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ABSTRACT

We find conditions for the Friedman rule to be optimal in three standard models of money. These conditions are homotheticity and separability assumptions on preferences similar to those in the public finance literature on optimal uniform commodity taxation. We show that there is no connection between our results and the result in the standard public finance literature that intermediate goods should not be taxed.

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In a classic contribution Friedman (1969) argued that optimal monetary policy requires setting nominal interest rates to zero. Phelps (1973) argued that in economies that must raise revenues with distorting taxes it is optimal to tax all goods, including the liquidity services derived from holding money. Hence, Phelps argued that there is no theoretical presumption that the Friedman rule is optimal when there are distorting taxes. Indeed he goes on to argue that "if, as is often maintained, the demand for money is highly interest-inelastic, then liquidity is an attractive candidate for heavy taxation—at least from the standpoint of monetary and fiscal efficiency" (Phelps 1973, p. 82). In this paper we analyze three standard monetary models with distorting taxes: a cash-credit model, a money in the utility function model, and a shopping time model. We show that in these three models the Friedman rule can still be optimal even though the government raises revenues only with distorting taxes. The common features of the requirements for optimality are simple homotheticity and separability conditions.

In the cash-credit model, the Friedman rule turns out to be optimal if consumers' preferences are homothetic in the cash and credit goods and weakly separable in leisure. These conditions are consistent with a wide range of elasticities of the demand for money function. Thus, there is no obvious connection between the elasticity of the demand for liquidity services and the desirability of taxing these services. It turns out that there is a connection between our results for this model and the optimality of uniform commodity taxation in the public finance literature (see Atkinson and Stiglitz 1972). As Lucas and Stokey (1983) pointed out, the cash credit goods economy is equivalent to a particular barter economy. The Friedman rule is optimal in the monetary economy if it is optimal to levy uniform commodity taxes in the barter economy. Our homotheticity and separability conditions are those required for uniform taxation in the barter economy and thus imply the Friedman rule is optimal in the monetary economy.

In the money in the utility function model, similar homotheticity and separability conditions on the utility functions imply that the Friedman rule is optimal. Again these conditions are consistent with a wide range of elasticities of money demand. The intuition underlying the result is somewhat different than in the cash-credit goods model. In the money in the utility function model, Friedman's original intuition that the economy should be satiated with money balances because they are free plays a key role. In related work Woodford (1990) analyzes the optimality of the Friedman rule for a deterministic version of the money in the utility function model and obtains conditions quite different from ours. His optimality criterion is the steady-state utility level of the representative consumer while ours is the welfare of the representative consumer, namely the discounted sum of single-period utilities. Not surprisingly, his conditions are different from ours.

In the shopping time model, we show that the Friedman rule is optimal if the shopping time technology is homogenous of degree at least one. Kimbrough (1986) and Faig (1988) proved the optimality of the Friedman rule for economies in which technology is homogenous of degree one. One can construct examples where the technology is homogenous of degree less than one in which the Friedman rule is not optimal. In fact, Guidotti and Vegh (1993) show that the Friedman rule is typically not optimal if the shopping time technology is homogeneous of degree less than one.

There has been extensive discussion in the literature of the connection between the optimality of the Friedman rule in economies with distorting taxes and the standard public finance result (see Diamond and Mirrlees 1971) that intermediate goods should not be taxed (see Kimbrough 1986, Faig 1988, and Guidotti and Vegh 1993). We show that our monetary economies can be reinterpreted as intermediate goods economies. We also construct versions of our monetary economies in which it is costly to maintain the money stock. In these versions, the Friedman rule is no longer optimal but the reinterpretation as intermediate goods economies continues to go through unchanged. In the intermediate goods economies, it is optimal not to tax intermediate goods but the Friedman rule is not optimal in the monetary economies. This result leads to the conclusion that there is no connection between the optimality of the Friedman rule and the taxation of intermediate goods.

1. A Monetary Economy

Consider a simple production economy populated by a large number of identical infinitely lived consumers. In each period $t = 0, 1, \ldots$, the economy experiences one of finitely many events s_t. We denote by $s^t = (s_0, \ldots, s_t)$ the history of events up through and including period t. The probability, as of period zero, of any particular history s^t is $\mu(s^t)$. The initial realization s_0 is given. This suggests a natural commodity space in which goods are differentiated by histories.

In each period t there are three goods, labor and two consumption goods: a cash good and a credit good. A constant returns-to-scale technology is available to transform labor $\ell(s^t)$ into output. The output can be used for private consumption of either cash good $c_1(s^t)$ or the credit good $c_2(s^t)$ or government consumption $g(s^t)$. Throughout, we will take government consumption to be exogenously specified.

The resource constraint is

$$(1.1) \quad c_1(s^t) + c_2(s^t) + g(s^t) = \ell(s^t)$$

where $g(s^t)$ is government consumption. The preferences of each consumer are given by

$$(1.2) \quad \sum_t \sum_{s^t} \beta^t \mu(s^t) U(c_1(s^t), c_2(s^t), \ell(s^t))$$

where U is strictly concave and satisfies the Inada conditions.

In period t, consumers trade money, assets, and goods in particular ways. At the start of period t, after observing the current state s_t, consumers trade money and assets in a centralized securities market. The assets are one-period state-noncontingent nominal claims. Let $M(s^t)$ and $B(s^t)$ denote the money and nominal bonds held at the end of the securities market trading. Let $R(s^t)$

denote the gross nominal return on these bonds payable in period $t + 1$ in all states $s^{t+1} = (s^t, s_{t+1})$. After this trading, each consumer splits into a worker and a shopper. The shopper must use the money to purchase cash goods. To purchase credit goods, the shopper issues nominal claims which are settled in the securities market in the next period. The worker is paid in cash at the end of each period.

