A note on doubly nonlinear SPDEs with singular drift in divergence form

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Abstract

We prove well-posedness for a class of second-order SPDEs with multiplicative Wiener noise and doubly nonlinear drift of the form $-\operatorname{div}\gamma(\nabla\cdot) + \beta(\cdot)$, where γ is the subdifferential of a convex function on $\mathbb{R}^d$ and β is a maximal monotone graph everywhere defined on $\mathbb{R}$, on which neither growth nor continuity assumptions are imposed.

1 Introduction

Let D be a bounded domain of $\mathbb{R}^d$ with smooth boundary and $T > 0$ a fixed number. We shall establish well-posedness in the strong sense for stochastic partial differential equations of the type

$$\begin{cases} du(t) - \operatorname{div}\gamma(\nabla u(t))\,dt + \beta(u(t))\,dt \ni B(t,u(t))\,dW(t) & \text{in } (0,T)\times D,\\ u = 0 & \text{in } (0,T)\times\partial D,\\ u(0) = u_0 & \text{in } D, \end{cases} \tag{1.1}$$

where $\gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ and $\beta \subset \mathbb{R}\times\mathbb{R}$ are everywhere-defined maximal monotone graphs, the first one of which is assumed to be the subdifferential of a convex function $k : \mathbb{R}^d \to \mathbb{R}$. Furthermore, W is a cylindrical Wiener process on a separable Hilbert space U, and B takes values in the space of Hilbert-Schmidt operators from U to $L^2(D)$. Precise assumptions on the data of the problem are given in §2 below.

Equations with drift in divergence type, both in deterministic and stochastic settings, have a long history and are thoroughly studied, especially because of their physical significance. From a mathematical point of view, they are particularly interesting because they are fully nonlinear, in the sense that they do not contain any "leading" linear term. For stochastic equations, the first well-posedness result is most likely due to Pardoux, as an application of his general results in [12] on monotone stochastic evolution equations in the variational setting (see also [4] for improved results under more general assumptions on B). In this case one needs to assume $\beta = 0$ and

$$\gamma(x) \cdot x \gtrsim |x|^p - 1, \qquad |\gamma(x)| \lesssim |x|^{p-1} - 1 \qquad \forall x \in \mathbb{R}^d,$$

with $p > 1$ (the centered dot stands for the usual Euclidean scalar product in $\mathbb{R}^d$). These are precisely the classical Leray-Lions conditions, well known in the deterministic theory (cf. [5]). In some special cases a simple polynomial-type β can be added: for instance, if γ corresponds to the p-Laplacian, i.e. $\gamma(x) = |x|^{p-2}x$, $p \geq 2$, one may consider $\beta(r) = |r|^{p-2}r$ (cf. [6, p. 83]). However, it is well known that if two nonlinear operators satisfy the conditions needed in the variational setting, their sum in general does not. This phenomenon already gives rise to severe restrictions on the class of semilinear equations with polynomial nonlinearities that can be solved by such methods.

In some recent works we have obtained well-posedness results for (1.1) under much more general hypotheses than those mentioned above. In particular, in [13] it is assumed that γ still satisfies the classical Leray-Lions assumptions, but no growth restriction on β is imposed: a very mild symmetry-like condition on its behavior at infinity is shown to suffice. On the other hand, in [9] we consider the case $\beta = 0$, with no hypotheses on the growth of γ, but with the additional requirement that γ is single-valued (a symmetry-like assumption on γ is needed in this case as well). Equations with more general, possibly multivalued γ, are treated in [10], where, however, less regular solutions are obtained.

Our goal is to unify and extend the above-mentioned well-posedness results for equation (1.1), thus treating the case where both γ and β can be multivalued, without any restriction on their rate of growth. We shall also show that we can do so without loosing any regularity of solutions with respect to the results of [9]. The approach we take, initiated in [11] and further refined and extended in [7]–[10], consists in a combination of (deterministic and stochastic) variational techniques and weak compactness in L^1 spaces. A key feature is the construction of a candidate solution as pathwise limit, in suitable topologies, of solutions to regularized equations. In particular, due to this type of construction, in order to obtain measurability properties of solutions, uniqueness of limits is crucial. Roughly speaking, we can prove that $-\operatorname{div}\gamma(\nabla u) + \beta(u)$ is unique, hence that it is measurable, but showing that each one of them is unique (hence measurable) seems difficult, if not impossible. This is the reason why γ was assumed to be single-valued in [9, 13]. In the general setting of this work we thus need different ideas: let u_λ, γ_λ, and β_λ be suitable regularizations of u, γ, and β, respectively, and set $\eta_\lambda := \gamma_\lambda(\nabla u_\lambda)$ and $\xi_\lambda := \beta_\lambda(u_\lambda)$. Comparing weak limits, obtained in different ways, of the image of the pair $(\eta_\lambda, \xi_\lambda)$ under a continuous linear map, we are going to prove that there exist two limiting processes η and ξ, "sections" of $\gamma(\nabla u)$ and $\beta(u)$, respectively, that are indeed predictable and satisfy suitable uniqueness properties. One may say that we restore uniqueness working in a suitable quotient space, although quotient spaces do not appear explicitly.

The well-posedness result obtained here may be interesting also in the deterministic setting, as our results extend to the doubly nonlinear case the sharpest results available for equations with $\beta = 0$ and $B = 0$, whose hypotheses on γ are identical to ours (cf. [2, p. 207-ff]).

