Empowering teachers conceptually and pedagogically through supporting them in seeing connections between school mathematics and relevant advanced mathematics knowledge

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Since 'advanced mathematics knowledge' (AMK) was first conceptualized by Zaskis and Leikin (2009), researchers have striven to determine whether teachers' ability to identify explicit connections between AMK and the mathematics taught in school is a rare gift of only a few teachers or whether specific prompting is needed to develop this ability in teachers. In this paper we provide empirical evidence showing that those teachers who attended a CPD designed to support them to 'see' and make explicit such connections, have increased their awareness of the implications for the teaching and learning of school mathematics topics in ways that allow for creating a solid foundation for development of further, more advanced ideas in the school mathematics curriculum. We thus propose that any mathematics teacher, irrespective of their academic background, could benefit from professional development opportunities where explicit guidance is provided in terms of the relevant AMK and how it informs school mathematics teaching and learning.

Keywords: Advanced mathematics knowledge, mathematics knowledge for teaching, specialized knowledge, continuous professional development (CPD).

Introduction

There is agreement amongst researchers all over the world that teachers need to have both subject knowledge of mathematics per se, and mathematical knowledge for teaching in order to teach effectively. There is also agreement that teachers must know in detail the school mathematics they are expected to teach and a bit more, beyond the level they are assigned to teach. But 'how much *more*?' and '*More* of what'?

This body of *more* mathematics knowledge acquired through studying mathematics beyond school level is referred to in literature as 'academic mathematics' or 'advanced mathematics'. Moreira and David (2008) refer to academic mathematics as that large part of the mathematics that a 'major' of mathematics is required to learn and which consists of that "scientific body of knowledge as produced and organized by the professional mathematicians" (p. 24). Similarly (and somehow a more influential terminology) is Advanced Mathematical Knowledge (AMK) put forward in 2009 by Zazkis and Leikin and defined as "*systematic formal mathematical knowledge beyond secondary mathematics curriculum, likely acquired during undergraduate studies"* (p.2368).

However, what and in which ways this body of knowledge of 'AMK' or 'academic mathematics' is necessary or useful to functioning effectively as a teacher of mathematics at school level is still under much debate, as there is little agreement amongst researchers worldwide about how completing these courses influences future teachers' instruction (Zaskis & Leikin, 2010) or improves their students' subsequent achievement in the subject (Darling-Hammond, 2000).

Theoretical Influences – brief overviews

Teachers' knowledge for teaching

The study of teachers' knowledge of subject matter and its relationship to the quality of classroom instruction has grown substantially since Lee Shulman launched a call for researching teachers' different components of a professional knowledge base for teaching (Shulman, 1986). While there is no agreement amongst the mathematics education community about the relationship between these components, research flourished in an effort to conceptualize mathematics teachers' professional knowledge base for teaching.

One of such effort which builds on and refines Shulman's (1986) initial categorization of types of knowledge of a teacher of any subject, namely subject matter knowledge and pedagogical content knowledge, and which has proven to be very to be influential is the mathematics specific framework advanced by Ball, Thames and Phelps (2008). Their Mathematics Knowledge for Teaching (MKT) framework lays the foundation for a practice-based theory for mathematical knowledge for teaching. The authors divided Shulman's second category of Pedagogical Content Knowledge (PCK) into two other sub-domains, Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT), while Shulman's third category of Curricular Knowledge (CK) was also relocated under PCK as Knowledge of Content and Curriculum.

Similarly, Shulman's category of Subject Matter Knowledge was divided into three sub-domains: Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Horizon Content Knowledge (HCK). Recently, a few other researchers (e.g., Zaskis & Leikin 2010, Even 2011) proposed positioning advanced mathematical knowledge (AMK) as an important aspect of MKT and in the following we will briefly review these categories and how they complement each other.

Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK)

CCK is knowledge used in the work of teaching, but also used in ways that correspond with how it is used in settings other than teaching. SCK encompasses knowledge of mathematics needed by teachers, but not necessarily used by others, such as knowledge of a particular mathematical model or representation useful for teaching a certain concept.