This environment leads to this constraint for the securities market:

$$(1.3)\quad M(s^t) + B(s^t) = R(s^{t-1})B(s^{t-1}) + M(s^{t-1}) - p(s^{t-1})c_1(s^{t-1}) - p(s^{t-1})c_2(s^{t-1}) + p(s^{t-1})(1 - \tau(s^{t-1}))\ell(s^{t-1}).$$

The left side of (1.3) is the nominal value of assets held at the end of securities market trading. The first term on the right side is the value of nominal debt bought in the preceding period. The next two terms are the shopper's unspent cash. The next is the payments for credit goods, and the last is the after-tax receipts from labor services. We will assume that the holdings of real debt $B(s^t)/p(s^t)$ are bounded above and below by some arbitrarily large constants. Purchases of cash goods must satisfy a cash-in-advance constraint:

$$(1.4)\quad p(s^t)c_1(s^t) \leq M(s^t).$$

We let $x(s^t) = (c_1(s^t), c_2(s^t), \ell(s^t), M(s^t), B(s^t))$ denote an allocation for consumers at s^t and let $x = (x(s^t))$ denote an allocation for all s^t. The initial stock of money M_{-1} and the initial stock of nominal debt B_{-1} are given.

Money is introduced into and withdrawn from the economy through open market operations in the securities market. The constraint facing the government in this market is

$$(1.5)\quad M(s^t) - M(s^{t-1}) + B(s^t) = R(s^{t-1})B(s^{t-1}) + p(s^{t-1})g(s^{t-1}) - p(s^{t-1})\tau(s^{t-1})\ell(s^{t-1}).$$

The terms on the left side of this equation are the assets sold by the government. The first term on the right is the payments on debt incurred in the preceding period, the second is the payment for government consumption, and the third is tax receipts. Notice that government consumption is bought on credit. We let $\pi(s^t) = (\tau(s^t), p(s^t), R(s^t))$ denote a policy for the government at s^t and let $\pi = (\pi(s^t))$ denote an allocation for all s^t.

Consider now the policy problem faced by the government. Suppose an institution or a commitment technology exists through which the government can bind itself to a particular sequence of policies once and for all at period zero. We model this by having the government choose a policy $\pi = (\pi(s^t))$ at the beginning of time and then having consumers choose their allocations. Since the government needs to predict how consumer allocations and prices will respond to its policies, consumer allocations and prices are described by rules that associate allocations with government policies. Formally, allocation rules are sequences of functions $x(\pi) = (x(s^t|\pi))$ that map policies π into allocations x.

A *Ramsey equilibrium* is a policy π and an allocation rule $x(\cdot)$ with the following conditions: (i) the policy π maximizes

$$\sum_{t,s^t} \beta^t \mu(s^t) U(c_1(s^t|\pi), c_2(s^t|\pi), \ell(s^t|\pi))$$

subject to (1.5) with allocations given by $x(\pi)$, and (ii) for every π', the allocation $x(\pi')$ maximizes (1.2) subject to the bounds on debt purchases and to (1.3) and (1.4) evaluated at the policy π'.

In this equilibrium the consumer maximizes (1.2) subject to (1.3) and (1.4) and the bounds on debt. Money earns a gross nominal return of one. If bonds earn a gross nominal return of less than one, then the consumer can make infinite profits by buying money and selling bonds. Thus, in any equilibrium, $R(s^t) \geq 1$. The consumer's first order conditions imply that $U_1(s^t)/U_2(s^t) = R(s^t)$; thus, in any equilibrium, this constraint must hold:

(1.6) $U_1(s^t) \geq U_2(s^t)$.

This feature of the competitive equilibrium constrains the set of Ramsey allocations.

The allocations in the Ramsey equilibrium solve a simple programming problem called the Ramsey allocation problem. As is well known, if the initial stock of nominal assets held by consumers is positive, then welfare is maximized by increasing the initial price level to infinity. If the initial stock is negative, then welfare is maximized by setting the initial price level so low that the government raises all the revenue it needs without levying any distorting taxes. To make the problem interesting, we set the initial sum of nominal assets of consumers, $M_{-1} + B_{-1}$, to zero. In terms of notation it will be convenient here and throughout the paper to let $U_i(s^t)$, $i = 1, 2, 3$, denote the marginal utilities at state s^t. Using standard techniques (see Lucas and Stokey 1983, and Chari, Christiano, and Kehoe 1991) we can establish

PROPOSITION 1 (The Ramsey Allocations). The consumption and labor allocations in the Ramsey equilibrium solve the Ramsey allocation problem

$$\max \sum_t \sum_{s^t} \beta^t \mu(s^t) U(c_1(s^t), c_2(s^t), \ell(s^t))$$

subject to (1.1), (1.6), and

(1.7) $$\sum_t \sum_{s^t} \beta^t \mu(s^t) [U_1(s^t) c_1(s^t) + U_2(s^t) c_2(s^t) + U_3(s^t) \ell(s^t)] = 0.$$

The proof of the proposition has two parts: Any competitive equilibrium allocation must satisfy (1.1), (1.6), and (1.7); conversely, any allocation satisfying (1.1), (1.6), and (1.7) can be decentralized as a competitive equilibrium. Thus the resource constraint (1.1), no return dominance constraint (1.6) and the implementability constraint (1.7) completely characterize the competitive equilibrium allocations. We use this characterization in Section 4.