Our interest for stochastic PDEs with singular monotone drift has been influenced by reading the article [1], which, however, deals with semilinear equations only.

The paper is organized as follows: in Section 2 we state the assumptions and the main result, which is then proved in Section 3.

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2 Main result

Before stating the main result, we fix notation and introduce the necessary assumptions.

As already mentioned, D stands for a bounded domain in $\mathbb{R}^d$ with smooth boundary. We shall denote the Hilbert space $L^2(D)$ by H, its norm and scalar product by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. We shall denote the Dirichlet Laplacian on $L^1(D)$ (as well as on $L^2(D)$, without notationally distinguish them) by Δ. The space of Hilbert-Schmidt operators from the separable Hilbert space U to H is denoted by $\mathscr{L}^2(U,H)$. We shall write $a \lesssim b$ to mean that there exists a constant $N > 0$ such that $a \leq Nb$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, endowed with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ satisfying the so-called usual conditions, on which all random elements will be defined. Equality of stochastic processes is meant to be in the sense of indistinguishability, unless otherwise stated. We assume that the diffusion coefficient

$$B : \Omega \times [0,T] \times H \to \mathscr{L}^2(U,H)$$

is such that $B(\cdot,\cdot,h)$ is progressively measurable for all $h \in H$, and there exists a positive constant N_B such that

$$\begin{aligned} \big\|B(\omega,t,x)\big\|_{\mathscr{L}^2(U,H)} &\leq N_B\big(1+\|x\|\big), \\ \big\|B(\omega,t,x)-B(\omega,t,y)\big\|_{\mathscr{L}^2(U,H)} &\leq N_B\|x-y\| \end{aligned}$$

for all $(\omega,t) \in \Omega \times [0,T]$ and $x,y \in H$. Moreover, let the initial datum u_0 be $\mathscr{F}_0$-measurable with finite second moment, i.e. $u_0 \in L^2(\Omega,\mathscr{F}_0;H)$.

Let $k : \mathbb{R}^d \to \mathbb{R}_+$ be a convex function with $k(0) = 0$ such that

$$\limsup_{|x|\to+\infty} \frac{k(x)}{k(-x)} < +\infty, \qquad \lim_{|x|\to+\infty} \frac{k(x)}{|x|} = +\infty$$

(we shall call the second condition superlinearity at infinity). Then its subdifferential $\gamma := \partial k$ is a maximal monotone graph in $\mathbb{R}^d \times \mathbb{R}^d$. We assume that the domain of γ coincides with $\mathbb{R}^d$, which implies that k^*, the convex conjugate of k, is superlinear at infinity as well. Moreover, let $j : \mathbb{R} \to \mathbb{R}_+$ be a further convex function with $j(0) = 0$ such that

$$\limsup_{|x|\to+\infty} \frac{j(x)}{j(-x)} < +\infty,$$

whose subdifferential $\beta := \partial j$ is an everywhere defined maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, so that j^* is superlinear at infinity. The condition on j involving the upper limit at infinity can be interpreted as a natural generalization of symmetry, in the sense that its rates of growth at $+\infty$ and $-\infty$ are assumed to be comparable. The corresponding condition on k has the same interpretation, after restricting k to any ray in $\mathbb{R}^d$. All notions of convex analysis and from the theory of maximal monotone operators used thus far and in the sequel are standard and are treated in detail, for instance, in [2].

We can now give the notion of solution to (1.1) that we are going to work with. Throughout the work, V_0 is a separable Hilbert space continuously embedded in both $W^{1,\infty}(D)$ and $H^1_0(D)$: for instance one can take, thanks to Sobolev embedding theorems, $V_0 := H^k_0(D)$ for $k \in \mathbb{N}$ sufficiently large. Moreover, the divergence operator is defined as

$$\begin{aligned} \operatorname{div} : L^1(D)^d &\longrightarrow V_0' \\ f &\longmapsto \big[g \mapsto -\langle f, \nabla g\rangle\big], \end{aligned}$$

which is thus linear and bounded. In fact, for any $f \in L^1(D)^d$ and $g \in V_0$,

$$\big|\langle f, \nabla g\rangle\big| \le \|f\|_{L^1(D)} \|g\|_{W^{1,\infty}(D)} \lesssim \|f\|_{L^1(D)} \|g\|_{V_0}$$

because V_0 is continuously embedded in $W^{1,\infty}(D)$.

Definition 2.1. *A strong solution to* (1.1) *is a triplet* (u, η, ξ)*, where* u*,* η*, and* ξ *are adapted processes taking values in* $W^{1,1}_0(D) \cap H$*,* $L^1(D)^d$*, and* $L^1(D)$*, respectively, such that* $\eta \in \gamma(\nabla u)$ *and* $\xi \in \beta(u)$ *a.e. in* $\Omega \times (0,T) \times D$*,*

$$\begin{aligned} u &\in L^0(\Omega; C([0,T]; H)) \cap L^0(\Omega; L^1(0,T; W^{1,1}_0(D))), \\ \eta &\in L^0(\Omega; L^1((0,T) \times D)^d), \\ \xi &\in L^0(\Omega; L^1((0,T) \times D)), \\ \nabla u \cdot \eta + u\xi &\in L^0(\Omega; L^1((0,T) \times D)), \end{aligned}$$

and

$$\langle u, \phi \rangle + \int_0^\cdot \langle \eta(s), \nabla\phi \rangle\, ds + \int_0^\cdot \langle \xi(s), \phi \rangle\, ds = \langle u_0, \phi \rangle + \left\langle \int_0^\cdot B(s, u(s))\, dW(s), \phi \right\rangle$$

for all $\phi \in V_0$.

