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Robust variable structure observer design for nonlinear large-scale systems with nonlinear interconnections*

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Abstract: In this paper, a variable structure observer is designed for a class of nonlinear large-scale interconnected systems in the presence of uncertainties and nonlinear interconnections. The modern geometric approach is used to explore system structure and a transformation is employed to facilitate the observer design. Based on the Lyapunov direct method, a set of conditions are developed such that the proposed variable structure systems can be used to estimate the states of the original interconnected systems asymptotically. The internal dynamical structure of the isolated nominal subsystems as well as the structure of the uncertainties are employed to reduce the conservatism. The bounds on the uncertainties are nonlinear and are employed in the observer design to reject the effect of the uncertainties. A numerical example is presented to illustrate the approach and the simulation results show that the proposed approach is effective.

Keywords: large-scale interconnected systems, variable structure observer, Lyapunov direct method, uncertainties

1. Introduction

The development of advanced technologies has produced many complex systems. An important class of complex systems, which is frequently called a system of systems or large-scale system, can usually be expressed by sets of lower-order ordinary differential equations which are linked through interconnections. Such models are typically called large-scale interconnected systems (see, e.g. Bakule (2008), Mahmoud (2011), Yan *et al.*(2003) and Yan *et al.*(2013)). Large-scale interconnected systems widely exist in practice, for example, power networks, ecological systems, transportation networks, biological systems and information technology networks (Lunze (1992) and Mahmoud (2011)). Increasing requirements for system performance have resulted in increasing complexity within system modelling and it becomes of interest to consider nonlinear large-scale interconnected systems. Such models are then used for controller design. In order to obtain good performance levels, a controller may benefit from knowledge of all the system states. This state information may be difficult or expensive to obtain and it becomes of interest to design an observer to estimate all the system states using only the subset

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of information available from the measured and known input and output of the system.

Large-scale interconnected systems have been studied since the 1970s (Sandell *et al.*(1978)). Early work focussed on linear systems. Subsequent results used decentralised control frameworks for nonlinear large-scale interconnected systems. In much of this work, however, it is assumed that all the system state variables are available for use by the controller (Bakule (2008), Guoliang *et al.*(2014), Mahmoud (2011) and Wu (2005)). However, this may be limiting in practice as only a subset of state variables may be available/measureable. It becomes of interest to establish observers to estimate the system states and then use the estimated states to replace the true system states in order to implement state feedback decentralised controllers. It is also the case that observer design has been heavily applied for fault detection and isolation (Reppa *et al.*(2014) and Yan & Edwards (2008)). This further motivates the study of observer design for nonlinear large scale interconnected systems.

The concept of an observer was first introduced by Luenberger (1964) where the difference between the output measurements from the actual plant and the output measurements of a corresponding dynamical model were used to develop an injection signal to force the resulting output error to zero. Later the approach was extended to nonlinear systems and an extended Luenberger observer for nonlinear systems is proposed in Zeitz (1987) where uncertainties are not considered. It should be noted that many approaches have been developed for observer design such as the sliding mode observer approach in Yan *et al.*(2013), the adaptive observer in Wu (2009) and an error linearisation approach in Xia & Gao (1989). However, results concerning observer design for interconnected systems are very few when compared with the corresponding results available on controller design for interconnected systems.

Sliding mode techniques have been used to design observers for nonlinear interconnected power systems in Modarres *et al.*(2012). In Li *et al.*(2015) state estimation and sliding mode control for a special class of stochastic dynamic systems which is semi Markovian jump systems is presented. The authors designed a state observer to generate the estimate of unmeasured state components, and then synthesize a sliding mode control law based on the state estimates. Wang, Y., & Fei, J. (2015) discussed the position regulation problem of permanent magnet synchronous motor (PMSM) servo system based on adaptive fuzzy sliding mode control (AFSMC) method. They used adaptive method to estimate the upper bound of the approximation error between the equivalent control law and the fuzzy controller are utilized in the paper. An adaptive observer is designed for a class of interconnected systems in Wu (2009) in which it is required that the isolated nominal subsystems are linear. Observer schemes for interconnected systems are proposed in Keliris *et al.*(2015), Reppa *et al.*(2014), Sharma & Aldeen (2011) and Yan & Edwards (2008) where the obtained results are unavoidably conservative as it is required that the designed observer can be used for certain fault detection and isolation problems. For example, it is required that the uncertainty can be decoupled with faults in Yan & Edwards (2008) and the considered system is not interconnected systems. Robust observer design is considered in Mohmoud (2012) for a class of linear large scale dynamical systems where it is required that the interconnections satisfy quadratic constraints. In Swarnakar (2007) a new decentralized control scheme which uses estimated states from a decentralised observer within a feedback controller is proposed. This uses a design framework based on linear matrix inequalities and is thus applicable for linear systems. A robust observer for nonlinear interconnected systems based on a constrained Lyapunov equation has been developed in Yan *et al.*(2003). A Proportional Integral observer is utilized for nonlinear interconnected systems for disturbance attenuation in Ghadami & Shafai (2011) and interconnected nonlinear dynamical systems are considered in Dashkoskiy & Naujok (2015) where the authors combine the advantages of input-to-state dynamical stability and use reduced order observers to obtain quantitative information about the state estimation error. This work does not, however, consider uncertainties. It should be noted that in all the existing work relating to observer design for large scale interconnected systems, it is required that either the isolated subsystems are linear or the interconnections are linear. Moreover, most of the designed observers are used for special purposes such as fault detection or stabilization and thus they impose specific requirements on the class of interconnected systems considered.

In this paper, a class of nonlinear interconnected systems with disturbances is considered where both the nominal isolated subsystems and interconnections are nonlinear. It is not required that either the nominal isolated subsystems or the interconnections are linearisable. A robust variable structure observer is established based on a simplified system structure by using Lyapunov analysis methodology. The structure of the internal dynamics, the structure of uncertainties and the bounds on uncertainties are fully used in the observer design to reduce the conservatism. These bounds are allowed to have a general nonlinear form. The observer states converge to the system states asymptotically. An example with simulation is given to demonstrate the proposed approach.

2. Preliminaries

Consider the single input single output nonlinear system

$$\dot{x}(t) = f(x) + g(x)u \tag{2.1}$$
$$y(t) = h(x) \tag{2.2}$$

where $x \in \Omega \subset R^n$ (Ω is a neighborhood of the origin), $y \in R$ and $u \in U \subset R$ (U is an admissible control set) are the state, output and input respectively, $f(x)$, $g(x) \in R^n$ are smooth vector fields defined in the domain Ω, and $h(x) \in R^m$ is a smooth vector in the domain Ω.

Firstly, recall some key elements of the geometric approach in Isidori (1995) which will be used in the later analysis. The notation used in this paper is the same as Isidori (1995) unless it is specifically defined.