Consider utility functions of the form

(1.8) $U(c_1,c_2,\ell) = V(h(c_1,c_2),\ell)$

where h is homothetic. We then have

PROPOSITION 2 (The Optimality of the Friedman Rule). For utility functions of the form (1.8), the Ramsey equilibrium has $R(s^t) = 1$ for all s^t.

Proof. Consider for a moment the Ramsey problem with constraint (1.6) dropped. Let λ denote the Lagrange multiplier on (1.7) and $\beta^t\mu(s^t)\gamma(s^t)$ the Lagrange multiplier on (1.1). The first order conditions for $c_i(s^t)$, i = 1, 2 in this problem are

$$(1.9)\quad (1+\lambda)U_i(s^t) + \lambda\left[\sum_{j=1}^{2} U_{ji}(s^t)c_j(s^t) + U_{3i}(s^t)\ell(s^t)\right] = \gamma(s^t).$$

Note that a utility function which satisfies (1.8) also satisfies

$$(1.10)\quad \sum_{j=1}^{2} U_{j1}(s^t)c_j(s^t)/U_1(s^t) = \sum_{j=1}^{2} U_{j2}(s^t)c_j(s^t)/U_2(s^t).$$

To see this recall that homotheticity implies for any $\alpha > 0$,

$$(1.11)\quad \frac{U_1(\alpha c_1(s^t),\alpha c_2(s^t),\ell)}{U_2(\alpha c_1(s^t),\alpha c_2(s^t),\ell)} = \frac{U_1(c_1(s^t),c_2(s^t),\ell(s^t))}{U_2(c_1(s^t),c_2(s^t),\ell(s^t))}.$$

Differentiating (1.11) with respect to α and evaluating it at $\alpha = 1$ gives (1.10). Next, dividing (1.9) by U_i and noting that $U_{3i}/U_i = V_{12}/V_1$ for i = 1, 2 we have that

$$(1.12)\quad (1+\lambda) + \lambda\left[\sum_{j=1}^{2}\frac{U_{ji}(s^t)c_j(s^t)}{U_i(s^t)} + \frac{V_{12}(s^t)}{V_1(s^t)}\ell(s^t)\right] = \frac{\gamma(s^t)}{U_i(s^t)}.$$

Using (1.10) we have that the left side of (1.12) has the same value for i = 1 and for i = 2. It follows that $U_1(s^t)/U_2(s^t) = 1$. Since the solution to the less constrained problem satisfies (1.6) it is also a solution to the Ramsey problem. From the consumer's first order condition, $U_1(s^t)/U_2(s^t) = R(s^t)$ and thus $R(s^t) = 1$. □

To make this intuition precise consider a real barter economy with the same preferences (1.2) and resource constraint (1.1) as the monetary economy and with commodity taxes on the two consumption goods. Consider a date 0 representation of the budget constraints. The consumer budget constraint is

$$(1.20)\quad \sum_t \sum_{s^t} q(s^t)[(1 + \tau_1(s^t))c_1(s^t) + (1 + \tau_2(s^t))c_2(s^t)] = \sum q(s^t)\ell(s^t)$$

and the government budget constraint is

$$(1.21)\quad \sum_t \sum_{s^t} q(s^t)g(s^t) = \sum_t \sum_{s^t} q(s^t)[\tau_1(s^t)c_1(s^t) + \tau_2(s^t)c_2(s^t)]$$

where $q(s^t)$ is the price of goods at date t and state s^t. A Ramsey equilibrium for this economy is defined in the obvious fashion. The Ramsey allocation problem for this barter economy is similar to that in the monetary economy except that there is no constraint (1.6).

The consumer's first order conditions imply

$$(1.22)\quad \frac{U_1(s^t)}{U_2(s^t)} = \frac{1 + \tau_1(s^t)}{1 + \tau_2(s^t)}.$$

Thus Ramsey taxes satisfy $\tau_1(s^t) = \tau_2(s^t)$ if and only if in the Ramsey allocation problem of maximizing (1.2) subject to (1.1) and (1.7) the solution has $U_1(s^t)/U_2(s^t) = 1$. It is then immediate to use the argument in Proposition 2 to show:

PROPOSITION 3. (Optimality of Uniform Commodity Taxation). For utility functions of the form (1.8) the Ramsey equilibrium has $\tau_1(s^t) = \tau_2(s^t)$ for all s^t.

Thus with homotheticity and separability in the period utility function the optimal taxes on the two consumption goods are equal at each state. Notice that this proposition does not imply that commodity taxes are equal across states (so $\tau_i(s^t)$ may not equal $\tau_j(s^r)$ for $t \neq r$, i, j = 1, 2).

We have shown that if the conditions for uniform commodity taxation are satisfied in the barter economy then in the associated monetary economy the Friedman rule is optimal. Of course, since the allocations in the monetary economy must satisfy (1.6) while those in the barter economy need not there are situations in which uniform commodity taxes are not optimal in the barter economy but the Friedman rule is optimal in the monetary economy. To see this consider preferences of the form

$$(1.23)\quad U(c_1,c_2,\ell) = \frac{c_1^{1-\sigma_1}}{1-\sigma_1} + \frac{c_2^{1-\sigma_2}}{1-\sigma_2} + V(\ell).$$

Consider the first order conditions to the Ramsey problem in the barter economy

$$(1.24)\quad (1+\lambda)c_i(s^t)^{-\sigma_i} + \lambda[-\sigma_i c_i(s^t)^{-\sigma_i}] = \gamma(s^t)$$

which imply

$$(1.25)\quad \frac{U_1(s^t)}{U_2(s^t)} = \frac{c_1(s^t)^{-\sigma_1}}{c_2(s^t)^{-\sigma_2}} = \frac{1 + \lambda(1-\sigma_2)}{1 + \lambda(1-\sigma_1)}.$$