The last identity in the above definition is equivalent to the validity in the dual of V_0 of the equality

$$u - \int_0^\cdot \operatorname{div} \eta(s)\, ds + \int_0^\cdot \xi(s)\, ds = u_0 + \int_0^\cdot B(s, u(s))\, dW(s).$$

Note that u, u_0 and the stochastic integrals take values in H and the third term on the left-hand side takes values in $L^1(D)$, hence also the second term on the right-hand side belongs to $L^1(D)$, so that the equality holds also in $L^1(D)$. The same reasoning implies that the sum of the second and third terms on the left-hand side take values in H, so that the above equality can also be seen as valid in H.

The main result of the paper is the following. The proof is given in §3 below.

Theorem 2.2. *There exists a strong solution* (u, η, ξ) *to equation* (1.1). *It is predictable and satisfies the following properties:*

$$\begin{aligned} u &\in L^2(\Omega; C([0,T]; H)) \cap L^1(\Omega; L^1(0,T; W_0^{1,1}(D))),\\ \eta &\in L^1(\Omega \times (0,T) \times D)^d,\\ \xi &\in L^1(\Omega \times (0,T) \times D),\\ \nabla u \cdot \eta &\in L^1(\Omega \times (0,T) \times D),\\ u\xi &\in L^1(\Omega \times (0,T) \times D). \end{aligned}$$

Moreover, the solution map

$$\begin{aligned} L^2(\Omega, \mathscr{F}_0; H) &\longrightarrow L^2(\Omega; C([0,T]; H))\\ u_0 &\longmapsto u \end{aligned}$$

is Lipschitz-continuous. In particular, if (u_1, η_1, ξ_1) *and* (u_2, η_2, ξ_2) *are any two strong solutions satisfying the properties above, then* $u_1 = u_2$ *and* $-\operatorname{div} \eta_1 + \xi_1 = -\operatorname{div} \eta_2 + \xi_2$ *in* $L^2(\Omega; C([0,T]; H))$ *and* $L^1(\Omega; L^1(0,T; V_0'))$, *respectively.*

3 Proof of Theorem 2.2

3.1 Itô's formula for the square of the H-norm

We establish a version of Itô's formula for the square of the H-norm in a generalized variational setting, which will play an important role in the sequel. The result is interesting in its own right, as it does not follow from the classical ones in [4, 12], and is apparently new for Itô processes containing a drift term in divergence form with minimal integrability properties.

Proposition 3.1. *Let Y, f, and g be measurable adapted processes with values in $H \cap W_0^{1,1}(D)$, $L^1(D)^d$, and $L^1(D)$, respectively, such that*

$$\begin{aligned}
&Y \in L^0(\Omega; L^\infty(0,T;H)) \cap L^0(\Omega; L^1(0,T;W_0^{1,1}(D))),\\
&f \in L^0(\Omega; L^1((0,T)\times D)^d),\\
&g \in L^0(\Omega; L^1((0,T)\times D)),
\end{aligned}$$

and there exists constants a, $b > 0$ such that

$$k(a\nabla u) + k^*(af) + j(bu) + j^*(bg) \in L^0(\Omega; L^1((0,T)\times D)).$$

Moreover, let $Y_0 \in L^0(\Omega, \mathscr{F}_0; H)$ and G be an $\mathscr{L}^2(U,H)$-valued progressively measurable process such that $G \in L^0(\Omega; L^2(0,T;\mathscr{L}^2(U,H)))$. If

$$Y - \int_0^\cdot \operatorname{div} f(s)\,ds + \int_0^\cdot g(s)\,ds = Y_0 + \int_0^\cdot G(s)\,dW(s)$$

as an identity in V_0', then

$$\begin{aligned}
&\frac{1}{2}\|Y\|^2 + \int_0^\cdot\!\int_D f(s)\cdot\nabla Y(s)\,ds + \int_0^\cdot\!\int_D g(s)Y(s)\,ds\\
&\qquad = \frac{1}{2}\|Y_0\|^2 + \frac{1}{2}\int_0^\cdot \|G(s)\|^2_{\mathscr{L}^2(U,H)}\,ds + \int_0^\cdot Y(s)G(s)\,dW(s).
\end{aligned}$$

Proof. The proof is essentially a combination of arguments described in great detail in [8, 9], hence we shall limit ourselves to a sketch only. Using a superscript δ to denote the action of $(I-\delta\Delta)^{-m}$, for a sufficiently large $m \in \mathbb{N}$, we have, thanks to Sobolev embedding theorems and classical elliptic regularity results,

$$Y^\delta - \int_0^\cdot \operatorname{div} f^\delta(s)\,ds + \int_0^\cdot g^\delta(s)\,ds = Y_0^\delta + \int_0^\cdot G^\delta(s)\,dW(s)$$

as an identity of H-valued processes. Itô's formula for Hilbert-space valued continuous semimartingales thus yields

$$\begin{aligned}
&\frac{1}{2}\|Y^\delta\|^2 + \int_0^\cdot\!\int_D f^\delta(s)\cdot\nabla Y^\delta(s)\,ds + \int_0^\cdot\!\int_D g^\delta(s)Y^\delta(s)\,ds\\
&\qquad = \frac{1}{2}\|Y_0^\delta\|^2 + \frac{1}{2}\int_0^\cdot \|G^\delta(s)\|^2_{\mathscr{L}^2(U,H)}\,ds + \int_0^\cdot Y^\delta(s)G^\delta(s)\,dW(s).
\end{aligned} \tag{3.1}$$

