design and the role of the teacher that will guide both the design and analysis of the collected data are presented. Following this, a discussion of the theoretical assumptions underlying participant observation, interviews and documentary analysis- the methodological approaches which I will use in my own research design- is provided by justifying their selection considering both their strengths and weaknesses. This general discussion will allow, in the following chapters of this thesis, the elaboration of the specific methods employed for the three phases of this study. The chapter then proceeds with a description of the data analysis process implemented in this study. This chapter progresses by exemplifying the ethical issues that needed to be taken into consideration for this study to maintain theoretical sophistication and methodological rigour.

5.2. Cultural-Historical Activity Theory and other research methodologies

As illustrated in previous sections of the methodological aspects of this research, Cultural-Historical Activity Theory is a theory both for conducting research and analysis. Employing Cultural-Historical Activity Theory as a conceptual tool enables the researcher to *'conduct very detailed data-driven analyses of the discursive processes, practical actions and mediating artefacts that are employed in the step-by-step production'* of an idea or solution (Engeström, 1999, p.377). Thus this structure and analysis leads to the understanding and interpretation of otherwise fragmented and confusing data. Going further, as Cultural-Historical Activity Theory is a multidisciplinary theory, it is utilized along with other research methodologies to understand, describe and analyse organizations, departments, practices as well as classroom environments. These studies use basic elements of Cultural-Historical Activity Theory and other research paradigms that are considered relevant for the purposes of their research.

The step following the utilization of CHAT as the framework that will inform this study concerns the research paradigm most suitable for capturing and making explicit the issues under investigation. To achieve this, key elements that will be taken into consideration by the fusion of both chosen frameworks had to be made explicit.

To begin with, this study aims to portray the way the activity of proving is constituted for upper primary school students in the classroom. Thus, the research will be undertaken in a classroom in a primary school over a period of time. Since the activity system is investigated against its history, this study will also attempt to fully understand the qualitative changes of this activity within the timeframe that will last. Even though the characteristics of the research population will be identified in a subsequent section of the methodology, it should be noted here that a part of this research will consist of the students and the teacher using a DGE. Neither the students nor the teacher have used this DGE before. This means that the teacher and I will collaborate in order to discuss the utilization of this environment and the design of tasks appropriate for the goals of the teaching sessions. What is more, this collaboration will entail to a certain degree iterative reflection.

Keeping in mind the qualitative approach of Cultural-Historical Activity Theory as well as the aforementioned key elements, the research methodology that initially appeared most suitable was iterative design, which is placed within the paradigm of experimental design and shares some commonalities with action research (Cobb et al, 2003). Would adaptation of this framework lead to accounting for the complexity of the educational setting of the classroom, the task design and the collaboration between the teacher and me? Design research is typically concerned about developing a design and generating new theory about that design process whereas the design process in my study acted more like a Trojan Horse, a means of gaining access to the teacher's objectives. Furthermore, the time constraints of this research study worked against deploying the cyclic nature of iterative design. Thus, an alternative approach had to be employed. In fact, the methodology thought to be taking into account the previously underlined issues that was eventually adopted was a collaborative design approach.

Below I give a brief description of collaborative design, as a more thorough account on how this approach informed my research plan will be presented in Section 5.5. However, due to the fact that a collaborative design approach has been applied mostly in fields related with architecture, engineering and construction where design was supported by information technology, I will also draw upon the literature available for design-based research in order to provide a holistic argument regarding the relevance of this approach to my research. Following this, potential drawbacks and limitations of this research paradigm will also be exemplified so as to tackle these and maintain validity and reliability at all stages of undertaking the study.

The fundamental facet of collaborative design is the notion of collaboration. Studies have attempted to exemplify what collaboration entails and have investigated the social, organisational and technical issues surrounding collaboration in design.

Kvan (2000), in discussing the degree of participation in computer-supported environments, makes the distinction between collaboration and co-operation. He considers collaboration as *'to work together with a shared goal'*, which is different from co-operation, which could be defined as *'to work side-by-side with mutual goals. This distinction indicates that 'design collaboration requires a higher sense of working together in order to achieve a holistic creative result',* (p.410).

Going further, in collaborative design the participants/designers bring into the design process their individual backgrounds, objectives, and motivations and by working with one another they make an effort to achieve a shared goal by making optimal use of each other's knowledge and experience (Simoff and Maher, 2000). This can be achieved by *'observing and understanding each other's moves, the reasoning behind them and the intentions. At any stage of the design, the observer cannot identify a discrete contribution to the design product from one participant or the other'* (Kvan, 2000, p.411). Thus, it can be argued that the design interactions and the exchange of design ideas are influenced by social roles, individual experience or level of expertise (Chamorro-Koc et al, 2009). Consequently, design involves the negotiation of multiple perspectives as the participants with different responsibilities, interests and competencies negotiate the object of the design.

This characterisation of collaboration in design indicates that it is a demanding activity. In order to maintain this collaboration when a team or group of people work together, the issue of compromising inevitably emerges. However, this issue just makes explicit that some of the decisions made might only partially satisfy the team members and it should not be understood as a core problem of collaboration. Through dialogue and negotiation a common ground can be found and without anyone being forced to accept a solution, a conclusion can be made (Détienne, 2006).

In addition, Maher et al (1997), in analysing participation in collaborative design environments, distinguish three categories of design collaboration: mutual collaboration, in which the participants are working together; exclusive collaboration, where the participants 'work on separate parts of the problem, negotiating occasionally by asking advice from the other' (Kvan, 2000), and dictator collaboration, where a decision is being made about the participant that is 'in charge' and guides the process. They conclude that exclusive collaboration is the most effective model to be followed.

Another characteristic of collaborative design approach is time. That is, due to its complex nature, design is not a simple process, but *'consists of a series of distinct events that occupy discrete and measurable periods of time'* (Kvan, 2000, p.412). Gero and McNeill (1998) develop tools in order to analyse design as a *'time sequence of activities'* and argue that being able to 'measure' the design process can provide further answers to questions related with differences that may exist among experienced/inexperienced designers, designers from different disciplines, as well as differences in designing with and without aids.

It should also be mentioned that due to the fact that adopting a collaborative design is a multifaceted process, educational courses are being developed in order to provide the opportunity of experiencing collaborative design. Van Leeuwen et al (2005) employ three approaches in order to address the organisational social and technical issues of collaboration in design projects and they conclude that creativity in teams, collective communication as well as process organisation are three important issues in learning collaborative design. These are aspects of the collaboration process that need to be taken into consideration for any difficult situations to be sufficiently resolved. This is also in accordance with Détienne (2006) who refers to the aforementioned issues while discussing managing task interdependencies and multiple perspectives in collaborative design.

Having presented the literature related to collaborative design, this section now proceeds by providing the necessary links between the aforementioned literature and the collaborative design approach adopted in this study before illustrating how this approach employed alongside Cultural-Historical Activity Theory provides the ground on which the research objectives of this study can be met. Even though a collaborative approach is mostly associated with big projects and generally involves differing groups of people working together, it can also be adopted within a small team as long as collaboration and a design process are involved. As the design process adopted in this research would function as a Trojan Horse, a means of gaining access to the teacher's objectives, the relevance of this approach in this study lies in the collaboration between the teacher and me. After establishing that the teacher and I would work together with the shared goal being the design of DGE-based tasks, it was decided that the nature of the participation in this collaborative design environment would be mutual collaboration. Due to the fact that the team of designers in this project would just be the teacher and I, exclusive collaboration was not the most appropriate model to be followed. By employing mutual collaboration and considering the challenges that may exist throughout collaborative design, it will be possible to unravel these issues and explore how they were resolved. Nevertheless, while mutual collaboration was conducted, how this collaboration evolved in differing phases of this process will be presented and analysed in Chapters VII and VIII. One more aspect of collaborative design that should be clarified for the purposes of this study is time. It has already been mentioned that time was one of the elements that influenced decisions that had to be made in terms of the methodological aspects of this study. However, this characteristic of collaborative design should not be understood as a constraint of this study. This characteristic of a collaborative design approach recognises that the design process includes several phases. By making explicit the sequence of the activities involved, one can better understand this process. This is in agreement with the purposes of this study. Nonetheless, the process of discussing the tasks that would be designed and that actual task design are presented in section 8.2.2.

Having chosen the research paradigms that will inform the study, the issues of generalisability, validity and reliability also need to be addressed. A pitfall that this research might have is generalisability. Due to its contextual focus as small-scale research, the results may not be capable of statistical generalisation and thus have low external validity. However, this issue will be addressed in this study by aiming at a ‘comprehensive data treatment’ (Silverman, 2010). By giving emphasis on the value of even this small-scale study and insisting on a more detailed analysis of all cases of data, then the outcomes can become transferable to other situations. Thus, transferability of results may be achieved.

In addition, given the facilitative role of the researcher, the relationships and working processes between researcher and participants are of central importance (Gray, 2004). The strong personal involvement on the part of the researcher and the participants may raise significant issues of subjectivity and unreliability. Nevertheless, reliability of findings and measures can be supported and retained through the triangulation of multiple resources and kinds of data, the repetition of analyses across the cyclical processes and the use of standardised measures or instruments (Cobb et al, 2003, p.7).

Going further, the issue of validation is as significant for these chosen paradigms as for any other research methodology. To be more precise, according to Gray (2004) the starting point for establishing claims for the validity of the research is with the researcher. The researcher needs to demonstrate publicly that he or she has followed a system of disciplined inquiry, including checking that any judgments made about the data are reasonably fair and accurate (McNiff et al., 2003). The validity of the findings can be achieved by providing the criteria for including a selection of instances illustrating in this way the representativeness of the instances and the findings generated from them, and presenting these material in their original form so as to increase the credibility of the inferences made (Silverman, 2010).

Furthermore, validation can be an informal process, but may also involve the use of formal groups, consisting of critical colleagues, advisers or mentors, or fellow research colleagues, especially selected to scrutinise the outcomes of the research project (Gray, 2004). Respondent validation is another technique that may support the validation of the research findings.

Lincoln and Cuba (1985, cited in Gray, 2004) argue that instead of validity, the aim, certainly of qualitative research, should be to establish the credibility of the research through forging confidence in the accuracy of its interpretations. Nevertheless, the process of validation in this study is strengthened by the elements in the collaborative task design with the teacher· planning, acting, observing and reflection (McNiff, 2003, Gray, 2004).

Considering the main principles and key ideas of both CHAT and a collaborative task design approach, it can be argued that a combination of these methods may lead to a thorough investigation of the research questions. Cultural Historical Activity Theory provides the research with the framework in which all components of the activity system operate at different levels with the possibility of contradictions between these levels. A collaborative design approach helps in identifying how the activity system is transformed over a period of time. It also provides a way of identifying the evolution of objects and other components of the activity system by engaging the teacher in the design of DGE-based tasks to be used with the children. By collaborating with the teacher on such tasks, the aim is to expose the nature of the teacher's objects at the beginning and how these objects change and maybe clash with objects at different levels, such as it is portrayed in the curriculum and expressed by children as they work on those tasks. With the spiral cycle of the design process, the aim is also to identify the factors that influence students' argumentation, either in a positive or negative way and point the activity system towards the elements that may have a positive impact on students' geometrical reasoning.

Therefore, it is important to use the most appropriate techniques for the data gathering, as well as a variety of methods to allow for triangulation. Keeping in mind the objectives of this study and the characteristics of the aforementioned methodological approaches, a potential framework for designing the research and collecting adequate and trustworthy data will be presented and elaborated on. Following this, the data gathering methods chosen for this research project will be discussed.

5.3. A Research Plan using CHAT

As exemplified in a preceding subsection of the methodology, this study aims to explore the way the activity of proving is constituted in the classroom for upper primary school students. It has also been concluded that, in order to achieve this, the research would take place in the classroom. Going further, since the classroom is a micro context within broader macro context levels, and it is the objective of this study to explore all the forces that drive the activity system of the classroom into transformation and change, the structure and organization of the school and the Ministry of Education as wider educational contexts will also be considered. Even though it was considered to undertake the research in two classrooms in two different primary schools, so as to investigate whether contradictions exist in all four levels, time constraints did not allow that. The research aims to consider all levels of the activity system requires extensive (across several components) and intensive (detailed and rich data within each component) data gathering and analysis. The scope of the study means that there is only time to consider such data in one classroom.

In order to be able to tackle the objectives of the study related with the object of developing proving in the classroom, three levels of analysis must be considered: the system level, the teacher level and the student level. The system level, which will remain the same throughout the study, in the broader sense, refers to the policy statements, curriculum, research about proof and proving and why it is important. The teacher level refers to the teacher's attitudes and perceptions concerning the role of proof in the curriculum and in the mathematics classroom, compared with what the teacher actually does in the everyday mathematics classroom. The student level is concerned with the students' activity in the classroom. This may be a less informed way of what is really going on in the classroom, or something completely different.

Keeping in mind the above, the first step towards researching the objectives of the study will be to map out the current situation of the mathematics classroom. Concerning the first level, I will identify and analyse the most relevant official documents from the Ministry of Education and Culture of Cyprus related with the learning and teaching of proof and geometry; the Cypriot curriculum for mathematics, the Cypriot primary school students' mathematics textbook, the Cypriot primary school teachers' guidance books and other documents will be analysed. This analysis will give a comprehensive description of the nature of geometry, geometrical tasks and proving in geometric contexts as they are illustrated in the above resources.

Going further, it will be very interesting to explore initially the personal beliefs and attitudes of the teacher concerning proof and proving as well as the perceptions regarding their importance in learning mathematics and specifically geometry. These will be compared against the outcomes of the analysis of the documents of the first level. Adding to this comparison will be the way these beliefs are reflected or not in everyday practice in the mathematics classroom and the way this perhaps influences students' attitudes and engagement in proving. To achieve this, the current situation of the classroom will be also mapped out. This will make possible the identification of tensions that may occur.

A potential divergence, that might be identified in the teacher's as well as the students' thinking about what is going on in the classroom, will be contrasted with the way the specific primary school is organized and interprets the national curriculum regarding mathematics. A possible contradiction between the teacher's practices in the classroom might be attributable to the expectations and beliefs of the principal of the primary school. In order to explore whether the identified differences are interpreted in relation to the principal's practices, information will also be gathered in relation to the organisation of the school.

The following phase will be concerned with the introduction of a DGE. At this point, it is considered crucial to provide a more detailed justification regarding the introduction of a DGE in the classroom. As exemplified in Chapter II, during the two exploratory studies that were conducted prior to the main data collection process of this study, pairs of students were exploring DGE-based tasks. The findings related with students' argumentation while interacting with this DGE were fruitful. It was also noted that in order to explore students' activity of proving, this could not be achieved in isolation of the classroom environment. By employing CHAT, the utilization of DGE and tasks designed can be explored within the socio-cultural aspects of the activity system of the classroom. Introducing a DGE would also make possible the identification of potential tensions and contradictions. Even though it was the researcher's objective to employ a DGE, it was after a discussion with the teacher that it was decided to specifically introduce Cabri. Nonetheless, it should be stated again that the tasks were the research vehicle, the window for generating data rather than any kind of curriculum intervention. Nevertheless, information regarding this decision will be provided in Section 8.2. where the way the teacher and I collaborated is portrayed.

Concerning the introduction of Cabri, since neither the teacher nor the students are familiar with this Dynamic Geometry Environment, I will collaborate with the teacher in order to design the tasks and organize the teaching episode. This is also why a collaborative design approach has been selected against other research paradigms. To be more precise, it is the goal of this study to engage the teacher in collaborative design with the researcher. As stated previously in reviewing the literature regarding collaborative design approach, collaboration will help making explicit the teacher's objectives and motivations. Despite the fact that the teacher is not familiar with Cabri, collaborating with me and making decisions related with the design of the Cabri tasks, reflecting on the tasks after their introduction in the classroom, and making, where necessary changes on the tasks will provide this study further data regarding the teacher and the overall objectives of the study. Considering the aforementioned shows that the design process of this mutual collaboration would entail two design actions; the design of the DGE-based tasks as well as the design and planning of the lessons.

Sketching all these different elements that influence, in one way or another, what is going on in the classroom will make possible the illustration of the components of this classroom's activity system. Even though the activity system of the mathematics classroom, where the teacher is the subject of the activity, will be portrayed after gathering all data, some of the elements that may characterize the components of the activity system are presented here. The teacher as the subject might be motivated by the need to cover the curriculum and make mathematics meaningful as a school subject and/or the belief that students need to able to reason and reflect on their actions.

The tools refer to both the psychological and material instruments that influence the transformation process. This study focuses on both material and psychological tools. Material tools employed in the classroom include technologies such as computers, Dynamic Geometry Environments (GeoGebra and Cabri), interactive whiteboard, worksheets, books and other physical educational aids such as geometrical instruments. Psychological tools include the students' mathematical knowledge, language, gestures, and the teacher's pedagogical content knowledge.

The object may have several dimensions and thus be associated with the curriculum content, behaviour regulation, developing motivation and technical skills, proving, students learning how to prove and developing students' understanding of mathematics.

As the community is composed by the subject and other individuals, it includes both the teacher and the students. It is also possible to include the school, the Ministry of

Education and other professional bodies such as the teacher's training institutes, networks of teachers' affiliations, communities of mathematics teachers.

The rules, as exemplified earlier in the literature, include implicit as well as explicit rules established by the community. As the community is composed by the classroom and the macro educational context, these rules refer to the sociomathematical norms as well as rules set by the school authorities and other professional bodies. These formal and informal rules are related with classroom and social rules, curriculum protocols, assessments, societal rules, and cultural norms. Thus, concerning the rules, the way the teacher encourages the establishment of the rules in the classroom and achieves collaborative work in teams/pairs will also be illustrated.

The division of labour is concerned with the group dynamics, the division of tasks, power and status among different actors of the system. The division of labour is horizontal in terms of tasks and vertical in terms of power and status. The division of labour for this particular activity system of the mathematics classroom may include the teacher's teaching and management of the classroom situations as well as the responsibilities assigned among the members of the community, students learning, studying, collaborative agreements, validation of solutions and curriculum managers and designers. The question to be asked about the division of labour is how the teacher manages the classroom and what the role of students is in that classroom. Thus, in this particular component of the activity system, the way the teacher intervened during the several phases of the lessons will be illustrated.

As the activity system of proving in the upper primary school consists of several activity triangles, and as exploring students' argumentation is among the objectives of this study, the activity triangle where the students are considered the subject component will also be illustrated and described.

Keeping in mind the aforementioned, the research study is divided into three phases. Phase I includes the analysis of the official documentation and the initial interview with the teacher. In order to map out the current situation of the classroom, Phase II will constitute the baseline observation of the classroom. Phase III includes collaborative task design and thus a Dynamic Geometry Environment is being introduced in the classroom.

Each phase constitutes a clear iteration. At this point, it is considered important to make explicit the iterative nature of the research design of this study. In section 5.2., it has been argued that the purpose of this research study worked against deploying the cyclic nature of iterative design. At the same time, in establishing the methodological approach most appropriate for investigating the research objectives, this study has also been influenced by design-based research. In design-based research, the iterative design process features cycles of invention and revision where *'the intended outcome is an exploratory framework that specifies expectations that become the focus of investigation during the next cycle of inquiry'* (Cobb et al, 2003, p.10). In this study, each phase of data collection is distinct as it corresponds to specific purposes. Simultaneously, conducting the three phases of this study will provide the necessary information towards exploring the activity of proving as developed and transformed in the classroom Thus, themes of interest, emerging from the ongoing analysis of each phase, will also inform the design, implementation and analysis of the subsequent phases. Keeping this in mind, the design of each iteration may differ compared with the phase already undertaken, according to the results of each iteration and the questions raised related with these results and the overall objectives of the study. To conclude, the connection between the iterative nature of this study's design and design based research lies in the emergent themes of interest informing the subsequent phase of data collection.

Considering the aforementioned issues related with the design of the research, the research techniques chosen to support the recognized objectives are participant observation, interviews and documentary analysis.

By critically reviewing the research literature concerning proving in the classroom, it has been argued that, in order to explore the way proving is constituted in the primary classroom, three levels of analysis will be taken into consideration; the system level, the teacher level and the student level. Furthermore, the theoretical assumptions of both CHAT and collaborative design approach, as well as the refinement of the main objective of the study direct the specific research design of this study towards the role of the teacher, Dynamic Geometry Environments, and task design among other crucial factors that impact on the evolving microculture of the classroom. Thus, before proceeding with the presentation of the research techniques, the design and implementation of the research, the theoretical assumptions that underlie these aspects need to be illustrated. That is, the role of tools and instruments in mediating student's understandings and constructions of proofs in geometry, the nature and types of tasks that are considered crucial in providing occasions to facilitate transitions to and from conjectures to proofs in DGE, as well as the mediating role of the teacher in such learning environments, are analysed.

5.4. Tools and instruments

A microworld can be a catalyst for intellectual experience and growth and can be exploited to enrich the social and psychological space of the individual (Noss and Hoyles, 1996). That is, the tool employed in the mathematical experience is not *'only a cognitive tool but also a genuine mediator of social interaction through which shared expressions can be constructed'* (Hoyles at al, 2004, p.317). This remark highlights the necessity to take into consideration the way the tools and instruments shape the students' experience when technology is integrated in the learning and teaching of mathematics (Noss and Hoyles, 1996, Vérillon and Rabardel, 1995, Artigue, 2002, Guin and Trouche, 2002). Even though it is beyond the scope of this study to elaborate on the theoretical frameworks that have been developed by a number of studies for use in technology environments dedicated to mathematics learning (Drijvers et al, 2010), the appropriation and transformation of the tool by the student and the effects of tool use in the students' activity will be taken into consideration.

Dynamic Geometry Environments, such as Cabri Geometry, include any technological medium that provides the user with tools for creating the basic elements of Euclidean geometry through direct motion via a pointing device and the means to construct geometric relations among these objects (Olive et al, 2010, p.147). Once these objects are constructed, they can be transformed simply by dragging.

The central idea of dragging implicates that when relationships have been set up among points, lines and circles, they are preserved even when one of the basic components of the construction is dragged (Hoyles and Noss, 1994, Hölzl, 2001, Arzarello et al, 2002). This real-time transformation feature of Dynamic Geometry allows users to freely move constituent parts of a construction and to observe how other elements *'respond dynamically to the altered conditions'* (Goldenberg and Cuoco, 1998, p.351). In comparison with paper-and-pencil Euclidean geometry constructions, a dynamic Euclidean geometry environment can 'expand our capacity for figure manipulation and address some of the practical limitations of paper and pencil while retaining the basic characteristics of the geometry represented' (Stylianides and Stylianides, 2005, p.38). Hölzl (1996) talks about a Cabri geometry. Even though dragging is a tool for exploring the various invariant relationships inherent in a geometric construction, implementing dragging involves *'new styles of consideration and reasoning'*, (p.171). For instance:

> Cabri does not permit one to drag constructed points; a distinction arises between 'dragable' and 'non-dragable' points. This distinction may be 'ungeometrical' and totally unknown (because unnecessary) in a paper-and-pencil environment but is nevertheless important for pupils working in a DGE (Hölzl, 1996, p.172).

Nevertheless, Cabri-geometre offers an environment which *'favours a stronger link between spatial-graphical and geometrical aspects since spatial invariants in the moving diagrams almost certainly represent geometrical invariants'* (Laborde, 2000, p.184).

In addition to this, the dragging mode can also be viewed as a mediator between the concepts 'drawing' and 'figure' (Hoyles, 1996). As illustrated in Section 2.3.1.1. of the review of the literature, a drawing refers to the material entity while figure refers to the theoretical object. The continuous feedback that a student can get by dragging points with a mouse provides a means to make a distinction between what is constructed by the student from the images on the screen. For example, if the constructed figure does not keep its shape as it supposed to through dragging, then the construction process was incorrect. This is also in accordance with 'messing up'; drawings that can be 'messed up' under dragging (Healy et al, 1994). Hence, for the properties to be kept in the drag mode, students must consider the theoretical features of the construction. The possibility to check visually the construction by dragging also offers opportunities for *'relating the theoretical concepts to visual effects and* *for linking the visual aspects and theoretical aspects of this notion of composition of transformations not only in a passive way but in an operative way'* (Laborde, 1993, p.47). Strässer (1996) concludes that dragging *'helps to bridge the dichotomy of empirical and theoretical aspects of the configuration, by bringing to light visually implicit properties or showing the absence of desired properties',* (p.214).

Having in mind the potentialities of Dynamic Geometry Environments for the validation of geometric constructions, a number of studies have developed different approaches that support students in producing proofs within such an environment.

Hoyles and Jones (1998) argue that dynamic geometry has the potential to promote links between empirical and deductive reasoning by supporting 'what if' and 'what if not' questions and by allowing the production of auxiliary constructions. Jones (2000) explores 12-year-old students' interpretations while they are dealing with problems related to the classification of quadrilaterals in Cabri. The data of this study illustrate that using the particular tool helped students to progress in their understanding of the dependence relationships among properties of a figure and amongst families of figures, and thus, advance towards progressive mathematisation.

Mariotti (2000, 2006) examines how geometrical constructions in Cabri can constitute the key to accessing the idea of theorem by helping students to move from a generic idea of justification toward a formal proof. She argues that the specific features of the software encompass a mediation function, related to the possibility of establishing a channel of communication between the teacher and the pupil, based on a shared language. She also stressed that such an evolution should not be expected to be simple and spontaneous.

Healy and Hoyles (2001) explored the role of software tools in geometry problem solving and how these tools, in the interaction with activities that embed the goals of teachers and students mediate the problem solving process. Through an analysis of successful student responses they concluded that dynamic software tools cannot only scaffold the solution process but can also help students move from argumentation to logical deduction. However, from an analysis of responses of less successful students they found that software tools that cannot be programmed to fit the goals of the students might, in fact, prevent them from expressing their mathematical ideas. Thus the individual behaviour and learning styles must be taken into account by the teacher in a dynamic geometry environment.

In relation to the elaboration of proof within a dynamic geometry environment, Laborde (2000) correlates the cognitive and social nature of proof and the learning environment. She states that on the one hand proof is *'a specific kind of discourse meant both for validating the truth of a statement and for convincing other for the validity of this assertion'*, (p.155). On the other hand, the organisation of these learning environments can be achieved in two ways: *'a cognitive way consisting of a progressive construction of mathematical statements by means of tasks and systematically reconsider and questioned by following tasks, and a social way consisting of a construction of social rules of acceptance of results in the classroom'*, (p.155).

Keeping in mind the students' conceptions of proving, as well as the psychological factors identified previously that influence the emergence of students' meaning for proof, it can be argued that DGE may provide a foundation for deductive reasoning. The aforementioned studies also indicate that the way tasks are designed and the teacher influence the mediating role of these tools. A discussion on these issues is presented subsequently.

5.5. Task design

In relation to the effective design of pedagogic tasks that are more meaningful for teaching and learning, Ainley et al (2006) argue about two contrasting situations that can be identified in the daily mathematical classrooms, a conflict that they call the planning paradox. If the mathematical tasks are determined by the targets identified in the national curriculum in a *'narrow and constrained way'*, (Ainley et al, 2006, p.24), they are not interesting and rewarding for the pupils but just a mechanical application of taught rules. On the other hand, *'planning from tasks may increase pupils' engagement but their activity is likely to be unfocused and learning difficult to assess'* (Ainley et al, 2006, p.24). In this case, the teacher fails to contextualize the tasks and while the tasks motivate pupils in becoming engaged with the activity, they appear unsuccessful in exploiting pupils' mathematical thinking (Ainley, 1999).

A way to address the planning paradox is by considering Ainley et al's argument which is concerned more specifically with task design in classroom settings. Ainley et al (2005) identify the need for purpose and utility (how this piece of mathematics can be powerful) in tasks. According to them a geometrical activity should point students towards the utility of construction and at the same time be seen as purposeful. They argue that a rich mathematical idea involves a pedagogic design based on the framework of purpose and utility, with the support of technology. The pedagogic tasks must have a purpose for students, so that they can have the opportunity to 'appreciate the utility of mathematical concepts and techniques', (Ainley et al, 2006). This position suggests that the task must have an explicit end product that the children care about, one that is perhaps based on an intriguing question. The task may involve making something for other students, containing opportunities for students to make meaningful decisions and involving them in arguing from a particular point of view. In so doing, emergent knowledge is imbued with utility in which the abstractions are seen as useful and the limitations of those abstractions are gradually discriminated.

In addition to what has been discussed, it is also worth conjecturing that the nature of the task is an extra fact that influences the extent to which students get involved with and enjoy working on the task. To be more accurate, open-ended tasks, complex and challenging tasks and tasks that build on cognitive conflict seem to be effective in promoting the development of pupils' mathematical understanding.

Open-ended tasks, which are problems with more than one acceptable answer or problems for which different approaches or strategies lead to the correct single result (Kabiri and Smith, 2003), avoid the mechanical application of taught rules, which is what constitute the traditional closed tasks. In contrast with routine tasks, open-ended tasks offer the opportunity for varied ability students to demonstrate their mathematical ability. In turning traditional textbook problems into open-ended problems, Kabiri and Smith (2003) support the argument that students with different ability levels are able to participate with the mathematical concept.

Furthermore, complexity is concerned with *'the number of subproblems that have to be solved to reach the final answer, along with the number of principles from which the solver has to make a choice when planning the solution'* (Berge et al, 2004, p.6).

While evidence supports the positive effect of complex tasks in avoiding boredom and motivating students, the complexity of the syntax, the amount of information given, the number of conditions and variables and the mathematical content must be considered when designing complex tasks in order not to upset or frustrate students, since this might result in the opposite to the objectives of the designed task.

It also seems preferable to engage students in tasks that use cognitive conflict (disagreement about ideas and approaches) in order for naïve conceptions to become more sophisticated. According to Wood (1999) *'the significance of argument to conceptual understanding in mathematics is related to the development of students' thinking and reasoning that occurs during the acts of challenge and justification'*, (p.189). Moreover, by using tasks that foster assimilation which involves incorporating new information into previously existing structures or schema and accommodation which involves the formation of new mental structures or schema when new information does not fit into existing structures, students are supported in learning geometry.

Considering geometry, the tasks that involve the construction of geometric objects allow students to identify relationships among different geometric properties and objects and reason mathematically (Wares, 2007, p.600). Concerning proving, in avoiding misunderstandings about mathematical deduction and reinforcing pupils in developing a constructive understanding of what a proof entails, Duval (2007) argues that the tasks given to students should be structured in the following way: *'a first stage of free exploration, a second stage of specific investigation into the deductive organisation of propositions in a non-discursive register, and a last stage of verbal description or of verbal explanation of the deductive organisation which has been discovered'*, (p.154).

Additionally, the use of computer based environments for fostering geometrical understanding affects the design of geometrical tasks. In Section 5.2.1. an initial discussion concerning the affordances of computer-based environments in designing tasks and allowing students to construct meaning for geometric construction was provided.

Adding to this discussion, Pratt and Davison (2003), investigating the use of Interactive White Board (IWB) with a dynamic geometry software, conclude that *'the visual and the kinaesthetic affordances of the IWB are insufficient to encourage the fusion of conceptual and visual aspects of children's figural concepts when these affordances are embodied in tasks that simply focus on the visual transformation of geometric figures'*, (p. 37). Therefore, in order for students to draw attention to the conceptual aspect of the figural concept, tasks that are based on *'the utilities of contrasting definitions'* (p.37) should be provided.

Laborde (2001, 2004), in elaborating the integration of technology in the design of geometrical tasks acknowledges that technology allows one to give tasks which would not be proposed in a paper and pencil environment and distinguishes four types of tasks in relation to the way Cabri is exploited: *'tasks in which the environment facilitates the material actions but does not change the task for the students, for example, producing figures and measuring their elements'* (Laborde et al, 2006, p.293); tasks that help students explore, analyse and make hypotheses about geometrical figures using the drag mode; tasks that, while they have a paper-and-pencil counterpart, are solved differently in a computer based environment as it provides several tools for tackling for example a construction task; and tasks that cannot be proposed in a paper and pencil environment but can be carried out only with mediation of the environment.

These types of tasks can be employed by the teacher in relation to the kind of mathematics he aims to develop. To elaborate on this, the first two types of tasks are *'facilitated rather than changed by the mediation of a dynamic geometry environment'* (Laborde et al, 2006, p.293). The last two types of tasks are changed as they allow well-organized strategies which are not possible in a paper-and-pencil environment, or can be carried out only in a dynamic geometry environment (Laborde, 2001).

5.6. The role of the teacher

Within the classrooms where students are encouraged to develop their own personally meaningful ways of knowing, the role of the teacher as a representative of the mathematical community becomes central and critical. The selection of studies presented across the thesis give emphasis on the role of the teacher in shaping the rules of discourse that get privileged in classroom activity, in helping students to construct mathematical ways of knowing that are compatible with those of wider society, and in the negotiation, acceptance and development of these rules of discourse among the students that would allow them to accept or reject an argument as proof (Yackel and Cobb, 1996).

On one hand, the teacher is expected to make the students familiar with patterns of argumentation and with terms such as assumptions, conjecture, example, counter example, refutation and generalisation. Then, through interaction with students, the students' responses contribute to the teacher's developing understanding of their mathematical activity and conceptual development, and the teacher's actions and responses can be interpreted as an implicit indicator of how the students' responses are valued mathematically. Without giving a direct response concerning students' solutions, students *'develop a sense of the teacher's expectations for their mathematical learning without feeling obliged to imitate solutions that might be beyond their current conceptual possibilities'* (Yackel and Cobb, 1996, p.465).

Considering the use of technology, the teacher is also responsible for the orchestration of mathematical situations. In a given mathematical task, the teacher should guide students' instrumental genesis through *'the intentional and systematic organisation and use of various artefacts available in the learning environment'* (Drijvers et al, 2009, p.1350).

Concerning the key place mathematical definitions have in reasoning and proving, the discussion developed in Section 2.4.1. highlighted the complexity of role of the teacher, not only at the secondary and tertiary level, but at the elementary level as well. This is in accordance with Ball et al (2008) who recognize that choosing and developing useable definitions as one of the challenges that are distinctive to the work of teaching mathematics. That is, teachers often face the challenge of how to handle the tension between using mathematically precise definitions and definitions that are appropriate for their students. Also, teachers need to judge the appropriateness and accuracy of the definitions presented in textbooks, as well as make sense of and evaluate definitions used by their students. Furthermore, teachers need to understand how definitions can be used to reconcile disagreements. Thus, teachers need to understand the power of definitions in mathematical reasoning. Understanding this challenge entails explicating the position of the student in relation to mathematics in general and to definitions and the act of defining in particular, as set by the mathematics curriculum and the available textbooks (Morgan, 2006).

This role that the teachers are now expected to have, highlights the significance of the teachers' own personal mathematical beliefs and values as well as their own mathematical knowledge and understanding (Yackel and Cobb, 1996). Yackel (2002) further supports this argument by explicating the need for the teacher to have *'both an in-depth understanding of students' mathematical conceptual development and a sophisticated understanding of the mathematical concepts that underlie the instructional activities being used',* (p.426). This is also highlighted by considering the fact that the studies researching teachers' knowledge about the logico-linguistic structure of proof are suggesting that the inadequacy identified in effectively cultivating proving in the classroom is reflected in the misconceptions students have about proof (Healy and Hoyles, 2000; Stylianides et al, 2004, 2013).

Considering the above it can be concluded that any interpretation of classroom events must also focus on the teacher's actions or pedagogical choices and their impact on the students' understanding. To elaborate more, Küchemann and Hoyles (2006) identified two teacher variables that influence proof scores and their improvement in geometry and not in algebra; the length of teaching experience and the involvement in continuing professional development (p.584). These can be explained by taking into account the recent changes that have taken place in the geometry curriculum in the National Curriculum of England and Wales. Teachers who studied as students little geometry might face difficulties in implementing the curriculum. This, in turn influences their teaching, and therefore their students' learning. The teacher's pedagogical choices are also influenced by the teacher's knowledge about proof (Stylianides and Ball, 2008).

The following section of the methodology chapter of this study describes the aforementioned techniques that will be employed so as to obtain the information needed for achieving the study's purposes.

5.7. Data Collection

As it has been declared formerly, the purpose of this study is to explore the way the activity of proving is constituted for upper primary school students. This section is concerned with the research instruments selected in order to undertake the research. Initially, methodological issues concerning the data gathering tools will be comprehended and discussed. To be more precise, the research procedures chosen will be analysed in consideration with both their strengths and weaknesses along with the advantages and disadvantages when using other research procedures. What is more, the issues of validity and reliability will be raised and explored in relation to the research methods selected. The specific methods will be discussed in the following chapters where the method for conducting each phase of data collection is presented.

5.7.1 The setting

The study was undertaken in a classroom in a mixed public primary school in Nicosia, the capital of Cyprus. This mainstream school is considered to be a dynamic school; it participates in European educational programs and endorses the goals set and published through the official newsletters of the Cyprus Ministry of Education and Culture. That is, the school actively encourages teachers and students in engaging at a deeper level with the educational experience. It also has a good reputation due to achievement and social order. Adding to this, the Principal of the school always welcomes young researchers to undertake their studies at the school as long as this does not disrupt the ordinary operation of the school. As the Principal of the school states, this is beneficial for all individuals involved as this can promote professional development and new ways of teaching and learning.

The teacher voluntarily agreed to take part in the research. This experienced teacher was one of the Deputy Principals of the school. She also participated in a program organised by the Ministry of Education and Culture regarding the integration of technology and the way the teachers in a school collaborate in order to support each other in integrating technology in their teaching. This was among the reasons why the teacher was allocated in a classroom with 12 computers and an interactive whiteboard. It is worth mentioning that this regular classroom was the only classroom at the school with ICT facilities. There were 22 students in the classroom, 15 boys and 7 girls. According to the teacher, the students had mixed abilities. What is more, since the classroom was equipped with technology, using computers was a flexible procedure and part of the classroom's routine. However, the students were not really familiar with DGE.

5.7.2 Participant Observation

Observation is considered to be one of the most important methods of data collection and it is a fundamental tool associated with design-based research. In observation the primary research instrument is the self, consciously gathering sensory data through sight, hearing, taste, smell and touch (Somekh and Jones, cited in Somekh and Lewin, 2005). In agreement with the latter Gray (2004) mentions that observation is a complex combination of sensation and perception.

Participant observation is largely qualitative and emphasises the meanings that people give to their actions. It is a research method most closely associated with ethnographic methodology and the central intent is to generate data though observing and listening to people in their natural setting, and to discover their social meaning and interpretations of their own activities (Gray, 2004). With participant observation, the researcher becomes a member of the group being researched and so begins to understand their situation by experiencing it. According to McKernan (1991), participant observation bears the highest fidelity with the methodological purpose of action research and is the foremost techniques for use in the study of classrooms and curriculum. It is not a single strategy but a methodology for field-work studies.

Participant observation has certain decided advantages as a research technique. In particular, participant observation can be contrasted with research using a questionnaire where it is often not possible to verify whether people are telling the truth. In contrast with participant observation it can be possible to interpret some of the subtleties of meaning in the data (Gray, 2004).

It is of great value that participant observers gain unique insights onto the behaviour and activities of those they observe because they participate in their activities and, to some extent are absorbed into the culture of the group (Somekh and Jones, cited in Somekh and Lewin, 2005). In addition, the greatest benefits of participant observation are in terms of collecting authentic accounts and verification of ideas through empirical observations (McKernan, 1991). It is also important that the study takes place in the 'natural' environment of the participants.

Furthermore, unlike the survey researcher, the observer can take as much time as is required to gain a representative sample of behaviour, wherever the time constraints of his/her research study allow him/her to do so. The advantage is that unlikely as well as likely occurrences will probably be sampled. Add to the latter, the observer can make notes of non-verbal behaviour, like facial and body movement and gestures, which are not available to the sample survey researcher (McKernan, 1991).

Nevertheless, participant observation has several limitations as a research technique. Initially, an important disadvantage is the enormous complexity of human behaviour, whether as individuals or in groups, and the impossibility of making a complete record of all the researcher's impressions.

Add to this is the subjectivity of the researcher who at the same time as collecting sensory data is actively engaged in making sense of impressions and interpreting the meaning of observed behaviour and events. In undertaking a participant observation one of the challenges is to maintain a balance between "insider" and "outsider" status. To gain a deep understanding it is essential that the researcher gets both physically but also emotionally close to the group being observed (Gray, 2004). However, in doing so it is difficult for the researcher to maintain a professional distance.

Considering the aforementioned, one can realise that observers always have some kind of impact on those they are observing who, at worst, may become tense and have a strong sense of performing, even of being inspected. However negative effects are reduced if the purposes of the observation, how the data will be used and who will be given access to them are made clear in advance (McKernan, 1991). Of course not too much information should be given about the research as this may have an impact on the subject's activity.

Furthermore, since unstructured modes of observation rely heavily on description rather than measurement and counting procedures, it is often difficult to impose a coding frame on massive amounts of qualitative data. Additionally, the small size of population observed may lead to difficulties in generalising the results to larger populations (McKernan, 1991).

Moreover, disadvantages include that the participant observers may be distracted from their research purpose by tasks given to them by the group, and note-making becomes much more difficult and may have to be done after the event, ideally in the same evening (Somekh and Jones, cited in Somekh and Lewin, 2005).

Considering all of the above, a participant observer-researcher should have in mind that observation needs to be a systematic, structured process, so that data can later be categorised and ready for analysis.

Given the fact that participant observation was one of the research techniques employed to gather the research data, my role in the classroom had to be clearly defined and followed. In doing so, decisions considering how much participation was really possible as well as ways to maintain sufficient distance to observe activity in the classroom had to be made. These decisions would also delimit the level of participant observation.

According to Spradley (1980) there are four levels of participation within the method of participant observation; passive, moderate, active and complete participation. For the purposes of this research study, my role as an observer was placed in the intermediate position between the two extremes of this dimension. However, it should be noted that despite the level of my participation, I tried to be as unobtrusive as possible at all times.

5.7.3 Interviews

Interview is one of the most effective methods of gathering data. It can be used in a variety of research contexts and it is used frequently within a design-based research project. It is significant to mention that because design-based research typically aims to be educational, interviews are likely to be informal discussions rather than formal interviews.

As Roberts-Holmes (2005) points out, the rich detail so often elicited during interviews is unlikely to be gained using questionnaires. In addition, interviews can provide a wide range of information that a written response would conceal (Bell, 2005). In contrast with other research techniques, it allows the interviewer to probe areas of interest as they arise during the interview. Included in the advantages is that interviews can be used to test out a hypothesis or to identify variables and their relationships (Cohen et al., 2005).

Furthermore, the interview is a personal contact situation in which one person asks another person questions, which are pertinent to some research problem. As such, it allows the focus to settle upon a specific issue which can be explored in some real depth and determines what an issue looks like from another's vantage point (McKernan, 1991).

Nevertheless, interviewer bias can creep into the interview situation in many subtle, and not so subtle, ways, during interview instructions, altering factual questions, rephrasing questions, etc. The skill of the interviewer, when he or she has to provide guidance or clarification, is to offer such explanation without influencing the answer of the respondent (Gray, 2004).

Regarding the different ways of conducting an interview, in the particular study semi-structured interviews will be employed. While a structured interview is similar to a questionnaire and an unstructured interview is more difficult to analyse, semi-structured interviews *'allow for a depth of feeling to be ascertained by providing opportunities to probe and expand the interviewee's responses'* (Opie, 2004, p.118). To elaborate more, probing on interviewees' opinions, in order to expand on their answers, may allow *'for the diversion of the interview into new pathways'* (Gray, 2004, p.217) which even though they might not have been considered as part of the interview schedule, they assist towards meeting the research objectives. Consequently, semi-structured interviews provide the interviewer with sufficient flexibility, in a way that the already determined questions help prevent aimless rambling and the interviewer can still ask supplementary questions and shape the flow of the information (Wilkinson and Birmingham, 2003).

5.7.4 Documentary Analysis

Documentary analysis is considered a research tool associated with the analysis of documents and texts, which seeks to quantify content in terms of predetermined categories and in a systematic and replicable way. It has also been referred to as a specific type of descriptive method, as it is an approach that attempts to describe data (Anderson, 1998). What is more, it has been described as 'content analysis' which is defined by Berelson (1952, cited in Gall et al, 1996) as *'a research technique for the objective, systematic and quantitative description of the manifest content of communication',* (p.357).

According to Anderson, (1998) documentary analysis is commonly used for describing the relative frequency and importance of certain topics, evaluating bias, prejudice or propaganda in print materials, assessing the level of difficulty in reading materials and for analysing types of errors in students' work.

Documentary analysis is a method that can be applied to many different kinds of documents. Therefore, it enables the researcher to comprise a great amount of textual information and identify its properties and structures. What is more, documents can usually be accessed more easily in contrast to human subjects and the data *'can be analysed directly through the page without having to collect it first through interviews or other processes and there is reduced danger of bias in its collection and interpretation'* (Verma and Malick, 1999, p.113).

In conducting documentary analysis, one must consider certain limitations, from which like all research techniques, documentary analysis suffers. Content analysis alone cannot provide answers to 'why' questions. Moreover, finding and analysing the volume of documents acquired can become an extremely time consuming procedure.

In order to choose the documents, on which the researcher will work, authenticity, credibility and representativeness of the assessed documents must be assured. This means that the researcher must be sure that the document under examination is what its purports to be and is representative of all possible relevant documents, since if certain kinds of documents are unavailable or no longer exist, generalisability will be jeopardized. In addition to this, an unrepresentative sample of documents would *'bias the study's findings and damage them'* (Verma and Malick, 1999, p.113).

The following section provides a description of the data analysis process implemented in this study.

5.8. Method of Analysis

After choosing the data collection processes, the approach most apposite for interpreting, understanding and explaining these data had to be chosen. This study followed the three concurrent flows of activity in data analysis, as proposed by Miles and Huberman (1984) *'data reduction, data display, conclusion drawing and verification'*, (p.21).

The overall process of analysis of the collected qualitative data was of progressive focusing which shares common features with both content analysis and grounded theory. While Parlett and Hamilton (1972) supported an approach where 'researchers systematically reduce the breath of their enquiry to give more concentrate attention to the emerging issues' (p.18), Stake (1981, 1995, 2000, 2004) formally described this perspective. According to Stake (1981), *'progressing focusing requires that the researcher be well acquainted with the complexities of the problem before going to the field, but not too committed to a study plan. It is accomplished in multiple stages: first observation of the site, then further inquiry, beginning to focus on the relevant issues, and then seeking to explain'*, (p.1). Adding to the above, Hammersley and Atkinson (1983) state that *'progressive focusing has two analytically distinct components. First, over time the research problem is developed or transformed, and eventually its scope is clarified and delimited and its internal structure explored'*, (p.175).

While progressive focusing describes the development of this study, the complexity and 'messiness' in the conducting and reporting of qualitative data should also be addressed. In tackling these issues, coming to know and validating that knowing in a *'reliable, robust and generative manner'* (Mason, 2002, p.212) can been achieved by getting agreement not only about the analysis but also about the phenomenon to be explained. That is, to draw a distinction between giving an 'account-of' and 'accounting-for' some incident or situation. According to Mason (2002), who elaborates on the discipline of noticing, to give an account-of is to describe or define something as objectively as possible. An account-of is a *'description of what was seen, heard, experience, described in terms which others can recognize, without elaboration, justification or explanation'* (Mason, 2002, p.52). An account-of 'attempts to draw attention to or to resonate with experience of some phenomenon' (p.40). Collecting accounts-of *'is one step towards creating a phenomenon, that is, identifying a type of situation, tension, issue or interaction which is exemplified in several different incidents or experiences'*, (p.41). To account-for something is to offer *'interpretation, explanation, value-judgement, justification, or criticism'* (Mason, 2002, p.41).