Clearly $U_1(s^t) \geq U_2(s^t)$ if and only if $\sigma_1 \geq \sigma_2$. For $\sigma_1 = \sigma_2$ these preferences satisfy condition (1.6) and both uniform commodity taxation and the Friedman rule are optimal. For $\sigma_1 > \sigma_2$ neither uniform commodity taxation nor the Friedman rule is optimal. It is optimal to tax good 1 at a higher rate than good 2. In the barter economy this is accomplished by setting $\tau_1(s^t) > \tau_2(s^t)$ while in the monetary economy it is accomplished by setting $R(s^t) > 1$. More interesting is that when $\sigma_1 < \sigma_2$ uniform commodity taxation is not optimal but the Friedman rule is. To see this note that with $\sigma_1 < \sigma_2$, the solution in the monetary economy ignoring the constraint $U_1(s^t) \geq U_2(s^t)$ violates this constraint. Thus this constraint binds and in the monetary economy $U_1(s^t) = U_2(s^t)$. Thus in the barter economy it is optimal to tax good 1 at a lower rate than good 2, and this is accomplished by setting $\tau_1(s^t) < \tau_2(s^t)$. In the monetary economy it is not feasible to tax good 1 at a lower rate then good 2, since $R(s^t) \geq 1$, and the best feasible solution is to set $R(s^t) = 1$.

In this section we have focused on the Lucas-Stokey (1983) cash-credit version of the cash-in-advance model. It turns out that in the simpler cash-in-advance model without credit goods the inflation rate and the labor tax rate are indeterminate. The first order conditions for a deterministic version of that model are the cash-in-advance constraint, the budget constraint, and

(1.26) $$-\frac{U_{1t}}{U_{2t}} = R_{t+1}(1-\tau_t)$$

(1.27) $$\frac{1}{\beta}\frac{U_{1t}}{U_{1t+1}} = \frac{R_{t+1}P_t}{P_{t+1}}$$

where the period utility function is $U(c_t,\ell_t)$ and R_{t+1} is the nominal interest rate from period t to period t + 1. Here, only the products $R_{t+1}(1-\tau_t)$ and $R_{t+1}P_t/P_{t+1}$ are pinned down by the allocations. Thus the nominal interest rate, the tax rate, and the inflation rate are not individually determined. There are a whole variety of ways of decentralizing the Ramsey allocation. In particular, trivially, the Friedman rule is optimal as well as arbitrarily high rates of inflation.

2. Money in the Utility Function Models

Consider the following monetary economy. Labor is transformed into consumption goods according to

(2.1) $$c(s^t) + g(s^t) = \ell(s^t).$$

The preferences of the representative consumer are given by

(2.2) $$\sum_t \sum_{s^t} \beta^t \mu(s^t) U(M(s^t)/p(s^t), c(s^t), \ell(s^t)).$$

In period t the consumer budget constraint is

(2.3) $$p(s^t)c(s^t) + M(s^t) + B(s^t) = M(s^{t-1}) + R(s^{t-1})B(s^{t-1}) + p(s^t)(1 - \tau(s^t))\ell(s^t).$$

The holdings of real debt $B(s^t)/p(s^t)$ are bounded below by some arbitrarily large constant and the holdings of money are bounded below by zero. Let M_{-1} and $R_{-1}B_{-1}$ denote the initial asset holdings of the consumer. The budget constraint of the government is given by

(2.4) $B(s^t) = R(s^{t-1})B(s^{t-1}) + p(s^t)g(s^t) - [M(s^t) - M(s^{t-1})] - p(s^t)(1 - \tau(s^t))g(s^t).$

A Ramsey equilibrium for this economy is defined in the obvious fashion. We set the initial stock of assets to zero for reasons similar to those given in the preceding section. Let $m(s^t) = M(s^t)/p(s^t)$ denote the real balances in the Ramsey equilibrium. Using logic similar to that in Proposition 1, we can show that the consumption and labor allocations and real money balances in the Ramsey equilibrium solve the Ramsey allocation problem:

(2.5) $\max \sum_t \sum_{s^t} \beta^t \mu(s^t) U(m(s^t), c(s^t), \ell(s^t))$

subject to (2.1) and

(2.6) $\sum \beta^t [m(s^t)U_1(s^t) + c(s^t)U_2(s^t) + \ell(s^t)U_3(s^t)] = 0.$

The resource constraint (2.1) and the implementability constraint (2.6) completely characterize the set of competitive equilibrium allocations.

We are interested in finding conditions under which the Friedman rule is optimal. Now the consumers first order conditions imply

(2.7) $\dfrac{U_1(s^t)}{U_2(s^t)} = 1 - \dfrac{1}{R(s^t)}.$

Thus for the Friedman rule to hold, namely $R(s^t) = 1$, it must be that

(2.8) $U_1(s^t)/U_2(s^t) = 0.$

Since the marginal utility of consumption goods will be finite (2.8) will hold only if $U_1(s^t) = 0$, namely the marginal utility of real money balances is zero. Intuitively, under the Friedman rule it is optimal to satiate the economy with real money balances.

We are interested in economies for which preferences are not satiated of any finite level of money balances and for which the marginal utility of real money balances converges to zero as the level of real money balances converges to infinity. That is, for each c and ℓ, $U_1(m,c,\ell) \to 0$ as $m \to \infty$. Intuitively, in such economies the Friedman rule holds exactly only if the value of real money balances is infinite and for such economies there would be no solution to the Ramsey allocation problem. To get around this technicality we consider an economy in which the level of real money balances are exogenously bounded by a constant. We will say the Friedman rule is optimal if as this bound on real money balances increases the associated nominal interest rates in the Ramsey equilibrium converge to one. With this in mind consider modifying the Ramsey allocation problem to include the constraint

(2.9) $m(s^t) \leq \bar{m}$.

