Nicely Embedded Curves in Symplectic Cobordisms

Alexandru Cioba

A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

of

University College London.

Department of Mathematics

University College London

April 24, 2018

I, Alexandru Cioba, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

In the following text we investigate the properties of moduli spaces of so-called "nicely embedded" curves in Liouville symplectic cobordisms. We exhibit a topological obstruction which occurs in contact manifolds cobordant to the tight 3-sphere, namely the presence of an unknotted Reeb orbit, with self-linking number -1. The same result is shown to apply in overtwisted manifolds. A similar proof establishes the result in reducible contact manifolds. Along the way we recall several classical results, and prove a series of auxiliary lemmas. Throughout most of the work we limit ourselves to the context of 4-dimensional cobordisms, where intersection theory for pseudoholomorphic curves is an indispensable tool.

Acknowledgements

This thesis would not exist without the continued guidance of my supervisor, Professor Chris Wendl. His care and support, his commitment in providing me with achievable goals and fondness of teaching me the ins and outs of the theory described herein are undoubtedly the foundation of all my achievements over these many years. He has generously relinquished many of his valuable ideas for my benefit and my one regret is that I was unable to bring to fruition all of the promising research topics he suggested.

In virtually equal measure, my second supervisor, Dr. Jonathan Evans, has invested an enormous amount of effort into my mathematical upbringing. It was the contact with his research, primarily, that invited me to explore topics outside my immediate area of focus, and the many hours he spent teaching and explaining to me the various facets of theories I was unfamiliar with I have found invaluable.

Many others made the endeavour of producing and writing up this text possible. Oldrich Spacil and Jacqueline Espina acted both as mentors and peers over the course of my stay at UCL. Countless conversations I have had with them have enriched my experience as a mathematician and as a student. Professors Michael Hutchings and Jacob Rasmussen took time to give me very intensive schoolings on their work, their help could not have been more appreciated. Professor Martin Guest kindly hosted me in Japan not once but twice, and has shown a humbling interest in my research. Over the final months of my write-up, Professor Terry Lyons has provided me with much needed encouragement and a wealth of valuable advice. I am indebted to him for his patience and for many enlightening conversations.

I would like to extend my warmest gratitude to my fellow aspiring PhD. students, many of whom I have had the pleasure of meeting only briefly, but who contributed regardless, in an essential fashion, to creating the stimulating community of peers one needs in order to grow through these formative years.

To my parents, Mianda and Gheorghe, I am grateful most of all. Through patience and all-encompassing love, they have stood beside me through all my joys and challenges.

The University College London has been my home over the past years, and have made this endeavour possible through their support, both financial and moral. The Alan Turing Institute graciously hosted me over my final months as a PhD. student and facilitated a good part of the writing process for this text.

Contents

List of Figures

Chapter 1

Introductory Material

1.1 Introduction

The following sections loosely describe the structure of the main text and set up the context of our results. Unless otherwise indicated, the treatment coincides with that of [CW].

1.1.1 Statement of the main results

Contact structures arise in the context of Hamiltonian dynamics via the notion of *convexity*: a convex hypersurface in a symplectic manifold naturally inherits a contact structure, and the orbits of its Reeb vector field then match the Hamiltonian orbits defined by any Hamiltonian function that has the hypersurface as a regular level set. In this text, we consider contact structures that are induced on convex and concave *boundaries* of symplectic manifolds, i.e. symplectic cobordisms. Our main theorem relates the existence of exact symplectic cobordisms between given contact manifolds to a dynamical condition on their Reeb vector fields. In particular, we will restrict attention to dimension three and discuss the existence of closed Reeb orbits $\gamma : S^1 \to M$ that are not only contractible but also **unknotted**, meaning

$$\gamma = f|_{\partial\mathbb{D}^2} \qquad \text{for some embedding } f : \mathbb{D}^2 \hookrightarrow M,$$

where $\mathbb{D}^2 \subset \mathbb{C}$ denotes the closed unit disk. All definitions relevant to the following statements may be found in §2.1.1).

Theorem 1.1.1. *Assume (M,ξ) is a closed contact 3-manifold that admits a Liouville cobordism to the standard contact 3-sphere (S^3,ξ_{std}). Then for every nondegenerate contact form α on (M,ξ), the Reeb vector field R_α admits a simple closed orbit γ whose image is the boundary of an embedded disk $\mathscr{D} \subset M$. Moreover, the Conley-Zehnder index and self-linking number of γ with respect to $\mathscr{D}$ satisfy*

$$\mu_{\text{CZ}}(\gamma;\mathscr{D}) \in \{2,3\} \quad \textit{and} \quad \text{sl}(\gamma;\mathscr{D}) = -1.$$

A minor variation on the same techniques in the spirit of [Hof93] will also imply the following:

Theorem 1.1.2. *Assume (M,ξ) is a closed contact 3-manifold and that either of the following is true:*

1. *M is reducible, i.e. it contains an embedded 2-sphere that does not bound an embedded ball;*

2. *(M,ξ) admits a Liouville cobordism to an overtwisted contact manifold.*

Then for every nondegenerate contact form α on (M,ξ), the Reeb vector field R_α admits a simple closed orbit γ whose image is the boundary of an embedded disk $\mathscr{D} \subset M$ such that

$$\mu_{\text{CZ}}(\gamma;\mathscr{D}) = 2 \quad \textit{and} \quad \text{sl}(\gamma;\mathscr{D}) = -1.$$

Recall that an oriented 3-manifold is reducible if and only if it is either $S^1 \times S^2$ or $M_1 \# M_2$ for a pair of closed oriented 3-manifolds that are not spheres. This condition is now known to be equivalent to the hypothesis $\pi_2(M) \neq 0$ used in [Hof93]: in one direction this follows from the sphere theorem for 3-manifolds, and in the other, from [Hat, Prop. 3.10] and the Poincaré conjecture. Note that both of the above theorems require nondegeneracy of the contact form α, but it is possible for the sake of applications to weaken this condition; see Theorem 1.1.12 below.

1.1.2 Context

The prototype for Theorems 1.1.1 and 1.1.2 is a 20-year-old result of Hofer-Wysocki-Zehnder [HWZ96c], which amounts to the case $(M,\xi) = (S^3,\xi_{\text{std}})$ of Theorem 1.1.1. The result in [HWZ96c] was in some sense far ahead of its time, as it required ideas from both the compactness theory [BEH$^+$03] and the intersection theory [Sie11] of punctured holomorphic curves, but it appeared several years before either of those theories were developed in earnest. In the mean time the available techniques have improved, and our proofs will make use of those improvements.

A weaker version of Theorem 1.1.1 can be shown to hold in all dimensions, namely:

Theorem 1.1.3. *If (M,ξ) is a closed $(2n-1)$-dimensional contact manifold admitting a Liouville cobordism to a standard contact sphere $(S^{2n-1},\xi_{\text{std}})$, then every contact form for (M,ξ) admits a contractible closed Reeb orbit.*

This result can largely be attributed to Hofer, as most of the ideas needed for its proof are present in [Hof93]. An alternative proof using symplectic homology has recently been announced by Albers, Cieliebak and Oancea (cf. [CO]), and a detailed proof of the 3-dimensional case has also been given by Geiges and Zehmisch [GZ13a, GZ13b]. We will sketch a proof for the general case in §1.1.4. Analogous results that may be viewed as higher-dimensional versions of Theorem 1.1.2 have appeared in [AH09, NR11, GZ16, GNW16]. The conclusions of our main results however are stronger and uniquely low dimensional: for instance in §1.1.3 below, we will see examples of contact 3-manifolds that always admit contractible but not necessarily unknotted Reeb orbits. Theorem 1.1.1 thus gives a new means of proving that these examples cannot be exactly cobordant to the standard 3-sphere.

We are aware of three general classes of contact 3-manifolds that satisfy the hypothesis of Theorem 1.1.1.

Example 1.1.4. If ξ is overtwisted, then a theorem of Etnyre and Honda [EH02] provides Stein cobordisms from (M,ξ) to any other contact 3-manifold, so in particular to (S^3,ξ_{std}). Of course, in this case Theorem 1.1.2 also applies and gives a slightly stronger result.

Example 1.1.5. Suppose (M,ξ) is subcritically Stein fillable, or equivalently, that it can be obtained by performing contact connected sums on copies of the tight S^3 and $S^1 \times S^2$. In this case, (M,ξ) is the convex boundary of a Weinstein domain W constructed by attaching 1-handles to a ball, and these 1-handles can then be cancelled by attaching suitable Weinstein 2-handles. This procedure embeds W into the standard 4-ball as a Weinstein subdomain and thus produces a Weinstein cobordism from (M,ξ) to (S^3,ξ_{std}). Note that Theorem 1.1.2 also applies in this case unless $M = S^3$.

The third class of examples was brought to our attention by Emmy Murphy.

Example 1.1.6. Suppose $L \subset [1,\infty) \times S^3$ is an *exact Lagrangian cap* for some Legendrian knot Λ in (S^3,ξ_{std}), i.e. L is a compact Lagrangian submanifold properly embedded in the top half of the symplectization $\mathbb{R} \times S^3$, such that $\partial L = \{1\} \times \Lambda$, L is tangent near its boundary to a globally defined Liouville vector field pointing transversely inward at $\{1\} \times S^3$, and the restriction of the corresponding Liouville form to L is exact. A result of Francesco Lin [Lin] guarantees that such caps always exist after stabilizing Λ sufficiently many times. Now suppose $\mathscr{U}_L$ is an open neighbourhood of L in $[1,\infty) \times S^3$, where the latter is viewed as sitting on top of the standard Weinstein filling B^4 of (S^3,ξ_{std}). This neighbourhood can be choosen such that, after smoothing corners, $B^4 \cup \overline{\mathscr{U}}_L$ is a Weinstein filling of some contact 3-manifold (M,ξ), and $\big([1,T] \times S^3\big) \setminus \mathscr{U}_L$ for suitable $T > 1$ defines a Liouville cobordism W_+ from (M,ξ) to (S^3,ξ_{std}), see Figure 1.1. Using a Morse function on L that has one index 2 critical point and an inward gradient at ∂L, one can find a Weinstein handle decomposition of $B^4 \cup \overline{\mathscr{U}}_L$ having exactly one 2-handle (see Remark A.0.2), thus $B^4 \cup \overline{\mathscr{U}}_L$ is not subcritical, and it follows from the uniqueness of Stein fillings in the subcritical case [CE12, Theorem 16.9(c)] that (M,ξ) is not subcritically fillable. For more details on this construction, see Appendix A.

One can now use a well-known result of Eliashberg [Eli90, CE12] to extract from this example contact 3-manifolds other than (S^3,ξ_{std}) to which Theorem 1.1.1 applies but Theorem 1.1.2 does not. Indeed, while $(M,\xi) = \partial(B^4 \cup \overline{\mathscr{U}}_L)$ could be reducible, it is Stein fillable and therefore tight, so Colin [Col97] (see also [Gei08,

§4.12]) provides a prime decomposition

$$(M,\xi) = (M_1,\xi_1)\#\ldots\#(M_k,\xi_k),$$

and Eliashberg's theorem implies that $B^4 \cup \overline{\mathscr{U}}_L$ must be Weinstein deformation equivalent to a domain obtained by attaching Weinstein 1-handles to Weinstein fillings of the summands. But the summands cannot all be $S^1 \times S^2$ since (M,ξ) is not subcritically fillable, so at least one of them is an irreducible tight contact 3-manifold admitting a Liouville cobordism to (S^3,ξ_{std}).

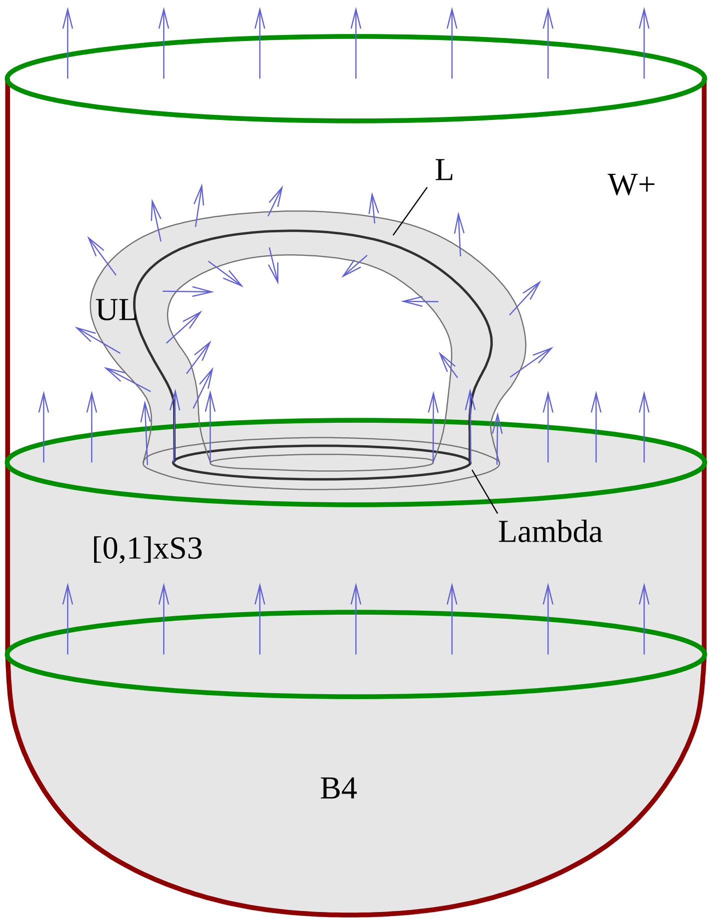

Figure 1.1: An exact Lagrangian cap for a Legendrian in (S^3,ξ_{std}) produces a Liouville cobordism W_+ from (M,ξ) to (S^3,ξ_{std}), where $(M,\xi) := \partial(B^4 \cup \overline{\mathscr{U}}_L)$ is not subcritically fillable.

Corollary 1.1.7. *The contact 3-manifolds (M,ξ) described in Example 1.1.6 and their prime summands all admit unknotted Reeb orbits with Conley-Zehnder index*

2 *or* 3 *and self-linking number* -1 *for every choice of nondegenerate contact form.*

The construction outlined in Example 1.1.6 also works in higher dimensions using the exact Lagrangian caps of Eliashberg-Murphy [EM13], cf. Appendix A. In this case it produces Weinstein subdomains of the standard ball which are presumably *flexible* in the sense of [CE12]. Recently, Murphy and Siegel [MS] have also found examples of nonflexible Weinstein subdomains in the standard ball, whose boundaries therefore also satisfy the hypothesis of Theorem 1.1.3.

Remark 1.1.8. It is not known whether any contact 3-manifolds satisfy the hypothesis of Theorem 1.1.2(2) without being overtwisted, though Andy Wand [Wan] has proved that the answer is no under the stronger condition that the cobordism is Stein. Theorem 1.1.2(2) may thus be interpreted as a small measure of support for the conjecture that Wand's theorem extends to Liouville cobordisms (cf. [Wena, Question 5]).

We remark that the word "Liouville" definitely cannot be dropped from the statements of any of the above theorems: for instance, any Lagrangian torus in the standard symplectic $\mathbb{R}^{2n}$ gives rise to a strong symplectic cobordism from the unit cotangent bundle of the torus to $(S^{2n-1}, \xi_{\text{std}})$, but one can easily find contact forms on the former that have no contractible Reeb orbits, corresponding to metrics on the torus with no contractible geodesics. The cobordism of course cannot be Liouville because, by a well-known theorem of Gromov [Gro85], the Lagrangian torus cannot be exact. Similarly, [Gay06] and [Wen13b] show that every contact 3-manifold with positive Giroux torsion is symplectically cobordant to something overtwisted, including e.g. the nonfillable tight 3-tori, which admit contact forms without contractible orbits.

1.1.3 Applications

Here is a specific situation in which Theorem 1.1.1 can be used to rule out the existence of exact symplectic cobordisms. Good candidates for manifolds that fail to satisfy the conclusion of the theorem are furnished by the universally tight lens spaces $L(p,q)$ for $p \neq 1$. Recall that $L(p,q)$ is defined as the quotient

$$L(p,q) = S^3/G_{p,q},$$

where $G_{p,q} \subset \mathrm{U}(2)$ denotes the cyclic group of matrices $\begin{pmatrix} e^{2\pi ik/p} & 0 \\ 0 & e^{2\pi ikq/p} \end{pmatrix}$ for $k \in \mathbb{Z}_p$, acting on the unit sphere $S^3 \subset \mathbb{C}^2$ by unitary transformations. This action preserves the standard contact form $\alpha_{\mathrm{std}} = \frac{1}{2}\sum_{j=1}^{2}(x_j\,dy_j - y_j\,dx_j)$ on S^3, written here in coordinates $(z_1,z_2) = (x_1 + iy_1, x_2 + iy_2)$, so the standard contact structure ξ_{std} on $L(p,q)$ is defined via this quotient.

Proposition 1.1.9. *For every relatively prime pair of integers $p > q \geq 1$, $L(p,q)$ admits a nondegenerate contact form with only two simple closed Reeb orbits, both of them nondegenerate and noncontractible.*

Proof. We present $(L(p,q),\xi_{\mathrm{std}})$ as a quotient of the so-called *irrational ellipsoid*. Let $\alpha_H := \frac{1}{H}\alpha_{\mathrm{std}}$ on S^3, where H is the restriction to the unit sphere $S^3 \subset \mathbb{C}^2$ of the function

$$H(z_1,z_2) = \frac{|z_1|^2}{a^2} + \frac{|z_2|^2}{b^2}$$

for some $a,b > 0$. The closed orbits for the Reeb flow on S^3 determined by α_H are then in bijective correspondence with the closed orbits on the ellipsoid $H^{-1}(1) \subset \mathbb{C}^2$ for the Hamiltonian flow of H on the standard symplectic $\mathbb{C}^2$. In particular, if a/b is irrational, then the only simple closed orbits of this flow are (up to parametrization) the embedded loops $\gamma_1,\gamma_2 : S^1 \to S^3 \subset \mathbb{C}^2$ defined by

$$\gamma_1(t) = (e^{2\pi it},0), \qquad \gamma_2(t) = (0,e^{2\pi it})$$

for $t \in S^1 = \mathbb{R}/\mathbb{Z}$, and moreover, these orbits and their multiple covers are all nondegenerate. Now since α_{std} and H are both invariant under the action of $\mathrm{U}(1)\times\mathrm{U}(1) \subset \mathrm{U}(2)$, which contains $G_{p,q}$, α_H descends to a well-defined contact form on $L(p,q)$, and this contact form is nondegenerate. But the orbits γ_1 and γ_2 project to orbits in $L(p,q)$ that are p-fold covered, so their underlying simple orbits lift to the universal cover $S^3 \to L(p,q)$ as non-closed paths since $p > 1$, hence they are noncontractible. □

Corollary 1.1.10. *For every pair of relatively prime integers $p > q \geq 1$, $(L(p,q), \xi_{\text{std}})$ admits no exact cobordism to (S^3, ξ_{std}).*

Remark 1.1.11. The Reeb flow on any universally tight $L(p,q)$ admits a contractible Reeb orbit since $\pi_1(L(p,q))$ is torsion, so previously known criteria for excluding such cobordisms do not apply. For the stronger case of Stein cobordisms, the same result was obtained by Plamenevskaya in [Pla12]

While the lens space example is relatively easy to work with, the nondegeneracy of a contact form is usually a rather difficult condition to check, and for this reason one might sometimes want to have the following technical enhancement of Theorems 1.1.1 and 1.1.2. It will be an immediate consequence of our proofs, requiring only that one pay closer attention to the relationship between periods of orbits and energies of holomorphic curves.

Theorem 1.1.12. *Assume (M, ξ) satisfies the hypotheses of either Theorem 1.1.1 or Theorem 1.1.2, and fix a contact form α_0 for (M, ξ). There exists a constant $T > 0$, dependent on α_0, such that the following holds: suppose $\alpha = f\alpha_0$ is a contact form on (M, ξ) such that*

1. *$f : M \to (0, \infty)$ satisfies $f < T$, and*
2. *All closed Reeb orbits for α with period less than T are nondegenerate.*

Then the Reeb flow of α satisfies the conclusions of Theorems 1.1.1 or 1.1.2 respectively, and the unknotted orbit can be assumed to have period less than T.

One could apply this in practice if e.g. α_0 is Morse-Bott and admits no unknotted Reeb orbits, as then one can define perturbations of α_0 as in [Bou02] whose orbits up to some arbitrarily large period are nondegenerate and still knotted—the topology of orbits with large period may be harder to control, but for Theorem 1.1.12 this does not matter.

Remark 1.1.13. We have chosen to adopt a mainly contact topological perspective on the main theorems of this thesis, but for other purposes (e.g. quantitative Reeb dynamics, cf. [GZ13a, §3.23]), one could also state more quantitatively precise versions of Theorem 1.1.12.

Note that no such enhancement is necessary for Theorem 1.1.3, which does not require nondegeneracy, see Remark 1.1.15.

1.1.4 Outline of proofs, part 1: seed curves and compactness

All proofs of theorems in this thesis follow a similar scheme, which in the case of Theorems 1.1.1 and 1.1.3 can be described as follows. Suppose $(W,d\lambda)$ is a Liouville cobordism from (M,ξ) to a standard contact sphere $(S^{2n-1},\xi_{\text{std}})$, and let $(\overline{W},d\lambda)$ denote the completion obtained by attaching cylindrical ends in the standard way (see §2.1.3). Then the positive end of $\overline{W}$ can be assumed to match the top half of the symplectization

$$\left(\mathbb{R}\times S^{2n-1},d(e^r\alpha_{\text{std}})\right),$$

where α_{std} is the standard contact form, defined by restricting the Liouville form $\lambda_{\text{std}} := \sum_{j=1}^n (x_j\,dy_j - y_j\,dx_j)$ to the unit sphere. We will assume also that the negative end matches $((-\infty,0]\times M,d(e^r\alpha))$ where α is (after a positive rescaling) an arbitrary nondegenerate contact form for (M,ξ). (The nondegeneracy assumption was not included in Theorem 1.1.3, but this assumption will be easy to remove in the final step, see Remark 1.1.15 below.)

