Thermodynamics and cosmological reconstruction in $f(T, B)$ gravity

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Recently, it was formulated a teleparallel theory called $f(T, B)$ gravity which connects both $f(T)$ and $f(R)$ under suitable limits. In this theory, the function in the action is assumed to depend on the torsion scalar T and also on a boundary term related with the divergence of torsion, $B = 2\nabla_\mu T^\mu$. In this work, we study different features of a flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology in this theory. First, we show that the FLRW equations can be transformed to the form of Clausius relation $\hat{T}_h S_{\rm eff} = -dE + WdV$, where $\hat{T}_h$ is the horizon temperature and $S_{\rm eff}$ is the entropy which contains contributions both from horizon entropy and an additional entropy term introduced due to the non-equilibrium. We also formulate the constraint for the validity of the generalised second law of thermodynamics (GSLT). Additionally, using a cosmological reconstruction technique, we show that both $f(T, B)$ and $-T + F(B)$ gravity can mimic power-law, de-Sitter and ΛCDM models. Finally, we formulate the perturbed evolution equations and analyse the stability of some important cosmological solutions.

I. INTRODUCTION

In the current scenario, dark energy (DE) is referred as an active agent which tends to accelerate the expansion in cosmos. The expanding paradigm of the Universe has been affirmed from various observational measurements. In 1998, observations of SNeIa accumulated by the high-redshift SN team [1] and SN cosmology project team [2] appearing as illuminating candles suggested an accelerating expansion of the Universe. The source for this observed cosmic acceleration may be an anonymous energy component entitled as dark energy. In spite of tremendous efforts, late-time cosmic acceleration is certainly a major challenge for cosmologists. The direct evidence for cosmic acceleration has strengthened over time with measurements from temperature anisotropies in the cosmic microwave background (CMB) [3] and Baryon acoustic oscillations (BAO) [4] which confirm the existence of DE. Dark energy is appeared as an enigmatic cosmic ingredient and the interpretation of its gravitational effects is a dynamic research field. The most likely theoretical campaigner of DE is the cosmological constant Λ characterized by a constant equation of state (EoS) $w = -1$ [5]. A number of alternative models have been proposed in this perspective to explain the role of DE in the present cosmic acceleration [6]. The other proposal for the construction of DE models is the modification of Einstein-Hilbert action which leads to modified gravity models. Some important alternative theories of gravity are $f(R)$ gravity [7], $f(R, \mathcal{T})$ gravity ($\mathcal{T}$ is the trace of energy-momentum tensor $\mathcal{T}_{\alpha\beta}$) [8], $f(R, \mathcal{T}, \mathcal{Q})$ gravity (where $\mathcal{Q} = R_{\alpha\beta}T^{\alpha\beta}$) [9, 10], Gauss-Bonnet gravity [11], teleparallel modifications [12–14], scalar-tensor theories [15, 16], among others.

In current scenarios, generalization of teleparallel theory has gained significant importance, which could provide alternative explanations for the cosmic acceleration [17]. A key problem in $f(T)$ gravity is that it breaks the invariance under local Lorentz transformations. Lack of local Lorentz symmetry implies that there is no freedom to fix any of the components of the tetrad [18]. Hence, loosing the Lorentz invariance means that two different tetrads corresponding to the same metric could give different field equations. A new approach done in [19], introduced a new way to construct a covariant formulation of $f(T)$ gravity. Basically, in this approach the spin connection is chosen to be non-zero and being pure-gauge. A more general theory containing the squares of the irreducible parts of torsion $f(T_{\rm ax}, T_{\rm vec}, T_{\rm ten})$ has been also introduced in [20] using its covariant formulation. In our work, we will not use this covariant approach, instead we will use the standard formulation of modified teleparallel theories of gravity where the spin connection is assumed to be zero. In despite of the loose of the Lorentz invariance, this standard teleparallel approach has been very used in the literature. One can somehow "alleviative" the covariant issue (only at the level of the field equations) by choosing the correct tetrads [21]. In FLRW cosmology, it is always possible to find "good tetrads" to obtain non-trivial

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cosmological solutions [21]. In flat FLRW cosmology, the diagonal tetrad in Cartesian coordinates is a good tetrad since it does not restrict the field equations being the general relativity case as the tetrad in spherical coordinates. In principle, at the level of the computations of the field equations, both approaches (covariant and non-covariant using the good tetrads) should give the same result [20]. In [13], the authors generalized $f(T)$ by introducing a new Lagrangian $f(T,B)$ which involves a boundary term B which is related to the divergence of the torsion tensor. This theory becomes equivalent to $f(R)$ gravity for the choice of special form $f(-T+B)$. The latter is the only case in which Lorentz invariance can be achieved for a zero spin-connection. In our work, we are interested on studying different cosmological properties of this theory as its thermodynamics laws, cosmological reconstruction method and the stability of some cosmological models.

Cosmological reconstruction is one of the most important tools that can be used in modified gravity to mimic realistic cosmological scenarios. The reconstruction scheme in $f(R)$ gravity and its modifications have been carried out under different scenarios [22–25] to find out realistic cosmology which can explain the transition of matter dominated epoch to DE phase. In this study, one interesting way is to consider the known cosmic evolution and use the field equations to find particular form of Lagrangian that can reproduce the given evolution background. Nojiri et al. [25] executed such reconstruction scheme in order to find some realistic models in $f(R)$ theory which was then applied in $f(R,\mathcal{G})$ modified Gauss-Bonnet theories [26] (where $\mathcal{G}$ is the Gauss-Bonnet term). The cosmic evolution based on power law solution of the scale factor has also been discussed in modified theories [27]. Dunsby et al. [28] explored that extra degrees of freedom to the matter component are necessary to reconstruct the Λ cold dark matter (ΛCDM) evolution in $f(R)$ gravity. Carloni et al. [29] set up a new method of reconstructing $f(R)$ gravity using the cosmic parameters rather than any form of the scale factor. In the context of modified theories, stability of cosmological solutions has been analysed for homogeneous perturbations [30–35]. In [31], the stability of $f(R,\mathcal{G})$ models is presented for power law and ΛCDM cosmology. In the context of teleparallel gravity, it was showed that one can reconstruct ΛCDM universes and describe holographic dark energy models for $f(T)$ gravity [32–34]. As $f(T,B)$ is a generalisation of $f(T)$ and $f(R)$ gravity, it is important to also find out how this theory can reconstruct or mimic different cosmological models. One goal of this work is to reconstruct different cosmological models for $f(T,B)$ gravity and also for the particular choice of the function where $f(T,B)=-T+F(B)$. After that, we will also study the stability of some of these cosmological models.

The connection between the FLRW equations and the first law of thermodynamics (FLT) at the apparent horizon was shown in [36] for $\tilde{T}=1/2\pi R_A$, $S=\pi R_A^2/G$, where R_A is the radius of the apparent horizon and $\tilde{T}$ is the temperature. The Friedmann equations in Gauss-Bonnet gravity and Lovelock gravity were also formulated by using the corresponding entropy formula of static spherically symmetric black holes. Eling et al. [37] found that one cannot find the correct field equations simply by using the Clausius relation in nonlinear theories of gravity. It was remarked that a non-equilibrium treatment of thermodynamics is required, whereby the Clausius relation is modified to $\tilde{T}dS=\delta Q+d_iS$, where d_iS is the entropy production term. Akbar and Cai [38] showed that the Friedmann equations in general relativity (GR) can be written as $dE=\tilde{T}dS+WdV$ (unified FLT on the trapping horizon suggested by Hayward [39, 40]) with the work term being $W=\frac{1}{2}(\rho-p)$. They also extended this work to Gauss-Bonnet gravity [38], Lovelock gravity [38, 41], braneworld gravity [42, 43], $f(R)$ gravity [44] and scalar-tensor gravity [45]. The generalised first and second laws of thermodynamics were also studied in the context of $f(T)$ gravity for different forms of the function [46–48]. As we have pointed out, the investigation about the validity of thermodynamical laws in modified theories has been carried out by numerous researchers in literature [49]. Here, we are interested to explore the validity of these laws in $f(T,B)$ gravity.

This paper is organised as follows: In Sec. II, we briefly introduce teleparallel equivalent of general relativity and then its generalisation, $f(T,B)$ gravity. Then, we present the basis equations for a FLRW cosmology. Sec. III is devoted to the study of the first and second laws of thermodynamics in this theory. Different reconstructions models are presented in Sec. IV for $f(T,B)$ and also for $-T+F(B)$ gravity. Using perturbation techniques, the stability of different cosmological models are studied in Sec. V. Finally, in Sec. VI we conclude our main results.

II. TELEPARALLEL EQUIVALENT OF GENERAL RELATIVITY AND ITS MODIFICATIONS

Let us briefly introduce the basis of the teleparallel equivalent of general relativity (TEGR). We will use the convention used in Ref. [13] where E_m^μ is the inverse of the tetrad e_μ^m and Greek and Latin indices refer to space-time and tangent space ones respectively. This theory lies in the idea that the manifold has a vanished curvature but a non-zero torsion. To ensure this kind of geometry, one needs to chose a specific connection where the space is globally flat, the so-called Weitzenböck connection $W_\mu{}^a{}_\nu$. One important fact it is that this alternative representation of gravity is equivalent (in the field equations) to general relativity. The dynamical variable is the tetrad field and it is related with the metric with the following equation,

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \,, \tag{1}$$

where η_{ab} represents the Minkowski metric $(-,+,+,+)$. Note that the tetrad fields are orthonormal vector at each point of the manifold, hence they obey the following orthogonality relationships

$$E^\mu_m e^n_\mu = \delta^n_m \,, \tag{2}$$
$$E^\nu_m e^m_\mu = \delta^\nu_\mu \,. \tag{3}$$

The torsion tensor is constructed by taking the anti-symmetric part of the Weitzenböck connection,

$$T^a{}_{\mu\nu} = W_\mu{}^a{}_\nu - W_\nu{}^a{}_\mu = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu \,. \tag{4}$$