Consider preferences of the form

(2.10) $U(m,c,\ell) = V(h(m,c),\ell)$

where h is homothetic. We then have

PROPOSITION 4 (Optimality of the Friedman Rule). If the utility function is of the form (2.10) then the Friedman rule is optimal.

Proof. The Ramsey allocation problem is to maximize (2.2) subject to (2.1), (2.6), and (2.9). Consider a less constrained version of this problem in which constraint (2.9) is dropped. Let

$\beta^t\mu(s^t)\gamma(s^t)$ and λ denote the Lagrange multipliers on constraints (2.1) and (2.6). The first order condition for real money balances and consumption are

$$(2.11)\quad (1+\lambda)U_1(s^t) + \lambda[m(s^t)U_{11}(s^t) + c(s^t)U_{21}(s^t) + \ell(s^t)U_{31}(s^t)] = 0$$

and

$$(2.12)\quad (1+\lambda)U_2(s^t) + \lambda[m(s^t)U_{12}(s^t) + c(s^t)U_{22}(s^t) + \ell(s^t)U_{32}(s^t)] = \gamma(s^t).$$

Since the utility function satisfies (2.10) it follows that

$$(2.13)\quad \frac{m(s^t)U_{11}(s^t) + c(s^t)U_{21}(s^t)}{U_1(s^t)} = \frac{m(s^t)U_{12} + c(s^t)U_{22}(s^t)}{U_2(s^t)}.$$

Using the form of (2.10) we can rewrite (2.11) and (2.12) as

$$(2.14)\quad (1+\lambda) + \lambda\left[\frac{m(s^t)U_{11}(s^t) + c(s^t)U_{21}(s^t)}{U_1(s^t)} + \ell(s^t)\,\frac{V_{21}(s^t)}{V_1(s^t)}\right] = 0$$

and

$$(2.15)\quad (1+\lambda) + \lambda\left[\frac{m(s^t)U_{12}(s^t) + c(s^t)U_{22}(s^t)}{U_2(s^t)} + \ell(s^t)\,\frac{V_{21}(s^t)}{V_1(s^t)}\right] = \frac{\gamma(s^t)}{U_1(s^t)}.$$

Using (2.13) it follows that

$$(2.16)\quad \frac{U_1(s^t)}{U_2(s^t)} = 0$$

in the less constrained problem. Hence the associated $m(s^t)$ is arbitrarily large and thus for any finite bound $\bar{m}$ the constraint (2.9) binds in the original problem. The result then follows from (2.7). □

Again restricting h to be homogenous does not reduce the generality of the result.

Clearly there are preferences which do not satisfy (2.10) for which the Friedman rule is optimal. Consider,

$$(2.17)\quad U(m,c,\ell) = \frac{m^{1-\sigma_1}}{1-\sigma_1} + \frac{c^{1-\sigma_1}}{1-\sigma_2} + V(\ell).$$

The first order condition in the Ramsey problem for money balances $m(s^t)$ is

$$(2.18)\quad [1 + \lambda(1-\sigma_1)]m(s^t)^{-\sigma_1} = 0.$$

Unless the endogenous Lagrange multiplier λ just happens to equal $(\sigma_1 - 1)^{-1}$, (2.17) implies that the Friedman rule is optimal. Note that for $\sigma_1 \neq \sigma_2$ (2.16) does not satisfy (2.10).

In related work Woodford (1990) considers the optimality of the Friedman rule in this model. He characterizes the policy that maximizes steady state utility rather than the one that maximizes the discounted value of utility. His Ramsey problem is

$$(2.19)\quad \max\ U(m,c,\ell)$$

subject to

$$(2.20)\quad c + g \leq \ell$$

$$(2.21)\quad U_1 m + U_2 c + U_3 \ell = (1-\beta)U_1 m.$$

Woodford shows that if consumption and real balances are gross substitutes the Friedman rule is not optimal. Of course, there are functions which satisfy our homotheticity and separability assumptions which are gross substitutes, for example,

$$(2.22)\quad U(m,c,\ell) = \frac{m^{1-\sigma}}{1-\sigma} + \frac{c^{1-\sigma}}{1-\sigma} + V(\ell).$$

The reason for the difference in the results arises from the difference in the implementability constraints. The first order conditions to our problem are similar to those of Woodford's problem except that his include derivatives of the right side of (2.21). Notice that in Woodford's problem if $\beta = 1$ and preferences satisfy our homotheticity and separability conditions then the Friedman rule is optimal.

3. A Shopping Time Model of Money

Consider a monetary economy along the lines of Kimbrough (1986). Labor is transformed into consumption goods according to

$$(3.1)\quad c(s^t) + g(s^t) \leq \ell(s^t).$$

The preferences of the representative consumer are given by

$$(3.2)\quad \sum_t \sum_{s^t} \beta^t \mu(s^t) U(c(s^t), \ell(s^t) + \phi(c(s^t), M(s^t)/p(s^t)))$$

where U is concave, $U_1 > 0$ and $U_2 < 0$ and $\phi_1 > 0$, $\phi_2 < 0$. The function $\phi(c_1, M/p)$ describes the amount of time it takes to obtain c units of the consumption good when the consumer has M/p units of real money balances. We assume $\phi_1 > 0$ so that with the same amount of money it takes more time to obtain more consumption goods and $\phi_2 < 0$ so that with more money it takes less time to obtain the same amount of consumption goods. The budget constraints of the consumer and the government are the same as (2.3) and (2.4).