At this point, let me further elaborate on the relevance of the above constructs in the development of this study. In order to portray the activity of proving in the mathematics classroom, this study was undertaken in three phases. Providing plain, objective accounts-of before interpreting the data is considered important as it provides a clear connection with the way the collected data were analysed (see Section 5.8.2. below). Furthermore, in many instances, these accounts encompass critical moments that will be analysed in several ways. Thus, providing a plain accounting-of the data will also lead to a systematic and coherent argument in the analysis and discussion that will follow. Furthermore, using accounts of and then accounting for also strengthens the validity of the assertions that will be made regarding the way argumentation is established in the classroom.

Keeping in mind the aforementioned, the analysis of the collected data began with informative questions of 'what' and 'how' (Silverman, 2010). This led to asking 'why', which further guided the analysis. The following sections describe in more detail this process.

5.8.1 Management and selection of data

As illustrated earlier, during the data collection process the following data were gathered: official documentation, the researcher's field notes from the observation and the informal discussions with the teacher, audio recording of the teacher's interview, audio recording of one pair of students while they were exploring tasks in a DGE, video recording of the lessons for both phases and students' paper work.

Initially, in preparing the report on the documentation, contextual information was collected and added to the documentary material. Furthermore, in order to maintain consistency, the selected documents were analysed several times.

Concerning the field notes, they were word processed and used at the beginning to get a general feeling of the progression of the lessons. During the reduction of data, the notes of the observational data were drawn on so as to recall instances that were not clearly apparent while analyzing the transcripts, but were thought of as important during the classroom observation. Of course, care was taken to avoid allowing these events to structure the data recording itself.

With reference to the video recordings from the classroom observations, the first step was to watch the videos. During this process I was keeping records of perceptible actions or occurrences. This was the first step towards emerging themes as well as attempting to develop classifications that illuminate the data. The subsequent step was to transcribe the video data. In order to make the video data easily accessible and suitable for analysis, it had to be decided what to transcribe and how. Specific transcription rules were not followed. However, all video data were transcribed in Greek at the beginning and then translated into English, by writing down precisely everything that was said, as well as gestures and other activities that seemed relevant for this analysis. The same process was followed for the collected audio data through the teacher's initial interview.

In order to make sure that the meaning of what was said was not changed during the translation, a fellow Greek Cypriot PhD student agreed to read the transcriptions and compare them with the data collected.

A complete, accurate and detailed lesson narrative for each of the classroom observation was achieved by including direct quotes from the classroom discussion, the field notes (in particular the informal discussions with the teacher) as well as screen shots from the teacher's technological display and other photographical material being utilized in key, as considered, classroom protocols.

At no point was any data discarded. That is, even if initially some data may have not appeared to shed light on the way proving is constituted in the classroom, they were required at a later stage of the process of analysis and discussion according to the main themes of the research questions.

The above process of preparing the data led to the following separate datasets; the report on the documentation, the teacher's story as well as the lesson narratives from the two phases of classroom observation. The following section provides the process of analysis that was followed in this research study.

5.8.2 Process of data analysis

To begin with, before proceeding, it should be made explicit that as the analysis and discussion of each dataset informed the subsequent phase, the analyses are presented in the chronological order in which they were carried out. This way, the overall analytical process can be perceived as a continuously developing organic whole.

Phase I

To begin with, it has been exemplified in Section 5.7., that documentary analysis and an initial interview with the teacher would be conducted in order to shed light on the system level and the teacher level accordingly. Sections 5.7.3. and 5.7.4. focus on the presentation and justification of the themes of interest that were taken into consideration in implementing the documentary analysis and the initial interview with the teacher. These are summarized in Table 5.1. below.

In order to analyse the report on the official documentation as well as the transcripts from the teacher's interview, specific steps were followed.

Regarding the official documentation, as the aim of this study is to portray the activity of proving in the mathematics classroom, the report was further scrutinized in conjunction with mathematical argumentation. This was achieved by employing the notions of explanation, justification, proof and proving as illustrated in the literature and elaborating on them by using the examples pinpointed by the data. Following this, by keeping in mind the elements that direct mathematical reasoning towards the ultimate goal of formal proving, the status of exploration and definition in the official documentation was also identified and further discussed.

Regarding the transcripts from the teacher's interview, the first step of analysis was to identify whether contradictory statements existed in the teacher's interview. Following this, the teacher's views and beliefs were compared against the report and analysis of the official documentation as well as the literature.

Table 5.1.: Initial themes for the documentary analysis and the interview with the teacher

Initial themes of interest emerging from the literature review and the exploratory studies	**Themes for the Documentary Analysis** Information was collected concerning: • The role of proving in primary education, informed by Chapter 2, section 2.2. • The objectives for teaching and learning geometry, informed by Chapter 2, section 2.3.1.1. • The geometrical tasks illustrated as important for developing geometric thinking and understanding, informed by Chapter 2, section 2.3.1.1. and Chapter 5, section 5.5. • The approaches the ICT offers in facilitating the teaching and learning of school geometry, informed by Chapter 5, section 5.4. **Themes for the Interview with the teacher** Teacher's beliefs and views regarding: • The nature of mathematics, informed by Chapter 2, (the nature of mathematics, the significance of geometry and the importance of proof and proving in mathematics and specifically in geometry). • The nature of teaching mathematics, informed by Chapter 2, section 2.3.1.1., Chapter 3, section 3.3.4 and Chapter 5, sections 5.4. and 5.6. (the mathematics curriculum, the factors the teacher takes into consideration in teaching geometry and proof, as well as the introduction of ICT and DGE in the teaching and learning of mathematics). • The nature of learning mathematics, informed by Chapter 2, section 2.3.1.2. (the teacher's perspectives regarding students learning geometry).

Analysis of the report on the official documentation and the teacher's initial interview led to the emergence of clear themes. Bringing together these insights led to a synthesised set of themes that were taken into consideration in accounting of the data of the classroom observation. An overview of the development of the themes in Phase I is presented in Table 5.2. below.

Table 5.2.: Map of themes Phase I

Post Phase I	**Insights from the documentary analysis** D1 Low level of expectation with regards to explaining in Year 6 (an explanation is given by providing the mathematical operations used to find the answer; the justification is provided by using the definition). D2 Low level of expectation regarding exploration and investigation in problem solving. D3 Proof and proving is not acknowledged as a key criterion, nor mentioned in the mathematics curriculum. D4 Definitions as approached by the official documentation are descriptive and extracted. **Insights from the interview with the teacher** I1 The transition from primary to secondary school mathematics influences her teaching. I2 The teacher endorses the use of precise mathematical language. I3 The teacher gave particular emphasis on mathematical definitions. I4 The teacher believes that exploration which leads to discoveries is very important. I5 The teacher considers the justification and proving processes as necessary for the geometry concepts that she teaches. I6 The teacher made contradictory statements regarding the integration of technology.
New synthesised set of themes	T1 Exploration (D2, I1, I4, I6) T2 Mathematical argumentation: explore the opportunities the students had to explain, justify and prove in the classroom (D1, D3, D4, I2, I3, I5)

This map of the themes, and those set below, will be elaborated fully when the findings and analysis are presented in Chapters VI-VIII of this thesis.

Phase II

By keeping in mind the themes which surfaced from Phase I of the study, going through the classroom accounts initial classifications were formed by a repetition of the teacher's actions that were either constant or diverse in comparison to what the teacher said during the interview or what the curriculum states about geometry. By bringing together the insights from the initial analysis of the lesson accounts and the informal discussion with the teacher, levels of actions describing the classroom activity emerged. An overview of the development of the themes in Phase II is presented in Table 5.3. below.

Table 5.3.: Map of themes Phase II

Post Phase II (acting as moderate observer)	**Insights from the classroom observation and the informal discussions with the teacher** MO1 Occasions where the teacher is making connections with parts of mathematics that the students would be taught in secondary school, were taught either recently or in the past are identified ('opening out' value). MO2 The parameters play/ learn are identified: there are occasions where the students' activity was being translated by the teacher as 'playing' instead of learning. MO3 The following rules of discourse being negotiated and established in the classroom are identified: 'doing mathematics requires us to justify our assertions', 'doing mathematics requires us to use precise language', 'we write coherent geometrical explanations'. MO4 The teacher is integrating technology in her teaching.
New synthesised set of themes	**Levels of action** L1 Level of exploration (T1, MO1, MO4) L2 Level of play (MO2) L3 Level of participation (MO3) L4 Level of intervention (T1, T2), L5 Level of proving (T2) L6 Level of collaboration

Phase III

The process of analysis of the classroom observation followed in Phase III of the study was undertaken by following a similar approach as in Phase II. Accounting of the data was initially guided by the previously identified levels of actions. This led to the exemplification of protocols that fall into the emergent levels of action. It also led to identifying connections between the emergent levels and thus, reconsidering them. Table 5.4. below provides an overview of the ongoing analysis and discussion as developed in Phase III.

Table 5.4.: Map of themes Phase III

Post Phase III (acting as participant observer)	**Insights from the classroom observation** PO1 The contradictory value open/close is identified: there are instances where the teacher is closing down an exploration activity. PO2 Exploration is related with the mathematical situations the teacher provided, the exploration of the Dynamic Geometry Environments as well as exploration for supporting mathematical connections. PO3 The value ‘play’ is related with both exploration and intervention. PO4 Mathematical argumentation is related with explanation and justification. PO5 Definitions and defining seem to be an integral aspect of this mathematics classroom, around which explaining and justifying developed. PO6 Sociomathematical norms are identified: ‘doing mathematics requires us to use precise language’, ‘doing mathematics requires us to justify our assertions’‘we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations’, ‘we write coherent geometrical explanations’. **Insights from the teacher-researcher collaboration** C1 A tension during the task design process is identified. This was related with the types of tasks to be designed. C2 A tension related to the utilization of the tasks throughout the lessons is identified: the teacher closed down the tasks.
New synthesised set of themes	**Activities of action** Activity of exploration (PO1, PO2, PO3, C1, L1, L2, L3, L4, L6). Activity of explanation (PO4, PO5, PO6, C1, L3, L4, L5, L6).

At this point it is considered important to make explicit that as a worksheet was provided to students while working on the computers, the transcripts from the students’ activity were analysed so as to find instances of students’ explanation and justification.

Putting it all together

The following phase of analyzing the classroom situation was achieved by conducting a retrospective analysis. The ongoing analysis conducted while the study was in progress led to a focus on several issues and events. In contrast, retrospective analysis seeks to place these in a broader theoretical context. By working systematically through the data sets generated through the three iterations, and being explicit about the evidence used when making particular inferences, the aim is for the resulting claims to be trustworthy.

Initially, the classroom data were analysed through the lens of CHAT. By employing the main aspirations of CHAT alongside the literature that informs the study, the activity systems of exploration and explanation were constructed. Achieving this led to better explore the interrelations of the components of each activity system. Portraying these activities also constituted the foreground in discussing the research questions. Even though the rationale for using CHAT has been elaborated on in Sections 4.4. 'Using CHAT as a lens for analyzing students' proving' and 5.3. 'A research plan using CHAT', it will be further presented in Chapter IX.

Achieving the above, led to a synthesis of the analysis. This process involved conducting retrospective analysis by drawing on the entire data set generated through the three phases of the study. That is, by portraying the three levels of analysis, the micro level of the classroom activity was contrasted against the broader macro context as well as the collaboration with the researcher.

To conclude, one cannot neglect the fact that as a single author-study, the findings are dependent on one person's interpretation, validated by the independent interpretations of my supervisor. Nonetheless, while acknowledging the issue of subjectivity, I consider it crucial to make explicit that I have endeavoured to clearly corroborate all claims made, and shared my findings in order to receive feedback at various stages.

5.9. Ethical Considerations

This research study was planned and carried out in compliance with the British Educational Research Association's (2004) *Revised ethical guidelines for* *educational research*. I also drew upon the British Psychological Society's (2010) *Ethical principles for conducting research with human participants*.

In conducting the research in a primary school in Cyprus, the permission of the Heads of the Departments of Primary and Secondary Schooling had to be obtained. In doing so, an application was completed and submitted in the Cyprus Pedagogical Institute requesting authorization to conduct the research. This application included a summary of the planned research, the methods employed and any ethical issues that would be taken into consideration. The approval letter for conducting the research (see Appendix I), included ethical considerations that should be taken into account in approaching the school and conducting the research. This also satisfied the Institute of Education's own ethical approval process (Institute of Education, 2206). It was also stressed that the activities used had to fall into the framework defined by the national curriculum and that the loss of teaching time should be kept to the minimum.

Following this, the school was approached in order for the researcher to get the approval firstly from the Principal of the school and secondly from the teacher. Subsequently, a letter was given to the students' parents requesting their informed consent for their children to take part in this study and videotaped during the lessons (see Appendix II). It was stressed that the video recording device would be placed in such position in the classroom that would not allow the recording of the faces of the students. They were also assured that confidentiality, anonymity and privacy of the participants' data would be sustained at all levels of this project. In addition, it was declared to all participants that the researcher would return with a summary of themes that would be elicited from the data in order to verify that the findings of the study are consistent with the participants' perceptions. In this way, respondent validation would be addressed. Furthermore, the findings would also be disseminated to the Ministry of Education and Culture in Cyprus, so as to be exploited at both a theoretical and practical level.

Ethical considerations were also taken into account while planning the interview schedule, interviewing the teacher, as well as during informal discussions at the end of each lesson. Concerning the interview schedule, a pilot study was undertaken in order to gain feedback on the validity and reliability of the questions. The interview and the informal discussions took place in a familiar environment of the teacher so that she would feel comfortable and avoid interruptions from outside, such as phone or people knocking on the door. In this way, the teacher would not become anxious and upset during these discussions but would concentrate and express her feelings and thoughts on the subject investigated.

5.10. Summary

This chapter was devoted to discussing and presenting the methodological aspects of this study. Initially, an argument why collaborative design alongside CHAT enables this study to systematically investigate the study's objectives was provided. That is, the relevance of this approach in this study lies in the collaboration between the teacher and me. Thus, the design process adopted in this research would function as a Trojan Horse, a means of gaining access to the teacher's objectives.

Following this, the research plan was presented by taking into consideration the theoretical aspects related with tools and instruments, task design and the role of the teacher, as they constitute elements that would guide the research. That is, the data collection process would be undertaken in three phases which include: (i) reporting the official documentation and interviewing the teacher; (ii) mapping the current situation of the classroom and (iii) introducing a DGE.

This chapter then proceeded by elaborating on the theoretical assumptions underlying participant observation, interviews and documentary analysis as this will support the discussion pertaining the specific details in conducting the three phases of the study.

By identifying the three iterations as well as the research techniques that will be employed so as to obtain the information needed for achieving the study's purposes, the development of the study can now be portrayed (see Table 5.5. below).

This chapter progressed by providing a thorough discussion regarding the method of analysis. Drawing upon progressive focusing and the constructs 'account-of' and 'accounting-for' led to describing how the management and selection of collected data, as well as the process of analysis was achieved. The ethical considerations that were considered in conducting the research have also been elaborated on.

Table 5.5.: An overview of the stages of the study

PRELIMINARY STUDIES	
Exploratory Study I	**Main Research Focus:** Exploring students' argumentation, experienced through a dynamic geometry environment **Research Instrument:** Observation, informal discussions with the students, DGE-based tasks **Role of the Researcher:** Teacher-researcher
Exploratory Study II	**Main Research Focus:** Investigating students' argumentation within the classroom setting **Research Instrument:** Classroom observation, DGE-based tasks **Role of the Researcher:** Teacher-researcher
MAIN STUDY	
Phase I The Documentary Analysis and the Initial Interview with the teacher	**Main Research Focus:** Identifying the system level and the teacher level **Research Instrument:** Documentary Analysis, Semi-Structured Interview
Phase II Baseline Observation	**Main Research Focus:** Mapping the current situation of the classroom **Research Instrument:** Classroom observation, informal discussions with the teacher **Role of the Researcher:** Moderate participation
Phase III Participant Observation	**Main Research Focus:** Collaborating with the teacher as a means of gaining access to the teacher's objectives **Research Instrument:** Classroom observation, informal discussions with the teacher, DGE-based tasks **Role of the Researcher:** Active participation

The remaining chapters of this thesis are dedicated to presenting the three phases of the study, analysing as well as describing the meanings of the findings.

CHAPTER VI

PHASE I

DOCUMENTARY ANALYSIS AND THE INITIAL INTERVIEW WITH THE TEACHER

6.1. Introduction

Chapter VI focuses on Phase I of the data collection process. By taking into consideration the general discussion of the theoretical assumptions underlying documentary analysis (see section 5.7.4), this chapter initially provides a concise description of the way the selection and analysis of the 'units' of analysis was achieved. Subsequently, the report on the official documentation is presented.

This chapter then proceeds by elaborating on the initial interview with the teacher. That is, by drawing upon the general discussion of the theoretical assumptions underlying interviews (see Section 5.7.3), the method for conducting the semi-structured interview is introduced. Following this, the responses the teacher gave to the questions asked during the interview as well as statements that were considered important are illustrated.

This chapter continues with the analysis of the official documentation and the extracts from the teacher's initial interview. Insights emerging through the analysis of the official documentation are elaborated on by taking into consideration the initial themes for conducting this analysis as well as the available research literature (see Section 6.2.1.). Insights emerging from the teachers' initial interview are discussed by taking into consideration the themes of interest that were taken into account in conducting the interview (see Section 6.3.1.).

This chapter progresses by describing how accounting of the data from Phase I of the study led to new themes being drawn together.

6.2. Documentary Analysis

This section presents the method for conducting the documentary analysis. It also provides the report from the analysis of the official documentation.

6.2.1 Method for the documentary analysis

As exemplified in Section 5.3. of the methodology chapter, one level of analysis that needs to be considered in picturing the activity of proving in the classroom is the system level.

In identifying the system level, documentary analysis is employed so as to give a comprehensive description of proof and proving as presented in the official educational documents of the Ministry of Education and Culture in Cyprus. To be more accurate, information will be collected concerning the role of proving in primary education, the objectives for teaching and learning geometry, the geometrical tasks illustrated as important for developing geometric thinking and understanding and the approaches the ICT offers in facilitating the teaching and learning of school geometry. Since this research is focusing on the upper primary school classroom, the analysis will be carried out in the existing documentation for Year 6.

Keeping in mind that documentary analysis offers the prospect of different kinds of 'units' of analysis to be considered, the documents used for identifying and comprehensively discussing the above patterns are:

- The National and Mathematics Curriculum of Cyprus.
- The Cypriot primary school students' mathematics textbooks.
- The Cypriot primary school teachers' guidance books.
- Official documents from the Cyprus Ministry of Education and Culture which are concerned with the teaching and learning of geometry and identify ways for enhancing the process of learning geometry and construct geometrical tasks, either by the mediation of technology, or by replacing the traditional ones.

Initially, the analysis of the mathematics national curriculum will make explicit the general objectives of primary education as well as the objectives to be met regarding the learning and teaching of geometry. Following this, the position of the official documentation in regards to the integration of technology will also be identified.

Furthermore, as the textbook constitutes an important element of the learning and teaching process, analysis of the textbooks will provide further information regarding the mathematical knowledge that is considered relevant in this particular historical moment.

Considering the aforementioned statement, the teachers' guidance books will be analysed in order to identify the specific goals of each geometric chapter that needs to be covered as well as the nature of the examples that are provided as a guidance and support for the teachers in their attempt to meet the national curriculum requirements for mathematics.

Going further, consideration of the aforesaid will guide the analysis of the activities presented in the students' mathematics textbooks. Research studies that focus on the analysis of mathematics textbooks examine a variety of aspects and issues. These studies used different methods as well as different units of analysis. In regards to proof and proving (including explaining and justifying), textbook analysis focused either only on the explanatory text presented in textbooks (Stacey and Vincent, 2009), the tasks intended for student work (Stylianides, 2008), or both textbook components (Hanna and de Bruyn, 1999). An important remark that needs to be made regarding the analysis of mathematics textbooks is related to the recognition of the challenge the researcher faces when deciding what inferences to make regarding the formulation of the task. That is, since there is no direct access to the objectives of the textbook developers and the way the tasks play out in the classroom activity is not available when analysing the textbooks, a careful analytical framework needs to be designed and utilized.

Building on the above studies and keeping in mind the purpose of the documentary analysis employed in this study, the analysis of the students' textbooks will examine (1) the justifications to mathematical statements offered in the textbooks, and (2) the opportunities provided for students to justify and explain their own mathematical work.

Reporting on these different units of analysis will then make possible the findings of the documentary analysis being brought together and further scrutinized in conjunction with mathematical argumentation (see Section 6.4.1.). This will be achieved by employing the notions of explanation, justification, proof and proving as illustrated in the literature and elaborating on them by using the examples pinpointed by the data. Adding to the above, by keeping in mind the elements that direct mathematical reasoning towards the ultimate goal of formal proving, the status of exploration and definition in the official documentation will also be identified and further discussed.

6.2.2 Findings from the documentary analysis

This section provides the findings from the official documentation. Initially, the characteristics of the Educational System of Cyprus are presented. Following this, the National and Mathematics Curriculum of Cyprus with a particular focus on Year 6 is reported, by taking into consideration the initial themes of interest for conducting the documentary analysis. Going further, a description of the teacher's guidance book is provided. This section proceeds by presenting the findings from the analysis of the students' textbooks.

6.2.2.1. The Educational System of Cyprus

The main characteristic of the Educational System of Cyprus is that it is highly centralized. The highest administrative body of the Government for education is the Ministry of Education and Culture, which is responsible for all educational institutions in Cyprus. The Ministry of Education and Culture is responsible for the preparation and enforcement of new legislation concerning education, as well as for the prescription of the syllabi, the national curriculum and the national textbooks (Pashiardis and Ribbins, 2003; Pashiardis, 2004). In regards to primary education it should be noted that it covers Years 1 to 6 (6 to 12 years old) and is compulsory and provided free in public schools. Generally, the same classroom teacher organizes the teaching and learning process in all subjects. There are no final examinations at the end of primary education. Upon successful completion of Year 6, pupils receive a primary school-leaving certificate which states that the pupil can enrol in a lower secondary school.

6.2.2.2. The National and Mathematics Curriculum of Cyprus

The National Curriculum of Cyprus is applied to all schools on both the primary and secondary level. The National Curriculum is written on very general lines and applies to all pupils attending public schools. The curriculum does not contain aims and objectives for each school year, but generally for all grades both in the primary and secondary schools. Of course, this gap is often completed by teachers' guides or other material offering instructions for teaching and in which the targets for each school year and units that are to be taught are clearly stated and analysed. The textbooks are prepared by practicing teachers working under the guidance of Departmental Committees consisting of members of the representatives of the Pedagogical Institute, a training and development unit of the Ministry of Education, and the teachers' union. Concerning primary education, it is customary that the school textbooks are accompanied by some supplementary material, often taking the form of teacher's guides. The textbooks for public secondary schools in Cyprus have never been accompanied by a teacher's book. Nevertheless, each elementary school specifies the objectives for each grade.

The general objectives of primary education, as stated in the introduction of the national curriculum in 1996 are to:

- Develop children's knowledge using modern technology to the greatest extent possible
- Ensure children's emotional and psycho-motor development
- Help children to successfully face problems of adaptation and other challenges in their school environment and in society
- Promote the gradual socialization of children, their sense of national identity and culture and respect for other countries and cultures
- Provide children with a positive attitude towards knowledge and human values
- Develop appreciation for beauty, creativity and love for life
- Develop a sense of respect and protection towards nature

Regarding the subject of mathematics, the general scope of mathematical teaching in the Cyprus primary education, as stated clearly in the mathematics curriculum, is to develop the mathematical thinking of the pupils and to help pupils solve mathematical problems that are useful in everyday life and in the sciences, as well as to appreciate the usefulness of mathematics and enjoy the disciplined thinking and harmony that exist in it (Cyprus Ministry of Education and Culture, 1996).

For Year 6 mathematics, the curriculum considers mathematics as a means to communicate, and emphasizes problem solving. In order for the students to be able to solve problems in mathematics, the students' textbooks offer activities that can be solved by employing several strategies like choosing the necessary operation, constructing a table, discovering a pattern. What is more, there is a section about students' assessment which is considered an integral part of teaching. The purpose of assessment, which is expected to be in line with the curriculum objectives is to provide support to pupils so that they can reach their full potential. The evaluation includes the initial evaluation, continuous and formative evaluation as well as the final evaluation and the results are not expressed by numerical grading.

Concerning proving in mathematics, there are no specific references regarding mathematical argumentation. However, opportunities to explain and justify one's work appear in some chapters both in the students' textbooks and teachers' guidance books. These examples will be presented and discussed in Section 6.4.1. of this chapter.

In relation to primary school geometry, the main teaching objectives for the first three grades are the visual recognition of shapes based on their outline and their name as well as their classification. In Year 4, the students are also expected to indicate characteristic properties of geometric shapes and record them. What is more, the students should start finding properties that are common in different shapes. In Year 5 and 6, students must be able to describe a geometric shape based on its properties and explain and justify the classification of shapes according to their properties. The construction of shapes by using their properties is an additional goal. There is also an attempt to utilize auxiliary instruments and for transformations only in on order to find the area of a geometric shape.

The areas in geometry that need to be covered in Year 6 include the recognition and construction of quadrilaterals, the identification of parallel and intersecting lines, the construction of 3D shapes and their representation in an isometric paper, angles (understanding the concept of angle, the process of measuring the angle, constructing, symbolizing, reading and comparing angles), triangles (classifying, naming, identifying and constructing angles, discovering the sum of the angles of a triangle), the identification of the distinctive features of trapeziums, the construction and naming trapeziums, perimeter, the area of rectangles, triangles, parallelograms and polygons (identification and exploration of the properties of polygons and the size of their internal angle), symmetry and circle (recognition of the elements of a circle and the relation between the radius-diameter, circumference and area of circle).

What is more, the mathematics curriculum encourages teachers to integrate ICT into their teaching of geometry. Giving the opportunity for students to use ICT may facilitate the development of positive attitudes towards mathematics. However, while the use of information communication technologies for efficient teaching has also been acknowledged by the Cypriot educational community, teachers seem to be unable to lean on the national curriculum or the teacher's guide book, in order to be supported in integrating technology in the teaching. This phenomenon appears because these educational materials do not provide the teacher with the variety of ways, in which computer-based environments can be employed, to effectively influence their teaching. As an illustration, it seems appropriate to mention that the opportunities for integrating technology in teaching geometry in Year 6 students are only mentioned once in the teacher's guide book. To be more accurate, students are encouraged to use a computer to construct drawings using only quadrilaterals (Cyprus Ministry of Education and Culture, 2003, p.39).

Nevertheless, the need to support teachers in integrating technology in their teaching was taken into consideration by the Pedagogical Institute. Hence, in 2008 a series of 3 additional books directed to teachers were published, focusing on the development and usage of educational supportive sources for the integration of ICT in the learning and teaching process. The philosophy underpinning the design of these material is to provide differing ways and examples of how one can use tools and environments in his/her teaching. That is, these books consist of educational scenarios, didactical recommendations, suggested lesson plans and activities as well as simple guidelines on how to use the proposed programs. However, while it is acknowledged that these books constitute the basis on which teachers rely at the beginning, it is expected that the teachers will gradually be able to design their own material and find new ways to integrate ICT in their teaching.

While the first book concentrates on the internet and the third one on the software suggested by the Pedagogical Institute of Greece, the second book is focused on Euclidraw and Sketchpad for the teaching and learning of Year 5 and Year 6 school geometry. In the introduction of this guidance book, it is stated that the basic characteristic of these software is the opportunity to drag, a process during which a diagram can be moved without changing the properties of its construction. What is more, by employing these Dynamic Geometry Environments, *'students have the opportunity to build their mathematical knowledge in a dynamic environment, explore freely, to investigate, to justify and reach higher levels of understanding compared to more traditional methods of teaching'*. It is also stressed that the teacher should be an active participant in this process of knowledge construction.

6.2.2.3. Teachers' Guidance Books

The objectives of the mathematics curriculum are in the same way addressed for the teaching and learning of geometry, and are further described in the guidance book for primary school teachers. For each area in geometry that needs to be covered, the teacher's guidance book states the goals and the cognitive content of the activities, along with their solutions. What is more, the teacher's guide book provides additional tasks, which the teacher can use for reinforcing the establishment of the geometrical knowledge being taught, as well as definitions that need to be taken into consideration when teaching specific concepts. Each task in the teacher's guide book is accompanied by the level of the difficulty and the curriculum context that is expected to be reached.

6.2.2.4. Students' textbooks

As only one series of mathematical textbooks is provided for primary mathematics education, these textbooks affect to a great extent what is taught in the classroom.

That is, the teachers use these textbooks as a daily guide for organizing their teaching, both with respect to the teaching content and the teaching methods, and rarely deviate from them.

The students' textbooks present a variety of tasks, which are organised in such a way that they foster the understanding of basic geometrical concepts, and the development of skills required for tackling different tasks. The textbooks set out to provide material for students of all levels of competence and include tasks scaled from the easiest to the most difficult ones; textbooks address the needs of the fairly competent and the top students, while there are simple tasks for the less competent students. While it is not the purpose of this study to analyse all geometrical tasks with regards to their characteristics, it is worth mentioning that these tasks can be grouped into the following categories: open/closed tasks, tasks that involved practical or paper-and-pencil work, tasks with everyday/classroom context and tasks for individual work.

It has been exemplified in Section 6.2.1. that the analysis of the students' textbooks would examine the justifications to mathematical statements offered in the textbooks, as well as the opportunities provided for students to justify and explain their own mathematical work. Considering the latter, the students' textbooks were analysed so as to find tasks that encourage students to investigate, hypothesise, explain or/and to justify their work. Thus, the unit of analysis is a task, a single numbered problem or exercise. In these activities the students were either asked to write their conclusions, explain, describe the way they worked and/or justify their answer. That is, activities consisting phrases like 'explain how', 'explain why', 'show the way you worked', 'what do you observe' and 'write your conclusions', were included in this analysis. In analyzing these activities, the position of each activity in the progression of the specific geometric topic was also explored. Whether the level of investigation, exploration and justification that takes place in classrooms corresponds to what is presented in the literature regarding proof and proving will be discussed in Section 6.4.1. Even though all activities presented in each geometric chapter were included in this analysis, only 25 out of 158 activities were found that share the aforementioned characteristics. The table illustrating these activities can be found in Table 6.1. below. For each activity, the solution presented in the teacher' guidance book is also provided. While a deeper analysis of the significance of the findings presented in the table will be presented in Section 6.4.1., it should be noted that exploration is aiming at reaching conclusions based on a small number of cases, and explaining and justifying, as encouraged in the students' textbook is reduced to an empirical level. What is more, even if an activity appears to encourage students to reason mathematically, again in the teacher's guidance book the answer provided demonstrates low expectations regarding explaining and justifying.

Table 6.1.: Analysis of students' textbooks: Geometrical Activities

	Topic	**Activity**	**Solution as presented in the teacher's guidance book**
1.	Types of triangles	Find the triangles in the adjacent shape. Study them and write your conclusions like the example: triangle ABC is right-angled because angle B is 90˚.	The answer is the same as the example. The students have to say the type of each triangle and justify that by using the definition of each triangle.
2.		Activity which involves the measurement of the angles of several triangles. Students are asked to write a conclusion.	The sum of the angles of a triangle is 180 °
3.		Construction of isosceles triangle. Find the degrees of each angle and compare them. Write a conclusion.	Two angles are equal.
4.	Trapezium	Write your observation about the relationships between the opposite sides of trapeziums.	Two of the sides are parallel. The other two are neither parallel nor equal.
5.		Explain why EFGH is called right-angled trapezium	It is called right-angled trapezium because one angle is 90°.
6.	Perimeter	The students are asked to find the perimeter of 3 different shapes. Then they are asked to show the way they worked.	The solution is presented with mathematical operations

7.	Perimeter	If the length and the width of a rectangle double, will its perimeter double? Justify your answer.	The perimeter will double because its dimensions will double. (Note: algebraic thinking is required)
8.	Perimeter and Area	Find the perimeter of the equilateral triangle, whose length of its side equals with 6.8cm. Show the way you worked.	Equilateral triangles have 3 equal sides. I multiply the length of one side times 3. 3 x 6.8 = 20.4cm
9.		Problem solving – Tania used 36m of wire to fence her garden. Demos' garden covers $56m^2$. Can their gardens you're the same dimensions? Justify your answer.	The gardens can have same dimensions as 4 x 14 = $56m^2$ (2 x 4) + (2 x 14) = 36m
10.	Area of rectangles	Find an easy way to calculate how many squares will be needed to cover the area of the yard. Show the way you worked.	I multiply the length with the width and I find the area.
11.		Solve the problem If the length and the width of a rectangle double, will its area double? Explain.	By trying several dimensions I discover that the area quadruples. (Note: algebraic thinking is required)
12.		Solve the problem If the length of one side of a square is doubled will its area quadruple? Explain.	The area of the square will quadruple. (Note: algebraic thinking is required)
13.		Find the dimensions of 2 rectangles whose perimeter is the same but their area is different.	Two examples are presented.
14.	Polygons	Study the table and draw out a conclusion about the sum of the angles of a polygon with n sides.	(V x 2) x 180º

15.	Area of Triangles	Two triangles have the same altitude. Do they have the same area?	It depends on the length of the basis. If the basis of the two triangles is the same so is the area.
16.	Area of parallelograms	What is the area if this new shape? Describe the way you worked.	The area is $45m^2$. I multiplied the length with the width.
17.		Find the area of the parallelogram with basis 3cm and height 5cm. Then double, triple, quadruple, quintuple the length of the basis. What do you observe about the area of the parallelogram?	The area of the parallelogram changes as the length of its basis changes.
18.		Find the area of the parallelogram with basis 3cm and height 5cm. Then double, triple, quadruple, quintuple the length of the altitude. What do you observe about the area of the parallelogram?	The area of the parallelogram changes as the length of its height changes.
19.		Compare the area of the two parallelograms and write what you observe.	The area of the two parallelograms is the same.
20.	Regular polygons	Study the table illustrating number of sides, length of sides and sizes of angles and write your conclusions	The size of the sides of the polygons is the same. The size of the angles of the polygons is the same.
21.	Symmetry	Write a conclusion about the number of axes of symmetry of a regular polygon.	The number of the axes of symmetry of a regular polygon is the same with the number of its sides.

22.	Symmetry	What do you observe about the points of the shapes that are symmetrical?	The points that are symmetrical are equidistant from the y axis which is the axis of symmetry.
23.	Circle	Table with the radius and diameter of several circles. Students are asked to study the table and write what they observe.	I notice that the length of the diameter of the circle is twice the length of its radius.
24.		Students are asked to use a thread and a ruler in order to find the circumference and the diameter of circle accordingly and complete a table. By using this information they are asked to find an equation for the circumference of a circle.	Circumference of circle = diameter x 3.14
25.		With a circle cut into pizza pieces, put the pieces next to each other and find the area of the shape you have constructed. Explain how you found your answer.	With the pieces of the circle I made a shape similar with a parallelogram and I found its area.

The following section focuses on the initial interview with the teacher.

6.3. The teacher's initial interview

As stated in Section 5.3. of the methodology chapter, the second level of analysis is the teacher. In order to map out the current situation of the classroom, and be able to identify whether any ruptures within and across levels are apparent, an initial interview with the teacher participating in this study would be carried out, so as to have a holistic view of the teacher's perceptions and beliefs.

The following section discusses the method for conducting the initial interview with the teacher.

6.3.1 Method for the teacher's initial interview

As stated in section 5.7.3, a semi structured interview seemed most appropriate for conducting the interview with the teacher participating in this study. In achieving this, an interview schedule would be prepared by moving from more structured questions to less structured ones, which could be followed by probes. Regarding the content of the questions, the path followed concerned the translation of the research questions into interview questions. That is, the schedule of questions to be asked was organized in such a way that it would help to delineate the activity system of proving in the mathematics classroom, exploring the way changes influence this system and identify potential ruptures. Thus, the questions would help to obtain an insight into the beliefs and views of the teacher.

By taking into consideration the literature used to inform this study (see Chapter V), as well as the research related with teachers and their beliefs (see Section 5.6.), the interview schedule was divided in three categories; the nature of mathematics, the nature of teaching mathematics and the nature of learning mathematics.

To be more precise, concerning the first category, the questions intended to reveal the teacher's views and beliefs concerning the nature of mathematics, the significance of geometry and the importance of proof and proving in mathematics and specifically in geometry. Concerning the teaching of mathematics, the interview schedule was centered on the teacher's views about the mathematics curriculum, the factors the teacher takes into consideration in teaching geometry and proof, as well as the introduction of ICT and DGE in the teaching and learning of mathematics. Concerning the learning of mathematics, the questions were focused on the teacher's perspectives regarding students learning geometry. Nevertheless, it should be noted that some of the questions fell into more than one category as these questions had sub-questions.

It is also worth mentioning that in order for the interview schedule to be effective and appropriate for exploring the teacher's perceptions, an initial exploration and analysing of the official documentation of the Cypriot educational system was undertaken prior to the interview. Notes were taken for each question according to what the research literature and the official documentation states about the issues under discussion. A pilot was also conducted. Piloting the interview schedule aimed at checking the clarity of the questions, eliminating difficulties in wording and estimating the time taken to complete the interview. Additionally, the pilot gave me the opportunity to better prepare for the interview and gain more confidence in conducting the interview. For instance, conducting the pilot study would assist in making a list of possible probe situations. Furthermore, I would improve my interview skills regarding informal and floating prompts. The aforementioned would increase the reliability and validity of the questionnaire.

The pilot study was carried out with a Greek Cypriot teacher who was also a PhD student. The interview was carried out in the same way as it would be undertaken in the main study. Furthermore, at the end of the interview the respondent provided her feedback regarding the interview schedule so as to identify ambiguities and difficult questions. The analysis of the answers the respondent provided established that the questions were not repetitive and that the replies could be interpreted in terms of the information that was required. Adding to the above, piloting the interview schedule revealed that the sequencing of the questions could indeed lead to building and establishing rapport. However, the analysis also illuminated that one item of the interview schedule had to be improved. To exemplify more, in conducting the pilot, it felt that one of the questions related to DGE (What do DGE involve?) had to be paraphrased in order for the wording to be less threatening and thus, gain more information. The interview schedule is presented in Appendix IV.

When the interview schedule was finalized, I proceeded with the interview. Before the interview was conducted I described to the teacher how the interview would be conducted, how long it would last, the general subjects that were to be covered. I also asked for permission to audio record the interview, I guaranteed confidentiality and I asked the teacher if she had any questions. The interview lasted 1 hour and 20 minutes and was audio-recorded. During the interview, the questions asked were not in the same sequence as in the interview schedule. Asking these questions was dependent and influenced by the teacher's responses. Despite this, all the areas included in the interview schedule were covered.

The following section focuses on the presentation of the responses the teacher gave during this initial interview.

6.3.2 Findings from the teacher's initial interview

Below, I give a plain account of the teacher's initial interview, selecting critical moments that will subsequently be analysed and cross-related to data from other levels. For this reason, TQ1, TQ2…TQ12 represent quotes extracted from the teacher's responses to the questions. The teacher's initial interview is presented in Appendix V.

The teacher first described the basic topics around geometry that she usually teaches in Year 6. Following this, the teacher was asked about the areas in geometry that she believes are more important than others. The teacher gave emphasis not to specific geometric areas but to specific skills the students need to develop. These skills include the skill to measure angles, constructing the height of several shapes as well as the skill to know how to find the area of shapes.

Teacher Quote 1:

'It is very important to discern and recognize these concepts, before proceeding in applying them in order to find various measurements. In order to proceed in high school is very important to know how to employ them. For this reason, I do not stop at what we only do in primary schools but also in what will follow and the skills they need to have in order to be able to continue in high school.'

Regarding the difficulties the students have in understanding the basic concepts of geometry, the teacher argued that the spiral arrangement of the curriculum, the numerous concepts and the time allocated to cover this geometric curriculum in Year 6 is not enough for the students to fully acquire the new knowledge. Despite this, the teacher stated that practical work assists students in overcoming these difficulties:

Teacher Quote 2:

‘These concepts are reappearing with the spiral arrangement of the curriculum, but there is a large amount of concepts and I think that perhaps some things need to be completely removed so as to give students the opportunity to work more and more often with basic central concepts which they will be using throughout their life for mathematics and help them with mathematical and critical thinking.’

The teacher concluded that this might change when the mathematics curriculum is introduced. The discussion that followed was related to the national curriculum and the teachers’ guidance books. The teacher stated that she never uses the teacher’s guidance book. She explained that, due to the fact that the teacher’s guidance books were published long after the publication of the student’s textbook, the teacher had to adjust the planning of the lessons. What is more, she does not find it practical:

Teacher Quote 3:

‘Yes the book talks about the goals, but mostly it gives answers … for some exercises there are teachers that may not be that confident with mathematics and have difficulties in understanding how to solve them …thus for this reason it is useful.’

Concerning students learning geometry, the teacher indicated that geometry is important for students to learn. She justified this view by saying that geometry permeates our world and that we use it in our everyday life.

The teacher argued that Year 6 students will acquire the appropriate geometrical concepts and knowledge that will allow them to proceed further on the subject of geometry in the secondary school. However, according to her, this depends to a great extent on the way someone teaches:

Teacher Quote 4:

'… for example, if you just use the book…they will not acquire many skills. You need to give them opportunities to work a lot and not just with the book…in the notebook, the worksheet, on the software…they need to have the chance to work more so as to accumulate the concepts.'

Following this, the teacher talked about the transition from primary to secondary school mathematics and the way this transition influences her teaching:

Teacher Quote 5:

'In primary school we do not really have definitions as in secondary school. These are functional definitions. For example when you bring the altitude, the altitude is a distance, the students must understand that it is distance, they need to distinguish that it is not a point as some did, it is the distance from one point to another in any line and forms 90 degrees. I insist in the phrasing of definitions as it will help them in secondary school. I do not insist on assessing the definitions. For example, when I asked a student to say what altitude is in a triangle, he said 'when we bring a line forming 90 degrees'. There is understanding but he cannot give the formal definition. The definitions in primary school are more functional and less theoretical. Nevertheless the use of precise mathematical language will help them in secondary school.'

While the above argument was focused on the transition from primary to secondary school, it also included the teacher's approach to mathematical definitions. Concerning definitions the teacher also stated:

Teacher Quote 6:

'I think we saw from the lesson that a mere memorization of the formula for the area of triangles is not enough. It is very easy to know the formula. But, what does 'altitude' mean? What does 'base' mean? For example, what happens when the shape is rotated in such a way that the students need to measure?'

The teacher stated that there is differentiation in her teaching. She also said that concerning her students' previous experience around geometry, she doesn't take as given something that has already been taught:

Teacher Quote 7:

'I sometimes tell my students 'theoretically you have learnt it' … essentially what is valid I do not know … most of them forget … the knowledge that they acquire is very specific … they need time to remember … you need to help them recall the previous knowledge … which means nothing should be taken for granted.'

Pertaining to the incorporation of technology in the teaching of mathematics and particularly in geometry, the teacher referred to DGE on many occasions while discussing many of the issues included in the interview schedule. The teacher declared that she integrates DGE in her teaching. Having said that, she argued that there is not enough time for the students to learn all the available tools and to fully explore these environments:

Teacher Quote 8:

'We do not use the software all the time ... we come back to it at a later stage ... there is no day that we worked in the software and not used something similar in a worksheet or the notebook; so that what they do in the software they can also apply elsewhere.'

Concerning teachers integrating technology in their teaching, the teacher said:

Teacher Quote 9:

'As long as they use at least one or two things that really help and have an additional value for the activity … If it does not offer that, there is no point in using the technology … The specific software... the fact that it gives you the opportunity to drag the points and the vertices of the triangles and change the triangle from right-angled to obtuse-angled and to acute-angled triangle is very essential. Whereas if you do it on paper, it is there and then you will have to erase it and draw it again.'

Following this, the teacher elaborated on what DGE involved as well as the benefits and constraints of such environments:

Teacher Quote 10:

‘The software is always a source, an aid, a tool. It doesn’t mean that the students will learn 100% just because they used the software. I use the software in the same way I use paper and pencil, and in the same way that I would use any other method that would help me to prove a mathematical relationship.’

Teacher Quote 11:

‘Yes but the point is that in this age, when you leave students completely free … in the end the result will not be the achievement of the goals set when employing the software. So at the end of the day you should not destroy the lesson in order to use technology.’

Regarding proof and proving the teacher said that she considers both the justification and proving processes as necessary for the geometry concepts that she teaches. She argued that it is easy for the students to memorize a certain formula without understanding the steps they need to follow so as to correctly apply this formula. However, she leaves her students to try to discover something on their own. What is more, she said that with this way, the students can reflect on the process they followed to do something, and, if for example they made a mistake, they can realize that, while trying to justify and explain the steps followed. She also provides the proof of a theorem or an axiom in order for her students to fully understand the argumentation behind it.

Concerning the role of proof in geometry the teacher stated:

Teacher Quote 12:

‘I believe that the students acquire knowledge better when they prove it rather than when it is just provided to them. And in geometry it is even more practical because they are things that they can see. They can prove them … they see that the area of the triangle is half the area of the square in which the triangle is inscribed, they can measure it and compare it and prove it. Thus, proving is actually what helps them practically to assimilate better the knowledge. And it is an important process that is essential. And our books are based on this.’

Teacher Quote 13:

'For the things that they prove, they do not ask questions because they do it themselves … it is when they are given something that they ask "why this is so"… but we do not stop at the group work … we have the classroom discussion so as to show and present the proof and accept it as a whole.'

The following section focuses on the analysis of the findings of Phase I of the study.

6.4. Ongoing Analysis

In this section, the insights from the documentary analysis as well as the initial interview with the teacher are presented and elaborated on.

6.4.1 Insights from the official documentation

The findings on the official documentation regarding proof and proving in geometry, as well as the opportunities provided in integrating technology in the teaching and learning of geometry, reveal a very low level of expectation with regards to explaining in Year 6 (see Section 6.2.2.2.).

As illustrated in Section 6.2.2.2., four objectives are to be met regarding the learning and teaching of primary school mathematics. Even though it can be argued that mathematical argumentation can be identified within these principles, this connection is not made explicit; proof and proving is not acknowledged as a key criterion, nor mentioned anywhere else in the mathematics curriculum. This is also apparent in the objectives of primary school geometry with argumentation being important only when students are expected to explain and justify the classification of shapes according to their properties. This is in contrast firstly with the research literature related to mathematical argumentation where the importance of mathematical argumentation throughout schooling is stressed and emphasized (Hanna, 2000; Yackel and Hanna, 2003; Stylianides, 2007a, 2007b). Secondly, this is also in contrast with the fact that in Year 1 in high school, students are expected to understand theorems and solve geometrical problems that require proving, a substantial leap if the foundations of proving have not been established in primary school.

The discussion regarding the analysis of the geometrical activities in the students' textbooks is focused on exploration and explanation. An initial analysis of the work that the student is asked to engage with and the answer the teacher should be expecting and validating indicates the emergences of two common themes; conclusions are to be drawn based on two or more cases and examples and explanations should be given by the usage of the definitions or the properties of the shapes.

