Supplement to Bounds On Treatment Effects On Transitions

Online Appendix C: Average treatment effect on survivors

In this appendix we consider the average effect when averaging over the subpopulation of individuals who would have survived until t under both treatment and no-treatment. We call this average effect the Average Treatment Effect on Survivors, ATES_t:

Definition 1 *Average Treatment Effect on Survivors (ATES)*

$$\text{ATES}_t = \mathbb{E}\left(Y_t^1 | \overline{Y}_{t-1}^1 = 0, Y_{t-1}^0 = 0\right) - \mathbb{E}\left(Y_t^0 | \overline{Y}_{t-1}^1 = 0, Y_{t-1}^0 = 0\right)$$

The bounds for ATES_t are given in Theorem 1.

Theorem 1 (Bounds on ATES) *Suppose that Assumption 1 holds. If* $\Pr\left(\overline{Y}_{t-1} = 0 | D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0 | D = 0\right) - 1 \leq 0$*, then* ATES_t *is not defined.*

If $\Pr\left(\overline{Y}_{t-1} = 0 | D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0 | D = 0\right) - 1 > 0$*, then we have the following sharp bounds*

$$\max\left\{0, \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | D = 1) + \Pr\left(\overline{Y}_{t-1} = 0 | D = 0\right) - 1}{\Pr\left(\overline{Y}_{t-1} = 0 | D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0 | D = 0\right) - 1}\right\} -$$

$$\min\left\{1, \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | D = 0)}{\Pr\left(\overline{Y}_{t-1} = 0 | D = 0\right) + \Pr\left(\overline{Y}_{t-1} = 0 | D = 1\right) - 1}\right\} \leq \text{ATES}_t \leq$$

$$\min\left\{1, \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | D = 1)}{\Pr\left(\overline{Y}_{t-1} = 0 | D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0 | D = 0\right) - 1}\right\} -$$

$$\max\left\{0, \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | D = 0) + \Pr\left(\overline{Y}_{t-1} = 0 | D = 1\right) - 1}{\Pr \overline{Y}_{t-1} = 0 | D = 1) + \Pr\left(\overline{Y}_{t-1} = 0 | D = 1\right) - 1}\right\}.$$

Proof: First, consider bounds on $\mathbb{E}\left[Y_t^1 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] = p_t^1(1|0,0)$. By Assumption 2

$$\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | D = 1) = \Pr(Y_t^1 = 1, \overline{Y}_{t-1}^1 = 0).$$

By the law of total probability

$$\Pr(Y_t^1 = 1, \overline{Y}_{t-1}^1 = 0) = p_t^0(1|0,0) p_{t-1}(0,0) + p_t^0(1|0,\neq 0) p_{t-1}(0,\neq 0)$$

Therefore,

$$\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | D = 1) = p_t^0(1|0,0) p_{t-1}(0,0) + p_t^0(1|0,\neq 0) p_{t-1}(0,\neq 0)$$

Solving for $p_t^1(1|0,0) = \mathbb{E}\left[Y_t^1 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right]$ gives

$$\mathbb{E}\left[Y_t^1 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] = \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | D = 1) - p_t^0(1|0,\neq 0) p_{t-1}(0,\neq 0)}{p_{t-1}(0,0)}$$

The expression on the right-hand side is decreasing in $p_t^0(1|0, \neq 0)$. The lower bound is obtained by setting $p_t^0(1|0, \neq 0)$ at 1 and the upper bound by setting $p_t^0(1|0, \neq 0)$ at 0.

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) - p_{t-1}(0, \neq 0)}{p_{t-1}(0,0)}$$

$$\leq \mathbb{E}\left[Y_t^1 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{p_{t-1}(0,0)}.$$

Because

$$\Pr(\overline{Y}_{t-1} = 0|D = 1) = p_{t-1}(0,0) + p_{t-1}(0, \neq 0)$$

we have

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) - \Pr(\overline{Y}_{t-1} = 0|D = 1) + p_{t-1}(0,0)}{p_{t-1}(0,0)}$$

$$\leq \mathbb{E}\left[Y_t^1 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{p_{t-1}(0,0)}.$$

The upper bound is decreasing and the lower bound is increasing in $p_{t-1}(0,0)$. From the proof of Theorem 1 we have

$$p_{t-1}(0,0) \geq \max\left\{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1, 0\right\}.$$

If $\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1 > 0$ then we are sure that there are survivors in both treatment arms. Upon substitution of this lower bound

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}$$

$$\leq \mathbb{E}\left[Y_t^1 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}.$$

By an analogous argument we have

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0) + \Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) - 1}{\Pr \overline{Y}_{t-1} = 0|D = 1) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}$$

$$\leq \mathbb{E}\left[Y_t^0 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0)}{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}.$$

Substitution of these results for $\mathbb{E}\left[Y_t^1 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right]$ and $\mathbb{E}\left[Y_t^0 | \overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right]$ and because both probabilites are bounded by zero and one gives the bounds on ATES_t.