Thanks to the assumptions on Y, f, g ad G, it easily follows that, $\mathbb{P}$-a.s.,

$$\begin{aligned}
Y_0^\delta &\longrightarrow Y_0 && \text{in } H,\\
Y^\delta(t) &\longrightarrow Y(t) && \text{in } H \quad \forall\, t \in [0,T],\\
f^\delta &\longrightarrow f && \text{in } L^1((0,T)\times D)^d,\\
g^\delta &\longrightarrow g && \text{in } L^1((0,T)\times D),\\
G^\delta &\longrightarrow G && \text{in } L^2(0,T;\mathscr{L}^2(U,H)).
\end{aligned}$$

Similarly, using simple properties of Hilbert-Schmidt operators and the dominated convergence theorem, it is not difficult to verify that the quadratic variation of $(Y^\delta G^\delta - YG) \cdot W$ converges to zero in probability, so that

$$\int_0^\cdot Y^\delta G^\delta \, dW \longrightarrow \int_0^\cdot YG \, dW$$

uniformly (with respect to time) in probability. Furthermore, thanks to the hypotheses on k and j, the families $(\nabla u^\delta \cdot Y^\delta)$ and $(g^\delta Y^\delta)$ are uniformly integrable in $(0,T) \times D$ $\mathbb{P}$-a.s., hence by Vitali's theorem we also have that, $\mathbb{P}$-a.s.,

$$\begin{aligned} f^\delta \cdot \nabla Y^\delta &\longrightarrow f \cdot \nabla Y \quad \text{in } L^1((0,T) \times D), \\ g^\delta Y^\delta &\longrightarrow gY \quad\quad\ \text{in } L^1((0,T) \times D). \end{aligned}$$

The proof is completed passing to the limit as $\delta \to 0$ in (3.1), in complete analogy to [7, § 4] and [9, § 3]. □

Corollary 3.2. *Under the assumptions of the previous proposition, one has*

$$Y \in L^0(\Omega; C([0,T]; H)).$$

Proof. Since $Y \in L^\infty(0,T;H) \cap C([0,T];V_0')$, the trajectories of Y are weakly continuous in H (see, e.g.,[14]). Moreover, by Itô's formula one has

$$\begin{aligned} \frac{1}{2}\|Y(t)\|^2 - \frac{1}{2}\|Y(r)\|^2 + \int_r^t \int_D f(s) \cdot \nabla Y(s) \, ds + \int_r^t \int_D g(s) Y(s) \, ds \\ = \frac{1}{2} \int_r^t \|G(s)\|^2_{\mathscr{L}^2(U,H)} \, ds + \int_r^t Y(s) G(s) \, dW(s) \end{aligned}$$

for every r, $t \in [0,T]$. This implies, by an argument analogous to the one used in [8, § 3], that the function $t \mapsto \|Y(t)\|$ is continuous on $[0,T]$. By a well-known criterion we thus conclude that Y has strongly continuous trajectories in H. □

3.2 Well-posedness in a special case

As a first step we prove existence of solutions to (1.1) assuming that the noise is of additive type and that

$$B \in L^2(\Omega; L^2(0,T; \mathscr{L}^2(U, V_0))).$$

For any $\lambda > 0$, let γ_λ and β_λ denote the Yosida approximations of γ and β, respectively, and consider the regularized equation

$$du_\lambda(t) - \lambda \Delta u_\lambda(t) \, dt - \operatorname{div} \gamma_\lambda(\nabla u_\lambda(t)) \, dt + \beta_\lambda(u_\lambda(t)) \, dt = B(t) \, dW(t), \qquad u_\lambda(0) = u_0.$$

Since γ_λ and β_λ are monotone and Lipschitz-continuous, it is not difficult to check that the operator

$$\phi \longmapsto -\lambda \Delta \phi - \operatorname{div} \gamma_\lambda(\nabla \phi) + \beta_\lambda(\phi)$$

is hemicontinuous, monotone, coercive and bounded on the triple $(H_0^1(D), H, H^{-1}(D))$, so that the classical results by Pardoux [12] provide existence and uniqueness of a variational solution

$$u_\lambda \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega; L^2(0,T; H_0^1(D))).$$

The a priori estimates on the solution u_λ contained in the next lemma can be obtained essentially as in [9, 10, 11, 13].

Lemma 3.3. *There exists a constant N independent of λ such that*

$$\begin{aligned}\|u_\lambda\|^2_{L^2(\Omega;C([0,T];H))} &+ \lambda\|\nabla u_\lambda\|^2_{L^2(\Omega;L^2(0,T;H))} \\ &+ \|\gamma_\lambda(\nabla u_\lambda)\cdot\nabla u_\lambda\|_{L^1(\Omega\times(0,T)\times D)} + \|\beta_\lambda(u_\lambda)u_\lambda\|_{L^1(\Omega\times(0,T)\times D)} < N\end{aligned}$$

for all $\lambda \in (0,1)$. Furthermore, there exists $\Omega' \in \mathscr{F}$ with $\mathbb{P}(\Omega') = 1$ such that, for every $\omega \in \Omega'$, there exists a constant $M(\omega)$ independent of λ such that

$$\begin{aligned}\|u_\lambda(\omega)\|^2_{C([0,T];H)} &+ \lambda\|\nabla u_\lambda(\omega)\|^2_{L^2(0,T;H)} \\ &+ \|\gamma_\lambda(\nabla u_\lambda(\omega))\cdot\nabla u_\lambda(\omega)\|_{L^1((0,T)\times D)} + \|\beta_\lambda(u_\lambda(\omega))u_\lambda(\omega)\|_{L^1((0,T)\times D)} < M(\omega)\end{aligned}$$

for all $\lambda \in (0,1)$.