The report of the official documentation illustrates the central focus on problem solving in teaching and learning mathematics. However, given the several kinds of problems (exercise, word problem, problem to put into equation form, problems to prove, problems to discover, real life problems, problematic situations), a more detailed argument is needed, as different problems have differing educational value. What is the goal of exploration and investigation in problem solving? Is the educational goal to reach a conclusion based on a small number of cases or does it aim at the formulation of questions, making and testing conjectures, and, eventually, proving them?

As exploration and investigation are considered important aspects of pre-proving, the analysis of the geometrical activities was also focused on these notions. This analysis indicates low degree of expectation regarding exploration. This also had an impact on the degree of explaining and justifying requested by the students.

A further level of analysis involved identifying the meaning of the words 'explain' and 'justify', as used in the activities and their solution as provided in the teacher's guidance book. Within this setting, an explanation is given by providing the mathematical operations used to find the answer. The justification is provided by using the definition.

What is the position of definitions and defining in the official documentation and how does this relate to proof and proving? It has been exemplified in Section 2.4.1 that definitions and defining may be approached in differing ways depending on the teacher's goals and the level of the students. The analysis raises some questions and hypotheses about the ways that definitions are presented at different levels. In the official documentation what is said about the nature or function of mathematical definitions lies in the assumptions that can be made when approaching its specific objectives. Furthermore, both in the students' textbook and the teacher's guidance book the notion of definition is more descriptive of an existing object rather than *'purposeful design of a definition for theory building'* (Morgan, 2005). For instance, when students are requested for the first time to draw the altitude in several triangles, the students' textbook provides the following statement: *'notice that the altitude forms 90° with the side that intersects'*. This statement is accompanied with an illustrative figural representation. A definition of the altitude in a given triangle is not provided and the description is based on the properties of the concept so as to apply the given criterion. An additional example is related to the content of the curriculum about circle. As shown in Appendix VIII, the word 'recognise' is employed to refer to circle and its properties and 'calculate' to refer to the circumference and area of circle. It can be argued that the words utilised do not indicate any act of definition or defining as a mathematical activity. The definition may be given by authority. The descriptions provided in the students' textbook are about the radius and diameter of circle ('the segment connecting the center of the circle to the circumference is called the radius of the circle' and 'the segment connecting two points of the circumference of the circle is called the diameter of the circle'). The position of the student in relation to the definition is not made explicit. A consideration of the above statements indicates that definitions as approached by the official documentation are descriptive and extracted (see Section 2.4.1). In contrast, one might imagine a curriculum in which experience with circles leads to a need for definitions of key elements such as its centre (for example of rotational symmetry) or its diameter (to define the 'width' of the circle), where a need for the definitions is emphasised and to some extent takes over the role of authority.

In the research literature it is argued that students should be encouraged to use mathematical language to explain their work as this is among the prerequisites for the transition to deductive reasoning (Yackel and Hana, 2003). Most of the activities presented in this analysis require students to use mathematical language through the definitions or the properties of the shapes (see Table 6.1.). However, even though this is supported in the students' textbooks, it is only reduced to an empirical level. What is more, even if an activity appears to encourage students to reason mathematically, again in the teacher's guidance book the answer provided demonstrates low expectations regarding explaining and justifying. An illustrative example that supports the aforementioned statement is the second activity on the area of parallelograms (see Table 6.1.).

The objectives of the mathematics curriculum of primary education, as well as the types of activities presented in the students' textbooks, have also been the focus of several research studies of the University of Cyprus. To be more precise, these studies explored the way representations, problem solving and technology, as appeared in primary schooling, support or hinder geometrical reasoning. The findings of these studies, as detailed below, support the aforementioned discussion related with the limitations of the mathematics curriculum and the textbooks.

Gagatsis, Tsakiri and Rousou-Mihailidou (2004), in investigating the way upper primary school students define, recognize and represent shapes in geometry, concluded that the classification of geometrical shapes depends on the perceptual aspects of the shapes instead of the conceptual aspects. Even though the students can respond to a great extent to activities that require the association of differing shapes in order to construct a new shape, they find difficulties when they are asked to divide, reconnect and transform the shape (p.214). They concluded that the way the geometrical concepts as they are being approached and taught at the moment does not encourage the development of perception of space.

In a similar way, Panagidis and Christodoulou (2004) explored the way upper primary school students solve problems in geometry. They argued that while the goal is the recognition of geometrical shapes according to their properties, the visual perception of shapes that has been formed in previous years of schooling prevails. Thus, the pictorial representation does not allow the identification of facts and unknown data can block the development of geometric thinking and students may fail when approaching the problem intuitively (p.215).

Despite this, research has also been conducted to explore the new perspectives in the teaching of geometry provided by the introduction of dynamic geometry environments and the interactive whiteboard. Mousoulides, Pittalis and Christou (2004) investigated the opportunities given by these environments for the teaching of the area of triangles. They suggested that the continuous interaction and immediate feedback provided by the software drove students to explain how they work, to reflect on the strategies they implemented and systematize their thinking (p.199).

At this point, it should also be noted that the Ministry of Education and Culture in Cyprus is now establishing a new curriculum for all subjects in all levels of schooling. The overall goal of the new curriculum is to promote the democratic citizen. Nevertheless, more information about the new mathematics curriculum will be provided in Chapter X, where the implications of this study in curriculum design and development will be elaborated on.

6.4.2 Insights from the teacher's initial interview

In this subsection of the analysis, I draw inferences based on the data reported in Section 6.3.2. by using the line numbers indicated within the thesis (rather than line numbers in the original transcript).

The analysis of the teacher's initial interview led to some very clear themes. To begin with, there is a contradiction in the comments the teacher made regarding technology. While the teacher believes that integrating technology in someone's teaching should be done only if it offers an additional value for the activity (TQ9), she then says that she uses technology in the same way that she would use any other method to prove a mathematical relationship (TQ10). This comment looks rather important as the teacher seems to see the software as supporting what has to be done with pencil and paper. This is in accordance with her statement that when she does not employ mathematical software in her teaching without using the students' textbook as well (TQ8). She does not seem to see that the software might provide a completely new approach that cannot be done with conventional media.

In discussing the importance of geometry, there is no reference to proof or to mathematics as a discipline. However, when the opinion of the teacher is requested regarding whether the justification and proving processes are necessary in geometry, the teacher stated that she considers them important for the geometry concepts that she teaches (TQ12).

Another interesting observation is that when discussing the difficulties the students may be faced with when learning the geometric concepts, the teacher focuses on the skills that according to her the students need to acquire in order to be able to do practical work and understand the concepts that are being taught. This is considered important as the teacher does not mention any misconceptions that the students may have, or difficulties related with the conceptual aspects of these concepts. Furthermore, she makes a clear connection regarding the importance of acquiring these skills and the transition to secondary school mathematics (TQ1, TQ5).

The teacher believes that exploration which leads to discoveries is very important. However, she provides the proof rather than allowing the students ways of working towards it themselves. Following this comment, her remarks are contradictory. She states that it is really important for the students to prove something so as to better understand it, instead of the proof to be provided to them (TQ12). Nevertheless, the teacher points to an important aspect of proof; that it needs to be shared by the classroom community (TQ13).

An important aspect of her teaching, as emerged through the interview data, is that the teacher endorses the use of precise mathematical language. Her rationale for this is twofold; it enhances students' mathematical understanding, and this is something that will help them in secondary school (TQ5).

The teacher's comment regarding the role of mathematical language is also related to her views about mathematical definitions (TQ5). Her statements illustrate the characteristics and functions of definitions and defining in her teaching practice. According to the teacher, mathematical definitions should have precision in terminology and be easily comprehended by students (TQ6). This is also in accordance with what is stated in the mathematics research literature. Understanding the definitions utilised in the classroom is important for the teacher. Furthermore, contrasting definitions as approached in the mathematics curriculum of elementary and secondary school shows that the teacher has an appreciation of the role of definitions in mathematics and mathematics education. Her comments show that even though she is expecting her students to use precise mathematical language she is aware that the level of mathematical knowledge of a class may not allow for something 'more precise'. This may lead the teacher sometimes to make some conscious decisions to let the class proceed with some definitions that she considers as being 'not that precise'. If this occurs, it can be argued that the teacher may just want to give the students a sense of a concept.

Furthermore, time seems to be a constant concern for the teacher. The time constraints may not allow the class to fully engage with the geometrical ideas under investigation (TQ2). Also, the time available for teaching each mathematical topic might not give the classroom the opportunity to fully exploit the possibilities offered by the dynamic geometry environments.

6.5. Summary and Discussion

This chapter has focused on Phase I of data collection. Reporting on the official documentation, and depicting critical moments from the teacher's story, provides the basis for determining an understanding of how proving is constituted in this classroom. All elements that influence the way proving may be established in the classroom will be portrayed by conducting a retrospective analysis (see Chapter IX) and further scrutinized in Chapter X. Simultaneously, it has been explicit that the analysis and discussion of each dataset informs the subsequent phase of data collection (see Section 5.8.2.). How a synthesis of the insights from the official documentation and the teacher's initial interview leads to the emergence of themes of interest that will inform Phase II of the study is discussed below and summarized in Table 6.2. The insights gained from the documentary analysis and from the initial interview of the teacher were summarized in Table 5.2. and are duplicated below in Table 6.2. for ease of reference.

Clear themes regarding the report of the official documentation and the teacher's initial interview have been identified and elaborated on (see Section 6.4). By keeping in mind the initial themes of interest that have emerged from the literature review and the exploratory studies (see Table 5.1.), in the light of the analysis pertaining to the system level as well as to the teacher level, two broad themes of interest are emerging: exploration and argumentation.

To be more elaborative, the information collected from the official documentation concerning the initial four themes of interest in conducting the documentary analysis (see Section 6.2.1.) points to low level of expectation with regards to exploration and investigation in problem solving in Year 6 (D2, Table 6.2). Despite this observation, the teacher believes that exploration which leads to discoveries is very important (I4, Table 6.2.).

Table 6.2.: Map of themes Phase I

Post Phase I	**Insights from the documentary analysis** D5 Low level of expectation with regards to explaining in Year 6 (an explanation is given by providing the mathematical operations used to find the answer; the justification is provided by using the definition). D6 Low level of expectation regarding exploration and investigation in problem solving. D7 Proof and proving is not acknowledged as a key criterion, nor mentioned in the mathematics curriculum. D8 Definitions as approached by the official documentation are descriptive and extracted. **Insights from the interview with the teacher** I7 The transition from primary to secondary school mathematics influences her teaching. I8 The teacher endorses the use of precise mathematical language. I9 The teacher gave particular emphasis on mathematical definitions. I10 The teacher believes that exploration which leads to discoveries is very important. I11 The teacher considers the justification and proving processes as necessary for the geometry concepts that she teaches. I12 The teacher made contradictory statements regarding the integration of technology.
New synthesised set of themes	T1 Exploration (D2, I1, I4, I6) T2 Mathematical argumentation: explore the opportunities the students had to explain, justify and prove in the classroom (D1, D3, D4, I2, I3, I5)

Going further, the teacher indicated that one factor that influences her teaching is the transition from primary to secondary school mathematics (I1, Table 6.2.). As this statement is related with the nature of teaching mathematics, the way it is reflected in the classroom's mathematical practices should be further investigated and identified.

It has also been illustrated that the teacher made contradictory statements regarding the integration of technology (I6, Table 6.2.). It has been established that DGE may provide a foundation of deductive reasoning (see Section 5.4.). Furthermore, the way tasks are designed influences the mediating role of these tools (see Sections 3.2.4., 3.3.4., 5.5.). Keeping in mind the above, along with the teacher's comments concerning exploration, this specific area of focus should be further investigated.

It can be argued that these differing insights are related with exploration and thus may influence the expectation and degree of exploration as portrayed in the mathematical activity of the classroom. Considering this, as well as the notion of pre-proving, in mapping the current situation of the classroom (Phase II of the study), exploration becomes a theme of interest that will guide the initial analysis of the collected data.

Considering the purposes of this study, proof and proving have also been identified among the initial themes that guided both the documentary analysis and the interview with the teacher (see Sections 6.2.1. and 6.3.1. accordingly). Analysis of the report of the official documentation shows that proof and proving is not being acknowledged as a key criterion, nor mentioned in the mathematics curriculum (D3, Table 6.2.). Furthermore, analysis of the official documentation concerning explaining and justifying points to an explanation being given by providing the mathematical operations used to find the answer and the justification being provided by using the definition (D1, Table 6.2.). The teacher considers the justification and proving processes as necessary for the geometry concepts that she teaches.

Another theme that has emerged is the fact that definitions as approached by the official documentation are descriptive and extracted (D4, Table 6.2.). On the other hand, the teacher stated that she gives particular emphasis to mathematical definitions (I3, Table 6.2.). The teacher also commented that she endorses the use of precise mathematical language (I2, Table 6.2.). This statement relates to the nature of learning mathematics and draws attention to the notion of sociomathematical norms.

These emerging themes are related with mathematical argumentation. They also highlight the role of the teacher in orchestrating mathematical argumentation. Thus, in exploring the opportunities the students had to explain, justify and prove in the classroom, the above themes should be taken into consideration. Consequently, argumentation becomes a theme of interest in understanding the baseline observational data of the classroom.

Concisely, exploration and argumentation become the synthesized set of themes that will be taken into consideration in mapping the current situation of this mathematics classroom. Accordingly, the following chapter presents Phase II of the study.

CHAPTER VII

PHASE II

BASELINE OBSERVATION OF THE CLASSROOM

7.1. Introduction

It has been exemplified in Chapter V that the main objective of Phase II of this study was to map the current situation of the classroom. This chapter begins by introducing the method for conducting the second phase of the study. That is, by taking into consideration the general discussion of the theoretical assumptions underlying participant observation and interviews (see Sections 5.7.2. and 5.7.3. accordingly), this chapter initially provides a concise description of the way the data gathered was achieved. My role as a researcher is also exemplified.

This chapter then proceeds by introducing the findings from Phase II of data collection. That is, a chronological overview of the lessons is provided by presenting episodes from the classroom observations as well as the informal discussions. The classroom protocols are accompanied by short commentaries that either point to a theme that has already been introduced or are a precursor to introducing a new theme.

This chapter continues with the analysis of the classroom protocols and the informal discussion with the teacher by taking into consideration the themes of interest as emerged from the documentary analysis and the interview with the teacher (see Section 6.5.). This chapter progresses by describing how accounting of the data from the classroom setting led to new themes being drawn together.

7.2. Method

In order to map the current situation of the classroom, the mathematics lessons of the classroom were observed for a week. To be more precise, 8 periods (40 minutes each) were observed and video recorded and field notes were kept. The video recorder was positioned in such a place that allowed the recording of both the teacher and the students.

The topic of the curriculum that was covered that week was the area of triangles. The teacher decided to employ a Dynamic Geometry Environment in her teaching for the first time, after the Advisor of Mathematics assigned by the Ministry of Education for this school did exemplary geometry lessons in her classroom using GeoGebra for the teaching of the area of parallelograms.

Before proceeding, it is considered important to provide the reader with essential information regarding GeoGebra. GeoGebra is a freely-available open source dynamic mathematics software that joins arithmetic, geometry, algebra and calculus. Adding to the above, this software can be used for demonstration and visualization, as a construction tool, as a tool for discovering mathematics and preparing teaching materials (Hohenwarter and Fuchs, 2004).

For the purposes of this research, a structured observation scheme was not employed in gathering observational data. Despite this, a list of possible data sources was used for the writing of field notes (Burgess, 1984, in Gray, 2004). Hence, the components of the field notes were the date and the time of the lesson noted, the layout of the classroom, the goals of the lesson, the various activities the participants were involved with, their actions, the sequence of the lesson, as well as the participants' and the researcher's comments.

During the observation week, informal discussions before each lesson would take place in order for the teacher to present to me the lesson plan for that day. Despite this, informal discussions after each lesson were not formally planned. However, the fact that the teacher was employing a Dynamic Geometry Environment in her teaching for the first time led her to initiate these discussions. These informal discussions had in a way an evaluative form. To be more specific, the teacher would justify why some things were done in a certain way. She would also evaluate the lesson with regards to the DGE used, the tasks that she designed, the structure of the classroom, and the classroom discussion that followed the students' engagement with the tasks.

7.2.1 My role as a researcher

As this phase of data collection was mainly focused on a baseline observation of the lessons, my involvement in the classroom could be described as moderate participation. That is, I was both an insider and outsider, and I would conduct some participation while observing the mathematics lessons of the classroom. However, this participation was kept to a minimum, as my goal was to observe the classroom as it normally is when I am not present and hence to be able to map the current situation of the classroom. The teacher was responsible for the teaching and the development of the learning situation. The instances in which I participated, involved pairs of students working on an activity in the Dynamic Geometry Environment. This involvement could also be characterized as moderate intervention.

After deciding on the level of participation in the classroom, my role within this specific group had to be established. In doing so, in order to avoid problems that can occur when using this specific intrusive technique, a researcher has to signal clearly his own professional ground rules and role boundaries (Simpson and Tuson, 1995, p.61). Thus, before establishing my role in the classroom, my role as an objective detached researcher had to be formed at the school. Following this, the teacher explained to the class that I would just observe their lessons.

As an observer, I decided to position myself at the front of the classroom and next to the teacher's desk and computer, in order to be able to observe the whole class from the teacher's angle during the lesson.

7.3. Findings

This section reports the findings of Phase II of the study. That is, a chronological overview of the lessons is provided by presenting episodes from the classroom observations as well as the informal discussions. The classroom protocols are accompanied by short commentaries that either point to exploration and argumentation (themes of interest that have been introduced in Section 6.5.) or are a precursor to introducing a new theme. These commentaries are for the benefit of the reader to highlight the potential significance of the protocol; they were not part of the original accounting of the data but reflect the development of old themes and sometimes the emergence of new themes which will become a significant part of the subsequent analysis when accounting for the data. For this reason, the commentaries are written in italics.

Regarding the classroom protocols, it should also be noted that all paragraphs are numbered in order to maintain a coherent system for their presentation and discussion. In the dialogues, T represents the teacher and S1, S2…Ss the students that participated in the discussion.

Day 1, Phase 2

At the beginning of the lesson the teacher informed the students about the general scope of the geometric lessons of the week; area of triangles. She said that the class would use a Dynamic Geometry Environment in their lessons. While making explicit that she was not that familiar with this specific DGE, she said that she designed the activities by herself, by using as reference what the Advisor planned for his lessons. The teacher continued by introducing me to the students. She said that I was there to observe their mathematics lessons for that week.

The teacher began by asking the students to identify and name the shapes illustrated on the interactive whiteboard (three rectangles, with triangles inscribed in them as well as three triangles). Following this, she asked them to work in pairs to find the area of the rectangles illustrated on the board, write their answer in their notebooks, present their answer in the classroom and explain how they found that answer. (2.1.1.)

In the discussion that follows, the students are expected to say the area of one rectangle presented on the board and make explicit the way they worked towards the answer. The teacher, looking for a reason, asks, “What is the area of the rectangle?” S1 replies, “9” and the teacher asks, “9 what?” S1 says “9 cm^2”. The teacher asks, “How did you find it?” S1 says, “We counted the boxes.” (2.1.2.)

The above protocol focuses on the teacher not accepting an answer that is not complete.

At this point, the teacher tried to guide the students from counting to calculating, T: “You counted the squares. Is there a team that found that using a different way?” S2 says, “Yes … 3x3”. The teacher says, “These are its dimensions … 3x3 … we can see that on the axis. The next one." S3 says, "12 squared …" The teacher says, "Centimeters." The teacher, still looking for the need for justification asks, "How did you find it?" S3 replies, "I counted the boxes, 3x4." The teacher asks, "What did she say?" The classroom replies, "Squares." The teacher continues, "She gave us two ways. I count the squares and …" S4 says, "3x4." The teacher says, "Width x length. Let's move on to the next one." S5 says, "12." (2.1.3.)

At this point, the teacher was concerned when one group was 'playing' with the computers, "12 again … but why are you playing? We are not doing something on the computers now. Stop." (2.1.4.)

In this protocol, the teacher is relating exploration of DGE with 'play'. That is, exploration was interpreted by the teacher as 'playing' instead of learning.

Following the previous discussion regarding the area of the rectangles, the teacher said: "Now I want you to find the area of the triangles inscribed in these three rectangles." After the teacher with the students named the triangles, the teacher helped the children to identify a general formula for the area: T: "You can count the squares or try another way." S1 says, "Base x altitude / 2". The teacher replies, "If you know how to apply the formula". S1 says, "I lost the altitude". At this point the teacher made the following comment, "This is the problem: how to find the altitude in a triangle." (2.1.5.)

The above protocol shows the class making a connection with a part of mathematics previously taught as well as recognizing the difficulty of finding the altitude.

The following discussion unfolds after finding the area of the triangles inscribed in the rectangles. In this discussion, the teacher helped students to reach a conclusion regarding the area. T: "Compare the area of the rectangles with the area of the triangles inscribed in them. What do you observe"? S1 replies, "The area of the triangle is half the area of the rectangle. The teacher says, "Well done. I will stop here for now." (2.1.6.)

In this protocol, the students with the teacher arrive at conclusions based on one or several drawings or examples. This way of making general comments is inductive generalization.

The discussion related with the area continues with triangles not being inscribed in a rectangle. The teacher asks: "What about the triangles below"? S2 says, "I know what you want to do". T: "Can I find an easy way that helps me how to find the area of the triangles"? S2 replies, "Yes, base x altitude / 2." The teacher says, "Ok. Can you do this for the first triangle?" At this point, the students remain quiet. The teacher made the following comment: "You remember one thing from the year before, but the essence is to be able to apply it." (2.1.7.)

This protocol shows the class making a connection with a part of mathematics previously taught and recognizing the difficulty of applying the mathematical formula for the area of triangles.

The subsequent part of the lesson involved students working on the computers. Before the class started exploring activities in the dynamic geometry environment, the teacher guided students' instrumental genesis, T: "Let's go now to the software to remember a few things. In the third box on the top, the one with the dots we can find the option 'segment between two points'. This is the option we were using all the time when we were doing the parallelogram. I am constructing the first one and then you will construct the remainder rectangles." (2.1.8.)

In this protocol, the teacher, adopting the role of instructor, recalled the way this environment was used for the lessons related with parallelograms by focusing on the technical aspects of the dynamic geometry environment.

After demonstrating on the interactive whiteboard how to construct a rectangle in which the triangle is inscribed, the students worked in pairs and constructed rectangles on GeoGebra. When the students finished, the teacher asked, "Now that you constructed the rectangles, can they help you to find the area of triangles"? The students reply, "Yes". T: "What is the area of triangle PRS?" S1 replies "3." At this point, the teacher interrupted the classroom discussion as she was concerned with a student 'playing' with the computers, "S2 you are still talking. You are playing all the time and I will move you from the computers." (2.1.9.)

In this protocol, the teacher is relating exploration of DGE with 'play'. That is, exploration was interpreted by the teacher as 'playing' instead of learning.

After the students divided the area of the rectangles by 2 in order to find the area of the triangles inscribed in them, the teacher guided students in constructing the definition of altitude based on its conceptual aspects, T: "What is the altitude in a given triangle"? S1: "Mrs …" The teacher says, "We need to remember what the altitude is. Or to understand what it is." S2 says, "Here where the vertex is." T: "From the vertex." S2 continues, "You have a dashed line." The teacher asks, "Any line?" S2 replies, "No, a straight line." T: "Any straight line, dashed or solid. Any random line?" S3 replies, "It has to make a right angle. The teacher says, "From the vertex you construct to the opposite base a line." While she is repeating what the students said, she is doing it simultaneously on the board. The line is not the altitude of the triangle. T: "Is this the altitude of the triangle? Is my construction correct?" The students reply, "No." The teacher says, "You say that it is not correct. Tell me what I should do in order for my construction to be correct." S2 responds, "To do it straight." The teacher insists, "Tell me what I should do in order to be correct. S4." S4 says, "To do a straight line." The teacher responds, "What I did is a straight line." S5 adds, "The line should make 90 degrees with the base." The teacher praises S1's response, "Well done. It creates a right angle." (2.1.10.)

In this protocol the teacher was constructing the altitude by following the students' definitions so as to make them realize that the way they phrase things has an impact on the final construction. The students, by focusing on the perceptual aspects of the construction, were making alterations to their definitions in order for the altitude constructed to be accurate.

Subsequently, the students worked again on the computer and tried to find the area of the triangles by employing the mathematical formula. When they finished the teacher asked, "What do we notice about the area of a triangle?" S1 replies, "For the area of a triangle we notice that we can find it with two ways. We can multiply the base with the altitude and divide it by 2 or we can count the squares that are inside the triangle." T: "Good.". (2.1.11.)

In this protocol the class reached a conclusion regarding the mathematical formula.

The teacher returns to the interactive whiteboard and moves point F along AD (see Figure 7.a.). In the discussion that follows, the teacher tried to enhance the generality of the mathematical formula. T: "I want in this rectangle to construct a triangle with its area half the area of the rectangle. How will I do that? I do not want it to be a right-angle triangle. This is the easiest to construct. Is the area half the area of the rectangle? Yes, no and why?" S2 says, "The area is the same when you move it." The teacher asks, "Why?" and S2 replies, "Because the base and the altitude stay the same." T: "Very nice." The teacher moves again point F along AD. T: "Is the area of this triangle half the area of the rectangle? Yes, no and why?" S3 says, "Yes the area of the triangle is half the area of the rectangle because it covers half the area of the rectangle." The teacher asks, "How do I know that? What stays the same?" S3 replies, "The base." The teacher says, "The base is equal with one side of the rectangle and the altitude with the other side." (2.1.12.)

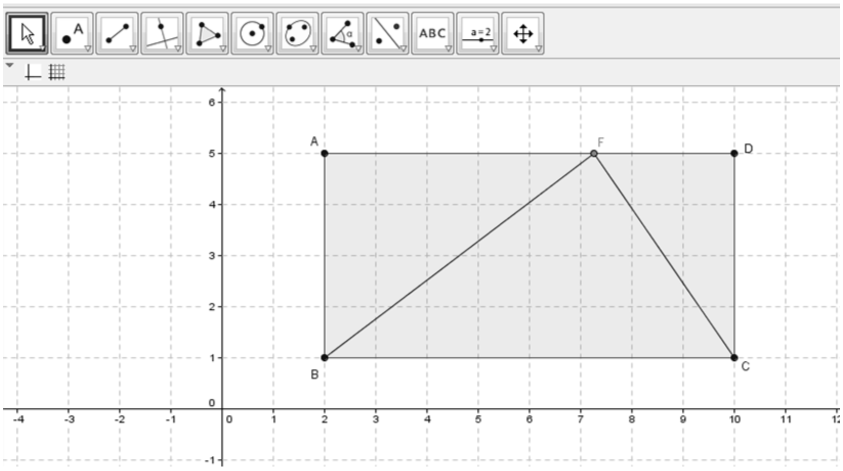


Figure 7.a: The area of triangles

T: "If I move the vertex here". (see Figure 7.b.). S4 says, "The altitude …" The teacher asks, "Is it still half the area of the rectangle?" S4 replies, "Yes." The teacher asks, "Why?" S4 says, "Because you didn't change …" The teacher asks, "What stays the same?" S4 says, "The base is the same, the altitude is the same." T: "If I move it here?" S5: "The same." (2.1.13.)

In protocols 2.1.12.-13, the teacher, by exploiting the opportunities dynamic geometry environments provide in mediating students' understanding, endorsed the gradual detachment from empirical arguments and the move towards formal mathematical reasoning.

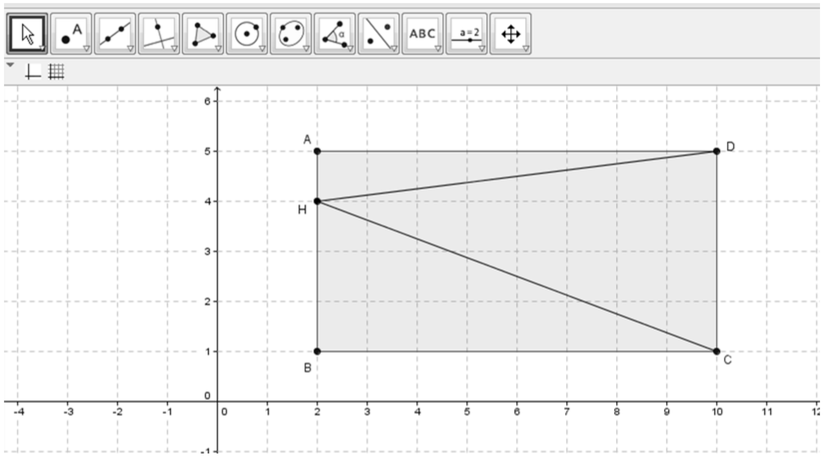


Figure 7.b: The area of triangles

Following this, the class engaged in elaborating the number of altitudes in a triangle. T: “Question: How many altitudes does a triangle have? 1, 2, 3?” S1 says, “2”, while S2 says, “One triangle.” The teacher repeats the question, “In one triangle how many altitudes can I have?” S2 says, “1.”, while S3 and S5 say “2.” Instead, S4 says, “3”. The teacher asks again, “I am asking … How many altitudes?” S6 says, “I think that because you can …” T: “In that one I constructed one altitude.” S6 says, “2.” The teacher repeats the question, “How many altitudes can I construct in one triangle?” S7 replies, “2.” The teacher asks, “Why”? and S7 says, “Because we have the altitude horizontally”. T: “In this triangle …” The students reply “Yes.” T: “Ok, let’s move to this one. From X I constructed one altitude. From where else can I construct an altitude?” S4 says, “We can do it from CO.” The teacher repeats the question, “From where else can I construct an altitude?” S8 says “From nowhere.” while S9 says, “From the vertex X.” The teacher says, “From the vertex X yes on CO. I did that.” S4 says, “They are three so far.” The teacher replies, “We constructed only one so far.” (2.1.14.)

The teacher moves from the rectangle and concentrates only on the triangle. T: “From the vertex X, I constructed an altitude …” S10 says, “From the vertex C on the base XO.” The teacher asks, “And where should it be on this base? S10 replies, “In the middle.” The teacher disagrees, “Not in the middle. It has to make a right angle.” S10 insists, “Then in the middle.” The teacher says, “It happened to be the middle. And this is because I have an isosceles triangle. Nice. Can I have another altitude?” The students reply “Yes.” T: “Nice, let’s go to this one.” S2 says, “From C.” The teacher says, “From the vertex C.” S2 continues, “On XO.” The teacher says, “On the base XO.” S2 says, “There aren’t any more.” At this point, the students conclude, “They all pass through the same point.” The teacher says, “Well done. This is what I was trying to achieve.” (2.1.15.)

The teacher continues, “All the altitudes pass … this is not part of our curriculum but part of the mathematics curriculum of secondary school, but it’s good for you to know it because it helps you. In a triangle I can have three altitudes. Tomorrow we will continue with other triangles. From one vertex … from each vertex I can create an altitude to the opposite side.” The students say, “3 altitudes.” The teacher concludes, “Yes, I have as many altitudes as the vertexes of our shape. Nice.” (2.1.16.)

In the above protocols (2.1.14.-2.1.16), the teacher encouraged a fusion between the concept image and concept definition regarding the altitude. That is, by making a connection with a part of secondary school mathematics, the students had the opportunity to make a hypothesis and investigate the validity of their assertions regarding the number of altitudes in a triangle.

The remainder of the lesson was devoted to engaging in exercises in the students’ textbooks, which according to their teacher were very similar with what they had been doing on the computer. The teacher was circulating among the students to make sure that they understood what they had to do. (2.1.17.)

During the informal discussion between the teacher and myself at the end of this lesson, while I was enthusiastic with the way the lesson progressed, as the students were exploring the DGE, the teacher was the opposite. She was not sure that the lesson was good, as she felt that more time was needed and that even if you have organized everything and you are prepared for your lesson, when you are working with computers, unforeseen events can happen that you cannot avoid, which have an impact on the lesson. (2.1.18.)

Day 2, Phase 2

On the second day, the lesson was initiated by revising the definition of the altitude. T: “Who is going to remind us what altitude is as this is something we found difficult in understanding yesterday". S1 says, "From the base to the vertex but … is right-angled". The teacher says, "It's … so … who is going to tell me again what is this … what do I bring from the vertex to the base and constructs a right angle?" S1 replies, "A line." The teacher says, "Instead of a line …" S2 says, "A segment …" and the teacher repeats, "A segment …" S1 says, "From the vertex to the base a segment that constructs a right angle." T: "Very good." (2.2.1.)

In this protocol the teacher guided the classroom discussion for the students to give the definition of the altitude as, according to her the students struggled to comprehend and appropriately use it. The teacher also appraises a definition that makes use of precise mathematical language.

Following this, the teacher encouraged explanation and justification that draws from the definition of the altitude. T: "If I draw a line to the middle of the base is it an altitude?" S2 replies, "No because it doesn't construct an altitude." while S3 seems to disagree, "It's not that." (2.2.2.)

At this point, the teacher, still looking for a reason said: "This is what I am asking. If from the vertex I draw a line to the middle of the base, is this an altitude? Yes, no and why?" S3 says, "No because for the line to be an altitude it has to start from the vertex, end up on the base and construct a right angle". T: "Well done. Thus, it needs to construct a right angle. It can be in the middle, it can be at the edge of the triangle, or it can be constructed outside the triangle. But it has to construct a right angle." (2.2.3.)

In this protocol the teacher appraises an explanation that makes use of precise mathematical language.

Following this, the students were drawing the altitudes of triangles presented in their textbook. In doing so, the students had to use a ruler, as the squared grid did not always prove to be a useful tool. The teacher was circulating among the students. Realizing that the students would not correctly use the ruler to draw the altitude, the teacher guided students in properly using the ruler to draw the altitude, T: "What do I need to place on the base to show that it is the altitude?" S1 says, "A line", while S2 says, "A right angle." The teacher repeats her question, "Where do I place the ruler?"

S2 replies, "To slide it on the side". The teacher concludes, "I use the perpendicular angle of the ruler and I place it on the side in order to draw a correct altitude." (2.2.4.)

In this protocol the teacher is asking the students to explain how and why they used the ruler by using the definition of the altitude.

After dedicating enough time, so as to making explicit the process that needs to be followed so as to correctly draw the altitude in a triangle, the students, in pairs, moved to the computers. The teacher told the students to find the area of the triangles shown on the GeoGebra screen. The students had the opportunity to explore the environment and decide which tools would help them in finding the area of the triangles. During classroom discussion, the students presented again the way they worked. (2.2.5.)

The following discussion illustrates the conclusion drawn regarding the area of specific triangles, T: "What is common in these triangles?" The students reply, "They are all the same." At this point, the teacher is guiding students in using precise mathematical language by giving a negative response to the students' answer, "This is not an answer." S1 says, "They all have the same area". (2.2.6.)

At this point, the teacher praises the complete answer: "Exactly. They all have the same area". S2 concludes, "They are not the same triangles but they have the same area." (2.2.7.)

Protocols 2.2.6. and 2.2.7. show the teacher following a variety of approaches that guide students towards the endorsement of this norm 'doing mathematics requires us to use precise language'.

At the end of the lesson, during the informal discussion the teacher commented that she could see her students gradually being more comfortable and confident with regards to exploring this dynamic geometry environment. She also said that this is happening 'with less noise and waste of time'. (2.2.8.)

Day 3, Phase 2

At the beginning of the lesson on the third day, the teacher asked the students to tell her the conditions that need to be applied in order to construct an altitude. T: "What is an altitude?" S1 says, "To be altitude it has to construct a right angle." At this point, the teacher praised the definition provided by the student, “Well done. Altitude of a point to any line, either horizontal, either perpendicular, either lateral … from the point to the line it needs to construct a right angle. And I showed the approach we need to follow so as to construct a right angle. We can use the triangle that had a right angle or the angle of the ruler.” (2.3.1.)

In this protocol the teacher praised the answer that made use of precise mathematical language.

Subsequently, the students had to draw the altitudes of triangles presented in their textbook. The triangles were presented on the interactive whiteboard, and the teacher was either following the students’ instructions to construct the altitude, or she was asking them to go to the board and do it themselves. (2.3.2.)

For the following exercise in the students’ textbook, the students had to write whether two triangles that have the same altitude have the same area. The teacher constructed on the interactive whiteboard two triangles with the same altitude. T: “I have these two triangles. Their altitude is 1,8cm. Do they have the same area? Yes, no and why? What did you write here?” S1 says, “No because they have different base.” The teacher says, “S1 says no because the base is different. Do we agree with S1?” S2: “Mrs …” T: “Wait, let me move it.” The teacher changes the base of the triangles so that they are the same. T: “Now? For the previous one, they had a different base, different area, now?” S3 says, “If they have the same base yes, if not …” The teacher says, “Well done. Thus it depends on the base, if they have the same base as in this example …” S4 says, “That’s what I wrote. No …” The teacher replies, “No. it depends on the base.” (2.3.3.)

This protocol points to the students justifying their answer by employing the mathematical formula for the area of triangles.

At this point, the teacher insists in providing complete answers, “Complete answer. If they have the same base and the same altitude then the area of the triangles is the same, if not they have different areas … Did you write that?” When S4 replies “Something like that.” The teacher says, “As long as the meaning is the same it is fine.” S5 says, “Mrs I wrote that when the base is not the same the area is not the same.” (2.3.4.)

At this point, the teacher makes the norm related with justification explicit. T: “Nice. We need to justify our answer.” (2.3.5.)

Protocols 2.3.3.-2.3.5. show the teacher following a variety of approaches that guide students towards the endorsement of the norms ‘doing mathematics requires us to use precise language’, ‘doing mathematics requires us to justify our assertions’ and ‘we write coherent geometrical explanations’.

At this point, the following episode developed, S1 asks, “But … Mrs … how did you find the area?” S2 repeats, S2 “How?” The teacher replies, “I didn’t.” S3 says, “We tried from the menu bar but …” The teacher says, “I didn’t manage … I didn’t find it ... I will ask and I will learn … I tried as well but it’s not that”. S4 says, “The same with me” and S5 agrees, “Me too”. The teacher says, “Let me see if I can do it this way. If I put area of ABC …” S6 says, “It might need a rectangle”. The teacher replies, “Not a rectangle, somewhere it must have the option to write the name of the triangle ABC, it wants to define it but I haven’t discovered that, I struggled a bit. But I will find it and I will tell you”. The students agree with the teacher, “Yes”. (2.3.6.)

This protocol indicates that regarding the integration of technology, both the teacher and the students are learning together how to use the tools available in the dynamic geometry environments.

With the above discussion the classroom completed the exercises in the textbook. Following this, the students would move to the computers. At this point, the teacher said: “Now we will go back to the software to play a while with its features … I will give you instructions and you will construct … to see that you understand how it operates, for 5-6 minutes and then we will move on to a game where you will play in pairs on the computers online.” (2.3.7.)

In this protocol, the teacher relates both exploration of the DGE, and playing a game on the computer with ‘play’, something encouraging and constructive.

In the following part of the lesson, the teacher asked the students to construct an obtuse-angle, acute-angle and right-angle triangle in the DGE whose area equals $4cm^2$. Before leaving the students to construct the triangles, the teacher demonstrated how to construct a triangle in this particular DGE, T: Look now how it works. I will construct a random triangle and then you will construct triangles. In order to construct a triangle I need to define the three vertices and close the construction. If I start from here and click for a first point, I move up and click for the second vertex and then I go down, the area is colored but my construction is not finished as I need to go back to first point for my construction to be complete. Now the triangle is red. Did you see that? The students reply, “Yes”, and the teacher says, “Well, nice. Construct now the three triangles”. (2.3.8.)

In this protocol, the teacher, as instructor, guided students’ instrumental genesis.

After the students constructed the triangles, the teacher asked them to construct the altitudes from the three vertices of the obtuse angle triangle IKL. T: “Do you agree with this? That the altitude is IM?” Some students say, “Yes” whereas others say “no”. (2.3.9.)

At this point, the teacher looking for an explanation asks, “Why not? Because I heard you saying no.” S1 says, “It should be on K … it can’t go there.” T: “Here?” The teacher is extending the side. T: “Here? Does it construct a right angle with KL?” The students reply “Yes.” T: “By extending KL.” Again, the students reply, “Yes”. The teacher continues, “Where did you say that I need to bring it? Here?” The teacher moves the ruler across KL. The students say “Yes.” The teacher continues, “But does it construct a right angle?” The students say “No.” (2.3.10.)

The teacher concludes, “No … so I cannot bring the altitude here. In the obtuse angle triangle, from some vertices I cannot bring the altitude inside the triangle, the altitude in some cases falls outside the triangle. Yes, the altitude is outside the triangle because it is there where it constructs the right angle.” (2.3.11.)

The protocols 2.3.9.-11, point to the teacher making a connection with a part of mathematics that the students will explore in secondary school.

Day 4, Phase 2

The lesson started with the students exploring a task in the DGE. On a blank DGE window, the students had to construct a right-angled triangle, an acute-angled triangle and an obtuse-angled triangle. Adding to the above, they had to write in their notebook the base and altitude of each triangle that they had constructed. (2.4.1.)

This protocol focuses on the exploration opportunity the teacher provided to the students. The students themselves had to make the necessary technical and theoretical connections and appropriate the technology according to their needs.

The classroom discussion that followed the students working on the computers was initiated by the teacher commenting on the way the groups worked. T: “Some groups worked really well with the software, some forgot and had some difficulties. However, there was a group that worked with a really nice way which I will demonstrate. This software helps me … of course in the notebook I will learn how to measure with the ruler … you had to construct a right-angled triangle. You all constructed this triangle correctly. The right-angle triangle was easy. The one that was a little challenging was mostly the obtuse-angle triangle. But the way some worked was quite clever. Some groups thought that in order to have an acute-angled triangle with area 8 cm^2, I need the base x altitude to be 16cm. Thus, they either had 4x4, or 2x8. And how did they do it? To be easy, they constructed initially a right-angle triangle that was 4x4 and moved the point (vertex) so as to become an acute-angled triangle. That was really clever. They did the same with the obtuse-angled triangle. They constructed the right-angle triangle, but because the angle had to be bigger than 90°, they moved the vertex on the opposite direction. The way other groups worked was more forward.” (2.4.2.)

This protocol shows the teacher supporting the solution of problems using a variety of approaches.

After revising what the class worked on throughout the week, the students played a game on the computer. A random triangle would appear in a squared framework on the computer screen. The students would write the area that they thought the triangle had, click ‘check answer’. If the answer was correct the students could move to a new triangle. If the answer the students gave was incorrect, they had to try again. The degree of difficulty in finding the area of the triangles was increasing gradually. (2.4.3.)

This protocol points to the activity the teacher refer to in protocol 2.3.7.

At some point, pairs of students complained that their answer was correct but on the screen it would show that it was incorrect. These students insisted that what they did was right, even after the teacher told them that they made a mistake in calculating the area of the triangles and encouraged them to try again. (2.4.4.)

Day 5, Phase 2

The lesson began with the students and the teacher discussing once more, the topic covered throughout the week. When the students said that one of the things they did was how to find the area of triangles, the following dialogue occurred, T: "And how do we find the area of a triangle S1?" S1 replies, "Firstly we need to find the altitude which is from the vertex to the opposite base and constructs a right angle." T: "Well done." S1 continues, "And then to choose a base so that when we multiply them and divide them by 2 we find the area." The teacher asks, "Multiply what?" and S1 replies, "The base and the altitude." T: "Perfect. A complete answer S1." (2.5.1.)

In this protocol the teacher praised the answer that made use of precise mathematical language.

The end of the lesson indicated the completion of the lessons related with the area of triangles. T: On Tuesday we will have a small …" The students said, "Test". At this point, the teacher said, "Game, test on the computer in order to construct triangles and parallelograms with specific areas". (2.5.2.)

This protocol focuses on the teacher relating assessment to 'play', indicating that the evaluation will take a more playful form.

The following section focuses on the analysis of the findings of Phase II of the study.

7.4. Ongoing analysis

As exemplified in Section 6.5., exploration and argumentation have been identified as the themes of interest that would initially guide analysis of the observation data and the data from the informal discussions with the teacher. How the themes from Phase I developed and how new themes emerged during Phase II is set out below and was summarized in Table 5.3., duplicated for convenience below in Table 7.1.

Initially, through classroom observation, it becomes apparent that the teacher is supporting mathematical connections. The 'opening value' points to the teacher making connections with parts of mathematics that the students would be taught in secondary school or that were taught recently or further in the past. These forward connections seemed to be a natural part of this teacher's mathematics lesson.

To be more accurate, protocols 2.1.14.-16, and 2.3.9.-11 indicate the teacher including things in the lesson that are being part of the mathematics curriculum of lower secondary school. For instance, a part of the curriculum concerned with triangles in Year 6 is constructing the altitude in a triangle. In protocols 2.1.14.-16 the teacher made a forward connection by exploring with the students the number of altitudes a triangle has. By constructing the three altitudes, the teacher's aim was for the students to see that the altitudes intersect in a single point, the orthocentre of the triangle.

Furthermore, in protocols 2.3.9.-11 the teacher moved forward by also exploring with the students how the altitude of an obtuse triangle can fall outside the triangle. Even though in the mathematics curriculum there is no specification concerning whether the students should be aware of the fact that there are cases where the altitude of a triangle lies outside the triangle, this is not supported by the students' textbook nor the teacher's guidance book. This is, however, part of the curriculum of secondary school mathematics.

The teacher's behaviour in the above protocols is in accordance with what she said in the initial interview. She makes these forward connections as she believes that these varying ways of exploration in the classroom will assist students in gaining a deeper understanding of the geometry under study, which will prove valuable for a more effective transition to secondary school geometry.

In addition to the above, in protocols 2.1.5. and 2.1.7. the teacher is making a connection with mathematics previously taught. That is, the teacher with the students make a connection regarding the mathematical formula for the area of triangles and recognize the difficulty of finding the altitude.

Going further, the teacher is supporting the solution of problems using a variety of approaches (protocol 2.4.2.).

The teacher is also integrating technology in her teaching. In order for the students to engage with the activities the teacher would provide, the teacher would illustrate how certain available tools could be used to explore a certain activity (protocols

2.1.8. and 2.3.8.). In the above protocols, the teacher's focus was mainly on the technical aspects of the dynamic environment. Protocols 2.2.5. and 2.4.1 point to exploration opportunities where the students themselves had to make the necessary technical and theoretical connections and appropriate the technology according to their needs. Protocols 2.1.12.-13 illustrate how that specific activity was employed by the teacher as a way to enhance the generality of the algebraic expression of the area of triangles. In order for this to be achieved, the argumentation process was guided by the teacher. Protocols 2.4.3.-4 describes the game the students played on the computer.

Adding to the above, the teacher makes connections between the dynamic geometry environment and the paper-and-pencil environment. In protocol 2.1.17., the teacher returned to the paper-and-pencil environment so as to make connections with the mathematics outside the microworld. Thus, it can be argued that making these connections support the students' instrumental genesis. Returning back to the mathematics textbook may also be interpreted by taking into consideration the contradictory statements the teacher made regarding the appropriation of technology in her teaching.

Keeping in mind the aforementioned, it can be concluded that these aspects of classroom activity that have emerged from the data, correspond with exploration (see Level of exploration in Section 7.5.). However, mapping out the current situation of the classroom reveals that the degree of expectation regarding exploration does not match the teacher's exemplification pertaining to the importance of exploration that leads to discoveries (see I4 in Table 5.2.).

