Odd dimensional manifolds with highly-connected covers

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Abstract

We show that for each integer $n \geq 3$ any finite group G acts smoothly and freely on a connected sum $\underbrace{(S^n \times S^{n+1})\,\#\ldots\ldots\#\,(S^n \times S^{n+1})}_{r}$ for some r.

Moreover, as a module over $\mathbf{Z}[G]$, the middle dimensional homotopy group can be specified in advance to belong to the stable syzygy $\Omega^G_{n+1}(\mathbf{Z})$.

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Let G be a finite group. We consider smooth, closed manifolds M of odd dimension $2n+1 \geq 5$ having fundamental group G and whose universal coverings $\widetilde{M}$ are *highly connected* in the sense of Wall [32]; that is, where

$$\pi_r(\widetilde{M}) \;=\; 0 \quad \text{for} \quad 0 \leq r < n.$$

The first significant invariant of $\widetilde{M}$ is then the middle dimensional homotopy group $\pi_n(\widetilde{M}) = \pi_n(M)$. As $\pi_n(M)$ is a module over the group ring $\Lambda = \mathbf{Z}[G]$ we ask:

Question : Which Λ-modules J can be realised as $J \cong_\Lambda \pi_n(M)$ where M is a smooth closed $(2n+1)$-manifold with highly connected universal cover ?

The modules J we consider belong to the *stable syzygy* $\Omega^G_{n+1}(\mathbf{Z})$, the definition of which is given in §1. Observing that for integers $n \geq 2$ the connected sum

$$\Sigma(2n+1;r) \;=\; \underbrace{(S^n \times S^{n+1})\,\#\ldots\ldots\#\,(S^n \times S^{n+1})}_{r}$$

is a highly connected manifold of dimension $(2n+1)$ we prove:

Theorem I: Let G be a finite group and let $J \in \Omega^G_{n+1}(\mathbf{Z})$; if $n \geq 3$ then there exists a smooth closed $(2n+1)$-manifold M with $\pi_1(M) \cong G$ such that

i) $\pi_n(M) \cong_\Lambda J$ and

ii) $\widetilde{M} \cong_{\mathrm{diff}} \Sigma(2n+1; \mathrm{rk}_{\mathbf{Z}}(J))$.

The argument carries through in the case $n = 2$ subject to an additional hypothesis; say that $J \in \Omega^G_{n+1}(\mathbf{Z})$ is *geometrically realisable* when there exists a finite connected n-dimensional complex K for which $\pi_1(K) = G$, $\pi_r(K) = 0$ for $1 < r < n$ and $\pi_n(K) \cong_\Lambda J$.

Theorem II: Let G be a finite group and let $J \in \Omega_3^G(\mathbf{Z})$; if J is geometrically realisable there exists a smooth closed 5-manifold M with $\pi_1(M) \cong G$ such that

i) $\pi_2(M) \cong_\Lambda J$

ii) $\widetilde{M} \cong_{\mathrm{diff}} \Sigma(5; \mathrm{rk}_{\mathbf{Z}}(J))$.

When $n = 2$ the question of whether *every* $J \in \Omega_3^G(\mathbf{Z})$ is geometrically realisable forms part of the D(2) problem (cf [14] Chap 10) which, to date, has been solved only for comparatively few groups G. This condition does not intervene in Theorem I as, when $n \geq 3$, every $J \in \Omega_{n+1}^G(\mathbf{Z})$ is geometrically realisable ([15], p.163).

By Theorem I any finite group acts smoothly and freely on some $\Sigma(2n+1, r)$. These manifolds thus occupy something of a universal position in regard to the general study of free actions of finite groups on highly connected manifolds. By contrast, whilst this general problem has a longstanding history (cf [6]), for the most part the existing literature has dealt either with obstructions to free actions [4], [19] or with actions on quite specific examples [7]. For an significant class of finite groups we can improve upon Theorem I as follows:

Theorem III : Let G be a finite group with free cohomological period $d \geq 4$; then G acts freely and smoothly on $S^{d-1} \times S^d$.

Historically (cf. [20], [29]), considerable effort has been invested in studying the extent to which finite groups G of periodic cohomology can act freely on spheres of odd dimension, these being the simplest type of highly connected manifold. A classic result of Milnor [22] shows that this cannot happen when G has a noncentral element of order 2. Consequently, the dihedral groups of order $2p$

$$D_{2p} = \langle x, y \mid x^p = 1,\, y^2 = 1,\, yx = x^{p-1}y \rangle$$

cannot act freely on any sphere. However, when p is prime D_{2p} has free period 4; in fact, an explicit such free resolution may be found in [16]. Hence each integer $4m$ is a free period of D_{2p} so that, in contrast to Milnor's prohibition, we have:

Corollary IV : D_{2p} acts freely and smoothly on $S^{4m-1} \times S^{4m}$ for each $m \geq 1$.

Our construction makes essential use of the theory of canonical neighbourhoods in the sense of Mazur [21]. However, rather than following Mazur directly, we found it technically easier to adapt an old approach of the author [11] based on Siebenmann's technique of end completion [24]. Likewise, although no explicit use is made of it, we have inevitably been influenced by Wall's more detailed study [31], especially in regard to the identification of universal coverings in §4.

We give no consideration to the even dimensional analogue of the question considered here. We shall pursue this aspect in a subsequent paper [17].

We wish to thank the referee for suggesting a number of notational clarifications.