Definition 1 (Isidori (1995)) System $(2.1)-(2.2)$ is said to have uniform relative degree r in the domain Ω if for any $x \in \Omega$,

(i) $L_g L_f^k h(x) = 0, \quad \text{for} \quad k = 1, 2, \cdots, r-1$

(ii) $L_g L_f^{r-1} h(x) \neq 0$

Now consider system $(2.1)-(2.2)$. It is assumed that system $(2.1)-(2.2)$ has uniform relative degree r in domain Ω. Construct a mapping $\phi : x \to z$ as follows:

$$\phi(\cdot) : \begin{cases} z_1 = h(x) \\ z_2 = L_f h(x) \\ \quad \vdots \\ z_r = L_f^{r-1} h(x) \\ z_{r+1} = \phi_{r+1} \\ \quad \vdots \\ z_n = \phi_n(x) \end{cases} \tag{2.3}$$

where $\phi(\cdot) = \text{col}(\phi_1(x), \phi_2(x), \cdots, \phi_n(x))$, $\phi_1(x) = h(x)$, $\phi_2(x) = L_f h(x), \cdots, \phi_r(x) = L_f^{r-1} h(x)$ and the functions $\phi_{r+1}(x), \cdots, \phi_n(x)$ need to be selected such that

$$L_g \phi_i(x) = 0, \qquad i = r+1, r+2, \cdots, N$$

and the Jacobian matrix

$$J_\phi := \frac{\partial \phi(x)}{\partial x}$$

is nonsingular in domain Ω.Then the mapping $\phi : x \to z$ forms a diffeomorphism in the domain Ω. For the sake of simplicity, let

$$\begin{aligned} \zeta &= \begin{bmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_r \end{bmatrix}^T := \begin{bmatrix} z_1 & z_2 & \cdots & z_r \end{bmatrix}^T \\ \eta &= \begin{bmatrix} \zeta_{r+1} & \zeta_{r+2} & \cdots & \zeta_n \end{bmatrix}^T := \begin{bmatrix} z_{r+1} & z_{r+2} & \cdots & z_n \end{bmatrix}^T \end{aligned}$$

Then, from Isidori (1995), it follows that in the new coordinates z, system $(2.1)-(2.2)$ can be described by

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r \\ \dot{\zeta}_r &= a(\zeta,\eta) + b(\zeta,\eta)u \\ \dot{\eta} &= q(\zeta,\eta) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} a(\zeta,\eta) &= L_f^r h(\phi^{-1}(\zeta,\eta)) \\ b(\zeta,\eta) &= L_g L_f^{r-1} h(\phi^{-1}(\zeta,\eta)) \end{aligned}$$

and

$$q(\zeta,\eta) = \begin{bmatrix} q_{r+1}(\zeta,\eta) \\ q_{r+2}(\zeta,\eta) \\ \vdots \\ q_n(\zeta,\eta) \end{bmatrix} = \begin{bmatrix} L_f\phi_{r+1}(\phi^{-1}(\zeta,\eta)) \\ L_f\phi_{r+2}(\phi^{-1}(\zeta,\eta)) \\ \vdots \\ L_f\phi_n(\phi^{-1}(\zeta,\eta)) \end{bmatrix}$$

It should be noted that the coordinate transformation (2.3) will be available if $\phi_i(x)$ are available for $i = r+1, \cdots, N$, and in this case, the system (2.4) can be obtained directly.

3. Large-Scale System Description and Problem Statement

Consider the nonlinear interconnected systems

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i + \Delta f_i(x_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} D_{ij}(x_j) \tag{3.1}$$

$$y_i(t) = h_i(x_i), \quad i = 1, 2, \cdots, N \tag{3.2}$$

where $x_i \in \Omega_i \subset R^{n_i}$ (Ω_i is a neighbourhood of the origin), $y_i \in R$ and $u_i \in U_i \subset R$ (U_i is an admissible control set) are the state, output and input of the i-th subsystem respectively, $f_i(x_i) \in R^{n_i}$ and $g_i(x_i) \in R^{n_i}$

are smooth vector fields defined in the domain Ω_i, and $h_i(x_i) \in R^{m_i}$ are smooth in the domain Ω_i for $i = 1, 2, \cdots, N$. The term $\Delta f_i(x_i)$ includes all the uncertainties experienced by the i-th subsystem. The term $\sum_{\substack{j=1 \\ j \neq i}}^{N} D_{ij}(x_j)$ is the nonlinear interconnection of the i-th subsystem.

Definition 2 The systems

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i + \Delta f_i(x_i) \tag{3.3}$$

$$y_i(t) = h_i(x_i), \qquad i = 1, 2, \cdots, N \tag{3.4}$$

are called the isolated subsystems of the systems (3.1)-(3.2), and the systems

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i \tag{3.5}$$

$$y_i(t) = h_i(x_i), \qquad i = 1, 2, \cdots, N \tag{3.6}$$

are called the nominal isolated subsytems of the systems (3.1)-(3.2).

In this paper, under the assumption that the isolated subsystems (3.5)-(3.6) have uniform relative degree r_i in the considered domain Ω_i, the interconnected systems (3.1)-(3.2) are to be analysed. The objective is to explore the system structure based on a geometric transformation to design a robust asymptotic observer for the interconnected system (3.1)-(3.2).

It should be noted that the following results can be extended to the case where the isolated subsystems are multi-input and multi-output using the corresponding framework to Section 2 for the multi-input and multi-output case provided in Isidori (1995).

4. System Analysis and Assumptions

In this section, some assumptions are imposed on the system (3.1)–(3.2) to facilitate the observer design.

Assumption 1. The nominal isolated subsystem (3.5)-(3.6) has uniform relative degree r_i in domain $x_i \in \Omega_i$ for $i = 1, 2, \cdots, N$.

Under Assumption 1, it follows from Section 2 that there exists a coordinate transformation

$$T_i : x_i \to \text{col}(\zeta_i, \eta_i) \tag{4.1}$$

where

$$\zeta_i = \begin{bmatrix} \zeta_{i1} \\ \zeta_{i2} \\ \vdots \\ \zeta_{ir_i} \end{bmatrix} = \begin{bmatrix} h_i(x_i) \\ L_f h_i(x_i) \\ \vdots \\ L_f^{r_i - 1} h_i(x_i) \end{bmatrix} \in R^{r_i} \tag{4.2}$$

and $\eta_i \in R^{n_i - r_i}$ is defined by

$$\eta_i = \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \vdots \\ \eta_{n_i - r_i} \end{bmatrix} = \begin{bmatrix} \phi_{i(r_i+1)}(x_i) \\ \phi_{i(r_i+2)}(x_i) \\ \vdots \\ \phi_{in_i}(x_i) \end{bmatrix} \in R^{n_i - r_i} \tag{4.3}$$

for $i = 1, 2, \cdots, N$. The functions $\phi_{i(r_i+1)}(x_i)$, $\phi_{i(r_i+2)}(x_i), \cdots, \phi_{in_i}(x_i)$ can be obtained by solving the following partial differential equations:

$$L_{g_i} \phi_i(x_i) = 0, \quad x_i \in \Omega_i, \quad i = 1, 2, \cdots, N. \tag{4.4}$$

From Section 2, it follows that in the new coordinate system (ζ_i, η_i), the nominal isolated subsystem (3.5)-(3.6) can be described by

$$\begin{aligned} \dot{\zeta}_i &= A_i\zeta_i + \beta_i(\zeta_i, \eta_i, u_i) && (4.5) \\ \dot{\eta}_i &= q_i(\zeta_i, \eta_i) && (4.6) \\ y_i &= C_i\zeta_i && (4.7) \end{aligned}$$

where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in R^{r_i \times r_i}, \qquad C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in R^{1\times r_i} \tag{4.8}$$

$$\beta_i(\zeta_i, \eta_i, u_i) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ L_{f_i}^{r_i} h_i(T_i^{-1}(\zeta_i, \eta_i)) + L_{g_i} L_{f_i}^{r_i - 1} h_i(T_i^{-1}(\zeta_i, \eta_i)) u_i \end{bmatrix} \tag{4.9}$$