Accounting of the data also led to identifying occasions where the teacher made use of the word 'play' to refer to the students' activity (see Level of play in Section 7.5.). To be more precise, in protocols 2.1.4. and 2.1.9., this word had a negative value when the exploration was interpreted by the teacher as messing around instead of focusing which would lead to learning. That is, while she would encourage students to explore the activity in order to reach to some conclusions, she would also make a negative remark about this exploration as something that had no didactical value. In protocol 2.3.7., 'play' had a positive value when it was used by the teacher to refer to the exploration of the activity. The teacher also made a positive use of the word when referring to assessment. In protocol 2.5.2., evaluation takes a more playful form.

It has been established in Section 2.3.1.2. that the rules of discourse established in the mathematics classroom practices influence the emergence of students' meanings for proof. Several protocols point to the teacher negotiating and establishing social and socio-mathematical norms (see Level of participation in Section 7.5.). To be more comprehensive, the teacher gave emphasis on the use of precise mathematics language and terminology. In addition, the teacher would not accept an answer (verbal or written one) unless it was complete. Instead, she would encourage students to develop both their verbal and written communication of their geometrical reasoning. Protocols 2.1.2., 2.2.1., 2.2.3., 2.2.6., 2.2.7., 2.3.1., 2.3.3.-2.3.5. and 2.5.1. refer to the norm 'doing mathematics requires us to use precise language'.

The fact that the teacher endorses the use of precise mathematical language has also emerged through the interview data. This can also be considered a forward connection. Even though many aspects of the mathematical language being used in the primary classroom are still informal, accurate expressions prove extremely helpful when first moving to the secondary classroom.

With regards to explanation and justification, protocols 2.3.3.-2.3.5. point to the sociomathematical norm 'doing mathematics requires us to justify our assertions'. Protocols 2.3.3.-2.3.5. also point to the sociomathematical norm 'we write coherent geometrical explanations'. What is more, protocol 2.3.6. seems to draw attention to the norm 'when we work with computers we learn together'. By considering the fact that the teacher is supporting the solution of problems using a variety of approaches (protocol 2.4.2.), it can be argued that the social norm 'we solve problems using a variety of approaches' is also negotiated in this classroom's practices.

Accounting of the data from the classroom observation provides an initial consideration of the mathematical argumentation as occurred in the classroom (see Level of proving in Section 7.5.). To begin with, it can be argued that the opportunity to explain and justify was provided to the students and thus, mathematical justification is encouraged in the classroom. The principles of regulation of action in this mathematics classroom which have been exemplified previously strengthen this remark. To be more elaborative, in protocol 2.1.10., explaining develops as an activity related with defining. In protocols 2.2.3. and 2.2.4. the students give reasons based on the definition of the altitude. In protocols 2.1.12-2.1.13, 2.3.3. and 2.3.4., the students are justifying their answer by employing the mathematical formula for the area of triangles. In addition, in protocol 2.1.6. the students with the teacher arrived at conclusions based on one or several drawings or examples. Protocols 2.1.14.-2.1.16 indicate that, to an extent, the students had the opportunity to make hypothesis and investigate the validity of their assertions. What seems interesting though is that while the teacher exemplified in the initial interview the importance of justification and proving in the teaching of geometry, the classroom does not engage often or very deeply in proving in the formal sense.

Keeping in mind the aforementioned, it becomes apparent that the teacher adopted various roles throughout the lessons (see Level of intervention in Section 7.5.). For instance, protocols 2.1.2., 2.1.12. and 2.3.3.-2.3.5. show the teacher following a variety of approaches that guide students towards the endorsement of socio-mathematical norms. The teacher either makes this rule explicit, rephrases what the students say, gives a negative feedback to a response that does not embrace the norm and/or appraises the response that is correct (protocol 2.5.1.). Regarding the appropriation of technology, the teacher would have the role of instructor, facilitator and mediator guiding students' instrumental genesis.

Regarding the informal discussions with the teacher (see Level of participation in Section 7.5.), during this phase of data collection, the teacher's comments were focused on the integration of technology in her teaching (protocols 2.1.18. and 2.2.8.). Initially, time seemed to be a concern for the teacher. However, allocating enough time for the students to explore the environment and engage with the activities provided was necessary. Gradually, as the students gained more confidence in exploiting the opportunities provided by the dynamic geometry environment, the teacher's comments appeared to be more positive. The impact of time constraints in this teacher's teaching practices also appeared in the initial interview. However, it should be made explicit that this concern is not only related with the appropriation of technology in the teaching. It is also related with the teacher's concerns regarding the mathematics curriculum.

Table 7.1.: Map of themes Phase II

Post Phase II (acting as moderate observer)	**Insights from the classroom observation and the informal discussions with the teacher** MO1 Occasions where the teacher is making connections with parts of mathematics that the students would be taught in secondary school, were taught either recently or in the past are identified ('opening out' value). MO2 The parameters play/ learn are identified: there are occasions where the students' activity was being translated by the teacher as 'playing' instead of learning. MO3 The following rules of discourse being negotiated and established in the classroom are identified: 'doing mathematics requires us to justify our assertions', 'doing mathematics requires us to use precise language', 'we write coherent geometrical explanations'. MO4 The teacher is integrating technology in her teaching.
New synthesised set of themes	**Levels of action** L1 Level of exploration (T1, MO1, MO4) L2 Level of play (MO2) L3 Level of participation (MO3) L4 Level of intervention (T1, T2), L5 Level of proving (T2) L6 Level of collaboration

7.5. Summary and discussion

In the light of the above discussion, it is only reasonable to reconsider the previously identified themes of interest. Levels of actions related with the classroom endeavour should be taken into consideration, so as to thoroughly portray the classroom level.

Exploration has been previously identified as a theme of interest (see Table 7.1.). The insights gained by considering the classroom protocols indicate that exploration continues to be a theme of interest. Keeping in mind the aspects of the classroom activity related with exploration as well as the initial exploration of the mathematical argumentation as occurred in the classroom, **the level of exploration**, is concerned with the degree of exploring the mathematics in the classroom. This level includes the 'opening out' value as well as the integration of technology.

Adding to the above, as this research study aims to understand how proving is constituted in the mathematics classroom, **the level of proving** aims to provide a portrait of the mathematical argumentation as occurred in the classroom. Instances of students' explanation and justification fall into this level. At this point, unfolding the argumentation as occurred in the classroom is considered helpful in also determining the way changes made in the activity system of the classroom in the subsequent phase of the research affected students' argumentation.

In order to further explore and understand the significance of 'play', **the level of play**, is concerned with occasions where the students' activity was being translated by the teacher as 'playing' instead of focusing on the undergoing endeavours of the classroom.

Keeping in mind the way the teacher expected the classroom to participate in the mathematics lesson, **the level of participation** includes the identified social and sociomathematical norms being negotiated and established in the classroom. Identifying these principles of regulation of action will aid in establishing their influence in the activity of proving.

The teacher's actions pointed out in the above discussion appear fundamental in determining overall the teacher's extent of involvement throughout the lessons. Thus, the way the teacher orchestrates the classroom situations should be further explored. The **level of intervention** explores the way the teacher intervened at different phases of the mathematics lesson.

As the teacher and I would collaborate in the subsequent phase of data collection, and informal discussions would be taking place after the lessons, **the level of collaboration** is also identified as a theme of interest. This level focuses on the collaboration between the teacher and me.

Ultimately, this chapter has outlined the outcomes of the second phase of this study. The insights from the analysis of the baseline observation of the classroom and the informal discussions with the teacher point to six levels of action (see Table 7.1.).

How these levels of action inform the design and development of Phase III of this study is presented in Chapter VIII, which follows.

CHAPTER VIII

PHASE III

PARTICIPANT OBSERVATION

8.1. Introduction

Chapter VIII focuses on Phase III of this study. Identifying the system level, exploring the teacher's story and mapping the current situation of the mathematics classroom have led to the identification of levels of action. Simultaneously, in understanding how proving is constituted in the primary classroom, the necessity of gathering additional data has also been stressed. In Chapter V, it has been argued that a combination of the ideas of CHAT and a collaborative design approach would lead to a thorough investigation of the research questions. That is, collaborative task design would function as a Trojan horse, a means of gaining access to the teacher's objectives. By engaging the teacher in the design of tasks to be used with the children, the aim is to expose the nature of the teacher's objects at the beginning and how these objects change and maybe clash with objects at different levels, such as it is portrayed in the curriculum and expressed by children as they work on those tasks.

This chapter begins by introducing the method for conducting the third phase of the study. That is, by focusing on the mutual collaboration between the teacher and me, this chapter initially provides a concise description of this collaboration and the justifications for the decisions taken. This discussion leads to the specifics of the design of the tasks. How the data were gathered is also presented. Additionally, my role as a researcher is also exemplified.

This chapter then proceeds by introducing the findings from Phase III of data collection. A chronological overview of the lessons is provided by presenting episodes from the classroom observations as well as the informal discussions.

This chapter continues with the analysis of the classroom protocols and the informal discussion with the teacher by taking into consideration the levels of actions (see Section 7.5.). Insights from the collaboration between the teacher and me that have emerged through the task design process and the lessons are presented and elaborated on. This chapter progresses by describing how accounting of the data from the classroom setting led to a new synthesised set of themes being drawn together.

8.2. Method

To begin with, the teacher and I undertook informal discussions prior to the data collection process. These discussions were related with the development of the research plan of this phase of the study.

Initially, the first decision that had to be taken was related to the part of mathematics with which the class would engage. By keeping in mind the purpose of this research, as well as the mathematics curriculum that the teacher had to follow, it was agreed that, for this phase of data collection, the content of the geometry curriculum that would be covered would be related with circle.

The following decision that was made was related with introducing Cabri in the classroom. At this point, it is crucial to clarify and justify this decision. Keeping in mind both the fact that the teacher had already employed a dynamic geometry environment in the classroom, as well as the potentials of DGEs, I suggested using Dynamic Geometry Software for circle. The teacher was in agreement with this and suggested employing Cabri. The teacher had the opportunity to be introduced to this Dynamic Geometry Environment in an informal way during the exploratory study I (see Section 3.3.). By considering this, the teacher stated that, in her opinion, these two Dynamic Geometry Environments shared many similarities. What is more, she said that by employing Cabri she would have the opportunity to learn more about this specific dynamic geometry environment. By doing this though, the teacher said that she would feel more comfortable with me doing the lessons, as she feared that she might not be able to answer students' questions regarding this DGE. Despite this, she said that she would be there to help with the parts of the lessons not being related with the exploration of the DGE-based tasks. However, this was in contrast with my research objectives. Nevertheless, after discussing this further, we agreed that I would design the tasks on this DGE, by drawing on her suggestions and we would do the lessons together.

Following this, the teacher informed me about how she usually teaches this area of geometry and what she would like me to do. The discussion related with the lesson plan and the task design process will be presented in more detail subsequently, in Section 8.2.1. Following our discussion, I designed the DGE-based tasks, emailed them to her and then we had a phone conversation in order to discuss them. We had another discussion prior to the data collection process.

The iteration phase lasted one week, during which 7 periods (40 minutes each) were allocated for the teaching of 'Circle'. The lessons were video recorded and field notes were kept. The video recorder was positioned in such a place that allowed the recording of both the teacher and the students. An audio recorder was also placed next to one pair of students while they were exploring the DGE-based tasks in order to get an insight into the way these students are interacting towards the exploration of these tasks. The structure of the field notes was the same as in Phase II of the study (see Section 7.2.).

Compared to the baseline observation of the classroom, in Phase III of the study the informal discussions carried out after each lesson were more structured. This is due to the fact that both the teacher and I were doing the lessons together. The first step was to see whether the goals of the lesson were achieved. This also made explicit the content of the geometry curriculum that was covered that day. Following this, striking events that occurred during the lesson were pointed out either by the teacher or by the researcher. In addition, the DGE-based tasks designed for the specific lesson were evaluated, and suggestions were made, if applicable, often related to ways that would make these tasks more 'approachable' to students. This will be further elaborated on in a succeeding subsection related to the designing of the DGE-based tasks. Finally, a discussion was invariably scheduled to consider how the aforementioned might influence planning and organization of the lesson for the following day.

The following subsections further illustrate the sequence of the phases involved in the development of the collaborative design approach as conducted in this study.

8.2.1 Lesson plan for circle

The teacher said that when she teaches this topic in geometry, she follows the students' textbook. She mentioned that she doesn't use the teacher's guidance book as a reference as she is familiar with the educational goals of this topic. She exemplified the way she usually introduces students to 'circle' and requested that I would do the same. That is, she presents several shapes to the students asking them to specify the objects that are circles by also providing an explanation.

Nevertheless, since we would teach together this area of geometry, an exploration of both the students' textbook as well as the teacher's guidance book was considered essential. The exploration of the content of these two books related with 'circle' would guide me towards the design of the DGE-based tasks that would be used in the classroom, and would be in accordance with both the teacher's and the researcher's goals.

The general educational goals for 'Circle' (circumference and area of circle), as presented by the Indicatory Monthly Planning for Year 6, are the recognition of the elements of the circle (centre, radius, diameter, circumference), the solution of problems regarding circle, the recognition of the relation between radius-diameter, explorations for the number 'π', calculation of the circumference of the circle when the length of the radius is given and recognition of the area of the circle when the length of the radius is given. The specific cognitive goals of this topic in the mathematics curriculum, as presented in the teachers' guidance book, are presented in the Appendix VIII.

What is more, while the analysis of the students' textbooks has been presented and elaborated in the section of the report on the official documentation, and the design of the DGE-based tasks did not really rely on the activities presented in the textbooks, it is worth mentioning that these activities directed to reaching the aforementioned goals are mainly based on the students drawing conclusions about these relations by making generalizations from data added in a table.

The above goals in accordance with the activities in the textbooks and the teacher's input let to the construction of two DGE-based tasks. In the subsequent subsection, the rationale that guided the task design process is further explicated. Following this, the tasks are illustrated and elaborated on.

8.2.2 Task design

In order to design the DGE-based tasks in such a way that would be in accordance with the overall purpose of this research, specific steps were followed.

Initially, the first step was to take into consideration the research's objectives, that is, to investigate the activity of proving as constituted in the classroom. In order to achieve this, situations for argumentation had to be created and explored by taking into consideration the nature and types of tasks that are considered crucial in providing such occasions both generally and in DGE particular, as presented in the review of the research literature. That is, these tasks would be open so as to give the opportunity to students to dynamically explore, make hypotheses, test these conjectures and justify their work. This was also in accordance with the themes of interest as emerged from the classroom. The teacher also exemplified the importance of exploration that leads to discoveries, as well as the importance of justification and proving in the teaching of geometry during the initial interview.

Following this, as mentioned in section 8.2.1., the overall educational and cognitive goals of this topic in geometry were taken into consideration.

Adding to this, the teacher's contribution was very essential. While I was talking about tasks that would give the opportunity for argumentation, the teacher said that she did not want these tasks to be similar with the tasks used the year before, during the exploratory study that I conducted with her students (see Section 3.3.). To be more precise, the teacher wanted simple activities, where the students could explore circle, make a table and add numbers in a table concerning the radius, diameter, circumference and area of circle so as to identify the relationship that exists between these notions. These activities could be found in the students' textbook. Even though this was in a way in contrast with my expectations and this research's objectives, the teacher's recommendations could not be ignored. To be more accurate, even though I recognized the fact that the aforementioned area of the geometry curriculum had to be taught, I originally wished to draw on the 'circle' topic so as to explore, for example, the circle theorems. 'Opening out' the mathematics could also be related with the level of exploration (see Section 7.5.). Despite this, the teacher was very specific about the structure of the tasks. Nevertheless, our mutual collaboration led to resolve this challenge by considering designing tasks that would provide the opportunity to students to explore the mathematical ideas the teacher was interested in, and simultaneously encourage mathematical argumentation.

As a result of the aforementioned conditions, the following tasks were constructed:

DGE-based Task 1: Circumference of the circle

DGE-based Task 1 included two figures. The first figure included a circle centre I, radius IA, with a free radius point (A) that can moves along the line IA (see Figure 8.1a.).

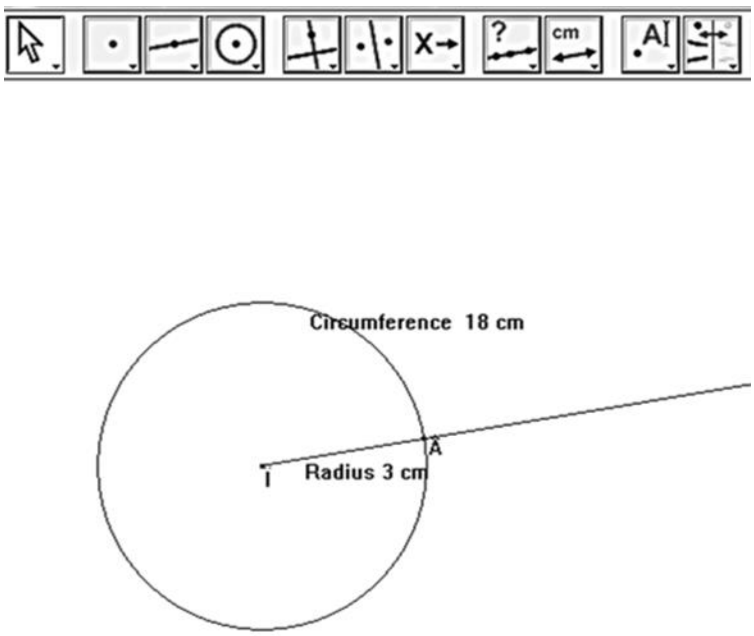


Figure 8.1a.: DGE-based Task 1 - The circumference of circle

The second figure included the circle with a free radius point (A) as before. As the free radius point (A) is moved along line IA, the circle changes size and a linear graph of the circumference (C) against its radius is plotted. In designing this graph the coordinate axis was made visible, values of the radius were plotted on the x axis and the circumference on the y axis. Nevertheless, even though the values of the radius and the circumference were plotted on the x and y axis accordingly, no points were given names or letters on the two axis (for example A, C). By default, the trace of the graph was off but this could be switched on at any time. Figure 8.1b. below shows a screenshot of the DGE-based Task 1.

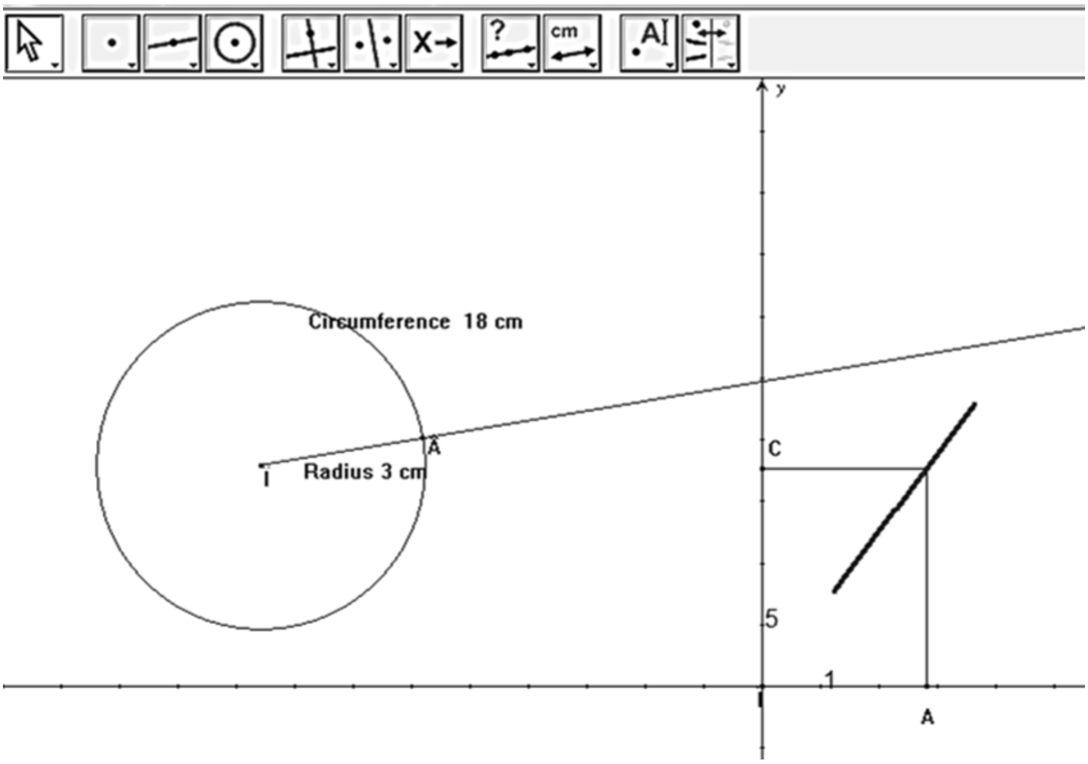


Figure 8.1b.: DGE-based Task 1 - The circumference of circle

The rationale for designing this task was for the students to have the opportunity to investigate the relationship that exists between the radius, the diameter and the circumference of the circle, by relating changes made in the circle with what may or may not change in the graph. What is more, DGE-based Task 1 as well as DGE-based Task 2 gave the opportunity for students to relate the numbers in a table, the diagram and the graph. Thus, the exploration of the task could encourage formulating hypotheses and conjectures and thus, mathematical argumentation.

Initially, the students would explore the first figure. That is, they would begin to drag point A along the path and make observations regarding changes made to the circle. That is, when the radius increases, the circle becomes larger and vice versa. Furthermore, the students could explore the relationship between the radius and the diameter of the circle. This could be done by also making a table in either their notebook or on the screen on which they would add the value of the radius and the diameter each time they moved the point. Going further, they could also explore the algebraic expression of the circumference of the circle by initially relating the properties of the circle with the graph. Furthermore, the students could conclude that the increase of the circumference of the circle is proportional to its diameter and radius.

DGE-based Task 2: Area of the circle

The task related with the area of circle was designed in a similar way as the circumference graph, though in this case the graph would be parabolic. This task gave students the opportunity to explore how the radius and the squared radius of a circle relate with its area.

This task included one figure (see Figure 8.2.); the circle, centre O with a radius OB that changes as the free radius point moves along the line OB, also generating a graph of the radius against the area of the circle. In designing this graph the coordinate axis was made visible, values of the radius were plotted on the x axis and the area on the y axis. Nevertheless, even though the values of the radius and the area plotted on the x and y axis accordingly, no points were given names or letters on the two axis (for example A, E, K).

Figure 8.2. below shows a screenshot of the DGE-based Task 2.

At this point it should be mentioned that for DGE-based Task 2, a worksheet was not given to students. That is, the students were familiar with the steps that they had follow in order to explore the relation between the radius, the diameter and the circumference of the circle, and were able to do the same with the task related with the area of the circle.

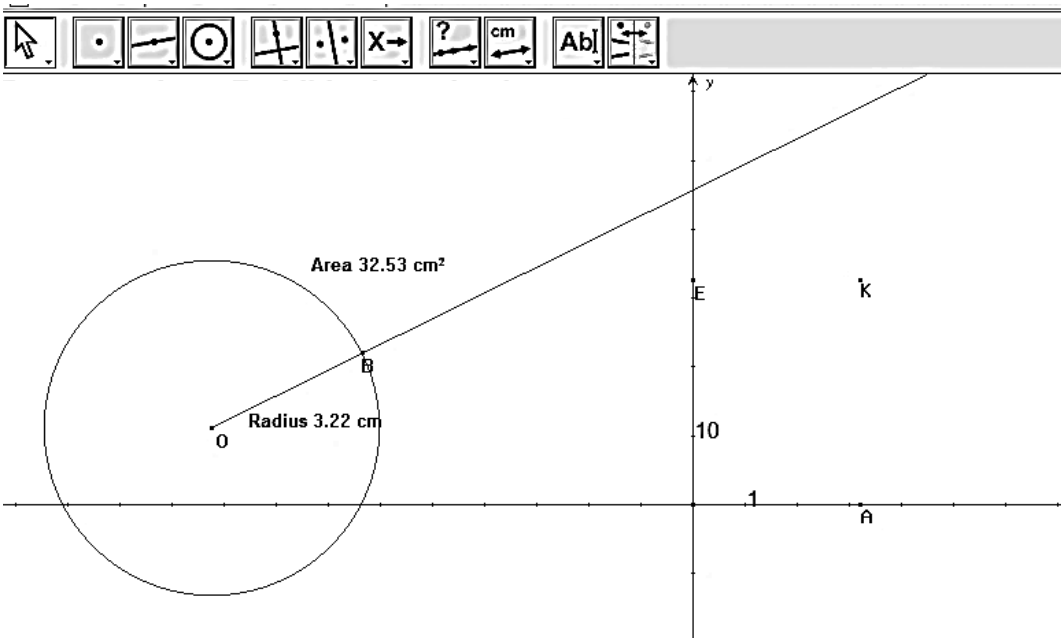


Figure 8.2.: DGE-based Task 2 – The area of circle

8.2.3 My role as a researcher

During the third phase of the study, I had an active role on the planning of the lesson and the activities; I was doing parts of the geometric lesson with the teacher, participating in many of the activities that I was observing, as well as discussing with the teacher the alterations that had to be made to the following stage of the lesson, which overall might be described as active participation.

As in this phase of data collection the level of my participation in the classroom shifted, my role within this specific group had to be re-established. The teacher informed the students that I would present to them a different Dynamic Geometry Environment that would be used in the classroom in a similar way to when they used GeoGebra for other lessons in geometry. This decision, made by the teacher, uncovered the role she preferred me to take on. This would be a familiar situation that made her feel more comfortable. This was another incident that stressed the complexities of my presence in the classroom.

In addition to all the above, I also had to keep a balance between my role as a researcher and my role as a co-teacher. The tensions, which arose during the data collection process as well as during the process of the data analysis, were related firstly with the fact that as a researcher I was collaborating with the teacher in planning the lesson and designing the DGE-based tasks. While collaborating with the teacher in designing the tasks and constructing a lesson plan, I could see myself dichotomized between the things I was doing as a researcher and my views and goals as a teacher. The fact that I am also a primary school teacher made this collaboration challenging. For instance, in making decisions regarding the design of the DGE-based tasks, tensions would be more concerned with the fact I was a researcher whereas the teacher was not, which meant that some issues that had to be considered which were not so important to her. What is more, as a teacher, there were some things that I would have done in a different way in the classroom.

The tensions were also related with the fact that in the classroom, during the second phase, I was both a researcher as well as a teacher. Trying to maintain stability between teaching and researching proved to be a multifaceted process.

Bearing in mind the challenges I would have to confront by being both a researcher and a supporting teacher, I decided to follow specific strategies that would ensure the balance between detachment and active involvement. Deciding which strategies to employ was not an easy task to achieve, as this was the first time that I was undertaking this role in the classroom. Despite this, the objectives of my research, my experience as a teacher and researcher, the discussions with fellow researchers concerning the steps that they followed to maintain this balance, as well as the literature available concerning the roles in school based research, enhanced my understanding of the situation and helped me to make the appropriate decisions.

My involvement was firstly concerned with answering questions related with the tools the software provided, which the students had to use in order to explore the tasks. This was more apparent at the beginning of the iteration phase, even though an introductory lesson was given with an emphasis on the differing functions this specific DGE provided when exploring different tasks. This was also one of the reasons why at the end of the first day, the teacher and I agreed to provide students a worksheet including steps that they could use to explore the tasks. Furthermore, on several occasions the students would seek explanations concerning their observations. On this occasion, my involvement was related with probing questions.

8.3. Findings

This section reports the findings of Phase III of the study. That is, the chronological overview of the lessons is provided by presenting episodes from the classroom observations as well as the informal discussions. The classroom protocols are accompanied by short commentaries that either point to a theme that has already been introduced or are a precursor to introducing a new theme. These commentaries are for the benefit of the reader to highlight the potential significance of the protocol; they were not part of the original accounting of the data but reflect the development of old themes and sometimes the emergence of new themes which will become a significant part of the subsequent analysis when accounting for the data. For this reason, the commentaries are written in italics.

Regarding the classroom protocols, it should also be noted that all paragraphs are numbered in order to maintain a coherent system for their presentation and discussion. In the dialogues, T represents the teacher and S1, S2…Ss the students that participated in the discussion.

Day 1, Phase 3

At the beginning of the lesson the teacher informed the students of the content of mathematics with which the class would engage i.e. the circle. The teacher began by asking the class for the definition of a circle, T: "What is circle?" S1 replies, "It is a shape that does not have sides or angles." (3.1.1.)

This protocol focuses on the question with which the lesson begun.

The teacher probed for a definition that was precise. T: "S1 says that a circle is a shape without sides or angles. I draw a shape according to this definition". The teacher draws a non-regular shape with curved lines. T: "According to what S1 said this is a circle." The students reply "This is not a circle". S2 says, "A straight shape" while S3 says, "Without curves." (3.1.2.)

The teacher still insisting for an accurate definition says, "I want an accurate definition. S4". Responding with a precise definition was not easy for the students, who looked instead for analogies. S4 says, "We call a circle the shape that … it has the shape of a sphere." The teacher asks, "What is the difference between a circle and a sphere?" S5 says, "The sphere has volume." The teacher says, "The sphere has volume, it is 3-dimensional, whereas a circle is …" The students say, "Flat." The teacher continues, "Flat … Thus, a circle is a flat shape whereas a sphere is a 3-dimensional shape. Which shape do we call circle S6?" S6 says, "The shape that does not have angles." The teacher agrees, "Yes." S6 continues, "And has a curve as a side … a curve." The student draws a circle in the air with his hand. T: "Like this?" The teacher draws an ellipse on the whiteboard. The students' response to the teacher's drawing is "No." S6 says, "No, I mean … it is like …" The student is using again his hands to show what he means. (3.1.3.)

At this point, the teacher makes explicit this specific characteristic of mathematical definitions, T: "We said that in mathematics, our definitions must be accurate. Is there a detail that is missing?" S7 says, "A circle is a flat shape." T: "Correct." S7 continues, "That …" (3.1.4.)

In this protocol the teacher makes explicit one of the functions of mathematical definitions.

Having made the point that it is not so easy to be precise, the teacher turned the class's attention towards properties. T: "A circle has some characteristics." S8 says, "You take the compass …" T: "Yes." S9: "Oh I know." S8 says, "The center of the circle." S9 says, "When you fold it the two parts are equal." The teacher says, "This applies for this shape as well *(the ellipse)."* S9 says, "It's more circular." S8 agrees, "I know." S10 says, "Because the distance from the center to the …" T: "Circumference." S10 says, "It's the same." T: "Exactly." S11 says, "Mrs I am trying to remember what we call this line *(he means the radius)."* (3.1.5.)

Protocols 3.1.2.-5 show the teacher intervening in an attempt to encourage students to give explanations in terms of the properties of the concept discussed.

Finally, the teacher summarized what she felt had been important in the prior discussion. T: "We will talk about that later. Thus, S11 says that a circle is the shape that, according to S8 has a center, has a circular circumference and all the points of the circumference have the same distance from the center. Right?" The students say "Yes." (3.1.6.)

This protocol focuses on the definition of circle formulated by the classroom.

Following the classroom discussion on defining circle, the teacher told students to look at several shapes illustrated on the interactive whiteboard. She then asked students to determine whether these shapes were circles by also justifying their answer. (3.1.7.)

This protocol focuses on the teacher, expecting explaining and justifying by discriminating between instances and noninstances of the specific concept.

T: "Based on what we have said so far, look at these shapes that Mrs Maria has on the screen. Are these circles?" The students reply "No." The teacher asks, "S1, is this a circle?" S1 replies, "No." At this point, the teacher made the following comment: "I don't accept your answer." S1 says, "No its not." (3.1.8.)

In the above protocol the teacher is not satisfied by an answer which has no justification.

At this point, the teacher guided students in making the definition operable: "Why?" S1 says, "They are not circles because …" The teacher says, "Because … you were not paying any attention earlier … S2." S2 says, "We have one circle here …" The teacher asks, "Which one is the circle?" S2 replies, "There … on the right … the other shapes are not circles because their centre does not have the same distance from their circumference." The teacher in effect affirms the importance of a justification by accepting the response with an explanation and not accepting the previous responses: "Yes." S2 continues, "The others are not circles because their centre … isn't in the middle … the centre is not equidistant from the circumference." (3.1.9.)

This protocol points to justifying by using the definition of circle.

At this point, the teacher asked the students to write the definition of circle in their notebook. After that she asked them to draw a circle by using either a coin or a compass. By drawing a segment from the center to the circumference and a segment that meets two points of the circle and passes through the center, the class engaged in a discussion related with naming these segments. After establishing what radius and diameter is in a circle, and writing the definitions in the notebook, the teacher commented that the succeeding part of the lesson would be devoted to investigating these terms. In doing so, the students in pairs moved to the computers, also taking with them their notebook. (3.1.10.)

At this point, the teacher said that I would introduce them to a new dynamic geometry environment commenting that is very similar to the dynamic geometry environment employed before in the mathematics lesson. Initially, exploring this dynamic geometric environment aimed at looking at the available tools and their uses. This was followed by a more focused exploration related to circle; constructing a circle, constructing the radius and the diameter, naming and dragging the different points on the circle. During this exploration the teacher would repeatedly remind students that Cabri has common similarities with GeoGebra. (3.1.11.)

Eventually, the teacher made the following comment, “I can make it bigger or smaller.” The teacher is referring to the circle that is on the screen. T: Look what we will do next. We will play later. Construct a circle first and move it. Click on the center. Did you all do it? Nice. Now stop.” (3.1.12.)

The above protocol focuses on the teacher is relating exploration to ‘play’. In this instance, exploration has a positive value.

Following the introduction to this DGE, the students moved to DGE-based Task 1. Initially, the class commented on what was shown on the screen. The teacher said that that there is a relationship between the radius and the diameter of the circle and encouraged the students to explore this relationship. In doing so, the students were taking measures of the radius and the diameter of different circles in Cabri. The teacher told them that she expected them to write two mathematical relationships. Throughout this exploration my role focused on answering technical questions related to the exploration of this Dynamic Geometry Environment. (3.1.13.)

This protocol focuses on the exploration opportunity the teacher provided to the students. The students could make inferences by relating the changes made in the circle with the measurements in the table.

During classroom discussion below, one student suggested that the radius is half the diameter but the teacher wanted this to be formalized. S1 says, “The radius is half the diameter.” T: “Excellent. Now I want to write this relationship mathematically.” S1 continues by saying “Radius = ½ of the diameter.” The teacher asks, “Which sign do I use for 'of the diameter'?" S2 replies, "times." T: "Thus, A = ½ x D. What should I write for the relationship between the diameter and the radius?" S3 says, "Double." The teacher says, "It is correct but …" S3 says, "The diameter is double." The teacher says, "Double the radius. How do I write it mathematically?" S3 says, "Times 2." The teacher says, "Thus, the diameter is times 2 the radius. From our measurements we conclude that the diameter is twice the radius or the radius is half the diameter." (3.1.14.)

The above protocol the class reached to a conclusion regarding the mathematical relationship that exists between the radius and the diameter.

In fact, there was a small discrepancy which, as shown below, one student picked up. The teacher pointed out the anomalies of measurement in DGEs due to the rounding errors. S4 says, "But it is not that …" The teacher says, "But I explained. It depends on the millimeters. For some the division is exact and for others not." (3.1.15.)

In the above protocol the teacher and the students pointed out the anomalies of measurement in DGEs; the rounding errors.

Following this, the teacher asked the students if they remembered what the distance around a circle is called. Having established that it is called 'perimeter' for other shapes, in a circle is called 'circumference', the teacher asked how they could find the circumference of a circle. Given the fact that the students were making hypotheses by relating the mathematical formula with either the radius or the diameter of the circle multiplied by 3 or 4, the teacher encouraged them to investigate the relationship between the circumference of the circle and its radius on the DGE screen, by tabulating the diameter, the radius and the circumference of the circle. While the students had time to do that, and make a similar table in their notebooks, there was no time to discuss their ideas as the bell rang and it was break time. The teacher told them to use the measurements from their notebooks in order to investigate this relationship at home. (3.1.16.)

This protocol focuses on the exploration opportunity the teacher provided to students. The students could make inferences by relating the changes made in the circle with the measurements in the table.

The informal discussion between the teacher and myself at the end of this lesson focused on the interaction of students with the dynamic geometry environment. According to the teacher, while the students were working on DGE-based Task 1, they were asking questions in order to clarify aspects of the task that they did not understand fully. They were also asking questions about what they had to do. Trying to answer all these questions simultaneously was not the ideal scenario. Confusion was created and valuable lesson time was lost. For these reasons, the teacher suggested that providing a worksheet for the pairs of students to work on would assist both the teacher and the students. This worksheet would include steps that students had to follow so as to investigate the relationship between the diameter and the circumference of the circle and questions related to what they would observe each time they followed a step, as well their conclusions regarding these observations. The worksheet is shown in Appendix IX. (3.1.17.)

Day 2, Phase 3

At the beginning of the lesson on the second day, the teacher and the students revised the previous lesson. In revising what circle is, the teacher guided students in providing a definition for circle, emphasizing the use of precise mathematical language. T: "Which shape do we call circle?" S1 says, "Circle we call the shape that does not have …" T: "The flat shape …" S1: "The flat shape that does not have sides or angles, and the …" T: "The circumference is a curved line." S1: "And the center …" T: "All the points …" S1: "All the points have the same distance from the center." T: "Are equidistant from the center." S2: "And 360°." T: "Yes. We need to be able communicate mathematically." (3.2.1.)

The above protocol focuses on the teacher endorsing the sociomathematical norm 'doing mathematics requires us to use precise language'.

After revising the definition of a circle, the teacher asked the students to tell her the mathematical relationships explored the day before. At this point, several students could not give an answer. The following comment comprises the teacher's interpretation of the hesitation these students had in participating in the classroom discussion, "You shouldn't only play but concentrate and listen in the classroom." (3.2.2.)

In this protocol, the teacher is relating exploration of DGE with 'play'. That is, exploration was interpreted by the teacher as 'playing' instead of learning.

Following this comment, the teacher asked the students if they found a way to measure the circumference of circle. After noticing that only a few students explored the relationship between the circumference and the diameter at home, the teacher told them to divide the circumference of several circles by their diameter, write the answers in their notebook and discuss their observation. She asked each pair of students to give measurements for different circles, which she then transferred on a table she made on the whiteboard. Students noticed that the values were not exact. However, it was observed that the average ratio of the circumference to its diameter came out quite close to 3.14. The teacher explained what this mathematical constant is and asked them to use this to find the mathematical formula for the circumference of circle. During the classroom discussion, it was concluded that the circumference of the circle is $2\pi\alpha$ (α is the first letter of the Greek word for radius). (3.2.3.)

At this point, the teacher seemed to be concerned with the letter representing the radius in the mathematical formula of the circumference of circle and made the following comment, "In our book they use the letter α for the radius. Internationally though they don't use this letter. The radius is symbolized with the letter r. Thus the circumference equals $2\pi r$. We just have a different sequence of the symbols. This is the formula that you will find the textbooks in secondary school. In multiplication the commutative property can be applied. Thus, it will be the same, no matter what the sequence of these letters is. Write now the formulas in your notebook." (3.2.4.)

In this protocol the teacher is 'opening out' the mathematics explored in the classroom by helping the pupils to connect what they are doing to what they will do in the future.

Following this, each pair was provided with the worksheet so as to explore the second figure in DGE-based Task 1. During this exploration, the students were asking both the teacher and myself to confirm their observations. They also waited for us to verify their answer before completing the worksheet. In these instances, my involvement was related with probing questions. At some point, even though not all pairs had enough time to complete the worksheet, the teacher decided that they have to move to the next part of the lesson. Thus, she asked them to stop exploring this task and said that they would discuss this task later. (3.2.5.)

The above protocol focuses on the exploration opportunity (DGE-based Task 1).

She then asked them to think of ways that they could help them calculate the area of circle. One of the ideas the students had in order to find the area of the circle was to count the squares inside a circle. However, it was concluded that this would be quite difficult. Some students hypothesized that the area might be equal to circumference times radius. Others said that the area could be equal to circumference times diameter. The teacher encouraged them to investigate and test these hypotheses while exploring DGE-based Task 2, by making the following comment, "I will leave you for a while to play." (3.2.6.)

In this protocol, the teacher is relating exploration of DGE with 'play'. That is, exploration is translated as something encouraging and constructive.

While the students were exploring DGE-based Task 2, the teacher gave each pair a circle divided in either 8 or 10 pizza pieces. She commented that they could also use the pizza slices to explore the area of circle using pizza slices by making a shape whose area they knew how to find (see Figure 8.3. below). Below, one group of students followed this suggestion and, through exploration discovered the mathematical formula for the area of circle. S1: "Mrs, we made a rectangle. Radius times circumference … wait …" The teacher asks, "How do I find the area of a rectangle?" S1 replies, "Length times width." T: "Nice. Do I know the length and the width here? Write it in your notebooks." S1 responds, "Radius times diameter." The teacher asks, "Is this the diameter?" S1: "Aaa … its half the circumference." The teacher says, "Write it down." S2 says, "So … what are we going to write? We will write … area equals …" The teacher left this group to write their conclusion and she moved on to other pairs of students (3.2.7.).

Soon, another group called me. S3: "Mrs Maria we finished. Can we tell you? Radius time half the circumference. It's a rectangle thus the length is the radius and the width is half the circumference because its half." R: "Thus …" S3 says, "We wrote it down. Radius x circumference/2." R: "Nice. Now replace the circumference with the formula." S3 says, "Radius x radius x π…2radii x π." R: "2 radii?" At this point, the teacher makes the following comment, "Children, radius x radius is squared radius, not 2 radii." (3.2.8.)

At the end of the lesson, the pairs presented their work, and justified their answer. It was concluded that the area of circle is πr^2. (3.2.9.)

Protocols 3.2.7.-9 focus on the exploration opportunity the teacher provided to the students. This activity exploration opportunity encompassed making and testing hypothesis, explaining and justifying their proving process.

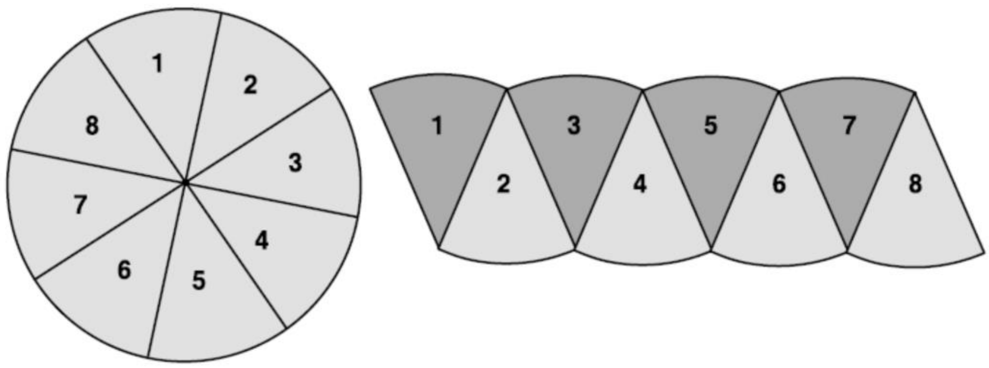


Figure 8.3.: The area of circle - A pizza demonstration

During the informal discussion between the teacher and me, I commented that in my opinion the students had been more quiet and collaborative. The teacher agreed with me. She argued that this is because the day before was the first day back to school after the Easter break. She also justified this observation by saying that the students seemed to be more familiar with the dynamic geometry environment as in a way they repeated things that they did the day before. What is more, she said that the worksheet might have also contributed to this 'positive change'. (3.2.10.)

Day 3, Phase 3

On the third day, the classroom engaged in going through the exercises the students had as homework. T: **"**The circumference of the circle is equaled with …" S1 replies, "2r times … 3,14." T: "2 times 3,14 times … open your notebook." S2 makes the following comment, "I wrote r²." The teacher says, "The circumference of the circle not the area." S3 says, "2πr." The teacher continues, "So … what do I write here? What is missing?" S4 says, "2π." and S3 adds, "α." The teacher says, "α … in the notebook we wrote α for the radius. And this is 2πr." S5 says, "I wrote …" (3.3.1.)

The above protocol shows the teacher trying to make the mathematical formula for the circumference of circle operable for students.

At this point, the teacher seemed to be concerned with the letter representing the radius in the mathematical formula of the circumference of circle and made the following comment, "I insist to use 'r' instead of 'α' because from next year you will only be using 'r'. You will never see α again and wrongly they use that in the textbooks of primary school." (3.3.2.)

The above protocol focuses on the teacher making a forward connection.

After revising what circumference is, the teacher asked, "What does 'the circumference of a circle is 30cm' mean?" S1 says, "That the round of the circle is 30cm". T: "Meaning?" S2 says, "If I start from one point and I make a full round, it will be 30cm". T: "Well done. Now, I have the following question. If this the wheel of a bicycle, what is the distance that it will cover when it makes a full rotation? S2 says, "30." T: "The distance will be 30cm. Well done." (3.3.3.)

In this protocol, the teacher appraises an explanation that is correct.

Following this, the teacher was holding a circle and a ruler and asked the students to tell her the circumference of the circle. She gave the circle and the ruler to one student asking him to do it while everyone was watching. Below, the teacher guided the student in using precise mathematical language. T: "What did S3 do?" S1 replies, "He placed the ruler …" T: "We put a point here yes …" S1: "We put the point …" T: "He started from zero." S1: "He made a whole round with the circle." T: "He made a full rotation." S1: "He stopped when he reached the point again and counted." T: "He started from zero up to the point and found 32cm. Thus, 32cm is the circumference of the circle." (3.3.4.)

The above protocol points to the teacher guiding the student in using precise mathematical language.

Following this, the teacher asked the students to open their notebooks, write 'circumference-area of circle'. T: "I gave you a circle whose circumference is 32cm. I want you to find its radius and its area." S1 asks, "The radius?" The teacher repeats, "I want you to find the radius and the area of the circle." S2 asks, "But how?" The teacher replies, "I do not know. We have some formulas … Your calculator on the desk." S3: "Mrs I did not understand." (3.3.5.)

At this point, the teacher interpreted the queries some students had as a result of 'playing' with the computers, "We came up to some conclusions. We have been working on the computers for two days now. We should not only play but also find …" Through classroom discussion, the students were able to use the mathematical formula, separate the variables, divide and find the radius of the circle with a given circumference. (3.3.6.)

The above protocol focuses on the teacher translating exploration as playing, something that has no didactical value.

Day 4, Phase 3

At the beginning of the lesson on the fourth day, the teacher asked the students to find the relationship that exists between the area and the squared radius of a circle. T: "Which operation do you need to use in order to find this relationship?" S1 says, "Division. Area divided by squared radius." The teacher replies, "Nice. C says that we will find the ratio … you remember that the ratio helps us to find the relation between two measurements, two units. Do the division." The teacher gives the students time to find the ratio Area/r^2. She then asks each pair to give their answer. All pairs say that the ratio is 3.14. T: "What do you observe?" The students reply, "That they are all 3.14." The teacher continues, "What is the relationship between the area and the squared radius of a circle?" S2 says, "That the quotient is 3.14." The teacher says, "Right." (3.4.1.)

In this protocol the class reached a conclusion regarding the relationship between the area and the squared radius of a circle.