Development of such SCK usually starts in teacher education programmes. Indeed, teachers acquire SCK through learning about how to use number lines, hot-air balloons or any other representations and metaphors that would enable them to teach about operations with negative numbers. For instance, while an engineer knows that the product of two negative numbers is a positive number, s/he does not need to know or give a mathematical reason for why this rule works or be able to provide a conceptually-sound explanation for the 'minus and minus make plus' metaphor. This kind of knowledge and reasoning should be an intrinsic part of a teacher's everyday classroom teaching, knowledge of the mathematics underlying rules, approaches, representations. However, far too often the implicit assumption is that prospective teachers already know the mathematics, to include the what and the why. But this is not the case. Many recent studies (Ball, 1991; Tirosh, Fischbein, Graeber, & Wilson, 1999) have revealed that school teachers possess a limited knowledge of mathematics, including the mathematics they teach. The mathematical education they received, both as pupils in school education and in teacher preparation, more often than not did not provide them with appropriate or sufficient opportunities to learn mathematics relationally (Skemp, 1976) and as a result, teachers themselves may know the facts and procedures that they teach but often have little or weak understanding of the conceptual basis underlying those rules and procedures (Ball, 1990; Taylor, 2002).

Horizon Content Knowledge

HCK (horizon content knowledge), the third sub-domain of subject matter knowledge in the MKT framework was tentatively defined by Ball and colleagues as 'an awareness of how mathematical topics are related over the span of mathematics included in the curriculum' (Ball et al., 2008, p. 403). Defined, interpreted and re-interpreted, HCK means different things to different researchers; what is common ground amongst these interpretations is that it is knowledge that goes beyond that included in school mathematics curriculum, that influences teaching!

Ball et al. (2008) themselves describe HCK domain of knowledge as "a kind of mathematical 'peripheral vision' needed in teaching, a view of the larger mathematical landscape that teaching requires" (p.1), including "the vision useful in seeing connections to much later mathematical ideas" (p.403). The authors acknowledged that "we do not know how horizon knowledge can be helpfully acquired and developed" (ibid, 2009, p.11)

Advanced Mathematics Knowledge

While engaging with the MKT framework, Zaskis and Mamolo (2011) proposed to view HCK through the notion of viewing elementary (school) mathematics from an advanced standpoint, thus positioning advanced mathematical knowledge (AMK) as an important aspect of MKT. The notion of horizon content knowledge is given by Zazkis and Mamolo in terms of the application of the notion of 'advanced mathematical knowledge', which they define as the "knowledge of the subject matter acquired during undergraduate studies at colleges or universities" (Zazkis & Leikin, 2010 p. 264). Wasserman (2016), and later Stockon and Wasserman (2017) narrowed down the description of AMK to knowledge outside the typical scope of what a school mathematics teacher would likely teach, in that AMK is <u>relevant</u>, the advanced mathematical ideas are connected to the content of school mathematics, but also that these forms of knowledge of advanced mathematics are in some way productive for the teaching of school mathematics content.

Since Zaskis and Leikin's (2009) first conceptualisation of AMK, the authors also launched a call for further research to determine whether teachers' ability to identify explicit connections between AMK and the mathematics taught in school is a rare gift of only a few teachers or whether specific prompting is needed to bring this ability to surface.

The study: the England, UK context

Rather than relying on individual teachers' gift to identify explicit connections between AMK and the mathematics they are expected to teach, in this paper we propose that teachers of mathematics are explicitly supported in becoming aware of these connections.

However, unlike other countries where teacher training is undertaken alongside undergraduate mathematics studies, and were such opportunities could be offered to prospective teachers as part of their undergraduate studies (e.g., Wasserman et al., 2017), the UK context is different. The teachers in the UK complete their training in a one-year postgraduate course, meaning that they would have studied advanced mathematics as part of their undergraduate studies not related with teacher education. Thus, for them the study of advanced mathematics had no explicit relation to school mathematics content.

Hence the study presented in this paper is a step forward in response to Zaskis and Leikin's (2009) call, but suited to the UK approach to teacher training, and where such interventions are offered to teachers after they completed their undergraduate studies, either as part of their initial teacher training postgraduate course or as a professional development opportunity after they qualify as teachers.