It is clear to see that the pair (A_i, C_i) is observable. Thus, there exists a matrix L_i such that $A_i - L_iC_i$ is Hurwitz stable. This implies that, for any positive definite matrix $Q_i \in R^{r_i \times r_i}$, the Lyapunov equation

$$(A_i - L_iC_i)^T P_i + P_i(A_i - L_iC_i) = -Q_i \tag{4.10}$$

has a unique positive-definite solution $P_i \in R^{r_i\times r_i}$ for $i = 1, 2, \cdots, N$.
Assumption 2. The uncertainty $\Delta f_i(x_i)$ in (3.1) satisfies

$$\frac{\partial T_i}{\partial x_i}\Delta f_i(x_i) = \begin{bmatrix} E_i\Delta\Psi(x_i) \\ 0 \end{bmatrix} \tag{4.11}$$

where $T_i(\cdot)$ is defined in (4.1), $E_i \in R^{r_i\times r_i}$ is a constant matrix satisfying

$$E_i^T P_i = H_iC_i \tag{4.12}$$

for some matrix H_i, with P_i satisfying (4.10), and $\|\Delta\Psi_i(x_i)\| \leqslant \kappa_i(x_i)$, where $\kappa_i(x_i)$ is continuous and Lipschitz about x_i in the domain Ω_i for $i = 1, 2, \cdots, N$.
Remark 1. Solving the Lyapunov equation (4.10) in the presence of the constraint (4.12) is the well known constrained Lyapunov problem (Galimidi, A., & Barmish, B. (1986)). Although there is no general solution available for this problem, associated discussion and an algorithm can be found in Edwards *et al.*(2007).
Remark 2. Assumption 2 is a limitation on the uncertainty $\Delta f_i(x_i)$, and this is necessary to guarantee the existence of asymptotic observers. Denote the nonlinear uncertain term $\Delta\Psi_i(x_i)$ in (4.11) in the new coordinate frame (ζ_i, η_i) by $\Delta\Phi_i(\zeta_i, \eta_i)$ i.e.

$$\Delta\Phi_i(\zeta_i, \eta_i) = [\Delta\Psi_i(\zeta_i, \eta_i)]_{x_i = T_i^{-1}(\zeta_i, \eta_i)} \tag{4.13}$$

From Assumption 2, there exists a function $\rho_i(\zeta_i,\eta_i)$ such that

$$\|\Delta\Phi_i(\zeta_i,\eta_i)\| \leqslant \rho_i(\zeta_i,\eta_i) \tag{4.14}$$

and $\rho_i(\zeta_i,\eta_i)$ satisfies the Lipschitz condition in $T_i(\Omega_i)$. Thus for any (ζ_i,η_i) and $(\hat{\zeta}_i,\hat{\eta}_i)\in T_i(\Omega_i)$,

$$\|\rho_i(\zeta_i,\eta_i)-\rho_i(\hat{\zeta}_i,\hat{\eta}_i)\| \leqslant l_i^a\|\zeta_i-\hat{\zeta}_i\|+l_i^b\|\eta_i-\hat{\eta}_i\| \tag{4.15}$$

where both l_i^a and l_i^b are nonnegative constants.

Consider the interconnections $D_{ij}(x_j)$ in system (3.1). Partition the term $\frac{\partial T_i}{\partial x_i}D_{ij}(x_j)$ as follows

$$\frac{\partial T_i}{\partial x_i}D_{ij}(x_j)\big|_{x_j=T_j^{-1}(\zeta_j,\eta_j)} = \begin{bmatrix} \Gamma_{ij}^a(\zeta_j,\eta_j) \\ \Gamma_{ij}^b(\zeta_j,\eta_j) \end{bmatrix} \tag{4.16}$$

where $\Gamma_{ij}^a(\zeta_j,\eta_j)\in R^{r_i}$, $\Gamma_{ij}^b(\zeta_j,\eta_j)\in R^{n_i-r_i}$ for $i=1,2,\cdots,N$ and $i\neq j$.

Assumption 3. The nonlinear terms $\Gamma_{ij}^a(\zeta_j,\eta_j)\in R^{r_i}$ and $\Gamma_{ij}^b(\zeta_j,\eta_j)\in R^{n_i-r_i}$ in (4.16) satisfy the Lipschitz condition in $T_i(\Omega_i)$.

Assumption 3 implies that there exist nonnegative constants α_{ij}^a, α_{ij}^b, μ_{ij}^a and μ_{ij}^b such that

$$\|\Gamma_{ij}^a(\zeta_i,\eta_i)-\Gamma_{ij}^a(\hat{\zeta}_i,\hat{\eta}_i)\| \quad\leqslant\quad \alpha_{ij}^a\|\zeta_j-\hat{\zeta}_j\|+\alpha_{ij}^b\parallel\eta_j-\hat{\eta}_j\| \tag{4.17}$$

$$\|\Gamma_{ij}^b(\zeta_i,\eta_i)-\Gamma_{ij}^b(\hat{\zeta}_i,\hat{\eta}_i)\| \quad\leqslant\quad \mu_{ij}^a\|\zeta_j-\hat{\zeta}_j\|+\mu_{ij}^b\parallel\eta_j-\hat{\eta}_j\| \tag{4.18}$$

for $i=1,2,\cdots,N$ and $i\neq j$. From $(4.5)-(4.7)$ and the analysis above, it follows that under Assumption 2, in the new coordinate system (ζ_i,η_i), the system (3.1)-(3.2) can be described by

$$\dot{\zeta}_i = A_i\zeta_i+\beta_i(\zeta_i,\eta_i,u_i)+E_i\Delta\Psi_i(\zeta_i,\eta_i)+\sum_{\substack{j=1\\ j\neq i}}^{N}\Gamma_{ij}^a(\zeta_j,\eta_j) \tag{4.19}$$

$$\dot{\eta}_i = q_i(\zeta_i,\eta_i)+\sum_{\substack{j=1\\ j\neq i}}^{N}\Gamma_{ij}^b(\zeta_j,\eta_j) \tag{4.20}$$

$$y_i = C_i\zeta_i \tag{4.21}$$

where A_i and C_i are given in (4.8), $\beta_i(\cdot)$ is defined in (4.9) and $\Gamma_{ij}^a(\cdot)$ and $\Gamma_{ij}^b(\cdot)$ are defined in (4.16).

Remark 3. Since $\beta_i(\cdot)$ is continuous in the domain $T_i(\Omega_i)$, it is straightforward to see that there exists a subset in the domain $T_i(\Omega_i)$ such that the function $\beta_i(\cdot)$ is Lipschitz in the subset

$$\parallel\beta_i(\zeta_i,\eta_i,u_i)-\beta_i(\hat{\zeta}_i,\hat{\eta}_i,u_i)\parallel\leqslant v_i^a(u_i)\parallel\zeta_i-\hat{\zeta}_i\parallel+v_i^b(u_i)\parallel\eta_i-\hat{\eta}_i\parallel \tag{4.22}$$

where $v_i^a(u_i)$ and $v_i^b(u_i)$ are nonnegative functions of u_i for $i=1,2,\cdots,N$.

Assumption 4. The function $q_i(\zeta_i,\eta_i)$ in equation (4.20) has the following decomposition

$$q_i(\zeta_i,\eta_i)=M_i\eta_i+\theta_i(\zeta_i,\eta_i) \tag{4.23}$$

where $M_i\in R^{(n_i-r_i)\times(n_i-r_i)}$ is a Hurwitz matrix and $\theta_i(\zeta_i,\eta_i)$ are Lipschitz in domain $T_i(\Omega_i)$.