§1 : Syzygies and algebraic complexes:

Let Λ denote the integral group ring $\Lambda = \mathbf{Z}[G]$ where G is finite. By a Λ-*lattice* we mean a Λ-module whose underlying additive group is free abelian of finite rank. If J, J' are Λ-modules we say that J, J' are *stably equivalent*, written $J \sim J'$, when $J \oplus \Lambda^a \cong J' \oplus \Lambda^b$ for some integers $a,\ b \geq 0$. Let

$$(\mathcal{F}) \qquad \cdots \xrightarrow{\partial_{n+2}} F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} \ldots\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbf{Z} \to 0$$

be a resolution over Λ of the trivial module $\mathbf{Z}$ in which each F_r is a finitely generated free module. The *syzygy modules* $(J_r)_{1 \leq r}$ of $\mathcal{F}$ are the intermediate modules

$$J_r \quad = \quad \mathrm{Im}(\partial_r) \quad = \quad \mathrm{Ker}(\partial_{r-1}).$$

The *stable syzygy* $\Omega_r^G(\mathbf{Z})$ is then defined to be the stable class $[J_r]$ of any such J_r. It is a standard consequence of Schanuel's Lemma that $\Omega_r^G(\mathbf{Z})$ is independent of the particular choice of $\mathcal{F}$. We note for future reference that:

(1.1) If $J \in \Omega_{n+1}^G(\mathbf{Z})$ then J is a Λ-lattice.

By an *algebraic n-complex* over Λ we mean an exact sequence of Λ-modules

$$\mathbf{E}_* \;=\; (0 \to J \to E_n \xrightarrow{\partial_n} E_{n-1} \xrightarrow{\partial_{n-1}} \ldots\cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \to \mathbf{Z} \to 0)$$

in which each E_r is finitely generated projective and stably free. For such a complex $\mathbf{E}_*$ it is clear that $J \in \Omega_{n+1}^G(\mathbf{Z})$. We define[(†)] $H_n(\mathbf{E}) = \mathrm{Ker}(\partial_n : E_n \to E_{n-1})$. As G is finite then the Eckmann-Shapiro Lemma shows that $\mathrm{Ext}^{n+1}(\mathbf{Z}, \Lambda) = 0$. Consequently, it follows from (8.18) of [15] (p.159) that:

(1.2) For any $J \in \Omega_{n+1}^G(\mathbf{Z})$ there is an *algebraic* n-complex $\mathbf{E}_*$ for which $J \cong H_n(\mathbf{E})$.

We say that $J \in \Omega_{n+1}^G(\mathbf{Z})$ is *geometrically realizable* when there exists a finite, connected n-dimensional cell complex K such that $\pi_1(K) = G$, $\pi_r(K) = 0$ for $1 < r < n$ and $\pi_n(K) = J$; any such complex K is called a *realising complex* for J. In consequence, if K^n is a realising complex for $J \in \Omega_{n+1}^G(\mathbf{Z})$ then $\widetilde{K}^n$ is homotopy equivalent to a wedge of n-spheres; in fact:

$$\textbf{(1.3)} \qquad \widetilde{K}^n \quad \simeq \quad \underbrace{S^n \vee \ldots\cdots \vee S^n}_{\mathrm{rk}_{\mathbf{Z}}(J)} \;.$$

(1.4) If $n \geq 3$ then every $J \in \Omega_{n+1}^G(\mathbf{Z})$ is geometrically realizable.

The conclusion of (1.4) is a consequence of a general theorem of Wall [30] ; see also [15] ((8.27), p. 163) for a more direct proof. By contrast, for reasons associated with the $D(2)$-problem ([13], [14], [30]), the geometric realisability of elements of $\Omega_3^G(\mathbf{Z})$ is problematic. Note that a finite presentation $\mathcal{G} = \langle X_1, \ldots, X_g \,|\, W_1, \ldots, W_r \rangle$ of G

† At the referee's suggestion we have here written $H_n(\mathbf{E})$ rather than $\pi_n(\mathbf{E})$ which is the notation used in [15]. In the geometric context the two notations are equivalent via the Hurewicz Theorem.

gives rise to a geometrically realisable 2-complex, the Cayley complex, thus:

$$C_*(\mathcal{G}) \;=\; (0 \to \pi_2(\mathcal{G}) \to \Lambda^r \to \Lambda^g \to \Lambda \to \mathbf{Z} \to 0).$$

We then have:

Proposition 1.5: $J \in \Omega_3^G(\mathbf{Z})$ is geometrically realisable if and only if $J \cong \pi_2(\mathcal{G})$ for some finite presentation $\mathcal{G}$.

Only in a small number of cases ([14], Chap. 10) is it known that *every* $J \in \Omega_3^G(\mathbf{Z})$ is realisable. By contrast, there are cases where geometric realisability seems highly problematic [13]. The difficulty is concentrated at the lower levels of $\Omega_3^G(\mathbf{Z})$. A theorem of Browning [1] shows:

(1.6) For all $J \in \Omega_3^G(\mathbf{Z})$ there exists $n \geq 1$ such that $J \oplus \Lambda^N$ is geometrically realisable whenever $N \geq n$.