Methodology

In order to support teachers (re)engage with AMK and make explicit connection between advanced mathematics and the school mathematics, in ways that are relevant to their teaching practices, I trialled a number of activities as part of a series of CPD workshops aimed at supporting the teaching of specific areas of the school curriculum.

In this paper, I will be reporting on a two-hour CPD workshop designed to support the teaching of specific areas of the school curriculum, namely functions, aimed at increasing teachers' familiarity with a variety of representations of functions in the school mathematics and their awareness of how these representations interconnect. The tasks attempted were of both mathematical and pedagogical nature, and aimed at developing a deeper conceptual and pedagogical understanding of this topic.

Each workshop was designed to start by posing a school mathematics question or a problem situation that teachers could do but where they may encounter some difficulties in answering it correctly and completely; in order to overcome the difficulties, the teachers will be guided towards recalling/reengaging with some relevant AMK; this will then be followed by a classroom-inspired scenario, of a pedagogical nature, where the teachers will be applying their new learning and become explicitly aware how their new learning supports their teaching.

The Research Question

I was interested to find out if and in what ways does (re)engagement with relevant AMK empower teachers conceptually and pedagogically, in ways that could support pupils' learning of this (and related) school mathematics topic.

The participants

The participants were eight early career teachers of mathematics attending a two-hour CPD course designed and taught by the author of this paper. They were practicing mathematics teachers who wanted to refresh their knowledge about functions, especially given the high profile of this topic in the new re-vamped mathematics curriculum in England, UK. All teachers gained their qualified teacher status as a result of studying on a one-year teacher training course. As expected, all teachers studied some mathematics at undergraduate level: six teachers studied for mathematics degrees, one had an engineering background, while one other teacher had an economics background and introduced himself as a non-specialist mathematics teacher.

Data sources

Textual data was collected through field notes that detailed some of the group interactions. Post-session written reflections were solicited and collected at the end of the session. The teachers were asked to comment on the activities in relation to their own learning, their pupils' mathematical learning, and their preparedness of teaching this topic.

Results and Discussion

The workshop started by posing a school mathematics question or a problem situation that teachers could do but where it was envisaged they may encounter some difficulties in answering it correctly and completely. The teachers were provided with an activity which required sketching the graph of functions that shared the same function rule $f(x) = x^2$. The teachers worked in pairs, and each pair was provided with a different domain for the function. The domains were the whole set of real numbers, open and closed intervals, and discrete sets of real numbers. At the start of the activity, the teachers were not aware of the functions allocated to the other pairs.

The graphs produced by each pair looked more or less the same; a smooth curve in the shape of a parabola, carefully drawn to look symmetric about the y-axis. When the graphs were shared with the whole group, the teachers became aware of the similarities, but also the differences in their tasks; despite all sharing functions described by the same rule, the domains for each function were different, and so the discussion led naturally to a discussion about what a function was.

In the discussion, some teachers seemed to recall having studied about the formal definition of a function in their undergraduate studies, while others did not seem to have such a recollection or even an awareness of ever encountering such a definition. In what followed, disparate suggestions from teachers were recalled and put forward such as domains, co-domains, ranges, one-to-one, correspondence, notation conventions, and with some guidance from me, they reached the formal definition of a function of one real variable, which some recognised as having encountered them in their undergraduate mathematics course.

This is evidence of teachers reaching for *more* advanced mathematics knowledge related to functions, in the need to complete the task successfully. The 'starter' activity provided them with the impetus to reach for *more* than the school mathematics they were all too familiar with. Tapping into that knowledge, and once recalled (or newly learned, in the case of two teachers) there was evidence that it supported teachers completing the activity successfully. The teachers revisited the graphs they initially produced and each pair produced different graphs: either a smooth continuous parabola, or a pointwise graph, or a piece-wise graph, depending of the given domain of definition.

Even though a few of the teachers had an awareness of the formal definition of functions and were able to recall some 'bits' of it, they commented that "it did not occur to me to relate this activity with the formal definition", and "that was high level mathematics not much used after the [undergraduate] course". The teachers seemed to be much more influenced by the current limited description of functions in the school curriculum, where domains and ranges of functions are not explicitly considered.