Under Assumption 4, there exist nonnegative constants τ_i^a and τ_i^b such that.

$$\parallel\theta_i(\zeta_i,\eta_i)-\theta_i(\hat{\zeta}_i,\hat{\eta}_i)\parallel\leqslant\tau_i^a\parallel\zeta_i-\hat{\zeta}_i\parallel+\tau_i^b\parallel\eta_i-\hat{\eta}_i\parallel \tag{4.24}$$

for $i=1,2,\cdots,N$. Further, from the fact that M_i is Hurwitz stable for $\Lambda_i>0$, the following Lyapunov equation has a unique solution $\Pi_i>0$

$$M_i^T\Pi_i+\Pi_iM_i=-\Lambda_i, \qquad i=1,2,\cdots,N. \tag{4.25}$$

5. Nonlinear Observer Synthesis

In this section, an observer is designed for the transformed systems $(4.19)-(4.21)$ which provides asymptotic estimation of the states of the interconnected systems $(4.19)-(4.21)$.

For system $(4.19)-(4.21)$, construct dynamical systems

$$\dot{\hat{\zeta}}_i = A_i\hat{\zeta}_i + L_i(y_i - C_i\hat{\zeta}_i) + \beta_i(\hat{\zeta}_i, \hat{\eta}_i, u_i) + K_i(y, \hat{\zeta}_i, \hat{\eta}_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} \Gamma_{ij}^a(\hat{\zeta}_j, \hat{\eta}_j) \tag{5.1}$$

$$\dot{\hat{\eta}}_i = M_i\hat{\eta}_i + \theta_i(\hat{\zeta}_i, \hat{\eta}_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} \Gamma_{ij}^b(\hat{\zeta}_j, \hat{\eta}_j) \tag{5.2}$$

where the term $K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i)$ is defined by

$$K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i) = \begin{cases} \frac{P_i^{-1}C_i^T(y_i - C_i\hat{\zeta}_i)}{\|y_i - C_i\hat{\zeta}_i\|} \parallel H_i \parallel \rho_i(\hat{\zeta}_i, \hat{\eta}_i), & y_i - C_i\hat{\zeta}_i \neq 0 \\ 0, & y_i - C_i\hat{\zeta}_i = 0 \end{cases} \tag{5.3}$$

where P_i and H_i satisfy (4.10) and (4.12) respectively.

Remark 4. System (5.1)-(5.2) is called a variable structure system throughout this paper due to the discontinuous terms defined in (5.3). It should be noted that this observer is different from a sliding mode observer as the proposed observer (5.1)-(5.2) may not produce a sliding motion.

The following results are ready to be presented.

Theorem 1. Suppose Assumptions $1-4$ hold. Then, the dynamical system $(5.1)-(5.2)$ is a robust asymptotic observer of system (4.19)-(4.21), if the function matrix $W^T(\cdot) + W(\cdot)$ is positive definite in the domain $T(\Omega) \times U := T(\Omega_1) \times U_1 \times T(\Omega_2) \times U_2 \times \cdots \times T(\Omega_N) \times U_N$, where the matrix $W(\cdot) = [w_{ij}(\cdot)]_{2N\times 2N}$, and its entries $w_{ij}(\cdot)$ are defined by

$$w_{ij} = \begin{cases} \lambda_{\min}(Q_i) - 2\lambda_{\max}(P_i)v_i^a - 2l_i^a\|C_i\|\|H_i\|, & i = j, \quad 1 \leqslant i \leqslant N \\ -2\lambda_{\max}(P_i)\alpha_{ij}^a, & i \neq j, \quad 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant N \\ \lambda_{\min}(\Lambda_{i-N}) - 2\lambda_{\max}(\Pi_{i-N})\tau_{i-N}^b, & i = j, \quad N+1 \leqslant i \leqslant 2N, \\ -2\lambda_{\max}(\Pi_{(i-N)})\mu_{(i-N)(j-N)}^b, & i \neq j, \quad N+1 \leqslant i \leqslant 2N, N+1 \leqslant j \leqslant 2N \\ -2[\lambda_{\max}(P_i)v_i^b + l_i^b\|C_i\|\|H_i\| + \lambda_{\max}(\Pi_i)\tau_i^a], & j-i = N, \quad 1 \leqslant i \leqslant N, N+1 \leqslant j \leqslant 2N \\ -2\lambda_{\max}(P_i)\alpha_{i(j-N)}^b, & j-i \neq N, \quad 1 \leqslant i \leqslant N, N+1 \leqslant j \leqslant 2N \\ 0, & i-j = N, \quad N+1 \leqslant i \leqslant 2N, 1 \leqslant j \leqslant N \\ -2\lambda_{\max}(\Pi_{i-N})\mu_{(i-N)j}^a, & i-j \neq N, \quad N+1 \leqslant i \leqslant 2N, 1 \leqslant j \leqslant N \end{cases}$$

Proof. Let $e_{\zeta_i} = \zeta_i - \hat{\zeta}_i$ and $e_{\eta_i} = \eta_i - \hat{\eta}_i$ for $i = 1, 2, \cdots, N$. Compare systems $(4.19)-(4.20)$ and

$(5.1)-(5.2)$. It follows that the error dynamical systems are described by

$$\begin{aligned} \dot{e}_{\zeta_i} &= (A_i - L_iC_i)e_{\zeta_i} + \beta_i(\zeta_i,\eta_i,u_i) - \beta_i(\hat{\zeta}_i,\hat{\eta}_i,u_i) + E_i\Delta\Psi_i(\zeta_i,\eta_i) - K_i(y_i,\hat{\zeta}_i,\hat{\eta}_i) \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{N} \Gamma_{ij}^{a}(\zeta_j,\eta_j) - \sum_{\substack{j=1\\ j\neq i}}^{N} \Gamma_{ij}^{a}(\hat{\zeta}_j,\hat{\eta}_j) \qquad (5.4) \\ \dot{e}_{\eta_i} &= M_ie_{\eta_i} + \theta_i(\zeta_i,\eta_i) - \theta_i(\hat{\zeta}_i,\hat{\eta}_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} \Gamma_{ij}^{b}(\zeta_j,\eta_j) - \sum_{\substack{j=1\\ j\neq i}}^{N} \Gamma_{ij}^{b}(\hat{\zeta}_j,\hat{\eta}_j) \qquad (5.5) \end{aligned}$$

Now, for the system (5.4) and (5.5) consider the following candidate Lyapunov function

$$V = \sum_{i=1}^{N} e_{\zeta_i}^T P_i e_{\zeta_i} + \sum_{i=1}^{N} e_{\eta_i}^T \Pi_i e_{\eta_i} \qquad (5.6)$$

Then, the time derivative of the candidate Lyapunov function can be described by

$$\dot{V} = \sum_{i=1}^{N} [(\dot{e}_{\zeta_i}^T P_i e_{\zeta_i} + e_{\zeta_i}^T P_i \dot{e}_{\zeta_i}) + (\dot{e}_{\eta_i}^T \Pi_i e_{\eta_i} + e_{\eta_i}^T \Pi_i \dot{e}_{\eta_i})] \qquad (5.7)$$

Substituting both $\dot{e}_{\zeta_i}$ in (5.4) and $\dot{e}_{\eta_i}$ in (5.5) into equation (5.7), it follows by direct computation that the time derivative of the function V in (5.6) can be described by