Using this, the teacher helped the children to construct the mathematical formula for the area, T: "Can you use this to end up with a formula, to a conclusion? What links the area of the circle with its radius?" S3 says, "That the area with the squared radius …" T: "The area equals …" S3: "3.14 times …" S4: "Times r." S5: "Times r^2." T: "Times r^2. Do you agree with that?" The students reply, "Yes." The teacher asks, "What have we called 3.14?" S5 says, "π." T: "So what is our formula for the area of circle?" S5 replies, "π times r^2." T: "Yes. Write it in your notebook." (3.4.2.)

In this protocol the class reached a conclusion regarding the mathematical formula for the area of circle.

After revising the two mathematical formulas that are used to find the circumference and the area of circle, the teacher asked the students to find a relationship that related the circumference and the area of the circle. A student asked if they should write it in their notebook. Then the teacher said immediately, "I want you to find the ratio of Area / Circumference. (3.4.3.)

In this protocol, the teacher closed down the exploration opportunity provided to students by telling them exactly what to do.

A few minutes later the teacher said: "By using these two relationships, I want you to prove that ratio of the area / circumference of circle equals r/2, half the radius of the circle." S1 says, "Mrs repeat what you said about the radius." The teacher replies, "Tell me what to explain. What?" S2 replies, "No, no …" The teacher says, "Prove mathematically, algebraically. Who wants to come to the notice board, to replace Area / Circumference with the formulas and prove whether the ratio is always half the radius. S3 on the board. Look the stages that S3 follows. Yes." S3 writes on the board the two formulas. (3.4.4.)

The teacher is now guiding the students in proving that A / C equals r / 2. T: "Do we agree so far?" The students reply, "Yes." The teacher says, "He replaced the area with πR^2, its equal, and circumference with $2\pi r$. Do we agree so far?" Again, the students reply, "Yes." The teacher continues, "We now have a fraction. Instead of a fraction with numbers, I have a fraction with letters. I am allowed to simplify even if I have letters?" The students reply, "Yes." The teacher says, "Nice. Do the simplification then. In order to help you, instead of r^2, you can put r times r. Just to help you for the simplification. I know how to do it with indices, but you don't know how to do that yet … Yes … Is there any other simplification to do?" S3 replies, "No." The teacher asks, "What is left?" S4 replies, "1." The teacher says, "Look again." S4 says, "1 times r." The teacher says, "It's r." S5 adds, "Divided by 2." The teacher says, "Seconds". (3.4.5.)

The above protocol points to proving.

At this point, the teacher makes the following comment: “Do you realize now why I insisted with simplifications with numbers? So that you can also do it with letters. So, what we have proved with the numbers after observations, we also did algebraically. You will do that next year in high school.” (3.4.6.)

This protocol points to the teacher making a connection with a part of mathematics that the students will explore in secondary school.

Following the above discussion, the teacher says, “This is how you will be proving in secondary school.” S1 asks, “We will only be doing that with letters?” S2 asks, “Mrs … why do we learn with letters in secondary school?” The teacher, focusing on the general structure of the proof, says, “We don’t solve with letters, we replace the numbers.” (3.4.7.)

The above protocol points to the teacher making a forward connection involving the formal aspects of proving in mathematics.

Following this, the teacher and the students returned to the DGE-based tasks to explore the circumference and the area graph. T: “What is constructed?” S1 says, “A line.” while S2 says, “A curve.” The teacher asks, “Can we understand why while we increase the radius, the circumference increases and a line is being constructed?” S3 replies, “It’s like what we do in science, proportional figures.” (3.4.8.)

The previous discussion continues with the teacher moving to the area graph. The teacher moves the point along the path. T: “What is being constructed?” S1 says, “A curve.” The teacher asks, “Why is this curve constructed?” S2 asks, “Why?” S3 replies, “Because it is radius times itself.” The teacher agrees, “Yes.” S4 says, “It doubles.” The teacher asks, “It doubles?” S5 says, “No, it’s squared.” (3.4.9.)

Following the exploration of the two graphs the teacher helped the children to relate the mathematical formulas with the graphs, T: “Now I want to see what I should put in the axis *(the teacher means the x axis)* so that while I increase the radius, the area increases as well and makes a line.” S1 says, “Radius.” The teacher says, “No … To have a line. What should I have on the x axis?” S2 asks, “Circumference?” The teacher says, “No, I want to compare it with the radius.” S5 says, “Diameter.” S3 and S4 say “r^2.” The teacher replies, “r^2.” (3.4.10.)

Protocols 3.4.8.-10 focus on the students being able to relate the graphs with the DGE figures so as to come up to conclusions regarding the formula of the circumference and the area of circle.

The end of the lesson indicated the completion of the lessons related with circle. In the following discussion, the teacher guided students in identifying the mathematical relationships explored throughout the week. T: "What have we learnt by using this software?" S1 says, "We learnt how to find the circumference and the area of circle." The teacher says, "It helped us to make observations, comparisons and to end up with some mathematics relations. Which are these?" S2 says, "That the area of the circle is πr^2." The teacher adds, "It is radius times itself. Do we understand that?" The students say, "Yes." The teacher continues, "What else?" S3 says, "The circumference equals double radius times π." The teacher asks, "How do we call this differently?" S4 replies, "Diameter." The teacher continues, "We also explored another relationship." S5 says, "That radius / 2 is area / circumference." The teacher agrees, "Yes." (3.4.11.)

In this protocol, the teacher guided students in identifying the mathematical relationships explored throughout the week.

The following section focuses on the analysis of the findings of Phase III of the study.

8.4.Ongoing analysis

In this section, the insights from the classroom observation as well as the collaboration between the teacher and the researcher both during the task design and throughout the lessons are presented and elaborated on. How the themes developed and new themes emerged in Phase III is discussed below and summarised in Table 5.4, duplicated below as Table 8.1. for ease of reference.

8.4.1 Insights from the classroom observation

Bringing together the initial themes identified in Chapter VII (see Section 7.5) and the findings of Phase III of data collection, leads to the exemplification of protocols that fall into the emergent levels of actions. It also leads to identifying connections between the emergent levels and, thus, reconsidering them. In achieving this the insights that have emerged from Phase I of the study are also taken into consideration.

Regarding the level of exploration, the first step towards identifying the degree of exploring mathematics in the classroom is to take into account the exploration opportunities provided by the teacher. Consideration of the activities the class engaged with indicates that the teacher provided opportunities for exploration and investigation of mathematical situations. For instance, exploring the relationship between the radius and the diameter of a circle (protocol 3.1.13.) and between the circumference and the radius of a circle (protocol 3.1.16.) involved exploration on a dynamic geometry environment, where the students could make inferences by relating the changes made in the circle with the measurements in the table. The 'pizza demonstration' activity (protocols 3.2.7.-9.) as well as proving that the ratio of the area/ circumference of circle equals r/2 (protocol 3.4.5.) encompassed making and testing hypothesis, explaining and justifying their proving process. In addition, protocol 3.4.1. points to a mathematical situation encouraging students to explore a recently taught mathematical formula in a different way. Furthermore, the students had the opportunity to explore the two DGE-based tasks.

While instances that support exploration and investigation in the classroom are identified, occasions where the teacher is closing down an exploration opportunity are also emerging through the data. In protocol 3.4.3. the teacher did not give the opportunity to students to explore the activity and come up with ideas or hypotheses, but instead she told them the steps that needed to be followed. Furthermore, the teacher closed down the exploration of the DGE-based tasks (protocols 3.1.13. and 3.2.5.). These tasks were fully explored towards the end of the lessons related with circle (protocols 3.4.8.-10). Keeping in mind the above, the contradicting values open/close may be considered as another parameter that determines how mathematics was explored in the classroom. These values illustrate the teacher closing down a task that was open, and investigating the reasons that let the teacher behaving the way she did.

Through classroom observation, it also becomes apparent that the teacher is supporting mathematical connections. To be more accurate, in protocols 3.2.4. and 3.3.2. the teacher is opening out the mathematics by suggesting to students to use the letter 'r' instead of 'a' in order to represent the radius of the circle as this is how the radius is represented internationally in the mathematics community and this is how it is being used in secondary school. In addition, the simplification of fractions when including letters (protocol 3.4.6.) as well as making simplifications with indices in mathematics (protocol 3.4.5.) are part of the curriculum of secondary school mathematics. Once again the teacher is opening out the part of mathematics being explored in the classroom. Furthermore, in protocol 3.4.7. the teacher is making a forward connection involving the formal aspects of proving in mathematics. However, how this forward connection influences the emergence of students' meaning for proof will be discussed in Chapter X of the thesis.

The teacher also supports students' instrumental genesis (protocol 3.1.15.).

By keeping in mind the above analysis and discussion regarding exploration, the level of exploration seems to be related with the mathematical situations the teacher provided, the exploration of the Dynamic Geometry Environments, as well as exploration for supporting mathematical connections.

Regarding the level of play, it is being noticed that the teacher referred to the word play in Phase III as well. As in Phase II, the word 'play' had either a positive or a negative value. In protocols 3.1.12. and 3.2.6. the teacher is translating exploration as something encouraging and constructive positive value. On the contrary, in protocol 3.2.2. the teacher translated the fact that some students could not really summarize the work that was done previously as 'playing'. In addition, in protocol 3.3.6. the teacher interpreted her students' queries as a result of 'playing' with the computers.

Accounting of the protocols where 'play' appeared in the teacher's words leads to the conclusion that the level of play falls into the level of exploration as it is related with the degree of exploring mathematics in the classroom. It also reveals an aspect of the way the teacher intervened at specific aspects of the lesson. Thus, occasions where this critical moment of interest occurred should be further explored and compared with the occasions where it did not.

In relation to the level of participation, the following social and sociomathematical norms are being negotiated and established in this mathematics classroom: 'doing mathematics requires us to use precise language', 'doing mathematics requires us to justify our assertions', 'we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations', 'we write coherent geometrical explanations'. Furthermore, the teacher either, makes this rule explicit (protocols 3.1.4. and 3.2.1.), rephrases what the students say (protocol 3.3.4.), gives a negative feedback to a response that does not embrace the norm (protocol 3.1.8.), and/or appraises the response that is correct (protocol 3.1.9.) in order to guide students towards the endorsement of socio-mathematical norms.

Concerning the level of proving, providing exploration and investigation opportunities that encourage students to explain, justify and/or prove, leads to considering that mathematical justification is encouraged in the classroom. However, the fact that the classroom argumentation was, in many instances, followed by closing down the exploration opportunity provided to students, contradicts the aforementioned argument. This, unavoidably, leads to tensions. Thus, the impact of these tensions on the way proving is constituted in the classroom should be further explored.

Despite the aforementioned observation, it becomes obvious that the classroom argumentation was related with explanation and justification. It is being observed that attempting to capture instances of generation of general statements that request proofs of their truths is indeed not enough in understanding how proving is constituted in the classroom. On the contrary, through the protocols, instances of the generation of ideas that have the potential at some point to inform the customary notion of proof can be identified.

To be more elaborative, it is being observed that a great part of the lessons is devoted for the definition of mathematical concepts and formulas. That is, the teacher does not provide the definitions of the mathematical concepts and formulas explored in the classroom. On the contrary, the teacher, by encouraging the move from a definition of perception to a definition that involves properties, engages students in defining activity (protocols 3.1.2.-5, 3.1.14., 3.2.1.). Once the students appropriate these definitions, they become tools for them employed in explaining and justifying mathematical situations (protocols 3.1.9., 3.3.3., 3.3.4., 3.4.5., 3.4.8.-10).

In Section 7.3., instances where explaining and justifying were developed around mathematical definitions and formulas, were also identified. While not being considered as a distinct theme of interest that would guide the initial analysis of the classroom observation data, this emphasis on mathematical definitions was also recorded in the teacher's initial interview (see I3 in Table 8.1.). What seems interesting though is the tension that arises when considering that definitions as approached by the official documentation are descriptive and extracted (see D4 in Table 8.1.).

Keeping in mind the above, it becomes clear that definitions and defining as activity was an integral aspect of this mathematics classroom, around which, explaining and justifying developed. However, the connection that exists between definitions and explanation needs to be exemplified.

Considering the elements that direct mathematical reasoning towards the ultimate goal of formal proving, the systematization of the classroom data leads to the generation of two broad activities of action: (i) the activity of exploration and (ii) the activity of explanation. As the activity of explanation unfolds and expands mainly around mathematical definitions, defining as activity is inherent in the activity of explanation.

The following section illustrates and discusses the themes emerging through the collaboration between the teacher and the researcher.

8.4.2 Insights from the teacher-researcher collaboration

In this section, the collaboration established between the teacher and I both during the task design and throughout the lessons is analysed. As exemplified earlier the collaborative design approach employed in this study aimed at making explicit the teacher's motivations and perceptions. Even though during my initial discussion with the teacher we reached an agreement concerning the way our collaboration would develop, during this process of collaboration, tensions would arise. These tensions had two dimensions; the task design process and the utilization of the tasks throughout the lessons. An initial description of the collaboration between the teacher and me as well as instances of tensions occurred has been presented in

Section 8.2.2. as well as in the findings. Below, the analysis of this collaboration is further explored.

To begin with, a tension occurred during the decision process regarding the design of the DGE-based tasks. On several occasions of this process, the teacher would use the tasks employed in the exploratory study as an example of what she did not want the tasks to look like. Both as a researcher and a teacher, I would find myself disagreeing with the teacher's views on the tasks. I would explain my thoughts and make an argument towards open tasks that support exploration. In this specific occasion, we both understood each other's backgrounds and objectives and through dialogue we reached a common ground. Nevertheless, her attitude towards task design will be discussed later on in the thesis.

There is a tension related with the way the tasks were used. Even though the teacher and I agreed on the tasks to be employed and the general lesson plan, the teacher was the one deciding when to stop, and what aspects of the task not to use until later when she felt it was more appropriate. For example, for the two DGE-based tasks, the graphs were not really explored until towards the end of the week, as it seemed that this was just an additional thing that the students could know for 'circle'. Even though the goal was, throughout the week to explore all aspects of the tasks that would allow the students to investigate in a different way the relationship that exists between the radius, the circumference, and the area of circle, it was not until the end that this happened holistically. I was feeling frustrated throughout the week, thinking that my objectives were not being addressed and that the teacher was doing the lesson like any other day.

Going further, as stated earlier, during our informal discussion at the end of the first day of the iteration phase, the teacher felt that again the tasks were not that structured. That is, the teacher felt that due to the fact that the students were asking questions related mostly with the technical aspects of this specific DGE, and the classroom was quite noisy, the students should be given a worksheet with specific instructions. This tension was first related to the teacher's interpretation of 'open' and 'closed' tasks. The teacher's understanding was related to the degree of independence in exploration. This again was in contrast with the teacher's statement of supporting exploration and investigation in the classroom. At this point of our collaboration, compromising was inevitable. Nevertheless, this tension was resolved by considering providing students with a worksheet as an opportunity to explore the students' written responses regarding the explanations and justifications given about their observations (protocol 3.1.17.)

The teacher's actions led to the conclusion that even though she supported employing the tasks, the time constraints that existed in covering this specific part of the mathematics curriculum, as well as the way she usually teaches this particular area of the mathematics curriculum, directed her decisions. Specific aspects of the tasks were initially used so as to achieve the educational goals of this geometric topic, and at a later stage, in concluding what was explored throughout the week, the tasks were fully explored. The teacher's actions in regards to this area are in contrast with the fact that she was supporting mathematical connections.

Nevertheless, the discussions we both had at the end of each lesson constitute indications of collaboration between the teacher and me. For example, the fact that the teacher agreed to employ a DGE in the classroom was a positive move for our collaboration. What is more, one cannot neglect the fact that, as the teacher of that classroom, she was in the end responsible.

8.5. Summary and Discussion

This Chapter has focused on Phase III of the study. Initially, this chapter exemplified how consideration of the synthesized themes that emerged through Phase II, influenced the design process. Following this, the findings of this iteration phase were presented. Furthermore, it has been illustrated how insights from the classroom observation and the collaboration between the teacher and the researcher, led to considering the activity of exploration and explanation as broad activities that may encapsulate the classroom activity (see Table 8.1.).

It has been established that CHAT is the most appropriate tool that offers the means for handling this complexity in coming to understand how proving might be constituted in the classroom. As will become clear in the retrospective analysis (see Chapter IX), interpreting the activity of exploration and explanation through the lens of CHAT by generating the activity triangles, indicates that the level of participation is concerned with the rules component and the level of intervention with the division of labour.

Table 8.1.: Map of themes Phase III

Post Phase III (acting as participant observer)	**Insights from the classroom observation** PO1 The contradictory value open/close is identified: there are instances where the teacher is closing down an exploration activity. PO2 Exploration is related with the mathematical situations the teacher provided, the exploration of the Dynamic Geometry Environments as well as exploration for supporting mathematical connections. PO3 The value 'play' is related with both exploration and intervention. PO4 Mathematical argumentation is related with explanation and justification. PO5 Definitions and defining seem to be an integral aspect of this mathematics classroom, around which explaining and justifying developed. PO6 Sociomathematical norms are identified: 'doing mathematics requires us to use precise language', 'doing mathematics requires us to justify our assertions''we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations', 'we write coherent geometrical explanations'. **Insights from the teacher-researcher collaboration** C1 A tension during the task design process is identified. This was related with the types of tasks to be designed. C2 A tension related to the utilization of the tasks throughout the lessons is identified: the teacher closed down the tasks.
New synthesised set of themes	**Activities of action** Activity of exploration (PO1, PO2, PO3, C1, L1, L2, L3, L4, L6). Activity of explanation (PO4, PO5, PO6, C1, L3, L4, L5, L6).

The level of collaboration, concerned with the way the teacher and I collaborated during this phase of data collection, puts particular emphasis on the design and utilisation of the DGE-based tasks. Even though the collaboration between the teacher and me is not an inherent part of the activity systems, making explicit the teacher's motivations and perceptions will shed more light towards understanding the activity of the classroom.

The following chapter is concerned with the analysis of the findings of the three phases of data collection through the lens of CHAT.

CHAPTER IX

RETROSPECTIVE ANALYSIS

9.1 Introduction

The ongoing analysis conducted while the study was in progress has led to focusing on several issues and events. Chapter IX constitutes the final phase of the analysis of the classroom situation. That is, this chapter aims to place these in a broader theoretical context by conducting a retrospective analysis on the entire data set generated from the three phases of this study. This is being achieved by employing the main aspirations of CHAT, alongside the available research literature that informs this study.

It has been illustrated that CHAT is the most appropriate tool that offers the means for handling this complexity in coming to understand how proving might be constituted in the classroom (see Sections 4.4. and 5.3.). Furthermore, as stated in Section 8.5., the systematisation of the classroom data led to the evolution of two broad activities that may encapsulate the classroom activity: (i) the activity of exploration including the exploration of mathematical situations, exploration for supporting mathematical connections and exploration of DGE and (ii) the activity of explanation which focuses on clarifying aspects of one's mathematical thinking to others, and, sometimes, justifying for them the validity of a statement.

This chapter initially analyses the activity of exploration and explanation through the lens of CHAT. To be more precise, by introducing the triangle model of human activity, the activity systems of exploration and explanation will be generated by drawing on protocols from both phases of classroom observation. Portraying the activity systems will also lead to considering the potential emergence of points of contradictions. Before proceeding, it should be made explicit that while these tensions will be elaborated by taking into consideration the notion of contradictions proposed by Engeström (see Section 3.2), unravelling whether these systemic imbalances emerged within or/and between components of the activity system, or/and across entire activity systems will be presented in Section 10.1.2. of the concluding chapter of the thesis.

In addition, it has been illustrated that an in-depth understanding of the outcome of the activity might not be possible if it is comprehended in isolation from the social context in which it emerges (see Section 4.4.). Considering this remark, Chapter IX will proceed by identifying the way the activity of the mathematics classroom is influenced and dependent upon the structure and organization of the school and the Ministry of Education and Culture as wider educational contexts. That is, by having identified the three levels of analysis (the system level, the teacher level, and the classroom level), the micro level of the classroom activity will be contrasted against the broader macro context as well as the collaboration with the researcher.

9.2 Interpreting the micro context/the classroom through Cultural-Historical Activity Theory

This section focuses on the analysis of the activity of exploration and explanation through the lens of CHAT. At this point, it should be made explicit that even though the activity of exploration and explanation can be distinguished from each other, they are simultaneously related as one can help understand the other. In this sense, protocols presented in the findings of Phase II and III may fall into more than one activity.

In addition to the above, it is also worth noting that the structure of the sections regarding the activity of exploration and that of explanation differ slightly. As the discussion concerning the activity of exploration is concentrated around the exploring opportunities provided by the teacher, these opportunities are first presented and elaborated on. As the activity of explanation focuses mainly on definitions, it was considered most appropriate for this section to introduce initially the way explaining was developed in each phase of classroom observation. This, for the purposes of the discussion, will allow the reader to understand more comprehensively how the activity system of explaining is portrayed.

Furthermore, it should also be noted that even though it has been recognised in the analysis that in the activity system of developing proving in the classroom the subject might be the teacher, the students or the researcher. For the purposes of this discussion, I will concentrate on a focal instance; the activity system in which the subject only involves the teacher. Even though the discussion will be centred on the teacher as the subject, in order for the line of reasoning to be thorough, information related with both the students and I as a participant observer will also be drawn upon.

9.2.1 The activity of exploration

The activity of exploration is concerned with the extent that exploring mathematics takes place in the classroom. Analysis of the classroom activity has pointed to three levels that correspond with the activity of exploration: (i) exploration of mathematical situations, (ii) exploration for supporting mathematical connections and (iii) exploration of DGE (see Section 8.4.1.). As exploration has been identified among the elements that direct mathematical reasoning towards the ultimate goal of formal proving (see Section 2.4.), the aforementioned levels should be considered as distinct activities within the nest of activities related with the activity of exploration. That is, by portraying and thoroughly analyzing each element of the activity system of these three activities, inferences regarding how exploring, as developed in the classroom, impact on the way proving is constituted in the classroom, will be made possible.

In the following sections, the activity of exploration is interpreted through the main aspirations of CHAT. Portraying the activity of exploration will also reveal how the value 'play', previously identified as a critical moment of interest, is related with the degree of exploring mathematics in the classroom as well as the way the teacher intervened at specific aspects of the mathematics lesson (see Section 8.4.1.).

9.2.1.1. Exploration of mathematical situations

One dimension that needs to be considered in identifying the degree of exploring mathematics in the classroom is the exploration opportunities provided by the teacher (see Section 8.4.1.). In achieving this, the activity of exploration of mathematical situations constitutes one activity of the nest of activities related with the activity of exploration.

The interpretation of the activity of exploring mathematical situations will be undertaken using the standard terminology of CHAT, namely tools, community, rules, division of labour and object as introduced in Section 4.2., before identifying points of contradictions.

Tools

Several mediating artefacts are used for the activity of exploration. Most of the activities involved exploration on dynamic geometry environments. As this activity system focuses on the exploration of mathematical situations, the tools that are considered to play a catalytic role in how the activity unfolds and expands are the exploring opportunities provided by the teacher. To be more precise, the teacher is posing questions that require the allocation of sufficient time for the students to consider these questions. The questions asked show that the teacher desires to provide opportunities for exploration and investigation. A record of the questions posed by the teacher, illustrating how she attempts to provide these exploring opportunities, are presented in Table 9.1. below. These opportunities are also accompanied by an analysis so as to indicate how they constitute instances that support exploration and investigation.

Table 9.1.: Exploring opportunities provided by the teacher

	Exploring opportunity	**Analysis of the activity**
1.	How many altitudes does a triangle have? (Phase II, Day 1)	This question provides the opportunity for generating hypotheses, testing these hypotheses, explaining and justifying.
2.	Find the area of the triangles. (Phase II, Day 2)	On a blank DGE window, the teacher is asking the students to find the area of triangles. The students have the opportunity to explore this mathematical situation on a DGE and decide for themselves which tools should be utilized that would assist them in finding the area of the triangles.

3.	Explore the relationship between the radius and the diameter of a circle. (Phase III, Day 1)	The students can make inferences and come up with conclusions by relating the properties of the circle with the measurements in the table.
4.	Explore the relationship between the circumference and the radius of a circle. (Phase III, Day 1)	The students can make inferences and come up with conclusions regarding the algebraic expression of the circumference of the circle, by relating the properties of the circle with the measurements in the table.
5.	Find the formula of the circumference of the circle. (Phase III, Day 2)	The teacher does not provide the formula of the circumference of the circle. By relating the observations made regarding π as well as the relationship between the radius and the diameter of the circle the students can make inferences and come up with conclusions regarding the formula of the circumference of the circle.
6.	Use the pizza slices to find the formula of the area of circle (Phase III, Day 2)	The students are encouraged to use the 'pizza slices' so as to construct a shape whose area they know how to find. Through this exploration the students are initially able to construct a rectangle. Following this, the students can use the formula for the area of a rectangle, replace its components with those corresponding in the circle and find the area of circle. With this activity, the students have the opportunity to relate the practical work and their observations with formulas and numbers as well as explain and justify their answer. What is more, the students are provided the opportunity to find on their own the formal mathematical formula used for the area of the circle. This activity encompassed making and testing hypothesis, explaining, justifying and 'proving' the mathematical formula.

7.	Find the relationship between squared radius and the area of circle. (Phase III, Day 4)	This mathematical situation encourages the students to explore a recently taught mathematical formula in a different way.
8.	Prove that ratio of the area/ circumference of circle equals r/2. (Phase III, Day 4)	The students were in a position to prove mathematically that the ratio area/circumference of a circle is r/2. By replacing the area and circumference with the mathematical formulas, the students were able to prove algebraically thus ratio. The proving process encompasses explaining and justifying the steps followed towards the proof of the statement.
9.	Exploration of the DGE-based tasks. (Phase III, Day 5)	The students are encouraged to relate the graphs with the DGE figures so as to come up with conclusions regarding the formula of the circumference and the area of circle. The students have the opportunity to discuss the two graphs, the curves that are constructed and make inferences regarding what the x and y axis in the second graph could represent in order for the curve to be a straight line.

The **community** in this activity system is the classroom community, made up of the teacher and the students. It has been acknowledged that the school and the wider educational and social community are also elements of the community, as they share to a certain degree the object. However, a discussion concerning how these broader macro context levels drive the activity system will be presented in Section 9.3.

Rules

The social norms, as identified through the analysis of the protocols include an expectation that: (i) ‘we solve problems using a variety of approaches’, (ii) ‘we raise the hand before answering the teacher’s question’, (iii) ‘we collaborate and work in pairs while exploring a task either on a dynamic geometry environment or with other artefacts'. .

The sociomathematical norms, as identified through the analysis of the episodes include an expectation that: (i) 'doing mathematics requires us to use precise language', (ii) 'doing mathematics requires us to justify our assertions', (iii) 'we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations', (iv) 'we write coherent geometrical explanations'.

Division of labour

Providing opportunities for exploration and investigation indicates a mediating role for the teacher. However, even though the teacher is endeavouring to provide opportunities for exploration, it is being noticed that the teacher does not always give adequate time for the students to formulate a response to the question asked.

This observation is reinforced by counting the time from the moment the students began the exploration up to the point where the teacher interrupted the exploration. Following this interruption, the teacher's intervention was explored so as to identify the motives behind this intervention. This was achieved by analyzing this protocol not in isolation but by looking into what was said when this exploring situation was revisited in subsequent lessons. This interruption indicated that the teacher was either translating students' exploration as playing and/or was providing the step that needed to be followed.

In protocol 2.4.1., where the teacher asked the students to construct triangles in a DGE window, which was an empty screen, the students had to decide themselves which of the tools available would enable them to successfully engage with the activity. The students themselves had to make the necessary technical and theoretical connections and appropriate the technology according to their needs. For instance, after adding the ruler and the squared grid, the students tried to construct triangles. In this particular DGE, while constructing a triangle, the area of the triangle is coloured, even though only two sides are constructed. The students would stop after constructing the first two sides, seeing that they constructed a triangle. However, something unexpected happened. Their construction was not complete. The students asked the teacher why this happened. The teacher asked them to recall the definition of a triangle and compared that with what they were doing on the computer. Some pairs realised the mistake they made and finished their construction whereas other pairs focused on the coloured area of the triangle and insisted that they succeeded in constructing the triangle but the computer was doing something wrong. In these cases, the teacher would talk to each pair. In this exploration opportunity, the students were provided enough time to investigate the activity. The teacher had the role of observer and moderator.

Regarding circle, before the students investigated the exploring opportunities provided by the teacher, a new DGE was introduced in the classroom (see protocol 3.1.11.). The initial exploration of this dynamic geometric environment is interrupted by the teacher who is informing the students that they will play later (protocol 3.1.12.). It appears that in this protocol, exploring the environment by following the teacher's instructions has more value to the teacher than the exploration that will follow by the students, which is called 'playing'.

For the exploration opportunity where students would investigate the formula of the area of circle using the pizza slices, the teacher gave students enough time to investigate the activity (protocol 3.2.7.). For this particular exploration, the teacher had the role of mediator. While the pairs were engaging with this activity, the teacher's intervention involved evaluating the students' responses so as to facilitate their interaction with the task. Through classroom discussion the students shared their proof and explained and justified their responses. It can also be concluded that with this exploration opportunity, proof is also used for enhancing the students' understanding of this particular mathematical formula. To elaborate more, the fact that the students themselves moved from the visual to the theoretical aspects of the area of circle, by discovering the formula of the area of circle, strengthens their understanding.

While the teacher asked the students to find the relationship between the radius and the diameter of a circle (protocol 3.1.13.) as well as the relationship between the radius and the circumference of the circle (protocol 3.1.16. and 3.2.3.), she did not give them enough time to investigate these relationships. Instead, she started guiding them towards this mathematical relationship. Even though it was the purpose of the teacher to give time for the students to make their observations and come up to conclusions regarding these relationships, she soon after that explored these relationships with the students through guided classroom discussion (protocol 3.1.14.). Going further, the day after, while revising what the class worked on the day before, the teacher commented that the students should not only play but also concentrate (protocol 3.2.2.). In this protocol, the teacher translated the fact that some students could not really summarize the work that was done previously as 'playing'. Here, exploring the first task was understood as 'play' instead of learning.

In protocol 3.2.6., where the teacher encouraged the students to explore the environment so as to find a way to calculate the area of the circle, her immediate comment was that she would give them some time to play. In this protocol, the teacher translated this exploration as 'play'.

When the teacher asked the students to find the relationship between the circumference and the area of a circle, she once again closed down the exploration opportunity by telling them the exact mathematical operation that they had to use in order to find the way these notions are related (protocol 3.4.3.). That is, the teacher's intervention controlled and limited the exploration. That is, initially the teacher asked the students to individually find the relationship that exists between the circumference and the area of a circle. Soon after that, she asked the students to individually prove that this ratio equals r/2 (protocol 3.4.4.). Following this, she asked a student to go to the whiteboard (protocol 3.4.5.).

During the lesson that followed the exploration of the mathematical formulas of the circumference and area of circle, the students were asked to find the radius and area of a circle with a given circumference (protocol 3.3.5.). The teacher interpreted her students' queries as a result of 'playing' with the computers (protocol 3.3.6.).

Despite this, the exploration of the DGE-based tasks (Phase III of the study) is an instance where the teacher decided instead of closing the students' engagement with a task, to return to the task at a later time for further exploration and discussion. The DGE-based tasks were partly explored throughout the week. To elaborate more, the tasks were initially explored so as to investigate the formula of circumference and area of circle. The tasks were fully explored and discussed during the last lesson related with this particular content of the curriculum. Nevertheless, how these tasks were fully explored will be illustrated in the activity of explanation (see Section 9.2.2.).

Object

It has been illustrated that the object of an activity system has several dimensions (see Sections 4.2. and 5.3.). The object for the teacher is associated with the curriculum content, behaviour regulation, the computer and proving. It can be argued that the object of the activity of exploration is the investigation of situations that lead to conclusions related to a given part of the curriculum.

Keeping in mind the analysis of the exploration opportunities provided to the students, it can also be argued that, concerning proving, the object of the activity is, through this investigation, the engagement with the elements that direct mathematical reasoning towards the ultimate goal of formal proving. The engagement of the students in these incidents indicates that they share this object with the teacher. Thus, this object is a collective one. However, even though the students share the teacher's object, the students share an object as this is being merely transformed by the teacher. Nevertheless, a more detailed analysis regarding the object of this activity system will be presented in the discussion that follows the portrait of the activity system.

By analysing the exploration of mathematical situations, as developed in the classroom through the lens of Cultural-Historical Activity Theory, a snapshot of the activity system can now be portrayed (see Figure 9.1. below).

Tools

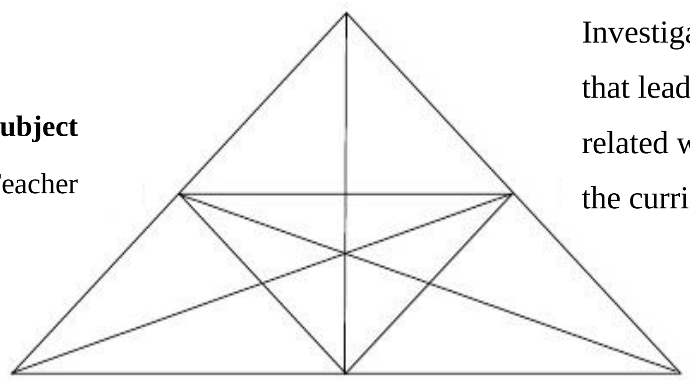
Opportunities for exploration and investigation that require the allocation of sufficient time for the students to consider these questions

Object

Investigation of situations that lead to conclusions related with a given part of the curriculum

Subject

Teacher



Community

Teacher, students

Division of labour

Teacher's intervention: mediating role VS closing down the exploration

Rules

Social norms: 'we solve problems using a variety of approaches', 'we raise the hand before answering the teacher's question', 'when we use computers/artefacts we work in pairs' .

Sociomathematical norms: 'doing mathematics requires us to use precise language', 'doing mathematics requires us to justify our assertions''we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations', 'we write coherent geometrical explanations'

Figure 9.1.: The activity system of exploring mathematical situations

Identifying points of contradictions

Play/learn

A consideration of the analysis of the aforementioned protocols related with the exploring opportunities provided by the teacher leads to the conclusion that the teacher’s intervention has an impact on how the activity of exploration of mathematical situations unfolds. What is more, it becomes apparent that closing down such an exploration opportunity was followed by the teacher using the word ‘play’. However, this word was used differently when not associated with exploration. This leads to the emergence of two contrasting values, play/learn.

Regarding the activity of exploration, the word ‘play’ when used by the teacher had a negative connotation. It had a negative value when this exploration was interpreted by the teacher as ‘playing’ instead of learning. That is, while the teacher would encourage the students to explore the activity in order to reach to some conclusions, she would also make a negative remark about this exploration as something that had no didactical value (protocols 2.1.4. and 2.1.9.).

However, concerning the ‘play’ value, the word ‘play’ is not only being used by the teacher with negative connotations. This parameter had a positive value when it was used by the teacher to refer in a more general way to the exploration of the activity.

To be more explicit, in protocol 2.3.7., the value ‘play’ is being employed with its positive value. In this protocol, the teacher is informing the students that the following part of the lesson will involve them collaborating in groups in a DGE. Exploring the features of this dynamic geometry environment and working in pairs for the construction of triangles is translated as something encouraging and constructive. Moving on, again in protocol 2.3.7., the teacher also states that the students will have the opportunity to ‘play’ with a game on the computer. In this instance, the word ‘play’ is used with its authentic meaning, even though, from an educational and didactical perspective, it can be considered as a form of reflection, evaluation and further understanding.

Additionally, the last remark the teacher, made at the end of the observation week, demonstrates that even when the teacher is talking about the test the students will have since they finished this topic in geometry, she uses the word 'play'. In this protocol (2.5.2.), the teacher changes the word 'test' that the students used with 'play' to show that since the assessment will not be from the assessment book but will be undertaken in a DGE, the evaluation takes a more playful form. While assessment constitutes an important element for the teacher's work, the teacher attempts to remove from the word its formal and sometimes negative value. However, considering the aforementioned protocols, it can be also argued that the teacher views the assessment on the computer less formal than assessing students using a test on paper.

The tension concerning the values play/learn was also made apparent during the initial interview of the teacher as well as in one of the informal discussions. That is, the teacher's approach in employing technology in the classroom, as made explicit in the interview, was in contrast with her objectives about the tasks that would be designed for Phase III, as well as what she would actually do during the lesson. To be more accurate, while discussing the tasks that would be designed for the iteration phase, the teacher commented that she did not want the tasks that would be employed to be like the ones I used the year before during the exploratory study. The teacher felt that if the tasks were 'too open' then the teacher would end up losing the students and the goals of the lesson would not be achieved. In addition, even though she stated that she employs technology the way she uses a textbook, through the observations, it becomes obvious that she evaluated differently the work students did on the computer from that done on paper-and-pencil.

The 'play' dichotomy relates to the notion of the play paradox (Hoyles and Noss, 1992) and the notion of the planning paradox (Ainley et al, 2006). Hoyles and Noss (1992) introduce the notion of the *play paradox* to describe the multiplicity of paths that are available to students when using a tool in an exploration related with a mathematical task. That is, the students, through their exploration, might not encounter the mathematical ideas that were perceived as the objectives set by the teacher or the curriculum materials. Thus, the teacher may decide to close down an exploration opportunity as she may interpret student's exploration as shifting away from her own objectives. In a similar way, Ainley et al (2006) call the conflict that may occur in the daily mathematical classrooms, due to a failure to contextualize tasks, as the *planning paradox* (see Section 5.5).

Keeping in mind the above, it can be argued that while a play-like exploration can facilitate learning, this can prove quite challenging for the teacher as the roles shift. Thus, the teacher may find it difficult to take advantage of such opportunities.

Computers/face to face lesson

A comparison of the above episodes where exploring opportunities were provided indicates that division of labour when using computers differed from the face-to-face lessons. That is, while exploring a task on the computer, it is not automatically clear that it is the teacher who decides what counts as meaningful. As the teacher's role shifts when computers are employed, it can be argued that, due to the fact that the teacher cannot assist all students with the computer task, contradictions may arise. The norm 'when we are using computers we learn together' reinforces this observation. The fact that the teacher thought that a worksheet should be provided to the pairs can be perceived as a way to resolve this tension, as a way for the teacher to feel more comfortable with not losing control of the situation. That is, having a worksheet may work as a reassurance for the teacher as the students can rely on it if the teacher is not available. This is the teacher's resolution of the planning paradox. This tension can also be related with the play/planning paradoxes, as well as linked with the notion of ownership as perceived by Papert (1993) in his formulation of Constructionism. That is, while the students are provided with the necessary tools to participate and to take ownership of the learning process, the teacher is at the same time attempting to avoid facing these paradoxes.

Division of labour/object

A close examination of the way the teacher was intervening in the above incidents leads to a tension between the object and the division of labour. That is, closing down a task clashes with the object of the activity system. By closing down such investigation, the students do not have the opportunity to initiate a solution on their own and complete the task. Going further, by closing down such tasks, the students have few opportunities for explanation and justification. As these notions are important aspects of proving, it is reasonable to argue that closing down a task works against proving.

This observation is reinforced by comparing the teacher's actions in the lessons with the teacher's initial interview. The teacher's behaviour was in contrast with what she shared during the initial interview. To elaborate more, the teacher exemplified that she gives time to the students to try to discover something on their own. If these incidents concerning the exploring opportunities are considered in isolation, they might not look as striking events that need further consideration. There are occasions where time constraints and the feedback from the students might force the teacher to close a task down. However, the fact that the teacher declared that she wanted her students to explore the mathematics and for this reason she employed open tasks indicates that this behaviour was contradictory to that aim.

Discussion

To conclude, the way the teacher intervenes is what drives the activity of exploration. The main actor is the teacher, who decides what is meaningful and thus orchestrates the students' activity. The teacher is encouraging the exploration that leads to hypothesising, explaining and justifying assertions. However, it seems that while the teacher is supporting exploration, at the same time she closes down such opportunities. While the content of the mathematics curriculum is being covered, the object of this activity system is partially met. Translating the exploration as playing puts a negative value to an activity important to proving. This tension leads to the conclusion that closing down exploration opportunities works against proving.

9.2.1.2. Exploration for supporting mathematical connections

This activity is concerned with the teacher supporting mathematical connections. To be more accurate, incidents of the teacher making connections with parts of mathematics that the students would be taught in secondary school or that were taught recently or further in the past have been identified in both phases of classroom observation (see Sections 7.4. and 8.4.1. accordingly).

As indicated below, there were many instances where the teacher would include things in the lesson that are part of the mathematics curriculum of lower secondary school. The teacher would give particular emphasis on some things that they were doing in the classroom saying that they will do these things in secondary school. These forward connections seemed to be a natural part of the mathematics lesson; the teacher would make explicit that what was explored and discussed was something that the students would further explore in secondary school.

To be more elaborative, an area in geometry as part of the mathematics curriculum that needs to be covered in Year 6 is concerned with the area of triangles (see Section 6.2.2.2.). A prerequisite in achieving this is the successful construction of the altitude in a triangle. In protocols 2.1.14.-16, one can see that the teacher made a forward connection by exploring with the students the number of altitudes a triangle has. By constructing the three altitudes, the teacher's aim was for the students to see that the altitudes intersect in a single point, the orthocentre of the triangle. The teacher, without giving any formal mathematical language, said that this is what she wanted to achieve (protocol 2.1.15.) and explained that this is something that the students will do in secondary school (protocol 2.1.16.). The teacher added that it is helpful for the students to be aware of that now.

Regarding the altitudes of a triangle, the curriculum concerning primary school mathematics does not specify whether the students should be aware of the fact that there are cases where the altitude lies outside the triangle. This is not supported by the students' textbook nor the teacher's guidance book. However, this is included in the curriculum of secondary school mathematics. Nevertheless, once again, the teacher moved forward by also exploring with the students how the altitude of an obtuse triangle can fall outside the triangle. In protocols 2.3.9.-11, the construction of altitudes in several triangles led to investigating the construction of altitudes in an obtuse angle. The students with the teacher concluded that as the altitude, which according to its definition is a line that passes through the vertex and is perpendicular to the opposite side, can meet the extended base outside the triangle.

Regarding the area of geometry related with circle, it can be argued that the teacher made a connection between the properties of the circle with everyday life. That is, through classroom discussion the students related circle with the wheels of the bicycle (protocol 3.3.3.). As a result, they concluded that the distance covered by the full rotation of the wheel of the bicycle indicates the circumference of the circle. This strengthens the understanding of the conceptual aspects related with circle.

In protocols 3.2.4. and 3.3.2., the teacher encouraged students to use the letter ‘r’ instead of ‘a’ in order to represent the radius of the circle as this is how the radius is represented internationally in the mathematics community and this is how it is being used in secondary school. In these protocols the teacher again is making a forward connection.

At a first glance, even though it may appear that the aforementioned protocols do not have any obvious deep pedagogical value, the teacher’s reasoning behind this remark is the continuity that should exist from primary to secondary mathematics education. Despite this, the teacher did not extend this discussion to the use of any letter in mathematics. That is, even though the teacher’s intention was to make a forward connection regarding the way letters are used in mathematics, she did not emphasise the interpretation of letters as generalized or even as specific unknown numbers instead of shorthand of names or measurements labels.

The teacher exploited another exploring opportunity provided to students (protocol 3.4.6.), in order to make a forward connection. This connection involves the formal aspects of proving in mathematics. By asking students to prove that the ratio of area/circumference of a circle is r/2, the students had to replace the area and the circumference of a circle with their mathematical formulas and recognize that this is a fraction which would then mean that they could make the necessary simplifications. However, the simplification of fractions when including letters is not part of the mathematics curriculum of primary education. Making simplifications with indices in mathematics is also in the curriculum of secondary school mathematics. Once again the teacher is opening out the area of mathematics being explored in the classroom. The teacher acknowledged to her students that even though they explored indices in previous lessons, simplifications with indices was not included in their curriculum and demonstrated how this is done, so as to assist students (protocol 3.4.5.). However, the teacher’s emphasis in this particular incident was proving. At the end of this task the teacher said that this is how the students will be proving in secondary school (protocol 3.4.6.).

An observation that can be made in this incident is that, even though the teacher embraced opening out the mathematics explored in the classroom, this was mainly realized by her instigating these explorations. That is, these explorations would not unfold after a comment made by students. For example, in protocol 3.4.7., the student's comment could further the discussion regarding proving with numbers in secondary school. The teacher closed down this exploration opportunity by commenting that we do not prove with numbers, we just replace numbers with letters. This is another incident that could be used to initiate a discussion regarding the usage of numbers in mathematics.

Going further, the teacher also makes connections between the dynamic geometry environments and the paper-and-pencil environment. For instance, after concluding what an altitude is, and constructing the altitudes in various triangles in the DGE, the teacher asked the students to construct the altitudes in triangles on paper (protocol 2.1.17.). The teacher returned to the paper-and-pencil environment so as to make connections with the mathematics outside the microworld. While there were instances where the teacher would do the same things in both environments, showing that she was not comfortable if things were only done in a dynamic geometry environment, making these connections supported the students' instrumental genesis. Nevertheless, this will be further discussed when interpreting the activity of exploration of DGEs (see Section 9.2.1.3.).

In addition to the above, in protocols 2.1.5. and 2.1.7., the teacher is making a connection with mathematics previously taught. That is, the teacher with the students make a connection regarding the mathematical formula for the area of triangles and recognize the difficulty of finding the altitude.

Additionally, a close examination of the incident shows that the teacher is also encouraging mathematical connections between classes of problems.

The teacher's behaviour in the above incidents is in accordance with what she stated in the initial interview. The teacher makes connections with what was previously explored in the classroom as she wants to reinforce what the students have learnt. She also makes these forward connections as she believes that these varying ways of exploration in the classroom will assist students in gaining a deeper understanding of the geometry under study, which will prove valuable for a more effective transition to secondary school geometry.

Keeping in mind the above, it can be argued that making forward connections works in favour of proving.

The interpretation of the activity of exploration for supporting mathematical connections will be undertaken using the standard terminology of CHAT, namely tools, community, rules, division of labour and object as introduced in Section 4.2., before identifying points of contradictions.

Tools

In this activity system the prominent tools used by the teacher to illustrate the discussion around mathematical connections is talk and the whiteboard. Another tool that mediates the discussion in making these connections is the exploring opportunity that emerges during classroom activity.

Rules

The sociomathematical norms, as identified through the analysis of the classroom lessons include the students using the 'right' mathematical language, collaborating and working in pairs while exploring a task, raising the hand before answering the teacher's question and presenting their solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations. In regards with explanation and justification, the norm established is that the students are expected to justify their answer and explain the steps they followed in order to find what was required from them.

Division of labour

The way the teacher manages the exploration the goal of which is to make forward connections, differs from the way the teacher intervenes when exploring opportunities of mathematical situations are provided. The main actor in these protocols is the teacher, who determines what counts as meaningful mathematical knowledge. The teacher guides and mediates the classroom discussion. She uses questions to guide the students' engagement and facilitate their interaction with the content of mathematics which is under investigation.

Object

What becomes clear from the above incidents is that the teacher is concerned with getting the students to make connections between the content of mathematics, with which they are engaged, with parts of mathematics that the students would be taught in secondary school or that were taught either recently or in the past. The engagement of the students in these protocols indicates that they share this object with the teacher. Thus, being responsive to the teacher's ideas can act as an indicator that the object is a collective one.