§2 : Canonical neighbourhoods :

We review the notion of canonical neighbourhoods of tamely imbedded polyhedra. Thus suppose that K^k is a compact connected polyhedron, X^{k+c} is a topological manifold of dimension $k+c$, and that $i : K \hookrightarrow X$ is a continuous imbedding such that $i(K) \cap \partial X = \emptyset$. When no confusion is caused we shall write $K = i(K)$. We say such an imbedding is *tame* when $X - K$ is locally 1- connected at infinity (cf [5], [11]). By a *canonical neighbourhood* of K we mean a compact submanifold $\mathcal{N}$ of codimension zero in X having properties (2.1) - (2.3) below:

(2.1) $K \subset \mathrm{Int}\,\mathcal{N}$ and the boundary $\partial\mathcal{N}$ of $\mathcal{N}$ is locally bicollared in X;

(2.2) the inclusion $i : K \hookrightarrow \mathcal{N}$ is a simple homotopy equivalence;

(2.3) there is a homeomorphism $\eta : \partial\mathcal{N} \times [0,1) \xrightarrow{\simeq} \mathcal{N} - K$ such that $\eta(x,0) = x$.

We note that if $j : \partial N \to \mathcal{N}$ is the inclusion it is a consequence of the definition that

(2.4) K is a strong deformation retract of $\mathcal{N}$; that is, there exists a continuous mapping $r : \mathcal{N} \to K$ such that $r \circ i = \mathrm{Id}_K$ and $i \circ r \simeq_{i(K)} \mathrm{Id}_M$.

General dimensional arguments imply relations between the homotopy groups $\pi_r(\partial\mathcal{N})$ and $\pi_r(\mathcal{N})$ provided $r \leq c-1$. Although these are well known, at the referee's suggestion we recall them briefly for the sake of completeness. By general position, a mapping $f : S^r \to \mathcal{N}$ can be deformed by homotopy to a mapping $f' : S^r \to \mathcal{N} - K$ provided that $k + r < k + c$. It follows that the inclusion $\iota : \mathcal{N} - K \hookrightarrow \mathcal{N}$ induces a surjection $\iota_* : \pi_r(\mathcal{N} - k) \twoheadrightarrow \pi_r(\mathcal{N})$ provided $r \leq c-1$. Taken in conjunction with the homeomorphism $\partial\mathcal{N} \times [0,1) \cong \mathcal{N} - K$ of (2.3) we see, in particular, that

(2.5) $j_* : \pi_{c-1}(\partial\mathcal{N}) \xrightarrow{\simeq} \pi_{c-1}(\mathcal{N})$ is surjective.

When $r < c-1$ this can be improved upon. Given a homotopy $F : S^r \times [0,1] \to \mathcal{N}$ in which $F(S^r \times \{t\}) \subset \mathcal{N} - K$ for $t = 0, 1$, then, provided $r < c-1$, F can be a deformed, leaving the ends fixed, to a mapping $F' : S^r \times [0,1] \to \mathcal{N} - K$ so that $\iota_* : \pi_r(\mathcal{N} - k) \xrightarrow{\simeq} \pi_r(\mathcal{N})$ is an isomorphism $r < c - 1$. Hence it follows that:

(2.6) $j_* : \pi_r(\partial\mathcal{N}) \xrightarrow{\simeq} \pi_r(\mathcal{N})$ is an isomorphism for $r < c - 1$.

Moreover, denoting by ρ the restriction of the retraction r to $\partial\mathcal{N}$ we have :

(2.7) $\rho_* : \pi_r(\partial\mathcal{N}) \xrightarrow{\simeq} \pi_r(K)$ is an isomorphism for $r < c - 1$ and

(2.8) $\rho_* : \pi_{c-1}(\partial\mathcal{N}) \xrightarrow{\simeq} \pi_{c-1}(K)$ is surjective.

When X is a combinatorial manifold, a canonical neighbourhood $\mathcal{N}$ is called a PL-*canonical neighbourhood* when $\mathcal{N}$ is a PL-submanifold of X and $\partial\mathcal{N}$ imbeds as a piecewise linear submanifold in X. Likewise if X is a smooth manifold then a canonical neighbourhood $\mathcal{N}$ is called a DIFF-*canonical neighbourhood* when $\mathcal{N}$ is a smooth submanifold of X and $\partial\mathcal{N}$ imbeds in X as a smooth submanifold. Now suppose that $\mathcal{C}$ is one of the categories TOP, PL, DIFF. We have the following existence and uniqueness theorem.

Theorem 2.9 : Let K^k is a compact connected polyhedron, let X^{k+c} be a $\mathcal{C}$-manifold of dimension $k+c$ and suppose that $i : K \hookrightarrow X$ is a tame imbedding such that $i(K) \cap \partial X = \emptyset$; if $k + c \geq 6$ and $c \geq 3$ then

i) any open neighbourhood U of K in X contains a $\mathcal{C}$-canonical neighbourhood $\mathcal{N}$;

ii) if $\mathcal{N}_1$, $\mathcal{N}_2$ are $\mathcal{C}$-canonical neighbourhoods of K in X there exists a $\mathcal{C}$-canonical neighbourhood $\mathcal{N}_0$ of K such that $\mathcal{N}_0 \subset \mathrm{Int}(\mathcal{N}_1 \cap \mathcal{N}_2)$ and a $\mathcal{C}$-isomorphism $h : \mathcal{N}_1 \xrightarrow{\simeq} \mathcal{N}_2$ such that $h_{|\mathcal{N}_0} = \mathrm{Id}$.