The Ramsey equilibrium is defined in the obvious fashion. Letting $m(s^t) = M(s^t)/p(s^t)$ and setting the initial nominal assets to zero we can show the consumption, labor allocations and real money balances in the Ramsey equilibrium solve the problem

$$\max \sum_t \sum_{s^t} \beta^t \mu(s^t) U(c(s^t), \ell(s^t) + \phi(c(s^t), m(s^t)))$$

subject to (3.1) and

$$(3.3)\quad \sum_t \sum_{s^t} \beta^t \mu(s^t)[c(s^t)(U_1(s^t) + \phi_1(s^t)U_2(s^t)) + \ell(s^t)U_2(s^t) + m(s^t)\phi_2(s^t)U_2(s^t)] = 0.$$

The resource constraint (3.1) and the implementability constraint (3.3) completely characterize the set of competitive allocations.

From the consumer's first order conditions it follows that $R(s^t) = 1$ if $\phi_2 = 0$. We then have

PROPOSITION 5 (Optimality of the Friedman Rule). If ϕ is homogenous of degree k, $k \geq 1$ then the Friedman rule is optimal.

Proof. The first order conditions to the Ramsey problem with respect to $m(s^t)$ and $\ell(s^t)$ are given by

(3.4) $$U_2\phi_2 + \lambda[cU_{12}\phi_2 + U_{22}\phi_2(\phi_1 c+\phi_2 m+\ell) + U_2\phi_2 + U_2(\phi_{12}c+\phi_{22}m)] = 0$$

and

(3.5) $$U_2 + \lambda[cU_{12} + U_{22}(\phi_1 c+\phi_2 m+\ell) + U_2] + \mu = 0$$

where μ is the multiplier on the resource constraint.

Suppose first that $\phi_2 \neq 0$ so that the optimal policy does not follow the Friedman rule. Then, from (3.4) and (3.5) we have that

(3.6) $$-\frac{\lambda U_2(\phi_{12}c+\phi_{22}m)}{\phi_2} + \mu = 0.$$

Now, under the condition that $\phi(c,m)$ is homogenous of degree k and $k \geq 1$, we have that $\phi_2(\gamma c,\gamma m) = \gamma^{k-1}\phi_2(c,m)$. Differentiating with respect to γ and evaluating at $\gamma = 1$ we have $c\phi_{12} + m\phi_{22} = (k-1)\phi_2$ and thus,

(3.7) $$\frac{c\phi_{12} + m\phi_{22}}{\phi_2} \geq 0.$$

Since $\lambda \geq 0$, $U_2 < 0$, and $\mu \geq 0$, (3.6) and (3.7) contradict each other. □

Note that this proof does not go through if $\phi(c,m)$ is homogenous of degree less than unity. It is easy to construct counterexamples when $\phi(c,m) = c/m$. In this case, the shopping time technology is homogenous of degree zero and the Friedman rule is nonoptimal for a large class of utility functions.

Guidotti and Vegh (1993) independently proved that the Friedman rule is typically not optimal if shopping time technology is homogenous of degree less than one. They use the so-called dual approach to solving the Ramsey problem rather than our primal approach. It turns out that their method of proof also implies that the Friedman rule is optimal if the technology is homogenous of degree greater than one. They also argue that zero degree homogeneity is a more reasonable requirement of the shopping time technology than the constant returns to scale requirement of Kimbrough.

4. Interpreting Monetary Models as Models With Intermediate Goods

There has been some discussion in the literature suggesting that, for the shopping time model, the optimality of the Friedman rule follows the public finance result that it is not optimal to tax intermediate goods (see Kimbrough 1986, Faig 1988, Woodford 1990, and Guidotti and Vegh 1993). In this section we elaborate on the connections between the optimality of the Friedman rule and the intermediate goods result for all three models. For simplicity we consider deterministic versions of our models.

Consider first the shopping time model. We will construct a simple real economy with intermediate goods whose competitive equilibrium allocations coincide with those of the monetary economy. Consider a real economy in which firms produce private consumption c and government consumption g using labor n and a (real) intermediate good m according to

$$(4.1) \quad c_t + g_t + \phi(c_t, m_t) = n_t$$

where ϕ satisfies CRS. Consumer preferences are given by

$$(4.2) \quad \sum_{t=0}^{\infty} \beta^t U(c_t, n_t).$$

Consumers maximize (4.2) subject to

(4.3) $\sum q_t c_t = \sum q_t(1-\tau_t)w_t n_t$

where q_t is the date 0 price of consumption, w_t is the wage rate and τ_t is the tax rate on labor income. Firms maximize

(4.4) $\sum q_t(c_t+g_t-w_t n_t-v_t m_t)$

subject to (4.1) where v_t is the price of the intermediate good m_t. In addition to setting the tax policy τ_t, the government sets the price v_t of the intermediate good, produces it at zero cost, and supplies it to firms. The government's budget constraint is given by

(4.5) $\sum q_t(\tau_t w_t n_t+v_t m_t-g_t) = 0.$

Using the same techniques as in Proposition 1, it is easy to show that the Ramsey problem is to maximize (4.2) subject to

(4.6) $c_t + g_t + \phi(c_t,m_t) = n_t$

(4.7) $\sum \beta^t[c_t U_{1t}+n_t U_{2t}] = 0.$

Since m_t only enters (4.6) it is clear that any Ramsey allocation has $\phi_{mt} = 0$ for all t. Thus the corresponding equilibrium has the price of the intermediate good $v_t = 0$ for all t.