The teleparallel action is then constructed with the torsion scalar which is defined as a contraction of the super potential

$$S^{abc} = \frac{1}{4}(T^{abc} - T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c) \,, \tag{5}$$

with the torsion tensor $T = S_a{}^{bc}T^a{}_{bc}$. Here, the torsion vector is defined contracting the first two indices of the torsion tensor $T_\mu = T^\nu{}_{\nu\mu}$. Explicitly, the action reads

$$S_{\rm TEGR} = \frac{1}{\kappa^2}\int e\, T\, d^4x + S_{\rm m} \,, \tag{6}$$

where e denotes the determinant of the tetrad which is equal to $\sqrt{-g}$ and $\kappa^2 = 8\pi G$. Here, $S_{\rm m}$ is the action of the matter content. It is possible to prove that the torsion scalar is related with the Ricci scalar directly by

$$R = -T + \frac{2}{e}\partial_\mu(eT^\mu) = -T + B \,, \tag{7}$$

where B is a boundary term. The Einstein-Hilbert action is constructed with the Ricci scalar R, so that it differs only by a boundary term with respect to the TEGR action. Hence, tetrad variations of the action (6) are equivalent to metric variations of the Einstein-Hilbert action. Therefore, if one varies the action (6) with respect to the tetrad, the corresponding field equations will be identical to the Einstein field equations.
A well-studied modification of the action (6) is obtained by changing the torsion scalar T to an arbitrary function $f(T)$ which depends smoothly on T. This generalisation then has the following action

$$S_{f(T)} = \frac{1}{\kappa^2}\int e\, f(T)\, d^4x + S_{\rm m} \,, \tag{8}$$

which gives rise to the $f(T)$ field equations which is a second order theory. This theory is in this sense, analogous to $f(R)$ gravity. However, these two theories are not equivalent. With the aim to combine both $f(R)$ gravity and $f(T)$ gravity, in Ref. [13] it was proposed the following action

$$S_{f(T,B)} = \frac{1}{\kappa^2}\int dx^4\, e\, f(T,B) + S_{\rm m} \,, \tag{9}$$

which is a modified teleparallel theory of gravity where now $f(T,B)$ also depends on the boundary term B. In [13] it was proved that by choosing $f = f(T)$ and $f = f(-T+B) = f(R)$ it is possible to recover both $f(T)$ and $f(R)$ gravity respectively. The field equations of this theory are obtained by varying the action with respect to the tetrad giving us,

$$\begin{aligned} &2e\delta^\lambda_\nu \Box f_B - 2e\nabla^\lambda\nabla_\nu f_B + eBf_B\delta^\lambda_\nu + 4e\Big[(\partial_\mu f_B) + (\partial_\mu f_T)\Big]S_\nu{}^{\mu\lambda} \\ &\qquad + 4e^a_\nu\partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef\delta^\lambda_\nu = 16\pi e\mathcal{T}^\lambda_\nu \,, \end{aligned} \tag{10}$$

where $\mathcal{T}^\lambda_\nu = e^a_\nu \mathcal{T}^\lambda_a$ is the standard energy momentum tensor and $\Box = \nabla^\mu\nabla_\mu$. In general, this theory is a fourth-order one and in pure tetrad formalism, it is not invariant under local Lorentz transformations (since T and B are not invariant under local LT). Indeed, the only theory which is invariant under these transformations is obtained by taking $f(T,B) = f(-T+B) = f(R)$, i.e., in the $f(R)$ case.

A. $f(T,B)$ Cosmology

In this section, the basic equations for a flat FLRW cosmology in $f(T,B)$ will be introduced. The metric which describes this space-time in Cartesian coordinates is given by

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,, \tag{11}$$

where $a(t)$ is the scale factor of the universe. In these coordinates, the tetrad field can be expressed as follows

$$e^a_\mu = \mathrm{diag}\,(1, a(t), a(t), a(t))\;. \tag{12}$$

If we assume that the content of the universe is a perfect fluid and we use the above FLRW tetrad, the $f(T,B)$ cosmology field equations (10) become

$$-3H^2(3f_B + 2f_T) + 3H\dot{f}_B - 3\dot{H}f_B + \frac{1}{2}f(T,B) = \kappa^2\rho_{\rm m}\,, \tag{13}$$

$$-3H^2(3f_B + 2f_T) - \dot{H}(3f_B + 2f_T) - 2H\dot{f}_T + \ddot{f}_B + \frac{1}{2}f(T,B) = -\kappa^2 p_{\rm m}\,. \tag{14}$$

Here, $H = \dot{a}/a$ is the Hubble parameter and dots are differentiation with respect to t. Additionally, $\rho_{\rm m}$ and $p_{\rm m}$ are the energy density and pressure of the matter content. It is easy to prove that the Ricci scalar is $R = -T+B = 6(2H^2+\dot{H})$, where the torsion scalar and the boundary term are $T = 6H^2$ and $B = 6(\dot{H} + 3H^2)$ respectively. Moreover, by setting $f = f(T)$ or $f = f(-T + B)$ in the above equations, we recover the standard $f(T)$ and $f(R)$ flat FLRW equations. One needs to be very careful with different metric signature notations. In other $f(T)$ papers, some authors used a different signature notation where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ changing the sign of the torsion scalar $T \to -T = -6H^2$. For example, Eqs. (2.9)-(2.10) reported in [50] used the other signature metric notation. To match those equations, one needs to change $T \to -T = -6H^2$ and hence $f_T \to -f_T$ and $f_{TT} \to f_{TT}$.

Eqs. (13) and (14) can be also represented in a fluid form,

$$3H^2 = \kappa^2_{\rm eff}\left(\rho_{\rm m} + \rho_{TB}\right)\,, \tag{15}$$

$$2\dot{H} = -\kappa^2_{\rm eff}(\rho_{\rm m} + p_{\rm m} + \rho_{TB} + p_{TB})\,. \tag{16}$$

The above equations are analogous to standard FLRW equations as in GR, the quantities appearing in these equations are defined in terms of $f(T,B)$ gravity as follows:

$$\rho_{\rm TB} = \frac{1}{\kappa^2}\Big[-3H\dot{f}_B + (3\dot{H} + 9H^2)f_B - \frac{1}{2}f(T,B)\Big]\,, \tag{17}$$

$$p_{\rm TB} = \frac{1}{\kappa^2}\Big[\frac{1}{2}f(T,B) + \dot{H}(2f_T - 3f_B) - 2H\dot{f}_T - 9H^2 f_B + \ddot{f}_B\Big]\,, \tag{18}$$

where we have defined $\kappa^2_{\rm eff}$ as follows

$$\kappa^2_{\rm eff} = -\frac{\kappa^2}{2f_T}\,. \tag{19}$$

Now, we have all the basis ingredients to study some properties in $f(T,B)$ cosmology as its thermodynamics and reconstructs some cosmological models.

III. THERMODYNAMICS OF $f(T,B)$ GRAVITY

A. Non-Equilibrium Description of Thermodynamics

Here, we intend to explore the validity of thermodynamic laws in generalized teleparallel theory of gravity in the non-equilibrium description. In the following section, we determine the restriction on parameters and model of $f(T,B)$ gravity for the validity of first and second laws of thermodynamics at the apparent horizon of FLRW model. Also, we will show that for the total energy of the system to be positive, it is necessary that graviton is not a ghost in the sense of quantum gravity. We would like to mention that the results of non-equilibrium thermodynamics in $f(R)$ and $f(T)$ theories can be retrieved for some specific cases in this modified gravity.

The energy momentum tensor of additional geometric components satisfy the following conservation equation

$$\dot{\rho}_{\rm TB} + 3H(\rho_{\rm TB} + p_{\rm TB}) = \frac{T}{2\kappa^2}(2\dot{f}_T)\,, \tag{20}$$

Here, the energy conservation equation is not trivially satisfied since $2\dot{f}_T \neq 0$.

1. First Law of Thermodynamics in $f(T,B)$ gravity

In order to discuss the thermodynamics of $f(T,B)$ gravity, we can find the dynamical apparent horizon by using the relation $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r}=0$. For the flat FLRW metric the radius of the apparent horizon is

$$\tilde{r}_A=\frac{1}{H}\,. \tag{21}$$

The time derivative of the above equation gives us

$$-\frac{d\tilde{r}_A}{\tilde{r}_A^3}=\dot{H}Hdt\,. \tag{22}$$

Using Eq. (16) in the above equation, one gets

$$\frac{f_T d\tilde{r}_A}{2\pi G}=-\tilde{r}_A^3H(\rho_{\rm eff}+p_{\rm eff})dt\,. \tag{23}$$

Here, $\rho_{\rm eff}=\rho_{\rm TB}+\rho_{\rm m}$ and $p_{\rm eff}=p_{\rm TB}+p_{\rm m}$, are the total density and pressure of the universe.

Now we need to define the Bekenstein-Hawking horizon entropy in $f(T,B)$ gravity. For this purpose, we provide the review of such definition in GR as well as in some non-standard theories. In GR, Bekenstein-Hawking horizon entropy is defined by $S_h=A/(4G)$, where $A=4\pi\tilde{r}_A^2$ is the area of the apparent horizon [51]. In modified theories of gravity like $f(R)$ gravity, the horizon entropy S_h associated with the Noether charge, the so-called Wald entropy, can be defined by $S_h=A/(4G_{\rm eff})$ [52], where $G_{\rm eff}=G/f_R$ with $f_R=df(R)/dR$. We would like to mention that this definition of Wald entropy in $f(R)$ gravity is valid for both metric and Palatini formulism [53]. In [54], Brustein et al. showed that the Noether charge entropy is equal to a quarter of the horizon area in units of the effective gravitational coupling on the horizon defined by the coefficient of the kinetic term of a specific metric perturbation polarization on the horizon. They proposed that Wald's entropy can be expressed as

$$S_h=\frac{A}{4G_{\rm eff}}\,.$$

Similarly, in this notation, Wald entropy in $f(T)$ gravity is defined as $S_h=2A/(4G_{\rm eff})$, where $G_{\rm eff}=G/f_T$ [50]. Hence in newly proposed modified teleparalell gravity theory, we define the Wald entropy as $S_h=A/(4G_{\rm eff})$, where $G_{\rm eff}=-G/(2f_T)$. The Wald entropy in $f(T,B)$ then reads as follows

$$S_h=-\frac{A(2f_T)}{4G}\,. \tag{24}$$

Clearly, if we set $f=f(-T+B)=f(R)$ and $f=f(T)$ we recover the standard $f(R)$ and $f(T)$ Wald entropy relationships respectively. From Eqs. (23) and (24), we get

$$-\frac{dS_h}{2\pi\tilde{r}_A}=\frac{\tilde{r}_A}{G}df_T-4\pi\tilde{r}_A^3H(\rho_{\rm eff}+p_{\rm eff})dt\,. \tag{25}$$