Theorem 2.9 is well known. In its purely topological form it is a special case of the main result (Theorem 3.5) of [11]. As the proof in [11] requires only handle theory for topological manifolds of dimension ≥ 6, the translation to the categories PL and DIFF is straightforward. From (2.9) it is straightforward to derive

Proposition 2.10 : Let $\mathcal{N}, \mathcal{N}'$ be closed $\mathcal{C}$-neighbourhoods of K in X and suppose that there exists a $\mathcal{C}$-isomorphism equivalence $h : \mathcal{N} \to \mathcal{N}'$ such that $h_{|K} = \mathrm{Id}_K$. If $\mathcal{N}$ is a $\mathcal{C}$-canonical neighbourhood of K then so also is $\mathcal{N}'$.

Using the S-cobordism theorem we obtain the following recognition criterion for canonical neighbourhoods.

Proposition 2.11 : Let $i : K^k \hookrightarrow \mathcal{N}^{k+c}$ be a tame imbedding into a the interior of a connected $\mathcal{C}$-manifold with connected boundary for which the induced map $\pi_1(\partial\mathcal{N}) \xrightarrow{\simeq} \pi_1(\mathcal{N})$ is an isomorphism. Suppose that $k + c \geq 6$ and $c \geq 3$; if i is a simple homotopy equivalence then $\mathcal{N}$ is a $\mathcal{C}$-canonical neighbourhood of K.

Proof : Choose $\mathcal{C}$-canonical neighbourhoods $\mathcal{N}_0$ and $\mathcal{N}_1$ of K such that

$$\mathcal{N}_0 \subset \operatorname{Int}\mathcal{N}_1 \subset \operatorname{Int}\mathcal{N}.$$

By the S-cobordism theorem there is $\mathcal{C}$-isomorphism

$$\eta : \mathcal{N} - \operatorname{Int}\mathcal{N}_0 \xrightarrow{\simeq} \partial\mathcal{N}_0 \times [0,1]$$

which maps $\partial\mathcal{M}_0$ to isomorphically to $\partial\mathcal{M}_0 \times \{0\}$ via the mapping $\mathbf{z} \mapsto (\mathbf{z}, 0)$ and $\partial\mathcal{M}$ isomorphically to $\partial\mathcal{M}_0 \times \{1\}$. Consequently we may represent $\mathcal{N}$ in the form

$$\mathcal{N} = \mathcal{N}_0 \bigcup\nolimits_\eta \partial\mathcal{N}_0 \times [0,1]$$

Likewise we may represent $\mathcal{N}_1$ in the form $\mathcal{N}_1 = \mathcal{N}_0 \bigcup_{\eta'} \partial\mathcal{N}_0 \times [0,1]$. Thus there is a $\mathcal{C}$-isomorphism $\mathcal{N} \to \mathcal{N}_1$ which extends the identity on $\mathcal{N}_0$. Hence $\mathcal{N}$ is $\mathcal{C}$-canonical neighbourhood of K by (2.10) □

Corollary 2.12 : $\mathcal{N}$ be a compact connected $\mathcal{C}$-manifold with connected boundary $\partial\mathcal{N}$ in which the inclusion $\partial\mathcal{N} \hookrightarrow \partial\mathcal{N}$ induces an isomorphism $\pi_1(\partial\mathcal{N}) \xrightarrow{\simeq} \pi_1(\mathcal{N})$. Let K_1, K_2 be connected polyhedra and let $K_1^k \overset{i_1}{\hookrightarrow} \mathcal{N}$ and $K_2^k \overset{i_2}{\hookrightarrow} \mathcal{N}$ be tame imbeddings with codimension $c \geq 3$. Suppose that $h : K_2 \to K_1$ is a simple homotopy equivalence such that $i_2 \simeq i_1 \circ h$; then

$\mathcal{N}$ is a canonical neighbourhood of $K_1 \iff \mathcal{N}$ is a canonical neighbourhood of K_2

§3 : Neighbourhood covering theorem:

If we work purely within the PL-category then canonical neighbourhoods exist without the dimensional restrictions of (2.9) and can be constructed directly.

Proposition 3.1: Let $i : K^k \hookrightarrow X$ be a piecewise linear imbedding of a compact, connected polyhedron in a combinatorial manifold X satisfying the condition that $i(K) \cap \partial X = \emptyset$. Then $i(K)$ has a canonical neighbourhood in X.

Proof : Take K^k to be a compact polyhedron, X^{k+c} to be a combinatorial manifold and suppose that the imbedding $i : K^k \hookrightarrow X^{k+c}$ is piecewise linear; then we may assume (cf [10] p.84) that X is triangulated by a simplicial complex in such a way that $i(K)$ is a finite subcomplex.Taking $\mathcal{N}$ to be the star neighbourhood of $i(K)$ in the second derived subdivision of X, we claim that $\mathcal{N}$ is a canonical neighbourhood of $i(K)$. A theorem of Whitehead [33] then shows that there is a retraction $r : \mathcal{N} \to K$ which is a composition $r = c_m \circ c_{m-1} \circ \ldots c_2 \circ c_1$ where each c_i is an elementary simplicial collapse. Furthermore (cf [2]) there is a piecewise linear equivalence of triples $(\mathcal{N}; \partial\mathcal{N}, i(K)) \cong_{PL} (C_\rho; \partial\mathcal{N}\times\{0\}, i(K))$ where $\rho : \partial\mathcal{N} \to K$ is the restriction of the retraction r to the boundary ∂N and where $C_\rho = \partial\mathcal{N} \times [0,1] \cup_\rho K$ is the mapping cylinder of ρ. The properties (2.1) - (2.3) of canonical neighbourhoods now follow from this description. □