We now establish the equivalence between this real economy and the monetary shopping time economy when ϕ satisfies CRS. Recall that for the shopping time economy the competitive equilibrium allocations are completely characterized by (3.1) and (3.3). When there is no uncertainty these are simply

(4.8) $c_t + g_t = \ell_t$

(4.9) $\sum \beta^t(c_t U_{1t} + (\ell_t+c_t\phi_{1t}+m_t\phi_{2t})U_{2t}) = 0.$

When ϕ satisfies constant returns-to-scale $c_t\phi_{1t} + m_t\phi_{2t} = \phi(c_t,m_t)$. Letting $n_t = \ell_t + \phi_t$ it is clear that (4.8) and (4.9) are equivalent to (4.6) and (4.7) and thus the resource constraints and the implementability constraints for the two economies are the same. We then have

PROPOSITION 6. When ϕ satisfies CRS the set of equilibrium allocations for the real model with intermediate goods coincides with those in the monetary shopping time economy.

When ϕ is homogenous of degree $k > 1$, as we have shown, the Friedman rule still holds but it is not possible to interpret the monetary model as one with intermediate goods since there are increasing returns which are inconsistent with competitive equilibrium in the intermediate goods model. Thus the optimality of the Friedman rule in the monetary model does not require that the monetary model be equivalent to a real model with intermediate goods.

For the money in the utility function economy and the cash-in-advance economy we can also construct real intermediate good economies which have the same set of equilibrium allocations. For the money in the utility function economy we construct the real economy as follows. Firms produce final consumption x_t, government consumption g_t using labor ℓ_t and intermediate goods c_t and m_t according to

(4.10) $x_t = h(m_t,c_t)$

(4.11) $c_t + g_t = \ell_t$

where h satisfies CRS. Consumer preferences are

(4.12) $\sum_{t=0}^{\infty} \beta^t V(x_t,\ell_t).$

Consumers maximize (4.12) subject to

$$(4.13)\quad \sum_{t=0}^{\infty} q_t x_t \leq \sum_{t=0}^{\infty} q_t(1-\tau_t)w_t \ell_t$$

where q_t is the date 0 price of x_t, w_t is the wage rate, and τ_t is the tax rate on labor income. Firms maximize

$$(4.14)\quad \sum_{t=0}^{\infty} q_t(x_t - w_t\ell_t - v_t m_t)$$

where the intermediate good m_t is purchased from the government at v_t and the intermediate good c_t is produced within the firm from labor. The government's budget constraint is

$$(4.15)\quad \sum q_t(\tau_t w_t \ell_t + v_t m_t = \sum q_t g_t.$$

The Ramsey problem is to maximize (4.12) subject to

$$(4.16)\quad x_t = h(m_t, c_t)$$

$$(4.17)\quad c_t + g_t = \ell_t$$

$$(4.18)\quad \sum \beta^t[x_t V_{1t} + \ell_t V_{2t}] = 0.$$

Since m_t only enters (4.16) any Ramsey allocation has $h_{1t} = 0$ for all t and the corresponding equilibrium has the price of the intermediate good $m_t = 0$.

To establish the equivalence between this real economy and the money in the utility function economy, recall that the competitive equilibrium allocations in the latter economy are completely characterized by

$$(4.19)\quad c_t + g_t = \ell_t$$

$$(4.20)\quad \sum \beta^t(m_t h_{1t} V_{1t} + c_t h_{2t} V_{1t} + \ell_t V_{2t}) = 0.$$

When h is CRS, $m_t h_{1t} + c_t h_{2t} = h(m_t, c_t) = x_t$ and (4.19) and (4.20) are equivalent to (4.17)–(4.18). Thus we have the analogue of Proposition 6: When h is CRS the set of equilibrium allocations for the real model with intermediate goods coincides with those in the money in the utility function model.

Finally, for the cash-in-advance economy we construct the real economy as follows. Intermediate goods c_{1t} and c_{2t} together with labor ℓ_t produce final private consumption c_t and government consumption g_t according to

(4.21) $c_{1t} + c_{2t} + g_t = \ell_t$

(4.22) $c_t = h(c_{1t}, c_{2t})$

(4.23) $h_1(c_{1t}, c_{2t}) \geq h_2(c_{1t}, c_{2t})$

where (4.23) is just some constraint on the technology set. Consumer preferences are

(4.24) $\sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t)$

and the consumer budget constraint is

(4.25) $\sum_{t=0}^{\infty} q_t c_t \leq \sum_{t=0}^{\infty} q_t (1-\tau_t) w_t \ell_t$

where q_t is the date 0 price of c_t, w_t is the wage rate, and τ_t is the tax rate on labor income. Firms maximize

(4.26) $\sum q_t [c_t - w_t \ell_t - v_t m_t]$

subject to (4.21)-(4.23). The government budget constraint is

(4.27) $\sum q_t (\tau_t w_t \ell_t + v_t m_t) = \sum q_t g_t.$

The Ramsey problem is to maximize (4.24) subject to (4.21)-(4.23) and

(4.28) $\sum_{t=0}^{\infty} \beta^t [c_t V_{1t} + \ell_t V_{2t}] = 0.$

Consider a less constrained problem with constraint (4.23) dropped. It is clear that for this problem it is optimal to set $h_1 = h_2$. Since this allocation satisfies the dropped constraint it is optimal for the Ramsey problem.