Proof. It is an immediate consequence of the (proofs of the) [9, Lemmata 4.3–4.7], for the part involving γ, and [11, Lemmata 5.3–5.6], for the part involving β. □

Since

$$\begin{aligned}k^*(\gamma_\lambda(\nabla u_\lambda)) \le k^*(\gamma_\lambda(\nabla u_\lambda)) + k((I+\lambda\gamma)^{-1}\nabla u_\lambda) &= \gamma_\lambda(\nabla u_\lambda)\cdot(I+\lambda\gamma)^{-1}\nabla u_\lambda \\ &\le \gamma_\lambda(\nabla u_\lambda)\cdot\nabla u_\lambda\end{aligned}$$

and

$$j^*(\beta_\lambda(u_\lambda)) \le j^*(\beta_\lambda(u_\lambda)) + j((I+\lambda\beta)^{-1}u_\lambda) = \beta_\lambda(u_\lambda)(I+\lambda\beta)^{-1}u_\lambda \le \beta_\lambda(u_\lambda)u_\lambda,$$

we infer that the families $(k^*(\gamma_\lambda(\nabla u_\lambda)))$ and $(j^*(\beta_\lambda(u_\lambda)))$ are uniformly bounded in $L^1(\Omega\times(0,T)\times D)$. Therefore, recalling that k^* and j^* are superlinear, thanks to the de la Vallée-Poussin criterion and the Dunford-Pettis theorem we deduce that the families $(\gamma_\lambda(u_\lambda))$ and $(\beta_\lambda(u_\lambda))$ are relatively weakly compact in $L^1(\Omega\times(0,T)\times D)^d$ and $L^1(\Omega\times(0,T)\times D)$, respectively. Analogously, the families $(\gamma_\lambda(u_\lambda(\omega)))$ and $(\beta_\lambda(u_\lambda(\omega)))$ are relatively weakly compact in $L^1((0,T)\times D)^d$ and $L^1((0,T)\times D)$, respectively, for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Let Ω' be as in the previous lemma and take $\omega \in \Omega'$. Then we have, along a subsequence λ' of λ depending on ω,

$$\begin{aligned}u_{\lambda'}(\omega) &\longrightarrow u(\omega) &&\text{weakly* in } L^\infty(0,T;H),\\ \nabla u_{\lambda'}(\omega) &\longrightarrow \nabla u(\omega) &&\text{weakly in } L^1((0,T)\times D)^d,\\ \lambda' u_{\lambda'}(\omega) &\longrightarrow 0 &&\text{in } L^2(0,T;H_0^1(D)),\\ \gamma_{\lambda'}(u_{\lambda'}(\omega)) &\longrightarrow \eta(\omega) &&\text{weakly in } L^1((0,T)\times D)^d,\\ \beta_{\lambda'}(u_{\lambda'}(\omega)) &\longrightarrow \xi(\omega) &&\text{weakly in } L^1((0,T)\times D),\end{aligned}$$

hence, by passage to the weak limit in the regularized equation taking test functions in V_0, we have

$$u - \int_0^\cdot \operatorname{div}\eta(s)\,ds + \int_0^\cdot \xi(s)\,ds = u_0 + \int_0^\cdot B(s)\,dW(s). \tag{3.2}$$

Moreover, by the lower semicontinuity of convex integrals, it also follows that

$$k(\nabla u(\omega)) + k^*(\eta(\omega)) + j(u(\omega)) + j^*(\xi(\omega)) \in L^1((0,T)\times D).$$

Arguing as in [11, pp. 27–28] and [9, pp. 18–19], one can show that the process u constructed in this way is unique in the space $L^2(\Omega; C([0,T]; H))$. This ensures in turn that the convergences of (u_λ) to u hold along the entire sequence λ, which is independent of ω. In particular, we have that

$$u_\lambda(\omega) \longrightarrow u(\omega) \quad \text{weakly in } L^2(0,T;H) \quad \forall\,\omega \in \Omega'.$$

Since (u_λ) is bounded in $L^2(\Omega \times (0,T) \times D)$, we deduce that u_λ converges weakly to u also in $L^2(\Omega \times (0,T); H)$. Hence, by a direct application of Mazur's lemma, we infer that u is a predictable process with values in H. Unfortunately a similar argument does not apply to η and ξ. In fact, by uniqueness of u, we can only infer from (3.2) that $-\operatorname{div}\eta + \xi$ is unique: namely, assume that $(\eta_i(\omega), \xi_i(\omega))$, $i = 1, 2$, are weak limits in $L^1(0,T; L^1(D))^{d+1}$ of $\big(\gamma_\lambda(\nabla u_\lambda(\omega)), \beta_\lambda(u_\lambda)\big)$ along two subsequences of λ (depending on ω). Then

$$\int_0^t \big(-\operatorname{div}(\eta_1 - \eta_2) + (\xi_1 - \xi_2)\big)\,ds = 0 \qquad \forall t \in [0,T],$$