In regard of (3.1) we note that piecewise linear imbeddings are necessarily tame. Canonical neighbourhoods constructed in the manner of (3.1) via simplicial collapsing are usually referred to *regular neighbourhoods*. Beyond the existence of regular neighbourhoods, in [33] Whitehead also considered their uniqueness based only on the technique of simplicial collapsing. Accounts of this purely combinatorial theory can be found in the texts both of Hudson [10] and Stallings [27]. The most general account is that of M.M. Cohen [3]. The following is straightforward:

Proposition 3.2 : Let K be a subcomplex of a finite simplicial complex L and suppose that $L \searrow K$. Suppose also that $\widehat{L}$ is a finite simplicial complex admitting a surjective simplicial mapping $\pi : \widehat{L} \to L$ for which the induced map on geometrical realisations $|\pi| : |\widehat{L}| \to |L|$ is a fibre bundle with finite fibres; then $\widehat{L} \searrow \pi^{-1}(K)$.

Now suppose that K^k is a compact, connected k-dimensional polyhedron imbedded piecewise linearly in $\mathbf{R}^{c+k}$ and that N is a closed regular neighbourhood of K.

Theorem 3.3 : Let $p : \widehat{N} \to N$ be a connected regular covering of N having finite degree $d \geq 2$ and let $\widehat{K} = p^{-1}(K)$; if $k < c$ then $\widehat{N}$ is combinatorially equivalent to a PL-canonical neighbourhood of $\widehat{K}$ with respect to some piecewise linear imbedding $j : \widehat{K} \hookrightarrow \mathbf{R}^{c+k}$.

Proof : Let T be a finite simplicial complex which triangulates N in such a way that K is triangulated by a subcomplex S of T and such that the inclusion $i : N \hookrightarrow \mathbf{R}^{c+k}$ is affine on each simplex of T. For each $r \geq 1$, let $T(r)$ (resp. $S(r)$) be the r^{th} barycentric subdivision of T (resp. S) and let $N(r)$ be the star neighbourhood of $i(S(r))$ in $i(T(r))$. In particular, $\{N(r)\}_{1\leq r}$ is a fundamental system of closed neighbourhoods of K in $\mathbf{R}^{c+k}$.

Clearly $\widehat{N}$ is a polyhedron. As the degree d is finite then $\widehat{N}$ is compact. For $r \geq 1$, let $\widetilde{T}(r)$ be the simplicial covering of $T(r)$ induced from $p : \widehat{N} \to N$ and let $\widehat{N}(r)$ be the geometric realisation of the star neighbourhood of $\widehat{S}(r)$ in $\widehat{T}(r)$; then:

i) each $\widehat{N}(r)$ is a compact subpolyhedron and

ii) $\{\widehat{N}(r)\}_{1\leq r}$ is a fundamental system of closed neighbourhoods of $p^{-1}(K)$ in $\widehat{N}$.

Let $\{u_r\}_{1\leq r\leq m}$ denote the vertices of T. For any choice of points $\{f(u_r)\}_{1\leq r\leq m}$ in $\mathbf{R}^{c+k}$ let $f : N \to \mathbf{R}^{c+k}$ be the map defined on each simplex by affine extension of the assignment $u_i \mapsto f(u_i)$. It follows by Mazur's Stability Theorem (cf [27], p.53) that

iii) there exists $\epsilon > 0$ such that if $f(u_1), \ldots, f(u_m) \in \mathbf{R}^{c+k}$ are chosen so that $|f(u_r) - i(u_r)| < \epsilon$ for all r then f is also an imbedding.

Let $\{w_1, \ldots, w_{dm}\}$ be a labelling of the vertices of $\widehat{T}$ such that $\{w_1, \ldots, w_\mu\}$ is a labelling of the vertices of $\widehat{S}$. For each r, $1 \leq r \leq dm$ choose $v_r \in \mathbf{R}^{c+k}$ such that

iv) $|v_r - ip(u_r)| < \epsilon$ for all r and $\{v_1, \ldots, v_{dm}\}$ are in general position.

Thus we have a mapping $j : \widehat{N} \to \mathbf{R}^{c+k}$ defined on each simplex by affine extension of the assignment $j(w_i) = v_i$. It follows easily from iii) that

v) $j : \widehat{N} \to \mathbf{R}^{c+k}$ is locally injective.

Moreover, as $k < c$ and $\{v_1, \ldots, v_\mu\}$ are in general position then

vi) $j : \widehat{K} \to \mathbf{R}^{c+k}$ is injective.

As $\{\widehat{N}(r)\}_{1 \leq r}$ is a fundamental system of closed neighbourhoods of $\widehat{K}$ in $\widehat{N}$ it follows from v) and vi) that for some $s \quad j : \widehat{N}(s) \to \mathbf{R}^{c+k}$ is injective. The interior of $N(s)$ is an open subset of $\mathbf{R}^{c+k}$. As $\widehat{N}(s)$ is locally homeomorphic to $N(s)$ it follows from Brouwer's Open Mapping Theorem and the injectivity of j on $\widehat{N}(s)$ that:

vii) $\quad j(\mathrm{Int}(\widehat{N}(s))$ is an open subset of $\mathbf{R}^{c+k}$.