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} \Big\{ e_{\zeta_i}^T[(A_i - L_iC_i)^T P_i + P_i(A_i - L_iC_i)]e_{\zeta_i} + 2e_{\zeta_i}^T P_i[\beta_i(\zeta_i,\eta_i,u_i) - \beta_i(\hat{\zeta}_i,\hat{\eta}_i,u_i)] \\ &\quad + 2[e_{\zeta_i}^T P_i E_i \Delta\Psi_i(\zeta_i,\eta_i) - e_{\zeta_i}^T P_i K_i(y_i,\hat{\zeta}_i,\hat{\eta}_i)] + 2e_{\zeta_i}^T P_i \sum_{\substack{j=1\\ j\neq i}}^{N} [\Gamma_{ij}^{a}(\zeta_j,\eta_j) - \Gamma_{ij}^{a}(\hat{\zeta}_j,\hat{\eta}_j)] \\ &\quad + e_{\eta_i}^T(M_i^T\Pi_i + \Pi_iM_i)e_{\eta_i} + 2e_{\eta_i}^T\Pi_i[\theta_i(\zeta_i,\eta_i) - \theta_i(\hat{\zeta}_i,\hat{\eta}_i)] \\ &\quad + 2e_{\eta_i}^T\Pi_i \sum_{\substack{j=1\\ j\neq i}}^{N} [\Gamma_{ij}^{b}(\zeta_j,\eta_j) - \Gamma_{ij}^{b}(\hat{\zeta}_j,\hat{\eta}_j)] \Big\} \qquad (5.8) \end{aligned}$$

From (4.12), (4.14), (4.15) and (5.3), it follows that:

(i) If $y_i - C_i\hat{\zeta}_i = 0$, then from (4.12) and $e_{\zeta_i}^T C_i^T = (y_i - C_i\hat{\zeta}_i)^T$

$$e_{\zeta_i}^T P_i E_i \Delta\Phi_i(\zeta_i,\eta_i) - e_{\zeta_i}^T P_i K_i(y_i,\hat{\zeta}_i,\hat{\eta}_i) = e_{\zeta_i}^T C_i^T H_i^T \Delta\Phi_i(\zeta_i,\eta_i) = (H_i(y_i - C_i\hat{\zeta}_i))^T \Delta\Phi_i(\zeta_i,\eta_i) = 0$$

(ii) If $y_i - C_i\hat{\zeta}_i \neq 0$, then from (4.12), (4.14), (4.15) and (5.3)

$$\begin{aligned}
& e_{\zeta_i}^T P_i E_i \Delta\Phi_i(\zeta_i,\eta_i) - e_{\zeta_i}^T P_i K_i(y_i,\hat{\zeta}_i,\hat{\eta}_i) \\
= \; & e_{\zeta_i}^T C_i^T H_i^T \Delta\Phi_i(\zeta_i,\eta_i) - e_{\zeta_i}^T P_i \frac{P_i^{-1}C_i^T(y_i - C_i\hat{\zeta}_i)}{\|y_i - C_i\hat{\zeta}_i\|} \parallel H_i \parallel \rho_i(\hat{\zeta}_i,\hat{\eta}_i) \\
= \; & (C_i e_{\zeta_i})^T H_i^T \Delta\Phi_i(\zeta_i,\eta_i) - \frac{e_{\zeta_i}^T C_i^T C_i e_{\zeta_i}}{\|C_i e_{\zeta_i}\|}\|H_i\|\rho_i(\hat{\zeta}_i,\hat{\eta}_i) \\
\leqslant \; & \|C_i e_{\zeta_i}\|\|H_i\|\big\{\rho_i(\zeta_i,\eta_i) - \rho_i(\hat{\zeta}_i,\hat{\eta}_i)\big\} \\
\leqslant \; & \|C_i e_{\zeta_i}\|\|H_i\|\big\{l_i^a\|\zeta_i - \hat{\zeta}_i\| + l_i^b\|\eta_i - \hat{\eta}_i\|\big\}
\end{aligned}$$

Then, from (i) and (ii) above, it follows that

$$e_{\zeta_i}^T P_i E_i \Delta\Phi_i(\zeta_i,\eta_i) - e_{\zeta_i}^T P_i K_i(y_i,\hat{\zeta}_i,\hat{\eta}_i) \leqslant \|C_i e_{\zeta_i}\|\|H_i\|\big(l_i^a\|e_{\zeta_i}\| + l_i^b\|e_{\eta_i}\|\big) \tag{5.9}$$

Substituting (4.17), (4.18), (4.22), (4.24), and (5.9) into (5.8) yields

$$\begin{aligned}
\dot{V} \leqslant \; & \sum_{i=1}^{N}\Big\{ -e_{\zeta_i}^T Q_i e_{\zeta_i} + 2\|e_{\zeta_i}\|\|P_i\|\big[\nu_i^a\|e_{\zeta_i}\| + \nu_i^b\|e_{\eta_i}\|\big] + 2\|e_{\zeta_i}\|\|C_i\|\|H_i\|\big[l_i^a\|e_{\zeta_i}\| + l_i^b\|e_{\eta_i}\|\big] \\
& + 2\|e_{\zeta_i}\|\|P_i\|\sum_{\substack{j=1\\ j\neq i}}^{N}\big[\alpha_{ij}^a\|e_{\zeta_j}\| + \alpha_{ij}^b\|e_{\eta_j}\|\big] - e_{\eta_i}^T \Lambda_i e_{\eta_i} + 2e_{\eta_i}^T\|\Pi_i\|\big[\tau_i^a\|e_{\zeta_i}\| + \tau_i^b\|e_{\eta_i}\|\big] \\
& + 2e_{\eta_i}^T\|\Pi_i\|\sum_{\substack{j=1\\ j\neq i}}^{N}\big[\mu_{ij}^a\|e_{\zeta_j}\| + \mu_{ij}^b\|e_{\eta_j}\|\big]\Big\}
\end{aligned}$$

$$\begin{aligned}
\leqslant \; & \sum_{i=1}^{N}\Big\{ -e_{\zeta_i}^T Q_i e_{\zeta_i} + 2\nu_i^a\|e_{\zeta_i}\|^2\|P_i\| + 2\nu_i^b\|e_{\zeta_i}\|\|e_{\eta_i}\|\|P_i\| + 2l_i^a\|e_{\zeta_i}\|^2\|C_i\|\|H_i\| + \\
& 2l_i^b\|e_{\zeta_i}\|\|e_{\eta_i}\|C_i\|\|H_i\| + \sum_{\substack{j=1\\ j\neq i}}^{N}\big[2\alpha_{ij}^a\|e_{\zeta_i}\|\|e_{\zeta_j}\|\|P_i\| + 2\alpha_{ij}^b\|e_{\zeta_i}\|\|e_{\eta_j}\|\|P_i\|\big] \\
& - e_{\eta_i}^T \Lambda_i e_{\eta_i} + 2\tau_i^a\|\Pi_i\|\|e_{\zeta_i}\|\|e_{\eta_i}\| + 2\tau_i^b\|\Pi_i\|\|e_{\eta_i}\|^2 \\
& + \sum_{\substack{j=1\\ j\neq i}}^{N}\big[2\mu_{ij}^a\|\Pi_i\|\|e_{\zeta_j}\|\|e_{\eta_i}\| + 2\mu_{ij}^b\|\Pi_i\|\|e_{\eta_i}\|\|e_{\eta_j}\|\big]\Big\}
\end{aligned}$$