The associated temperature of the apparent horizon is defined through the surface gravity $\kappa_{\rm sg}$ as

$$\tilde{T}_H=\frac{|\kappa_{\rm sg}|}{2\pi}\,, \tag{26}$$

where $\kappa_{\rm sg}$ is given by [55]

$$\kappa_{\rm sg} \;=\; \frac{1}{2\sqrt{-h}}\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta\tilde{r}_A)=-\frac{1}{\tilde{r}_A}(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A})=-\frac{\tilde{r}_A}{2}(2H^2+\dot{H})\,. \tag{27}$$

By multiplying the term $\left(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)$ on both sides of Eq. (25), we get

$$\tilde{T}_HdS_h=4\pi\tilde{r}_A^3H(\rho_{\rm eff}+p_{\rm eff})dt-2\pi\tilde{r}_A^2(\rho_{\rm eff}+p_{\rm eff})d\tilde{r}_A-\frac{\pi\tilde{r}_A^2\tilde{T}_H}{G}d(2f_T)\,. \tag{28}$$

In GR, the Misner-Sharp energy is defined as $E = \tilde{r}_A/(2G)$. For modified theories this relation is extended of the form $E = \tilde{r}_A/(2G_{\rm eff})$. In $f(T,B)$ gravity, this definition can be extended as

$$E = -\frac{\tilde{r}_A(2f_T)}{2G} \,. \tag{29}$$

From Eqs.(21) and (29), we then get that the Misner-Sharp energy is

$$E = -V\frac{3H^2(2f_T)}{8\pi G} = V\rho_{\rm eff} \,, \tag{30}$$

where $V = (4/3)\pi\tilde{r}_A^3$, is the volume of the interior region of the apparent horizon. From the above equation, we find that E is the total intrinsic energy of the system. Also, we need to have $f_T < 0$, so that $E > 0$. For this restriction on $f_T < 0$, the effective gravitational coupling $G_{\rm eff} = -G/(2f_T)$ needs to be positive. We would like to mention that the condition $f_T < 0$, is necessary condition to ensure that graviton is not a ghost in the sense of quantum gravity [56].

From Eqs. (15) and (29), one finds

$$dE = -\frac{\tilde{r}_A}{G}df_T + 4\pi\rho_{\rm eff}\tilde{r}_A^2 d\tilde{r}_A - 4\pi H\tilde{r}_A^3(\rho_{\rm eff} + p_{\rm eff})dt \,. \tag{31}$$

By combining Eqs. (28) and (31), one obtains

$$\tilde{T}_H dS_h = -dE + 2\pi\tilde{r}_A^2(\rho_{\rm eff} - p_{\rm eff})d\tilde{r}_A - \frac{\tilde{r}_A}{G}(2\pi\tilde{r}_A\tilde{T}_H + 1)df_T \,. \tag{32}$$

By defining the work density, we get

$$W = -\frac{1}{2}\left(\hat{T}^{(\rm M)\alpha\beta}h_{\alpha\beta} + \hat{T}^{(\rm DE)\alpha\beta}h_{\alpha\beta}\right) = \frac{1}{2}(\rho_{\rm eff} - p_{\rm eff}) \,. \tag{33}$$

Here, $\hat{T}^{(\rm DE)\alpha\beta}h_{\alpha\beta}$ and $\hat{T}^{(\rm M)\alpha\beta}h_{\alpha\beta}$ are the energy-density of the dark components and matter respectively.. Using the above definition of the work density in Eq. (32), one arrives at

$$\tilde{T}_H dS_h = -dE + WdV - \frac{\tilde{r}_A}{G}(2\pi\tilde{r}_A\tilde{T}_H + 1)df_T \,, \tag{34}$$

which can be re-written as

$$\tilde{T}_H dS_h + \tilde{T}_H d\bar{S} = -dE + WdV \,, \tag{35}$$

where $d\bar{S} = (\tilde{r}_A/(G\tilde{T}_H))(2\pi\tilde{r}_A\tilde{T}_H + 1)df_T$. The extra term $d\bar{S}$ defined in Eq. (35) can be treated as an entropy production term in non-equilibrium thermodynamics. Such additional term marks to the non-equilibrium treatment of thermodynamics and is produced internally due to the Lagrangian dependence both on the torsion scalar and the boundary term. The results in $f(R)$ and $f(T)$ theories can be retrieved for some specific cases in this modified gravity. For the choice of $f = f(T)$, we can reproduce the results of $f(T)$ gravity and the entropy production term is the same reported in [50]. Similarly for the choice of $f(T,B) = f(-T+B)$, we find the results of $f(R)$ gravity [59]. In literature, it has been shown that the theories involving non-minimal matter geometry coupling also produce additional entropy production term (for review see [49]).

In $f(T)$ theory, the additional entropy term depends only on T, whereas in $f(T,B)$ gravity we have the contribution both from torsion and boundary terms. In $f(T,B)$ gravity $d\bar{S} \neq 0$, due to $d(2f_T) \neq 0$. In GR and alternative theories including Gauss-Bonnet and Lovelock gravities [57], the usual FLT is satisfied by the respective field equations. In fact these theories do not involve any surplus term in universal form of FLT *i.e.*, $\tilde{T}dS = -dE + WdV$. Here, we may define the effective entropy term being the sum of horizon entropy and entropy production term as $S_{\rm eff} = S_h + \bar{S}$ so that Eq. (35) can be rewritten as

$$\tilde{T}_h dS_{\rm eff} = -dE + WdV \,, \tag{36}$$

where $S_{\rm eff}$ is the effective entropy related to the contributions from torsion scalar and boundary term at the apparent horizon of FLRW spacetime.

2. Generalized Second Law of Thermodynamics

In order to investigate the second law of thermodynamics in $f(T,B)$ gravity, one can start with the Gibbs equation in terms of matter and dark energy components, given by

$$\tilde{T}_{tot} dS_t = d(\rho_{\rm eff} V) + p_{\rm eff} dV = V d(\rho_{\rm eff}) + (\rho_{\rm eff} + p_{\rm eff}) dV \,, \tag{37}$$

where S_t denotes the total entropy of the system inside the horizon. It is natural to assume that the total temperature of energy source inside the horizon is proportional to the temperature of the apparent horizon *i.e.*, $\tilde{T}_{\rm tot} = b\tilde{T}_h$ where $0 < b < 1$. It may result in local equilibrium by setting the proportionality constant as unity as mentioned in [58]. Generically, the horizon temperature differs from the temperature of all energy sources inside the horizon and the systems must experience interaction for some interval of time ahead of attaining the thermal-equilibrium. Furthermore, mutual coupling of matter and curvature components in this theory may result in spontaneous energy flow between the horizon and matter contents.

The validity of the generalised second law of thermodynamics (GSLT) requires the condition

$$\Omega \equiv \frac{dS_h}{dt} + \frac{d(d\bar{S})}{dt} + \frac{dS_t}{dt} \geq 0 \,. \tag{38}$$

Now using the FLRW equations together with Eqs. (35) and (38), one finds the following condition for validity of GSLT:

$$\frac{1}{2GH^4}\{(2-b)(2f_T)\dot{H}^2 - 2(1-b)(2f_T)\dot{H}H^2 - (1-b)H^3(2\dot{f}_T)\} \geq 0 \,. \tag{39}$$

One can recover the expression of GSLT in $f(R)$ gravity presented in [59] under the transformation $f(T,B) \to 2f(-T+B) = 2f(R)$[1]. Similarly, one can retrieve the results of $f(T)$ gravity if $f(T,B) \to -2f(-T)$ [50]. Note that in the later reference, the authors used the other signature notation for the metric so that the scalar torsion is equal to $T = -6H^2$ and not $T = 6H^2$ as in our notation.

If $b = 1$, i.e., temperature between outside and inside the horizon remains the same then the GSLT is valid only if

$$\frac{3\dot{T}^2 f_T}{2GT^3} \geq 0 \,. \tag{40}$$

For flat FLRW metric, one can define the effective components as $\rho_{\rm eff} = \rho_{\rm m} + \rho_{\rm TB}$ and $p_{\rm eff} = p_{\rm m} + p_{\rm TB}$ so that EoS is defined as $w_{\rm eff} = -1 - 2\dot{H}/3H^2$. Here $\dot{H} < 0$ corresponds to quintessence region while $\dot{H} > 0$ represents the phantom phase of the universe. It follows that form (40) that GSLT in $f(T,B)$ gravity is satisfied in phantom era of cosmos. This result is compatible with [60] according to which entropy may be positive even at the phantom era.

B. Equilibrium Description of Thermodynamics

In previous section, an additional entropy term $d\bar{S}$ in formulation of laws of thermodynamics was found, which can be considered as the result of non-equilibrium description of the field equations. In literature [48–50, 53], it has been shown that the equilibrium description does exist in modified theories of gravity and one can eliminate the additional entropy term. Here, we discuss whether the equilibrium description of thermodynamics in $f(T,B)$ gravity can be achieved or not.

We can rewrite the equations (15)-(18)

$$3H^2 = \kappa^2\left(\rho_{\rm m} + \rho_{\rm TBE}\right) \,, \tag{41}$$

$$2\dot{H} = -\kappa^2(\rho_{\rm m} + p_{\rm m} + \rho_{\rm TBE} + p_{\rm TBE}) \,, \tag{42}$$

where now we have defined the new quantities as

$$\rho_{\rm TBE} = \frac{1}{\kappa^2}\left[\frac{T}{2}(3f_B + 2f_T + 1) - 3H\dot{f}_B + 3\dot{H}f_B - \frac{1}{2}f(T,B)\right] , \tag{43}$$

$$p_{\rm TBE} = \frac{1}{\kappa^2}\left[\frac{1}{2}f(T,B) - \frac{T}{2}(3f_B + 2f_T + 1) - \dot{H}(3f_B + 2f_T + 2) - 2H\dot{f}_T + \ddot{f}_B\right] . \tag{44}$$

The above equations are analogous to standard FLRW equations as in GR plus the contribution of $f(T,B)$ gravity. Now we can check the validity of the first and second laws of thermodynamics in this scenario.