As $\widehat{N}(s)$ is compact then $j(\widehat{N}(s))$ is closed in $\mathbf{R}^{c+k}$. Thus $j(\widehat{N}(s))$ is a closed polyhedral neighborhood of $j(\widehat{K})$ in $\mathbf{R}^{c+k}$. By (3.2) it follows that $\widehat{N}(s)$ collapses onto $\widehat{K}$ so that $\widehat{N}(s)$ is a canonical neighbourhood $\widehat{K}$ under the imbedding $j : \widehat{K} \hookrightarrow \mathbf{R}^{c+k}$. The stated conclusion follows as $\widehat{N}$ is combinatorially equivalent to $\widehat{N}(s)$. $\square$

The conclusion of (3.3) is well known as a 'folk theorem' but difficult to locate in the literature in this precise form. The above proof is a simplification of an argument of Spivak (cf Proposition 4.6 of [26]). Spivak's argument is complicated by allowing the degree of the covering to be infinite. In such cases the space $\widehat{K}$ is non-compact and the inclusion $\widehat{K} \hookrightarrow \mathbf{R}^{k+c}$ requires a further stabilisation, by increasing the codimension, in order to imbed $\widehat{K}$ as proper polyhedral subset and so construct a genuine infinite regular neighbourhood in the sense, for example, of [25]. This elaboration is unnecessary when the degree of the covering map is finite. Consequently we require only that the codimension be at least $k + 1$.

We next consider the direct construction of canonical neighbourhoods of imbeddings $i : K^k \hookrightarrow X^{k+c}$ where X is a smooth manifold. To do this we note that any smooth manifold X has a well defined class of C^∞ triangulations $\tau : |L| \to X$ (cf [23] , [34]) where L is a simplicial complex whose geometric realisation $|L|$ is a combinatorial manifold. If K is a compact polyhedron we say that an imbedding $i : K \to X$ is *piecewise smooth* when X admits a C^∞ triangulation $\tau : |L| \to X$ such that $\tau^{-1} \circ i$ is piecewise linear. Hirsch ([9]) has shown that for such piecewise smooth imbeddings $i(K)$ admits a neighbourhood $\mathcal{N}$ in which $\mathcal{N}$ is a compact smooth submanifold of codimension zero in X and $\partial\mathcal{N}$ is a smooth submanifold of X and where X admits a C^∞ triangulation $\tau : |L| \to X$ for which $\tau^{-1}(\mathcal{N})$ is a regular neighbourhood of $\tau^{-1} \circ i(K)$ in the combinatorial manifold $|L|$. Such a neighbourhood $\mathcal{N}$ is called a *smooth regular neighbourhood.*

Corollary 3.4 : Let K^k be a compact, connected k-dimensional polyhedron imbedded piecewise linearly in $\mathbf{R}^{c+k}$ and let $\mathcal{N}$ be a smooth regular neighbourhood of K. Let $p : \widehat{\mathcal{N}} \to \mathcal{N}$ be a connected regular covering of $\mathcal{N}$ having finite degree $d \geq 2$ and let $\widehat{K} = p^{-1}(K)$; if $k < c$ then $\widehat{N}$ diffeomorphic to a DIFF-canonical neighbourhood of $\widehat{K}$ with respect to some tame imbedding $j : \widehat{K} \hookrightarrow \mathbf{R}^{c+k}$.

§4 : Decomposing canonical neighbourhoods as connected sums:

Let $N_1,\ N_2$ be topological $(n+1)$-manifolds each with nonempty boundary and let $D_i \subset \partial N_i$ be a closed n-disc possessing an open neighbourhood $U_i \subset \partial N_i$ such that there is a homeomorphism of pairs $h_i : (U_i, D_i) \overset{\cong}{\to} (B^n, D^n)$ where

$$B^n \;=\; \{\mathbf{x} \in \mathbf{R}^n \mid \ ||\mathbf{x}|| \;<\; 2\} \quad ; \quad D^n \;=\; \{\mathbf{x} \in \mathbf{R}^n \mid \ ||\mathbf{x}|| \;\leq\; 1\}.$$

We define $$N_1 \diamond N_2 \;=\; (N_1 \coprod N_2) \,/ \sim$$

where $$\mathbf{x} \;\sim\; h_2^{-1} \circ h_1(\mathbf{x}) \;\text{ for }\; \mathbf{x} \in D_1.$$

$N_1 \diamond N_2$ is called the *connected sum in the boundary* of $N_1,\ N_2$. Up to homeomorphism the description is independent of the pairs (U_i, D_i) and the homeomorphisms h_i. Moreover, it is straightforward to see that:

(4.1) $\partial(N_1 \diamond N_2) \;\cong\; \partial N_1 \,\#\, \partial N_2.$

Here '#' denotes 'connected sum' in the usual sense. The above construction is defined for topological manifolds. The analogous construction for differential manifolds however requires a slight elaboration. Thus suppose that $N_1,\ N_2$ are smooth $(n+1)$-manifolds and that $h_1,\ h_2$ are diffeomorphisms. As it stands the construction $N_1 \diamond N_2$ is a 'smooth $(n+1)$-manifold with singularity in the boundary', the singularity occurring along the set

$$\Sigma \;=\; h_1^{-1}(S^{n-1}) \;=\; h_2^{-1}(S^{n-1}).$$

In fact putting $V_- \;=\; U_1 - \mathrm{Int}(D_1)$ and $V_- \;=\; U_2 - \mathrm{Int}(D_2)$ then although here are diffeomorphisms

$$\Sigma \times (-1, 0] \;\cong\; V_- \quad ; \quad \Sigma \times [0, 1) \;\cong\; V_+$$

the homeomorphism $\Sigma \times (-1, 1) \;\cong\; V_- \cup V_+$ obtained by glueing fails to be differentiable at $\Sigma \times (-1, 1)$. Using the standard technique of 'corner smoothing' one may make a variation in the homeomorphisms h_i so as to give a diffeomorphism

$$V_- \cup V_+ \;\cong\; \Sigma \times (-1, 1) \;\cong\; S^{n-1} \times (-1, 1).$$

With this qualification, $N_1 \diamond N_2$ is then a manifold with smooth boundary. Up to diffeomorphism the result is again independent of the above choices.