The first step in the proof is then to choose a suitable almost complex structure J on this symplectization that admits a foliation by a $(2n-2)$-dimensional family of J-holomorphic planes, so-called "seed curves," which are asymptotic to a fixed Reeb orbit γ for α_{std} that has the smallest possible period. We will be able to verify explicitly that these planes are *Fredholm regular* for the moduli problem with fixed asymptotic orbit, hence the moduli space is cut out transversely, and moreover, there exist no other curves in $\mathbb{R}\times S^{2n-1}$ with a single positive end approaching γ. Once these curves are understood, we can regard them as living in the cylindrical end $[0,\infty) \times S^{2n-1} \subset \overline{W}$, so after extending J to a compatible almost complex structure on the rest of $(\overline{W}, d\lambda)$, they generate a nonempty moduli space $\mathscr{M}(J)$ of unparametrized J-holomorphic planes in $\overline{W}$, all asymptotic to the same simply covered Reeb orbit in the sphere, and this moduli space is a smooth $(2n-2)$-dimensional manifold for generic extensions of J since all curves in $\mathscr{M}(J)$ are somewhere injective. Our main task is then to understand the natural compactification $\overline{\mathscr{M}}(J)$ of $\mathscr{M}(J)$, that is to say, the closure of $\mathscr{M}(J)$ in the space of J-holomorphic buildings in the sense of [BEH+03]. The uniqueness of the seed curves in the positive end implies the following:

Lemma 1.1.14. *If $u \in \overline{\mathscr{M}}(J)$ is a holomorphic building with a nontrivial upper level, then it has exactly one upper level, which consists of one of the seed curves in $\mathbb{R} \times S^{2n-1}$, and all its other levels are empty.* □

The lemma means that the only way for a sequence of planes in $\mathscr{M}(J)$ to "degenerate" with something nontrivial happening at the positive end is if the planes simply escape into the positive end and become seed curves; in particular, this cannot happen to any sequence of planes that have points falling into the negative end. Theorem 1.1.3 can now be proved as follows. Let $\mathscr{M}_1(J)$ denote the smooth $2n$-dimensional moduli space consisting of curves in $\mathscr{M}(J)$ with the additional data of an interior marked point, hence there is a well-defined evaluation map

$$\mathrm{ev} : \mathscr{M}_1(J) \to \overline{W}.$$

Choose a smooth properly embedded 1-dimensional submanifold $\ell \subset \overline{W}$ with one end in $[0,\infty) \times S^{2n-1}$ and the other in $(-\infty, 0] \times M$, and perturb it to be transverse to the evaluation map. Then

$$\mathscr{M}_\ell(J) := \mathrm{ev}^{-1}(\ell)$$

is a smooth 1-dimensional manifold, and it has a unique connected component $\mathscr{M}_\ell^0(J) \subset \mathscr{M}_\ell(J)$ that contains seed curves in the positive end. This component has a noncompact end consisting of a family of seed curves that escape to $+\infty$, thus it is manifestly noncompact and therefore diffeomorphic to $\mathbb{R}$. We claim now that

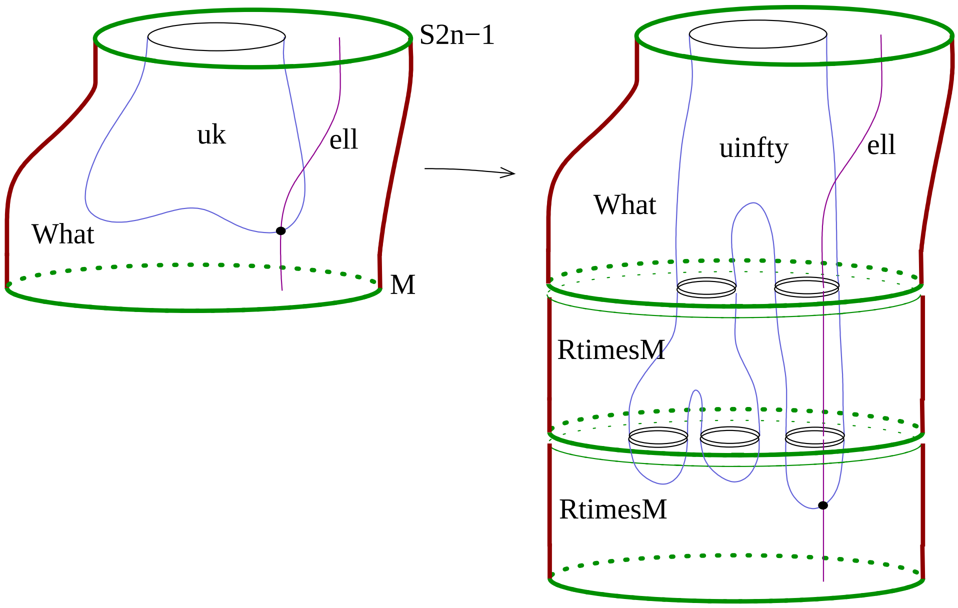

Figure 1.2: When holomorphic planes in an exact cobordism converge to a holomorphic building with nontrivial lower levels, at least one of them must include a plane.

$\mathscr{M}_\ell^0(J)$ must also contain curves with points that descend arbitrarily far into the negative end. Indeed, the SFT compactness theorem would otherwise imply that every sequence in $\mathscr{M}_\ell^0(J)$ has a subsequence convergent to either an element of $\mathscr{M}_\ell^0(J)$ or a holomorphic building of the type described in Lemma 1.1.14. But the latter can only happen if the sequence escapes through the neighbourhood of $+\infty$ in which all curves are seed curves. In particular, we obtain a contradiction by considering a noncompact sequence escaping to the *opposite* end of $\mathscr{M}_\ell^0(J) \cong \mathbb{R}$ from the one consisting of seed curves, and this proves the claim. It follows that one can find a sequence $u_k \in \mathscr{M}_\ell(J)$ of curves converging to a holomorphic building $u_\infty \in \overline{\mathscr{M}}(J)$ with a nontrivial lower level (see Figure 1.2). Since the cobordism is exact, every component curve in u_∞ must have exactly one positive end, and it follows that at least one of the curves in a lower level of u_∞ is a plane, whose asymptotic orbit is the contractible Reeb orbit promised by Theorem 1.1.3.

Remark 1.1.15. To remove the nondegeneracy assumption from Theorem 1.1.3, one can take advantage of the fact that due to the exactness of the cobordism, the contractible orbit found in the above argument comes with an a priori bound on its period. Then if α is a degenerate contact form on (M,ξ) approximated by a sequence α_k of nondegenerate contact forms, the above argument gives a sequence γ_k of contractible Reeb orbits with respect to α_k whose periods are uniformly bounded, so by Arzelà-Ascoli, these have a subsequence convergent to a contractible Reeb orbit with respect to α. Note that if the orbits γ_k are also unknotted, it is not so clear whether the limiting orbit will also be unknotted, hence the need for the more technical Theorem 1.1.12.

1.1.5 Outline of proofs, part 2: intersections

The argument described thus far is quite standard and, as mentioned earlier, is largely attributable to Hofer [Hof93] (though the use of the path $\ell \subset \overline{W}$ to define a 1-dimensional submanifold of the moduli space is borrowed from Niederkrüger [Nie06]). The arguments required for finding an orbit that is not only contractible but also *unknotted* are significantly subtler, and here we must make liberal use of Siefring's intersection theory [Sie11] in the low-dimensional setting.

To explain the idea, we briefly recall the notion of **nicely embedded** holomorphic curves, introduced by Chris Wendl in [Wen10a, Wen10b]. The precise definition will be reviewed in §2.1.4.5, but in essence, a holomorphic curve $u : \dot{\Sigma} \to \overline{W}$ in a completed 4-dimensional symplectic cobordism $\overline{W}$ is nicely embedded if it has the necessary intersection-theoretic properties to guarantee that it *does not intersect its neighbors* in the moduli space. This condition implies that the moduli space near u can be at most 2-dimensional, and in the 2-dimensional case the curves near u form the leaves of a foliation on a neighbourhood of $u(\dot{\Sigma})$ in $\overline{W}$. If $\overline{W}$ is a symplectization $\mathbb{R} \times M$ or the image of u is confined to a cylindrical end, then being nicely embedded has the additional implication that u projects to an embedding into M, i.e. u can be written as

$$u = (u_{\mathbb{R}}, u_M) : \dot{\Sigma} \to \mathbb{R} \times M,$$

where the map $u_M : \dot{\Sigma} \to M$ is also an embedding. It is easy to show that the seed curves we find in the symplectization of (S^3, ξ_{std}) are nicely embedded, and the homotopy invariance of the intersection theory then implies that the same is true for all curves in $\mathscr{M}(J)$.

The fundamental principle behind the proof of Theorems 1.1.1 and 1.1.2 is then the notion that "nice curves degenerate nicely," i.e. if a sequence $u_k \in \mathscr{M}(J)$ converges to a holomorphic building $u_\infty \in \overline{\mathscr{M}}(J)$, then we should expect the component curves in levels of u_∞ to be nicely embedded. This statement as such is false in full generality (see [Wen10b, Example 4.22 and Remark 4.23] for counterexamples), but we will show that it is true in the present situation. As a consequence, the plane we find in a lower level of u_∞ has the form $(u_\mathbb{R}, u_M) : \mathbb{C} \to \mathbb{R} \times M$, where $u_M : \mathbb{C} \to M$ is an embedding asymptotic to a contractible Reeb orbit.

There remains one complication: the fact that $u : \mathbb{C} \to \mathbb{R} \times M$ is nicely embedded does not guarantee that its asymptotic orbit must be simply covered, i.e. the image of $u_M : \mathbb{C} \to M$ might look like an immersed disk that is embedded on the interior but multiply covered on its boundary. We will show in fact that this can happen, but only in very specific ways, and to prove it, we develop a "local adjunction formula" for holomorphic annuli breaking along a Reeb orbit.

1.1.6 Local adjunction

We now briefly interrupt the outline of the proof to describe a tool of more general applicability. To set the stage, suppose that $\alpha_k \to \alpha_\infty$ is a $\mathscr{C}^\infty$-convergent sequence of contact forms on a 3-manifold M, and $J_k \to J_\infty$ is a corresponding sequence with each J_k belonging to the usual space (see §2.1.1) of admissible translation-invariant almost complex structures on the symplectization $(\mathbb{R} \times M, d(e^r \alpha_k))$. Assume then that

$$u_k : ([-k,k] \times S^1, i) \to (\mathbb{R} \times M, J_k)$$

is a sequence of pseudoholomorphic annuli which are converging in the sense of SFT compactness to a broken curve

$$u_k \to (u_\infty^+ | u_\infty^-),$$

where

$$u_\infty^+ : ((-\infty, 0] \times S^1, i) \to (\mathbb{R} \times M, J_\infty),$$
$$u_\infty^- : ([0, \infty) \times S^1, i) \to (\mathbb{R} \times M, J_\infty)$$

are J_∞-holomorphic half-cylinders each asymptotic to a nondegenerate Reeb orbit γ with covering multiplicity $m(\gamma)$; see Figure 1.3. This is intended as a local picture of the neighbourhood of a breaking orbit as a sequence of smooth finite energy curves converges to a holomorphic building as in [BEH$^+$03]. Recall from [Sie08] that for any finite energy punctured holomorphic curve that is not a multiple cover, sufficiently small neighbourhoods of each puncture are always embedded, hence if u_∞^+ and u_∞^- are not multiply covered then we are free to assume without loss of generality that both are embedded. This implies that each u_k is also embedded near the boundary of $[-k, k] \times S^1$ for sufficiently large k, but if $m(\gamma) > 1$, then u_k can have finitely many double points and critical points that "disappear into the breaking orbit" in the limit. See §2.1.4.5 for precise definitions of each of the quantities discussed below. We let

$$\delta(u_k) \geq 0$$

denote the algebraic count of double points and critical points of u_k: this is a non-negative integer that equals zero if and only if u_k is embedded. The half-cylinders $u_\infty^\pm$ are embedded by assumption, but if $m(\gamma) > 1$, then they may have "hidden double points at infinity" in the sense of [Sie11], i.e. double points that must emerge from infinity under generic perturbations of the curves. We denote the algebraic counts of these hidden double points by

$$\delta_\infty(u_\infty^\pm) \geq 0;$$

they are nonnegative integers that vanish if and only if generic perturbations of $u_\infty^\pm$ remain embedded. We denote by

$$\bar{\sigma}_\pm(\gamma) \geq 1$$

the so-called **spectral covering numbers** of γ as in [Sie11]: these are covering multiplicities of certain asymptotic eigenfunctions of γ, and are thus positive integers that equal 1 if and only if those eigenfunctions are simply covered (which is always the case e.g. if $m(\gamma) = 1$). For one last piece of notation, we let

$$p(\gamma) \in \{0, 1\}$$

denote the **parity** of γ, i.e. its Conley-Zehnder index modulo 2. The result we will prove in §3.1 can now be stated as follows.

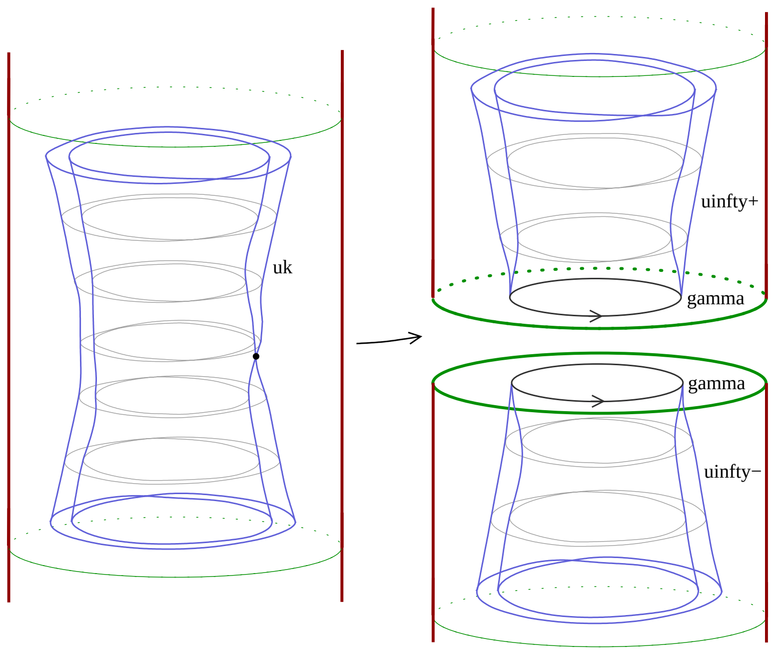

Figure 1.3: A sequence of pseudoholomorphic annuli u_k converging to a broken annulus consisting of embedded half-cylinders $u_\infty^\pm$ asymptotic to a doubly covered breaking orbit γ. In this case, u_k can have double points that disappear in the limit.

Theorem 1.1.16 (local adjunction)**.** *In the setting described above, assume $u_k \to (u_\infty^+|u_\infty^-)$ is a sequence of holomorphic annuli in $\mathbb{R} \times M$ converging to a broken pair of half-cylinders, where u_∞^+ and u_∞^- are both embedded and asymptotic to a nondegenerate Reeb orbit γ with covering multiplicity $m(\gamma)$, parity $p(\gamma)$ and spectral* *covering numbers $\bar{\sigma}_{\pm}(\gamma)$. Then for all k sufficiently large,*

$$2\delta(u_k) = 2[\delta_\infty(u_\infty^+) + \delta_\infty(u_\infty^-)] + [\bar{\sigma}_+(\gamma) - 1] + [\bar{\sigma}_-(\gamma) - 1] + [m(\gamma) - 1]\, p(\gamma).$$

The usefulness of this theorem lies in the fact that every bracketed term on the right hand side of the formula is known a priori to be nonnegative, so if we also know that the annuli u_k are embedded, then all these terms must vanish. In that case, we will easily be able to deduce the following consequence:

Corollary 1.1.17. *In the setting of Theorem 1.1.16, if u_k is embedded for every k, then one of the following is true:*

- *γ is a simply covered orbit;*
- *γ is a double cover of a simply covered orbit γ' such that $p(\gamma') = 1$ and $p(\gamma) = 0$, and both of the half-cylinders $u_\infty^\pm$ have no hidden double points at infinity.*

1.1.7 Outline of proofs, conclusion

In the situation at hand, our degenerating curves are all embedded, so Corollary 1.1.17 applies and we conclude that the breaking orbit is always either simply covered or a double cover of a negative hyperbolic orbit, what is known in the SFT literature (cf. [EGH00]) as a **bad orbit**. In the first case we are done, and in the second, we will show that degenerations of this form can always be glued back together so that they are interpreted as *interior points* of the compactified moduli space, and the moduli space must therefore have additional degenerations besides this. In other words, breaking along bad orbits can happen, but it cannot be the only type of breaking that happens, so there is still guaranteed to be some breaking along a simple orbit somewhere, producing a nicely embedded curve asymptotic to an unknotted orbit. The resulting constraints on the Conley-Zehnder index and self-linking number of the orbit then follow by a straightforward and essentially standard topological computation.

The major differences between the above summary and the proof of Theorem 1.1.2 are as follows. For the first statement in the theorem, the symplectic cobordism W is taken to be symplectically trivial, i.e. its completion has the form $(\mathbb{R} \times M, d\lambda)$, where λ is a Liouville form matching $e^r \alpha_\pm$ near $\{\pm\infty\} \times M$, and $\alpha_\pm$ are two nondegenerate contact forms for (M, ξ), of which α_- is given but α_+ is carefully chosen. The assumptions of the theorem then allow us to choose α_+ and a compatible almost complex structure J_+ near $+\infty$ so that we find a smooth 1-dimensional moduli space of seed curves. Since this moduli space is only 1- and not 2-dimensional, it does not form a foliation, but the curves are still nicely embedded and the same principles therefore apply: a variation on the same argument described above leads to a nicely embedded plane asymptotic to a simple Reeb orbit for α_-.

Here is an outline of the remainder of the text. In §2.1, we clarify the essential definitions and review the necessary facts about punctured holomorphic curves and their intersection theory in dimension four. The purpose of §2.2 is then to specify the data at the positive ends of our symplectic cobordisms, construct the seed curves and prove that they are Fredholm regular and nicely embedded. Theorem 1.1.16 and Corollary 1.1.17 on local adjunction for breaking holomorphic annuli are proved in §3.1. Finally, §3.2 carries out the main compactness arguments, and §3.3 completes the proofs of the main theorems.

Chapter 2

Preliminary Results

2.1 Preparation

The purpose of this section is to fix definitions and review some known results that will be needed in the rest of the text.

2.1.1 Contact manifolds and symplectic cobordisms

We begin by reviewing some basic definitions from contact geometry and the precise way in which contact manifolds arise as hypersurfaces or boundary components of symplectic manifolds.

Suppose (W, ω) is a $2n$-dimensional symplectic manifold, and $M \subset W$ is a smooth oriented hypersurface. We say that M is **convex** if there exists a Liouville vector field near M that is positively transverse to M: here a vector field V is called **Liouville** if its flow dilates the symplectic form, meaning $\mathscr{L}_V \omega = \omega$. This is equivalent to the condition that the dual 1-form $\lambda := \omega(V, \cdot)$ satisfies $d\lambda = \omega$, and being positively transverse to M then means that the restriction $\alpha := \lambda|_{TM}$ satisfies

$$\alpha \wedge (d\alpha)^{n-1} > 0.$$

This makes α a (positive) **contact form** on M, and the induced (positive and co-oriented) contact structure is the co-oriented hyperplane field $\xi := \ker \alpha \subset TM$. It follows from Gray's stability theorem that if V is replaced with any other Liouville vector field positively transverse to M, then the induced contact structure is isotopic to ξ, hence the contact form can be regarded as an auxiliary choice, but the contact structure is canonical up to isotopy.

Remark 2.1.1. In this text, every contact structure is assumed to be *co-oriented* and *positive* (with respect to a given orientation of the manifold), and contact forms are always assumed compatible with the given co-orientation.

Example 2.1.2. We denote by $\xi_{\text{std}} \subset TS^{2n-1}$ the **standard contact structure** on the sphere, which arises as the convex boundary of the standard symplectic unit ball with a Liouville vector field pointing radially outward. In coordinates $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$, the **standard contact form** α_{std} is the restriction to $S^{2n-1} \subset \mathbb{R}^{2n}$ of the Liouville form $\frac{1}{2}\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j)$.

Any choice of contact form α determines a **Reeb vector field** R_α on M via the conditions

$$d\alpha(R_\alpha, \cdot) \equiv 0, \qquad \alpha(R_\alpha) \equiv 1.$$

If M is a convex hypersurface in a symplectic manifold (W, ω), then the orbits of R_α are precisely the orbits on M of any Hamiltonian vector field defined by a Hamiltonian function on (W, ω) with M as a regular level set; moreover, convexity implies that a neighbourhood of M is foliated by other convex hypersurfaces that have the same Reeb obits. See [Gei08] for more on contact structures, and [HZ94] for more on the convexity condition in Hamiltonian dynamics.

Given two closed contact manifolds (M_-, ξ_-) and (M_+, ξ_+), a **strong symplectic cobordism** from (M_-, ξ_-) to (M_+, ξ_+) is a compact symplectic manifold (W, ω) whose boundary can be identified with $-M_- \sqcup M_+$ such that M_- and M_+ are both convex hypersurfaces and the contact structures they inherit are isotopic to ξ_- and ξ_+ respectively. Note that the orientation reversal for M_- means that the Liouville vector field points *inward* at M_- (for this reason we sometimes call M_- the **concave** boundary component), whereas it points outward at M_+. Additionally, (W, ω) is called a **Liouville** (or **exact symplectic**) cobordism from (M_-, ξ_-) to (M_+, ξ_+) if the transverse Liouville vector field defined near ∂W can be assumed to extend to a global Liouville vector field. This is equivalent to requiring $\omega = d\lambda$ for some 1-form λ that restricts to the boundary as contact forms $\alpha_\pm := \lambda|_{TM_\pm}$ for $\xi_\pm$.

The **symplectization** of a contact manifold $(M, \xi = \ker\alpha)$ is the open symplectic manifold $(\mathbb{R}\times M, d(e^r\alpha))$, where r denotes the coordinate on $\mathbb{R}$. Its symplectic structure is independent of the choice of α up to isotopy, but α determines a special class of compatible almost complex structures $\mathscr{J}(\alpha)$ on $(\mathbb{R}\times M, d(e^r\alpha))$ such that $J \in \mathscr{J}(\alpha)$ if and only if:

- J is $\mathbb{R}$-invariant (i.e. invariant under the flow of ∂_r);
- $J\partial_r = R_\alpha$;
- $J(\xi) = \xi$;
- $d\alpha(\cdot, J\cdot)|_\xi$ is a bundle metric on ξ.

Given a symplectic cobordism (W, ω) from (M_-, ξ_-) to (M_+, ξ_+) with induced contact forms $\alpha_\pm$ at $M_\pm$, the corresponding Liouville vector fields defined near M_+ and M_- determine collar neighbourhoods $(-\varepsilon, 0]\times M_+$ and $[0,\varepsilon)\times M_-$ respectively in which $\omega = d(e^r\alpha_\pm)$. One then defines the **symplectic completion**

$$\overline{W} = ((-\infty, 0]\times M_-)_{M_-} \cup W \cup_{M_+} ([0,\infty)\times M_+)$$

by extending ω over the cylindrical ends as $d(e^r\alpha_\pm)$. We shall denote by

$$\mathscr{J}(W, \omega, \alpha_+, \alpha_-)$$

the (nonempty and contractible) space of almost complex structures on $\overline{W}$ that are ω-compatible on W and restrict to the cylindrical ends as elements of $\mathscr{J}(\alpha_\pm)$. Almost complex structures of this type will be referred to simply as **admissible** whenever the corresponding symplectic and contact data is fixed.