$$\begin{aligned}
\leqslant\ & \sum_{i=1}^{N}\Big\{-\big[\lambda_{\min}(Q_i)-2\lambda_{\max}(P_i)v_i^a-2l_i^a\|C_i\|\|H_i\|\big]\|e_{\zeta_i}\|^2+\big[\sum_{\substack{j=1\\j\neq i}}^{N}2\lambda_{\max}(P_i)\alpha_{ij}^a\big]\|e_{\zeta_i}\|\|e_{\zeta_j}\| \\
&+\big[2\lambda_{\max}(P_i)v_i^b+2l_i^b\|C_i\|\|H_i\|+2\lambda_{\max}(\Pi_i)\tau_i^a\big]\|e_{\zeta_i}\|\|e_{\eta_i}\|+2\sum_{\substack{j=1\\j\neq i}}^{N}\lambda_{\max}(P_i)\alpha_{ij}^b\|e_{\zeta_i}\|\|e_{\eta_j}\| \\
&+\sum_{\substack{j=1\\j\neq i}}^{N}2\lambda_{\max}(\Pi_i)\mu_{ij}^a\|e_{\zeta_j}\|\|e_{\eta_i}\|-\big[\lambda_{\min}(\Lambda_i)-2\lambda_{\max}(\Pi_i)\tau_i^b\big]\|e_{\eta_i}\|^2 \\
&+\big[\sum_{\substack{j=1\\j\neq i}}^{N}2\lambda_{\max}(\Pi_i)\mu_{ij}^b\big]\|e_{\eta_i}\|\|e_{\eta_j}\|\Big\} \\
\leqslant\ & -\sum_{i=1}^{N}\Big\{\big[\lambda_{\min}(Q_i)-2\lambda_{\max}(P_i)v_i^a-2l_i^a\|C_i\|\|H_i\|\big]\|e_{\zeta_i}\|^2-\big[\sum_{\substack{j=1\\j\neq i}}^{N}2\lambda_{\max}(P_i)\alpha_{ij}^a\big]\|e_{\zeta_i}\|\|e_{\zeta_j}\| \\
&-\big[2\lambda_{\max}(P_i)v_i^b+2l_i^b\|C_i\|\|H_i\|+2\lambda_{\max}(\Pi_i)\tau_i^a\big]\|e_{\zeta_i}\|\|e_{\eta_i}\big]\|-2\sum_{\substack{j=1\\j\neq i}}^{N}\lambda_{\max}(P_i)\alpha_{ij}^b\|e_{\zeta_i}\|\|e_{\eta_j}\|\big] \\
&-\sum_{\substack{j=1\\j\neq i}}^{N}2\lambda_{\max}(\Pi_i)\mu_{ij}^a\|e_{\zeta_j}\|\|e_{\eta_i}\|+\big[\lambda_{\min}(\Lambda_i)-2\lambda_{\max}(\Pi_i)\tau_i^b\big]\|e_{\eta_i}\|^2 \\
&-\big[\sum_{\substack{j=1\\j\neq i}}^{N}2\lambda_{\max}(\Pi_i)\mu_{ij}^b\big]\|e_{\eta_i}\|\|e_{\eta_j}\|\Big\}
\end{aligned}$$

Then, from the definition of the matrix $W(\cdot)$ and the inequality above, it follows that

$$\dot{V}\leqslant-\frac{1}{2}X^T[W^T(\cdot)+W(\cdot)]X$$

where $X=[\|e_{\zeta_1}\|,\|e_{\zeta_2}\|,\cdots,\|e_{\zeta_N}\|,\|e_{\eta_1}\|,\|e_{\eta_2}\|,\cdots,\|e_{\eta_N}\|]^T$. Since $W^T(\cdot)+W(\cdot)$ is positive definite in the domain $T(\Omega)\times U$, it is clear that $\dot{V}|_{(5.1)-(5.2)}$ is negative definite. Therefore, the error system $(5.4)-(5.5)$ is asymptotically stable , that is,

$$\lim_{t\to\infty}\|\zeta_i(t)-\hat{\zeta}_i(t)\|=0 \quad\text{and}\quad \lim_{t\to\infty}\|\eta_i(t)-\hat{\eta}_i(t)\|=0 \tag{5.10}$$

Hence, the conclusion follows. △

Remark 5 Theorem 1 shows that variable structure system (5.1)-(5.2) is an asymptotic observer of the interconnected system (4.19)-(4.21). It is called a variable structure observer in this paper.

Now, consider interconnected system $(3.1)-(3.2)$. Assume that $\frac{\partial T_i(\zeta_i,\eta_i)}{\partial(\zeta_i,\eta_i)}$ is bounded in $T_i(\Omega_i)$ for $i=1,2,\cdots,N$. There exists a positive constant γ_i such that

$$\left\|\frac{\partial T_i(\zeta_i,\eta_i)}{\partial(\zeta_i,\eta_i)}\right\|\leqslant\gamma_i,\qquad (\zeta_i,\eta_i)\in T_i(\Omega_i),\ i=1,2,\cdots,N$$

Define $\hat{x}_i = T_i^{-1}(\hat{\zeta}_i, \hat{\eta}_i),\ i = 1,2,\cdots,N$. Then,

$$\|x_i - \hat{x}_i\| = \|T_i^{-1}(\zeta_i, \eta_i) - T_i^{-1}(\hat{\zeta}_i, \hat{\eta}_i)\| \leqslant \gamma_i(\|\zeta_i - \hat{\zeta}_i\| + \|\eta_i - \hat{\eta}_i\|) \tag{5.11}$$

From (5.10) and (5.11), it follows that

$$\lim_{t\to\infty} \|x_i(t) - \hat{x}_i(t)\| = 0$$

This implies that $\hat{x}_i$ is an asymptotic estimate of x_i for $i = 1,2,\cdots,N$. Therefore,

$$\hat{x}_i = T_i^{-1}(\hat{\zeta}_i, \hat{\eta}_i)$$

provide an asymptotic estimation for the states x_i of system $(3.1)-(3.2)$, where $\hat{\zeta}_i$ and $\hat{\eta}_i$ are given by (5.1)-(5.2) for $i = 1,2,\ldots,N$.

Remark 6 From the analysis above, it is clear to see that, in this paper, it is not required that either the nominal isolated subsystems or the interconnections are linearisable. The uncertainties are bounded by nonlinear functions and are fully used in the observer design in order to reject the effects of the uncertainties, and thus robustness is enhanced. The designed observer is an asymptotic observer and the developed results can be extended to the global case if the associated conditions hold globally.

6. Numerical example

Consider the nonlinear interconnected systems:

$$\dot{x}_1 = \underbrace{\begin{bmatrix} x_{12} \\ -0.1\sin x_{12} \\ -3x_{11}^2 - 3.25x_{13} - 2x_{12} \end{bmatrix}}_{f_1(x_1)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{g_1(x_1)} u_1 + \underbrace{\begin{bmatrix} \Delta\sigma_1 \\ 0.5\Delta\sigma_1 \\ -2\Delta\sigma_1 \end{bmatrix}}_{\Delta f_1(x_1)} + \underbrace{\begin{bmatrix} 0.2(x_{21}^2 + x_{22}) \\ 0 \\ 0.1\sin x_{21} \end{bmatrix}}_{D_{12}(x_2)} \tag{6.1}$$

$$y_1 = \underbrace{x_{11}}_{h_1(x_1)} \tag{6.2}$$

$$\dot{x}_2 = \underbrace{\begin{bmatrix} -x_{21} \\ -x_{21}^2 - 3x_{22} + \cos(x_{21}^2 + x_{22}) - 1 \\ -2x_{23} + 0.2x_{21}^2 \end{bmatrix}}_{f_2(x_2)} + \underbrace{\begin{bmatrix} 1 \\ -2x_{21} \\ 0 \end{bmatrix}}_{g_2(x_2)} u_2 + \underbrace{\begin{bmatrix} -\Delta\sigma_2 \\ 2x_{21}\Delta\sigma_2 \\ 0 \end{bmatrix}}_{\Delta f_2(x_2)}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0.1\sin(x_{13} + 2x_{11}) \\ 0 \end{bmatrix}}_{D_{21}(x_1)} \tag{6.3}$$

$$y_2 = \underbrace{x_{21}}_{h_2(x_2)} \tag{6.4}$$

where $x_1 = \mathrm{col}(x_{11}, x_{12}, x_{13})$ and $x_2 = \mathrm{col}(x_{21}, x_{22}, x_{23})$, $h_1(x_1)$ and $h_2(x_2)$, and $u_1(t)$ and $u_2(t)$ are the system state, output and input respectively, $D_{12}(\cdot)$ and $D_{21}(\cdot)$ are interconnected terms and $\Delta f_1(x_1)$ and