If $\Sigma_1,\ \ldots,\ \Sigma_{d-1},\ \Sigma_d$ are topological spaces with respective base points $*_r \in \Sigma_r$ we define $\Sigma_1 \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow \Sigma_d$ to be the identification space

$$\Sigma_1 \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow \Sigma_d \quad = \quad (\Sigma_1 \coprod \ldots\ldots \coprod \Sigma_{d-1} \coprod \Sigma_d \coprod \ [1, d]) / \sim$$

where $*_r \sim r$ for $1 \leq r \leq d$. We concentrate on the case when $\Sigma_1, \ldots, \Sigma_{d-1}, \Sigma_d$ are smooth, closed connected manifolds each of dimension $k \geq 2$; then we can regard $\Sigma_1 \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow \Sigma_d$ as a 'smooth, compact, stratified set' (cf [12], [28]) whose singularities are of such an elementary nature that it should no confusion if we adopt an extension of standard terminology as follows. Given a continuous imbedding

$$f : \Sigma_1 \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow \Sigma_d \hookrightarrow M$$

to a smooth manifold M; we describe f as a *smooth imbedding* when

i) f is injective :

ii) each restriction $f : \Sigma_r \to M$ is a smooth imbedding ; and

iii) the restriction $f : [1, d] \to M$ is also a smooth imbedding.

It is an elementary exercise in approximation to show that:

(4.2) If $k < c$ then any continuous mapping $f : \Sigma_0 \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow \Sigma_d \longrightarrow \mathbf{R}^{k+c}$ can be approximated arbitrarily closely by a smooth imbedding.

Now suppose that for $1 \leq r \leq d$ we are given a smooth imbedding $f_r : \Sigma_r \hookrightarrow \mathbf{R}^{k+c}$ where $3 \leq c$. By translating images we may suppose, without loss of generality, that

$$\mathrm{Im}(f_r) \cap \mathrm{Im}(f_s) = \emptyset \qquad \text{for } r \neq s.$$

That is, we may construct a smooth imbedding $f_0 : [1, d] \to R^{k+c}$ with the property that $\mathrm{Im}(f_0) \cap \mathrm{Im}(f_r) = *_r$. Then $f_0, f_1, \ldots, f_d$ together define a smooth imbedding

$$f : \Sigma_1 \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow \Sigma_d \longrightarrow M$$

We claim that :

Proposition 4.3 : For any smooth canonical neighbourhood $\mathcal{N}$ of $\mathrm{Im}(f)$ in $\mathbf{R}^{k+c}$ there is a diffeomorphism $\mathcal{N} \cong_{\text{diff}} \mathcal{N}_1 \diamond \cdots \diamond \mathcal{N}_{d-1} \diamond \mathcal{N}_d$ where $\mathcal{N}_r$ is smooth canonical neighbourhood of $\mathrm{Im}(f_r)$ in $\mathbf{R}^{k+c}$.

We now focus attention on the special case where each $\Sigma_r = S^k$ where $k \geq 2$. Recall ([8], [18]) that Kervaire showed that the normal bundle to a smooth imbedding $i : S^k \hookrightarrow \mathbf{R}^\mu$ is trivial provided $3k + 1 < 2\mu$. On writing $\mu = k + c$, Kervaire's condition becomes $k < 2c - 1$. This is clearly satisfied when $c = k + 2$.

Proposition 4.4: Let $\mathcal{N}$ be a smooth canonical neighbourhood of $\mathrm{Im}(f)$ where $f : \underbrace{S^k \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow S^k}_{d} \hookrightarrow \mathbf{R}^{2k+2}$ is a smooth imbedding; if $2 \leq k$ then

$$\mathcal{N} \cong_{\text{diff}} \underbrace{(S^k \times D^{k+2}) \diamond \ldots \cdots \diamond (S^k \times D^{k+2})}_{d}$$

Proof : f is a smooth imbedding of $\Sigma_1 \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow \Sigma_d$ where each $\Sigma_r \cong_{\text{diff}} S^k$. Taking $\mathcal{N}_r$ to be a closed canonical neighbourhood of Σ_r, it follows from (4.3) that $\mathcal{N} \cong_{\text{diff}} \mathcal{N}_1 \diamond \cdots \diamond \mathcal{N}_{d-1} \diamond \mathcal{N}_d$. However, by Kervaire's Theorem it follows that $\mathcal{N}_r \cong_{\text{diff}} S^k \times D^{k+2}$, whence the conclusion. □

§5: Proof of Theorems I and II:

We now specialise the above discussion by fixing the following notation:

K^k	:	a realising complex for some $J \in \Omega^G_{k+1}(\mathbf{Z})$ where $2 \leq k$;
$i : K^k \hookrightarrow \mathbf{R}^{2k+2}$	:	a piecewise linear imbedding
$\mathcal{N}$	:	a smooth canonical neighbourhood of $i(K^k)$. .