By analysing the activity of exploration for supporting mathematical connections, as developed in the classroom through the lens of Cultural-Historical Activity Theory, a snapshot of the activity system can now be portrayed (see Figure 9.2. below).

Discussion

Concerning exploration for supporting mathematical connections, it can be argued that no obvious tensions seem to exist in this particular activity system. However, in order to either strengthen or refute this claim, I will now extend the analysis concerning this activity system by initially analysing the content of these forward connections and investigating whether other opportunities for exploring mathematics could have been provided by the teacher.

Taking a particular mathematical idea to build connections can be achieved in a number of different ways. It can be connected: to a real world example; to another subject area; to another mathematical topic that the pupil has previously studied; to a pupil's way of thinking; or, to a pedagogical principle (Wood, 1993).

Tools

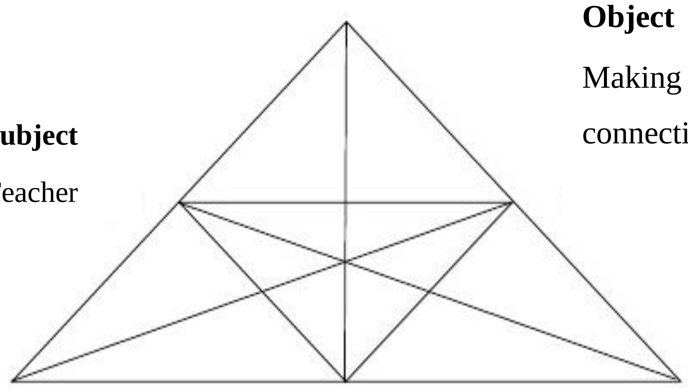
Talk, whiteboard, exploring opportunities

Object

Making mathematical connections

Subject

Teacher



Community

Teacher, students

Division of labour

Teacher as facilitator

Rules

Social norms: 'we solve problems using a variety of approaches', 'we raise the hand before answering the teacher's question', 'when we use computers/artefacts we work in pairs' .

Sociomathematical norms: 'doing mathematics requires us to use precise language', 'doing mathematics requires us to justify our assertions''we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations'

Figure 9.2: The activity system of exploration for supporting mathematical connections

To begin with, these forward connections seem to be connected to contextual and procedural knowledge rather than conceptual knowledge. That is, even though these connections constitute an expansion of the concepts that are part of the secondary school mathematics curriculum, they are not always related to relationships and interconnections that explain and give meaning to mathematical procedures. What is more, it seems that the connections attempted in the classroom are mostly originated by the teacher. When a student is attempting to make a link with mathematics in general (protocol 3.4.7.) or with another discipline (protocol 3.4.8.), the teacher is being unsuccessful in making connections with the students' way of thinking. Nevertheless, it can also be argued that these forward connections support the connection of the students' concept image to the concept definition.

Considering the aforementioned comments, it is also acknowledged that the teacher shapes the mathematics that is being taught in the classroom. That is, the teacher's actions shape the way students think about mathematics. An illustrative example is protocol 3.4.7. The discussion that developed in this protocol followed the students proving that A/C is r/2. What seems striking in this episode is that the teacher does not further discuss the student's statement. The student's statement could have provoked discussion about the nature and features of mathematical proof. Nevertheless, what this protocol reveals regarding the way the subject engages with proving in the classroom will be further elaborated on in Section 10.1.3.

9.2.1.3. Exploration of DGE

As DGEs were employed in designing and implementing the mathematics lessons, the activity of exploration is also concerned with the degree of exploring technology in the classroom.

By keeping this in mind, the interpretation of the activity of exploration of DGE will be undertaken using the standard terminology of CHAT, namely tools, community, rules, division of labour and object as introduced in Section 4.2., before identifying points of contradictions.

Tools

The prominent tools for this activity are the two DGEs utilized in the classroom, GeoGebra for the area of triangles and Cabri for the circumference and area of circle. The exploring opportunities that support explaining and justifying have been illustrated in the exploration of mathematical situations. However, it should be mentioned that in this activity other opportunities for exploration are provided. For the purposes of this discussion, the way the teacher generally employed DGE in the lesson is explored, as this may have had an impact on the students' exploration. What is more, other opportunities where the students were encouraged to explore DGE are provided. However, it should be made explicit, that in these opportunities, the tasks were also presented in the students' textbooks.

Nevertheless, it should be noted that DGEs were employed for exploration, construction, classroom discussion and demonstration by the teacher.

Division of labour

Regarding the appropriation of technology, the teacher would adapt different roles while the students were working on the computers. There were instances where the teacher would have the role of instructor, facilitator and mediator. However, even though there were instances where the teacher enacted these roles during Phase III, it was obvious that most of this time was spent trying to help the students managing the tool, instead of supporting them in reasoning mathematically.

By following the progression of the lessons, one can see that the teacher was guiding the students' instrumental genesis. During Phase II, where Geogebra was employed in the lessons, the students with the teacher recalled initially the way this environment was used for the lessons regarding parallelograms. Following this, while exploring the first activities related to the topic under investigation, the teacher's focus was mainly on the technical aspects of the dynamic geometry environment. That is, through these activities the teacher illustrated how certain available tools could be used to explore several activities.

For instance, in protocol 2.1.8., the teacher demonstrated in the whole classroom how a rectangle can be constructed in the DGE, as this was something new for the students. Another example of the teacher having the role of instructor is in protocol 2.3.8., where the teacher demonstrated to the students how a triangle can be constructed in the DGE.

When, though, the teacher asked students to construct triangles in a DGE window which was an empty screen, the students had to decide themselves which of the tools available would enable them to do the activity. The students themselves had to make the necessary technical and theoretical connections and appropriate the technology according to their needs.

In protocol 2.4.1. after adding the ruler and the squared grid, the students tried to construct triangles. In this DGE, while constructing a triangle, the area of the triangle is coloured, even though only two sides are constructed. The students would stop after constructing the first two sides, seeing that they constructed a triangle. However, something unexpected happened; they failed to construct a triangle. Thus, their construction was not complete. The students asked the teacher why this happened. The teacher asked them to recall the definition of a triangle and compared that with what they were doing on the computer. Some pairs realised the mistake they made and finished their construction whereas other pairs focused on the coloured area of the triangle and insisted that would construct the triangle but the computer was doing something wrong. In these cases, the teacher would talk to each pair.

Bretscher (2009), in investigating the crucial role of the teacher in facilitating the students' instrumental genesis, argues that one technique that can be used so as to promote students' mathematical thinking is to highlight not only the potentials but also the limitations of these environments. For instance below the teacher and the students pointed out the anomalies of measurement in a DGE; the rounding errors.

In protocol 3.1.15**,** the students with the teacher compared the measurements of the radius and the diameter of several circles by dividing each diameter by the radius to conclude that the radius is half the diameter. Some pairs complained that not all of their divisions showed that, deducing that this conclusion is not correct. The teacher had to explain why this occurred.

Another technique the teacher adopted in facilitating students' instrumental genesis is the connections made between the dynamic geometry environments and the paper-and-pencil environment (protocol 2.1.17.). That is the teacher is making connections between the two environments. By providing students multiple windows to construct mathematical ideas, the students can build connections within a web of ideas (Guin and Trouche, 1999). For instance, after concluding what an altitude is, and constructing the altitudes in various triangles in the DGE, the teacher asked the students to construct the altitudes of several triangles on paper (protocol 2.3.2.). The teacher returned to the paper-and-pencil environment so as to make connections with the mathematics outside the microworld. While there were instances where the teacher would do the same things in both environments, showing that she was not comfortable if things were only done in a dynamic geometry environment, making these connections supported the students' instrumental genesis.

Rules

For this activity to unfold the rules established were: 'we work in pairs when working with computers' and 'when we work with computers we learn together'.

The students were expected to collaborate and work in pairs while exploring a task either on a dynamic geometry environment or with other artefacts. When the teacher told students to move to the computers, the students would immediately take their notebook and pencil and go to the computer to which they were appointed. However, even though the teacher expected students to collaborate while working in pairs, when a worksheet was given to each pair on the second day, in Phase III, the pairs looked rather confused in terms of who was supposed to do what. That is, the students, despite working in pairs, would always write their conclusions individually and not in a shared worksheet.

However, during the Phase III, each pair was given one worksheet before moving to the computers. When the pairs of students had to complete the worksheet while exploring the DGE-based task, the students were repeatedly saying that they did not know what they were supposed to do. This was in a way quite straightforward. In this respect, this articulated behaviour indicates that collaborative wok did not include working on the same worksheet. Since the pairs would use a shared worksheet throughout the week, a new regulation of action had to be accomplished between the students. Each pair had to decide how to work collaboratively with a joint worksheet. As a norm concerning collaboration while working on a computer was already established with the students, the teacher expected the students to succeed in sharing the worksheet without having to intervene. As a result of this attitude of the teacher, the pairs adopted differing roles concerning their collaboration. To be more precise, some pairs decided to work in turns; one would read the instructions and write where indicated while the other student would explore the task and then the roles would change. For other pairs these roles were assumed individually.

Regarding the integration of technology both the teacher and the students are learning together how to use the tools available in different dynamic geometry environments. The teacher made explicit to the students that she was not familiar with all the tools available in the software being employed in the classroom. This did not affect the teacher's authority regarding the knowledge she had concerning the usage of software in the classroom. On the contrary, the students and the teacher together would discuss how the tools available could be used to explore specific aspects of the activities with which the class was engaged.

To be more accurate, in protocol 2.3.6., in finding the area of triangles in Geogebra, the teacher expected students to use either the axis, or the squared grid in order to find the altitude and base of the triangle, as these are the tools that were demonstrated in previous parts of the lessons. However, the students tried to use the 'area' from the menu in order to find the area of the triangle. Their attempt was not successful. The teacher said that she also failed, but she would try again as she knew what she needed to do even though she did not know exactly how.

Object

The object of the activity of exploration of DGE was twofold: exploring DGE, the purpose of which was the familiarization of the class with the environment, and exploring DGE so as to explore a specific mathematical situation. It should also be mentioned that exploring DGE would also be employed by the teacher in such a way so as to take the form of assessment.

In the instances where the object of the activity is to develop students' technical skills, it can be argued that the outcome is the establishment of technologically literate students. In the instances where the teacher's object was the development of students' mathematical understanding, the outcome was dependent upon the exploration of the task. Where the teacher closed down the activity, it can be argued that, despite the intended outcome, the actual outcome of the specific computer-based incident was the development students' technical skills.

By analysing the activity of exploring DGE, as developed in the classroom through the lens of Cultural-Historical Activity Theory, a snapshot of the activity system can now be portrayed (see Figure 9.3. below).

Identifying points of contradictions

Employing a new mediational tool in the activity system, inevitably leads to tensions. Handing out a worksheet to the pairs steered changes in the activity system. The change made in this activity system was the utilization of a different DGE in Phase III. Sharing a handout was something new that led to the emergence of tensions. However, what assisted students in resolving this tension was the fact that the rule regarding collaboration was already established in the classroom. This led to assuming differing ways of collaboration between the pairs. Thus, an emergent rule led to the resolution of this tension. Nevertheless, a more thorough discussion regarding how introducing a new mediational tool in the classroom led to the emergence and resolution of contradictions will be presented in Section 10.2.2..

Conclusion

How does exploring, as developed in the classroom, impact on the way proving is constituted in the classroom? In order to be able to answer this question, I now return to the ways exploration may promote the construction of mathematical proof, as exemplified in the literature: exploration reveals information necessary to prove; exploration facilitates the understanding of proof; exploration encourages the generation of conjectures; exploration supports justification for the process of proving (Hsieh et al, 2012).

Tools

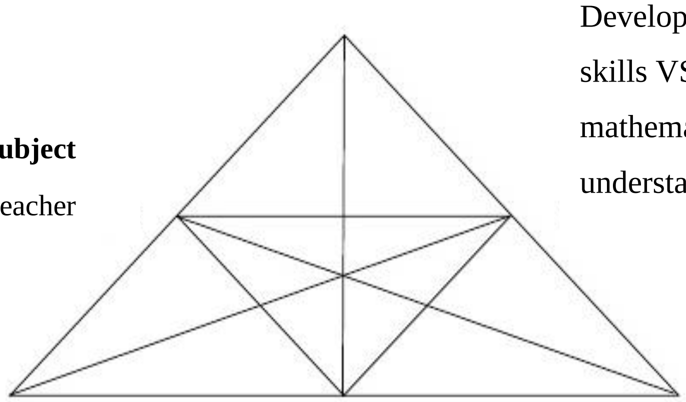
Dynamic Geometry Environments

Object

The object is twofold: Development of technical skills VS development of mathematical understanding

Subject

Teacher



Community

Teacher, students

Division of labour

Teacher as facilitator, instructor, mediator

Rules

Social norms: ‘we solve problems using a variety of approaches’, ‘we raise the hand before answering the teacher’s question’, ‘when we use computers/artefacts we work in pairs’, ‘when we work with computers we learn together’. .

Sociomathematical norms: ‘doing mathematics requires us to use precise language’, ‘doing mathematics requires us to justify our assertions’‘we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations’, ‘we write coherent geometrical explanations’

Figure 9.3.: The activity system of exploring DGE

By considering the exploring opportunities provided by the teacher, as well as the three levels of exploration that fall into the activity of exploration, it can be argued that in this particular classroom setting, the way exploration is attempted may correspond to the aforementioned classifications as it provides a point of reference for proof production. However, the way the teacher intervenes cannot be neglected. Closing down an exploration activity may potentially contribute to students having difficulties in initiating proofs. This also has an impact on the conceptualisation students establish regarding the structure of proofs. Nevertheless, a more thorough discussion related with the aforementioned remarks will be presented in Chapter X.

9.2.2 Activity of explanation

The act of communication is an integral part of the mathematics lesson. That is, instances where mathematical ideas are communicated emerge throughout the lesson. When mathematical reasoning is communicated to others, it is usually accompanied by explanation, which therefore emerges and develops through the various activities in a mathematics lesson that require reasoning. Keeping in mind the meaning of explanation, as perceived in this research study and exemplified earlier in Chapter II (see Section 2.4.), the activity of explanation focuses on clarifying aspects of one's mathematical thinking to others, and sometimes, justifying for them the validity of a statement.

Adding to the above, in Section 8.4.1., through the initial analysis of the classroom observations, it has been concluded that definitions and defining as activity was an integral aspect of this mathematics classroom, around which, explaining and justifying developed. While not being considered originally as a theme of interest that would guide the analysis of the classroom data, this remark will now be further elaborated on.

To be more precise, the activity of explanation in Phase II unfolds and expands mainly around mathematical definitions. That is, a great part of the lessons of this week is dedicated to definitions related with the area of triangles. Initially, establishing what an altitude is in a triangle is a prerequisite for understanding and properly applying the formula of the area of triangles. Furthermore, in order for the students to be able to successfully apply the mathematical formula of the area of triangles in differing mathematical situations, opportunities must be provided where discussion and reflection regarding this particular formula is encouraged. This is achieved through the activity of explanation.

This observation regarding definitions characterizes Phase III as well. To be more comprehensive, sufficient time is allocated for the definition of the circle, as well as for the formulas of the circumference and area of circle. Nevertheless, it is being acknowledged that explaining and justifying occurs throughout the lessons. Other instances that fall into the activity of explanation are also taken into consideration.

This section initially sets out the activity of explanation through the classroom discussions that are related with the definitions of the aforementioned mathematical notions. That is, incidents where explaining is related with this content of the mathematics curriculum will be presented and analysed. Subsequently, building on the analysis of these protocols, the components of the activity system will be portrayed and, successively, the activity system of explanation will be generated. A discussion will follow.

However, before proceeding, the connection that exists between definitions and explanation must be made explicit (see Section 8.4.1). Definitions are conventions that require no explanation. However, the teacher wants reference to the attributes that involve properties. That is, the move from a definition involving only perception to a definition that involves properties needs explaining.

Analysis of the classroom episodes

Content of the mathematics lessons: Area of triangles

An initial analysis of the classroom observations that are related with this content of the mathematics curriculum shows that a great part of the lessons is dedicated to definitions. In order to be able to justify this observation, the time allocated to explaining related to these definitions was counted.

In protocol 2.1.10. the class attempted, for the first time, to give a definition of the altitude in a triangle. The teacher was following the students’ instructions in order to construct triangles and their altitudes, so as to make them realize that the way they phrase things has an impact on the final construction. In this incident, the teacher wanted to see whether the students remembered and understood what 'altitude' is in a triangle. She was constructing the altitude of a triangle on the interactive whiteboard by following the students' definitions of the altitude. In this classroom discussion, the students, by focusing on the perceptual aspects of the construction, they were making alterations to the definition they were giving in order for the altitude constructed to be accurate. In this protocol, the classroom discussion is guided by the students' responses to the question. The activity continued in this way until an acceptable definition was given. The formulated definition captured and synthesised the mathematical essence of the concept. This is in accordance with what Borasi (1992) identifies as one of the functions of mathematical definitions.

At the beginning of the lesson on the second day, the teacher with the students revised the definition of the altitude (protocol 2.2.1.). This protocol has many commonalities as the aforementioned protocol. To elaborate more, the teacher guided the classroom discussion for the students to give the definition of the altitude. She gives emphasis to this definition as, according to her, the students struggled to comprehend and appropriately use it.

In protocol 2.2.3., S1 gives a definition that satisfies the teacher.

In protocol 2.2.4. the students are using the ruler to draw altitudes in triangles. At a first glance, it can be argued that the main objective of this activity is for the students to develop their technical skills. However, the fact that the teacher was asking the students to explain how and why they used the ruler by using the definition of the altitude strengthens the move to a definition that involves properties needs explaining.

On the third day, the teacher asks again for the students to give her the definition of the altitude (protocol 2.3.1.). By looking at the answer the student gave, it can be argued that this response does not count as a complete answer. However, this answer pleases the teacher as the student's response constitutes the condition that needs to be applied in order for a segment to be considered as an altitude of a triangle. The teacher's comment indicates this.

Protocols 2.1.12.-13 constitute extracts of the classroom discussion that followed the formation of the definition of an altitude and the algebraic expression for the area of a triangle. After a synopsis of the ways with which one can find the area of triangles, a DGE task that was designed by the teacher is introduced for the first time as a way to enhance the generality of the algebraic expression of the area of triangles. By exploiting the opportunities dynamic geometry environments provide in mediating students' understanding, this activity that unfolds is also employed for strengthening the understanding of the formula of the area of triangles. For this task, the triangle BFC was inscribed in rectangle ABCD, with point F moving along AD. The teacher shows several triangles to students by moving point F and asks repeatedly whether the area of the triangle is half the area of the rectangle and why. The teacher does not accept answers that only rely on what is shown on the screen, endorsing the gradual detachment from empirical arguments and the move towards formal mathematical reasoning. In order for this to be achieved, the argumentation process was guided by the teacher.

Protocol 2.5.1. is an extract of the discussion centred on revising what was learnt throughout the week. In this discussion, the teacher appraises the complete answer S1 gives regarding the area of triangles. As the teacher expects her students to use the mathematical language and give complete answers, her comment shows that the response S1 provides, satisfies these criteria.

Concerning the area of triangles, protocols 2.3.3.-5 constitutes the discussion that followed the comparison of the area of triangles whose altitudes had the same length. In order for the students to be able to come to conclusions, they had to use the formula for the area of triangles. During the classroom discussion, the students had to justify their answer. That is, they had to give an explanation using the formula for the area of triangles. This was an exercise from the students' textbook that required a written response. The students had the opportunity to share their responses. The teacher accepted the responses that, according to her, were complete answers and drew from the specific formula.

Content of the mathematics lessons: Circle

As exemplified in Section 8.2.1., this area of the mathematics curriculum is concerned with the definition of circle, the relationship between the radius and diameter of circle, as well as the circumference and area of circle (see also Appendix VIII).

The first lesson concerned with circle begun with a question (protocol 3.1.1.). The teacher does not provide the definition of circle. On the contrary, the students are expected to explain what circle is. The teacher was drawing on the whiteboard following the students' responses. In this classroom discussion, the students, by focusing on the perceptual aspects of the teacher's drawing, were making alterations to the definition they were giving in order for the drawing to be a circle. In this incident, the classroom discussion is guided by the students' responses to the question. The activity continued in this mode until an acceptable definition was given. The formulated definition captured and synthesised the mathematical essence of the concept. This is in accordance with what Borasi (1992) identifies as one of the functions of mathematical definitions.

However, protocol 3.1.7. differs from the previously mentioned incident. In this incident, the students had to say which of the shapes shown on the interactive whiteboard were circles and say why. In this incident, the students could not rely only on perception but had to distance themselves from the 'geometrical drawing', and use the properties of a circle in order to say why the shapes were or were not circles. At first glance, the question 'which of these shapes are circles?' seemed quite simple for the students (protocol 3.1.8.). However, the students had to put effort in explaining why this was the case. Perception was not enough as the teacher would not accept their answers otherwise. The students had to draw on the definition of circle and its properties so as to justify why the presented shapes were not circles. This incident with the function mathematical definitions should fulfil (Borasi, 1992). The definition students formulated in the previous incident allowed them to discriminate between instances and noninstances of the specific concept. This can also be considered the first instance where it is attempted to make the definition of circle operable for the students (protocol 3.1.9.).

As exemplified in the activity of exploring mathematical situations (see Section 9.2.1.1.), the students were provided enough time to explore the relationship between the radius and the diameter (protocol 3.1.13.) as well as the relationship between the radius and the circumference of the circle (protocol 3.1.16.). Despite this, we can see the teacher attempting to engage the students in a process where they can discover the relationship between these concepts themselves. The teacher does not expect students to prove the formula of the circumference of the circle. However, she expects students to engage in reasoning so as to arrive at a conclusion based on a set of observations. Even not a valid method of proof, it shows that the mathematical formula is true.

Protocols 3.2.7.-8 differ from the previous protocol. In these extracts of the classroom observation, the students had the opportunity to use the pizza demonstration to discover the area of circle. This consists of an illustrative example of exploration that reveals information necessary to prove, facilitates the understanding of proof, encourages the generation of conjectures as well as supports justification for the process of proving. This can be also characterised as constructive defining. The fact that differing numbers of pizza slices where given to pairs of students strengthens the generality of the formula of the area of circle. The students had to demonstrate the way they worked as the teacher would not accept their answer otherwise.

Moving further, in protocol 3.4.1., the teacher asks the students to find the relationship between the area of the circle and squared radius. After they conclude that this ratio equals π, she then asks them to reach a conclusion regarding the area of the circle. That is, she is asking them to work backwards. This is used as a way to enhance the understanding of the generality of this formula.

In protocol 3.4.3., after exploring the circumference and the area of circle, the students were in a position to prove mathematically that the ratio area/circumference of a circle is r/2. By replacing the words area and circumference with their formulas, the students were able to find the algebraic proof. Even though the teacher does not give adequate time to the students to prove this statement individually, she is asking the students to give feedback to what one student is doing on the whiteboard (protocol 3.4.4.). This gives the opportunity to the students to reflect and engage in their proving process. Through the classroom argumentation the students share this proof (protocol 3.4.5.).

In protocols 3.4.8.-10, the students were able to relate the graphs with the DGE figures so as to come up to conclusions regarding the formula of the circumference and the area of circle. The students with the teacher had the opportunity to discuss the two graphs, the curves that are constructed and make inferences regarding what should be different in the second graph in order for the curve to be a straight line.

In protocol 3.4.8. the student relates his observation with elements of the science curriculum taught in Year 6. The student appears to compare the radius and circumference of the circle by reference to the constant of proportionality, previously encountered in science experiments. This may also be considered as reasoning by analogy as involves making an assertion which is based on similarities between two situations, one well-known and another less well understood. What seems striking in this episode is that the teacher does not further discuss this statement.

The three students in protocol 3.4.9. are responsive to the idea of a non-linear relationship. These students had no experience with parabolas or quadratics from school. However, they have already been introduced to the formulas for the circumference and area of a circle and they have previously discussed the circumference graph. It could be argued that there may be some guesswork by the students in recognising that this is about πr^2, and that because the radius is squared in the formula the curve is produced. Nevertheless, it can reasonably be argued that the experience of seeing dynamically the movement of the radius point and the corresponding non-linear movement of the area point in a DGE has supported the students making a connection. Thus they may be using knowledge of the formula πr^2, to gain a dynamic graphical appreciation of the formula.

In protocol 3.4.10., the teacher seeks to advance the discussion by encouraging students to compare the two graphs and formulas and make an assertion. Two students recognise that it is radius squared that produces the curve. However, the teacher just agrees with the students and moves on, realising that the situation is too difficult for these students. This is an example of how the teacher still puts emphasis on explanation and justification, but it does not always work as sometimes the knowledge cannot be shared by the classroom community (Stylianides, 2007a).

The interpretation of the activity of exploring mathematical situations will be undertaken using the standard terminology of CHAT, before identifying points of contradictions.

Tools

Several mediating artefacts are used for the activity of explanation. Even though the activity of explanation is mainly focused on definitions, both psychological and material instruments influence the transformation process. To elaborate more, for the students to be able to engage in explaining centred on definitions, an exploration phase preceded where material instruments were employed. In this activity system the prominent tools used by the teacher are talk, the whiteboard as well as the interactive whiteboard.

Among the tools used by the students are terms, definitions and the formulas related to the area of triangles as well as the circumference and area of circle. However, it should be emphasized that regarding mathematical terms and definitions talk is initially employed so as to define them in such a way that are appropriate for the students before becoming tools for the classroom activity. To elaborate more, through the activity of explanation the classroom formed the following definitions: (i) the altitude of a point to any line, either horizontal, either perpendicular, either lateral, is the segment that from the point to the line it constructs a right angle and (ii) a circle is the shape that has a centre, has a circular circumference and all the points of the circumference are equidistant from the centre. Thus, the definitions, terms and formulas were the indented outcome in specific instances of the activity of explanation. Once the students appropriated the definitions, they become tools for them employed in explaining and justifying mathematical situations.

Regarding the area of triangles, the ruler also appears to constitute an important artefact, as the proper use of the ruler leads to proper formation of the altitude in a triangle.

Rules

Through the analysis of the episodes four sociomathematical norms (‘doing mathematics requires us to use precise language’, ‘doing mathematics requires us to justify our assertions', 'we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations' and 'we write coherent geometrical explanations') and two social norms were identified ('we solve problems using a variety of approaches', 'we raise the hand before answering the teacher's question').

To be more comprehensive, the teacher gave emphasis on the use of precise mathematics language and terminology. In addition, the teacher would not accept an answer (verbal or written one) unless it was complete. Instead, she would encourage students to develop both their verbal and written communication of their geometrical reasoning. Thus, the sociomathematical norm that was established in the classroom was that the students were expected to use the 'right' mathematical language, give definitions regarding shapes and terms and explain mathematical formulas. By embracing this norm, the teacher's objective is also that the students understand the importance of providing statements using the correct mathematical units after for example calculating the area using the formula required.

Another sociomathematical norm identified in the classroom is that the teacher encouraged students to compare the strategies used to solve problems looking for mathematically similarities and differences.

In regards with explanation and justification, the norm established is that the students are expected to justify their answer and explain the steps they followed in order to reach the conclusion/solution that was required from them.

The way the teacher intervened in the classroom in order to shape these rules of discourse, is further discussed in the component of the activity system related with the teacher's teaching and management of the classroom situations.

Division of labour

When the classroom discussion is concerned with definitions (altitude of a triangle, circle), the teacher intervenes in an attempt to encourage students to give explanations in terms of the properties of the concept discussed. The teacher is encouraging a fusion of both the figural and conceptual properties of the geometric concepts, the concept image and the concept definition.

In order for the norm ‘doing mathematics requires us to use precise language’ to be established in the classroom, the teacher follows a variety of approaches that guide students towards the endorsement of this norm. The teacher either makes this rule explicit (protocols 175, 3.1.3. and 3.2.1.), rephrases what the students say (protocols 3.2.1. and 3.3.4.), gives a negative feedback to a response that does not embrace the norm (protocols 2.1.12, 2.2.6. and 3.1.8.) and/or appraises the response that is correct (protocols 2.2.1., 2.2.3., 2.2.7., 2.3.1, 2.5.1. and 3.3.3.). Using precise mathematical language is also related with the norm ‘we write coherent mathematical explanations’. There are instances where the teacher, after accepting a student’s response, she would tell the class to write this explanation in the students’ textbook (protocol 2.3.4.).

Keeping in mind the above, it can be concluded that it is the teacher’s task to ask questions and the students’ task to answer these questions. Even though the teacher was the main ‘actor’ in orchestrating the mathematical argumentation, the students had opportunities to investigate either in groups or individually the validity of certain assertions.

Regarding explanation, the teacher’s questioning is crucial. Do the teacher’s questions encourage mathematical justification? Protocols 2.1.12.-13, 2.3.3., 3.1.9., 3.4.9., show the teacher asking students to justify their answer, whereas in protocol 2.3.5. the teacher is making this sociomathematical norm explicit.

By considering the above instances, it can be argued that by endorsing the socio-mathematical norm ‘doing mathematics requires us to justify our assertions’, mathematical justification is encouraged in the classroom. However, the fact that the classroom argumentation was, in many instances, followed by closing down the exploration opportunity provided to students, contradicts the aforementioned argument. This leads to tensions between activities. These tensions will be presented and discussed after portraying the activity system of explanation.

Object

It has been illustrated previously that the object of an activity system has several dimensions. The object for the teacher is associated with the curriculum content, behaviour regulation, the computer (development of technical skills) and proving.

Initially, one dimension of the object of this activity system is concerned with the content of the mathematics curriculum. That is, the object the teacher is working on is the development of students' understanding of the area of triangles as well as the circumference and area of circle. To achieve this, the formulas of these notions need to become tools for the students. Thus, an emphasis is put on these formulas. What is more, it can be argued that another dimension of the object is the preparation of the students for secondary school.

Keeping in mind the fact that the teacher is endorsing the socio-mathematical norm regarding explanation and justification, it can be argued that the object of the activity system of explanation is related with the establishment of this norm in the classroom. However, identifying the object of the activity system of explanation is not straightforward. That is, the object of explaining in the above incidents is not straightforward. It seems that there are competing objects related with explaining. To elaborate more, the object of explaining appears to be twofold; explaining mathematical procedures and explaining related with 'proving'. This leads to tensions regarding the object of the activity system. The teacher puts emphasis on the mathematical formulas.

By interpreting the classroom protocols related with the activity of explanation through the lens of Cultural-Historical Activity Theory, a snapshot of the activity system can now be graphically represented (see Figure 9.4. below).

Identifying points of contradiction

The tensions that have been identified and are being presented and elaborated on below are concerned with those things that have worked, in a certain way, against explanation. In addition, it is being noticed these emergent points of contradictions are related with the object of the activity system of explanation.

Explaining mathematical procedures/justification

It has been previously concluded that a primary contradiction seems to appear regarding the object of the activity system. This tension involves the two dimensions of the object of explaining, as identified through the analysis of the classroom observations.

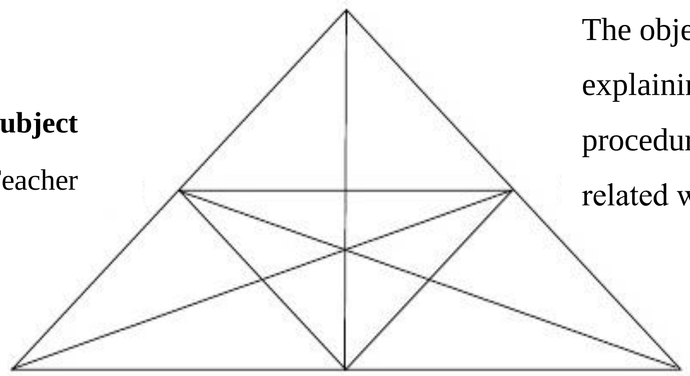


Figure 9.4.: The activity system of explanation

That is, putting emphasis on the explanation of mathematical procedures works against justification. Even though the teacher exemplifies that the students should be able to communicate their work and justify the steps followed to reach a conclusion, she puts more emphasis on procedures than on justification. When the goal set by the teacher is reached by the students, the teacher does not ask additional questions to engage students in a process of justification. Explaining why an assertion is true is not always requested. Nevertheless, this tension will be further analysed in a Section 9.3., where the system level will be contrasted with the classroom level.

Exploration/explanation

In discussing the way the teacher intervened in the specific aspects of the lessons related to activity of explanation, a tension between the activity of explanation and exploration has been identified. This contradiction can be considered a secondary contradiction between the object of the activity system and the division of labour, and simultaneously as a tension between two adjacent activity systems.

It can be argued that what restricted the process of explaining in some of the aforementioned incidents is the fact that the classroom argumentation followed the closing down of the exploration activities. As illustrated in the activity of exploration, closing down opportunities for investigation and exploration limits the opportunities for making hypotheses, testing these hypotheses, and explaining and justifying them. Analysis of the incidents where the exploring opportunity was closed down by the teacher reinforces this claim. The explanation process in these incidents was guided by the teacher.

Keeping in mind the identification of the above points of contradictions, it can be argued that the exploration opportunities were narrowed down by the teacher, because the teacher put more emphasis on explanation related with mathematical procedures.

Discussion

As illustrated in Chapter II, the notion of pre-proving is being introduced, as the aspect of mathematical reasoning that might nurture proving (see Section 2.4.). Keeping this in mind, the activity of explanation plays a vital role in directing mathematical reasoning towards the ultimate goal of formal proving. In addition, it has been illustrated that the teacher is providing opportunities for exploration that may lead to explanation and justification. However, it has also been shown that, even though these opportunities are provided, other constraining factors may clash with this effort.

To begin with, the teacher is encouraging the formulation of stipulated definitions. That is, even though the altitude and circle can be considered concepts known to the students, the teacher is expecting students to model them as new objects. However, defining the altitude of a triangle and formulating the definition for the concept of circle differs as an activity from formulating the mathematical formula for the circumference and area of circle. Constructing definitions is difficult especially when there is no a concept image. The DGE-based tasks are employed as tools for exploring these mathematical formulas, as active investigation and reflection promotes their development. Nevertheless, explanation and justification contribute to the defining activity. In the same way, when these definitions become operable for the students they contribute to the development of explaining and justifying in the classroom. This shows the interplay between defining and pre-proving activity.

Adding to the above, as giving mathematical adequate answers is a prerequisite for providing proofs, this approach that the teacher follows is positive for the establishment of proof in the classroom.

It is noticeable that the students are not independently trying to seek explanations and justifications. However, in order for the sociomathematical norm of explaining and justifying to be established in the classroom, the teacher has to explicate this process and demonstrate to the students what the nature of justifying is and why it is important.

A deeper consideration of the way the teacher acted in the aforementioned levels shows that the teacher was seeking procedural approaches in regards to proving more than focusing on explanation and justification.

The teacher’s behaviour in the above incidents is in agreement with her views towards the way mathematics language should be used in the classroom. During the interview the teacher exemplified that the students should be able, even informally, to talk about mathematical ideas. In addition to this, using accurate mathematical language and terminology will be more beneficial for students, especially when they go to secondary school. This can also be considered as a forward connection. Even though many aspects of the mathematical language being used in the primary classroom are still informal, accurate expressions prove extremely helpful when first moving to the secondary classroom.

What is more, the fact the teacher gives emphasis on the classroom discussion where the way the groups worked are presented and/or the whole class participates in the proving process of a mathematical situation, is in agreement with what she has stated in the interview. That is, the teacher considers the classroom discussion as an opportunity for presenting a proof and accepting it as a whole. The acceptance and understanding of a proof by the classroom community is in accordance with the definition of proof (Stylianides, 2007a; 2007b).

9.3 Bringing together the micro context of the classroom and the broader macro contexts

As demonstrated in Section 3.4., the activity of a mathematics classroom is influenced and dependent upon the structure and organization of the school and the Ministry of Education and Culture as wider educational contexts in the activity system. This is reinforced by the fact that only one series of mathematics textbooks is utilized in the teaching and learning of mathematics in the Cypriot primary classroom (see Section 6.2.2.4.). It has also been argued that, a combination of collaborative design approach and CHAT may lead to a thorough exploration of the research questions (see Section 5.2.).

In this section, the findings that have emerged through the analysis of the three phases of the study are being brought together. That is, the micro level of the classroom activity will be contrasted against the broader macro context as well as the collaboration with the researcher. To be more comprehensive, the activity of exploration and explanation will be explored in the light of the analysis of the official documentation. Going further, these activities will be discussed against the emergent themes from the mutual collaboration with the researcher so as to gain a deeper understanding regarding what drives the teacher's teaching decisions (see Section 8.4.2.).

9.3.1 Activity of exploration VS broader macro context

As illustrated in section 6.2.1. the goal of exploration as emerged through the textbook analysis is related with inferences made based on small cases of examples. This is in contrast with the research literature that explores the ways exploration can be approached in the classroom. As illustrated in the activity of exploration, differing ways are presented in which exploration may facilitate the construction of proof.

Concerning the opportunities provided by the teacher for exploration of mathematical situations, it becomes obvious that the teacher moves beyond the textbooks. However, even though the teacher provides opportunities for exploration that support explaining and justifying, the way she intervenes shows that she does not detach herself from the textbook. That is, the educational goals need to be reached. This argument is also reinforced by the fact that the teacher felt the need to cover the pages in the student's textbook. This is more obvious where the activities explored in the DGE were also presented in the textbook. In these instances, even though the students would write these conclusions in their notebook, they would also repeat this action in the textbook.

Concerning exploring mathematics, it can be concluded that the teacher recognizes the gap that exists between primary and secondary school mathematics. Even though opportunities for making these connections are not provided in the textbooks, not encouraged in teacher's guidance book, and not made explicit in the official documentation, the teacher makes this forward step.

Regarding the exploration of DGEs, it is acknowledged that in the curriculum only one statement is made regarding employing technology in the classroom. However, it has also been illustrated that the teachers are encouraged to integrate technology in the classroom (see Section 6.2.2.2.). The teacher is incorporating technology in the classroom. However, when allocating the didactical periods concerning mathematics, the integration of technology was not taken into consideration. Does the integration of technology in the classroom affect the successful coverage of the curriculum? If yes, is this one reason why the teacher would decide to close down an exploration activity?

An additional point of discussion regarding technology concerns the factors that influence the teacher to utilize technology in her teaching practice. Even though it is not explicitly stated that it is mandatory to use technology in the classroom, one cannot disregard these expectations. However, the fact that the teacher puts time and effort for class preparation and management shows that the teacher's main objective was to enhance instruction and learning. This is also shown in the initial interview where the teacher embraced the utilisation of technology only if it helps attain instructional objectives.

9.3.2 Activity of explanation VS broader macro context

This section is concerned with the impact of the broader macro context on the activity of explanation.

It has been illustrated that the mathematical activities related with geometry require students to use mathematical language through the definitions or properties of the shapes. This, to an extent, is in accordance with the activity of explanation. A theme that emerged in the activity of explanation is the explaining based on definitions. However, what distances the activity of explanation from what is required from the curriculum is the fact that the teacher engages students in a process of formulating terms and definitions as well as gives emphasis on explanations that are developed based on the conceptual aspects of the definitions and/or the shapes. While the official documentation consists of descriptive definitions, the classroom's goal is the formulation of stipulated definitions.

It can also be argued that the emphasis that is put on definitions has an impact on the socio-mathematical norm established in the classroom regarding definitions. A great part of the lessons is devoted to giving definitions using precise mathematical language. This is in line with the mathematics curriculum.

What is more, given the fact that the curriculum does not specify the role of proof and proving in primary mathematics, one can speculate that it is up to the teacher to approach proof and proving in the classroom. Firstly, this is in contrast with what is being explicated in the research literature. Secondly, this is also in contrast with what the teacher actually does in the classroom. The fact that the teacher makes use of the word 'proof', and her actions that follow this usage, have an impact on the way the students appreciate proof and proving. An illustrative example of how students' perceptions may be shaped from the teacher's actions is protocol 3.4.7. In this protocol, the connection the students make regarding proving is that one proves with letters.

These tensions between the official documentation and the teacher's classroom practice also shed light into the tensions that occurred in the activity of explanation. The instances where the teacher would close down an activity, or not seek further explanations from the students, may be interpreted as a struggle between what is requested from the curriculum, and the desire to further provide opportunities for explaining and justifying.

9.3.3 Classroom activity VS teacher/researcher collaboration

It has been established that collaborative design approach functions as a Trojan horse, a means of gaining access to the teacher's objectives (see Section 5.2.). Additionally, in discussing the challenges faced through the process of collaboration between the teacher and me, it was argued that time constraints, the role of the teacher in the learning environment, as well as the trust the teacher shows in her long teaching experience, may lead to a gap between the two collaborators. In this section, these tensions are further analysed by bringing into the discussion the activities of exploration and explanation as emerged through the data. That is, the way the teacher intervened throughout the lessons will be contrasted against this collaboration, so as to gain a deeper understanding regarding what drives the teacher's teaching decisions.

One challenge that emerged through our collaboration was related to the way the DGE-based tasks were utilized in the classroom (see Section 8.4.2.). Even though an agreement was reached regarding the goals underpinning the design of the tasks (see Section 8.2.1.), it has been pointed out that the teacher exploited specific aspects of the tasks in different parts of the lesson. That is, the graphs were not initially exploited, until the concluding lesson related with circle. One may argue that the teacher made this decision in order, perhaps, to satisfy the researcher. However, the teacher devoted enough time to fully explore the tasks and engage students in a process of explaining and justifying. The question regarding the opportunities that perhaps were missed by not fully exploiting the tasks from the beginning remains. By considering the fact that the teacher was making forward connections, one may also argue that the teacher fully exploited the DGE-based tasks to meet her own objectives. However, making connections did not act as an additional thing that the students could know. The teacher embraced the idea of investigating in a different way the relationship that exists between the radius, the circumference, and the area of circle.

What is more, the fact that the teacher considered it necessary to provide students with a handout led to an additional emerging tension. During our informal discussion, the teacher stated that the students were noisy and asking technical questions. This, for the teacher, was an indicator of the students not properly engaging with the task. This, for myself, was an indicator of the students engaging with the task. Nevertheless, it was agreed to provide the students with a handout (protocol 3.1.17.). While I was feeling that this action was against my objectives as a researcher, an opportunity arose to exploit this decision to fully understand the teacher's objectives. It was noticed that, even though this handout was given to the students for the first task, it was not given for the DGE-based Task 2. Analysis of the way the teacher intervened, as well as what she stated during the informal discussion, shows that the teacher gradually felt more comfortable in giving more freedom to students to work independently. Thus, she did not feel the necessity to provide an additional artefact to aid the students' activity. Therefore, it can be concluded that her unfamiliarity with this environment guided her in making the original decision to use a handout.

Nevertheless, while the above comment shows the teacher gradually detaching herself from the concern of losing authority while exploring the DGE-based tasks, this is in contrast with her closing down the exploration of mathematical situations that missed the opportunity for the students to either initiate a solution or test a hypothesis made.

An additional inference that can be made is related to the negative connotation of the word 'play', as used by the teacher. This comment made by the teacher may be associated with her concern regarding the ‘openness’ of the task. The teacher would make this negative remark about this playing with computers. As this comment was made mainly when the students were exploring the DGE-based tasks, it can be argued that the teacher valued exploration more when this was to a certain degree guided.

Regarding the activity of explanation, the emphasis the teacher put on definitions, was also evident during our collaboration, as well as during the initial interview.

9.4. Summary

This chapter has been devoted in conducting a retrospective analysis on the entire data set generated from the three phases of this study. The findings of the three phases of data collection have been analysed by employing the main aspirations of Cultural-Historical Activity Theory alongside the literature that informs the study.

Systematization of the research findings enabled the illumination of two broad activities of action; (i) the activity of exploration and (ii) the activity of explanation. By focusing on a focal instance where the subject is the teacher, the activity system of explanation and exploration was portrayed through the lens of CHAT. This analysis demonstrated that exploration and explanation are interrelated in a way that has impact on the way that proving is constituted in the classroom. Analysis of the activity of exploration and explanation has also indicated the emergence of a range of tensions emerging both within each activity as well as between activities.

By interpreting the classroom activity through the lens of CHAT, the situation of the classroom regarding proving activity was further scrutinized by contrasting the outcome of the activity with the social context in which it emerges. Instances of both congruence and diversion exist between the micro and macro level.

The impact of the aforementioned themes on the way proving activity is constituted in the classroom will be explicated in the concluding chapter of this thesis, which further discusses what the analysis revealed in relation to the main research questions of the study.

CHAPTER X

DISCUSSION AND CONCLUSION

10.1. Introduction

Chapter X concludes this thesis by reviewing the purpose of the study conducted and presenting its key outcomes. That is, this chapter focuses on discussing the findings of this research according to the main themes of the research questions. This synthesis of the findings follows, which leads to an outline and articulation of the main contributions of the research. In doing so, links are made between the research findings and those of the key studies outlined within the review of the literature. In addition, it discusses its limitations, summarises the issues raised that are worthy of further investigation, and outlines its implications for educational policy and practice.

10.2. Returning to the research questions

My overriding research question, as stated previously, is 'How is the activity of proving being constituted in the Cypriot classroom for 12 year old students?'

In the light of my discussion of Cultural-Historical Activity Theory, I previously restated this research question as three sub-questions (see Section 4.4.). That is, within the Activity Theory approach, the purpose of the study was threefold: to explore the object of developing proving in the classroom, to investigate the emergence of possible contradictions, and identify the way the classroom community engages with the classroom proving activity. In the light of the insights of the analysis of the findings, these three questions can now be more developed as below:

1. What is the object of developing proving in the classroom?

   Is this object shared by the students and the teacher? If no, how does a shared object develop?

   How is the object of developing proving within the classroom supported or hindered within the activity system?

   How do changes in the components of the activity system influence the motivation of students in approaching the object of the activity?

2. Are any types of contradictions identified?

   Do any of these changes in the components of the activity system lead to tensions? If yes, are these contradictions solved and how?

   Do any contradictions occur across systems?

3. How does the subject engage with proving in the classroom?

   How does the subject engage with DGEs and other instruments?

   What sense does the classroom of students make of proving?

Before proceeding, it should be made explicit that these sub-questions provide an orientation towards discussing the main research questions of this study. That is, as these main themes will be elaborated through the lens of CHAT, it should be noted that some arguments may fall into more than one research question. Despite this, a coherent and clear line of reasoning will be provided through each research objective this study aimed to investigate.

10.2.1 What is the object of developing proving in the classroom?

As stated in Section 4.2. the object of a collective activity is something that is constantly in transition and under construction, has both a material entity and is socially constructed and its formation and transformation depends on the motivation and actions of the subject indicating that it proves challenging to define it. In this respect, identifying the evolving object of the activity *'entails a dialogical interaction between aspects of the subject's personal experience and his/her* *relationship to the community of significant others with whom the object is pursued, and cultural-historical properties of the object. In other words, an individual's construction of an object is both facilitated and constrained by historically accumulated constructions of the object'* (Foot, 2002, p.135). It should also be taken into consideration that a distinction may exist between *'a generalized object of a historically evolving activity system'* and a specific object as it appears to a particular subject at a given moment (Engeström et al, 2003, p.181). Keeping in mind the aforementioned, the discussion that will follow pertaining to the identification of the evolving object of proving in the mathematics classroom begins by establishing its origin.