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[1] We used the Lagrangian $ef(T,B)\kappa^2$, whereas in [59] authors used the Lagrangian $\sqrt{-g}f(R)/(2\kappa^2)$

1. First Law of Thermodynamics

In this representation of the field equations, the time derivative of radius $\tilde{r}_A$ at the apparent horizon is given by

$$2d\tilde{r}_A = \kappa^2 \tilde{r}_A^3 (\rho_{\rm m} + p_{\rm m} + \rho_{\rm TBE} + p_{\rm TBE}) H dt \,. \tag{45}$$

Using the Bekenstein-Hawking entropy relation $S_h = A/(4G)$, one gets

$$\frac{1}{2\pi \tilde{r}_A} dS_h = 4\pi \tilde{r}_A^3 (\rho_{\rm m} + p_{\rm m} + \rho_{\rm TBE} + p_{\rm TBE}) H dt \,, \tag{46}$$

and then by multiplying both sides of the above equation by $1 - \dot{\tilde{r}}_A/(2H\tilde{r}_A)$, implies that

$$\tilde{T}_h d\hat{S}_h = -4\pi \tilde{r}_A^3 (\rho_{\rm m} + p_{\rm m} + \rho_{\rm TBE} + p_{\rm TBE}) H dt + 2\pi \tilde{r}_A^2 (\rho_{\rm m} + p_{\rm m} + \rho_{\rm TBE} + p_{\rm TBE}) d\tilde{r}_A \,. \tag{47}$$

Introducing the Misner-Sharp energy

$$\hat{E} = \frac{\tilde{r}_A}{2G} = V(\rho_{\rm m} + \rho_{\rm TBE}) \,, \tag{48}$$

one obtains

$$dE = -4\pi \tilde{r}_A^3 (\rho_{\rm m} + p_{\rm m} + \rho_{\rm TBE} + p_{\rm TBE}) H dt + 4\pi \tilde{r}_A^2 (\rho_{\rm m} + p_{\rm m} + \rho_{\rm TBE} + p_{\rm TBE}) d\tilde{r}_A \,. \tag{49}$$

Now, by replacing Eq. (49) into (48), one gets

$$\tilde{T}_h d\hat{S}_h = d\hat{E} - \hat{W} dV \,, \tag{50}$$

where we have used the work density $\hat{W} = (1/2)(\hat{\rho}_{\rm eff} - \hat{p}_{\rm eff})$ [39, 40]. Thus, as we have proved above, the equilibrium description of thermodynamics can be derived by redefining the energy density $\rho_{\rm TB}$ and the pressure $p_{\rm TB}$. Here, we find that the traditional first law of thermodynamics $\tilde{T}_h d\hat{S}_h = d\hat{E} - \hat{W} dV$ can be met in equilibrium thermodynamic description of $f(T, B)$ gravity. Hence, we can achieve first law of thermodynamics in similar fashion as in GR as well with modified theories including Gauss-Bonnet gravity [38], Lovelock gravity [38, 41] and braneworld gravity [42, 43], $f(R)$ and $f(T)$ theories [48, 50, 53].

2. Generalized Second Law of Thermodynamics

To establish the GSLT in this formulation of $f(T, B)$ gravity, one can consider the Gibbs equation in terms of all matter field and energy contents,

$$\tilde{T}_\nu dS_\nu = d(\rho_{\rm eff} V) + p_{\rm eff} dV \,, \tag{51}$$

where $\tilde{T}_\nu$ denotes the temperature within the horizon. The second law of thermodynamics can expressed as

$$\dot{S}_h + \dot{S}_\nu \geq 0 \,, \tag{52}$$

where S_h, S_ν are the horizon entropy and the entropy due to energy sources inside the horizon respectively. Now, we will assume a relation between the temperature within the horizon and the temperature of the apparent horizon given by

$$\tilde{T}_\nu = \tilde{T}_h \,. \tag{53}$$

Substituting (50) and (51) in (52), one obtains the condition for the validity of GSLT which is

$$\frac{\dot{H}^2}{2GH^4} \geq 0 \,, \tag{54}$$

which shows that GSLT can be satisfied in equilibrium description of thermodynamics in similar pattern as in GR. The reason behind the equilibrium description of thermodynamics is the validity of standard energy conservation as compared to previous section of non-equilibrium thermodynamics. Moreover, in this scheme entropy is defined by the Bekenstein-Hawking entropy relation $S_h = A/(4G)$, where entropy being proportional to horizon area. Here, we find that the second law of thermodynamics can be met in both non-phantom and phantom phases as that in $f(R)$ and $f(T)$ theories [50, 53, 59]. It is remarked that this result is valid only if we have the same temperature of the universe outside and inside the apparent horizon [61].

IV. RECONSTRUCTION METHOD IN $f(T,B)$ COSMOLOGY

In this section, the usual reconstruction method will be used to find specific form of the function $f(T,B)$ which mimics different cosmological models. Hereafter, we will assume that the matter pressure is $p_{\rm m}=w\rho_{\rm m}$ where w is the state parameter. Therefore, by using the matter conservation equation, one finds

$$\rho_{\rm m}(t) \;=\; \rho_0 a(t)^{-3(w+1)}\,. \tag{55}$$

A. Power-law Cosmology

It would be interesting to explore the existence of exact power solutions in $f(T,B)$ gravity theory corresponding to different phases of cosmic evolution. Let us consider a model described by a power-law scale factor given by

$$a(t) \;=\; \left(\frac{t}{t_0}\right)^h, \tag{56}$$

where t_0 is some fiducial time and h is greater than zero. These solutions help to explain the cosmic history including matter/radiation and dark energy dominated eras. Further, these solutions provide the scale factor evolution for the standard fluids such as dust ($h=2/3$) or radiation ($h=1/2$) dominated eras of the Universe. Also, $h>1$ predicts a late-time accelerating Universe. We would like to mention that h is arbitrary constant in the following form of power law solutions. For the above scale factor, the scalar torsion and boundary read as follows

$$T \;=\; \frac{6h^2}{t^2}\,, \tag{57}$$

$$B \;=\; \frac{6h(3h-1)}{t^2}\,. \tag{58}$$

Now, for simplicity, we will assume that the function can be written in the following form

$$f(T,B) \;=\; f_1(T)+f_2(B)\,. \tag{59}$$

By inverting (57) and (58), the 00 equation given by (13) becomes

$$\frac{1}{2}f_1(T)-Tf_{1,T}(T)-\kappa^2\rho_{\rm m}(t)=K\,, \tag{60}$$

$$-2B^2f_{2,BB}(B)+(1-3h)Bf_{2,B}(B)+(3h-1)f_2(B)=(2-6h)K\,. \tag{61}$$

Here, K is a constant for the method of separation variable and $f_{1,T}=df_1/dT$ and $f_{2,B}=df_2/dB$. We can directly solve the above equations obtaining

$$f_1(T) \;=\; \frac{2\kappa^2\rho_0}{1-3h(w+1)}\left(\frac{t_0}{\sqrt{6}h}\sqrt{T}\right)^{3h(w+1)}+C_1\sqrt{T}+2K\,, \tag{62}$$

$$f_2(B) \;=\; C_2B^{\frac{1}{2}(1-3h)}+C_3B-2K\,. \tag{63}$$

Note that this is one specific form of the function which mimics a power-law cosmology. There are other possible functions that also will represent this model. The separation of variable can be done either by choosing that $\rho_{\rm m}$ depends on T or B. The later comes from the fact that $\rho_{\rm m}=\rho_{\rm m}(t)$ and also $T=T(t)$ an $B=B(t)$. Hence, in principle, the energy density is $\rho(t)=\rho_0(\frac{t}{t_0})^{-3h(w+1)}$ and by using (57) and (58) one can rewrite the energy density in two ways, namely

$$\rho_{\rm m}(T) \;=\; \rho_0\left(6^{h/2}\left(\frac{h}{\sqrt{T}t_0}\right)^h\right)^{-3(w+1)}\,, \tag{64}$$

or

$$\rho_{\rm m}(B) \;=\; \rho_0\left(6^{h/2}\left(\frac{\sqrt{h(3h-1)}}{\sqrt{B}t_0}\right)^h\right)^{-3(w+1)}\,. \tag{65}$$

Hence, one has a freedom to choice either $\rho_{\rm m}=\rho_{\rm m}(T)$ or $\rho_{\rm m}=\rho_{\rm m}(B)$ in the separation of variables. In our case, we chose $\rho_{\rm m}=\rho_{\rm m}(T)$ but in principle, another kind of solution for the reconstruction method can be found by choosing $\rho_{\rm m}=\rho_{\rm m}(B)$. A similar approach was done in Sec. 5.1 in [62] and also in [63]. As it is stated in those references, in TEGR the density matter is usually described by T so that without loosing any generality, we will also use that approach. Then, for the following sections, we will use that $\rho_{\rm m}=\rho_{\rm m}(T)$ instead of $\rho_{\rm m}=\rho_{\rm m}(B)$. Note that from Eqs. (57) and (58), one can also express $T=T(B)$ or $B=B(T)$. In principle, one can try to solve the 00 equation (13) just by changing all in terms of T. However, this procedure makes the equation very complicated and it is almost impossible to find an analytical solution for the function f. As pointed out before, this kind of behaviour is something well-known in reconstruction techniques when one is considering two functions in f. See for example [62] and also [63] where they also can express either $T=T(T_G)$ or $R=R(G)$ in their theories. In those papers, one can also see the situation described above.

Now we explore the validity of GSLT for the power-law $f(T,B)$ model. By substituting Eqs. (62) and (63) into (40), one finds the following result for the validity of GSLT:

$$\frac{\dot{H}^2}{2TH^4}\left(C_1\sqrt{T}+\frac{6^{1-\frac{3}{2}h(1+w)}\rho_0 h\kappa^2(1+w)\left(\frac{\sqrt{T}t_0}{\sqrt{h}}\right)^{3h(1+w)}}{1-3h(1+w)}\right)\geq 0\,. \tag{66}$$

Here, the validity of GSLT depends on the constant C_1 and the power-law parameter h. On the left of Fig. 1, the evolution of GSLT is depicted by varying both C_1 and h. For $C_1\geq 0$, it is found that the validity period decreases when h increases. In the plot on the right in Fig. 1, we choose a particular value of h to show the validity of GSLT.