2.1.2 Reeb orbits and the Conley-Zehnder index

Given a contact form α on a contact manifold (M,ξ) of dimension $2n-1$, a closed Reeb orbit can be regarded as a smooth map

$$\gamma : S^1 := \mathbb{R}/\mathbb{Z} \to M$$

satisfying $\dot{\gamma} = TR_\alpha(\gamma)$ for some $T > 0$, which is the orbit's **period**. Indeed, setting $x(t) := \gamma(t/T)$, such a map is equivalent to a path $x : \mathbb{R} \to M$ that satisfies $\dot{x} = R_\alpha(x)$ and $x(t+T) = x(t)$ for all t. The number T need not generally be the **minimal period**, hence γ may be a multiple cover $\gamma(t) = \gamma_0(kt)$ of another closed Reeb orbit γ_0 for some integer $k \geq 2$; when this is not the case, we say γ is **simple**, and the map $\gamma : S^1 \to M$ is then an embedding. When γ is simple and $\dim M = 3$, it makes sense to ask whether γ is **unknotted**, meaning it is the boundary of an embedded disk, or more explicitly there exists an embedding

$$u : \mathbb{D}^2 \hookrightarrow M$$

whose restriction to the boundary coincides with the Reeb orbit:

$$u|_{\partial \mathbb{D}^2} = \gamma.$$

To every closed Reeb orbit one can associate an integer-valued invariant, the *Conley-Zehnder index*, which depends on a trivialization of the contact structure along the orbit. We will recall the definition of this invariant by way of a theorem regarding *asymptotic operators*.

Fix $J \in \mathscr{J}(\alpha)$ and suppose $\gamma : S^1 \to M$ is a closed orbit of R_α with period T. Given any symmetric connection ∇ on M, define $A_\gamma : \mathscr{C}^\infty(\gamma^*\xi) \to \mathscr{C}^\infty(\gamma^*\xi)$ by

$$A_\gamma \eta = -J(\nabla_t \eta - T \nabla_\eta R_\alpha). \tag{2.1.1}$$

This operator is independent of the choice of connection ∇, and it is symmetric with respect to the inner product on $\mathscr{C}^\infty(\gamma^*\xi)$ defined by

$$\langle \eta, \zeta \rangle = \int_{S^1} \omega_{\gamma(t)}\big(\eta(t), J(\gamma(t))\zeta(t)\big)\, dt.$$

It also extends to an unbounded self-adjoint operator on $L^2(\gamma^*\xi)$ with domain $W^{1,2}(\gamma^*\xi)$, referred to as the **asymptotic operator** associated to γ. Its spectral properties have been described in [HWZ95].

Proposition 2.1.3 ([HWZ95]). *With the notation above, let $\sigma(A_\gamma) \subset \mathbb{R}$ denote the spectrum of A_γ, and for any $\lambda \in \sigma(A_\gamma)$, denote the corresponding eigenspace by E_λ. Then:*

1. $0 \in \sigma(A_\gamma)$ *if and only if γ is degenerate;*

2. $\sigma(A_\gamma)$ *is a discrete subset;*

3. *For each $\lambda \in \sigma(A_\gamma)$, $1 \le \dim E_\lambda \le 2(n-1)$;*

4. *All nontrivial eigenfunctions of A_γ are everywhere nonzero.*

If $\dim M = 3$, then any trivialization Φ of $\gamma^*\xi$ defines a map $\Gamma(\gamma^*\xi) \to \mathbb{R}^2$, via the natural orientation of γ along the Reeb vector field. The last statement then implies that one can define winding numbers $\mathrm{wind}^\Phi(\eta) \in \mathbb{Z}$ of nontrivial eigenfunctions η relative to any fixed unitary trivialization Φ of $\gamma^*\xi$. The following statements then also hold:

5. *If $\eta, \zeta \in E_\lambda$ are two nontrivial elements of the same eigenspace, then $\mathrm{wind}^\Phi(\eta) = \mathrm{wind}^\Phi(\zeta)$, hence we can sensibly denote both by $\mathrm{wind}^\Phi(\lambda)$.*

6. *The map $\sigma(A_\gamma) \to \mathbb{Z} : \lambda \mapsto \mathrm{wind}^\Phi(\lambda)$ is 2-to-1 (counting multiplicity of eigenvalues) and increasing. Hence if two distinct eigenvalues have the same winding, they are consecutive and their eigenspaces are 1-dimensional.*

It follows that one can speak of the largest negative eigenvalue and the smallest positive eigenvalue associated to the asymptotic operator, and when $\dim M = 3$, their winding numbers relative to a chosen trivialization Φ are denoted by

$$\alpha_-^\Phi(\gamma), \quad \alpha_+^\Phi(\gamma) \in \mathbb{Z}$$

respectively. Proposition 2.1.3 implies that these two numbers differ by either 0 or 1 if γ is nondegenerate, and in this case, the **Conley-Zehnder index** (relative to the trivialization Φ of $\gamma^*\xi$) can be characterized via the relation

$$\mu_{CZ}^\Phi(\gamma) = \alpha_-^\Phi(\gamma) + \alpha_+^\Phi(\gamma) \in \mathbb{Z}, \tag{2.1.2}$$

and its **parity** (which does not depend on Φ) by

$$p(\gamma) = \alpha_+^\Phi(\gamma) - \alpha_-^\Phi(\gamma) \in \{0,1\}. \tag{2.1.3}$$

As these formulas indicate, $\mu_{CZ}^\Phi(\gamma)$ depends only on the asymptotic operator and can thus sensibly be written as

$$\mu_{CZ}^\Phi(A_\gamma) = \mu_{CZ}^\Phi(\gamma).$$

With this in mind, (2.1.2) can also be used to compute Conley-Zehnder indices in higher dimensions, via the relation

$$\mu_{CZ}^{\Phi_1 \oplus \ldots \oplus \Phi_m}(A_1 \oplus \ldots \oplus A_m) = \mu_{CZ}^{\Phi_1}(A_1) + \ldots + \mu_{CZ}^{\Phi_m}(A_m), \tag{2.1.4}$$

which holds for any collection of asymptotic operators A_j with trivial kernels on Hermitian line bundles trivialized by Φ_j for $j = 1, \ldots, m$.

While $\mu_{CZ}^\Phi(\gamma)$ depends generally on the choice of trivialization Φ, in certain situations one can make natural choices to remove this ambiguity. If γ is nullhomologous and forms the boundary of an immersed surface $\mathscr{D}$ in M, we define

$$\mu_{\mathrm{CZ}}(\gamma; \mathscr{D}) \in \mathbb{Z}$$

as $\mu_{CZ}^{\Phi}(\gamma)$ with Φ required to admit an extension to a unitary trivialization of ξ along $\mathscr{D}$. The index in this case still depends on the choice of surface $\mathscr{D}$, but this ambiguity also disappears if $c_1(\xi) = 0$, which is true e.g. on (S^3, ξ_{std}).

We require the following standard lemma on the behaviour of the index for multiply covered orbits in dimension three. Let

$$\gamma^k : S^1 \to M : t \mapsto \gamma(kt)$$

denote the k-fold cover of the orbit $\gamma : S^1 \to M$ for $k \in \mathbb{N}$, and note that any trivialization Φ of $\gamma^*\xi$ induces a trivialization Φ^k of $(\gamma^k)^*\xi$.

Lemma 2.1.4. *Suppose* $\dim M = 3$*, and that* γ *and all its multiple covers are nondegenerate. Then for any unitary trivialization* Φ *of* $\gamma^*\xi$*,*

$$\mu_{CZ}^{\Phi^k}(\gamma^k) = \begin{cases} k \cdot \mu_{CZ}^{\Phi}(\gamma) & \text{if } \gamma \text{ is hyperbolic} \\ 2\lfloor k\theta \rfloor + 1 & \text{if } \gamma \text{ is elliptic} \end{cases} \tag{2.1.5}$$

for every $k \in \mathbb{N}$*, where in the elliptic case,* $\theta \in \mathbb{R}$ *is an irrational number determined by* γ *and* Φ*.*

We remind the reader at this point, in light of the above lemma the definition of a **bad orbit**. A hyperbolic orbit, regardless of its parity, has even covers with even Conley-Zehnder index. In the SFT literature, double covers of simple odd hyperbolic orbits are called *bad*, and we will use the same terminology throughout §3.2, §3.3.

We will occasionally also need to deal with Reeb orbits γ that are degenerate but belong to Morse-Bott families, in which case the following definition will be convenient. If γ is degenerate, then $0 \in \sigma(A_\gamma)$ but one can find $\varepsilon > 0$ such that $(-\varepsilon, 0) \cap \sigma(A_\gamma) = \emptyset$. It follows that for any $\varepsilon > 0$ sufficiently small, $A_\gamma + \varepsilon$ is the asymptotic operator of a perturbed nondegenerate orbit, whose index we will denote by

$$\mu_{CZ}^{\Phi}(\gamma + \varepsilon) := \mu_{CZ}^{\Phi}(A_\gamma + \varepsilon). \tag{2.1.6}$$

This is independent of the choice as long as $\varepsilon > 0$ is sufficiently small, and this **perturbed Conley-Zehnder index** gives a sharp lower bound on the indices of possible nondegenerate perturbations of γ. The winding numbers $\alpha_\pm^\Phi(\gamma+\varepsilon) \in \mathbb{Z}$ are defined similarly after replacing A_γ by $A_\gamma + \varepsilon$, and they are then related to $\mu_{CZ}^\Phi(\gamma+\varepsilon)$ by the obvious analogue of (2.1.2). Notice that $\alpha_-^\Phi(\gamma+\varepsilon) = \alpha_-^\Phi(\gamma)$, but $\alpha_+^\Phi(\gamma+\varepsilon)$ and $\alpha_+^\Phi(\gamma)$ may differ if γ is degenerate.

Finally, here is a definition that will be needed for intersection theory when $\dim M = 3$. Observe that for integers $k \geq 2$, every eigenfunction in the λ-eigenspace of A_γ has a k-fold cover that belongs to the $k\lambda$-eigenspace of A_{γ^k}. In the three-dimensional case, one can use Proposition 2.1.3 to show that the covering multiplicity of an eigenfunction depends only on its winding number, thus all elements of the same eigenspace have the same covering multiplicity. The (positive and negative) **spectral covering numbers**

$$\bar{\sigma}_\pm(\gamma) \in \mathbb{N}$$

are defined as the covering multiplicity of the eigenspace that has winding $\alpha_\pm^\Phi(\gamma)$.

2.1.3 Holomorphic curves in completed symplectic cobordisms

In this subsection, fix a $2n$-dimensional symplectic cobordism (W, ω) with completion $\overline{W}$ and admissible almost complex structure $J \in \mathcal{J}(W, \omega, \alpha_+, \alpha_-)$, with the restrictions of J to the cylindrical ends denoted by $J_\pm \in \mathcal{J}(\alpha_\pm)$.

2.1.3.1 Asymptotics

We will consider asymptotically cylindrical pseudoholomorphic curves $u : (\dot{\Sigma}, j) \to (\overline{W}, J)$, where

$$\dot{\Sigma} = \Sigma \setminus \Gamma$$

is the result of removing finitely many punctures $\Gamma \subset \Sigma$ from a closed Riemann surface (Σ, j). The set of punctures is partitioned into sets of **positive** and **negative** punctures Γ^+ and Γ^- respectively, where $z \in \Gamma^\pm$ means that one can find a biholomorphic identification of a punctured neighbourhood of z with $[0, \infty) \times S^1$ or

$(-\infty,0] \times S^1$ respectively such that for $|s|$ sufficiently large, u in these coordinates takes the form

$$u(s,t) = \exp_{(Ts,\gamma(t))} h(s,t) \in [0,\infty) \times M_+ \text{ or } (-\infty,0] \times M_-$$

for some closed Reeb orbit $\gamma : S^1 \to M_\pm$ with period $T > 0$, where the exponential map is defined with respect to any choice of translation-invariant metric on the cylindrical ends, and $h(s,t)$ is a vector field along the trivial cylinder which satisfies $|h(s,t)| \to 0$ as $s \to \pm\infty$. We say in this case that u is (positively or negatively) **asymptotic to** γ at z, and $h(s,t)$ is called the **asymptotic representative** of u at z. The asymptotic behaviour of $h(s,t)$ is described by a formula proved in [HWZ96a, HWZ96b, Mor03, Sie08]: namely if the orbit γ is nondegenerate or Morse-Bott, then for $|s|$ sufficiently large, h is either identically zero or satisfies

$$h(s,t) = e^{\lambda s}(e_1(t) + r(s,t)), \tag{2.1.7}$$

where $r(s,t) \to 0$ uniformly in all derivatives as $s \to \pm\infty$, $\lambda \in \sigma(A_\gamma)$ is an eigenvalue of the asymptotic operator of γ with $\pm\lambda < 0$, and $e_1 \in \mathscr{C}^\infty(\gamma^*\xi_\pm)$ is a nontrivial element of the corresponding eigenspace.

2.1.3.2 Moduli spaces and compactness

It is a standard fact that every asymptotically cylindrical J-holomorphic curve $u : (\dot{\Sigma}, j) \to (\overline{W}, J)$ either is somewhere injective or is a multiple cover of a somewhere injective asymptotically cylindrical curve, and moreover, the set of *injective points* of a somewhere injective curve is open and dense. A complete proof of this statement may be found in [Nel15], using asymptotic results of Siefring [Sie08]. Recall that $z \in \dot{\Sigma}$ is called an **injective point** of u if $u^{-1}(u(z)) = \{z\}$ and $du(z) \neq 0$, and we call u a k-fold **multiple cover** of another curve $v : (\dot{\Sigma}' = \Sigma' \setminus \Gamma', j') \to (\overline{W}, J)$ if

$$u = v \circ \phi$$

for some holomorphic map $\phi : (\Sigma, j) \to (\Sigma', j')$ of degree k.

Fix finite ordered tuples of Reeb orbits $\gamma^+ = (\gamma_1^+, \dots, \gamma_{k_+}^+)$ and $\gamma^- = (\gamma_1^-, \dots, \gamma_{k_-}^-)$ in M_+ and M_- respectively (the case $k_\pm = 0$ is allowed), assuming that all of them are either nondegenerate or belong to Morse-Bott families. For an integer $m \geq 0$, the moduli space

$$\mathscr{M}_m(J, \gamma^+, \gamma^-)$$

of **unparametrized J-holomorphic spheres asymptotic to γ^+ and γ^- with m marked points** is defined as the set of equivalence classes of tuples $(\Sigma, j, \Gamma^+, \Gamma^-, u, (\zeta_1, \dots, \zeta_m))$ where (Σ, j) is a closed Riemann surface of genus zero, $\Gamma^+, \Gamma^- \subset \Sigma$ are disjoint finite sets, each equipped with an ordering, the **marked points** $\zeta_1, \dots, \zeta_m \in \dot{\Sigma} := \Sigma \setminus (\Gamma^+ \cup \Gamma^-)$ are all distinct, and

$$u : (\dot{\Sigma}, j) \to (\overline{W}, J)$$

is an asymptotically cylindrical J-holomorphic curve with positive punctures Γ^+ and negative punctures Γ^-, such that u is asymptotic at the ith puncture in $\Gamma^\pm$ to $\gamma_i^\pm$ for $i = 1, \dots, k_\pm$. Two such tuples are considered equivalent if one can be written as a reparametrization of the other via a biholomorphic diffeomorphism of their domains that maps marked points to marked points and punctures to punctures, with signs and orderings preserved. The topology of $\mathscr{M}_m(J, \gamma^+, \gamma^-)$ can be characterized by saying that a sequence converges if it has representatives with a fixed domain Σ and fixed sets of punctures and marked points such that the conformal structures converge in $\mathscr{C}^\infty(\Sigma)$ while the maps to $\overline{W}$ converge in $\mathscr{C}^\infty_{\text{loc}}(\dot{\Sigma})$ and also in $\mathscr{C}^0$ up to infinity (with respect to translation-invariant metrics on the cylindrical ends). We shall often abuse notation by referring to the entire equivalence class of tuples $[(\Sigma, j, \Gamma^+, \Gamma^-, u, (\zeta_1, \dots, \zeta_m))]$ forming an element of $\mathscr{M}_m(J, \gamma^+, \gamma^-)$ simply as u. In this text we will only consider the cases $m = 0, 1$, abbreviating the former by

$$\mathscr{M}(J, \gamma^+, \gamma^-) := \mathscr{M}_0(J, \gamma^+, \gamma^-).$$

For $m > 0$, the **evaluation map**

$$\begin{aligned} \mathrm{ev} : \mathscr{M}_m(J,\gamma^+,\gamma^-) &\to \overline{W}^m \\ [(\Sigma, j, \Gamma^+, \Gamma^-, u, (\zeta_1,\dots,\zeta_m))] &\mapsto (u(\zeta_1),\dots,u(\zeta_m)) \end{aligned}$$

is well defined and continuous by construction.

Recall that neighbourhoods in $\mathscr{M}_m(J,\gamma^+,\gamma^-)$ can be described as zero-sets of smooth Fredholm sections in suitable Banach space bundles (see e.g. [Wen10b]). A curve u is called **Fredholm regular** whenever it forms a transverse intersection of such a Fredholm section with the zero-section. The **virtual dimension** of $\mathscr{M}_m(J,\gamma^+,\gamma^-)$ at u is given by the Fredholm index of the linearized section at u minus the dimension of the group of automorphisms of the domain, and in the case $m = 0$ is also called the **index** of u. If the orbits are all nondegenerate, it is given by the formula

$$\mathrm{ind}(u) = (n-3)\chi(\dot{\Sigma}) + 2c_1^{\Phi}(u^*T\overline{W}) + \sum_{i=1}^{k_+} \mu_{CZ}^{\Phi}(\gamma_i^+) - \sum_{i=1}^{k_-} \mu_{CZ}^{\Phi}(\gamma_i^-). \qquad (2.1.8)$$

Here Φ is an arbitrary choice of unitary trivializations of $\xi_\pm$ along each of the asymptotic orbits, which naturally induce asymptotic trivializations of the complex vector bundle $u^*T\overline{W} \to \dot{\Sigma}$, and $c_1^{\Phi}(u^*T\overline{W}) \in \mathbb{Z}$ then denotes the **relative first Chern number** of $u^*T\overline{W}$ with respect to these asymptotic trivializations. This term ensures that the total expression is independent of the choice Φ. We will also need a special case of the index formula under Morse-Bott assumptions: if all positive asymptotic orbits are Morse-Bott (but possibly degenerate) and all negative orbits are nondegenerate, then

$$\mathrm{ind}(u) = (n-3)\chi(\dot{\Sigma}) + 2c_1^{\Phi}(u^*T\overline{W}) + \sum_{i=1}^{k_+} \mu_{CZ}^{\Phi}(\gamma_i^+ + \varepsilon) - \sum_{i=1}^{k_-} \mu_{CZ}^{\Phi}(\gamma_i^-), \qquad (2.1.9)$$

where $\varepsilon > 0$ is assumed sufficiently small (see (2.1.6)). Note that this is the virtual dimension of the moduli space of curves near u with *fixed* asymptotic orbits, i.e. the orbits are not allowed to move continuously in their respective Morse-Bott families.

The index without this constraint would be larger; see [Wen10b], §3.2 for an explanation of (2.1.9) and the constrained/unconstrained distinction. Adding a marked point generally increases the virtual dimension by 2, so $\mathcal{M}_m(J,\gamma^+,\gamma^-)$ has virtual dimension $\mathrm{ind}(u)+2m$ on any component that includes the curve $u \in \mathcal{M}(J,\gamma^+,\gamma^-)$.

A standard application of the implicit function theorem implies that the open subset consisting of Fredholm regular curves in $\mathcal{M}_m(J,\gamma^+,\gamma^-)$ admits the structure of a smooth finite-dimensional orbifold whose dimension locally equals its virtual dimension, and it is a manifold near any curve that is somewhere injective. Moreover, a standard argument via the Sard-Smale theorem (see [MS04] or [Wenc]) shows that after perturbing J generically in $\mathcal{J}(W,\omega,\alpha_+,\alpha_-)$ on some open subset $\mathcal{U} \subset W$ with compact closure, one can assume that all somewhere injective curves passing through $\mathcal{U}$ are Fredholm regular. Similarly, Dragnev [Dra04] (see also [Wenb]) has shown that on a symplectization $(\mathbb{R}\times M, d(e^r\alpha))$, generic perturbations within $\mathcal{J}(\alpha_\pm)$ suffice to make all somewhere injective curves regular, and this result can also be applied to any curves in the cobordism $\overline{W}$ that are contained in a cylindrical end.

If the Reeb flows on M_+ and M_- are both globally nondegenerate or Morse-Bott, then $\mathcal{M}_m(J,\gamma^+,\gamma^-)$ has a natural compactification

$$\overline{\mathcal{M}}_m(J,\gamma^+,\gamma^-)$$

defined in [BEH$^+$03], consisting of **stable holomorphic buildings** of arithmetic genus zero with m marked points. An example of a holomorphic building (with higher arithmetic genus) is shown in Figure 2.1. We shall write holomorphic buildings using the notation

$$(v^+_{N_+}|\dots|v^+_1|v_0|v^-_1|\dots|v^-_{N_-}),$$

where $N_+,N_- \geq 0$ are integers, $v^\pm_1,\dots,v^\pm_{N_\pm}$ are each (possibly disconnected and/or nodal) $J_\pm$-holomorphic curves in the symplectizations $\mathbb{R}\times M_\pm$, forming the **upper** and **lower levels** respectively, and v_0 is a (possibly disconnected and/or nodal) J-

holomorphic curve in $\overline{W}$, the **main level**. Note that by convention, the main level is allowed to be empty (i.e. v_0 is a curve with domain the empty set) if N_+ or N_- is nonzero. Each upper or level is defined only up to $\mathbb{R}$-translation, and the same is true of all levels when $\overline{W}$ is a symplectization, in which case there is no distinguished "main" level or distinction between "upper" and "lower" levels. The evaluation map extends continuously over $\overline{\mathscr{M}}_m(J,\gamma^+,\gamma^-)$ if we also compactify $\overline{W}$ by adding $\{\pm\infty\} \times M_\pm$ to the top and bottom of the cylindrical ends, i.e. marked points in upper or lower levels are mapped to $\{\infty\} \times M_+$ or $\{-\infty\} \times M_-$ respectively.

Our notation for buildings is convenient but suppresses an additional detail that will sometimes be quite important: the data also includes a one-to-one correson-dence between the positive punctures of each level (other than the topmost) and the negative punctures of the level above it, such that corresponding punctures have matching asymptotic orbits, the so-called **breaking orbits**. Additionally, each pair of corresponding punctures is equipped with a choice of a rotation angle for gluing the corresponding positive and negative ends along the breaking orbit—this choice is unique if the orbit is simple, but in general there are $m \in \mathbb{N}$ distinct choices if the orbit has covering multiplicity m. All of this data together is called a **decoration** of the building. Different choices of decoration often produce buildings that are biholomorphically inequivalent to each other and thus represent distinct elements of $\overline{\mathscr{M}}_m(J,\gamma^+,\gamma^-)$.

Whenever (W,ω) is a Liouville cobordism (and in particular if $\overline{W}$ is a symplec-tization), Stokes' theorem prevents the existence of curves with no positive ends, sometimes referred to as **holomorphic caps**. The following standard result is then immediate from the definition of convergence in [BEH$^+$03].