The ‘starter’ activity was then followed with a discussion of how the concept of function develops in the school mathematics curriculum. Representations of functions as they chronologically appear in the school curriculum were discussed: One-to-one or many-to-one mappings, Input/output machines, Relations between particular x-values and y-values; Expressions to calculate the y-values from given x-values, and Graphs. Each time, the teachers were encouraged to relate these representations with their recalled or newly learned AMK about functions. In doing so, the teachers came to realise that each of these representations explains particular aspects/features of the concept without being able to describe it completely! And a realization that overreliance on one representations or lack of connections between such representations gives way to misconceptions when working with functions, just as pointed out by Ayalon, Lerman and Watson (2013).

Indeed, the teachers themselves became explicitly “aware of stages of building up to the definition of a function”. A teacher in particular was able to illustrate this new learning eloquently. She stated that she learned about: “Different representations of functions – I’ve always seen them as disconnected representations, but they complement each other nicely towards understanding functions” and exemplified with how in the lower secondary school curriculum, functions are portrayed as a computational process and are seen as an input-output machine that processes input values into output values. Such representation emphasizes the rule aspect in the definition of functions, seeing thus functions as an instruction to calculate one numerical set from another. This view leads to a perception of graphical representation of functions amongst pupils as points (usually with integer coordinates) plotted on the set of axes, which are then joined up with segments, with no explicit awareness that any other point lying on those segments could be just as good a candidate in the table of values. And in fact, the teacher herself realized that she never discussed or pointed this out in her teaching when plotting graphs.

For the final task of this workshop, the teachers were asked to think about how they introduce and teach pupils of different ages about square roots. Sharing of own experiences immediately led into disputes over the numerical value of the square root of a number: a positive, a negative or a $\pm$ value. Most teachers defended the $\pm$ value, and attempted to justify their answers in a variety of ways: “this is how I was taught myself”, “this is how it is presented in textbooks”, “this is how it appears in marking schemes of examination board”, “This has always been the case.”.

I then prompted the teachers to think about how this school maths concepts related to their learning about functions gained in the session up to that point and I suggested that perhaps it could help them settle the inconsistency in their answers.

The teachers were indeed able to call upon their recollection of the more advanced knowledge about functions (extended at this point to inverse functions and relationships between domains and ranges) and agreed on the positive answer only, despite feeling uncomfortable about dismantling a long-held belief about a piece of knowledge about square roots, they themselves inherited from their secondary school maths education and unfortunately still perpetuated by current mathematics school textbooks (Crisan, 2008, 2014).

Concluding remarks

The teachers on this CPD course had revisited some of their AMK of functions, which provided them with a better understanding and an awareness of the developmental trajectory in learning about functions. They became aware of how representations in the school curriculum are particular instances of the concept itself, and how teaching towards a complete/full understanding of functions requires teachers to be aware of what each of these representations contribute to the complete understanding of the concept of functions: “Today’s session helped me understand how I could have addressed the [pupils’] errors and how I can clarify things in the future.”, while another teacher shared his learning in the session: “What I have learnt today? About advanced mathematics knowledge and its place in classroom and planning.”.

On a pedagogical level, AMK of functions empowered the teachers to justify why pupils make mistakes, and thus increased their knowledge for teaching this topic in the future.

The task in which the participants were involved in this workshop provided a context in which they recalled AMK related to functions, while in the case of some other teachers, they acquired new AMK (the teachers with an engineering and economics background). The teachers gained conceptually, as the mathematics tasks created some instability in what they knew about functions and their graphical representations, and in order to address the differences in their tasks, the teachers need to engage with *more* advanced mathematics knowledge of functions was brought out into the open.

However, this study has shown that even when teachers posses the AMK, they are not necessarily aware of manifestations of AMK in the school mathematics curriculum, thus they need to be supported in develop such awareness and make it explicit. One cannot simply assume that teachers will make connections without some intervention. This paper proposes that, in the UK context this should be the remit of courses preparing teachers of mathematics (either initial teacher training and/or CPD). All teachers, irrespective of their academic background, should be supported to look at school mathematics from an advanced standpoint and to examine school mathematics topics by engaging with advanced mathematics knowledge, while guidance is provided in terms of what AMK is relevant and how it informs school mathematics.

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