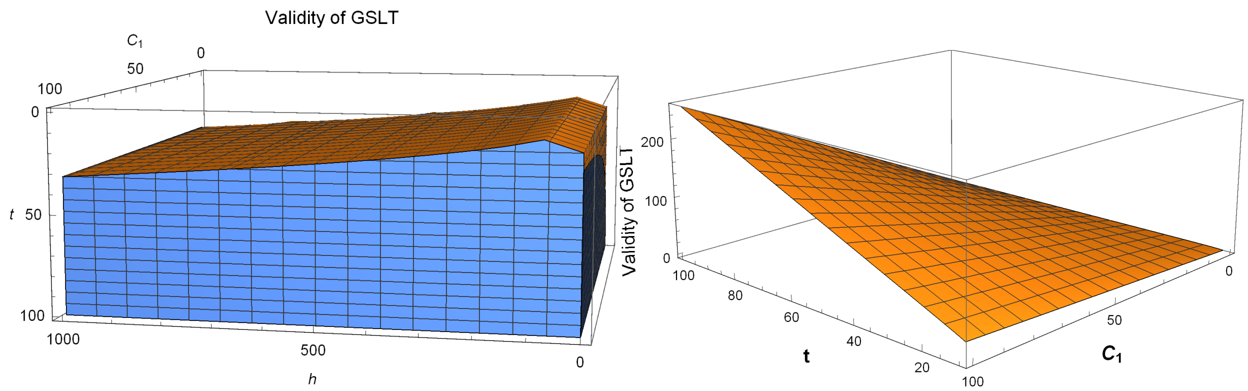

FIG. 1: The figure on the left represents the regions where GSLT is valid for $h>1$ and $C_1\geq 0$ whereas the figure on the right shows behavior of GSLT for $h=2$. Herein, we set $w=0$ and $H_0=67.3$.

B. de-Sitter reconstruction

If one assumes that the universe is governed by a de-Sitter form, i.e., the scale factor of the universe is an exponential $a(t)\propto e^{H_0 t}$, both the torsion scalar and the boundary term are constants. Explicitly they are given by $T=6H_0^2$ and $B=18H_0^2$ respectively. This kind of evolution of the universe is very well known and important since it correctly describes the expansion of the current universe. In GR, for this kind of universe, is known that the universe must be filled by a dark energy fluid whose state parameter $w=-1$ and hence the energy density is also a constant. From our modified theory, a priori $w=-1$ does not need to describe De-Sitter universes. Hence, to find de-Sitter reconstruction we must set $H=H_0$. From Eq. (13), it is easily to see that any kind of functions of $f(T,B)$ can admit de-Sitter solution if the following constraint is satisfied,

$$H_0^2\left(9f_B(T_0,B_0)+6f_T(T_0,B_0)\right)-\frac{1}{2}f(T_0,B_0)=0\,. \tag{67}$$

For instance, by assuming that the function is separable as $f(T,B)=f_1(B)+f_2(T)$, a possible reconstruction function which describes a de-Sitter universe is given by

$$f(T,B)\;=\;2(\kappa^2\rho_0+2K)+f_0e^{\frac{B}{18H_0^2}}+\tilde{f}_0e^{\frac{T}{12H_0^2}}\,, \tag{68}$$

which of course is a constant function. Here, f_0 and $\tilde{f}_0$ are integration constants. In case of de-Sitter model, it can found that GSLT is trivially satisfied.

C. ΛCDM reconstruction

Here, the reconstruction of the $f(T,B)$ function for a ΛCDM cosmological evolution will be discussed in the absence of any cosmological constant term in the modified Einstein field equations. This model was firstly formulated by Elizalde et al. [64] in $f(R,\mathcal{G})$ modified theory of gravity. The cosmological effects of the cosmological constant term in the concordance model is exactly replaced by the modification introduced by $f(T,B)$ function with respect to the usual Einstein-Hilbert Lagrangian.

For simplicity, instead of working with all the variables depending on the cosmic time t, the e-folding parameter defined as $N = \ln(a/a_0)$ will be used. By using $a(t) = a_0/(1+z)$, the e-folding parameter can be also written depending on the redshift function z as $N = -\ln(1+z) = \ln(1/(1+z))$. In terms of this variable, one can express $a(t)$, $H(t)$ and time derivatives as

$$a = a_0 e^N, \quad H = \frac{\dot{a}}{a} = \frac{dN}{dt}, \quad \frac{d}{dt} = H\frac{d}{dN}.$$

Therefore, one can rewrite equation (13) in terms of N, yielding

$$-3H^2(3f_B + 2f_T) + 18H\Big[(H^2H'' + HH'^2 + 6H^2H')f_{BB} + 2H^2H'f_{BT}\Big] - 3HH'f_B + \frac{1}{2}f(T,B) = \kappa^2\rho_{\rm m}(t). \quad (69)$$

Here, primes denote differentiation with respect to the e-folding N. Additionally, in term of the e-folding, the scalar torsion and the boundary term are $T = 6H^2$ and $B = 6H(3H + H')$ respectively. Now, for convenience, we introduce a new variable $g = H^2$ making that the above equation becomes

$$-\frac{3}{2}(g' + 6g)\,f_B + 18gg'f_{TB} + 9gf_{BB}\,(g'' + 6g') - 6gf_T + \frac{1}{2}f(T,B) = \kappa^2\rho_{\rm m}(t). \quad (70)$$

It is easily to compute that the torsion scalar and the boundary term written in this variable are $T = 6g$ and $B = 3(g' + 6g)$ respectively. Now, it will be also assumed that the function $f(T,B)$ is separable as Eq. (59). Using these assumptions, the above equation becomes

$$\begin{aligned} -\frac{3}{2}(g' + 6g)\,f_{1,B}(B) + 9gf_{1,BB}(B)\,(g'' + 6g') + \frac{1}{2}f_1(B) = \\ \kappa^2\rho_{\rm m}(t) - \frac{1}{2}f_2(T) + 6gf_{2,T}(T). \end{aligned} \quad (71)$$

Let us now reconstruct the ΛCDM model whose function $g = g(N)$ is given by [64]

$$g = H_0^2 + le^{-3N}, \quad l = \frac{\kappa^2\rho_0 a_0^{-3}}{3}. \quad (72)$$

In this model, the e-folding can be expressed depending on the boundary term and the torsion scalar as follows

$$N = \frac{1}{3}\log\left(\frac{9l}{B - 18H_0^2}\right) = \frac{1}{3}\log\left(-\frac{6l}{6H_0^2 - T}\right). \quad (73)$$

Therefore, one can rewrite Eq. (71) as follows

$$2\left(27BH_0^2 - 162H_0^4 - B^2\right)f_{1,BB}(B) - Bf_{1,B}(B) + f_1(B) = K, \quad (74)$$

$$\kappa^2\rho_0\left(\frac{6H_0^2 - T}{-6la_0^3}\right)^{w+1} - \frac{1}{2}f_2(T) + Tf_{2,T}(T) = \frac{K}{2}, \quad (75)$$

where K is a constant since the r.h.s. of (71) depends only on T and the l.h.s. only on B. Note that the energy density can be expressed depending on T or B so that, the above equations are one of the possible options to reconstruct a ΛCDM Universe. Thus, by solving the above equations, one gets that one way to reconstruct ΛCDM is by taking the following functions,

$$f_1(B) = \frac{C_2\left(3H_0\sqrt{B - 9H_0^2} - B\arctan\left(\frac{\sqrt{B-9H_0^2}}{3H_0}\right)\right)}{54H_0^3} + BC_1 + K, \quad (76)$$

$$f_2(T) = K + C_3\sqrt{T} + \frac{\kappa^2\rho_0}{3}H_0^{2w}(a_0^3 l)^{-(w+1)}\left(6H_0^2\,{}_2F_1\left(-\frac{1}{2}, -w; \frac{1}{2}; \frac{T}{6H_0^2}\right) + T\,{}_2F_1\left(\frac{1}{2}, -w; \frac{3}{2}; \frac{T}{6H_0^2}\right)\right), \quad (77)$$

where $H_0 \neq 0$ and for the case where $H_0 = 0$ we find

$$f_1(B) = BC_1 + \frac{C_2}{\sqrt{B}} + K\,, \tag{78}$$

$$f_2(T) = -\frac{2^{-w}\kappa^2\rho_0}{2w+1}\left(\frac{T}{3a_0^3 l}\right)^{w+1} + C_3\sqrt{T} - K\,. \tag{79}$$

Here, C_1, C_2 and C_3 are constants and ${}_2F_1$ represents the hypergeometric function of the second kind. The case where $H_0 = 0$ represents a power-law solution with $h = 2/3$. The above solution is consistent with Eqs. (62) and (63) in that limit. Note that the case $T = 6H_0^2, B = 18H_0^2$ which represents de-Sitter universes can not be recovered directly from the above equations. However, these models can be recovered directly from (71) by imposing $g = H_0^2$ (with $l = 0$) which actually gives us the same result obtained in the previous section (see Eq. (68)). This issue comes from the fact that one expresses the e-folding depending on B or T, one needs to assume $l \neq 0$. The same issue can be seen in Sec. 2.1 in [26].

For the above model (76)-(77), the validity constraint for GSLT takes the following form

$$\frac{3(g-H_0^2)^2}{8g^2(la_0^3)^{1+w}}\left((la_0^3)^{1+w}\frac{\sqrt{6g}}{C_3} + 2H_0^{2w}\kappa^2\left[\left(1-\frac{g^2}{H_0^2}\right)^w\right.\right. \tag{80}$$

$$\times \quad \left.\left.(g^2-H_0^2) + H_0^2 2F_1\left(-\frac{1}{2},-w;\frac{1}{2};\frac{g^2}{H_0^2}\right) + g2F_1\left(\frac{1}{2},-w;\frac{3}{2};\frac{g^2}{H_0^2}\right)\right]\rho_0\right) \geq 0\,.$$

Fig. 2 shows the validity of GSLT (above equation) as a function of redshift z for the specific case where $C_3 = 0.1$. It can be seen that for this case, expression (80) is always positive ensuring that GSLT is valid for any redshift. (see Figure 2).

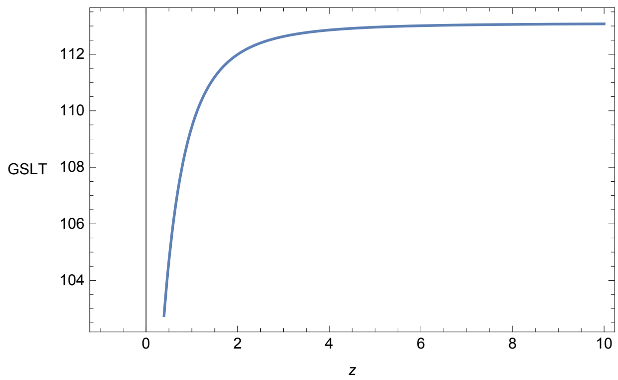

FIG. 2: Plot of GSLT (defined in Eq. (80)) versus the redshift function z for ΛCDM model. Here, $C_3 = 0.1$ is assumed.