Proposition 2.1.5. *Suppose $\overline{W}$ is either a symplectization or the completion of a Liouville cobordism, and $u_k \in \mathscr{M}_m(J,\gamma,\emptyset)$ is a sequence of J-holomorphic planes converging to a holomorphic building. Then the limiting building has the following properties:*

- *Each connected component of each level is a punctured sphere with precisely one positive puncture.*

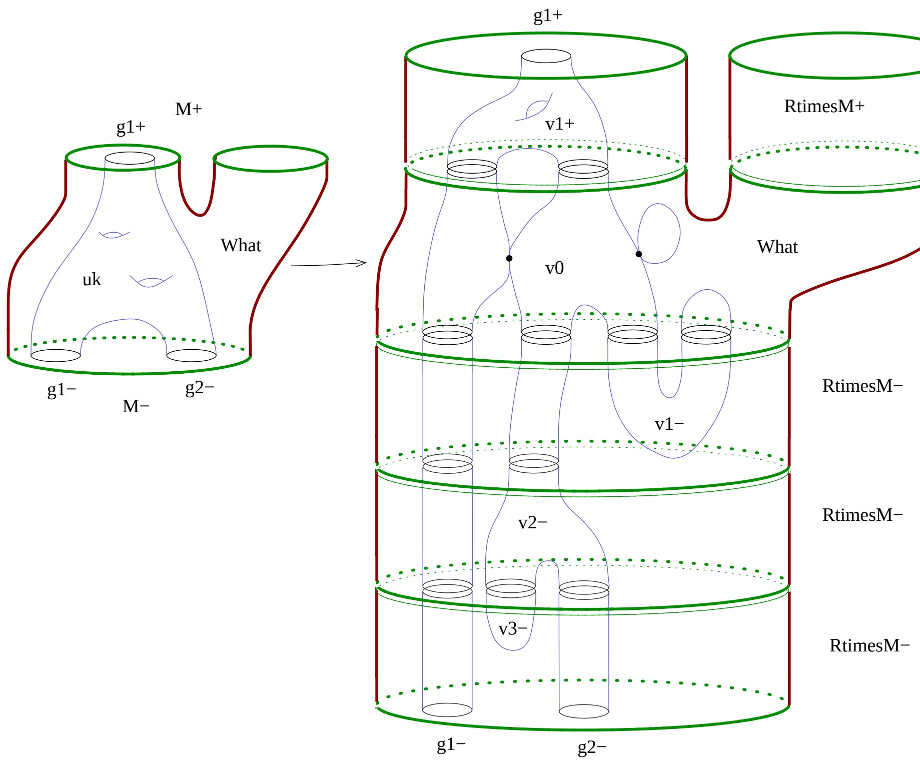

Figure 2.1: The picture shows the degeneration of a sequence of punctured curves of genus 2 into a building with a main level, one upper level and three lower levels. We label the building as $(v_1^+|v_0|v_1^-|v_2^-|v_3^-)$, where each $v_i^\pm$ is in general a disconnected nodal curve in a single level. The arithmetic genus of the building is still 2, and the levels match along their respective asymptotic orbits.

- *The lowest level has no negative punctures (so it is a disjoint union of planes).*
- *The top level is connected.*
- *There are no nodes.*

□

We shall refer to the components without negative ends in the above lemma as **capping planes**; they are not to be confused with "holomorphic caps," which have *only* negative ends.

The converse of compactness is gluing, as discussed e.g. in [Nel13, Chapter 7]. We will only need the following special case.

Proposition 2.1.6. *Assume γ_∞ is a Morse-Bott Reeb orbit in M_+, γ is a nondegenerate orbit in M_-, $m \geq 0$ is an integer, and $(v_0|v_1^-) \in \overline{\mathscr{M}}_m(J,\gamma_\infty,\emptyset)$ is a (decorated)* *stable J-holomorphic building such that* $v_0 \in \mathcal{M}_m(J,\gamma_\infty,\gamma)$ *and* $v_1^- \in \mathcal{M}(J_-,\gamma,\emptyset)/\mathbb{R}$ *are both somewhere injective and Fredholm regular. Then there exist neighborhoods*

$$v_0 \in \mathcal{U}_0 \subset \mathcal{M}_m(J,\gamma_\infty,\gamma)$$
$$v_1^- \in \mathcal{U}_- \subset \mathcal{M}(J_-,\gamma,\emptyset)/\mathbb{R}$$

and a smooth embedding

$$\Psi : [0,\infty) \times \mathcal{U}_0 \times \mathcal{U}_- \hookrightarrow \mathcal{M}_m(J,\gamma_\infty,\emptyset)$$

such that for any sequences $[0,\infty) \ni r_k \to +\infty$, $u_k \to u_\infty \in \mathcal{U}_0$ *and* $u_k^- \to u_\infty^- \in \mathcal{U}_-$,

$$\Psi(r_k,u_k,u_k^-) \to (u_\infty|u_\infty^-) \in \overline{\mathcal{M}}_m(J,\gamma_\infty,\emptyset)$$

in the SFT topology. Moreover, every smooth curve in $\mathcal{M}_m(J,\gamma_\infty,\emptyset)$ *sufficiently close to* $(v_0|v_1^-)$ *in the SFT topology is in the image of* Ψ. □

Remark 2.1.7. The notation for buildings used in Proposition 2.1.6 implicitly assumes that if multiple buildings can be constructed out of u_∞ and u_∞^- via different choices of decoration, then $(u_\infty|u_\infty^-)$ is the unique choice that is close to $(v_0|v_1^-)$ in the SFT topology.

2.1.4 The low-dimensional case

We now specialize to the case where the cobordism (W,ω) is 4-dimensional, so all contact manifolds under consideration will be 3-dimensional.

2.1.4.1 Indices of covers

We begin with a pair of convenient numerical observations. The first is borrowed (along with its proof) from [Hut02].

Proposition 2.1.8. *Suppose* $J \in \mathcal{J}(\alpha)$ *for a contact* 3*-manifold* $(M,\xi = \ker\alpha)$, *and* $u : (\dot{\Sigma},j) \to (\mathbb{R}\times M,J)$ *is a J-holomorphic branched cover of a trivial cylinder over a Reeb orbit whose covers are all nondegenerate. Then* $\mathrm{ind}(u) \geq 0$, *and equality can hold only when the cover is unbranched or the orbit is elliptic.*

Proof. If the underlying orbit γ is hyperbolic, then the index formula gives $\mathrm{ind}(u) = -\chi(\dot{\Sigma}) \geq 0$ due to Lemma 2.1.4, which is an equality if and only if $\dot{\Sigma}$ is the cylinder, in which case the Riemann-Hurwitz formula implies that the cover is unbranched. If the orbit is instead elliptic, we can make our lives slightly easier with the observation that u has the same index as that of some holomorphic building whose connected components are all thrice-punctured spheres that are also branched covers of the same trivial cylinder. It therefore suffices to prove that the inequality holds for thrice-punctured spheres. If for instance u has two positive punctures at γ^k and γ^ℓ and a negative puncture at $\gamma^{k+\ell}$, then Lemma 2.1.4 gives

$$\mathrm{ind}(u) = -\chi(\dot{\Sigma}) + (2\lfloor k\theta \rfloor + 1) + (2\lfloor \ell\theta \rfloor + 1) - (2\lfloor (k+\ell)\theta \rfloor + 1),$$

where $\chi(\dot{\Sigma}) = -1$, and the index is thus nonnegative due to the relation $\lfloor a+b \rfloor \leq \lfloor a \rfloor + \lfloor b \rfloor + 1$. In the inverse case with one positive puncture and two negative, we get the same result using $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a+b \rfloor$. □

Proposition 2.1.9. *Suppose* $\dim W = 4$ *and* $u = v \circ \phi : (\dot{\Sigma}, j) \to (\overline{W}, J)$ *is a k-fold cover of a somewhere injective J-holomorphic curve* $v : (\dot{\Sigma}', j') \to (\overline{W}, J)$ *whose asymptotic orbits are all nondegenerate and hyperbolic. Then*

$$\mathrm{ind}(u) \geq k\,\mathrm{ind}(v),$$

with equality if and only if the cover $\phi : (\dot{\Sigma}, j) \to (\dot{\Sigma}', j')$ *has no branch points in the punctured surface* $\dot{\Sigma}$.

Proof. This is a direct consequence of the index formula (2.1.8) together with Lemma 2.1.4 and the Riemann-Hurwitz formula $Z(d\phi) = -\chi(\dot{\Sigma}) + k\chi(\dot{\Sigma}')$, where $Z(d\phi) \geq 0$ denotes the algebraic count of zeroes of the holomorphic section $d\phi \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, \phi^* T\dot{\Sigma}'))$, and thus vanishes if and only if the cover is unbranched. □

2.1.4.2 Asymptotic defect

Suppose $u : \dot{\Sigma} \to \overline{W}$ is asymptotic at $z \in \Gamma^\pm$ to a T-periodic orbit $\gamma : S^1 \to M_\pm$ and has an asymptotic representative $h(s,t)$ at this puncture that is not identically zero. Then the asymptotic formula (2.1.7) provides a nonzero eigenfunction $e_1 \in \mathscr{C}^\infty(\gamma^*\xi_\pm)$, and given a trivialization Φ of $\gamma^*\xi_\pm$, one can define

$$\text{wind}^\Phi_\infty(u;z) := \text{wind}^\Phi(e_1) \in \mathbb{Z}.$$

If $z \in \Gamma^+$, then $\alpha^\Phi_-(\gamma)$ is the winding of the greatest negative eigenvalue of A_γ, thus $\text{wind}^\Phi_\infty(u;z) \le \alpha^\Phi_-(\gamma)$, and similarly, $\text{wind}^\Phi_\infty(u;z) \ge \alpha^\Phi_+(\gamma)$ if $z \in \Gamma^-$. The difference $\alpha^\Phi_-(\gamma) - \text{wind}^\Phi_\infty(u;z)$ or $\text{wind}^\Phi_\infty(u;z) - \alpha^\Phi_+(\gamma)$ for a positive or negative puncture respectively is denoted $d_0(u;z) \ge 0$ and called the **asymptotic defect** of u at $z \in \Gamma$. Notice that it does not depend on the trivialization. The total asymptotic defect of u is then a nonnegative integer

$$d_0(u) = \sum_{z\in\Gamma} d_0(u;z).$$

This is well defined for any curve u that is not identical to a trivial cylinder in some neighbourhood of any of its punctures; in particular, if $\overline{W}$ is a symplectization $(\mathbb{R}\times M, d(e^r\alpha))$ with $J \in \mathscr{J}(\alpha)$, then $d_0(u)$ is well defined for every curve other than covers of trivial cylinders.

2.1.4.3 The normal Chern number and $\text{wind}_\pi(u)$

The **normal Chern number** of a curve $u \in \mathscr{M}(J,\gamma^+,\gamma^-)$ with all asymptotic orbits nondegenerate is defined by

$$c_N(u) = c_1^\Phi(u^*T\overline{W}) - \chi(\dot{\Sigma}) + \sum_{i=1}^{k_+} \alpha^\Phi_-(\gamma_i^+) - \sum_{i=1}^{k_-} \alpha^\Phi_+(\gamma_i^-),$$

where Φ is again an arbitrary choice of unitary trivializations of $\xi_\pm$ along the asymptotic orbits, and the sum does not depend on this choice. The index formula and relations between Conley-Zehnder indices and winding numbers imply

$$2c_N(u) = \text{ind}(u) - 2 + 2g + \#\Gamma_0, \tag{2.1.10}$$

where $g \geq 0$ is the genus of the domain (zero in our case) and $\Gamma_0 \subset \Gamma$ denotes the set of punctures of u that have even parity. In the Morse-Bott setting of (2.1.9), the definition of $c_N(u)$ given above remains valid, and so does (2.1.10) after interpreting Γ_0 as the set of punctures for which the *perturbed* Conley-Zehnder index (see (2.1.6)) is even. One can interpret $c_N(u)$ as "c_1 of the normal bundle" when u is immersed; in particular, $c_N(u)$ then predicts the number of zeroes for a generic section in the kernel of the linearized normal deformation operator at u, see e.g. [Wen10b].

For curves in the symplectization $\mathbb{R} \times M$ of a contact manifold $(M, \xi = \ker \alpha)$, there is a further invariant related to $c_N(u)$ and the asymptotic defect. Let $\pi : TM \to \xi$ denote the fibrewise linear projection along the Reeb vector field. Then the nonlinear Cauchy-Riemann equation for $u : \dot{\Sigma} \to \mathbb{R} \times M$ implies that $\pi \circ du \in \mathscr{C}^\infty(\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, u^*\xi))$ locally satisfies a linear Cauchy-Riemann type equation, so zeroes of $\pi \circ du$ are isolated and positive by the similarity principle unless $\pi \circ du \equiv 0$. The latter is the case if and only if u is a cover of a trivial cylinder, and otherwise, we define

$$\mathrm{wind}_\pi(u) \geq 0$$

to be the algebraic count of zeroes of $\pi \circ du$. The asymptotic formula (2.1.7) implies that zeroes of $\pi \circ du$ cannot accumulate near infinity, so $\mathrm{wind}_\pi(u)$ is always finite. It equals 0 if and only if $u = (u_{\mathbb{R}}, u_M) : \dot{\Sigma} \to \mathbb{R} \times M$ has the property that $u_M : \dot{\Sigma} \to M$ is an immersion transverse to the Reeb vector field. From [HWZ95, Prop. 5.6], we have

$$c_N(u) = \mathrm{wind}_\pi(u) + d_0(u). \tag{2.1.11}$$

In particular, this implies

$$c_N(u) \geq d_0(u) \geq 0 \quad \text{and} \quad c_N(u) \geq \mathrm{wind}_\pi(u) \geq 0 \tag{2.1.12}$$

for any curve that is not a cover of a trivial cylinder, so $c_N(u) = 0$ gives a homotopy-invariant sufficient condition for both the asymptotic defect and $\mathrm{wind}_\pi(u)$ to vanish.

In 3.2 we will need an extra result concerning the additive behaviour of the normal Chern number as a sequence degenerates to its asymptotic limit. Suppose

$$u_k : (\Sigma, j) \to (W, J)$$

is a sequence of J-holomorphic curves which converges in the sense of 2.1.3.2 to a building

$$u_\infty = (v_{N_+}^+ | \dots | v_1^+ | v_0 | v_1^- | \dots | v_{N_-}^-).$$

Since $c_N(u_k)$ is eventually constant, the following formula holds.

Lemma 2.1.10. *Denote by $\Gamma(u_\infty)$ the union of the sets of breaking orbits of all levels of u_∞.*

$$c_N(u_k) = c_N(v_0) + \sum_{j=1}^{N_+} c_N(v_j^+) + \sum_{j=1}^{N_-} c_N(v_j^-) + \sum_{\gamma \in \Gamma(u_\infty)} p(\gamma),$$

Proof. This follows from the definition of the normal Chern number, together with the additivity of the relative Chern numbers under a consistent choice of trivialization across all levels, as well as the additivity of the Euler characteristics. The sum of parities across breaking orbits comes from the fact that each breaking orbit appears exactly twice, once as the positive orbit of some level, and once as the negative orbit of another.

□

2.1.4.4 Self-linking numbers

Let γ be a nullhomologous transverse knot in a closed contact 3-manifold (M, ξ), let $\Sigma \subset M$ be a Seifert surface and X a framing of γ, i.e. a non-zero section of $\gamma^*\xi$. The **self-linking number** of γ with respect to X is then the algebraic count of intersections between Σ and a generic push-off of γ in the direction of X:

$$\mathrm{sl}(\gamma, X) = (\exp_\gamma X) \cdot \Sigma \in \mathbb{Z}.$$

Note that this depends on X up to homotopy, but not on Σ, as a different choice of Seifert surface changes $\mathrm{sl}(\gamma, X)$ by the homological intersection number of γ with a closed 2-cycle, which vanishes since γ is nullhomologous. Replacing X with another framing changes $\mathrm{sl}(\gamma, X)$ by the relative winding of the two framings,

$$\mathrm{sl}(\gamma, X_1) - \mathrm{sl}(\gamma, X_2) = \mathrm{wind}(X_1, X_2), \tag{2.1.13}$$

where $\mathrm{wind}(X_1, X_2) \in \mathbb{Z}$ denotes the winding number of the section X_1 along γ in the trivialization induced by X_2. Note that the Seifert surface determines a canonical homotopy class of framings X_Σ via the condition that X_Σ should extend to a trivialization of ξ along Σ, so with this choice we shall denote

$$\mathrm{sl}(\gamma; \Sigma) := \mathrm{sl}(\gamma, X_\Sigma).$$

This depends on Σ since X_Σ does, but the dependence vanishes if $c_1(\xi) = 0$.

With this definition in mind, suppose γ is an unknotted Reeb orbit and $u = (u_\mathbb{R}, u_M) : \mathbb{C} \to \mathbb{R} \times M$ is a J-holomorphic plane asymptotic to γ for which $u_M : \mathbb{C} \to M$ is embedded. The closure of $u_M(\mathbb{C})$ is then a Seifert disk $\mathscr{D} \subset M$ for γ, and we claim

$$\mathrm{sl}(\gamma; \mathscr{D}) = \mathrm{wind}(X_\mathscr{D}, e_1(u)), \tag{2.1.14}$$

where $X_\mathscr{D}$ is the canonical framing determined by $\mathscr{D}$ as discussed above, and $e_1(u)$ is the nonzero eigenfunction appearing in the asymptotic formula (2.1.7) for the approach of u to γ. Indeed, $e_1(u)$ gives the direction of the approach of u to γ and is thus homotopic to the Seifert framing of γ, implying $\mathrm{sl}(\gamma, e_1(u)) = 0$, so

$$\mathrm{sl}(\gamma; \mathscr{D}) = \mathrm{sl}(\gamma, X_\mathscr{D}) = \mathrm{sl}(\gamma, e_1(u)) + \mathrm{wind}(X_\mathscr{D}, e_1(u)) = \mathrm{wind}(X_\mathscr{D}, e_1(u)).$$

2.1.4.5 Siefring intersection theory

Let (W, J) be a symplectic cobordism from (M_+, ξ_+) to (M_-, ξ_-) with and admissible almost complex structure as in 2.1.1. Let $u_1 : \dot{\Sigma}_1 \to \overline{W}$ and $u_2 : \dot{\Sigma}_2 \to \overline{W}$ be J-holomorphic maps such that both their asymptotic orbits are nondegenerate or Morse-Bott. The non-compactness of their domains implies their algebraic intersection number, denoted by $u_1 \cdot u_2$, is not a homotopy invariant (indeed it is not even a priori well defined). $u_1 \cdot u_2$ is however, well defined if one can guarantee their intersections lie in a compact subset. Often we will use this algebraic count after one of the two curves has been perturbed along a fixed trivilization Φ in its cylindrical ends; in such a case the algebraic count is independent of arbitrary compact perturbations and is well defined. We will denote it by

$$\mathrm{int}(u_1, u_2)$$

However, in [Sie11], Siefring associates to any pair of (not necessarily J-holomorphic) asymptotically cylindrical maps an integer

$$u_1 * u_2 \in \mathbb{Z},$$

which matches the homological intersection number $[u_1] \cdot [u_2]$ if both curves have no punctures, and in general has the following properties. First, the pairing is symmetric

$$u_1 * u_2 = u_2 * u_1,$$

and it is invariant under homotopies of asymptotically cylindrical maps with fixed asymptotic orbits; in fact, $u_1 * u_2$ depends only on the asymptotic orbits of u_1 and u_2 and their relative homology classes. If both maps are J-holomorphic and their images are non-identical, then the relative asymptotic results of [Sie08] imply that all intersections between u_1 and u_2 are isolated and contained in a compact subset, so by positivity of intersections, the algebraic count of intersections $u_1 \cdot u_2$ is finite and satisfies

$$u_1 \cdot u_2 \geq \left| \left\{ (z_1, z_2) \in \dot{\Sigma}_1 \times \dot{\Sigma}_2 \;\middle|\; u_1(z_1) = u_2(z_2) \right\} \right|,$$

with equality if and only if all intersections are transverse. This is then related to $u_1 * u_2$ by

$$u_1 * u_2 \geq u_1 \cdot u_2,$$

so the condition $u_1 * u_2 = 0$ gives a homotopy-invariant sufficient condition for u_1 and u_2 to be disjoint.

We briefly recall the construction of the pairing, as some of the terms involved reappear in subsequent computations in other sections.

First, for the collection of complex line bundles $\gamma^*\xi_\pm \to S^1$ for every asymptotic orbit $\gamma : S^1 \to M_\pm$ choose a collection of trivializations which we indiscriminately refer to as Φ which are compatible with the pullbacks of all covering maps $\gamma^m \to \gamma$. Once such as choice is made, we can define the *relative intersection number*

$$u_1 *_\Phi u_2$$

as the count of algebraic intersections between u_1 and a perturbation of u_2 along Φ at $\pm\infty$ such that this count is finite. Such a perturbation is constructed first in neighbourhoods of each asymptotic end as follows. For $z \in \Gamma(u_2)$ one can express the cylindrical end model of u_2 asymptotic to an orbit which we call γ_z^m after a coordinate transformation, $\psi(s,t)$

$$u_2(\psi(s,t)) = (mTs, \exp_{\gamma_z^m(t)} h(s,t)) \in [0,\infty) \times M_+ \text{ or } (-\infty, 0] \times M_-.$$

Then we define the perturbed end to be

$$u_2^{\Phi,\varepsilon}(\psi(s,t)) = (mTs, \exp_{\gamma_z(t)}[h(s,t) + \beta(\psi(s,t))\Phi(mt)\varepsilon])$$

where β is a smooth cut-off function which equals 1 in a neighbourhood of z.

If we rewrite $u_2^{\Phi,\varepsilon}$ to be the curve u_2 with such a replacement made in a neighbourhood of each end, it follows that one can find an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ the intersection number

$$u_1 *_\Phi u_2 = u_1 \cdot u_2^{\Phi,\varepsilon}$$

is well-defined, and only depends on the homotopy class of Φ.

The difference between $u_1 *_\Phi u_2$ and a true homotopy invariant turns out to be an asymptotic correction term which depends only on the common asymptotic orbits of the two curves. Define for any simple orbit γ

$$\Omega_{\pm}^{\Phi}(\gamma^k,\gamma^m) = \min\{\mp k\alpha_{\mp}^{\Phi}(\gamma^m), \mp m\alpha_{\mp}^{\Phi}(\gamma^k)\}$$

Then $u_1 * u_2$ is defined to be

$$u_1 * u_2 = u_1 *_{\Phi} u_2 - \sum_{(z,\zeta)\in\Gamma_{u_1}^{\pm}\times\Gamma_{u_2}^{\pm}} \Omega_{\pm}^{\Phi}(\gamma_z^{m_z},\gamma_{\zeta}^{m_{\zeta}}) \tag{2.1.15}$$

where the terms of the sum are understood to be 0 if $\gamma_z \neq \gamma_\zeta$.

What the sum on the right hand side represents is a theoretical lower bound for the intersections that arise from ∞ which we can write as

$$u_1 *_{\Phi} u_2 = u_1 \cdot u_2 + i_{\infty}^{\phi}(u_1,u_2)$$

whenever the $u_1 \cdot u_2$ is well defined. The quantity $i_{\infty}^{\phi}(u_1,u_2)$ has an independent definition in terms of asymptotic winding numbers of the ends of the two curves, see [Sie11] for full exposure.