hence $-\operatorname{div}(\eta_1 - \eta_2) + (\xi_1 - \xi_2) = 0$, or, equivalently, $-\operatorname{div}\eta_1 + \xi_1 = -\operatorname{div}\eta_2 + \xi_2$ in V_0' for a.a. $t \in [0,T]$. However, this allows us to claim, setting $\eta_\lambda := \gamma_\lambda(\nabla u_\lambda)$ and $\xi_\lambda := \beta_\lambda(u_\lambda)$, that

$$-\operatorname{div}\eta_\lambda + \xi_\lambda \longrightarrow -\operatorname{div}\eta + \xi \quad \text{weakly in } L^1(0,T;V_0') \quad \forall \omega \in \Omega'$$

along the whole sequence λ, thanks to the same uniqueness argument already used for u. In fact, let us set, for notational convenience,

$$\begin{aligned} \Phi : L^1(D)^{d+1} &\longrightarrow V_0' \\ (v, f) &\longmapsto -\operatorname{div} v + f \end{aligned}$$

and $\zeta_\lambda := (\eta_\lambda, \xi_\lambda)$, $\zeta := (\eta, \xi)$. Note that Φ, being a linear bounded operator, can be extended to a linear bounded operator from $L^1((0,T)\times D)^{d+1} \simeq L^1(0,T; L^1(D)^{d+1})$ to $L^1(0,T;V_0')$, also when both spaces are endowed with the weak topology. Then $\zeta_\lambda \to \zeta$ weakly in $L^1((0,T)\times D)^{d+1}$ implies that $\Phi\zeta_\lambda \to \Phi\zeta$ weakly in $L^1(0,T;V_0')$ for all $\omega \in \Omega'$. Such a convergence, however, does not allow to infer that $-\operatorname{div}\eta + \xi$ is predictable as a V_0'-valued process. The reason is that we may certainly find, by Mazur's lemma, a convex combination of $-\operatorname{div}\eta_\lambda + \xi_\lambda$ converging strongly to $-\operatorname{div}\eta + \xi$ in $L^1(0,T;V_0')$ for all $\omega \in \Omega'$, but such a convex combination would depend on ω, bringing us back to the same problem we are trying to solve. We could just say that $-\operatorname{div}\eta + \xi$ is weakly measurable with respect to $\mathscr{F}$ and the Borel σ-algebra of $L^1(0,T;V_0')$. Since this space is separable, by Pettis' theorem we also have strong measurability. This observation, however, does not seem to imply the desired result.

In order to show that $-\operatorname{div}\eta + \xi$ is indeed predictable, we are first going to prove that

$$-\operatorname{div}\eta_\lambda + \xi_\lambda \longrightarrow -\operatorname{div}\eta + \xi \qquad \text{weakly in } L^1(\Omega \times (0,T);V_0').$$

We have just shown that

$$\int_0^T \bigl\langle \Phi\zeta_\lambda(\omega,t), \phi(t) \bigr\rangle\,dt \longrightarrow \int_0^T \bigl\langle \Phi\zeta_\lambda(\omega,t), \phi(t) \bigr\rangle\,dt$$

for all $\phi \in L^\infty(0,T;V_0)$, for all $\omega \in \Omega'$, where $\langle\cdot,\cdot\rangle$ stands for the duality between V_0' and $V_0'' = V_0$. Let $\psi \in L^\infty(\Omega \times (0,T);V_0)$. Then $\psi(\omega,\cdot) \in L^\infty(0,T;V_0)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Indeed, the set

$$A := \bigl\{(\omega,t) \in \Omega \times [0,T] : \ \|\psi(\omega,t)\|_{V_0} > \|\psi\|_{L^\infty(\Omega\times(0,T);V_0)}\bigr\}$$

belongs to $\mathscr{F} \otimes \mathscr{B}([0,T])$, and, by Tonelli's theorem,

$$|A| = \int_\Omega \int_0^T 1_A\,dt\,d\mathbb{P} = \int_\Omega \operatorname{Leb}(A_\omega)\,\mathbb{P}(d\omega),$$

where $|A|$ denotes the measure of A and A_ω stands for the section of A at ω, i.e.

$$A_\omega := \bigl\{t \in [0,T] : \ (\omega,t) \in A\bigr\},$$

which belongs to $\mathscr{B}([0,T])$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Since $|A| = 0$, it follows that $|A_\omega| = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. This implies, by definition of A, that $\psi(\omega,\cdot) \in L^\infty(0,T)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Consequently, we have

$$\int_0^T \bigl\langle \Phi\zeta_\lambda(\omega,t), \psi(\omega,t) \bigr\rangle\,dt \longrightarrow \int_0^T \bigl\langle \Phi\zeta(\omega,t), \psi(\omega,t) \bigr\rangle\,dt$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. To complete the argument it is then enough to show that the left-hand side, as a subset of $L^0(\Omega)$ indexed by λ, is uniformly integrable. To this end, we collect some simple facts about uniform integrability in the following lemma.