D. Phantom behaviour

As an another example, let us now assume that the Hubble parameter and the energy density are [64]

$$\sqrt{g} = H = h_0 e^{mN}\,,\ \rho_{\rm m} = b_0 + b_1 e^{2mN} + \frac{96(m+1)}{5}b_2 e^{5mN}\,, \tag{81}$$

where h_0, b_0, b_1, b_2 and m are constants. Cosmologically speaking, this model represents a super accelerated universe phase with a phantom regime $w_{\rm eff} < -1$, making that the universe could end in a singularity. If one uses Eq (71), one can split the cosmological equations depending only on T and B as follows ($m \neq -3$)

$$2B^2 m f_{1,BB}(B) - B(m+3)f_{1,B}(B) + (m+3)f_1(B) = K(m+3)\,, \tag{82}$$

$$\kappa^2\left(b_0 + \frac{b_1 T}{6h_0^2} + \frac{4}{15}\sqrt{\frac{2}{3}}b_2(m+1)\left(\frac{T}{h_0^2}\right)^{5/2}\right) + Tf_{2,T}(T) - \frac{f_2(T)}{2} = \frac{K}{2}\,, \tag{83}$$

where K is a constant. Thus, one of the possible representation which produces a super accelerated universe is given by the following functions,

$$f_1(B) = K + C_1 B^{\frac{3+m}{2m}} + C_2 B\,, \tag{84}$$

$$f_2(T) = 2b_0\kappa^2 - K + C_3\sqrt{T} - \frac{b_1\kappa^2 T}{3h_0^2} - \frac{2\sqrt{\frac{2}{3}}b_2\kappa^2(m+1)T^{5/2}}{15h_0^5}\,, \tag{85}$$

where C_1, C_2 and C_3 are integration constants. For the above model, let us now explore the constraint for the validity of GSLT which becomes of the form

$$\frac{m^2}{2}\left(-\frac{2\kappa^2 b_1}{3h_0^2} - \frac{8(1+m)\kappa^2 b_2 g^3}{h_0^8} + \frac{C_3}{\sqrt{6g}}\right) \geq 0\,. \tag{86}$$

One can notice that GSLT is always satisfied for any values of C_3 and h_0 if both b_1 and b_2 are assigned negative values. As an example, in Fig. 3, the validity region of GSLT is depicted for the parameters b_1, b_2 and z, where it was set $h_0 = 0.1$, $m = 1$, and $C_3 = 1$.

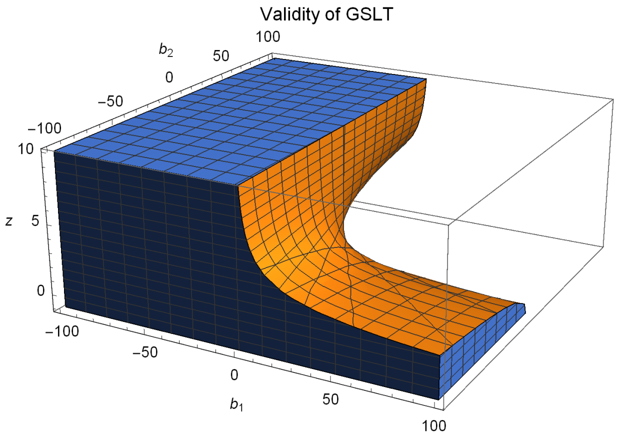

FIG. 3: Plot of the validity of GSLT for the phantom dominated model (defined in (86)). Here, $h_0 = 0.1$, $m = 1$, and $C_3 = 1$ are set.

E. Reconstruction method in $f(T,B) = -T + F(B)$ cosmology

In this section, the specific case where the function takes the form $f(T,B) = -T + F(B)$ will be studied, which is similar to models of the form $f(R) = R + F(R)$ and $f(T) = -T + f(T)$ studied in $f(R)$ and $f(T)$ gravity respectively [65]. This theory is equivalent to consider a teleparallel background (or GR) plus an additional function which depends on the boundary term which can be also understood as $F(B) = F(T+R)$. It is important to mention that even though the case $f(T,B) = f_1(B) + f_2(T)$ studied in the previous section is more general and in principle could contain the case $f(T) = -T + F(B)$, one might get a different reconstruction solution. The later comes from the fact that the case $f(T) = -T + F(B)$ is a very specific choice of the function and also that all the functions found before in Sec. IV are one of the possible choices for reconstructing the corresponding models. Moreover, due to the mathematics technique employed before, i.e., the method of separation of variables, if one tries to recover the case $f(T) = -T + F(B)$ from the solution, one might not get the same answer. As an example, for the power-law case is not possible to recover $f(T,B) = -T + F(B)$ unless we restrict our model with $C_1 = 0$ and $h = \frac{2}{3(w+1)}$ which is only a kind of power-law model (see Eqs. (62) and (63)). Hence, it is interesting and important to also study if it is possible to reconstruct these cosmological models within this particular theory.
In this model, the 00 field equation (13) becomes

$$-3H^2(3F_B - 2) + 3H\dot{F}_B - 3\dot{H}F_B + \frac{1}{2}(F(B) - 6H^2) = \kappa^2\rho_{\rm m}(t)\,, \tag{87}$$

where the energy density is given by (55). Equivalently, from (69), it is easily to rewrite the above equation in term of the e-folding,

$$-3H^2(3F_B - 2) + 18H\Big[(H^2H'' + HH'^2 + 6H^2H')F_{BB}\Big] - 3HH'F_B + \frac{1}{2}(-6H^2 + F(B)) = \kappa^2\rho_{\rm m}(t)\,. \tag{88}$$

Let us now perform a reconstruction method for all the same models studied in Secs. IV A-IV D.
For a power-law cosmology described in Sec. IV A, Eq. (87) can be written as follows,

$$\frac{B\left(h-2BF_{BB}(B)\right)}{6h-2}-\frac{1}{2}BF_B(B)+\frac{F(B)}{2}=\kappa^2\rho_0\left(\frac{6h(3h-1)}{Bt_0^2}\right)^{-\frac{3}{2}h(w+1)}\,, \tag{89}$$

which can be directly solved, yielding the following solution

$$\begin{aligned} F(B) \;&=\; C_1B^{\frac{1-3h}{2}}+B\left(C_2-\frac{2h}{(3h+1)^2}\right)+\frac{hB\log(B)}{3h+1}+\frac{Bh\log(9h+3)}{3h+1} \\ &\quad -\frac{\kappa^2\rho_0\left((3h-1)^{1-\frac{3}{2}h(w+1)}2^{2-\frac{3}{2}h(w+1)}\right)\left(\frac{Bt_0^2}{3h}\right)^{\frac{3}{2}h(w+1)}}{(3h(w+1)-2)(3h(w+2)-1)}\,, \end{aligned} \tag{90}$$

where C_1 and C_2 are integration constants.
Now, for a de-Sitter reconstruction, the scale factor behaves as $a(t)=a_0e^{H_0t}$, then $B=18H_0^2$ and hence from (87) we directly find that the function takes the following form,

$$F(B) \;=\; C_1e^{\frac{B}{18H_0^2}}-2\left(3H_0^2-\kappa^2\rho_0\right)\,. \tag{91}$$

Here, C_1 is an integration constant. Let us now reconstruct a ΛCDM universe where $g=H_0^2+le^{-3N}$. In this theory, Eq. (71) becomes

$$\begin{aligned} -\left(B-18H_0^2\right)\left(B-9H_0^2\right)F_{BB}(B)-\frac{1}{2}BF_B(B)+\frac{F(B)}{2}+\frac{1}{3}\left(B-9H_0^2\right)= \\ 3^{-2(w+1)}\kappa^2\rho_0\left(a_0\sqrt[3]{\frac{l}{B-18H_0^2}}\right)^{-3(w+1)}\,, \end{aligned} \tag{92}$$

where we have used Eq. (73) to express all in term of the boundary term B. The above equation is difficult to solve analytically for all values of w, so that for simplicity we assume the cold dust case $w=0$, which gives us

$$\begin{aligned} F(B) \;&=\; +BC_1+\frac{\log\left(B-18H_0^2\right)\left(6a_0^3Bl-2B\kappa^2\rho_0\right)+6a_0^3l\left(B-9H_0^2\right)+\kappa^2\rho_0\left(B-36H_0^2\right)}{27a_0^3l} \\ &\quad +\frac{C_2\left(3H_0\sqrt{B-9H_0^2}-B\arctan\left(\frac{\sqrt{B-9H_0^2}}{3H_0}\right)\right)}{54H_0^3}\,, \end{aligned} \tag{93}$$

where C_1 and C_2 are integration constants.
Finally, let us reconstruct the phantom behaviour scenario where the energy density and the Hubble parameter are given by Eq. (81). By using that $N=\frac{1}{2m}\ln(B/(6h_0^2(3+m)))$, Eq. (88) becomes

$$\begin{aligned} \frac{2B^2mF_{BB}(B)}{m+3}-BF_B(B)+F(B)+\frac{B}{m+3}= \\ 2\kappa^2\left(\frac{B\left(8\sqrt{6}B^{3/2}b_2(m+1)+15b_1h_0^3(m+3)^{3/2}\right)}{90h_0^5(m+3)^{5/2}}+b_0\right)\,, \end{aligned} \tag{94}$$

and then the corresponding reconstruction function is

$$\begin{aligned} F(B) \;&=\; -\frac{B\left((m-3)\log(B)\left(3h_0^2-b_1\kappa^2\right)+2b_1\kappa^2m-3h_0^2\left(C_2m^2-6C_2m+9C_2+4m\right)\right)}{3h_0^2(m-3)^2} \\ &\quad +\frac{16\sqrt{\frac{2}{3}}B^{5/2}b_2\kappa^2(m+1)}{45h_0^5(m+3)^{3/2}(4m-3)}+C_1B^{\frac{m+3}{2m}}+2b_0\kappa^2\,, \end{aligned} \tag{95}$$

where C_1 and C_2 are integration constants. Let us stress here that the final expressions for the function $F(B)$ becomes less complicated that in $f(T,B)$ gravity. For example, in the reconstruction of ΛCDM, for the $f(T,B)$ model, it was found that the expression contains hypergeometric expressions (see Eq. 78) where in the $-T+F(B)$ reconstruction (see Eq. 93), the expression does not have such complicated terms. This is another reason why we studied the reconstruction method in a general setting and then in a specific theory such as $-T+F(B)$ gravity.