The following computation is an easy consequence of the definition (cf. [Sie11, Prop. 5.6]):

Proposition 2.1.11. *Suppose $J \in \mathscr{J}(\alpha)$ for a contact 3-manifold $(M,\xi = \ker\alpha)$, and u and v are both J-holomorphic covers of the same trivial cylinder in $(\mathbb{R}\times M, J)$ over a nondegenerate Reeb orbit with even parity. Then $u * v = 0$.* □

The intersection product also has a natural extension to holomorphic buildings such that homotopy invariance holds for all continuous deformations in the SFT topology. We will need a particular result about this extension:

Proposition 2.1.12. *If $v = (v_{N_+}^+|\dots|v_1^+|v_0|v_1^-|\dots|v_{N_-}^-)$ is a holomorphic building in a 4-dimensional completed symplectic cobordism, we have*

$$v * v \geq \sum_{j=1}^{N_+} v_j^+ * v_j^+ + v_0 * v_0 + \sum_{j=1}^{N_-} v_j^- * v_j^- + \sum_{\gamma} m(\gamma)p(\gamma),$$

where the last sum is over all orbits γ that occur as breaking orbits in v, with covering multiplicities denoted by $m(\gamma) \in \mathbb{N}$.

Proof. Notice first that the relative intersection numbers $v_i^{\pm} *_{\Phi} v_i^{\pm}$ as defined, are additive and as a result

$$v * v - \sum_{j=1}^{N_+} v_j^+ * v_j^+ - v_0 * v_0 - \sum_{j=1}^{N_-} v_j^- * v_j^-$$

is a sum of terms of the form

$$\Omega^{\Phi}_{\pm}(\gamma_z^{m_z}, \gamma_\zeta^{m_\zeta})$$

as in 2.1.15 over all pairs of common breaking orbits. Among all of these terms, whenever γ^m is a breaking orbit, we can find

$$\Omega^{\Phi}_{+}(\gamma^m, \gamma^m) + \Omega^{\Phi}_{-}(\gamma^m, \gamma^m) = m\alpha^{\Phi}_{+}(\gamma^m) - m\alpha^{\Phi}_{-}(\gamma^m) = mp(\gamma^m)$$

It remains to see that all other terms pair up to positive quantities on the right hand side of the equation. Namely, we consider

$$\Omega^{\Phi}_{+}(\gamma^m, \gamma^k) + \Omega^{\Phi}_{-}(\gamma^m, \gamma^k) \tag{2.1.16}$$

If γ is an even orbit, then, $\alpha^{\Phi}_{+}(\gamma^m) = \alpha^{\Phi}_{+}(\gamma^m)$ and one easily sees that the term in 2.1.16 is 0.

If γ is odd hyperbolic, then similarly to Lemma 2.1.4,

$$\alpha^{\Phi}_{\pm}(\gamma^m) = ma_{\pm}(\gamma) \mp \frac{1}{2}(m - p(m))$$

and 2.1.16 becomes, after a short computation

$$\min\{mp(\gamma^n), np(\gamma^m)\} \geq 0.$$

Finally, if γ is elliptic, then there is an irrational number θ such that

$$\alpha_-^\Phi(\gamma^m) = \lfloor m\theta \rfloor$$

$$\alpha_+^\Phi(\gamma^m) = \lfloor m\theta \rfloor + 1$$

Then, 2.1.16 becomes

$$\min\{-n\lfloor m\theta\rfloor, -m\lfloor n\theta\rfloor\} + \min\{n\lfloor m\theta\rfloor + n, m\lfloor n\theta\rfloor + m\} \geq -\lfloor mn\theta\rfloor + \lfloor nm\theta\rfloor = 0,$$

which completes the claim. □

If $u : \dot{\Sigma} \to \overline{W}$ is somewhere injective and J-holomorphic, then the relative asymptotic results of [Sie08] also imply that it is embedded outside a compact subset, so there is a finite singularity count $\delta(u) \in \mathbb{Z}$, defined as the algebraic count of double points $\{(z_1, z_2) \in \dot{\Sigma} \times \dot{\Sigma} \mid u(z_1) = u(z_2) \text{ and } z_1 \neq z_2\}$ after perturbing u in a compact subset to make it immersed. Standard local results due to Micallef and White [MW95] imply that $\delta(u) \geq 0$ with equality if and only if u is embedded, but in contrast to the closed case, $\delta(u)$ is not generally homotopy invariant. Instead, it satisfies the generalized adjunction formula

$$u * u = 2\delta_{\text{total}}(u) + c_N(u) + [\bar{\sigma}(u) - \#\Gamma], \tag{2.1.17}$$

where

$$\delta_{\text{total}}(u) = \delta(u) + \delta_\infty(u)$$

includes an additional contribution $\delta_\infty(u) \geq 0$ counting double points that can emerge from infinity under generic perturbations, and the term $\bar{\sigma}(u) \in \mathbb{N}$ is a sum of the spectral covering numbers (see §2.1.2) of all asymptotic orbits, hence $\bar{\sigma}(u) - \#\Gamma$ is also nonnegative.

We will need a couple of precise definitions when proving the local adjunction formula in §3.1. For a curve u such as above, in the neighbourhood of a puncture, z, for a certain parametrization, which we omit, we can write

$$u_z(s,t) = (m_z T s, \exp_{\gamma_z^{m_z}(t)} h(s,t))$$

and

$$u(s,t)_{z,\Phi,\varepsilon} = (m_z T s, \exp_{\gamma_z^{m_z}(t)}(h(s,t) + \Phi(mt)\varepsilon)$$

as before, and then the count of zeros at infinity is defined as

$$i_\infty^\Phi(u,z) = \mathrm{int}(u_z, u_{z,\Phi,\varepsilon})$$

For an embedded end, it is a result from [Sie11] that this count is equal to the count of zeros of

$$F_j(s,t) = h(s,t+j/m_z) - h(s,t) - \varepsilon\Phi(mt)$$

over each j

$$i_\infty^\Phi(u,z) = \mp \sum_{j=1}^{m_z-1} \mathrm{wind}^\Phi(h(s,\cdot + j/m_z) - h(s,\cdot))$$

These winding numbers are bounded above (respectively, below) by the extremal winding, that of the largest negative (smallest positive) eigenvalue. In fact, a slightly stronger result holds, namely, since each the approach of $h(s,t)$ is controlled by the eigenfunction f of $A_{\gamma^{m_z}}$,

$$i_\infty^\Phi(u,z) \geq \mp(m_z - 1)\,\mathrm{wind}^\Phi(f)$$

and if $\mathrm{wind}^\Phi(f)$ is not extremal, we have:

$$i_\infty^\Phi(u,z) \geq \mp(m_z - 1)\alpha_\mp^\Phi(\gamma^{m_z}) + m_z - 1$$

If $\mathrm{wind}^\Phi(f)$ is extremal, and hence equal to $\alpha_\mp^\Phi(\gamma^{m_z})$, a better bound can be obtained by considering the covering multiplicity of f, which we denote $\bar{\sigma}_\mp(\gamma^{m_z})$ in the formula for $i_\infty^\Phi(u,z)$ and expressing the winding in terms of $\mathrm{wind}^\Phi(g)$ where $f = g^{\bar{\sigma}_\mp(\gamma^{m_z})}$. The lower bound thus obtained is denoted by

$$\Omega_\pm^\Phi(\gamma^{m_z}) = \mp(m_z - 1)\alpha_\mp^\Phi(\gamma^{m_z}) + [\bar{\sigma}_\mp(\gamma^{m_z}) - 1]$$

Finally, we define $\delta_\infty(u)$. In [Sie11] it is proved to be a non-negative integer, and a true homotopy invariant of asymptotically cylindrical maps, independent of the trivialization Φ.

$$\delta_\infty(u) = \frac{1}{2}\left[\sum_{z,\zeta\in\Gamma^\pm, z\neq\zeta} i_\infty^\Phi(u,z;u,\zeta) - \Omega_\pm^\Phi(\gamma_z^{m_z},\gamma_\zeta^{m_\zeta}) + \sum_{z\in\Gamma^\pm} i_\infty^\Phi(u,z) - \Omega_\pm^\Phi(\gamma_z^{m_z})\right]$$

Returning to the adjunction formula, 2.1.17 implies that $\delta_{\text{total}}(u)$ is homotopy invariant, and since $\delta_\infty(u) \geq 0$, the condition $\delta_{\text{total}}(u) = 0$ then suffices to ensure that all somewhere injective curves homotopic to u are embedded. The converse is false in general: a curve can still be embedded with $\delta_{\text{total}}(u) > 0$ due to hidden intersections, which can emerge from infinity under perturbations—but this can only happen if u has at least one multiply covered asymptotic orbit or at least two punctures of the same sign that approach covers of the same orbit, thus giving the following useful criterion:

Lemma 2.1.13. *If u is a somewhere injective curve whose asymptotic orbits are all distinct and simple, then $\delta_\infty(u) = \bar{\sigma}(u) - \#\Gamma = 0$.* □

The following is a minor improvement on a definition originating in [Wen10a, Wen10b].

Definition 2.1.14. An asymptotically cylindrical J-holomorphic curve $u : \dot{\Sigma} \to \overline{W}$ is called **nicely embedded** if it is somewhere injective and satisfies $u * u \leq 0$ and $\delta_{\text{total}}(u) = 0$.

It is clear from the above discussion that if u is nicely embedded, then so is any other somewhere injective curve u' in the same component of the moduli space, and moreover, u and u' must then be disjoint. Nicely embedded curves arise naturally in the study of finite energy foliations, initiated in [HWZ03]. Their most important properties for our purposes are the following.

Lemma 2.1.15. *If $u \in \mathscr{M}(J,\gamma^+,\gamma^-)$ is nicely embedded then $c_N(u) \leq 0$ and* $\text{ind}(u) \leq 2$.

Proof. The first inequality follows directly from the definition and the adjunction formula (2.1.17), and this implies the second via (2.1.10). □

Proposition 2.1.16. *If* $u \in \mathscr{M}(J, \gamma^+, \gamma^-)$ *is a nicely embedded curve with* $\mathrm{ind}(u) \in \{1,2\}$*, then u is Fredholm regular.*

Proof. Since u is immersed by assumption and, by Lemma 2.1.15, satisfies $c_N(u) \le 0$, it satisfies the criterion $\mathrm{ind}(u) > c_N(u)$ for automatic transversality given in [Wen10b]. □

Proposition 2.1.17. *Suppose* $\mathscr{M}^{\mathrm{nice}} \subset \mathscr{M}_1(J, \gamma^+, \gamma^-)$ *is a (not necessarily connected) component of the space of nicely embedded index* 2 *curves, equipped with the extra data of a marked point, such that all curves in* $\mathscr{M}^{\mathrm{nice}}$ *represent the same relative homology class. Then* $\mathscr{M}^{\mathrm{nice}}$ *is a smooth* 4*-manifold, and the evaluation map*

$$\mathrm{ev} : \mathscr{M}^{\mathrm{nice}} \to \overline{W}$$

is an embedding onto an open subset of $\overline{W}$*.*

Proof. This is a mild generalization of a similar result proved in [HWZ99] for planes with simply covered asymptotic orbits. We know every $u \in \mathscr{M}^{\mathrm{nice}}$ is Fredholm regular by Prop. 2.1.16, and $c_N(u) = 0$ due to (2.1.10) and Lemma 2.1.15. It follows that $\mathscr{M}^{\mathrm{nice}}$ is smooth and has dimension $\mathrm{ind}(u) + 2 = 4$, and since $u * u \le 0$ (which becomes $u * u = 0$ when $c_N(u) = 0$), invariance of the intersection number implies that no two curves in $\mathscr{M}^{\mathrm{nice}}$ can intersect, hence $\mathrm{ev} : \mathscr{M}^{\mathrm{nice}} \to \overline{W}$ is injective. To see that it is also an immersion, observe that for a given curve $u_0 : \dot{\Sigma} \to \overline{W}$ and marked point $\zeta_0 \in \dot{\Sigma}$ with the pair (u_0, ζ_0) representing an element of $\mathscr{M}^{\mathrm{nice}}$, the tangent space $T_{(u_0,\zeta_0)}\mathscr{M}^{\mathrm{nice}}$ is naturally identified with the direct sum of $T_{\zeta_0}\dot{\Sigma}$ and the kernel of the linearized Cauchy-Riemann operator acting on the normal bundle of u_0. The condition $c_N(u_0) = 0$ then implies via [Wen10b, Equation (2.7)] that sections in this kernel are nowhere zero, hence the derivative of the evaluation map $\mathrm{ev}(u, \zeta) = u(\zeta)$ at (u_0, ζ_0) is injective. □

Proposition 2.1.18. *Suppose $\overline{W}$ is a symplectization $(\mathbb{R}\times M, d(e^r\alpha))$ and $J \in \mathscr{J}(\alpha)$. Then for any nicely embedded J-holomorphic curve $u = (u_{\mathbb{R}}, u_M) : \dot{\Sigma} \to \mathbb{R}\times M$ that is not a trivial cylinder, the map $u_M : \dot{\Sigma} \to M$ is embedded.*

Proof. Since $c_N(u) \le 0$ by Lemma 2.1.15, $\text{wind}_\pi(u) = 0$ due to (2.1.11) and u_M is therefore immersed and transverse to the Reeb vector field. To show that u_M injective, observe that any double point $u_M(z_1) = u_M(z_2)$ can be interpreted as an intersection of u with one of its $\mathbb{R}$-translations $u^c := (u_{\mathbb{R}} + c, u_M)$ for some $c \in \mathbb{R}$, and c must be nonzero since $\delta_{\text{total}}(u) = 0$ implies that u itself is embedded. By homotopy invariance of the intersection product, $u * u = u * u^c \le 0$, so such an intersection is possible only if u and u^c are the same curve up to parametrization. But this would imply that u is also equivalent to u^{kc} for every $k \in \mathbb{N}$, so taking $k \to \infty$, we conclude from the asymptotically cylindrical behaviour of u that its image lies in an arbitrarily small neighbourhood of a collection of trivial cylinders. This can only happen if u itself is a trivial cylinder, so we have a contradiction. □

Lemma 2.1.19. *Under the assumptions of Prop. 2.1.18, suppose $u = (u_{\mathbb{R}}, u_M) : \mathbb{C} \to \mathbb{R}\times M$ is a nicely embedded plane asymptotic to a simply covered orbit γ and $\text{ind}(u) \in \{1,2\}$. Then if $\mathscr{D} \subset M$ denotes the Seifert surface with interior $u_M(\mathbb{C})$, we have*

$$\mu_{\text{CZ}}(\gamma;\mathscr{D}) = \begin{cases} 2 & \text{if } \text{ind}(u) = 1, \\ 3 & \text{if } \text{ind}(u) = 2, \end{cases}$$

and in both cases $\text{sl}(\gamma;\mathscr{D}) = -1$.

Proof. If Φ is the trivialization of $\gamma^*\xi$ that extends over $\mathscr{D}$, then the relative c_1 term in the index formula vanishes and gives the stated relation between $\text{ind}(u)$ and $\mu^{\Phi}_{\text{CZ}}(\gamma)$. By (2.1.2) and (2.1.3), this implies $\alpha^{\Phi}_{-}(\gamma) = 1$. Moreover, $c_N(u) \le 0$ by Lemma 2.1.15, thus (2.1.12) implies that u has zero asymptotic defect, so the nonzero eigenfunction $e_1(u)$ appearing in the asymptotic formula (2.1.7) satisfies

$$\text{wind}^{\Phi}(e_1(u)) = \alpha^{\Phi}_{-}(\gamma) = 1.$$

Now by (2.1.14),

$$\mathrm{sl}(\gamma;\mathscr{D}) = -\,\mathrm{wind}^{\Phi}(e_1(u)) = -1.$$

□

2.2 Seed curves in the positive end

In this section we describe the seed curves that will generate the moduli spaces required for proving Theorems 1.1.1, 1.1.2 and 1.1.3.

2.2.1 The standard sphere

The following construction is for the proofs of Theorems 1.1.1 and 1.1.3.

Regarding S^{2n-1} as the unit sphere in $\mathbb{C}^n$, fix the standard contact form α_{std} described in Example 2.1.2, along with the unique admissible complex structure $J_{\mathrm{std}} \in \mathscr{J}(\alpha_{\mathrm{std}})$ on $\mathbb{R} \times S^{2n-1}$ that restricts to $\xi_{\mathrm{std}} \subset TS^{2n-1} \subset \mathbb{C}^n$ as the standard complex structure i. Recall that the diffeomorphism

$$(\mathbb{R} \times S^{2n-1}, J_{\mathrm{std}}) \to (\mathbb{C}^n \setminus \{0\}, i) : (r,x) \mapsto e^{2r}x \tag{2.2.1}$$

is then biholomorphic, so we can regard holomorphic curves in $\mathbb{C}^n \setminus \{0\}$ as J_{std}-holomorphic curves in the symplectization of $(S^{2n-1}, \xi_{\mathrm{std}})$. With this understood, define for each $w \in \mathbb{C}^{n-1} \setminus \{0\}$ the holomorphic plane

$$u_w : (\mathbb{C}, i) \to (\mathbb{C}^n \setminus \{0\}, i) : z \mapsto (z, w).$$

As a curve in $\mathbb{R} \times S^{2n-1}$, each u_w is asymptotic at ∞ to the same closed Reeb orbit in $(S^{2n-1}, \alpha_{\mathrm{std}})$, namely

$$\gamma_\infty : S^1 \to S^{2n-1} : t \mapsto (e^{2\pi i t}, 0, \dots, 0).$$

This orbit belongs to a $(2n-2)$-dimensional Morse-Bott family of closed embedded Reeb orbits with period π, which foliate S^{2n-1}; indeed, they form the fibres of the Hopf fibration $S^1 \hookrightarrow S^{2n-1} \to \mathbb{C}P^{n-1}$.

Lemma 2.2.1. *For each* $w \in \mathbb{C}^{n-1} \setminus \{0\}$, $\mathrm{ind}(u_w) = 2n-2$.

Proof. Abbreviate $\overline{W} = \mathbb{R} \times S^{2n-1}$. The fibres of the contact bundle along γ_∞ are naturally identified with $\{0\} \oplus \mathbb{C}^{n-1} \subset TS^{2n-1} \subset \mathbb{C}^n$, so $\gamma_\infty^* \xi_{\mathrm{std}}$ has a natural trivialization, which we will denote by Φ, and it extends to a natural trivialization of the normal bundle $N_{u_w} \to \mathbb{C}$ of u_w. The latter implies $c_1^\Phi(N_{u_w}) = 0$, so writing $u_w^* T\overline{W} = T\mathbb{C} \oplus N_{u_w}$ gives

$$c_1^\Phi(u_w^* T\overline{W}) = \chi(\mathbb{C}) + c_1^\Phi(N_{u_w}) = 1.$$

To compute $\mu_{CZ}^\Phi(\gamma_\infty + \varepsilon)$, we observe that the asymptotic operator A_{γ_∞} splits with respect to the obvious decomposition

$$\gamma_\infty^* \xi_{\mathrm{std}} = S^1 \times \mathbb{C}^{n-1} = L_2 \oplus \ldots \oplus L_n$$

into trivial complex line bundles, so we can write $A_{\gamma_\infty} = A_2 \oplus \ldots \oplus A_m$, and the trivialization Φ is now also a direct sum $\Phi_2 \oplus \ldots \oplus \Phi_m$ of trivializations of these line bundles. The kernel of A_{γ_∞} is a complex $(n-1)$-dimensional space of sections along γ_∞ that point in the directions of other Hopf fibres, and its intersection with each of the summands L_j for $j = 2, \ldots, n$ is a complex 1-dimensional space spanned by a section of the form

$$\eta_j : S^1 \to L_j : t \mapsto (0, \ldots, 0, e^{2\pi i t}, 0, \ldots, 0).$$

We thus have $\mathrm{wind}^{\Phi_j}(\eta_j) = 1$, and $A_j + \varepsilon$ therefore has a real 2-dimensional eigenspace with the smallest positive eigenvalue ε and winding 1. By Proposition 2.1.3, the largest negative eigenvalue $A_j + \varepsilon$ must then have winding 0, so by (2.1.2),

$$\mu_{CZ}^{\Phi_j}(A_j + \varepsilon) = 1,$$

and (2.1.4) then implies

$$\mu_{CZ}^{\Phi}(\gamma_\infty + \varepsilon) = \sum_{j=2}^{n} \mu_{CZ}^{\Phi_j}(A_j + \varepsilon) = n - 1.$$

Finally, (2.1.9) gives

$$\begin{aligned}
\mathrm{ind}(u_w) &= (n-3)\chi(\mathbb{C}) + 2c_1^{\Phi}(u_w^* T\overline{W}) + \mu_{CZ}^{\Phi}(\gamma_\infty + \varepsilon) \\
&= (n-3) + 2 + (n-1) = 2n - 2.
\end{aligned}$$

□

Lemma 2.2.2. *The J_{std}-holomorphic planes u_w are all Fredholm regular.*

Proof. Note that the standard genericity arguments do not apply here since J_{std} is very far from being generic. But in this case we can check regularity explicitly. Recall that by [Wen10b, Theorem 3], it suffices to check that the linearized normal operator

$$\mathbf{D}_{u_w}^N : W^{1,p,\delta}(N_{u_w}) \to L^{p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\mathbb{C}, N_{u_w}))$$

is surjective, where $\mathrm{ind}(\mathbf{D}_{u_w}^N) = \mathrm{ind}(u_w)$. Here $p \in (2,\infty)$, and $\delta > 0$ is a small exponential weight, meaning that if sections $\eta : \mathbb{C} \to N_{u_w}$ in the domain of $\mathbf{D}_{u_w}^N$ are written near ∞ in cylindrical coordinates $(s,t) \in [0,\infty) \times S^1$ corresponding to $z = e^{2\pi(s+it)} \in \mathbb{C}$, then the section $e^{\delta s}\eta(s,t)$ must be of class $W^{1,p}$ on $[0,\infty) \times S^1$. This definition also assumes a translation-invariant metric on $\mathbb{R} \times S^{2n-1}$ for computing L^p-norms of sections along u_w. Note that since $p > 2$, sections of class $W^{1,p}$ are continuous, and we can therefore assume

$$\eta(s,t) \to 0 \quad \text{as} \quad s \to \infty \tag{2.2.2}$$

for $\eta \in W^{1,p,\delta}(N_{u_w})$.