$\Delta f_2(x_2)$ are the uncertainties experienced by the system which satisfy

$$||\Delta f_1(x_1)|| = 0.1|x_{13}+2x_{11}|\sin^2 t \tag{6.5}$$

$$||\Delta f_2(x_2)|| = 0.1x_{21}^2|\cos t| \tag{6.6}$$

The domain considered is

$$\Omega = \left\{(x_{11},\, x_{12},\, x_{13},\, x_{21},\, x_{22},\, x_{23}),\ \Big|\ |x_{11}|<3,\ |x_{21}| \leqslant 1.3,\ x_{12},\, x_{13},\, x_{22},\, x_{23} \in R\right\} \tag{6.7}$$

By direct computation, it follows that the first subsystem has a uniform relative degree 2, and the second subsystem has a uniform relative degree 1. The corresponding transformations are obtained as follows:

$$T_1 : \left\{ \begin{array}{rcl} \zeta_{11} & = & x_{11} \\ \zeta_{12} & = & x_{12} \\ \eta_1 & = & x_{13}+2x_{11} \end{array} \right. , \qquad T_2 : \left\{ \begin{array}{rcl} \zeta_2 & = & x_{21} \\ \eta_{21} & = & x_{21}^2+x_{22} \\ \eta_{22} & = & x_{23} \end{array} \right.$$

In the new coordinates, the system (6.1) – (6.4) can be described by:

$$\dot{\zeta}_1 = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} \zeta_{11} \\ \zeta_{12} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -0.1\sin\zeta_{11}+u_1 \end{bmatrix}}_{\beta_1} + \underbrace{\begin{bmatrix} \Delta\sigma_1(\zeta_1,\eta_1) \\ 0.5\Delta\sigma_1(\zeta_1,\eta_1) \end{bmatrix}}_{E_1\Delta\Psi(\zeta_1,\eta_1)} + \underbrace{\begin{bmatrix} 0.2\eta_{21} \\ 0 \end{bmatrix}}_{\Gamma_{12}^a} \tag{6.8}$$

$$\dot{\eta}_1 = \underbrace{-3.25\eta_1+0.25\zeta_{11}^2}_{q_1(\zeta_1,\eta_1)} + \underbrace{0.4\eta_{21}+0.1\sin\zeta_2}_{\Gamma_{12}^b} \tag{6.9}$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \zeta_{11} \\ \zeta_{12} \end{bmatrix} \tag{6.10}$$

$$\dot{\zeta}_2 = \underbrace{-}_{A_2}\zeta_2 + \underbrace{u_2}_{\beta_2} - \underbrace{\Delta\sigma_2(\zeta_2,\eta_2)}_{E_2\Delta\Psi(\zeta_2,\eta_2)} \tag{6.11}$$

$$\dot{\eta}_2 = \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \eta_{21} \\ \eta_{22} \end{bmatrix} + \begin{bmatrix} \cos\eta_{21}-1 \\ 0.2\zeta_2^2 \end{bmatrix}}_{q_2(\zeta_2,\eta_2)} + \underbrace{\begin{bmatrix} 0.1\sin\eta_1 \\ 0 \end{bmatrix}}_{\Gamma_{21}^b} \tag{6.12}$$

$$y_2 = \zeta_2 \tag{6.13}$$

where $\zeta_1 = (\zeta_{11},\ \zeta_{12})^T$, $\eta_1 \in R$, $\zeta_2 \in R$, and $\eta_2 = (\eta_{21}, \eta_{22})^T$.

From (6.5) and (6.6)

$$||\Delta\Psi_1(\zeta_1,\eta_1)|| \leqslant ||\Delta\sigma_1(\zeta_1,\eta_1)|| \leqslant \underbrace{0.1|\eta_1|\sin^2 t}_{\rho_1(\cdot)}$$

$$||\Delta\Psi_2(\zeta_2,\eta_2)|| \leqslant ||\Delta\sigma_2(\zeta_2,\eta_2)|| \leqslant \underbrace{0.1\zeta_2^2|\cos t|}_{\rho_2(\cdot)}$$

Then, for the first subsystem, choose $L_1 = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$ and $Q = I$. It follows that the Lyapunov equation (4.10) has a unique solution:

$$P_1 = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

and the solution to equation (4.12) is $H_1 = 0.25$. As $M_1 = -3.25$, let $\Lambda_1 = 3.25$. Thus the solution of equation (4.25) is $\Pi_1 = 0.5$. Now, for the second subsystem, choose $L_2 = 0$ and $Q_2 = 2$. It follows that the Lyapunov equation (4.10) has a unique solution $P_2 = 1$ and the solution to equation(4.12) is $H_2 = -1$. As

$$M_2 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}, \qquad \text{let} \qquad \Lambda_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$\Pi_2 = \begin{bmatrix} 0.1667 & 0 \\ 0 & 0.25 \end{bmatrix}$$

By direct computation, it follows that the matrix $W^T + W$ is positive definite in the domain Ω defined in (6.7). Thus, all the conditions of Theorem 1 are satisfied. This implies that the dynamical system

$$\begin{aligned}
\dot{\hat{\zeta}}_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\zeta}_{11} \\ \hat{\zeta}_{12} \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} (y_1 - C_1 \hat{\zeta}_1) + \begin{bmatrix} 0 \\ u_1 \end{bmatrix} + K_1(\cdot) + \begin{bmatrix} 0.2\eta_{21} \\ 0 \end{bmatrix} && (6.14) \\
\dot{\hat{\eta}}_1 &= -3.25\hat{\eta}_1 + 0.25\hat{\zeta}_{11}^2 + 0.4\hat{\eta}_{21} + 0.1\sin\hat{\zeta}_2 && (6.15) \\
\dot{\hat{\zeta}}_2 &= -\hat{\zeta}_2 + u_2 + K_2(\cdot) && (6.16) \\
\dot{\hat{\eta}}_2 &= \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \hat{\eta}_{21} \\ \hat{\eta}_{22} \end{bmatrix} + \begin{bmatrix} \cos\hat{\eta}_{21} - 1 \\ 0.2\hat{\zeta}_2^2 \end{bmatrix} + \begin{bmatrix} 0.1\sin\hat{\eta}_1 \\ 0 \end{bmatrix} && (6.17)
\end{aligned}$$

is a robust observer of the system (6.8) – (6.13) where $\hat{\zeta}_1 = \mathrm{col}(\hat{\zeta}_{11}, \hat{\zeta}_{12})$, $\hat{\eta}_2 = \mathrm{col}(\hat{\eta}_{21}, \hat{\eta}_{22})$, and $K_1(\cdot)$ and $K_2(\cdot)$ defined in (5.3) are as follows

$$K_1(y_1, \hat{\zeta}_1, \hat{\eta}_1) = \begin{cases} \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix} \frac{(\zeta_{11} - \hat{\zeta}_{11})}{\|\zeta_{11} - \hat{\zeta}_{11}\|} |\eta_1| \sin^2 t), & \zeta_{11} - \hat{\zeta}_{11} \neq 0 \\ 0, & \zeta_{11} - \hat{\zeta}_{11} = 0 \end{cases}$$

$$K_2(y_2, \hat{\zeta}_2, \hat{\eta}_2) = \begin{cases} 0.1 \frac{(\zeta_2 - \hat{\zeta}_2)}{\|\zeta_2 - \hat{\zeta}_2\|} \zeta_2^2 |\cos t|, & \zeta_2 - \hat{\zeta}_2 \neq 0 \\ 0, & \zeta_2 - \hat{\zeta}_2 = 0 \end{cases}$$