In Sections 8.4.1. and 8.5., it was made explicit that the systematization of the classroom data led to the identification of two broad activities of action: (i) the activity of exploration and (ii) the activity of explanation. Furthermore, it has been illustrated that pre-proving activity is closely connected with exploration and explanation. That is, those aspects of reasoning that appear to have the qualities of proving, even though they may not be proving in themselves, entail exploration and explanation that provide a point of reference for proof production. Correspondingly, the object of developing proving in the classroom is related with these notions. Furthermore, it has also been established in Section 9.2. that at given times of the mathematics lesson the object is also related with curriculum content, behaviour regulation as well as acquiring technical skills on the computer. How the object of the activity of exploration and explanation corresponds to the overall object of pre-proving activity, and is shaped in the light of the aforementioned intermediate goals, is analysed below.

Throughout the classroom activity, we have seen the teacher providing opportunities that may promote the construction of a mathematical proof (see Section 9.2.1.). We have also seen the teacher attempting to make explicit those crucial elements that should be taken into account when attempting the proof of a statement. Explorations and investigations have a role to fulfil in mathematics teaching, contributing to achieving specific objectives. Exploring triangles and circles entails a process of discovering relationships, formulas and definitions, making connections with other parts of mathematics, and thus, explaining and justifying procedures, assertions and ways of working. Additionally, explanation entails a process where mathematical definitions are being formulated. It also entails a process where the sociomathematical norms related with explanation and justification need to be established in the classroom. Students are expected to use precise mathematical language when communicating their ideas as well as when writing coherent geometrical explanations, clarifying aspects of their mathematical thinking to others, as well as justifying for them the validity of a statement. Thus, exploration and explanation is also directed to the fusion between the visual image and the properties of the figural concept and the concept image and concept definition of the concepts under development.

Furthermore, through these two broad activities, it can be argued that the classroom also attempts to make connections with the conceptualization of proof as proposed by Stylianides (2007a; 2007b). Even though there is no clearly defined connection with proof that is encouraged by the proposed mathematics curriculum, the classroom is engaging in a process through which the characteristics of this conceptualization can be identified. Analysis of the classroom episodes indicates several modes of reasoning employed in differing aspects of the lesson, and the 'proof' must be accepted and shared by the classroom community. It can also be argued that the definition construction process, with which the class engages, results in a set of accepted statements that can be utilized in a proving process (see Section 2.3.). Thus, pre-proving activity is also oriented towards those elements that are described as crucial in endorsing this conceptualization of the meaning of proof in school mathematics.

An additional remark can be made regarding the way the conceptualization of proof is related with pre-proving as manifested in the classroom activity. This connection lies on the activity of defining. Can this conceptualization be identified throughout the process of formulating mathematical definitions of concepts and formulas? To elaborate this question, I return to the incidents related to the definition of the altitude in a triangle and circle. The classroom discussion was initiated by a question (protocols 2.1.10. and 3.1.1.). This can be perceived as the starting point of the classroom argumentation. The ensuing argument is the formulation of the definition. The statements are expressed in a form that is known and is within the conceptual reach of the community. The outcome is a definition that is accepted by the classroom community. Even though the activity of formulating a definition is not proving in itself, it has the qualities of proving. This adds to the relevance of formulating definitions in promoting proving in the mathematics classroom.

In the light of the above information, it can be argued that the object has multiple manifestations for the participants engaged in the activity. This is congruent to the orientation towards pre-proving activity. Exploration is related with the pre-proving activity when information is revealed through the immediate feedback students get from the manipulation of objects. Pre-proving activity emerges while the students discover the definitions and formulas, as through discussion the generality and applicability of these concepts is being accepted by the classroom community. The discussion pertaining to the discovery and further exploration of these definitions and formulas entails a process where the aforementioned sociomathematical norms are being negotiated. The students cannot rely only on perception as a definition in this particular classroom is considered more what a concept really is rather than a description of how a concept is used. As proofs begin with an accepted set of definitions and axioms, it can be argued that ultimately all proofs depend on the underlying definitions and the earlier results derived from these definitions. Thus, understanding and explaining these definitions is a prerequisite when approaching a proof. Thus, pre-proving activity can be identified when explaining definitions and mathematical formulas. Considering this, making forward connections provides more information and knowledge about the axiomatic system in which the classroom community is working. Forward connections also strengthen the formulated definitions.

Consideration of the aforementioned manifestations of the object leads to the conclusion that the object of developing proving in the classroom is exploration and explanation that provide a point of reference for proof production. By identifying the object of developing proving in the classroom, the activity system of the classroom can now be portrayed. In the central system of pre-proving activity, explanation and exploration, which are interrelated, constitute its focal elements. Thus, the activity of exploration, which encompasses a nest of three distinct activities (exploration of mathematical situations, exploration for supporting mathematical connections and exploration of DGE) and explanation constitute sub-systems to this central activity. This is in accordance, with the idea of networks of activity, a characteristic of the third generation of CHAT (see Section 4.2.). That is, activity systems interact and overlap in such a way that the elements of an activity system are always produced by some other activity. By keeping this in mind, as the activity of exploration and explanation are interrelated, the elements of the central system of pre-proving activity are produced by the elements identified when portraying snapshots of the activity systems of both exploration and explanation (see Sections 9.2.1. and 9.2.2. accordingly). Table 10.1. provides a snapshot of the activity system of the classroom. A tabular form rather than the familiar triangle is being used, as it allows more details to be included. It is considered important to make explicit once again that the details were derived from the classroom observation data which were supported by the data from the discussions with the teacher as well as from the analysis of the official documentation and the school system.

Before proceeding, it should be clarified that the object should not be mistaken with the outcome of the classroom activity. From the teacher's perspective, the community of this activity system is engaged with the content of the mathematics curriculum, and this is the intended outcome of the classroom activity; the students, need to demonstrate attainment of curricular objectives, that is, developing understanding about triangles and circles.

A consideration of the discussion that followed the analysis of the activity of exploration and explanation indicates that the object of proving, even though it is supported by the teacher and other components of the activity, is also hindered within the activity system. In a broad sense, the activity of developing proving in the classroom is supported by the exploration activities provided by the teacher, the rules established in the classroom as well as the division of labour. It is also shared by the students. Even though the data regarding the students' perspectives and motives is rather limited, the way they engaged in the classroom discussion and collaborated with their classmates is an indication that they share the teacher's object. While the teacher would interpret students' exploration as 'play', this, simultaneously, indicates that the students were engaged in the activity. Despite the tension the teacher was encountering due to the duality of the object, the students did not face this tension and were engaging towards reaching the object.

Table 10.1.: The activity system of the classroom

Subject	Teacher
Tools	Prominent tools used by the teacher: talk, whiteboard, IW, DGS, mathematical definitions and formulas, exploration opportunities Teacher's pedagogical content knowledge and awareness of where the students are in terms of knowledge
Rules	**Social norms:** 'we solve problems using a variety of approaches', 'we raise the hand before answering the teacher's question', 'when we use computers/artefacts we work in pairs', 'individual ideas are welcome, respected and valued'. **Sociomathematical norms:** 'doing mathematics requires us to use precise language', 'doing mathematics requires us to justify our assertions', 'we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations', 'we write coherent geometrical explanations'. Conventions set by the school authorities (including the way the students are assessed as well as the homework requirements within the school).
Division of labor	The teacher has the authority to set tasks. Students are expected to engage with the lesson.
Community	Classroom community (teacher and students); school community (including other teachers and students); wider educational community (including The Ministry of Education and Culture); wider social community.
Object	Exploration and explanation that provide a point of reference for proof production
Outcome	Some evidence of achievement of object

This finding may also be supported by the following statement. Throughout the classroom observation it was noted that most students would devote a few minutes of their break time to either do the appointed homework, and wait to get feedback from the teacher, or ask further questions if what was required from the homework was not clear to them. It can also be argued that it might be that the students wanted to avoid having any homework, or felt pressure in being successful. However, expressing their enthusiasm and helping those classmates that needed assistance contradicts the above argument. The students were engaged in the classroom activity and thus appeared eager to engage in developing their understanding.

Nevertheless, the object is simultaneously hindered due to the dichotomies, tensions and conflicts identified in Chapter IX. The discussion that follows regarding emerging contradictions further elaborates this finding. It will also provide further information in understanding the way proving activity is constituted in the classroom.

10.2.2 Are any types of contradictions identified?

Throughout Chapter IX, it has been made obvious that a number of tensions have arisen in different aspects of the mathematics lesson, the classroom level as well as the system level. In elaborating these tensions, the notion of contradictions proposed by Engeström was taken into consideration. This analysis though did not make explicit whether these tensions were manifestations of contradictions. Thus, it did not unravel whether these systemic imbalances emerged within or/and between components of the activity system, or/and across entire activity systems. In answering this research question more thoroughly, I now return to the four levels of contradictions as have been portrayed in CHAT (see Section 4.2). By taking a deeper look at these tensions, it is my intention as a first step to unearth their origin and categorise them, and further, to explore whether they were resolved and if so, how. Even though a functional correspondence there exists between them, for the purposes of the discussion, they are presented and described as distinct from each other.

10.2.2.1. Primary contradictions

A primary contradiction resides on each component of the activity system. As it is impossible to have direct access to the primary contradiction due to the particularity of this study, it is approached through its manifestations in the discourse and actions of the participants. One primary contradiction that has emerged is inherent in the component related with the object of the activity system. This contradiction applies for both the activity systems of exploration and explanation. In the activity of exploring as part of pre-proving, it has been illustrated that the object for the teacher is related with exploring triangles and circles. At a first glance, this object seems to be clear and distinct. However, this object is multifaceted. To be more precise, the object for the teacher is related with the investigation of situations that lead to conclusions related with the aforementioned parts of the mathematics curriculum. The teacher on one hand understands the importance of providing enjoyable exploring opportunities that keep students' motivation and interest to engage with the problem. As a result, the teacher provides opportunities that can be approached by the students in their own way. On the other hand, students, through the exploration of these opportunities are expected to reach those conclusions regarding triangles and circles as pre-determined by the teacher. The two poles of the object lead to a constant struggle in the teacher's everyday practice. This primary contradiction that emerges within the object is manifested through the other components of the activity system, as it is mediated by actions. That is, even though in the division of labour, the teacher's role is mediational, there are instances where the teacher is closing down the exploration activity. The teacher, due to this multifaceted object, is faced with the play/learn dichotomy and thus the play and the planning paradoxes. That is, students' free exploration may lead to paths other than those expected by the teacher. This initially shows that the students share the teacher's object. Thus, the object related with exploring is being reached. However, if the exploration moves away from the teacher's motive, the teacher will inevitably close down the exploration opportunity and guide the students towards the exploration that leads to the conclusions that satisfy her. Time management and the pressure of the coverage of the curriculum further highlight this tension. Inevitably, even though closing down the exploration is necessary, the object will not be met because of this contradiction.

However, it should be noted that this contradiction is occurring in the activity where exploring is related with the exploration of mathematical situations. The duality of the object seems not to interfere with the exploration for supporting connections.

This is due to fact that even though making mathematical connections requires exploring, this exploration, even though it entails a degree of openness, simultaneously, it is strictly directed towards the specific connection. This, however, is not the case when the opportunity for a connection is originated by the student and not the teacher (for instance see protocol 3.4.8.). Nevertheless, the purpose for these forward connections, as argued by the teacher, was to make links between primary and secondary school mathematics. While this is crucial, would this activity differ if these forward connections were explicitly related with a fusion between the concept image and the concept definition? It might be that making this connection clear, would lead the teacher to providing differing opportunities to the students for making connections, as these connections would be related generally with mathematics, and not specifically focused on the transition to secondary school mathematics. However, this is dependent on the teacher's content knowledge.

In a similar way, the object of the activity of explaining as part of pre-proving, which is related with understanding definitions pertaining to triangles and circles, is multifaceted and thus leads to a primary contradiction. The object for the teacher in this activity is not straightforward. This duality of the object which has been identified in Section 9.2.2. is now further elaborated. Understanding definitions entails a process of argumentation which involves explaining, the goal of which is to move from a definition based on perception to a definition that involves properties. In this process, justification of statements is also required. This inherent duality of the object may also be related to the dichotomy explaining mathematical procedures/justification, which has been identified in Section 9.2.2. However, this needs further clarification. On one hand, the teacher's object is related with engaging students in formulating definitions (of concepts and formulas) in the same way that mathematicians do. In order for these definitions to become operable for the students, they need to focus on the properties required. Thus, this process includes a continuous interplay between the concept image and the concept definition, promoting the characteristics of definitions as identified in Section 2.4.1. and making the distinction between 'ordinary' and mathematical definitions. Even though the above facilitate the justification of statements, a tension within the object arises. That is, ensuring that the classroom engages in the construction of stipulated definitions and that these definitions are not just descriptive for the students seems to be competing with moving to justification based on these definitions. Furthermore, even though the teacher is embracing this object, she is simultaneously faced with the play/learn dichotomy due to the play and planning paradoxes, influencing the way she intervenes while this process of explaining and justifying develops in the classroom. If the students' argumentation leads to a discussion that diverts from the teacher's object, the teacher may decide not to take advantage of the opportunity that arises for further engaging students in explaining and justifying. This is related with the instances where the students' comments are within the conceptual reach of the classroom community. As a result, the teacher is faced with a dilemma. To what extent does she feel comfortable to give students the freedom to guide the classroom discussion? These two poles lead to an inherent contradiction within the object. Correspondingly, this primary contradiction that emerges within the object is manifested through the other components of the activity system.

The duality of the object is manifested as a primary contradiction in the rules established in the classroom. Even though the sociomathematical norms encouraged in the classroom support the pre-proving activity, at the same time the social rule of raising the hand hinders the free expression of mathematical ideas and argumentation. That is, allowing and encouraging students to socialise and discuss mathematical concepts within small groups and within the whole class may have an impact on the quality and development of classroom argumentation. Furthermore, the above tension leads to a primary contradiction within the rules when using precise mathematical language seems to be more valued than justifying assertions.

This primary contradiction is also manifested in the tools component of the activity system. This is more obvious in the instances where the classroom activity shares both the objects of exploring and explaining. This will be elaborated on subsequently. Nevertheless, it should also be noted that this contradiction is not occurring when the students are asked to give a definition for the circle and the altitude in a triangle. Throughout these instances the students are engaged in actually formulating definitions.

It is acknowledged that the primary contradiction, unlike the other types, generally remains unresolved (Engeström, 2001; Foot and Groleau, 2011). The primary contradiction cannot be eliminated due to the fact that it is fundamentally embedded within each constituent component of the activity itself (Engeström, 1987). Nonetheless, the primary contradiction constitutes the fundamental threshold for the other types of contradiction to be conceptualised. Thus, understanding the contradiction within the teacher's object of the classroom activity is crucial as it gives an indication of the manifestation of the contradictions that the teacher encounters in the classroom context, and, subsequently, what is attainable in terms of innovation and change (Engeström, 2001; Foot and Groleau, 2011).

10.2.2.2. Secondary contradictions

Secondary contradictions occur between the components that reside at the corners of the triangle of an activity system: the tool, rules and division of labour. These secondary contradictions are related with the systemic tensions that are occurring between the object of the activity. Unlike primary contradictions, a manifestation of secondary contradictions may be resolved when new elements are incorporated into the activity.

Introducing a new mediational tool in the classroom led to the emergence of secondary contradictions. That is, the primary contradiction is translated into a secondary contradiction in which the two poles of the object are opposed. Before proceeding, it is considered important to remind the reader that as the teacher would in general employ DGEs in her teaching, this secondary contradiction resides on the design of the DGE-based tasks. This is also in accordance with the rule established in the classroom where both the teacher and the students are learning together from the computers. The DGE-based tasks were co-developed by the teacher and the researcher by taking into consideration the teacher's objects. Thus, it can be argued that these mediational means are compatible with the object of the activity and the teacher's practice as a whole. However, the inherent contradiction within the object of exploration indicates that the tool is not corresponding to the teacher's object. As a result, a secondary contradiction between division of labour and tool as well as between the rules and tool component of the activity system occurred. The teacher's concern on how the tool would affect the object of the activity led her to deciding to hand out a worksheet to the students. Providing a worksheet to the students is arguably an action not consistent with attention to purpose and utility in task design (Ainley et al, 2005). Of course, in general it would depend on the nature of the worksheet but a danger is that a worksheet might be too prescriptive and constraining to facilitate the level of ownership a student needs in order to engage in purposeful activity around the task. The fact that some pairs felt that they had to follow in a strict way the steps in the worksheet and that if an answer was not provided in the worksheet, they could not move to next part of exploring the task strengthens the above statement. This observation may also be related with what Berge et al (2004) describe as complexity in tasks (discussed in Section 5.5.). Nevertheless, offering a worksheet was this teacher's way to resolve the planning paradox. Even though providing a worksheet to students automatically might restrict the openness of the exploration task, this approach, from the teacher's perspective, meant that the students wouldn't 'play'. Instead, they would explore the environment in such a directed way that a discussion could follow regarding the mathematical relationship under investigation. The teacher used the worksheet as a reassurance that the students would work towards achieving that component of her objective.

As a result, a tension occurred between the tool and rules. The students had to assume a new rule as they were not accustomed to sharing a worksheet in pairs. The tension between tool and rules was resolved by the students. That is, the students established a new social norm related with the collaboration between pairs of students when sharing a worksheet (see Section 9.2.2.3.).

The play/learn dichotomy also reflects the contradiction related to the introduction of the new mediational tool. The action of closing down the exploration activity indicates that the motives underlining the design of the tasks could not be reached. Inevitably, the outcome of this exploration activity was not the one intended. Regardless, this contradiction was resolved by the teacher (see Section 9.3.3.). The classroom fully explored the tasks as a way to make forward connections with other areas of mathematics. This, perhaps, reveals another challenge that the teacher faces regarding definition construction. Other than the fact that definition construction is difficult, definition construction for the teacher may only be related with exploration closely linked with the term or formula that is to be formulated. Thus, due to the duality of the object, bringing together exploration that is directed towards revealing information about the specific mathematical formula with the process of formulating this formula may be difficult for the teacher.

The above discussion constitutes another indication that the teacher does not reject technology. The concerns of the teacher related with the integration of technology in the mathematics classroom have been identified (see Sections 6.3.2. and 6.4.2.). Would a consideration of the emerging tensions due to the duality of the object of the activity direct towards rethinking the difficulties teachers encounter when employing technology in their teaching practices? I would consider that it is essential that teachers have an opportunity to make sense of these tensions (if any). There is a danger that manifestations of these tensions that are not followed by reflection may reinforce and establish the teachers' concern influencing their confidence in utilizing technology.

10.2.2.3. Tertiary contradictions

Tertiary contradictions appear between the object of activity in a central activity and the 'culturally more advanced' activities. Compared with other studies investigating tertiary contradictions, this study takes a rather different approach in discussing the tertiary contradiction that has emerged within this particular activity system. The objective for introducing a new tool has thoroughly been described in Section 5.2. That is, the collaborative task approach assisted in exposing the teacher's object. Even though the new mediational tool resulted in new actions being brought into the activity for the resolution of the secondary contradiction, this did not affect the object of proving as a cultural historical activity system. Thus, when elaborating on tertiary contradictions, this discussion focuses on a possible clash between the micro and macro level of this activity system.

Analysis of the micro activity system as a classroom which is nested within the system level such as the institutional level in which the school is part of, as well as the cultural-historical level which is involved with the available research literature results into identification of a contradiction. A tertiary contradiction appears across these three levels. While tensions have been identified between these levels (see Section 9.3.), a tertiary contradiction occurs due to a differentiated object.

The two poles of the object of the central activity of the classroom related with pre-proving activity will unavoidably clash with the object of pre-proving activity as identified in the system level. Initially, a contradiction between the classroom level and the institutional level resides in the fact that there is no clear identification of an object related to proving. That is, analysis of the official documentation indicates a general object of mathematical activity that is not necessarily in accordance with the object of the teacher related to pre-proving activity. Furthermore, there is no formal requirement regarding definitions. This is not in accordance with the teacher's practice where definitions play a vital role. It is acknowledged that one may argue that a consideration of the educational objectives, as pre-determined by the mathematics curriculum, leads to the conclusion that the outcome of the teacher's practice is the one intended by these objectives. However, in order for this to be achieved, the pre-proving activity is narrowed down. Thus, for instance, providing answers based on definitions and properties of shapes clashes with providing explanations based on the conceptual aspects of the definitions and the shapes (see Section 7.6). In a similar way, this tertiary contradiction concerns the cultural-historical level as well. Even though at a first glance the teacher's objects seem to be in line with the established research literature related with proving, the dilemmas the teacher needs to confront, as well as the ambiguity of the notion of proving existing at the institutional level, clash with the cultural-historical level.

One may argue that the teacher's actions show that she is, to an extent, aware of this contradiction and that she attempts to resolve it. She is trying to overcome the gap existing between primary and secondary school mathematics and moves beyond the mathematics textbooks. However, the fact that she is faced with paradoxes and dilemmas is an indication that this contradiction may be at moments suppressed within the activity system, due to the teacher's intentional actions, but not resolved.

A consideration of the above rationally points to the inference that the advanced form of the central activity object is not yet the dominant form of the activity. Thus, it can be argued that a first step towards a unification of these activities should be the resolution of the tension that exists within the macro system. Would providing a mathematics curriculum, which defines its object concerning proving and defining activity by incorporating crucial elements from the research literature, lead to a desired outcome? This question will be elaborated thoroughly in the concluding paragraphs related to the identified contradictions.

10.2.2.4. Quaternary contradictions

Quaternary contradictions exist between the central activity system and adjacent activities. These neighboring activity systems are called 'instrument-producing activity, subject-producing activity, rule-producing activity and object-activity' (see Figure 4.5.). In this central system of pre-proving activity, explanation and exploration, which are interrelated, constitute its focal elements. At the same time, the activity of exploration and explanation constitute sub-systems to this central activity which include the aforementioned neighbouring activity systems. Despite this fact, concluding that the exploring and explaining that occurs within these activities falls within the central activity system would be rather simplistic. On the contrary, the fact that there are inherent contradictions within explaining and exploring contradicts the above statement. Thus, manifestation of the contradictions within these activities leads to a clash firstly between them and, subsequently, with the way pre-proving activity occurs in the classroom. As a result, a quaternary contradiction arises.

To be more comprehensive, in answering research question 1, it was established that the object of the central system of pre-proving activity is related with exploration that leads to explaining and justifying for a specific part of the mathematics curriculum.

If there are instances where the two activities are in line with each other, then they are also corresponding to the central pre-proving activity. That is, the two poles of the activity of exploration are consistent with the two poles of the activity of explanation. The exploration of the mathematical situations leads to conclusions that are related with understanding definitions. As a result, exploration leads to explanation and justification. In these situations, the teacher has a mediational role and the students are the main actors. However, closing down the exploration has an impact on how explanation and justification are established in the classroom. By considering the analysis of the incidents where the teacher closed down the exploring opportunity it seems that explanation is developed to reach the teacher's objectives by, simultaneously, establishing this as a sociomathematical norm in the classroom. The tension between the two activities was identified in Section 9.2.2. Additionally, this clashes with the overall pre-proving activity. Closing down an exploration opportunity may have a negative impact on the students' ability to approach the construction of a proof. Referring to exploration as 'play' may also have a negative impact on students' confidence in relying on their intuition when exploring a situation. Furthermore, this may have an impact of the formulation of mathematical definitions. Even though it is difficult to argue how much 'enough' time should be provided to students to explore a mathematical situation, not having the opportunity to create a concept image or even relate the concept image to the definition under construction, entails the danger that the definition to be descriptive for the student. Nevertheless, exploring the sequence of the observed mathematical lessons points to an activity where despite these challenges, the students have the opportunity to evaluate the formulated terms, formulas and definitions and provide, eventually, more generalizable definitions.

The above discussion regarding contradictions reveals the value of this concept in understanding systems of activity. By identifying the manifestation of contradictions through the materialized tensions, a holistic view of the phenomenon under investigation emerges. That is, contradictions do not serve as points of failure or problems that need to be fixed. *'Rather than ending points, contradictions are starting places'* (Foot, 2014, p.337). It is also accepted that not all emergent contradictions can be resolved simultaneously. While a resolution exists for some contradictions, others are suppressed. It has been established that the primary contradiction is continually present, surfaces in the teacher's everyday practice in various forms and is foundational to the other levels of contradiction. However, since the primary contradiction remains, the discussion should be centred on the means that the teacher can turn to for a possible and fruitful resolution of the contradictions that emerge in the other levels.

Elaboration of the emergent tertiary contradictions led to asking whether a possible balance between the macro system would be an aid in the resolution of the tensions manifested as contradictions in the micro level. Due to the way the aforementioned forces impact on the classroom activity, providing a straightforward answer will not be an easy task. Undoubtedly, as it has been exemplified in the review of the mathematics education literature in Chapter II, proof and proving might be encouraged in all school levels. This indicates that exemplification of the role of proof, explanation, exploration and definitions might be included in the mathematics curriculum and the relevant curriculum material. Perhaps, a clear connection between the aforementioned would relieve, to an extent, the teacher from paradoxes. That is, knowing that the above aspects of mathematical reasoning might not be necessarily competing with each other and may be the way for a resolution of the play and planning paradoxes, as the purpose and utility underlining the task design would not clash with the object of the central activity system (Ainley et al, 2006).

Adding to the above, the fact that the official documentation is implemented in the classroom by the teacher points once again to the crucial role of the teacher. Specifically, the above further highlight the role of the teacher's knowledge about proof in mathematics teaching (elaborated in Section 5.6.). This study adds to the literature related with this aspect of teacher's content knowledge. Keeping in mind the key finding of this study related with definition construction as part of pre-proving activity, it is consider essential that the types, the characteristics and functions of mathematical definitions should be taken into account when understanding and describing the mathematical knowledge for teaching when engaging students in proving activity. Would this element of knowledge enable mathematics teaching to support desirable student learning outcomes in the domain of proof and in mathematics more broadly? The role of definitions in mathematical reasoning and proving has been exemplified in Section 2.4.1. Engaging students in formulating stipulated definitions entails a process where aspects of proving can be identified. It is accepted that constructing definitions is a challenging activity. However, engaging students, as early as the elementary school level, in a process where what is related with mathematical definitions is negotiated and communicated in the classroom provides an additional platform for proving-related activity. A further indication for this argument was provided in Research Question 1 (see Section 10.2.1.).

Keeping in mind the aforementioned, the following research question will further illuminate the way the classroom community engages with proving. Elaboration of Research Question 3 will then make possible discussing what is eventually possible for pre-proving activity, given the aforementioned issues related with the various forces that impact on the classroom activity.

10.2.3 How does the subject engage with proving in the classroom?

The above discussion, related to the object of proving and the identified contradictions, points to a teacher who is trying to create a classroom environment where the students, through exploration and communication, are engaging in a process of explaining and justifying (see Section 10.2.1. and 10.2.2.). However, this proves a challenging task. The discussion in this research question aims to further illuminate the struggle the teacher faces in attempting to successfully engage the classroom with proving. In doing so, the analysis of the findings as well as the role of the teacher as identified in the research literature is again taken into consideration.

The teacher's engagement with the activity of proving in the classroom is related with providing those exploring opportunities and establishing these sociomathematical norms that will allow students to successfully engage in a process of explaining and justifying. This activity is related with triangles and circles. The teacher recognizes the limitations of the current mathematics curriculum and textbooks (see Section 6.3.2.). As a result, she differentiates her teaching (as observed through the classroom observation). Instead of presenting the formulas and definitions related with triangles and circles, she encourages the students to discover them through exploration and investigation. In order to achieve this, the teacher provides opportunities for exploration of mathematical situations, exploration for supporting mathematical connections that may lead to formulating and checking hypothesis as well as explaining and justifying. Thus, for the teacher, the place for proof in school mathematics goes beyond constituting just a formal tool for verifying truths. This is also achieved by engaging students in formulating these definitions.

Thus, the analysis of the classroom observation indicates that the emphasis the teacher puts on accurately stating and explaining definitions and mathematical formulas constitutes an integral part of her teaching practice. For the teacher, it is not given that providing an acceptable definition will automatically eliminate purely intuitive responses. She is aware that the students may place more value in prior experiences they have had with these concepts or formulas. Her actions also indicate that she acknowledges the fact that in the absence of prior experience, the concept image is in many instances being formed and controlled by the concept definition. Thus, her approach encourages a continuous interplay between the concept image and the concept definition, even in the case where the definition is simply a mathematical formula. Additionally, the discussion pertaining to the activity of explanation further explicates the way the teacher approaches the activity of defining. The purpose of defining for the teacher is understanding the very nature of mathematical definitions and not just the content of the definitions. As a result, the goal of defining can be perceived as the formulation of stipulated definitions. Despite the challenges, the definitions formulated by the classroom were both mathematically precise and appropriate for the students (see Section 9.2.2.).

However, in this discussion, related to the way the teacher engages in the defining activity in a way that it contributes to pre-proving activity, the fact that she does not make explicit the role of mathematical definitions and defining in mathematics in general and in proving in particular is also being acknowledged. This may be due to the fact that defining is not formally expected from the official documentation. Perhaps, this has been explicated in earlier mathematical practices of this classroom. However, there is insufficient evidence to feel confident that this is the case. While more evidence exists regarding the teacher's beliefs about the role of definitions in mathematics, less evidence exists regarding her views about the interplay between defining and proving. Nevertheless, the teacher's comment regarding mathematical definitions (see protocol 3.1.4.) indicates that the teacher embraces the characteristics and functions of mathematical definitions as identified in the mathematics education research literature. These are reinforced by also considering the identified sociomathematical norms negotiated in the classroom.

What is the teacher's role in establishing a classroom environment that embraces the above teaching and learning context? The teacher adapts differing roles in this classroom activity. She has the role of mediator, instructor, and facilitator. This role requires the teacher to evaluate the students' responses, giving them access to appropriate mathematical ways of knowing as well as providing them with the rules with which to arrive at the intended outcome of the activity. This also includes structured instruction together with questions to guide students' engagement. Despite engaging in exploring and explaining as part of pre-proving activity, the inherent duality of the object of these two activities hinders this effort. As a result, the way the teacher eventually intervenes in specific aspects of the lesson is not the one desired or intended.

This inner conflict within the object of the activity of exploration and explanation is also manifested in the way the teacher engages with the tools component of the activity system. This tension is observed and identified through the analysis of the classroom episodes where differing artefacts were employed in her teaching. Returning to the textbook after the classroom was engaged in an activity on the computer or with other artefacts may be a manifestation of this contradiction. This may be another way for the teacher to make sure that the objectives set are reached. Despite this tension, the teacher is employing technology in her teaching in such a way that according to her provides an additional educational value. For instance, the DGE-based tasks designed by teacher can be considered more 'open' compared to the textbook activities that offer the opportunity to students with different ability levels to demonstrate their mathematical ability. This is congruent with what is stated in the literature concerning task design (see Section 5.5.).

How does the way the teacher engages with proving relate to the teacher's role as illustrated in the available research literature? As part of the Methodology Chapter, Section 5.6. was devoted to illustrating the multidimensional role of the teacher. Emphasis was placed on the teacher's role in fostering proving practices in the mathematics classroom. This study shows that even if the teacher in this particular classroom shares the qualities demonstrated in the literature, this only provides a partial portrait of how the surrounding forces impact on the teacher's practices.

My response to the research question relates to what sense the classroom of students makes of proving, the sociomathematical norms, the conceptualization of proof as proposed by Stylianides, as well as the notion of pre-proving.

To begin with, in portraying the activity system of the classroom, the rules through which the teacher establishes a feasible didactical contract with the students were identified (discussed in Section 9.2.). As the emergence of students' meanings for proof and proving is also influenced by elements of the classroom mathematics practices, the sociomathematical norms established in the classroom cannot be neglected (this was discussed in detail in Section 2.3.1.). The taken-as-shared mathematical meanings established in the classroom related with proof and proving have been identified as ‘doing mathematics requires us to use precise language’, ‘doing mathematics requires us to justify our assertions’. This is in accordance with what is being stated in the available mathematics education research literature concerning the establishment of these norms. However, as there is also an inherent concern that these norms may as well have a negative impact in motivating proving and hindering mathematical learning, this discussion now needs to be expanded. In the discussion that follows these norms are seen as a characteristic of the community of the classroom. However, it is acknowledged that not all norms desirable by the teacher are efficient for all students. In Section 9.2. it has been illustrated how the aforementioned sociomathematical norms are negotiated in the classroom. A close examination of the way these norms are negotiated shows that the students were participating in the classroom discussion without the fear of embarrassment. They were trying to communicate their thoughts using their own words and through the teacher’s immediate feedback, they would reconsider, re-evaluate and alter their responses. This is also evident in the activity related with defining, where the students were confident in expressing their concept image (see for instance protocols 3.1.2.-5 and 3.1.8-9). Understanding the importance of using precise mathematical language is an important aspect when formulating a definition and when approaching a proof. This, in turn, assists in the explanation to be more coherent. Furthermore, the fact that the students were expected to ensure the truth of a mathematical statement by pointing out its correspondence to facts indicates that the students are engaged in a process of argumentation that supports verification and mathematical understanding.

In addition, in establishing the aforementioned sociomathematical norms, even though students were expected to justify their answer, this was not enough. That is, the justification or ‘proof’ provided had to be accepted by the classroom community. Keeping this in mind, it can be argued that the students in a broad sense had the opportunity to make connections with Stylianides’ conceptualization (2007a, 2007b). Additionally, the students had the opportunity to experience the differing functions of proof and engage with patterns of argumentation. This can be considered a positive step towards acquiring the ability and confidence in evaluating and constructing proofs. This argument was rationalized in a preceding discussion related with Research Question 1.

At this point, an additional remark needs to be made. While the teacher would stress in several ways the importance of explaining and justifying, it was also indicated from the classroom observation that the students related proving as a process that involves letters (see protocol 3.4.7.). It may be argued that the students misunderstand the actual meaning of the letters. However, a more detailed exploration reveals why the students made this inference. Each time the teacher would use the word 'prove', this was related with an exploration activity where the students were requested to prove a statement. This process entailed using mathematical formulas and mathematical operations with proof being the culminating stage of mathematical activity. The students made a connection between these two. For the students proof and 'to prove' entailed a more technical and formal process that consists of formal descriptions of properties of geometrical concepts and the relationships between them. There is insufficient evidence to feel confident that this should be considered a sociomathematical norm emerging through the classroom discussion. However, this is an indication of how the teacher's beliefs and actions shape students' mathematical understanding. The teacher's response shows that the purpose of this proof was to show the students how one can prove these types of statements, so that they can use this form to produce similar proofs. Even though the teacher closed down the exploration process, she focused on the general structure of the proof. Thus, the result itself had less value than the argument the students were expected to develop.

Nevertheless, by closing down investigation opportunities, the students did not have the opportunity either to initiate a solution, nor to test the hypothesis made, thus limiting their explaining and justifying. It might be therefore conjectured that the students will have few opportunities in the near future to engage with proving related activity in a more independent way.

By considering what has been stated regarding students' activity, the activity system of the students can also be portrayed. However, it is recognized that some details are to some extent conjectural rather than evidential. Nevertheless, the details were derived from the classroom observation data which were supported by the data from the discussions with the teacher as well as from the analysis of the official documentation and the school system.

Table 10.2.: The students' activity system

Subject	Students
Tools	talk, mathematical definitions and formulas, exploration opportunities
Rules	Social and sociomathematical norms
Division of labour	The teacher has the authority to set tasks. Students are expected to engage with the lesson.
Community	Classroom community (teacher and students); school community; friends; home; wider social groups
Object	Classroom survival, and success in terms of understanding the curriculum content
Outcome	Some evidence of achievement of object

This section has provided a thorough discussion on the main themes of the research questions of the thesis. By identifying the evolving object of the activity of the classroom, this study was able to isolate the emergent contradictions. As a result, the way the teacher and the students engaged in this pre-proving activity was illustrated. The following sections provide a summary of this thesis, discuss its main contributions and limitations and overview possible directions for further research.

10.3. Towards a theoretical framework

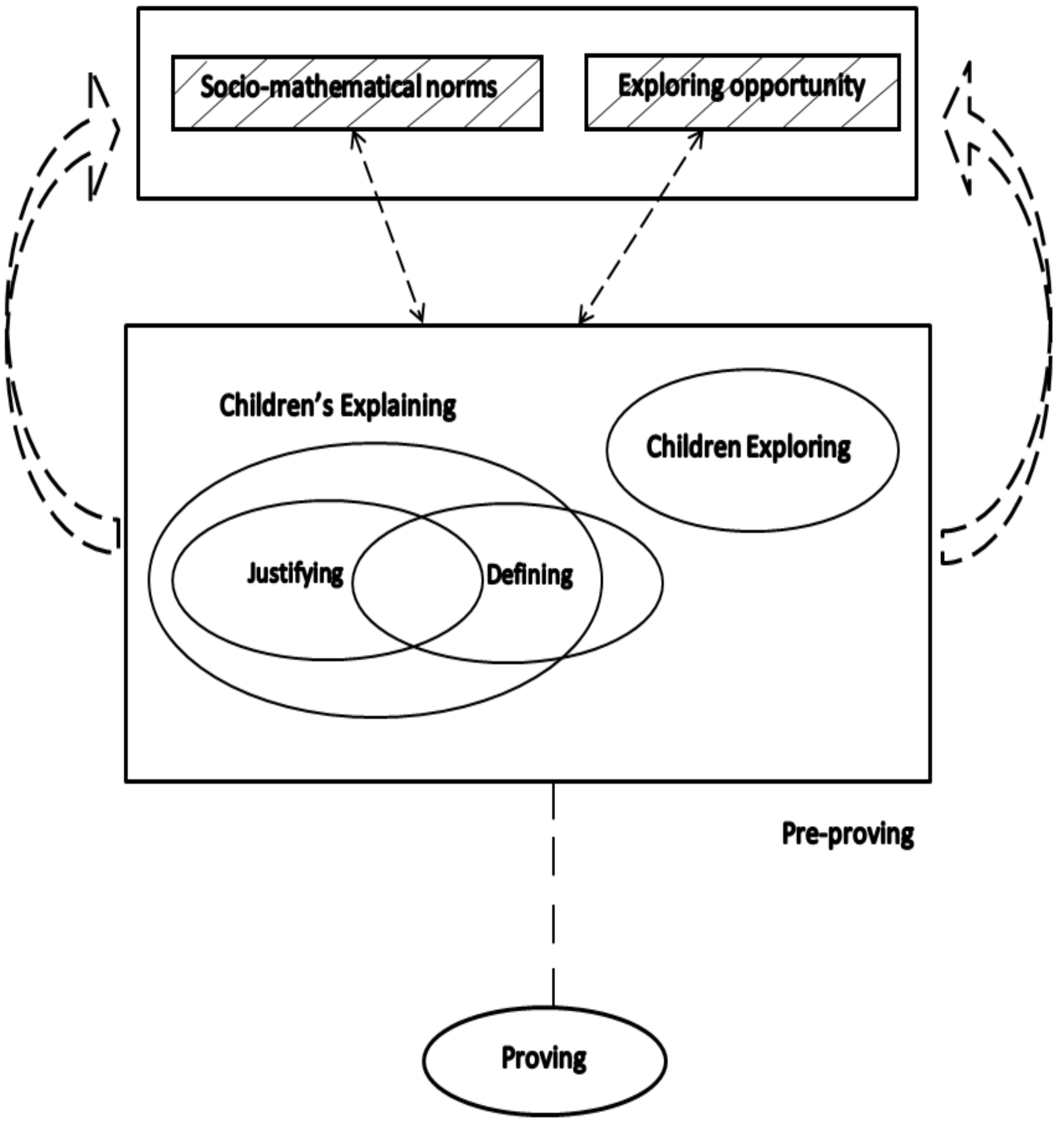
In Section 2.4. of this thesis, the notion of pre-proving was introduced as an orientation towards student activity, focusing on those aspects of reasoning that appear to have the qualities of proving, even though they may not be proving in themselves. This study revealed that exploration and explanation constitute key elements in establishing pre-proving activity in the classroom. Thus, the key findings of this thesis strengthen the adaptation of this conceptualisation. That is, when investigating proof and proving in the naturalistic setting of the classroom, one must not only consider the instances that fall into the conceptualisation of the notion of proof as proposed by Stylianides (2007). To be more precise, instances of students proving statements have been identified in this classroom community but instances where the argument was not in the conceptual reach of the classroom have also been identified. However, this study also points to those elements that play a catalytic role on how proving may be established in the classroom; exploring, explaining, justifying and defining. The relationship between these notions in regards to proving, as taken upon this study has been illustrated in section 2.4. I now turn to the way the elements that play the catalytic role are connected in regards to pre-proving activity.

A first attempt to schematise this model is illustrated in Figure 10.1. below. This figure illustrates both aspects of the setting initially shaped by the teacher and subsequently reinforced by the whole community, as well as the students' activity.

Mathematical argumentation, as explicated in section 2.4, is a discursive activity based on reasoning that supports or disproves something. However, mathematical reasoning is not always directed to proving.

Exploration of mathematical situations that provide the opportunities to hypothesise, make conjectures and test these conjectures might give rise to explaining and justifying. However, it has been illustrated through the analysis and discussion of the findings that difficulties may arise. Exploring may be closed down. Alternatively, the exploring opportunity may be exploited so as to negotiate and establish socio-mathematical norms in the classroom. Then again, exploration may also be initiated in a more direct way with the purpose to negotiate these norms.

As the sociomathematical norms established in the classroom are related with the very nature, functions and characteristics of proof, proving and defining, their establishment strengthens the activity of explanation.



Aspect of setting initially shaped by the teacher and subsequently reinforced by the whole community

Children's activity

– – – – – – – – 'might give rise to'

Figure 10.1.: A tentative model of the role of pre-proving in the argumentation process.

Thus, the socio-mathematical norms related with explaining and justifying ('doing mathematics requires us to use precise language', 'doing mathematics requires us to justify our assertions', 'we present our solution methods by describing actions on mathematical objects rather than simply accounting for calculating manipulations', 'we write coherent geometrical explanations') might give rise to explaining and justifying and thus, defining.

The diagram also shows the way explaining and justifying are connected. While this relationship was illustrated in Section 2.4. of the review of the literature, in the light of the findings of this study, this relationship is further elaborated on. As exemplified in Section 2.4, this study considers mathematical explanation an act of communication, the purpose of which is to clarify aspects of one's mathematical thinking that might not be apparent to others (Yackel and Cobb, 1996). Justification is *'the discourse of an individual who aims to establish for somebody else the validity of a statement'* (Balacheff, 1988a, p.2). In pre-proving, explaining and justifying take place alongside each other. Once appropriate socio-mathematical norms are established, each argument requires explanation and each assertion needs justification. The distinctions between argument and assertion and between explanation and justification become blurred but are key elements of the way proving becomes part of the classroom culture. This close relationship between explaining and explaining why is represented in the diagram in the way justifying constitutes an internal part of explaining.

Furthermore, it has been argued that defining is an activity integrated in the activity of explanation as it entails explaining and justifying (see Section 9.2.2.). While accepting this, in the diagram defining is presented in a separate circle to better illustrate the way it is connected with the other aspects of classroom activity. That is, exploration can lead to explaining and justifying necessary for defining. Exploration can also lead to conclusions that strengthen the fusion of the concept image and the concept definition. Additionally, defining can also be initiated by a question of the format 'what is'. This close relationship between explaining, justifying and defining is represented in the diagram in the way the three are overlapped.

Let me further explicate this relationship. When formulating a definition and negotiating what one wants a definition to be, defining is explaining. Definition construction entails justifying when one explains why. When discriminating between instances and noninstances of a concept and checking whether a potential candidate satisfies all the properties stated in the definition, the defining activity can entail both explaining and justifying. Evidence from the data also point to instances where defining might not be explaining but can be considered pre-proving. That is, once a definition is formulated, the need for this definition to be enriched and developed so as to become part of students' explaining emerges. When the students are explicitly asked to repeat the formulated definition, it can be argued that the definition remains active. By revising or revisiting a formulated definition, the students are provided with the opportunity to revise their concept image and thus broaden their concept definition. Consequently, the students can take control of the definition in their explaining.

For instance, the question 'What is the altitude in a triangle?' can direct the classroom towards formulating the definition of the altitude in a triangle (see protocol 2.1.10.). The verbal definition may not necessarily be linked with the students' concept image. However, by exploring the number of altitudes in a triangle (see protocols 2.1.14.-16), as well as the circumstance where an altitude can fall outside the triangle (see protocols 2.3.9.-11), the students have the opportunity to reflect on their concept image. The concept image and concept definition are being merged. This can result in a broader definition which the students can utilise at a later stage in their explaining.

Thus, in mathematical argumentation pre-proving is coming out of reasoning through exploring and explaining and can lead to proving.

The discussion that is developed in the following section further examines this study's contributions.

10.4. Other significant contributions to the field

Another significant contribution of this thesis is concerned with CHAT. Cultural-Historical Activity theory is a developing body of knowledge, where ideas and concepts continue to be debated and empirically tested. In reviewing the research literature related with the adaptation of CHAT in the mathematics classroom, it was demonstrated that research in mathematics education that takes into consideration all key ideas of activity theory is still limited (see Section 4.3.). This thesis contributes to this area. That is, this study has demonstrated that only when utilizing all constructs of CHAT, can one get a holistic view of the system of activity under investigation. This is in accordance with Bonneau (2013) who states that *'even if the activity system is the unit of analysis of the activity theory, it cannot be studied as an isolated, autonomous system, since each of its elements is the product or aim of a neighboring activity system'.*

In addition, this study showed how proving activity can be explored within this framework. That is, proving as a cultural historical activity can be thoroughly explored through the lens of CHAT. Furthermore, this study has also shown how a qualitative methodological framework can be designed employing CHAT both as a framework for conceptualising the research and formulating the research design. To be more accurate, this study has shown that employing collaborative task design alongside CHAT provides a research framework within which the subject's objects can be identified and explored (see Section 5.2.). In addition, the methodological approach focused on using documentary analysis, interviews, and observations as the instruments that would enable capturing this information and eliciting these elements and thus providing a detailed description of the classroom as a social system.

Adding to the aforementioned, it is also acknowledged that identifying the object of the activity system is not a task easily achieved. This study also contributes to this area. In Section 4.2. the way certain studies engaged in unfolding the evolving object of the classroom activity was demonstrated. While these studies focused mainly on observational data in order to identify the teacher's object, the present study has illustrated that tracking the evolving object of the activity was possible when the macro level was taken into consideration.

In conclusion, a CHAT based methodology may appear complex as it examines social systems. However, this study has outlined how CHAT has enabled the researcher to understand more clearly the social system of the mathematics classroom.

10.5. Looking Backwards: Limitations of this research study

Any classroom-based study will inevitably have limitations beyond the researcher's control and the current study is no exception. Therefore, acknowledging the limitations of the current study serves as a threshold for ideas regarding conducting future research studies. The remainder of this section addresses the possible limitations of the methods and theory employed by the research.

10.5.1 Practical limitations

At the earlier stages of this study, one of the objectives was to undertake the project in two classrooms. Overall, I used three classrooms, though only one was possible in the main study given the limited scope of the project. The need to consider a variety of data sources in order to pursue the CHAT approach meant the study in that one classroom was very intensive, limiting scope to use more than one classroom at that stage. That is, while a longer period of time may have led to the emergence of other meanings, it is also possible that some findings would have not been considered as significant as proved by the interpretation given in this account.