From a different perspective, however, $\mathbf{D}_{u_w}^N$ is an extremely simple operator: sections η of the normal bundle to $u_w : \mathbb{C} \to \mathbb{C}^n \setminus \{0\}$ can be identified canonically with functions $\tilde{\eta} : \mathbb{C} \to \mathbb{C}^{n-1}$ using the obvious trivialization of N_{u_w}, and since $\mathbf{D}_{u_w}^N$ is the linearization of the standard (and thus already linear) Cauchy-Riemann operator $\bar{\partial}$, $\eta \in \ker \mathbf{D}^N_{u_w}$ implies that $\tilde{\eta}$ is a $\mathbb{C}^{n-1}$-valued holomorphic function. Under the transformation (2.2.1), the condition (2.2.2) then implies

$$\frac{|\tilde{\eta}(z)|}{|z|} \to 0 \quad \text{as} \quad z \to \infty,$$

so the growth of $\tilde{\eta}$ at infinity is strictly smaller than that of an affine function. Picard's Theorem then implies that the singularity of $\tilde{\eta}$ at ∞ is removable, so $\tilde{\eta}$ is constant, proving

$$\dim_{\mathbb{C}} \ker \mathbf{D}^N_{u_w} = n - 1.$$

The real dimension of the kernel of $\mathbf{D}^N_{u_w}$ is thus equal to its index according to Lemma 2.2.1, so $\mathbf{D}^N_{u_w}$ has trivial cokernel. □

Remark 2.2.3. In the case $n = 2$, the automatic transversality results of [Wen10b] also apply and prove that the curves u_w are regular.

Lemma 2.2.4. *Up to parametrization, every asymptotically cylindrical J_{std}-holomorphic curve in $\mathbb{R} \times S^{2n-1}$ with a single positive puncture asymptotic to γ_∞ and arbitrary negative punctures is either the trivial cylinder over γ_∞ or one of the planes u_w.*

Proof. Since no Reeb orbit in $(S^3, \alpha_{\text{std}})$ has period smaller than that of γ_∞, any curve $u : \dot{\Sigma} \to \mathbb{R} \times S^{2n-1}$ of the specified type with a nonempty set of negative punctures would satisfy $\int_{\dot{\Sigma}} u^* \alpha_{\text{std}} = 0$ by Stokes' theorem, and since the positive asymptotic orbit is simple, u in this case could only be a trivial cylinder. If u has no negative punctures, then it defines via (2.2.1) a proper holomorphic map $u = (u_1, \ldots, u_n) : \dot{\Sigma} \to \mathbb{C}^n$ such that $u_2, \ldots, u_n : \dot{\Sigma} \to \mathbb{C}$ are all bounded holomorphic functions that decay to 0 at the unique puncture, so these all extend to holomorphic functions on the compact domain Σ and are therefore constant. The remaining function $u_1 : \dot{\Sigma} \to \mathbb{C}$ has a pole of order 1 at its unique puncture, thus it extends to a nonconstant holomorphic map $\Sigma \to S^2$ of degree 1, implying that $\Sigma = S^2$ and, after a suitable reparametrization, $\dot{\Sigma} = \mathbb{C}$ with $u_1 : \mathbb{C} \to \mathbb{C}$ an affine map. □

Lemma 2.2.5. *In the case* $\dim M = 3$*, the planes* u_w *satisfy* $c_N(u_w) = 0$ *and are nicely embedded.*

Proof. We saw in the proof of Lemma 2.2.1 that $\mu_{CZ}^{\Phi}(\gamma_\infty + \varepsilon)$ is odd and $\mathrm{ind}(u_w) = 2$, so (2.1.10) implies $c_N(u_w) = 0$. Since u_w is embedded and has only a single simple asymptotic orbit, $\delta_{\mathrm{total}}(u_w) = \bar{\sigma}(u_w) - 1 = 0$ by Lemma 2.1.13. Thus by Siefring's adjunction formula,

$$u_w * u_w = 2\delta_{\mathrm{total}}(u_w) + c_N(u_w) + [\bar{\sigma}(u_w) - 1] = 0.$$

□

2.2.2 Overtwisted contact 3-manifolds

If (M, ξ) is overtwisted, then Eliashberg's appendix to [Yau06] uses the following geometric picture to prove vanishing of contact homology. There is a (nondegenerate) contact form α_+ and an almost complex structure $J_+ \in \mathcal{J}(\alpha_+)$ admitting an embedded Fredholm regular J_+-holomorphic plane

$$u^\infty = (u_{\mathbb{R}}^\infty, u_M^\infty) : \mathbb{C} \to \mathbb{R} \times M$$

with index 1, asymptotic to a simple Reeb orbit

$$\gamma_\infty : S^1 \to M$$

with even Conley-Zehnder index, such that u^∞ is (up to parametrization and $\mathbb{R}$-translation) the only nontrivial J_+-holomorphic curve in $\mathbb{R} \times M$ with one positive puncture asymptotic to γ_∞ (and arbitrary negative punctures).

Proof. The construction in [Yau06] begins by considering a typical torus model for the overtwisted contact structure given by a Lutz twist. Under the (θ, ρ, ϕ) coordinate system in the solid torus where $\theta \in \mathbb{R}/\mathbb{Z}$, $\phi \in S^1$, define a contact form as

$$\alpha = f(\rho)d\theta + g(\rho)d\phi$$

where

$$f(\rho) = cos(n\rho) \qquad g(\rho) = \frac{1}{n}sin(n\rho)$$

away from the origin and such that

$$D(\rho) = f(\rho)g'(\rho) - f'(\rho)g(\rho) > 0$$

and

$$f(0)g''(0) > 0.$$

See [Wen05], for details of the construction. A general orbit of the Reeb vector field is given by

$$\gamma(t) = \left(\theta_0 + \frac{g'(r)}{D(r)}t, r, \phi_0 - \frac{f'(r)}{D(r)}t\right)$$

The $r = \pi/2n$ torus then admits fully horizontal orbits, which come in a 2-dimensional Morse-Bott family. Increasing n, we can make the action of these orbits as small as necessary. As in [Wen10c] and [Wen13a] we define the constrained Conley-Zehnder index for this orbit $\mu_{CZ}(\gamma \pm \varepsilon)$ computed in the trivialization induced by the coordinates (call it Φ). This is equal to the Conley-Zehnder index of a non-degenerate orbit, under a small perturbation. Switching the sign of ε has the effect of changing the value above by $\dim\ker(A_\gamma) = 1$ so that we may choose values for which the asymptotic winding numbers of the extremal eigenvalues are equal, and equal to the winding of an eigenfunction in $\ker(A_\gamma)$. Now $\ker(A_\gamma)$ is spanned by sections η which point in the direction of the Morse-Bott fibres on the torus, hence their winding with respect to the trivialization Φ of $\gamma^*\xi$ is 0. From our choice of ε, we have

$$\mu_{CZ}^{\Phi}(\gamma + \varepsilon) = 2\,\mathrm{wind}^{\Phi}(\eta) = 0$$

.

Also following [Wen10c] or [Wen05], by making an a priori assumption on the formulas of J and u, one can define an almost complex structure J on $\mathbb{R} \times M$ and an embedded J-holomorphic plane u asymptotic to γ. The trivialization Φ extends to a trivialization of the normal bundle over N_u away from the boundary, and thus

$c_1(N_u) = 0$. This shows

$$\mathrm{ind}(u) = \chi(D) + c_1(N_u) + \mu_{CZ}(\gamma + \varepsilon) = 1.$$

Remark 2.2.6. The original claim of Eliashberg is that a plane can be found asymptotic to a non-degenerate orbit, a fact which we need not use here, as in section 2.2.1. A Morse-Bott asymptotic orbit suffices for our claims in later sections. At the end of our analysis, a perturbation can be performed to make the contact form non-degenerate, with two Reeb orbits in our torus model, one with odd, one with even Conley-Zehnder index, the even one supporting an embedded plane of index 1. Further in this section we adopt the notation of u^∞, γ_∞, for a one of the planes above together with its asymptotic orbit and α_+ and J_+, for the contact form and associated almost complex structure.

Now since u^∞ is asymptotic to a simple orbit, Lemma 2.1.13 implies $\delta_\infty(u^\infty) = \bar{\sigma}(u^\infty) - 1 = 0$, and since it is also embedded, $\delta_{\mathrm{total}}(u^\infty) = 0$. Moreover, by (2.1.10), $c_N(u^\infty) = 0$, u^∞ is embedded, and satisfies $1 = \mathrm{ind}(()u^\infty) > c_N(u^\infty) = 0$, and thus satisfies automatic transversality. The adjunction formula (2.1.17) now gives

$$u^\infty * u^\infty = 2\delta_{\mathrm{total}}(u^\infty) + c_N(u^\infty) + [\bar{\sigma}(u^\infty) - 1] = 0,$$

implying that u^∞ is nicely embedded.

It remains to justify the uniqueness statement, namely that any curve with a single positive orbit asymptotic to γ_∞ and arbitrary negative orbits is either u^∞ or $\mathbb{R} \times \gamma$. Since we work in dimension 4, we may appeal to intersection theory.

This begins with a standard argument, which we learned from Chris Wendl, and in its current form states: if u is a plane in $[0,\infty) \times M$ with a positive puncture at a simple orbit γ and v_1 and v_2 are any two other curves, with a single positive puncture at the same orbit, and arbitrary negative punctures, then $u * v_1 = u * v_2$. This is due to the fact that the intersection product is homotopy invariant through homotopies of asymptotically cylindrical maps, and hence, since u has no negative ends, one can use the $\mathbb{R}$-translation to generate such a homotopy for v_1 that pushes it down such that its intersection with $[0,\infty)\times M$ looks like $\mathbb{R}\times\gamma$. Then

$$u*v_1 = u*(\mathbb{R}\times\gamma) = u*v_2.$$

In our case, since $u^\infty * u^\infty = 0$, it follows that $u^\infty * v = 0$ for any other curve as above. Notice, first, that if v has any negative ends, it must be a trivial cylinder, since γ_∞ has minimal period and by Stokes' theorem, $\int_{\Sigma(v)} v^*\alpha_+ = 0$. So v must be a plane itself. Let λ, $e_1(u^\infty)$ be the eigenvalue - eigenfunction pair, and since u^∞ is nicely embedded and γ_∞ is even, we have that

$$\mathrm{wind}^\Phi(e_1(u^\infty)) = \alpha_-^\Phi(\gamma_\infty)$$

and

$$\dim E_\lambda = 1$$

Now, if $e_1(v) \notin E_\lambda$, we must have $\mathrm{wind}^\Phi(e_1(v)) < \alpha_-^\Phi(\gamma_\infty)$ and there must be geometric intersections between the projections of v and u^∞ induce, via $\mathbb{R}$-translation $v*u_\infty > 0$. Now, if $e_1(u^\infty) = ke_1(v)$ we may have k either positive or negative. The family of planes homotopic to u_∞ foliate a solid torus in M and and hence, if $k<0$, $e_1(v)$ points inside the torus $\{\rho = \frac{2\pi}{n}\}$ (otherwise, the projection of v and that of a neighbouring plane in the foliation, call it w_∞, would intersect, giving $0 < v*w_\infty = v*u_\infty$). If the projection of v never leaves $\{\rho \leq \frac{2\pi}{n}\}$, that would make $0 = [\gamma_\infty] \in H_1(\{\rho \leq \frac{2\pi}{n}\})$. Otherwise the projection of the plane v must intersect one of the members of the foliation, again, call it w_∞ as before, so such a plane cannot exist. However, if $k>0$, Theorem 2.5 in [Sie11], tackles this exact case and assures us that either $v*u_\infty > 0$ or $v = u_\infty$ up to reparametrization and $\mathbb{R}$-translation.

□

Chapter 3

Main results

3.1 A local adjunction formula for breaking holomorphic annuli

The aim of this section is to prove Theorem 1.1.16 and derive Corollary 1.1.17. We start by recalling some of the definitions that come up in the statement of the theorem. Throughout this section, we work with a 3-dimensional contact manifold (M,ξ_M) and a sequence of contact forms $\alpha_k \to \alpha_\infty$. Correspondingly, a sequence of admissible almost complex structures

$$J_k \in \mathscr{J}(\alpha_k)$$

and for α_i a nondegenerate periodic orbit γ^m of multiplicity m.

The setup is as follows. Let u_n be a sequence of somewhere injective J_n-holomorphic compact cylinders

$$u_n : ([-R_n, R_n'] \times S^1, i) \to (\mathbb{R}\times M, J_n)$$

and assume that in the limit they tend to a building consisting of two open half-cylinders,

$$v_\pm : ([\pm R_\pm, \mp\infty) \times S^1, i) \to (\mathbb{R}\times M, J).$$

$$u_n \to \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$$

Suppose the common breaking orbit is denoted by γ^k where γ is simply covered, and that γ^k is nondegenerate. Suppose furthermore that the limit half-cylinders are simply covered. Without loss of generality, we will assume that they are both **embedded** as this can always be arranged in this case, by restricting to a small enough neighbourhood of the puncture. Then pick a trivialization ϕ of $\gamma^*\xi_M$. We denote the induced trivialization in γ^k by ϕ as well.

Recall, that by the definition of convergence to a building in [BEH$^+$03], there are sequences of diffeomorphisms of the domains

$$\Phi_n : [-R_n, R_n'] \times S^1 \to [-R_n, R_n'] \times S^1$$

such that

$$u_n(-R_n, t) \circ \Phi_n \to v_+(-R_+, t)$$

uniformly. One may then choose simultaneous perturbations of these loops which are transverse to the contact structure, denoted by $\hat{u}_n(-R_n,t)$ and $\hat{v}_+(-R_+,t)$. Then one may choose trivializations of the contact structure along the perturbed loops which are homotopic. We choose ϕ as a representative of such a homotopy class of trivializations both for the sequence of curves and for the limit, both for the upper half and the lower. With respect to this trivialization, one may compute self-intersection numbers as follows.

Definition 3.1.1. The relative self-intersection number of u_n denoted by $u_n *_\phi u_n$ is the count of transverse intersections between u_n and a generic perturbation pushed along in the direction of ϕ at the boundary. Thus it depends only on the homotopy class of ϕ.

Definition 3.1.2. The relative self-intersection number of $v_\pm$ denoted by $v_\pm *_\phi v_\pm$ is the count of transverse intersections between $v_\pm$ and a generic perturbation pushed in the direction of ϕ at the boundary and at the asymptotic limit. Thus it depends only on the homotopy class of ϕ.

Remark 3.1.3. Without loss of generality, one can assume that u_k and $v_\pm$ are sufficiently close to a trivial cylinder at their boundaries and asymptotic limits such that the trivialization ϕ serves as a trivialization of the normal bundles of these curves in neighbourhoods of the boundary and the asymptotic limits.

Given a complex vector bundle $E \to \dot{\Sigma}$ over a punctured surface (with boundary), and a choice of trivialization ϕ over the boundary and punctures one can define a *relative first Chern number*

$$c_1^\phi(E)$$

as the sum of the orders of the zeros of a generic section $s : \dot{\Sigma} \to E$ which is a non-zero constant at the boundaries with respect to ϕ. This number is well defined, does not depend on s but only on the homotopy class of ϕ.

One observes a certain similarity in these definitions to standard ones in the closed case. We do not recall the proof of the adjunction formula in the closed case here, see for example [Wend]. For a surface with boundary, such as u_n, we have the following version of the adjunction formula:

$$u_n *_\phi u_n = 2\delta(u_n) + c_1^\phi(u_n^* T(\mathbb{R} \times M)). \tag{3.1.1}$$

Proof. Recall the definition of $\delta(u_n)$ as a sum of intersection indices across each double point, plus a sum of orders of each critical point. If u_n is immersed, there are no critical points and there is a well-defined normal bundle N_{u_n}. A push-off u_n^ε of u_n along a generic section η of its normal bundle which is constant with respect to ϕ at the boundaries, produces transverse intersections with u_n for each zero of the section η, and two intersections corresponding to each double point, each with the appropriate intersection index. Thus

$$u_n *_\phi u_n = u_n * u_n^\varepsilon = 2\delta(u_n) + c_1^\phi(N_{u_n}) = 2\delta(u_n) + c_1^\phi(u_n^* T(\mathbb{R} \times M)), \tag{3.1.2}$$

since $\chi(u_n) = 0$ Now, if u_n is not immersed, there is an immersed perturbation $\tilde{u_n}$

in small neighbourhoods of each critical point, which has extra double points in each of these neighbourhoods with sum of intersection indices equal to $\delta(u)$ at each critical point. Now $\tilde{u_n}$ satisfies the previous formula, and as $c_1^\phi(u_n^*T(\mathbb{R}\times M))$ is invariant under homotopies of maps which are constant on a neighbourhood of the boundary, so does u_n. □

Remark 3.1.4. Note that, as for the closed case, one does not require u_n to be immersed for this formula to hold.

Now if $k=1$, γ is a simple cover, and the exact same formula holds for the self intersection numbers of $v_\pm$. However, if $k>1$, pushing off in the direction of ϕ can produce extra intersections in a neighbourhood of the limit. This phenomenon has been studied in [Hut02, §3.2] and [Sie11]. In the latter text it is expressed as a sum of relative winding numbers of the eigenvalues of the asymptotic operator A_{γ^k} which control the relative approach of the $k-1$ branches of the asymptotic representative. This gives rise to an adjunction formula for asymptotic ends of the form:

$$v_\pm *_\phi v_\pm = 2\delta(v_\pm)+c_1^\phi(v_\pm^*T(\mathbb{R}\times M))+i_\infty^\phi(v_\pm). \tag{3.1.3}$$

Of course, since we are assuming the limits are embedded, $\delta(v_\pm)=0$.

As a consequence of the convergence of

$$u_n \to \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$$

we have:

Lemma 3.1.5. *The following equations hold;*

$$\lim_{n\to\infty} u_n *_\phi u_n = v_+ *_\phi v_+ + v_- *_\phi v_- \tag{3.1.4}$$

$$\lim_{n\to\infty} c_1^\phi(u_n^*T(\mathbb{R}\times M)) = c_1^\phi(v_+^*T(\mathbb{R}\times M))+c_1^\phi(v_-^*T(\mathbb{R}\times M)) \tag{3.1.5}$$

Proof. Let $v_+\odot v_-$ be the continuous concatenation along the asymptotic orbits determined by whatever choice of decoration is specified in the definition of covergence of u_k. Then as continuous maps into $[0,1]\times M$ there exists a diffeomorphism $\psi_n : [-R_n, R_n] \to [-R_n, R_n]$ such that

$$u_n \circ \psi_n \to v_+ \odot v_-$$

uniformly. This already implies the second equation. To see that the first holds notice that if $v_\pm^\phi$ represent generic perturbations of $v_\pm$ from the definition of the relative intersection numbers, that one can form simultaneous concatenations $v_+^\phi \odot v_-^\phi$ and $v_+ \odot v_-$. Then

$$\mathrm{int}(v_+ \odot v_-; v_+^\phi \odot v_-^\phi) = v_+ *_\phi v_+ + v_- *_\phi v_-$$

On the other hand the intersection number on the left hand side is, again by the definition of convergence, equal to $u_n *_\phi u_n$ for large enough n.

□

From Lemma 3.1.5 we can easily read off the proof of Theorem 1.1.16.

Theorem 3.1.6 (Local adjunction formula)**.** *For all sufficiently large k,*

$$2\delta(u_k) = 2[\delta_\infty(v_+) + \delta_\infty(v_-)] + \left[\bar{\sigma}_+(\gamma^k) - 1\right] + \left[\bar{\sigma}_-(\gamma^k) - 1\right] + (k-1)\,p(\gamma^k).$$

Proof. As in §2.1.4.5

$$i_\infty^\phi(v_\pm) = 2\delta_\infty(v_\pm) + \Omega_\pm^\phi(\gamma^k) = \\ = 2\delta_\infty(v_\pm) + (\bar{\sigma}_\pm(\gamma^k) - 1) \pm (k-1)\alpha_\pm^\phi(\gamma^k)$$

Hence,

$$\begin{aligned}
2\delta(u_k) &= i_\infty^\phi(v_+) + i_\infty^\phi(v_-) = \\
&= 2\delta_\infty(v_+) + (\bar\sigma_+(\gamma^k) - 1) + (k-1)\alpha_+^\phi(\gamma^k) + \\
&\qquad + 2\delta_\infty(v_-) + (\bar\sigma_-(\gamma^k) - 1) - (k-1)\alpha_-^\phi(\gamma^k) = \\
&= 2[\delta_\infty(v_+) + \delta_\infty(v_-)] + [\bar\sigma_+(\gamma^k) - 1] + [\bar\sigma_-(\gamma^k) - 1] + (k-1)p(\gamma^k).
\end{aligned}$$

□

Lemma 3.1.7. *Let u_n and $v_\pm$ be as above and suppose that for n large enough, the u_n are restrictions of a sequence of nicely embedded curves to a subset of their domains. Then*

$$i_\infty^\phi(v_+) + i_\infty^\phi(v_-) = 0$$

Proof. This is self evident from the proof of 1.1.16, since such annuli are embedded and produce no self intersections under perturbations. Hence $\delta(u_n) = 0$, and the result follows. □

We can now prove Corollary 1.1.17.

Proof. The adjunction formula implies that if $\delta(u_k) = 0$ all the nonnegative terms on the right hand side must be 0. In particular $(k-1)p(\gamma^k) = 0$ so the orbit is either even or simply covered and $[\bar\sigma_+(\gamma^k) - 1] + [\bar\sigma_-(\gamma^k) - 1] = 0$ so their spectral covering numbers are equal. Let $\lambda_\pm$ be the smallest positive and the largest negative eigenvalues of the asymptotic operator A_{γ^k} respectively. Then since they each have the same winding, their corresponding eigenspaces must be 1-dimensional. Now the action of the cyclic group of order k by $j \cdot e_1(t) \to e_1(t + j/k)$ fixes the eigenspaces so it must be true that $e_1(t + j/k) = Ke_1(t)$, for a real constant K. $K \neq 1$ unless $k = 1$ since $\mathrm{cov}(e_1) = 1$. However $K^k = 1$, and since K is real, we are left with $K = -1$. Therefore there can be only one j that gives a nontrivial action on $e_1(t)$ and we have either $k = 2$ or $k = 1$, so the orbit is either even and simply covered, or double covers of an odd hyperbolic orbit. It cannot be a double cover of an even simple orbit since, in this case $\mathrm{cov}(e_1(v_+)) \geq 2$. □

3.2 Compactness

The current section is devoted to the study of what particular degeneracies may appear as limits of our family of seed curves. We use numerical constraints to reduce the complexity of the theoretical set of possible degeneracies defined in [BEH$^+$03]. We work with a Liouville cobordism $(W, d\lambda)$ with convex boundary (M_+, ξ_+) and concave boundary, (M_-, ξ_-), where $\xi_\pm = \ker \alpha_\pm$, such that α_+ is Morse - Bott and α_- is nondegenerate. We choose an almost complex structure J that is compatible with this data and generic both on the level of the cobordism and at the level of the cylindrical ends. Suppose that α_+ admits a closed simple orbit γ_∞.