Lemma 3.4. *Let $(X,\mathscr{A},m)$ be a finite measure space and I an arbitrary index set.*

(a) *Let* $(f_i)_{i\in I}$, $(g_i)_{i\in I} \subset L^0(X;\mathbb{R}^n)$ *be such that* $|f_i| \leq |g_i|$ *for all* $i \in I$ *and assume that* (g_i) *is uniformly integrable. Then* (f_i) *is uniformly integrable.*

(b) *Let* $(f_i) \subset L^0(X;\mathbb{R}^n)$ *be uniformly integrable and* $\phi \in L^\infty(X;\mathbb{R}^n)$. *Then* $(\phi \cdot f_i) \subset L^0(X)$ *is uniformly integrable.*

(c) *Let* $F : \mathbb{R}^n \to \mathbb{R}$ *with* $F(0) = 0$ *be convex and superlinear at infinity, and* $(f_i) \subset L^0(X;\mathbb{R}^n)$ *be such that* $(F \circ f_i)$ *is bounded in* $L^1(X)$. *Then* (f_i) *is uniformly integrable.*

(d) *Let* $(Y, \mathscr{B}, n)$ *be a further finite measure space. If* $(f_i) \subset L^0(X \times Y, \mathscr{A} \otimes \mathscr{B}, m \otimes n;\mathbb{R}^n)$ *is uniformly integrable, then* $(g_i) \subset L^0(X;\mathbb{R}^n)$ *defined by*

$$g_i := \int_Y f_i(\cdot, y)\, n(dy)$$

is uniformly integrable.

Proof. (a) is an immediate consequence of the definition of uniform integrability.
(b) Let $\varepsilon > 0$. By assumption, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\int_A \big|f_i\big|_{\mathbb{R}^n}\, dm < \frac{\varepsilon}{\|\phi\|_{L^\infty}} \qquad \forall A \in \mathscr{A},\ m(A) < \delta.$$

Then

$$\int_A \big|\phi \cdot f_i\big|\, dm \leq \|\phi\|_{L^\infty} \int_A \big|f_i\big|_{\mathbb{R}^n}\, dm < \varepsilon.$$

(c) is a variation of the classical criterion by de la Vallée-Poussin. A detailed proof (which is nonetheless very close to the one in the standard one-dimensional case) can be found in [9].
(d) Let $\varepsilon > 0$. By assumption, there exists $\delta' = \delta'(\varepsilon) > 0$ such that

$$\int_C \big|f_i\big|_{\mathbb{R}^n}\, dm \otimes n < \varepsilon \qquad \forall C \in \mathscr{A} \otimes \mathscr{B},\ m \otimes n(C) < \delta'.$$

Let $\delta := \delta'/n(Y)$ and $A \in \mathscr{A}$ with $m(A) < \delta$. Then

$$\int_A \bigg| \int_Y f_i(x,y)\, n(dy) \bigg|_{\mathbb{R}^n} m(dx) \leq \int_{A\times Y} \big|f_i(x,y)\big|_{\mathbb{R}^n}\, m(dx)\, n(dy) < \varepsilon$$

because $m \otimes n(A \times Y) = m(A)n(Y) < \delta n(Y) = \delta'$. $\square$

Let us now resume with the main reasoning. Since

$$\int_0^T \big\langle \Phi\zeta_\lambda, \psi \big\rangle \lesssim \|\psi\|_{L^\infty(\Omega\times(0,T);V_0)} \bigg(\int_0^T\!\!\int_D |\eta_\lambda| + \int_0^T\!\!\int_D |\xi_\lambda| \bigg),$$

by parts (a), (b) and (d) of the previous lemma it is sufficient to show that (η_λ) and (ξ_λ) are uniformly integrable in $\Omega \times (0,T) \times D$. But this is true, in view of part (c) of the previous lemma, because $k^*(\eta_\lambda)$ and $j^*(\xi_\lambda)$ are uniformly bounded in $L^1(\Omega \times (0,T) \times D)$. Vitali's theorem then yields

$$\int_0^T \big\langle \Phi\zeta_\lambda(\omega,t), \psi(\omega,t)\big\rangle\,dt \longrightarrow \int_0^T \big\langle \Phi\zeta(\omega,t), \psi(\omega,t)\big\rangle\,dt \qquad \text{in } L^1(\Omega),$$

hence, in particular,

$$\Phi(\eta_\lambda, \xi_\lambda) \longrightarrow \Phi(\eta, \xi) \quad \text{weakly in } L^1(\Omega \times (0,T); V_0').$$

Furthermore, from the uniform integrability of (η_λ) and (ξ_λ) in $\Omega \times (0,T) \times D$ it also follows that, along a subsequence μ of λ,

$$(\eta_\mu, \xi_\mu) \longrightarrow (\bar\eta, \bar\xi) \quad \text{weakly in } L^1(\Omega \times (0,T) \times D)^{d+1},$$

hence also

$$\Phi(\eta_\mu, \xi_\mu) \longrightarrow \Phi(\bar\eta, \bar\xi) \quad \text{weakly in } L^1(\Omega \times (0,T); V_0').$$

An application of Mazur's lemma yields, in complete analogy to the case of u, that $\bar\eta$ and $\bar\xi$ are predictable processes with values in $L^1(D)^d$ and $L^1(D)$, respectively. Since μ is a subsequence of λ, by uniqueness of the weak limit we have that $\Phi(\eta,\xi) = \Phi(\bar\eta,\bar\xi)$, i.e.

$$-\operatorname{div}\eta + \xi = -\operatorname{div}\bar\eta + \bar\xi.$$

This implies that the identity (3.2) remains true with η and ξ replaced by $\bar\eta$ and $\bar\xi$, respectively. In other words, modulo relabeling, we can just assume, without loss of generality, that η and ξ in (3.2) are predictable and that

$$(\eta_\lambda, \xi_\lambda) \longrightarrow (\eta, \xi) \quad \text{weakly in } L^1(\Omega \times (0,T) \times D)^{d+1}.$$