V. PERTURBATIONS AND STABILITY

In this section, we are interested to establish the stability conditions for cosmological solutions against linear isotropic homogeneous perturbations in $f(T,B)$ theory of gravity. The perturbation equations in FLRW universe will be formulated for a general framework and then de-Sitter and power-law solutions will be studied. We assume a general solution

$$H(t) = H_j(t)\,, \tag{96}$$

which satisfies the basic equations of motion of FLRW universe in $f(T,B)$ theory of gravity. In term of above solution, the torsion scalar and boundary B, can be written as follows

$$T_j \;=\; 6H_j^2(t)\,, \tag{97}$$
$$B_j \;=\; 6\dot{H}_j(t) + 18{H^2}_j(t)\,. \tag{98}$$

If one considers particular model of $f(T,B)$ that can generate solution (96), then, the following equations must be satisfied

$$-3H_j^2\Big(3f_B^j + 2f_T^j\Big) + 3H_j\dot{f}_B^j - 3\dot{H}_j f_B^j + \frac{1}{2}f^j = \kappa^2\rho_{\mathrm{m}j}\,, \tag{99}$$
$$\dot{\rho}_{\mathrm{m}j} + 3H_j(1+w)\rho_{\mathrm{m}j} = 0\,. \tag{100}$$

Now we define the perturbation for Hubble parameter and energy density as follows

$$H(t) = H_j(t)\Big(1+\delta(t)\Big)\,, \qquad \rho_m(t) = \rho_{\mathrm{m}j}\Big(1+\delta_{\mathrm{m}}(t)\Big)\,. \tag{101}$$

Here, our purpose is to make the perturbation analysis about the solution $H(t) = H_j(t)$, so that function $f(T,B)$ can be expressed in the powers of T and B as

$$f(T,B) = f^j + f_T^j(T-T_j) + f_B^j(B-B_j) + \mathcal{O}^2\,, \tag{102}$$

where the superscript j means the values of $f(T,B)$ and its derivatives are evaluated at $T = T_j$ and $B = B_j$. The term $\mathcal{O}^2$ includes all the terms which have power-square and higher powers of T and B, although we shall only consider the linear terms of the defined perturbation. Thus, by replacing Eqs. (101) and (102) in the FLRW equation (99) and in the continuity equation (100), we get the perturbation equations in terms of $\delta(t)$ and $\delta_{\mathrm{m}}(t)$, (in the linear approximation) in the form of the following differential equations

$$c_2\ddot{\delta}(t) + c_1\dot{\delta}(t) + c_0\delta(t) = c_{\mathrm{m}}\delta_{\mathrm{m}}(t)\,, \tag{103}$$
$$\dot{\delta_{\mathrm{m}}}(t) + 3H_j\delta(t) = 0\,. \tag{104}$$

The coefficients $c_{0,1,2,\mathrm{m}}$, are expressed in the Appendix (see (A.1)). These coefficients depend explicitly on $f(T,B)$ and its derivatives evaluated at background solutions $H = H_j$. In general it is not easy to solve the above equations analytically. In the coming sections we shall present some particular models for the solution of above equations.

A. Stability of de-Sitter Solution

Consider the de-Sitter solution with $H_j = H_0$ and $\rho_0 = 0$, then the perturbed equation takes the following form,

$$\Big(-18{H^2}_0 f_{TB}^0 T_0 + 324H_0^4 f_{BB}^0 - 36H_0^2 f_B^0 - 12H_0^2 f_{TT}^0 T_0 + 216H_0^4 f_{TB}^0$$
$$-24H_0^2 f_T^0 - 24H_0^2 f_B\Big)\delta(t) + \Big(-54H_0^3 f_{BB}^0 - 6H_0 f_B^0\Big)\dot{\delta}(t) + (-18H_0^2 f_{BB}^0)\ddot{\delta}(t) = 0\,. \tag{105}$$

Using the $f(T,B)$ model formulated in de-Sitter reconstruction, i.e., Eq. (68), one gets the following solution for $\delta(t)$

$$\delta(t) = C_1 e^{\mu_+ t} + C_2 e^{\mu_- t}\,, \tag{106}$$

where C_1 and C_2 are integration constants and

$$\mu_\pm = \frac{3H_0}{2f_0}\left(-3f_0 \pm \sqrt{f_0\left(f_0 - 28\sqrt{e}\tilde{f}_0\right)}\right) . \tag{107}$$

Here, f_0 and $\tilde{f}_0$ are the constants appearing in Eq. (68). Note that $\sqrt{e} = e^{1/2}$ is referring to the exponential e and not the determinant of the tetrad. The growth of the perturbation will depend both upon the overall sign of the parameters $\mu_\pm$ appearing in the expression (107) and also upon the real and imaginary character of the square root. Thus four different cases can be distinguished:

- $f_0 < 0$ and $f_0 > 28\sqrt{e}\tilde{f}_0$ with $\tilde{f}_0 < 0$, this implies that solutions are complex and $\Re(\mu_\pm) < 0$, thus solutions behave as a damped oscillator of decreasing amplitude. Hence, solutions are stable.
- $f_0 > 0$ and $f_0 < 28\sqrt{e}\tilde{f}_0$ with $\tilde{f}_0 > 0$, this implies that solutions are complex and $\Re(\mu_\pm) < 0$, thus solutions behave as a damped oscillator of decreasing amplitude. Hence, solutions are stable.
- $0 < 28\sqrt{e}\tilde{f}_0/f_0 < 1$ with $f_0 > 0$ and $\tilde{f}_0 > 0$, or $f_0 < 0$ and $\tilde{f}_0 < 0$, then both $\mu_\pm$ are real and $\mu_\pm < 0$, hence solutions are stable.

B. Stability of Power Law Solutions

In this section, the stability of the power-law solution described in (56) will be studied. For $0 < h < 1$, we have decelerated universe which may refer to dust dominated ($h = 2/3$) or radiation dominated ($h = 1/2$), while $h > 1$ results in accelerating picture of the universe. Here, we explore the stability of power law solutions for matter dominated, radiation dominated and late time accelerated eras.

- For matter dominated era with $h = 2/3$, and $w = 0$, Eqs. (62) and (63) result in

$$f(T,B) = \frac{C_2}{\sqrt{B}} + C_3 B + C_1\sqrt{T} - \frac{3}{4}\rho_0\kappa^2 T . \tag{108}$$

By substituting the above model in equations (103) and (104), one can find the required perturbation equations for matter dominated a power-law model. Here, we employ the numerical approach to solve these equations and present the evolution of perturbation parameters $\delta(t)$ and $\delta_{\rm m}(t)$. In this study we set $H_0 = 67.3$, $\Omega_{\rm m} = 0.23$, $C_2 = -0.2$, $C_1 = C_3 = 0.1$ and $\kappa^2 = 1$. Fig. 4 shows the oscillating behavior of $\delta(t)$ and $\delta_{\rm m}(t)$, however these do not decay in future evolution.

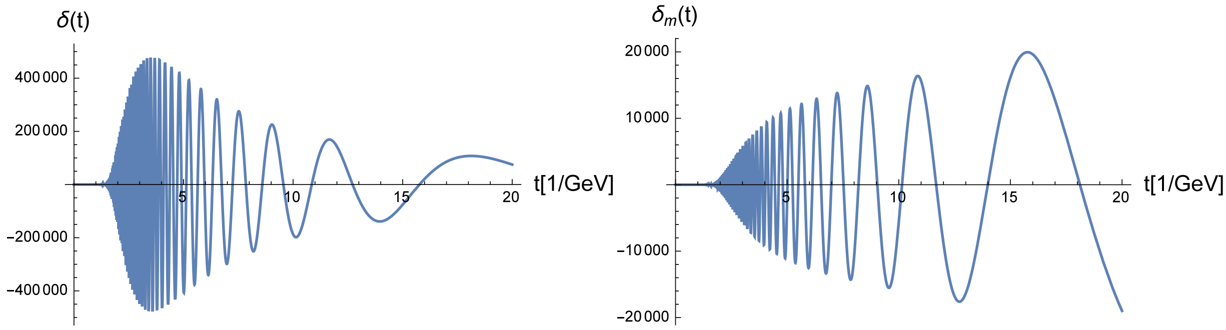

FIG. 4: Evolution of $\delta(t)$ and $\delta_{\rm m}(t)$ versus time t. Herein, we set the initial conditions $\delta'(1) = 0.2$, $\delta(1) = 0.1$ and $\delta_{\rm m}(1) = 0.1$. The figures show the evolution of perturbation parameters $\delta(t)$ and $\delta_{\rm m}(t)$ for the matter dominated solutions.

- For radiation dominated era with $h = 1/2$, and $w = 1/3$, Eqs. (62) and (63) result in

$$f(T,B) = \frac{C_2}{B^{\frac{1}{4}}} + C_3 B + C_1\sqrt{T} - \frac{4}{3}\kappa^2\rho_0 T . \tag{109}$$

One can substitute the model (109) in Eqs. (103) and (104) to find the required perturbation equations for radiation dominated power law model. The numerical scheme and the evolution of $\delta(t)$ and $\delta_{\rm m}(t)$ is depicted in Fig. 5. The figures show an oscillating behavior of $\delta(t)$ and $\delta_{\rm m}(t)$, however the oscillations of $\delta(t)$ and $\delta_{\rm m}(t)$ do not decay in future. Hence solutions are unstable as full perturbation around a cosmological solution is fully determined by the matter perturbations. This result is similar to matter dominated era with $h = 2/3$, and $w = 0$ which is shown in Fig. 4.