Therefore,

$$\begin{aligned} \hat{x}_{11} &= \hat{\zeta}_{11} \\ \hat{x}_{12} &= \hat{\zeta}_{12} \\ \hat{x}_{13} &= \hat{\eta}_1 - 2\hat{\zeta}_{11} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \hat{x}_{21} &= \hat{\zeta}_2 \\ \hat{x}_{22} &= \hat{\eta}_{21} - \hat{\zeta}_2^2 \\ \hat{x}_{23} &= \hat{\eta}_{22} \end{aligned}$$

with $\hat{\zeta}_1 = \mathrm{col}(\hat{\zeta}_{11},\ \hat{\zeta}_{12})$, $\hat{\eta}_1$, $\hat{\zeta}_2$ and $\hat{\eta}_2 = \mathrm{col}(\hat{\eta}_{21}, \hat{\eta}_{22})$ given by system (6.14) – (6.17), provide an asymptotic estimate for x_1 and x_2 of system (6.1)–(6.4).

For simulation purposes, the controllers are chosen as:

$$u_1 = -\zeta_{11} - 2\zeta_{12} \quad \text{and} \quad u_2 = \cos\zeta_2 + 5 \tag{6.18}$$

The initial conditions used in the simulation are chosen as $x_{10} = [-2\ \ 2\ \ -2]$, and $x_{20} = [1\ \ 4\ \ -5]$.

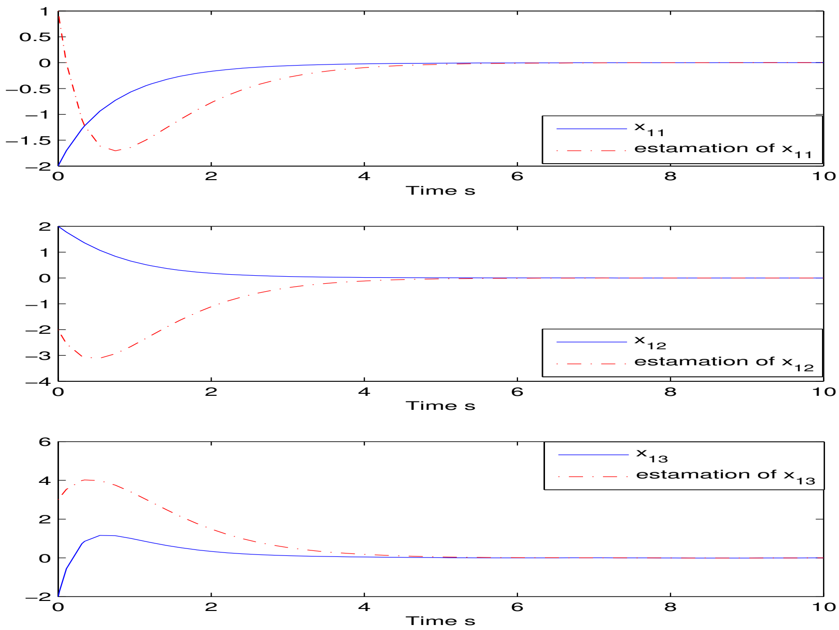

FIG. 1. The time response of the 1st subsystem states $x_1 = \text{col}\,(x_{11}, x_{12}, x_{13})$ and their estimates $\hat{x}_1 = \text{col}\,(\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{13})$

The simulation results in Figures 1 and 2 show that the designed observer estimates the states of the interconnected system $x_1 = \text{col}\,(x_{11},\, x_{12},\, x_{13})$ and $x_2 = \text{col}\,(x_{21},\, x_{22},\, x_{23})$ very well in $(6.1) - (6.4)$ even if the system is not asymptotically stable.

Remark 7. The aim of this paper is to design an observer for a class of nonlinear interconnected systems in the presence of uncertainties. Note that in this paper, it is not required that the considered systems are asymptotically stable. In order to guarantee the performance of the observer, it is only required that the error dynamical systems are asymptotically stable. The simulation results have shown that the errors between the estimated states and the actual states converge to zero even though the second subsystem is not asymptotically stable as shown in Fig.2.

7. Conclusions

In this paper, a class of nonlinear large scale interconnected systems with uniform relative degree has been considered. An asymptotic observer has been developed for nonlinear interconnected systems with uncertainties using the Lyapunov approach together with a geometric transformation which has been employed to exploit the system structure. It is not required that either the isolated nominal subsystems or the interconnections are linearisable. Robustness to uncertainties is enhanced by using the system structure and the structure of the uncertainties within the design framework.

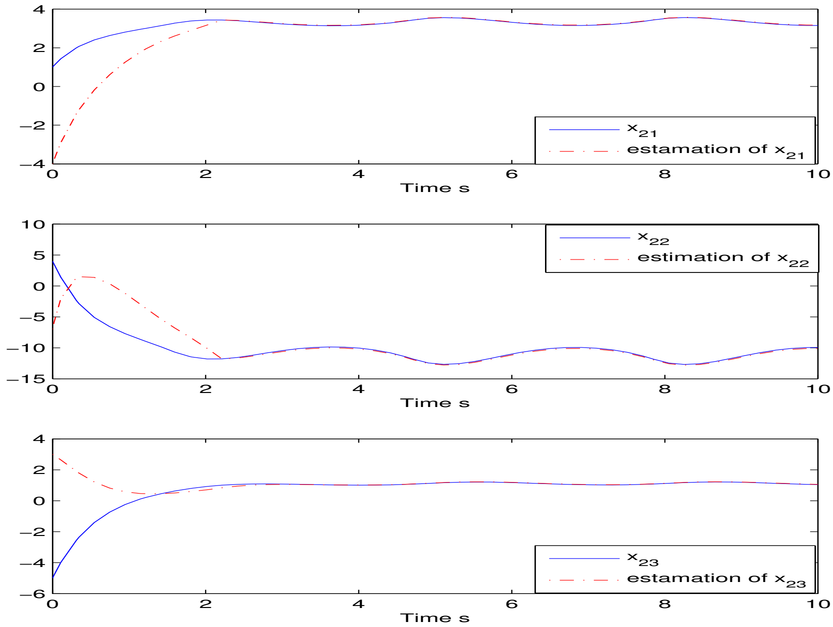

FIG. 2. The time response of 2nd subsystem states $x_2 = \text{col}\,(x_{21}, x_{22}, x_{23})$ and their estimates $\hat{x}_2 = \text{col}\,(\hat{x}_{21}, \hat{x}_{22}, \hat{x}_{23})$

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