10.5.2 Analytical limitations

As discussed both in Chapter II and in a previous section of this chapter, Cultural-Historical Activity Theory consists of a set of principles that are of course open to interpretation. I used CHAT as a framework for orienting my methodological approach. It gave me a way of conceptualising proving as an activity constituted through the role of the teacher, the endeavours of the children, the effect of curriculum and policy, and the influence of educational research. Nevertheless, my use of CHAT primarily inspired my general approach to analysis, which could seem limited when compared to using CHAT as a stance towards the role of culture and history.

Furthermore, being an active participant in the iteration phase in this study reasonably questions the effects of my role both as a teacher and a researcher on the collected data and articulated conclusions. In discussing my involvement as an active participant in Section 5.7.2, I exemplified the strategies that would be followed so as to maintain stability between teaching and researching. These strategies proved useful and valid in ensuring the balance between detachment and active involvement.

10.5.3 The generalisability of the research findings

It has been acknowledged in Section 5.2. in this thesis that, as a small-scale study, this research was undertaken in one classroom. How would the study have differed if carried out with other participants, and how does this affect the possibility of generalisation to other situations? A sceptic may argue that this is a narrow sample, and thus, the generalizability of the research findings is limited. However, throughout the thesis a consistency was kept both between the operationalisation of the concepts employed in differing parts of the data analysis stage as well as between the way these constructs were defined for the study and the way they are used in the available literature. Furthermore, the context, as well as the process is described in such detail that the reader may recognise and apply elements of it to another situation, even one superficially very different. In this sense, generalising is possible, although it must be done with particular care. Even though the findings are not generalizable in a statistical sense, they enable an increased understanding of how the various forces identified in this study might influence teaching and learning.

The above limitations of the research study, as well as the issues raised in this thesis, point the pathway for future research studies. These are discussed subsequently.

10.6. Looking forward: Directions for future research

The present study can be extended in a number of ways by considering the following dynamically productive lines of enquiry for further research. The first three involve recommendations based on this research study. The remainder is posed to the wider mathematics education research community.

Firstly, to extend the findings of this research, more teachers could be involved in a future study to include a wider range of participants and learning practices. This will enable more data to be collected and hence improve the generalisability of the findings.

In addition, as noted above, in a doctoral study, time is limited. Having more time to study pre-proving activity over a period of time as a longitudinal study would provide an insight into the wider variety of teachers' practices when approaching proving in the classroom. The role of the teacher with respect to the development of pre-proving activity needs to be studied in more detail.

It has been pointed out in Section 10.3. that data regarding students' perspectives and motives were rather limited. The study can be extended by also gathering more detailed data regarding students' activity, which would definitely provide more information regarding the way pre-proving activity is constituted in the classroom.

Finally, I offer the conceptualisation of pre-proving and the tentative model of the role of pre-proving in the argumentation process to the mathematics education research community as a construct to support research in the elementary school level within this domain. This may serve in expanding this conceptualisation's exemplifications. Future research can also look at the limitations of the conceptualisation of pre-proving introduced in this thesis.

The findings of this thesis have a number of implications for educational policy makers and educational practice. The educational significance of the findings is discussed in the following sections.

10.7. Implications for teaching

This study has demonstrated the importance of the sociomathematical norms established in the classroom in relation to pre-proving activity. A teacher, who seeks to establish sociomathematical norms related with explaining and justifying in their teaching practice, does not need to do so in opposition to exploration and investigation. At certain points of a mathematics lesson, the teachers' focus may be to negotiate these rules. As the classroom observations have demonstrated, in guiding students towards the endorsement of a sociomathematical norm, the teacher may request students to justify their assertions, rephrase what the students say, give a negative feedback to a response that does not embrace the norm as well as appraise the response that is correct. The teacher may also make explicit the rule. However, the important point here is for the teachers to be aware and mindful of this piece of evidence.

Concerning the integration of technology, evidence from this study shows that teachers do not need to consider technology as the object of their activity. That is, while there are instances where acquiring computer skills is in fact the object at a given point of mathematical activity, the general object of the activity does not have to be matched with the technology itself. If an activity is closed down this may be due to the clashing objects the teacher has, or it may be that the concept explored or the situation investigated is, at that time, difficult for the students. Doing this may result in removing to an extent the pressure and concerns teachers feel when employing technology in the classroom. For instance, the use of a DGE could facilitate a focus on pre-proving by drawing attention to relationships among geometric figures. However, if such a focus proves too difficult for some children the activity may need to be closed down and the object of emphasising justification might be approached with paper and pencil or other materials without the use of DGE, since the latter is seen as a means to an end rather than as the object itself.

The present study has noticed that students did not seek explanations and justifications in an independent way. With explanation and justification attracting considerable attention as two important aspects of mathematical reasoning that at a later stage result in a formal mathematical proof, teachers then play an important role in stimulating students to engage in explaining and justifying. Teachers could encourage students to explore and investigate mathematical situations in an independent way as well as take advantage of the situations where the students themselves provide opportunities for explaining and justifying. For instance, an opportunity may arise when the activity is related with exploration for supporting mathematical connections. This study has revealed that even though exploration for mathematical connections entails a degree of openness, simultaneously, it is strictly directed towards the specific connection. Despite this, by explicitly relating these forward connections with a fusion between the concept image and the concept definition, the students may explore themselves the opportunities provided, before elaborating their conclusions during whole class discussion. As an illustrative example, consider the definition of an altitude in a triangle. Exploring the number of altitudes in a triangle as well as the occasion where the altitude can fall outside the triangle may empower students to make hypotheses and investigate the validity of their assertions.

While acknowledging time as a constant issue for teachers, providing these opportunities does not need to be seen as opposing the objectives set by them. Providing these opportunities enhances initially the understanding of the mathematics explored as making connections encourages the fusion of the concept image and concept definition. Consequently, the students can take control of the definition in their explaining.

As the findings of this research have demonstrated, the classroom community constructed mathematical formulas. The classroom was engaged in a process where these relationships were articulated verbally or written in words before being expressed algebraically. Two points are to be made here. It would be helpful if teachers were to recognize the fact that the students' verbal or written descriptions of mathematical relationships do not necessarily ensure an easy or even successful translation of the relationships into their symbolic expressions. For instance, in this research study the students, in exploring circle, reached the conclusion that the radius is half the diameter, and were able to articulate a verbal and written description of the specific relationship. However, it was non-trivial to students to formalize this relationship and move from a description to a symbolic expression. The teacher facilitated this process of transforming the expression into an algebraically expression. In view of this realization, teachers might make explicit the use of symbolic notation in mathematics. That is, they could emphasise the interpretation of letters as generalized or even as specific unknown numbers instead of just shorthand for names or measurements labels.

Evidence from the literature and the findings of this study indicate that there might be benefits if teachers were to treat the nature of mathematical definitions and proving as concepts in their own right. While research concerning definitions in the elementary level is limited, this study adds to this argument. This study has demonstrated that the teacher made explicit various characteristics of mathematical definitions and proof. While there is insufficient evidence to feel confident in arguing that the teacher exposed the classroom community in a direct way to the nature of mathematical definitions and proving, the findings point to the educational significance of this recommendation. It would be reasonable to expect there to be an impact on the overall mathematical activity. Perhaps this will mean that the aforementioned implications may be incorporated in a more natural way into the teachers' mathematical practices.

10.8. Implications for teacher education

It is accepted that teachers' knowledge of content shapes their teaching practices. Furthermore, the teacher's practice is also influenced by the resources available to them through the textbooks and curriculum material. The outcomes of this study can be used to broaden teacher's content and pedagogical knowledge. Elaboration on the research questions as well as the implications for teaching lead to some clear themes regarding teacher education. This section further elaborates the implications for both pre-service teacher preparation and in-service teacher professional development.

Teacher education could expose teachers to the aspects of proving and pre-proving activity identified in this study; exploring, explaining, justifying, defining. The finding from this research regarding the role of defining in pre-proving and proving activity suggests that there might be advantage if teachers were further exposed to the role of definitions in mathematical reasoning initially, and their role in proving in particular. It is imperative that teachers appreciate the role of definitions of concepts, terms and even mathematical formulas so as to go beyond the ordinary usage of definitions in following procedures or reproducing standard arguments. This standpoint follows multiple studies on the role of definitions. Among the main roles attributed to definitions is that they constitute fundamental components for concept formation as they introduce the objects of a theory and capture the essence of a concept by conveying its characterizing properties. Adding to the above, definitions form a generative basis for logical deduction, not only of known properties of the concept but of new properties. Definitions may also facilitate the generation and construction of different types of theorems, proofs and solution methods. This is in agreement with Morgan (2005) who also highlights the idea of choice and purposeful formulation of definitions (Morgan, 2006). Given the multiple ways definitions can be approached in mathematics practice, it would be useful for teacher education to guide teachers so as to exploit these ways in designing meaningful teaching activities.

Teacher education might facilitate reflection among teachers on the tensions inherent in the multifaceted objects they face due to the various forces that impact on their activity. While this was posed as a question related to a possible resolution of contradictions, it is now further elaborated on. For instance, this study has shown how the multifaceted objects of explaining and justifying led to a constant struggle in the teacher's everyday practice. The object for the teacher is related with exploration that leads to conclusions related with parts of the mathematics curriculum. However, this object is being conflicted as, while a play-like exploration can facilitate learning, this can prove quite challenging for the teacher, as she wishes to maintain focus and is worried that exploring detracts from that focus. Keeping this in mind, it would be helpful if teacher education could expose teachers to the multidimensionality of objects of activity. Being aware of this may enable teachers to effectively develop coping strategies and find creative ways so as to overcome these dilemmas.

10.9. Implications for educational policy

The findings of this study have implications for curriculum design and development. In designing and developing a mathematics curriculum, one must consider those elements that play a catalyst role in shaping proving activity. While official endorsement from the curriculum guidance does not guarantee change in teacher's pedagogical practices, recognising the aspects of reasoning that may have the qualities of proving may create opportunities for teachers and students to develop and broaden their understanding in this domain.

The findings of this thesis have implications for the new mathematics curriculum currently being developed in Cyprus. These are of great importance when considering the fact that the entire classroom curriculum is only based on one textbook. Before elaborating on how the findings of this study relate with the implementation of the mathematics curriculum, I will provide the information I consider essential so as for the reader to be able to follow the argument that will follow. Thus, the purpose of the section that follows is to inform the reader about the system rather than criticise it.

10.9.1 Cyprus new mathematics curriculum

The Cypriot Educational System is currently under educational reformation. The Committee of Educational Reform, in examining the prospects of educational reform urged the MOEC for *‘the modernization of the content of curricula and textbooks in response to the contemporary trend of intercultural education’* (Committee of Educational Reform, 2004). The Educational Reform Programme was launched by the MOEC in 2005. The philosophical and ideological pillars of the ‘new’ curriculum were publicised in December 2008 via a policy document entitled ‘Curriculum for the Public Schools of the Republic of Cyprus’ (Cyprus Ministry of Education and Culture, 2008). In 2010 the new school curriculum was officially published. It was comprised of two concise volumes referring to the subject matters of K-12 and accompanied by separate volumes of each subject matter. In September 2011, the new mathematics curriculum started to be implemented on a pilot basis and it is now expected to be in full operation by June 2017. For instance, the Year 5 and Year 6 mathematics textbooks are still in the design process. At the moment, Year 5 and Year 6 primary school students use the textbooks as described in Sections 6.2.2. and 6.4.1.

The challenges in the transition to secondary school mathematics have been recognized (Sdrolias and Triandafillidis, 2008). As a result, the objective is to create an integrated continuum from early childhood to secondary and vocational education. That is, the aim is to *‘move from tangible and the familiar to the theoretical, distant and abstract and to shift from content-knowledge-orientated and teacher-centered to student-oriented and play-oriented curriculum’* (Christodoulou, 2013, p.158). The cornerstones of the curricular reform in Cyprus are the ‘democratic and humane school’. The official curriculum defines the democratic school as a school that includes and provides for all children, in spite of any differences they may have, and supports them in preparing for a common future. This school ensures equal educational opportunities for all and, most importantly, it is held responsible not only for the success but also for the disparity in the results of each and every individual child. The democratic school is organised in such a way that it will provide to all children the opportunity to achieve all educational objectives without any reductions in the quantity and quality of educational means. On the other hand, the humane school is a school that respects human dignity. It is a school where no child is marginalised, stigmatised or scorned. It is a school that celebrates childhood, acknowledging that this should be the most creative and happy period of human life (Cyprus Ministry of Education and Culture, 2010a).

As the subject of mathematics is concerned, the new mathematics curriculum has been designed based on the key principles that: (a) mathematical concepts should be explored in a way that enhance students' curiosity and interest, and are related to already existing knowledge, based on real life situations and interdisciplinary questions, (b) emphasis is placed on problem-solving, (c) technology and ICT constitute an integral part of mathematics education and (d) all students must enrich their experiences through pedagogically rich examples, that arise from the active engagement with meaningful mathematical problems and concepts (Cyprus Ministry of Education and Culture, 2010b).

The didactical model, as suggested by the new curriculum includes investigation and exploration activities, proposed activities for achieving the objectives of a specific mathematical content of the curriculum as well as enrichment activities. Thus, the design of the students' textbooks follows this proposed model. Thus, the introduction of a new mathematical topic is constituted by an exploration and investigation activity. According to this new philosophy, exploration activities are open-ended, in which students freely explore a mathematical concept, without specific common path or suffix. These activities contribute to the diversification and personalization of teaching, they provide motivation and the joy of learning, to the development of mathematical reasoning, creativity and imagination in mathematics as well as to the conceptual connection between mathematical concepts. Investigation activities are focused on the study concept through a specific and guided framework that provides students the opportunity to formulate hypotheses, to test the validity of their conjectures and justify their answers. According to this new philosophy, mathematical investigation should not be considered as identical to the starting point of a lesson.

New teaching material, textbooks, e-books, teacher aids and computer applications are being prepared based on the proposed curriculum and the experience gained from the three years of implementing the new concepts. Since the new curriculum of

Cyprus is still in the implementation process, it can be reasonably argued that we can still not examine its impact on educational practice in Cyprus. An initial form of assessment regarding the new mathematics curriculum will be the achievement scores of Year 4 students participating in TIMMS 2015.

Drawing on the current research finding regarding exploring and explaining as components of pre-proving activity, it can be argued that the educational reformation takes these notions into consideration. Exploration and investigation are considered important aspects of the mathematical didactical model. Exploration and investigation activities have an objective to be achieved. The teacher has the autonomy regarding how these activities can be exploited in the mathematics lesson. However, it should be made explicit that these should not constitute the only opportunities provided to students for exploration. Exploration and investigation as part of pre-proving should be considered as an integral part of the didactical model in general, so as for the students to have the opportunity to explore situations at several parts of the mathematics lesson.

Furthermore, while the new mathematics curriculum places emphasis on problem solving and states the goal towards continuity to all school levels, it is being noticed that no explicit reference is made regarding definitions. Perhaps mathematical definitions may constitute an integral part of the key principles of the new mathematics curriculum. Yet, this may also indicate that defining is not considered as a separate and autonomous mathematical activity. Nevertheless, the various ways students can be engaged with regarding definition construction should also be taken into consideration when designing the mathematics textbooks. This is considered imperative as the mathematics textbooks are the main educational aid the teachers use in their practices.

10.10. Concluding remarks

This present study has investigated the activity of proving as constituted in the naturalistic setting of the mathematics primary school classroom. The study has shed some light on this area. The key outcomes of this study, together with directions for future research studies as well as the implications for teaching, teacher education as well as curriculum development have been already illustrated and discussed in great detail above. The greatest strength of this present study lies in the identification of the elements that drive pre-proving activity and influence the way proving may be established in the classroom. That is, in mathematical argumentation pre-proving is coming out of reasoning through exploring, explaining, justifying and defining and can lead to proving. Hopefully, with a greater awareness of the way these elements shape pre-proving and proving activity, teaching and learning can be planned more effectively to support students when approaching proof. As a final point, it is hoped that the field related with proof and proving in mathematics education, though is undoubtedly a comprehensively well-researched area, will definitely benefit by expanded investigations in an attempt to further support the way proving is constituted in the classroom.

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APPENDIX I

AUTHORISATION FOR CONDUCTING RESEARCH BY THE MINISTRY OF EDUCATION AND CULTURE IN CYPRUS

Ministry of Education and Culture
Department of Primary Education
Tel: 22800661
Fax: 22428277
Email: dde@moec.gov.cy

25th February 2011

Subject: Authorization for conducting research in a Year 6 classroom at Saint John the Chrisostomos' Primary school in Lakatamia

Dear Ms Pericleous,

I have instructions to refer to your application to the Centre of Educational Research and Evaluation, submitted on the 21st of February 2011. I would like to inform you that you are given permission to conduct your research titled 'How is the activity of proving being constituted in the Cypriot classroom for 12 year old students?', in a Year 6 classroom at Saint John the Chrisostomos Primary school in Lakatamia this school year 2010-2011. However, you have to take into consideration the recommendations of the Centre of Educational Research and Evaluation.

You must get the permission of the Principal of the school so as to take all the necessary measures that will ensure its normal operation. The research needs to be conducted in such a way that the teachers' work, the families or the students are not offended. Also the activities used need to fall into the framework defined by the national curriculum. What is more, the loss of teaching time should be kept to the minimum. For the participation and video recording of the students, a signed consent statement from the parents should be provided. The parents should be well informed about this research study and its individual stages. It is also stressed that anonymity, confidentiality and privacy of the information gathered will only be used for the purposes of this research.

It is also expected that the results of this study will be made available to the Department of Primary Education of the Ministry of Education for appropriate use.

Kind regards,

Elpidoforos Neokleous

APPENDIX II

LETTER TO PARENTS REQUESTING INFORMED CONSENT

Maria Pericleous
1 John Kennedy
Pano Lakatamia
Nicosia 2314
Tel:

21st February 2011

Dear Parents,

My name is Maria Pericleous and I work as a teacher at the Brighton Greek School in the United Kingdom. I am an MPhil/PhD student. The title of my thesis is 'How is the activity of proving being constituted in the Cypriot classroom for 12 year old students'? In order to complete my postgraduate degree, it is requested to collect data from 12th year old primary school students. I would like to mention that this project has a lot to offer to students. Consequently, I would be grateful if you could allow your child/children to participate in this research project.

The methods of collecting data that will be employed include the video recording of the mathematics lesson. The data collected from the data will be confidential. I you want, I can inform you about the progression of this research project. The result of the study will be available to the head and inspector of the school.

In addition to the above, I consider it important to inform you that this particular research project has got a written agreement from the Head of Primary Education of the Ministry of Education and had the full support of the Headmistresses of the school.

I would be grateful if you could return the following document signed as soon as possible.

For any queries please do not hesitate to contact me.

Thank you very much in advance.

Yours sincerely,

Maria Pericleous

CONSENT FORM FOR PARTICIPATION IN RESEARCH

To: Maria Pericleous

The undersigned …………………………………………… father / mother / guardian

☐ I give consent to the student ……………………………………….. to participate in the research the title of which is 'How is the activity of proving being constituted in the Cypriot classroom for 12 year old students?

☐ My child has permission to be video recorded during the study.

☐ I do not give consent to the student ……………………………………….. to participate in the research the title of which is 'How is the activity of proving being constituted in the Cypriot classroom for 12 year old students?

Date: ………………………..

Signature

…………………………………..

APPENDIX III

EXPLORATORY STUDY I

AN ILLUSTRATIVE EXAMPLE OF A PAIR OF STUDENTS EXPLORING DGE-BASED TASK 1

The discussion follows students' exploration of the task where the condition should apply.

R: What do you observe?
S1: No matter where you drag it (point A) you need to keep the point stable.
S2: The angle.
R: Yes. What do you observe?
S1: What is constructed is like a clock.
S2: Like a semicircle.
S1: A semicircle. Yes.
S2: Like a curvy line *(the student points at the screen).* It's like a drawing with houses with a curve.
The students continue adding more points on the transparent slide.
S1: Now it seems that a circle is constructed *(the student points at the arc on the screen).*
R: What if we change the distance between points A and C?
S1: I think a circle will be constructed but inside the other one.
S2: A smaller circle. But why smaller?
S1: Because we moved the points closer to each other.
R: What are these points?
S1: The angles.
R: What did we change?
S1: The perimeter.
R: Investigate what happens.
The students add the points on the transparent slide.
S2: It will be smaller.
S1: But we need to change the angle.
R: What do you observe?
S1: It is not dependent … the distance is what matters. When we changed the distance, the distance was smaller and the shape constructed was smaller. Now that we changed the angles, the shape is still the same.
R: Thus, in these last two occasions the distance was …
S2: Kept the same.

R: Was there anything else that we took into consideration?

S1: Yes, the angle had to be half.

A new screen with the same diagram was presented to the students. For this construction though only point could be moved.

R: You can use the option 'Trace' to explore what happens.

S1: It will be a circle.

S2: Yes.

The students use the option and drag point .

S2: It's a circle.

R: Can we explain why a circle is constructed?

S1: The radius …

S2: There should be … we should see that the angle is half and it depends on the size of the circle. If I want the circle to be bigger, the distance should be bigger as well *(he points at the radii on the diagram).*

S1: I think that is the radius of the circle.

R: Yes.

S2: As the radius is getting smaller …

S1: So does the circle.

R: If we change the angle.

S1: The diameter…

R: What is the diameter in this shape?

S1: I think that …

R: How are these two angles related?

S1: The angles are in the same place.

S2: I think that …

S1: It depends on how big we want the circle to be.

S2: If we move the radius …

S1: It stays the same.

R: What stays the same?

S1: The angles.

S2: The condition will always hold.

R: Investigate your hypothesis.

The students move the points and so that AC is the diameter of the circle.

S1: Now we have the diameter.

R: Yes. *(S2 moves point B so that the condition holds).*

S2: Yes.

S2: The angle is half.

S1: And a right-angle triangle is constructed.

S2: But not always.

R: How many degrees is the angle?

S2: 90°.

S1: It will always be a right-angle triangle.

S2: The angle must … we need to draw a diameter, find the angle of the diameter…

S1: It will be 180°. The diameter is like a line. It will always be 180°.

S2: No, not always, it depends on how big the circle is.

R: You can check your hypothesis.

APPENDIX IV

THE INTERVIEW SCHEDULE

1. Could you describe some of the basic topics around geometry that you teach?

2. What areas of geometry do you find more important?

   Why are those areas important?

   Are there any difficulties for students to understand those basic concepts?

3. Do you think geometry is important for students to learn? Why?

4. Do you follow the instruction if the national curriculum? In what degree?

5. Do you think that students will acquire the appropriate geometrical concepts and knowledge that will allow them to proceed further on the subject of geometry in the secondary school?

6. What is the role of students' previous experience around geometry in your teaching? What is the nature of that experience?

7. Do you incorporate DGE in your teaching? Why? If yes, what DGE?

8. Do you know what DGE involve?

9. Do you think that justification and proving process are necessary for the geometry concepts that you teach?

10. What is the role of proof in geometry?

11. Do you provide the proof of a theorem or an axiom or leave the students to try to discover it on their own?

APPENDIX V

TRANSCRIPTS OF THE TEACHER'S INITIAL INTERVIEW

R: Could you describe some of the basic topics around geometry that you teach?

T: In Year 6 it's the distinction of shapes based on the numbers of sides in polygons, the types of quadrilaterals, the recognition of the types of triangles based on their angles and sides, the area of the rectangle, the square, the triangle and circle, diameter, radius, circumference and area. Also for the solid shapes we have the external surface area and volume of the rectangular parallelepiped and measuring angles.

R: What areas of geometry do you find more important? Why are those areas important?

T: There are some skills that are more important than others like measuring angles, being able to construct the altitude, the skill of what altitude means, what perpendicular means, the formation of 90 degrees, and then the formulas and how to find the area and apply it in order to find it, since I need to recognize what basis, height, length, width means. It is essential for the students to distinguish these concepts, and then to proceed in applying them so as to find the various measurements. To proceed in high school it is very crucial to know these concepts and apply them. For this reason I do not stop at what we have in primary school but also in what will follow and the skills that we need in order to able to continue in secondary school.

R: Are there any difficulties for students to understand those basic concepts?

T: What are the areas where the students struggle?

R: Yes.

T: There is always the support of practical work. That is to construct it themselves, to work by themselves helps them to assimilate some concepts. The problem is that even though the arrangement of the curriculum is in a spiral form, the content of it is so much that until you come back, we have to revise them. Sometimes they even forget what they have encountered. So the substance is that at times it feels that we start all from the beginning or we just name them and revise. For example like now when I asked them to draw some triangles, we had to remember what is rectangular, acute-angled and obtuse-angled triangle. But that was the purpose of the curriculum section. These concepts are reappearing with the spiral arrangement of the curriculum, but there is a large amount of concepts and I think that perhaps some things need to be completely removed so as to give students the opportunity to work more and more often with basic central concepts which they will be using throughout their life for mathematics and help them with mathematical and critical thinking. That is not to simply use operations mechanically but to be able to solve mathematical problems that I will also meet in my everyday life. I have a way of thinking and logic. It's what the students usually say: that mathematics is simple logic. I have to solve a problem and for example a get as an answer that on the bus there were 48.5 people. This means that I made a mistake. This is a far-fetched example, but actually, is an example which makes you puzzled. Is the answer reasonable based on these data? Or if I end up having more money than what I had before buying something … something went wrong. So there needs to be some logic in problem solving.

R: Should definitions and formulas be removed as well?

T: In primary school we do not really have definitions as in secondary school. These are functional definitions. For example when you bring the altitude, the altitude is a distance, the students must understand that it is distance, they need to distinguish that it is not a point as some did, it is the distance from one point to another in any line and forms 90 degrees. I insist in the phrasing of definitions as it will help them in secondary school. I do not insist on assessing the definitions. For example, when I asked a student to say what altitude is in a triangle, he said ‘when we bring a line forming 90 degrees’. There is understanding but he cannot give the formal definition. The definitions in primary school are more functional and less theoretical. Nevertheless the use of precise mathematical language will help them in secondary school. The point, the distance, the line are important geometrical concepts. And they must distinguish what we mean by ‘point. The corner of the triangles is a point, whereas the side is defined by points A and B. It is the side AB, not the side A and B. Particularly, to be to name the triangle ABC with letters, a vast amount of time needs to be devoted. This is the same when the students need to name squares and rectangles. This is very challenging and I insist in this. Still, there are students that say ‘this side is the side A and B’. When you teach in Year 4 the square or the rectangle ABCD, students work mechanically. If you tell them to define the shape, just by using the word 'appointed' the student will find it difficult. If you ask him to put letters … it is easier to do because naming is more practical. And what does it mean that the side AB which is parallel to DG … they are equal and parallel … these are concepts that students use often, but I insist that we should do not consider that they are conquered fully by the students. We have students who still have difficulties. Now with the new mathematics curriculum, the mathematics content will be reduced. This will give the teachers the opportunity to better integrate technology and new practices into their teaching. How will the mathematics content be reduced? We will see. It is expected that the curriculum will be ready by September so that we can use it to plan our teaching. We might be using the same textbooks by adjusting them to the new curriculum.

R: You mentioned that you started using the students’ textbooks before being provided the teacher’s guidance book.

T: Yes. It has been noticed that for the last 15 or so years that we have been using the textbooks, our performances have not changed. This has raised many concerns. There is an attempt to rearrange the content of the curriculum. In recent years this was being achieved by occasionally circulars and announcements made by the Ministry of Education. We have seen that teaching units were students had difficulties in secondary school, were being introduced in primary school. The teacher’s guidance book has been introduced in the last 4-5 years … as a book … there were guidelines with instructions on how to utilize the books.

T: I do not consider the teacher’s guidance book to be practical. Yes the book talks about the goals, but mostly it provides answers … for some exercises there are teachers that may not be that confident with the mathematics involved and have difficulties in understanding how to solve them … thus for this reason it is useful. The latest curriculum was written in mathematics since 1981. I do not know if there is one in the school. However, changes are being announced by the Ministry, amendments. So the organized curriculum, there is the book of the teacher for the subject of mathematics, were the teaching units are reported, the educational goals but mostly it has the activities for each unit and section that a teacher will teach.

R: Concerning technology…

T: There have been some efforts in recent years by the Pedagogical Institute which has issued specific books for each course, with activities for the utilisation of specific software and integration in teaching. This has also been done for the subject of mathematics ... how we can integrate specific software and use specific lesson plans. Two books were issued two years ago by the Pedagogical Institute. On the website of the Pedagogical Institute one can find integration courses. Of course, this still depends on each teacher … it's up to the teacher to get that book to get some ideas and use technology. Well, I am the computer coordinator at the school and I am participating in the program of the Pedagogical Institute that aims in assisting teachers in utilizing technology in their teaching … got instance, yesterday, I did the same lesson in the other Year class. The teacher is not familiar with the software and we did the lesson together. We will also do the lesson regarding triangles in Year 4. By doing this, many teachers get engaged … we help each other. Of course, we do not expect miracles. As long as they use at least one or two things that really help and have an additional value for the activity… if it doesn't offer that, there is no point in using the technology… the specific software ... the fact that it gives you the opportunity to drag the points and the vertices of the triangles and change the triangle from right-angled to obtuse-angled, to acute-angled is very essential. Whereas if you do it on paper, it's there and then you will have to erase it and draw it again. If you have the same basis, you change the altitude ... and you can see the types of triangles … So the software is very helpful. Indeed, it has an additional value. And it showed that the students that have some difficulties, the y also had difficulties when exploring the software. However, working on the software instead with paper-and-pencil did help them. If the assessment is on paper, the students may have difficulties. I am thinking that the assessment on Monday will be on GeoGebra. That is, to give each student a worksheet that states 'construct these 2 triangles'. They can do it in GeoGebra first and them on the paper.

R: Assessment using technology.

T: I sometimes use technology for assessment, mostly for mathematics and science. It is done either individually or it has a playful form; I have questions of multiple choice, the students choose their answer, they write it in a piece of paper, they hand it out as a coupon in a tombola style in order to win a prize. For the content of the curriculum related with the types of triangles with played 'Bingo'. The students had several answers and had to marked the squares in a line to win. For instance, the complementary angle of 56° … the students had to find the angle … the supplementary angle was also presented as a choice. Thus, the students had to remember what the complementary and supplementary angle is. The students do not consider it as assessment and do not have the stress of a test. For myself, I do not consider it as assessment but as a game. This is what I usually do even though my personal notes take the form of a report on assessment. The students think that they will win a prize. Some of them will not but that is fine as they comprehended the assessment as a game. Some students rush and do not win. They need to find the correct answer.

R: Do you think that students will acquire the appropriate geometrical concepts and knowledge that will allow them to proceed further on the subject of geometry in secondary school?

T: I consider that it depends on the way you teach. That is … for example, if you just use the book…they will not acquire many skills. You need to give them opportunities to work a lot and not just with the book … in the notebook, the worksheet, on the software…they need to have the chance to work more so as to accumulate the concepts. Of course, this takes more time … for example, even though I usually devote 3 days for ‘triangle’, now I devoted 4. It does not matter as long as they understand the meaning of what we are doing. And I mean no memorization, but proper understanding. They need to have the opportunity to work a lot. I think that practical work prepares them for secondary school.

R: What is the role of students’ previous experience around geometry in your teaching?

T: I take nothing as given, even when I am the one teaching it. I know that there are things that were taught in other lessons and previous school years. But when we come back for instance in the types of triangles, I will not take as given that they remember the types of triangles. We had to refer to them again. Thus, I always bring back previous knowledge before moving on to a new lesson.

R: What is the nature of this knowledge?

T: Most students don’t remember … but when we help them …. when we have revision questions like for instance ‘remember what we did’, then they can recall basic knowledge and progress. I sometimes tell my students ‘theoretically you learnt it’…essentially what is valid I do not know…most of them forget…the knowledge that they acquire is very specific … they need time to remember … you need to help them recall the previous knowledge… which means nothing should be taken as given.

R: For instance, when last year they students were exploring the task related with circle.

T: Yes. The students were saying that they did it the year before and were trying to remember what ‘this line’ is called, that is the radius and the diameter. Yes they saw it but needed time to remember … Because these will help them in secondary school. I think this may be one of the disadvantages of secondary schools. While there are mathematics teachers that remind students … there are also teachers that proceed to a new lesson without revision as they believe that students remember parts of the curriculum that they were taught previously. As a result, many students struggle. This is wrong as revision helps both the students but also the teacher to understand what the students know or/ and remember. For example, even though the mathematical formula for the area of square is on board, there were students that forgot how to find it. Even when I told them, some students needed more time to understand. Nothing is given. Indeed, there are teachers that remind students of what has been taught previously and the students say ‘oh yes we did this’ and recall previous knowledge. Nothing is given; everything needs to be continuously visualized.

R: Do you think that justification and proving process are necessary for the geometry concepts that you teach?

T: I think we saw from the lesson that a mere memorization of the formula for the area of triangles is not enough. It is very easy to know the formula. But, what does altitude mean? What does basis mean? For example, what happens when the shape is rotated in such a way that the students need to measure? For instance, the game we played today was at level 1 which is only involved with right-angled triangles. Level 2 consists of obtuse-angled and acute-angled triangles but again, it is quite easy can find their altitudes, as the sides fall in the lines of the squared grid. This is not the case for Level 3. In Level 3 students must use the Pythagorean Theorem to find the hypotenuse. There are activities that may be difficult but it is important to understand what these concepts mean. Measuring with the ruler is a basic skill … we do not use the software all the time...we come back to it at a later stage ... there is no day that we worked in the software and not used something similar in a worksheet or the notebook; so that what they do in the software they can also apply it elsewhere. For example, the software assisted students in drawing a right-angle. On paper it was more difficult. We need to insist in properly using the ruler in order to draw a right-angle.

R: What is the role of proof in geometry?

T: I believe that the students acquire knowledge better when they prove it rather than when it is just provided to them. And in geometry it is even more practical because they are things that they can see. They can prove them … they see that the area of the triangle is half the area of the square in which the triangle is inscribed to, they can measure it and compare it and prove it. Thus, proving is actually what helps them practically to assimilate better the knowledge. When they prove something they see that the rule stands. This way they do not ask ‘why’. And it is an important process that is essential. And our books are based on this.

R: Do you provide the proof of a theorem or an axiom or leave the students to try to discover it on their own?

T: For the things that they prove, they do not ask questions because they do it themselves … it is when they are given something that they ask why this is so … but we do not stop at the group work … we have the classroom discussion so as to show and present the proof and accept it as a whole.

R: Is argumentation and justification important in proving?

T: Yes, very important. For example, some students said that from a vertex they can draw many altitudes. Thus, they had to prove why the answer is one. Also, why three altitudes … because the triangle has three vertexes. The student that said many … when he was doing that on the whiteboard, he realized the mistake he made. He recalled the definition and altered his response. Thus, it is important to justify what I do. They also write their justifications. Of course there are students that their written communication does not include a justification. But the point is that justifying their assertions is an important process that we use in the classroom. That is, I write it and I justify what I write. I need to learn to justify my assertions.

R: Do you incorporate DGE in your teaching? Why? If yes, what DGE?

T: At the moment I am using GeoGebra as well as some other environments proposed by the Ministry of Education, which give you the opportunity to construct shapes with specific area, measure angles and distance. I can show them to you. They are very helpful for students because they can work with some relationships or even prove them. The software is always a source, an aid, a tool. It doesn't mean that the students will learn 100% just because they used the software. I use the software just as I would use the paper, just as I would use any other method that would help me to prove a mathematical relationship. The thing is that we do not have the luxury of time. The first exercises … the first day was devoted to learn the software, to explore the tools available in the screen, how to use the tools to work on the environment. That was before we started working with triangles. On the second day we showed the squared grid and the axis that would help them today as the activity would involve an empty screen. It is a matter of interacting with the software. When it is employed for exploring the types of triangles because there is the choice of the angle measurement, we prove that the sum of the angles of a triangle is 180 degrees. Dragging helps changing the angles. We notice that it is always 180.

R: For instance they can construct a square.

T: Yes but the point is that at this age, when you leave students completely free … in the end the result will not be the achievement of the goals set when employing the software. So at the end of the day you should not to destroy the lesson in order to use technology. We participated in a session with the Open University in England through the PI … using some tools ... well, it does not mean that by using technology you definitely build, the lesson can be a disastrous. There must be … Yes, with the software there are many possibilities but the issue is what we want and to decide what we want. We want these software to be exploited by the pupils at home. They should be uploaded on the Ministry's website or even the school's webpage so that the students can use them whenever they can. Time is very limited to only be exploited at school. I think that the software offers many possibilities, but it must be software that can also be exploited at home. If I did not have this software on my home computer, and be able to work, to explore, to see their features, how can I properly integrate them in my teaching? I would probably not use them, I could not use them. If I cannot design the activities here at the school where time is always an issue … myself and the students should have this technology at home. This is one of the suggestions I made in PI. We should be able to use these environments at home. I can devote a lesson in exploring a new environment. I can do that. But how am I going to be sure that the following the students will be able to use the tools and the opportunities this environment provides? Do not forget that in this age, if you give them the freedom they will start playing. This can be exploited … through playing we can explore. It might be positive, but also something really negative.

R: You have the possibility to hide some tools available.

R: Yes. What I am saying and thinking is that you must sacrifice a day or some days to interact with the software. This is our request for the new school year. To be able to do all these things, the proposed curriculum needs to have less objectives. Removing content from the curriculum does not mean that we will actually have the time to cover it. Will it affect me if I dedicate a day or two to learn the software? I will not sacrifice the lessons as this will have a purpose. But there is the issue of the pressure of time.

R: Do you know what DGE involve?

T: Instead of cutting, moving the triangle, I can drag it. I have been learning how to use GeoGebra the last two year. You can say that I am still a beginner as there are tools that I have not discovered yet. I also have this at home as exploring the environment only at the school is not enough. If I am teaching this lesson the following year, I will have more options as I will more acquainted with the environment. Now I only had one choice. Generally, there are numerous environments. The issue is how they are being utilized by the teachers. For instance, primary and secondary school teachers are expected to utilize DGE. In secondary school, teachers are being trained how to employ for example Cabri in the classroom. This means that the students do not interact themselves with the environment. And I am wondering whether this offers something to the students … if they are not working themselves on the computer.

R: Are the students given the opportunity to get familiar with DGE before being asked to engage with tasks?

T: Yes.

APPENDIX VI

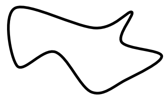
AN EXAMPLE OF THE TRANSCRIPT OF THE OBSERVATION OF THE CLASSROOM

PHASE 3, DAY 1, TEACHING PERIOD 1

T: What is circle?

S1: It is a shape that does not have sides or angles.

T: S1 says that a circle is a shape without sides or angles. I draw a shape according to this definition. According to what S1 said this is a circle.



Ss: This is not a circle.

S2: A straight shape.

S3: Without curves.

T: I want an accurate definition. S4

S4: We call a circle the shape that … it has the shape of a sphere.

T: What is the difference between a circle and a sphere?

S5: The sphere has volume.

T: The sphere has volume, it is 3-dimensional, whereas a circle is …

Ss: Flat.

T: Flat … Thus, a circle is a flat shape whereas a sphere is a 3-dimensional shape. Which shape do we call circle S6?

S6: The shape that does not have angles.

T: Yes.

S6: And has a curve as a side … a curve *(he draws a circle in the air with his hand).*

T: Like this? *(the teacher draws an ellipse on the whiteboard)*

Ss: No.

S6: No, I mean …it is like *(he is using his hands to show what he means)*

T: We said that in mathematics, our definitions must be accurate. Is there a detail that is missing?

S7: A circle is a flat shape.

T: Correct.

S7: That …

T: A circle has some characteristics.

S8: You take the compasses …

T: Yes.

S9: Oh I know.

S8: The center of the circle.

S9: When you fold it the two parts are equal.

T: This applies for this shape as well *(the ellipse).*

S9: It's more circular.

S8: I know.

S10: Because the distance from the center to the …

T: Circumference.

S10: It's the same.

T: Exactly.

S11: Mrs I am trying to remember what we call this line *(he means the radius).*

T: We will talk about that later. Thus, S11 says that a circle is the shape that, according to S8 has a center, has a circular circumference and all the points of the circumference have the same distance from the center. Right?

Ss: Yes.

T: Based on what we have said so far, look at these shapes that Mrs Maria has on the screen. Are these circles?

Ss: No.

T: Christo, is this a circle?

S: No.

T: I don't accept your answer.

S: No its not.

T: Why?

S: They are not circles because …

T: Because … you were not paying any attention earlier … Micaela

S: We have one circle here …

T: Which one is the circle?

S: There … on the right … the other shapes are not circles because their center does not have the same distance from their circumference.

T: Yes.

S: The others are not circles because their center … isn't in the middle … the center is not equidistant from the circumference.

S: Mrs, even my grandmother would know that this is the circle.

T: This shape is a circle because all the points of its circumference … I am using some words now … all the points of the circumference are equidistant from the center.

Ss: Yes.

(The teacher draws the radius in the circle)

S: I was about to say what we call this.

T: This? *(she points the radius).*

S: Yes, the line.

T: This is not important at the moment. We will work on that later. Why isn't this shape a circle? *(she points at the ellipse).*

S: Because if you draw a line …

T: If we put the center here …

S: And you draw a horizontal and perpendicular line …

T: Yes.

S: They are not the same.

T: They are not equidistant …

S: They are not equivalent.

S: You mean the length.

T: The distance. What about this shape?

S: Because it has an angle.

S: It has two sides and one angle.

S: A right angle.

S: It's an acute angle.

T: It's not a right angle. And it also has a curved side. Thus, it is not a circle since it has sides.

S: You can see that it is not a circle.

T: What about this shape?

S: It's a semicircle.

S: No, it is a half circle.

T: A half circle is called semicircle.

S: Because if you divide the circle in two …

T: But why is it a semicircle? How do I know that?

S: Because if you add the other half …

S: No.

S: We need to measure with the ruler.

S: We can use a mirror.

T: This is a nice idea. Your answer is related with symmetry. If you place the mirror you will have a whole shape.

S: Because where you have the point.

T: The center.

S: You can draw another line from the center to the circumference.

The teacher follows the student's instructions.

S: Yes.

T: If we do this, all the points of the circumference will be equidistant from the center. But for this shape to be circle, as M said, we need to continue drawing the shape.

S: Don't you need to measure the …

T: Why can't I call this a circle? Since the points of the circumference will be equidistant from the center.

S: Because it is not 360°.

T: This is one reason. The circle has 360°.

S: The circle doesn't have lines.

T: Since it has a line it is certainly not a circle. This shape is part of a circle. Thus, who can tell me the definition of a circle? A precise definition.

S: To be a circle …

T: Circle is …

S: It needs to be 360° and all the circumferences …

T: No, one circumference.

S: Well to be a circle it needs not to have a line.

T: To begin with, it is a plane geometric shape.

S: Yes, it doesn't have a line.

T: Yes. What does it have?

S: The distance needs to be … to have curves.

T: One curve. What do we call this curve?

S: Perimeter.

S: Circumference.

S: And the distance of the center to the circumference is equal.

T: All the points of the circumference are equidistant from the center. Right?

Ss: Yes.

T: It is a plane geometric shape, it is a curved line … when we say curved line we mean that it doesn't have a side or an angle … we did this in Year 5 when we were exploring the types of lines.

Ss: Oh yes.

T: It is a curved line. Write it in your notebook. When you finish, draw a circle using either a coin or a compasses.

S: Big or small?

T: It does not matter. *(The teacher draws a circle on the whiteboard).*

We name the center O and the point on the circumference B. What is OB called P?

S: Well … if I knew …

S: Radius.

T: Nice. Say is out loud.

S: Radius.

T: Well done. What does this word remind you?

S: The radius (spoke) of the bicycle.

T: Do we understand now why we call these lines of the wheels radii?

S: Because it is circular.

T: The wheel is a circle. And what is the relationship between the radii with the circle?

S: The radii start from the center and end up in the circumference.

T: Well done. The radii start from the center and end up in the circumference and are equidistant from the center. Now, if I extend BO? I name the point on the circumference A. What do I call this line? BA?

S: Radius.

T: AB. Notice that it starts from one point of the circumference and ends up on the opposite point of the circumference. What do we call this segment?

S: Diameter.

T: Diameter.

S: Ah, what divides the circle …

T: Yes it is called the diameter. It divides the circle in two equal parts. We will investigate this later. Write it down in your notebook. Don't forget to use your ruler.

APPENDIX VII

TRANSCRIPTS OF AN INFORMAL DISCUSSION WITH THE TEACHER

Phase 2: Mapping the current situation of the classroom

Day 3

-When I asked the teacher whether I could borrow the teacher's guidance books, she said that I can find them in the website of Primary Education. She then said that she does not use these books as she was teaching using the student's textbooks long before the teacher's guidance book became available to teachers.

-While I mentioned that the lesson was really good, as I was excited with the fact that the students were exploring tasks on GeoGebra, the teacher did not feel the same way. She was not sure as she said that valuable time is lost and that even though you might be well organized and have planned everything, when you are working with computers unexpected events might occur. Nevertheless, we both noticed that the students seemed to be more comfortable and confident to experiment while working on the computer. The teacher also said that compared to Tuesday where some students did not form the groups allocated by the teacher but made pairs with their 'friends', this day the teacher did not have to give the same instructions.

APPENDIX VIII

EEUCATIONAL GOALS FOR 'CIRCLE'

Cognitive goals as presented in the teacher's guidance book.

Section: Circle

With the completion of the lesson the students should be able to:

- Construct geometrical shapes by using several instruments
- Recognizing the elements of the circle (center, radius, diameter, circumference)
- Recognize the relationship that exists between the radius of the circle and its diameter
- Discover the number 'π' after estimating the circumference of circular objects
- Calculate the length of a circle
- Calculate the area of the circumference of a circle when they know its radius
- Solve problems related with the calculation of the length of the circumference of circles.

Additional activities (teacher's guidance book)
After introducing students to the concepts related with the circle, the students can do the crossroad. The goal of this exercise is the clarification of the terms that are being used in the lesson. The students can be given colorful sheets and be asked to construct decorations with circular shapes of different dimensions. This can be in groups of students and take the form of a project.

Section: Circle- investigation of the area of the circle

With the completion of the lesson the students should be able to:

- Estimate the area of circular discs by using squared paper
- Discover the number 'π' by estimating the area of circular objects
- Find the area of circular objects by using the area of the parallelogram (method of Archimedes)
- Find the area of circular objects when they know the radius

- Solve problems related with the calculation of the area of a circle

Additional activities (teacher's guidance book)

The teacher can give students exercises offered in the appendix in the teacher's book.

APPENDIX IX

WORKSHEET FOR DGE-BASED TASKS

1) We will explore the relation between the radius and the area of the circle.

   How can we achieve that? (in which ways?)

2) AB has the same length as the diameter of the circle.

a) Change the length AB. What do you observe?

b) What do you observe on the axis when you change the diameter?

c) What do we measure on the horizontal axis and what on the vertical axis?

d) Which part in the axis shows the area of the circle and which one the diameter?

3) Click 'Trace On/Off'

   Click on the point on the graph

   Change the length of AB

a) What do you observe?

b) What is your conclusion?

4) Calculate the ratio of the diameter to the circumference of the circle.

   What do you observe?

   What can we conclude for the relation between the radius and the area of the circle?