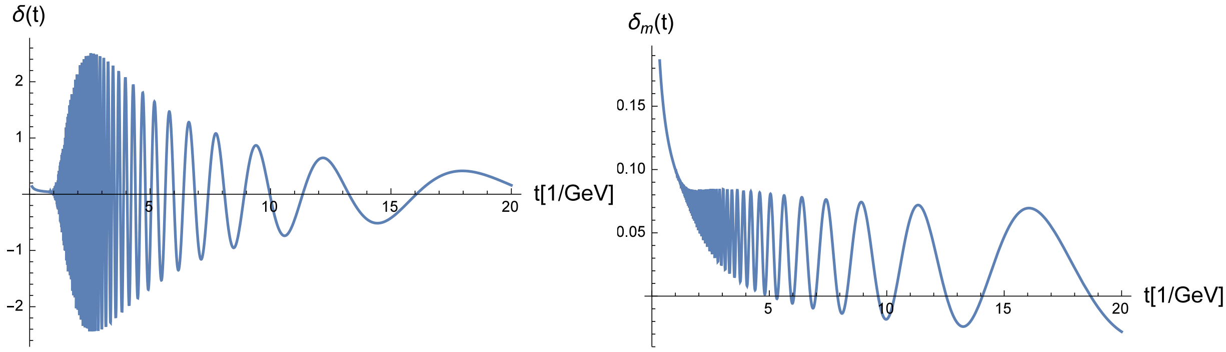

FIG. 5: Evolution of $\delta(t)$ and $\delta_{\rm m}(t)$ versus time t. Herein, we set the initial conditions $\delta'(1) = 0.2$, $\delta(1) = 0.1$ and $\delta_{\rm m}(1) = 0.1$. This Figure shows the evolution of perturbation parameters $\delta(t)$ and $\delta_{\rm m}(t)$ for the radiation dominated solutions.

- For the choice of $h > 1$, the universe is expanding. In our case, we set $h = 2$ with $w = -0.5$, so that the corresponding power law model is given by

$$f(T,B) = \frac{C_2}{B^{\frac{5}{2}}} + C_3 B + C_1\sqrt{T} - \frac{\kappa^2\rho_0 T^{\frac{3}{2}}}{48\sqrt{6}}\,. \tag{110}$$

Again following a similar approach, we show the results in Fig. 6. Here, we set $C_1 = 0.1$ and $C_2 = -10$, $C_3 = -1$. For this case it can be seen that $\delta(t)$ and $\delta_{\rm m}(t)$ decay in later times so that power-law model (110) for $h > 1$ is stable to some extent.

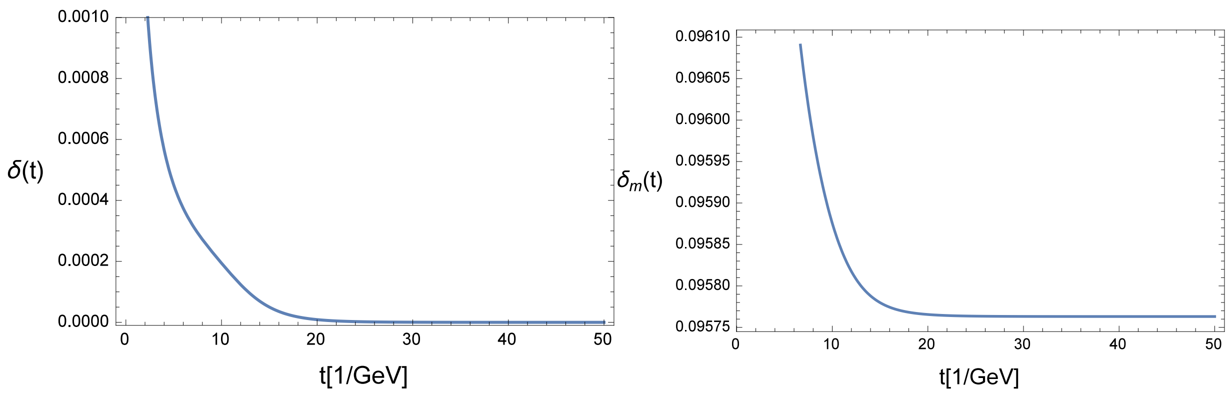

FIG. 6: Evolution of $\delta(t)$ and $\delta_{\rm m}(t)$ versus time t. Herein, we set the initial conditions $\delta'(1) = 0.2$, $\delta(1) = 0.1$ and $\delta_{\rm m}(1) = 0.1$. This Figure shows the evolution of perturbation parameters $\delta(t)$ and $\delta_{\rm m}(t)$ for the accelerated expansion solutions.

VI. CONCLUSIONS

Over the last years, teleparallel theories of gravity and its modifications have attained significant attention to address various issues in cosmology. These theories lie in a globally flat manifold endorsed with torsion. It is well-known that GR has an equivalent teleparallel representation (TEGR) based on the torsion (and tetrads) instead of curvature (and metric). In this perspective, different modified teleparallel theories have been proposed. The first one, is the so-called, $f(T)$ gravity, a natural generalisation of the TEGR action by changing the torsion scalar $T \to f(T)$ in the action. This approach is analogous with $f(R)$ gravity in the metric counterpart. These two theories have been very successful describing the cosmological behaviour of the universe. With the aim to unify both $f(R)$ and $f(T)$ gravity and see how these theories are connected, it was formulated a modified teleparallel theory of gravity named as $f(T, B)$ theory which under suitable limits can recover $f(T)$ or $f(R)$ gravity [13]. In this work, we have explored different cosmological features in $f(T, B)$ gravity as the establishment of laws of thermodynamics, reconstruction of some cosmological models and stability of some models corresponding to linear homogeneous perturbations.

In Sec. III, we have shown that the modified flat FLRW equations in this theory can be cast to the form of first law of thermodynamics, $\tilde{T}_h dS_h + \tilde{T}_h d\bar{S} = -dE + WdV$. Here, $d\bar{S}$ is the additional entropy term due to non-equilibrium thermodynamics which may be produced as a result of Lagrangian dependence both on the torsion scalar and the boundary term. The entropy production term in $f(T, B)$ gravity is more general and can reproduce the corresponding factor in $f(-T + B) = f(R)$ [53] and $f(T)$ [50] theories. It is worth mentioning that no such term is present in GR, Gauss-Bonnet gravity [38], Lovelock gravity [41, 57] and braneworld gravity [42, 43]. Moreover, in case of $f(R)$, $f(T)$ and scalar tensor theories different schemes have been suggested to avoid the auxiliary term in first law of thermodynamics [50, 66]. Bamba et al. [50] show that one can redefine the energy momentum tensor contributed from the modified theories so that the conservation equation is truly satisfied and hence results in omission of entropy production term. In case of $f(T, B)$ gravity we find that one can establish the equilibrium description of thermodynamics (presented in Sec. III-B) and remove the additional entropy production term. We also establish the GSLT which is found to be valid for the phantom era of cosmos.

We reconstructed the gravitational action of this model and have done a brief analysis of validity of GSLT and stability of reconstructed models. Here, we have used a more comprehensive approach for cosmological reconstruction of $f(T, B)$ gravity in terms of e-folding representing different eras of the universe. We have studied some important cosmological solutions in the standard cosmological concordance model around a spatially flat FLRW background namely dS expansion, power laws and the scale factor solutions as provided for the ΛCDM model and phantom dominated model. One can employ the reconstructed $f(T, B)$ to explore cosmic evolution in more consistent way. Additionally we have explored the specific case $f(T, B) - T + F(B)$, which is a GR background plus an additional function that depends on the boundary term. The reconstruction scheme is carried out for this specific case and we have also obtained the corresponding function $F(B)$ which mimics different cosmological models. One can employ the reconstructed $f(T, B)$ models to explore cosmic evolution in more consistent way.

We also examined the validity of GSLT: for the power solutions we find the validity constraints in case of specific model (62)-(63) which predicts a late-time accelerating universe. In this case we restrict the values of integration constant $C_1 \geq 0$ and vary values of h to see evolution of GSLT. In case of de-Sitter model, GSLT is trivially satisfied and for ΛCDM reconstruction (76)-(77), one needs to fix $C_3 = 0.1$. In case of phantom dominated model, GSLT is valid for negative values of parameters b_1 and b_2 (see Eq. (81)) together with all values of other parameters h_0, m and C_3. The study of stability/instability of various forms of Lagrangian is a useful tool to classify the modified theories on physical grounds. The linearised perturbed equations were derived by implementing the perturbation for the Hubble parameter and energy density. We analyzed the stability of de-Sitter and power law solutions, finding that de-Sitter model is found to be stable with some constraints on model parameters whereas as power law solution is stable only for $h > 1$ representing expanding behavior of universe. Hence, we conclude that power law solution is found to be more feasible as it validates GSLT and is stable against homogeneous perturbation.

It is stated that stability of linear homogeneous perturbations does not guarantee the stability of the reconstructed $f(T, B)$ models. In future project we will develop complete set of differential equations for the matter density perturbations and analyze the growth index to constrain the viable models.

Appendix A

$$
\begin{aligned}
c_0 &= \Big(-18{H^2}_j f^j_{TB} T_j + 324 H_j^4 f^j_{BB} + 54 H_j^2 \dot{H}_j f^j_{BB} - 36 H_j^2 f^j_B - 12 H_j^2 f^j_{TT} T_j \\
&\quad + 216 H_j^4 f^j_{TB} + 36 H_j^2 \dot{H}_j f^j_{TB} - 24 H_j^2 f^j_T + 6 H_j \dot{f}^j_{TB} T_j + 6 H_j f^j_{TB} \dot{T}_j - 108 H_j^3 \dot{f}^j_{BB} \\
&\quad - 18 \dot{H}_j H_j \dot{f}^j_{BB} - 216 H_j^2 \dot{H}_j f^j_{BB} - 18 H_j \ddot{H}_j f^j_{BB} + 6 H_j {\dot{f}^j}_B - 6 \dot{H}_j T_j f^j_{TB} \\
&\quad + 108 \dot{H}_j H_j^2 f^j_{BB} + 18 \dot{H}_j^2 f^j_{BB} - 9 \dot{H}_j f^j_B + f^j_B T_j - 18 f^j_B {H_j}^2 - 3 f^j_B \dot{H}_j \Big) \qquad \text{(A.1)} \\
c_1 &= 54 H_j^3 f^j_{BB} + 36 H_j^3 f^j_{TB} + 6 H_j f^j_{TB} T_j - 18 H_j^2 \dot{f}^j_{BB} - 108 H_j^3 f^j_{BB} \\
&\quad - 6 H_j f^j_B \,, \qquad \text{(A.2)} \\
c_2 &= -18 {H_j}^2 f^j_{BB} \,, \qquad \text{(A.3)} \\
c_{\mathrm{m}} &= \kappa^2 \rho_{\mathrm{m}} \,. \qquad \text{(A.4)}
\end{aligned}
$$

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