To establish the equivalence between this real economy and the cash-in-advance economy, recall that the competitive equilibrium allocations in the latter economy are characterized by

(4.29) $c_{1t} + c_{2t} + g_t = \ell_t$

(4.30) $U_{1t} \geq U_{2t}$

(4.31) $\sum \beta^t (U_{1t}c_{1t} + U_{2t}c_{2t} + U_{3t}\ell_t) = 0.$

When $U(m,c,\ell) = V(h(m,c),\ell)$ and h is CRS, (4.29)-(4.31) are the same as (4.21)-(4.23) together with (4.28) and thus the set of equilibrium allocations for the two economies coincide.

Recall that for preferences of the form given in (1.23) the Friedman rule is optimal when $\sigma_1 < \sigma_2$. It is easy to see, however, that such economies cannot be interpreted as intermediate goods economies. Thus the optimality of the Friedman rule does not require that the monetary model be equivalent to a real model with intermediate goods.

We have established an equivalence between a class of monetary economies and real intermediate good economies. The Friedman rule is optimal in the monetary economies. In the intermediate good economies it is optimal to supply the intermediate good for free. Since the cost of producing the intermediate good is zero this also means that it is optimal not to tax the intermediate good. From this equivalence one might be tempted to conclude that the optimality of the Friedman rule is closely connected to the well-known Diamond-Mirrlees result: Given a sufficiently rich set of final good taxes it is optimal not to tax intermediate goods. This temptation should be resisted as the following example shows.

Consider a variant of the deterministic shopping time model. Here it takes resources for the government to maintain the money stock. Specifically to maintain a stock of real money balances m_t takes αm_t units of labor, hence the technology is

(4.32) $\alpha m_t \leq \ell_{2t}$

where ℓ_{2t} is labor used in maintaining the money stock. Loosely, these resources may be thought of as those required to operate a central bank and a treasury. The rest of the technology is the same as before. Preferences are

$$\sum \beta^t U(c_t, \ell_{1t} + \ell_{2t} + \phi(c_t, m_t)),$$

where ℓ_{1t} is the labor input used to produce the consumption good. Proceeding along the same lines as earlier it is easy to show that, when ϕ is CRS, the Ramsey allocations satisfy $\phi_{mt} = -\alpha$ for all t. Thus the Friedman rule is *not* optimal in this economy.

We can interpret this economy as an intermediate goods economy with preferences given by $\Sigma\beta^t U(c_t, n_{1t} + n_{2t})$ and a technology given by

(4.33) $c_t + g_t + \phi(c_t, m_t) = n_{1t}$

(4.34) $\alpha m_t \leq n_{2t}$.

The only difference between this economy and the economy with $\alpha = 0$ is that the government hires labor to produce the intermediate good m. In this economy it is not optimal to tax the intermediate good and thus the government supplies the intermediate good to firms at cost.

We have constructed an example in which in the monetary economy the Friedman rule is not optimal but in the equivalent intermediate goods economy it is optimal to set intermediate goods taxes to zero. We have also pointed out that there are monetary economies for which the Friedman rule is optimal but there is no equivalent intermediate goods economy. Thus we conclude there is no connection between the optimality of the Friedman rule and the Diamond-Mirrlees result.

We have found that in our real economies when the intermediate good is costless to produce, the Friedman rule is optimal in the analogous monetary economies. In the example with costly intermediate good production, the Friedman rule turned out not to be optimal. This finding suggests that when the Friedman rule is optimal, it is not just because money is an intermediate good but because it is a *costless* intermediate good. This reasoning is suggestive of Friedman's original argument. It is not complete, however, because as we have demonstrated, there are monetary economies in which the Friedman rule is optimal but the monetary economy cannot be interpreted as a real economy with intermediate goods.

5. Conclusion

In this paper we analyzed the optimality of the Friedman rule in three monetary models with distorting taxes. We showed that there is no obvious connection between the optimality of the Friedman rule and the interest elasticity of money demand. We also showed that there is no connection between optimal taxation of intermediate goods and the optimality of the Friedman rule. In all three models the Friedman rule is optimal if preferences satisfy homotheticity and separability conditions similar to those in the literature on uniform commodity taxation. Moreover, Friedman's original intuition that consumers should be satiated with real balances when they are free plays an important role.

References

Atkinson, A. B., and Stiglitz, J. E. 1972. The structure of indirect taxation and economic efficiency. *Journal of Public Economics* 1 (April): 97-119.

Chari, V. V., Christiano, Lawrence J., and Kehoe, Patrick J. 1991. Optimal fiscal and monetary policy: Some recent results. *Journal of Money, Credit, and Banking* 23, 519-539.

Diamond, Peter A., and Mirrlees, James A. 1971. Optimal taxation and public production. *American Economic Review* 63, 8-27, 261-268.

Faig, Miquel. 1988. Characterization of the optimal tax on money when it functions as a medium of exchange. *Journal of Monetary Economics* 22 (July): 137-48.

Friedman, Milton. 1969. The optimum quantity of money. In *The optimum quantity of money and other essays*, pp. 1-50. Chicago: Aldine.

Guidotti, Pablo E., and Vegh, Carlos A. 1993. The optimal inflation tax when money reduces transactions costs: A reconsideration. *Journal of Monetary Economics* 31 (April): 189-205.

Kimbrough, Kent P. 1986. The optimum quantity of money rule in the theory of public finance." *Journal of Monetary Economics* 18 (November): 277-84.

Lucas, Robert E., Jr., and Stokey, Nancy L. 1983. Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics* 12 (July): 55-93.

Phelps, E. S. 1973. Inflation in the theory of public finance. *Swedish Journal of Economics* 75 (March): 67-82.

Woodford, Michael. 1990. The optimum quantity of money. In *Handbook of monetary economics*, vol. II, ed. Benjamin M. Friedman and Frank H. Hahn, pp. 1067-152. Amsterdam: North-Holland.