Theorem 3.2.1. *Suppose $u_k \in \mathscr{M}(J, \gamma_\infty, \emptyset)$ is a sequence of nicely embedded planes converging in the sense of* [BEH$^+$03] *to a holomorphic building $u_\infty \in \overline{\mathscr{M}}(J, \gamma_\infty, \emptyset)$ with no nontrivial upper levels but at least one nontrivial lower level. Then all components of the levels of u_∞ other than trivial cylinders are nicely embedded, all breaking orbits are either simply covered or are doubly covered bad orbits, and u_∞ fits one of the following descriptions (see Figure 3.1):*

- *Type (I): $(v_0|v_1^-)$, where v_0 is an index 0 cylinder, v_1^- is an index 1 plane, and the breaking orbit has even parity.*
- *Type (II): $(v_0|v_1^-)$, where v_0 is an index 0 cylinder, v_1^- is an index 2 plane, and the breaking orbit has odd parity.*
- *Type (III): $(v_0|v_1^-)$, where v_0 is an index 1 cylinder, v_1^- is an index 1 plane, and the breaking orbit has even parity.*
- *Type (IV): $(v_0|v_1^-)$, where v_0 has index 0 and two negative punctures, v_1^- is a disjoint union of two index 1 planes, and both breaking orbits have even parity.*
- *Type (V): $(v_0|v_1^-|v_2^-)$, where v_0 has index 0 and two negative punctures, v_1^- is the disjoint union of a trivial cylinder with an index 1 plane, and v_2^- is an additional index 1 plane, with all breaking orbits having even parity.*

- *Type (VI):* $(v_0|v_1^-|v_2^-)$, *where* v_0 *is an index* 0 *cylinder,* v_1^- *is an index* 1 *cylinder and* v_2^- *is an index* 1 *plane, the breaking orbit between* v_0 *and* v_1^- *has odd parity, and the breaking orbit between* v_1^- *and* v_2^- *has even parity.*

Towards this aim, we recall some of the properties of pseudoholomorphic buildings in $\overline{\mathscr{M}}(J,\gamma_\infty,\emptyset)$. By 2.1.5, it must have the structure of a *tree*, and since γ_∞ is simply covered, so must be v_0. Observe that if J is generic, Equation 2.1.10 and Lemma 2.1.15 imply that the curves u_k in our sequence can be assumed to satisfy either $\mathrm{ind}(u_k) \in \{1,2\}$ and $c_N(u_k) = 0$ or $\mathrm{ind}(u_k) = 0$ and $c_N(u_k) = -1$. The index 1 and 2 cases will be treated separately, in particular, curves of Type (I) only appear in the limit of a sequence of curves of index 1. We denote the limit of a sequence u_k by

$$u_\infty = (v_0|v_1^-|\dots|v_N^-)$$

and allow for each v_i^- to be a union of disjoint curves

$$v_{i,1}^-,\dots,v_{i,m_i}^-$$

each of which is a map from the punctured sphere with precisely one positive puncture. Recall Lemma 2.1.10.

One may now consider the sum of $c_N(v_{i,j}^-)$ with the parities of its negative orbits and denote it by $\hat{c}_N$. This gives

$$0 \geq c_N(u_k) = \hat{c}_N(v_0) + \sum_{j=1}^{N}\sum_{i=1}^{m_j} \hat{c}_N(v_{j,i}^-) \tag{3.2.1}$$

Lemma 3.2.2. *All the components* $v_{j,i}^-$ *in lower levels have* $c_N(v_{j,i}^-) = 0$*, and one of the following holds:*

1. *All breaking orbits in* u_∞ *have even parity and the main level* v_0 *satisfies* $c_N(v_0) = 0$*, or*

2. *The main level* v_0 *is a cylinder with* $c_N(v_0) = -1$ *whose negative asymptotic orbit has odd parity, and all other breaking orbits have even parity.*

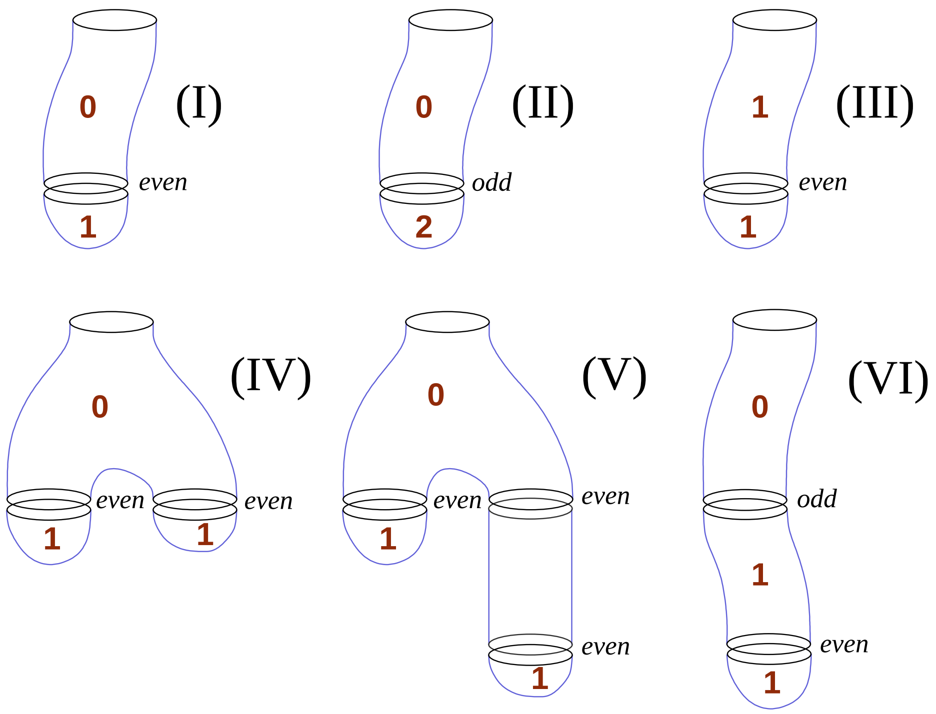

Figure 3.1: The six types of holomorphic buildings in Theorem 3.2.1.

Proof. Since γ_∞ is simply covered, so must be v_0 and, for generic J it must be Fredholm regular, thus $\operatorname{ind}(v_0) \geq 0$. Equality 2.1.10 now shows that

$$c_N(v_0) \geq -1$$

with equality if and only if all its negative punctures are asymptotic to odd orbits. Thus $\hat{c}_N(v_0) \geq 0$.

The curves $v_{i,j}^-$ in $\mathbb{R} \times M$ who are not covers of trivial cylinders satisfy $c_N(v_{i,j}^-) \geq 0$ by 2.1.12 hence also $\hat{c}_N(v_{i,j}^-)$. For covers of trivial cylinders, Proposition 2.1.8 applies and guarantees that $\operatorname{ind}(v_{i,j}^-) \geq 0$ with equality only if the cover is unbranched or the orbit is elliptic. Thus $c_N(v_{i,j}^-) \geq -1$ with equality only if all of it punctures are odd, and as such, $\hat{c}_N(v_{i,j}^-) \geq 0$ with with equality only if $v_{i,j}^-$ has at most one negative puncture. By Riemann-Hurewicz a cylinder can only be a cover of a trivial cylinder if it is unbranched, hence a trivial cylinder itself.

Now equation 3.2.1 implies all the components satisfy $\hat{c}_N = 0$. Thus all components that are not covers of trivial cylinders satisfy $c_N = 0$ and have only even orbits. The component in the main level either has only even orbits or is a cylinder of $\operatorname{ind}(v_0) = 0$ with a single negative orbit. All other curves are trivial cylinders, and it remains to prove they have only even orbits. If not, take one such cylinder that is maximal with respect to the SFT ordering of layers, say it lies in layer j. Notice that $j \geq 2$ since level 1 cannot consist of a single odd cylinder, by stability. Then, all curves in higher levels are not trivial cylinders with odd orbits. Thus, by the previous remark all of the negative orbits of layer $j-1$ are even. Hence so is the positive orbit of our trivial cylinder, and thus, such cylinders cannot exist and all curves in $\mathbb{R} \times M$ have only even negative orbits, which completes the proof. □

At this stage we are ready to justify the classification of buildings into the list presented in Theorem 3.2.1.

Notice that among the conclusions of the previous lemma is that there are no components of the building which are genuine branched covers of trivial cylinders. We shall appeal to this fact repeatedly in what follows.

Proof of Theorem 3.2.1. Consider the first case in which there are only even breaking orbits. Now curves $v_{i,j}^-$ that are not covers of trivial cylinders, are either somewhere injective themselves or a multiple cover of some somewhere injective curve $w_{i,j}^-$,

$$v_{i,j}^- = w_{i,j}^- \circ \theta_{i,j}^-,$$

where $\theta_{i,j}^-$ is a covering map of Riemann surfaces of multiplicity $k_{i,j}^-$. Due to genericity, $\operatorname{ind}(w_{i,j}^-) \geq 1$ and thus, by Proposition 2.1.9 $\operatorname{ind}(v_{i,j}^-) \geq k_{i,j}^-$. We have that

$$\sum_{i,j} k_{i,j}^- \leq \operatorname{ind}(u_\infty),$$

and since the index of the seed curves is either 2 or 1, this restricts the possible types of buildings that can occur. In the index 1 case, it immediately implies that $\operatorname{ind}(v_0) = 0$ since the negative levels contain at least one plane, which must have at least index 1. In particular, this plane must still exist, and it can only be in the first level in $\mathbb{R} \times M$ due to stability, hence we recover buildings of Type (I).

In the index 2 case, more cases can arise: we know there is at least one capping plane in $\mathbb{R}\times M$ and this means that v_0 can either have index 0 or 1. If it has index 1, then there is at most one plane in $\mathbb{R}\times M$, it is simply covered and all other curves in $\mathbb{R}\times M$ have index 0. Now v_1^- cannot be a trivial cylinder or a cover of one, so it must have index at least 1, and hence it is the aforementioned plane, and we recover buildings of Type (III).

If $\mathrm{ind}(v_0)=0$, equation 2.1.10 implies that it must have precisely 2 even punctures, and these occur by necessity at the negative end, call them z_1 and z_2. Therefore in $\mathbb{R}\times M$ there are two trees, each rooted at one of the orbits γ_{z_1} or γ_{z_2}, and each of these trees must contain at least one capping plane of index at least 1, and therefore in this case, at most 1. The same reasoning as in the previous paragraph proves one of these trees is merely a single plane of index 1, whereas, the second one may be either a single plane or a trivial cylinder (stability still applies for level 1, due to the plane at the other orbit), followed by a plane of index 1 in the next level. This recovers buildings of Types (IV) and (V).

What remains to be seen in this case is that all components are nicely embedded and to verify the claim on the breaking orbits. We can use Proposition 2.1.12 to write

$$0\geq u_k*u_k\geq v_0*v_0+\sum_{j=1}^{N}\sum_{i,\ell=1}^{m_j}v_{j,i}^-*v_{j,\ell}^-.$$

Positivity of intersections together with Prop. 2.1.11 implies that all terms on the right hand side are nonnegative, hence all of them vanish, including each $v_{j,i}^-*v_{j,i}^-$. Since $c_N(v_{j,i}^-)=c_N(v_0)=0$, the adjunction formula then gives $\delta_{\mathrm{total}}(v_{j,i}^-)=\delta_{\mathrm{total}}(v_0)=0$, hence all components are nicely embedded. The statement on the parities and multiplicities of the orbits follows now from Corollary 1.1.17.

Now for the case where v_0 has a single negative orbit γ_1, it has $\mathrm{ind}(v_0)=0$, $c_N(v_0)=-1$ and γ_1 is odd. Thus, v_1^- is a connected curve, and, as before, not a trivial cylinder, or a branched cover of one.

For curves in $\mathbb{R}\times M$ we have $v_{j,i}^-*v_{j,i}^-\geq 0$ as before, since regardless of the existence of an odd orbit, no trivial cylinders are allowed to tend to this orbit.

For v_0 the adjunction formula gives

$$v_0 * v_0 \geq -1$$

and Proposition 2.1.12 can be applied to give:

$$0 \geq u_k * u_k \geq -1 + m(\gamma_1),$$

which implies that $m(\gamma_1) = 1$ and thus v_1^- is a somewhere injective curve. Its index is either 1 or 2. If its index is 2, there can be no further curves in lower levels and v_1^- is thus a capping plane. This recovers buildings of Type (II). If $\text{ind}(v_1^-) = 1$, it must have a single even negative orbit, by Lemma 3.2.2 and equation 2.1.10. Level 2 thus cannot contain a trivial cylinder, and we must have at least one capping plane, which therefore resides here and we recover buildings of Type (VI). To see they are nicely embedded, we again apply Proposition 2.1.12 to deduce all components are nicely embedded. v_0 has negative self intersection number, but also $c_N(v_0) = -1$, so the adjunction formula again gives $\delta_{\text{total}}(v_0) = 0$. Corollary 1.1.17 applies now as before to determine the multiplicities of each orbit and this concludes the proof of Theorem 3.2.1.

□

3.3 Proof of Main Theorems

In the following, whenever we establish new notation, it is implicit that its definition expires when the context of the proof changes. We begin with the final details of the proof of Theorem 1.1.1, where we make use of Theorem 3.2.1 in the context of a Liouville cobordism from M to (S^3, λ_{std}).

Proof of Theorem 1.1.1. We have seen in the previous chapter what the quantitative structure of the boundary $\partial\mathscr{M} = \overline{\mathscr{M}}(J, \gamma_\infty, \emptyset) \setminus \mathscr{M}(J, \gamma_\infty, \emptyset)$ looks like, by enumerating the types of buildings that arise, and classifying them according to the topology of each level, their Fredholm indices and the Conley-Zehnder indices of the breaking orbits. Our aim, as stated previously is to find a simple Reeb orbit at the positive limit of a nicely embedded plane in the symplectization of M. To this end we ask ourselves what the set $U = \{p \in \overline{W} \mid \exists u \in \mathcal{M}(J, \gamma_\infty, \emptyset),\ p \in \mathrm{Im}(u)\}$ actually contains. Since the moduli space of smooth curves is smooth, this set is certainly open, every point in this set having a neighbourhood which is foliated by the simple planes in $\mathcal{M}(J, \gamma_\infty, \emptyset)$.

We will denote by $\mathcal{M}_u(J, \gamma_\infty, \emptyset)$ the connected component of $\mathcal{M}(J, \gamma_\infty, \emptyset)$ that contains the curve u. We will also need to allow marked points on our curves, and the moduli space of curves with one marked point will be denoted by $\mathcal{M}^1(J, \gamma_\infty, \emptyset)$. It has virtual dimension

$$2m + \mathrm{virdim}(\mathcal{M}(J, \gamma_\infty, \emptyset)) = 4.$$

A certain subset of this space consists of the nicely embedded curves in the same homotopy class as the seed curves u_w, and it is a union of connected components of the larger space. We denote it by $\mathcal{M}^1_{nice}(J, \gamma_\infty, \emptyset)$ and the evaluation map

$$\mathrm{ev} : \mathcal{M}^1_{nice}(J, \gamma_\infty, \emptyset) \to \overline{W}$$

is then an embedding onto an open subset of $\overline{W}$.

The SFT compactification of $\mathcal{M}^1_{nice}(J, \gamma_\infty, \emptyset)$ will be denoted by $\overline{\mathcal{M}}^1_{nice}(J, \gamma_\infty, \emptyset)$, and, except for buildings fully contained in the positive end (that is to say with no component in the main level), the components of its boundary have been classified in Theorem 3.2.1. The seed curves can be identified with buildings of type $(u_w|\emptyset)$.

The seed curves have index 2, and thus we need to account for degenerations of types (II) to (VI). Note that the breaking orbit for curves of Type (II) is odd and thus simple, with Conley-Zehnder index 3, whereas if the breaking orbit in a building of Type (III) is simple, it has Conley-Zehnder index 2.

In what follows we assume that in the boundary of $\overline{\mathcal{M}}^1_{nice}(J, \gamma_\infty, \emptyset)$, these kinds of buildings do not occur. In particular, if a building is of Type (III), its breaking orbit is a double cover of an odd hyperbolic orbit.

Under these assumptions it is clear that the remaining buildings of Type (III) occur generically in the boundary of $\overline{\mathscr{M}}^1_{nice}(J,\gamma_\infty,\emptyset)$.

Lemma 3.3.1. *Any connected component of $\overline{\mathscr{M}}^1_{nice}(J,\gamma_\infty,\emptyset)$ contains at least one boundary component consisting of buildings of Type (III).*

Proof. In the introduction, when we proved that closed orbits do in fact exist in M, we did so by obtaining a plane in $\mathbb{R}\times M$. In the absence of buildings of Types (III) and (II) it must mean that such a plane is a component in a level of a building of Types (IV), (V) or (VI). By Theorem 3.2.1 all such buildings are nicely embedded and satisfy a gluing theorem such as Theorem 2.1.6. But as all these buildings inhabit codimension 2 strata of the boundary, gluing across any 2 levels produces a codimension 1 building which is by necessity one of Types (II) or (III).

□

Proposition 3.3.2. *U is not dense in any neighbourhood of $-\infty$ in the negative end of $\overline{W}$.*

Proof. We proceed by contradiction. Under the assumption that Type (II) buildings never occur, all other types share the following property: the levels of such curves which lie in the negative end are rigid, when factoring out the $\mathbb{R}$ action. This is to say that their images lie in (at least) a codimension 1 subset of the negative end. Now there are finitely many of these families of buildings due to the SFT compactness theorem, and thus, overall there exists an ε-neighbourhood of each of these curves, such that their union represents a small open subset of $(-\infty,0]\times M$, which we denote by $N(\varepsilon)$.

Now pick a sequence of points $x_n\in(-\infty,0]\times M\setminus N(\varepsilon)$ such that

$$\pi_{\mathbb{R}}x_n\to-\infty,$$

where $\pi_{\mathbb{R}}$ is the projection onto the $\mathbb{R}$ coordinate. If U were dense, one would be able to pick x_k and a curve $u_k\in\mathscr{M}^{nice}_1(J,\gamma_\infty,\emptyset)$ such that $\mathrm{ev}(u_k)=x_k$. However, such a sequence of curves must have a subsequence tending to a building of Types (II) to (V). But this is impossible by our construction of x_k since it would not exhibit the appropriate convergence in the lower levels. □

For the rest of the chapter, we establish some notation. Curves of Type (III) with a doubly covered odd breaking orbit lie in a subset of the SFT compactification whose maps lie in the space

$$\bigcup_{\gamma\, bad} \mathscr{M}_{nice}(J,\gamma_\infty,\gamma^2)\times\mathscr{M}_{nice}(J,\gamma^2,\emptyset),$$

with the marked versions included in the union of the two products. We will call the union of the first components which lie in the main level, collectively, $\mathscr{M}_{(III)}(W)$ and those in the negative level by $\mathscr{M}_{(III)}(\mathbb{R}\times M)$.

Now, each connected component C of $\mathscr{M}^1_{nice}(J,\gamma_\infty,\emptyset)$ is mapped, via the evaluation map to an embedded open submanifold, $\mathrm{ev}(C)$. The closure of each of these sets is populated generically by points in

$$ev(\mathscr{M}^1_{(III)}(W))$$

since these have index 3. By Theorem 3.2.1, this set is itself a collection of embedded open 3-manifolds in $\overline{W}$. This immediately implies:

Lemma 3.3.3. *The set $\overline{W}\setminus\mathrm{ev}(\mathscr{M}_{(III)}(W))$ is a union of open subsets each of which is either densely foliated by smooth curves in $\mathscr{M}^1_{nice}(J,\gamma_\infty,\emptyset)$ or contains no such curves at all.*

Proof. □

The rest of this chapter is devoted to proving that all such subsets contain at least one smooth curve, contradicting Proposition 3.3.2.

Let X be a connected component of $\overline{\mathscr{M}}^1_{nice}(J,\gamma_\infty,\emptyset)\setminus\mathscr{M}^1_{nice}(J,\gamma_\infty,\emptyset)$ consisting of buildings of Type III breaking along a bad orbit and let

$$\Gamma_X = (\mathrm{ev}\times\mathrm{ev})(X\bigcap\bigcup_{\gamma\, bad}\mathscr{M}_{nice}(J,\gamma_\infty,\gamma^2)\times\mathscr{M}_{nice}(J,\gamma^2,\emptyset)).$$

Suppose without loss of generality that $\overline{W}\setminus\Gamma_X$ is disconnected, otherwise, move to a new component X'. Γ_X has the structure of a singular complex, which is smooth apart from at the breaking orbits. We shall show that, despite the fact that Γ_X disconnects, foliations from one connected component extent to the other via the different gluings along the hyperbolic orbits.

Let $p \in \Gamma_X$ be a point in the image of the broken curve $v = (v_0|v_1)$. Itself, it belongs to an equivalence class of buildings, which we denote by $\mathrm{v} = [S, j, v, D, \Phi]$ where, as before Φ is a set of *decorations*, orientation preserving diffeomorphisms from the circle compactifications at the positive puncture of one component to the negative puncture of another, if they tend to the same orbit. Now suppose without loss of generality $\phi = \phi_1 : \gamma \to \gamma$ is a decoration map on a double covered orbit γ. Then define a second decoration map $\tilde{\phi}(t) = \phi(t + \frac{1}{2})$. Let $\tilde{\Phi}$ represent the collection of decoration maps in Φ where we have changed ϕ to $\tilde{\phi}$. Let the two equivalence classes of buildings thus obtained be denoted by v and $\tilde{\mathrm{v}}$. Also, we denote by Γ_X^3 the 3 dimensional stratum of Γ_X. Since Γ_X disconnects $\overline{W}$, Γ_X^3 admits a collar neighbourhood, $\mathscr{N}(\Gamma_X^3)$.

Remark 3.3.4. The above statement is of course not correct in light of the classical Tubular Neighbourhood Theorem.

Lemma 3.3.5. *Under the conditions above, a certain collar neighbourhood $\mathscr{N}(\Gamma_X^3)$ admits a dense foliation by J-holomorphic curves in $\mathscr{M}(J, \gamma_\infty, \emptyset)$.*

Proof. We are in the case of the family of buildings v represented by broken curves $(v_0|v_1)$, where v_0 is a cylindrical component in W of Fredholm index 1, and v_1 is a plane in $\mathbb{R} \times M$ of Fredholm index 1. It is important to understand how gluing is performed under the change in decoration map.

If v and $\tilde{\mathrm{v}}$ were equivalent via a map $F = (f_0, f_1)$ then, $\tilde{\phi} \circ f_1 = f_0 \circ \phi$ and $\pi_M v_1 = \pi_M v_1 \circ f_1$. However, since v_1 is nicely embedded, its projection to M is an embedding, and thus $f_1 = \mathbb{1}$. f_2 is now an automorphism of the sphere fixing 2 points, acting like the identity around one and like $z \mapsto -z$ around the other. This is clearly impossible, so the two buildings are inequivalent.

The fact that $\tilde{\mathrm{v}}$ still represents an actual curve in the compactified moduli space is a consequence of Proposition 2.1.6.

The choices of pregluings w_R and $\tilde{w}_R$ corresponding to the two choices of decorations produce paths of actual J-holomorphic curves $u_R = \exp_{w_R}(\eta(R))$ and $\tilde{u}_R = \exp_{\tilde{w}_R}(\tilde{\eta}(R))$ for $\eta(R) \in W^{1,p}(w_R^* TW)$. The two paths of curves are not unique due to the possible choice of w_R and $\tilde{w}_R$. However, since any such pair of paths tend to two different points in the moduli space, and the compactified moduli space is Hausdorff, they must lie in non-intersecting open neighbourhoods of the space. Since the paths of curves tending to v foliate one half of $\mathscr{N}(\Gamma_X^3)$, what remains to see is that tTwo sequences u_R and $\tilde{u}_R$ as above, always have disjoint images in $\overline{W}$. This is a trivial matter, since they either have identical or disjoint images. If their images coincide, since they are both embeddings, $\tilde{u}_R^{-1} \circ u_R$ is a biholomorphic automorphism of the plane, and thus the two curves are equivalent and both represent the same point in the moduli space. But they are required to belong to non-intersecting neighbourhoods. Thus, the curves $\tilde{u}_R$ and u_R lie in different components of $\mathscr{N}(\Gamma_X^3)$. □

Thus, buildings of Type (III) cannot obstruct foliations from continuing past the codimension 1 hypersurface they define unless the breaking orbit is simple.