Put $\mathcal{M} = \partial\mathcal{N}$. The codimension c of the imbedding is then $c = k+2$ and evidently satisfies $3 \leq c$; hence:

(5.1) $$\pi_1(\mathcal{M}) \cong \pi_1(K) = G.$$

The middle dimension of $\widetilde{\mathcal{M}}$ is k as $\dim(\mathcal{M}) = 2k+1$. Writing $c-1 = k+1$, we see from (2.7) that $\pi_r(\widetilde{\mathcal{M}}) \cong \pi_r(\widetilde{K})$ for $r \leq k$. Consequently, with these conditions we have:

(5.2) $$\widetilde{\mathcal{M}} \text{ is highly connected and } \mu(M) \cong \pi_k(\widetilde{K}) = J \quad .$$

We again note that the condition $c = k+2$ implies the condition $k < 2c-1$ required by Kervaire's Theorem.

Theorem 5.3: With the above restrictions we have

$$\widetilde{\mathcal{M}} \cong_{\mathrm{PL}} \underbrace{(S^k \times S^{k+1}) \# \ldots\ldots \# (S^k \times S^{k+1})}_{d}.$$

Proof : By (3.3), $\widetilde{N}$ imbeds as a PL submanifold of $\mathbf{R}^{2k+2}$ and thereby induces an imbedding $j : \widetilde{K} \hookrightarrow \mathbf{R}^{2k+2}$ relative to which $\widetilde{\mathcal{N}}$ is a smooth canonical neighbourhood of $\widetilde{K}$. Let $\widetilde{i} : \widetilde{K} \hookrightarrow \widetilde{N}$ be the inclusion. As K is a realising complex then $\widetilde{K}$ is $(k-1)$-connected so that we have a homotopy equivalence $\widetilde{K} \simeq \underbrace{S^k \vee \ldots \cdots \vee S^k}_{d}$ where $d = \mathrm{rk}_{\mathbf{Z}}(J)$. Consequently there are homotopy equivalences $\underbrace{S^k \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow S^k}_{d} \simeq \underbrace{S^k \vee \ldots \cdots \vee S^k}_{d} \simeq \widetilde{K}$ Choosing a homotopy equivalence $h : \underbrace{S^k \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow S^k}_{d} \xrightarrow{\simeq} \widetilde{K}$ then $\widetilde{i} \circ h : \underbrace{S^k \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow S^k}_{d} \xrightarrow{\simeq} \widetilde{N}$ is also a homotopy equivalence which is necessarily simple as the spaces involved are simply connected. Under the hypothesis $k < c$, one may approximate $\widetilde{i} \circ h$ arbitrarily closely by a smooth imbedding

$$j : \underbrace{S^k \leftrightsquigarrow \cdots\cdots \leftrightsquigarrow S^k}_{d} \hookrightarrow \widetilde{N}$$

which is homotopic to $\widetilde{i} \circ h$ and hence also a simple homotopy equivalence. It follows that $\widetilde{N}$ is also a canonical neighbourhood of $\mathrm{Im}(j)$ so that, by (4.4), there is a diffeomorphism

$$\mathcal{N} \;\cong\; \underbrace{(S^k \times D^{k+2}) \diamond \ldots \cdots \diamond (S^k \times D^{k+2})}_{d}.$$

Hence, as claimed, we have a diffeomorphism

$$\widetilde{M} \;=\; \partial \widetilde{N} \;\cong \underbrace{(S^k \times S^{k+1}) \,\#\ldots\ldots\#\, (S^k \times S^{k+1})}_{d}. \qquad \square$$

In the above, the existence of a k-dimensional finite, connected complex K^k with the properties that $\pi_1(K) \cong G$ and that $\pi_r(\widetilde{K}) = 0$ for $r < k$ is assumed as given. The module $J = \pi_k(\widetilde{K})$ is thereby dependent on the choice of K. In the lowest case $k = 2$, the conclusions of (5.1) - (5.4) prove Theorem II of the Introduction.

When $k \geq 3$, however, we may specify $J \in \Omega^G_{k+1}(\mathbf{Z})$ in advance and, as noted in (1.4), then construct a realising complex K for J. The conclusions of (5.1)- (5.4) thereby also prove Theorem I.

§6: Proof of Theorem III:

If G is a finite group with free period d then there exists an exact sequence

$$0 \to \mathbf{Z} \to \mathbf{E}_d \to \mathbf{E}_{d-1} \to \cdots \to \mathbf{E}_1 \to \mathbf{E}_0 \to \mathbf{Z} \to 0$$

of modules over $\Lambda = \mathbf{Z}[G]$ in which each $\mathbf{E}_r$ is finitely generated and free. Consequently the trivial module $\mathbf{Z}$ belongs to $\Omega_d(\mathbf{Z})$. Provided $d \geq 4$ (which is necessarily the case when G is non-abelian) then, observing that $\mathrm{rk}(\mathbf{Z}) = 1$, Theorem I guarantees that G acts freely on $S^{d-1} \times S^d$. This proves Theorem III.

Finally suppose that p is an odd prime; as is well known, the dihedral group D_{2p} of order $2p$ has free period 4. (An explicit free resolution of period 4 may be found, for example, in [16]). Consequently D_{2p} acts freely and smoothly on $S^3 \times S^4$. More generally, each integer $4m \geq 4$ is a free period for D_{2p} so that D_{2p} acts freely and smoothly on $S^{4m-1} \times S^{4m}$. This proves Corollary IV.

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