□

The previous proof made crucial use of the fact that all the components of the moduli space $\mathscr{M}(J, \gamma_\infty, \emptyset)$ together with their respective SFT compactifications map to disjoint subsets of the target space and produce tractable foliations therein. The idea the author first acquired from Chris Wendl, which was also applied in the introduction to prove Theorem 1.1.3, which is to pull back generic "vertical" paths in the cobordism through the evaluation map, could have also been used. We shall not need it for the proof of Theorem 1.1.2(2). The ideas above apply in a far simpler fashion here due mostly to the fact that the index of the nicely embedded curves which appear is 1 instead of 2.

Proof of Theorem 1.1.1 (2). The setting is that of a cobordism $(W, d\lambda)$ between (M, ξ) at the negative end and (M_+, ξ_+) at the positive end where ξ_+ is overtwisted. M admits a nondegenerate contact form α and we can find a nondegenerate form α_+ on M_+ as in §2.2.2 such that both are compatible with λ. Pick a $J \in \mathscr{J}(W, d\lambda, \alpha_+, \alpha)$ which restricts to a generic element $J_- \in \mathscr{J}(\alpha)$ and to $J_+ \in \mathscr{J}(\alpha_+)$ defined in §2.2.2, such that J is also generic on W. In the same section, the construction of a nicely embedded plane $u^\infty \in \mathscr{M}(J_+, \gamma_\infty, \emptyset)$ of Fredholm index 1, and by automatic transversality, regular, implies that the moduli space of nicely embedded curves in the same homotopy class as u^∞ is non-empty. Denote it by $\mathscr{M}_{nice}(J) \subset \mathscr{M}(J, \gamma_\infty, \emptyset)$. u^∞ then gives rise to a 1-parameter family of nicely embedded curves in $[0, \infty) \times M_+$ which, as before, represent the limit of a sequence of curves in the moduli space $\mathscr{M}_{nice}(J)$ tending to $+\infty$. All other curves apart from u^∞ intersect the interior of W hence, since J is generic, they are also regular and thus $\mathscr{M}_{nice}(J)$ is a smooth 1-manifold. The construction of u^∞ also establishes the fact that it is the only non-trivial curve (up to $\mathbb{R}$ translation) in $\mathbb{R} \times M_+$ with a positive puncture asymptotic to γ_∞. Thus for any sequence of curves in $\mathscr{M}(J, \gamma_\infty, \emptyset)$ which degenerates in the limit to a building with a non-trivial level in $[0, \infty) \times M_+$, this building must be of the form $(u^\infty | \emptyset)$. Proposition 2.1.6 now characterizes a neighbourhood of this building in $\mathscr{M}(J, \gamma_\infty, \emptyset)$ hence also in $\mathscr{M}_{nice}$ since being nicely embedded is an open condition.

Theorem 3.2.1 now applies and serves to characterise $\overline{\mathscr{M}}_{nice}(J)$ as a subset of $\overline{\mathscr{M}}(J, \gamma_\infty, \emptyset)$. In particular, any non-smooth curve with a non-trivial main level must be $(v_0 | v_1^-)$ of Type (I), breaking a long an even breaking orbit, γ which is either a doubly covered odd hyperbolic or a simple orbit. Our aim is to argue that at least such a building breaks along a simple orbit.

The boundary points of $\overline{\mathscr{M}}_{nice}(J)$ are isolated in the SFT topology and once can thus count them mod 2 (a more refined argument could take into account orientations and produce an algebraic count - in either case, the result should be the same). To justify being able to count them notice that since all curves in $\mathscr{M}_{nice}(J)$ are in the same relative homotopy class, they satisfy a uniform energy bound, and thus the SFT compactness theorem applies. $\overline{\mathscr{M}}_{nice}(J)$ is thus an actual compact 1-manifold with boundary. The count of its boundary points then has to be 0 mod 2.

Now, by the same reasoning as in the proof of Theorem 1.1.1, buildings of Type (I) which break along doubly covered odd hyperbolic orbits come in pairs, distinct in the SFT topology and thus

$$0 \equiv |\overline{\mathscr{M}}_{nice}(J) \setminus \mathscr{M}_{nice}(J)| \equiv 1 + K \pmod 2,$$

where K is the number of boundary points corresponding to buildings of Type (I) breaking along a simple orbit.

This completes the proof. □

Notice that in the above we saved some space by not repeating arguments that were similarly given in the proof of Theorem 1.1.1. Theorem 1.1.21, also has a proof along the same lines, which we do not include but point the reader to the extensive treatment in [CW].

Chapter 4

Further Results and Conclusions

Here we collate, a couple of results in partial form, which arose from the study of moduli spaces of nicely embedded curves in other settings. While the statements below should be treated as being subject to change, given that the full scope of their proof is not yet achieved, wherever possible we state their most accurate version.

The following problem concerns the study of exact symplectic fillings of the universally tight lens space $(L(p,q),\xi_{std})$. Several details in the proof of compactness or in the appropriate construction of the holomorphic open book on this space prevent a proof of the following result from being complete.

The crux of the proof of the main Theorem 4.0.4 resides in Theorem 4.0.3, a technical but elementary device which attempts to recover the required family of J-holomophic planes to produce a compactness argument.

The following definition is due to Etnyre et. al. in [BEV12]

Definition 4.0.1. Let M be an 3-manifold, $L \subset M$ an oriented link, and $\pi : (M \setminus L) \Rightarrow S^1$ a fibration. Let N be a tubular neighbourhood of L and ∂N be its boundary as a union of tori. We call (L,π) a *rational open book* on M if

$$\pi^{-1}(\theta) \cap \partial N$$

is not a meridian for any of its components for any $\theta \in S^1$. The surface $\pi^{-1}(\theta)$ is referred to as a *page* of the rational open book.

As explained in [BEV12] one easily obtains rational open books on from any fibered knot in S^3 by Dehn surgery. In particular, for the unknot in S^3 one obtains a rational open book (K_0, π_0) on $L(p,q)$ by Dehn surgery along a (p,q)-torus knot.

Our aim is to construct a contact form in the style of Thurston and Wilkenkemper, and a foliation by pseudoholomorphic disks on the symplectization $\mathbb{R} \times L(p,q)$ with certain properties.

Definition 4.0.2. By analogy with the classical case, given a rational open book as above (L, π), on a manifold equipped with a contact form (M, λ) we call λ a *Giroux form* for (L, π) if:

1. $\lambda(v) > 0$ for all vectors v positively tangent to L
2. $d\lambda$ is a volume form on each page.

We say that the contact structure defined by $\ker \lambda$ *is supported* by the rational open book (L, π).

This definition may be restated in terms of the Reeb vector field of λ by requiring that it is positively transverse to the pages and positively tangent to the binding.

As it turns out, rational open books are a typical setting for the emergence of nicely embedded curves. To even begin to understand such curves in the symplectization $\mathbb{R} \times L(p,q)$ we must first build up a special contact form, for which we can then extend the rational open book as a local holomorphic foliation. The following aims to mirror a construction due to Thurston and Winkelnkemper.

Theorem 4.0.3. *For any $N \in \mathbb{N}$ and $\tau > 0$, $L(p,q)$ admits a contact form λ and a compatible almost complex structure on $\mathbb{R} \times L(p,q)$ such that:*

1. *λ admits a closed elliptic Reeb orbit γ_0 whose image coincides with the binding circle K_0.*
2. $\ker \lambda$ *is isotopic to ξ_{std} and λ is a Giroux form for (K_0, π_0).*
3. *All other orbits have period greater than τ^p while λ has period less than τ.*
4. *γ_0 satisfies $\mu_{CZ}(\gamma_0^n) = 1$ for all $n \leq N$ with respect to the trivialization induced by the pages.*

5. *All pages lift to a J-holomorphic curve, which is nicely embedded, of Fredholm index 2, and self-intersection number 0.*

A relatively simple compactness argument shows that there exists a family of nicely embedded planes, which does not break and which projects onto the rational open book.

Theorem 4.0.4. *For any exact filling $(W, d\lambda)$ of the universally tight lens space $(L(p,q), \xi_{std})$, the moduli space $\mathscr{M}_{nice}(J)$ for generic $J \in \mathscr{J}(d\lambda)$ is non-empty, a component of which consists of planes which foliate the interior of W and project onto a rational open book on $L(p,q)$, which supports the contact structure.*

Appendix A

Liouville cobordisms from exact Lagrangian caps

In this appendix, we provide the details behind Example 1.1.6, using a general construction that was explained to us by Emmy Murphy. This construction is reproduced from [CW].

Proposition A.0.1. *Suppose (M,ξ) is a closed contact manifold of dimension $2n-1 \geq 3$, $\Lambda \subset M$ is a closed Legendrian submanifold and $L \subset [1,\infty)\times M$ is an exact Lagrangian cap for Λ. Then L has an open neighbourhood $\mathscr{U}_L \subset [1,\infty)\times M$ such that, after smoothing corners,*

$$W_- := ([0,1]\times M)\cup \overline{\mathscr{U}}_L$$

admits the structure of a Weinstein cobordism from (M,ξ) to some contact manifold (M',ξ'), and for suitably large constants $T>1$,

$$W_+ := ([1,T]\times M)\setminus \mathscr{U}_L$$

is a Liouville cobordism from (M',ξ') to (M,ξ).

Proof. Being an exact Lagrangian cap means that for some choice of contact form α on (M,ξ) and some constant $T>1$, the trivial Liouville cobordism

$$(Z,d\lambda) := ([1,T]\times M, d(e^r\alpha))$$

contains L as a compact and properly embedded Lagrangian submanifold with $\partial L = \{1\} \times \Lambda$, such that the Liouville vector field ∂_t is tangent to L near ∂L and

$$\lambda|_{TL} = dg$$

for some smooth function $g : L \to \mathbb{R}$. Note that since Λ is Legendrian and λ annihilates its dual Liouville vector field, g must be constant near ∂L; we shall assume without loss of generality that it vanishes there. By a combination of the Lagrangian and Legendrian neighbourhood theorems, L has a symplectic neighbourhood $(\mathscr{U}_L, d\lambda)$ whose closure $\overline{\mathscr{U}}_L$ is symplectomorphic to the unit disk bundle in $\mathbb{D}T^*L \subset T^*L$ for some choice of Riemannian metric on L. Note that this disk bundle has boundary and corners, its boundary consisting of two smooth faces,

$$\partial_- \overline{\mathscr{U}}_L := \mathbb{D}T^*L|_{\partial L} \quad \text{and} \quad \partial_+ \overline{\mathscr{U}}_L := ST^*L,$$

where ST^*L is the unit cotangent bundle. We shall write points in T^*L as (q,p) for $q \in L$ and $p \in T_q^*L$, and use the metric and its induced Levi-Civita connection to identify $T_{(q,p)}(T^*L)$ with $T_qL \oplus T_q^*L = T_qL \oplus T_qL$, where the first splitting comes from the horizontal-vertical decomposition given by the connection, and the second uses the isomorphism $T_qL = T_q^*L$ determined by the metric. The canonical Liouville form λ_0 on T^*L can then be written as

$$\lambda_0 = -dF_0 \circ J,$$

where $F_0(q,p) = \frac{1}{2}|p|^2$ and J is the compatible almost complex structure on T^*L that acts on $T_{(q,p)}(T^*L) = T_qL \oplus T_qL$ as $\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$. In particular, F_0 is a J-convex function, and therefore so is

$$F_\varepsilon(q,p) := \varepsilon f(q) + \frac{1}{2}|p|^2$$

for any smooth function $f : L \to \mathbb{R}$ if $\varepsilon > 0$ is sufficiently small. Setting $\lambda_\varepsilon :=$

$-dF_\varepsilon \circ J$, $d\lambda_\varepsilon$ is then a symplectic form isotopic to $d\lambda_0$ on a suitable neighbourhood of the zero-section L, and since the antipodal map $(q,p) \mapsto (q,-p)$ is J-antiholomorphic but preserves F_ε, it also preserves the Liouville vector field V_ε dual to λ_ε, proving that V_ε is tangent to L.

Now choose $f : L \to \mathbb{R}$ in this construction to be a Morse function that is constant with inward-pointing gradient along ∂L. After possibly shrinking the neighbourhood $\overline{\mathscr{U}}_L \cong \mathbb{D}T^*L$ of L, we can then assume that V_ε points transversely inward at $\partial_-\overline{\mathscr{U}}_L$ and transversely outward at $\partial_+\overline{\mathscr{U}}_L$. Since the Liouville field of $(Z, d\lambda)$ is also tangent to L near ∂L and points inward at $\{1\} \times M \subset \partial Z$ (see Figure 1.1), we can now assume after an isotopy of $\overline{\mathscr{U}}_L$ that the two Liouville fields match near $\partial_-\overline{\mathscr{U}}_L$, meaning $\lambda = \lambda_\varepsilon$ on that region. We can therefore use λ_ε to extend λ from $[0,1] \times M$ over W_- so that the dual Liouville vector field remains gradient like, making W_- a Weinstein cobordism from (M,ξ) to the new contact manifold (M',ξ'), obtained by removing a neighbourhood of Λ from $\{1\} \times M$ and replacing it with ST^*L.

It is also immediate from the above construction that W_+ is a strong symplectic cobordism from (M',ξ') to (M,ξ), and the exactness of the cobordism follows from the fact that L is an exact Lagrangian. Indeed, let $\mathring{\mathscr{U}}_L := \overline{\mathscr{U}}_L \setminus \partial_-\overline{\mathscr{U}}_L \cong \mathbb{D}T^*L|_{\mathring{L}}$. Since λ and λ_ε match near $\partial_-\overline{\mathscr{U}}_L$ and are both primitives of the same symplectic form, $\lambda - \lambda_\varepsilon$ represents an element of the compactly supported de Rham cohomology $H^1_c(\mathring{\mathscr{U}}_L)$, which is isomorphic to $H^1_c(\mathring{L})$. But under restriction to L, λ_ε vanishes and λ is exact, so this cohomology class is zero, implying $\lambda = \lambda_\varepsilon + dh$ on $\mathscr{U}_L$ for some smooth function $h : \mathscr{U}_L \to \mathbb{R}$ that vanishes near $\partial_-\overline{\mathscr{U}}_L$. By multiplying h with a suitable cutoff function, we can then find a Liouville form on W_+ that matches λ_ε near L and matches λ outside a neighbourhood of L. □

Remark A.0.2. If W is a subcritical Weinstein filling of (M,ξ), then the Weinstein filling of (M',ξ') obtained by stacking W_- on top of W is never subcritical. To see this, note that the Morse function $f : L \to \mathbb{R}$ in the above proof can always be chosen to have exactly one critical point of index n, in which case F_ε also has exactly one critical point of index n. If W is subcritical, this produces a handle decomposition of $W \cup_M W_-$ that includes exactly one critical handle, so $H_n(W \cup_M W_-) \neq 0$.

Bibliography

[AH09] P. Albers and H. Hofer, *On the Weinstein conjecture in higher dimensions*, Comment. Math. Helv. **84** (2009), no. 2, 429–436.

[BEV12] K. L. Baker, J. B. Etnyre, and J. Van Horn-Morris, *Cabling, contact structures and mapping class monoids*, J. Differential Geom. **90** (2012), no. 1, 1–80.

[Bou02] F. Bourgeois, *A Morse-Bott approach to contact homology*, Ph.D. Thesis, Stanford University, 2002.

[BEH$^+$03] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799–888.

[CE12] K. Cieliebak and Y. Eliashberg, *From Stein to Weinstein and back: symplectic geometry of affine complex manifolds*, American Mathematical Society Colloquium Publications, vol. 59, American Mathematical Society, Providence, RI, 2012.

[CO] K. Cieliebak and A. Oancea, *Symplectic homology and the Eilenberg-Steenrod axioms*. Preprint arXiv:1511.00485.

[CW] A. Cioba and C. Wendl, *Unknotted Reeb orbits and nicely embedded holomorphic curves*. Preprint arXiv:1609.01660.

[Col97] V. Colin, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 6, 659–663 (French, with English and French summaries).

[Dra04] D. L. Dragnev, *Fredholm theory and transversality for noncompact pseudoholomorphic maps in symplectizations*, Comm. Pure Appl. Math. **57** (2004), no. 6, 726–763.

[Eli90] Y. Eliashberg, *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds, 2 (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 151, Cambridge Univ. Press, Cambridge, 1990, pp. 45–67.

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal., Special Volume (2000), 560–673.

[EM13] Y. Eliashberg and E. Murphy, *Lagrangian caps*, Geom. Funct. Anal. **23** (2013), no. 5, 1483–1514.

[EH02] J. B. Etnyre and K. Honda, *On symplectic cobordisms*, Math. Ann. **323** (2002), no. 1, 31–39.

[Gay06] D. T. Gay, *Four-dimensional symplectic cobordisms containing three-handles*, Geom. Topol. **10** (2006), 1749–1759.

[Gei08] H. Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics, vol. 109, Cambridge University Press, Cambridge, 2008.

[GZ13a] H. Geiges and K. Zehmisch, *How to recognize a 4-ball when you see one*, Münster J. Math. **6** (2013), 525–554.

[GZ13b] ______, *Erratum to: How to recognize a 4-ball when you see one [MR 3148221]*, Münster J. Math. **6** (2013), 555–556.

[GZ16] ______, *The Weinstein conjecture for connected sums*, Int. Math. Res. Not. **2** (2016), 325–342.

[GNW16] P. Ghiggini, K. Niederkrüger, and C. Wendl, *Subcritical contact surgeries and the topology of symplectic fillings*, J. Éc. polytech. Math. **3** (2016), 163–208.

[Gro85] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.

[Hat] A. Hatcher, *Notes on Basic 3-Manifold Topology*. Unpublished manuscript, available at `https://www.math.cornell.edu/~hatcher/3M/3Mdownloads.html`.

[Hof93] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993), no. 3, 515–563.

[HWZ95] H. Hofer, K. Wysocki, and E. Zehnder, *Properties of pseudo-holomorphic curves in symplectisations. II. Embedding controls and algebraic invariants*, Geom. Funct. Anal. **5** (1995), no. 2, 270–328.

[HWZ96a] ______, *Properties of pseudoholomorphic curves in symplectisations. I. Asymptotics*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), no. 3, 337–379.

[HWZ96b] ______, *Properties of pseudoholomorphic curves in symplectisations. IV. Asymptotics with degeneracies*, Contact and symplectic geometry (Cambridge, 1994), 1996, pp. 78–117.

[HWZ96c] ______, *Unknotted periodic orbits for Reeb flows on the three-sphere*, Topol. Methods Nonlinear Anal. **7** (1996), no. 2, 219–244.

[HWZ99] ———, *Properties of pseudoholomorphic curves in symplectizations. III. Fredholm theory*, Topics in nonlinear analysis, 1999, pp. 381–475.

[HWZ03] ———, *Finite energy foliations of tight three-spheres and Hamiltonian dynamics*, Ann. of Math. (2) **157** (2003), no. 1, 125–255.

[HZ94] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Verlag, Basel, 1994.

[Hut02] M. Hutchings, *An index inequality for embedded pseudoholomorphic curves in symplectizations*, J. Eur. Math. Soc. (JEMS) **4** (2002), no. 4, 313–361.

[Lin] F. Lin, *Exact Lagrangian caps of Legendrian knots*. Preprint arXiv:1309.5101.

[MS04] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society, Providence, RI, 2004.

[MW95] M. J. Micallef and B. White, *The structure of branch points in minimal surfaces and in pseudoholomorphic curves*, Ann. of Math. (2) **141** (1995), no. 1, 35–85.

[Mor03] E. Mora, *Pseudoholomorphic cylinders in symplectisations*, Ph.D. Thesis, New York University, 2003.

[MS] E. Murphy and K. Siegel, *Subflexible symplectic manifolds*. Preprint arXiv:1510.01867.

[Nel13] J. Nelson, *Applications of automatic transversality in contact homology*, Ph.D. Thesis, University of Wisconsin at Madison, 2013.

[Nel15] ———, *Automatic transversality in contact homology I: regularity*, Abh. Math. Semin. Univ. Hambg. **85** (2015), no. 2, 125–179.

[Nie06] K. Niederkrüger, *The plastikstufe—a generalization of the overtwisted disk to higher dimensions*, Algebr. Geom. Topol. **6** (2006), 2473–2508.

[NR11] K. Niederkrüger and A. Rechtman, *The Weinstein conjecture in the presence of submanifolds having a Legendrian foliation*, J. Topol. Anal. **3** (2011), no. 4, 405–421.

[Pla12] O. Plamenevskaya, *On Legendrian surgeries between lens spaces*, J. Symplectic Geom. **10** (2012), no. 2, 165–181.

[Sie08] R. Siefring, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Comm. Pure Appl. Math. **61** (2008), no. 12, 1631–1684.

[Sie11] ———, *Intersection theory of punctured pseudoholomorphic curves*, Geom. Topol. **15** (2011), 2351–2457.

[Wan] A. Wand, *Tightness is preserved by Legendrian surgery*. Preprint arXiv:1404.1705, to appear in Ann. of Math.

[Wen05] C. Wendl, *Finite energy foliations and surgery on transverse links*, Ph.D. Thesis, New York University, 2005.

[Wen10a] ______, *Compactness for embedded pseudoholomorphic curves in 3-manifolds*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 2, 313–342.

[Wen10b] ______, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*, Comment. Math. Helv. **85** (2010), no. 2, 347–407.

[Wen10c] ______, *Open book decompositions and stable Hamiltonian structures*, Expos. Math. **28** (2010), no. 2, 187–199.

[Wen13a] ______, *A hierarchy of local symplectic filling obstructions for contact* 3*-manifolds*, Duke Math. J. **162** (2013), no. 12, 2197–2283.

[Wen13b] ______, *Non-exact symplectic cobordisms between contact* 3*-manifolds*, J. Differential Geom. **95** (2013), no. 1, 121–182.

[Wena] ______. `https://symplecticfieldtheorist.wordpress.com/2014/11/17/`.

[Wenb] ______, *Some good news about the forgetful map in SFT*. Blog post, available at `https://symplecticfieldtheorist.wordpress.com/2015/07/16/`.

[Wenc] ______, *Lectures on holomorphic curves in symplectic and contact geometry*. Preprint arXiv:1011.1690.

[Wend] ______, *Contact 3-manifolds, holomorphic curves and intersection theory*. Lecture notes from the LMS Short Course on Topology in Low Dimensions, Durham, 2013, available at `http://www.homepages.ucl.ac.uk/~ucahcwe/Durham/`.

[Yau06] M.-L. Yau, *Vanishing of the contact homology of overtwisted contact 3-manifolds*, Bull. Inst. Math. Acad. Sin. (N.S.) **1** (2006), no. 2, 211–229. With an appendix by Yakov Eliashberg.