By weak lower semicontinuity and Lemma 3.3, this also implies, arguing as in [9, 10, 11, 13], that

$$\begin{aligned}
u &\in L^2(\Omega; L^\infty(0,T;H)) \cap L^1(\Omega; L^1(0,T; W_0^{1,1}(D))),\\
\eta &\in L^1(\Omega \times (0,T) \times D)^d,\\
\xi &\in L^1(\Omega \times (0,T) \times D),\\
k(\nabla u) + k^*(\eta) = \nabla u \cdot \eta &\in L^1(\Omega \times (0,T) \times D),\\
j(u) + j^*(\xi) = u\xi &\in L^1(\Omega \times (0,T) \times D).
\end{aligned}$$

In order to show that $\eta \in \gamma(\nabla u)$ and $\xi \in \beta(u)$ a.e. in $\Omega \times (0,T) \times D$, it suffices to prove, by the maximal monotonicity of γ and β, that

$$\limsup_{\lambda \to 0} \mathbb{E} \int_0^T \int_D \big(\eta_\lambda \cdot \nabla u_\lambda + \xi_\lambda u_\lambda\big) \le \mathbb{E} \int_0^T \int_D \big(\eta \cdot \nabla u + \xi u\big) \tag{3.3}$$

(cf. [9, pp. 17-18]). To this purpose, note that the ordinary Itô formula and Proposition 3.1 yield

$$\frac{1}{2}\,\mathbb{E}\|u_\lambda(T)\|^2 + \mathbb{E}\int_0^T\!\!\int_D \big(\eta_\lambda\cdot\nabla u_\lambda + \xi_\lambda u_\lambda\big) = \frac{1}{2}\,\mathbb{E}\|u_0\|^2 + \frac{1}{2}\,\mathbb{E}\int_0^T \big\|B(s)\big\|^2_{\mathscr{L}^2(U,H)}\,ds$$

and

$$\frac{1}{2}\,\mathbb{E}\|u(T)\|^2 + \mathbb{E}\int_0^T\!\!\int_D \big(\eta\cdot\nabla u + \xi u\big) = \frac{1}{2}\,\mathbb{E}\|u_0\|^2 + \frac{1}{2}\,\mathbb{E}\int_0^T \big\|B(s)\big\|^2_{\mathscr{L}^2(U,H)}\,ds,$$

respectively (the stochastic integrals appearing in both versions of Itô's formula are in fact martingales, not just local martingales, hence their expectation is zero). Since $u_\lambda(T)\to u(T)$ weakly in $L^2(\Omega;H)$, one has $\mathbb{E}\|u(T)\|^2 \le \liminf_{\lambda\to 0}\mathbb{E}\|u_\lambda(T)\|^2$, hence, by comparison, (3.3) follows.

Finally, the strong pathwise continuity (in H) of u is an immediate consequence of the corollary to Proposition 3.1.

Remark 3.5. Another way to "restore" uniqueness of limits for the pair $\zeta_\lambda = (\eta_\lambda,\xi_\lambda)$ is to view it as element of the quotient space $L^1(D)^{d+1}/M$, where $M := \ker\Phi$. Note that M is a closed subset of L^1 (we suppress the superscript as well as the indication of the domain just within this remark), as the inverse image of the closed set $\{0\}$ through a continuous linear map, hence L^1/M is a Banach space. However, working with the spaces $L^1(0,T;L^1/M)$ and $L^1(\Omega\times(0,T);L^1/M)$ present technical difficulties due to the fact that their dual spaces are hard to characterize. A bit more precisely, this has to do with the fact that the dual of $L^1(0,T;E)$ is $L^\infty(0,T;E')$ if and only if E has the Radon-Nikodym property. This property is enjoyed by reflexive spaces, but not by L^1 spaces (see, e.g., [3]).

3.3 Well-posedness in the general case

Let us consider now equation (1.1) with general additive noise, i.e. with

$$B\in L^2(\Omega;L^2(0,T;\mathscr{L}^2(U,H))).$$

Thanks to classical elliptic regularity results, there exists $m\in\mathbb{N}$ such that the $(I-\delta\Delta)^{-m}$ is a continuous linear map from $L^1(D)$ to $W^{1,\infty}(D)\cap H^1_0(D)$ for every $\delta>0$. Setting then $V_0 := (I-\Delta)^{-m}(H)$ and $B^\delta := (I-\delta\Delta)^{-m}B$, we have $B^\delta\in L^2(\Omega;L^2(0,T;\mathscr{L}^2(U,V_0)))$, hence, by the well-posedness results already obtained, the equation

$$du^\delta - \operatorname{div}\gamma(\nabla u^\delta)\,dt + \beta(u^\delta)\,dt \ni B^\delta\,dW, \qquad u^\delta(0)=u_0,$$

admits a strong solution $(u^\delta,\eta^\delta,\xi^\delta)$. Arguing as in [9, 10, 11, 13], one can show using Itô's formula that (u^δ) is a Cauchy sequence in $L^2(\Omega;C([0,T];H))$ and that

(∇u^δ), (η^δ), and (ξ^δ) are relatively weakly compact in $L^1(\Omega\times(0,T)\times D)$, so that

$$\begin{aligned} u^\delta &\longrightarrow u \quad && \text{in } L^2(\Omega; C([0,T];H)),\\ u^\delta &\longrightarrow u \quad && \text{weakly in } L^1(\Omega\times(0,T); W^{1,1}_0(D)),\\ \eta^\delta &\longrightarrow \eta \quad && \text{weakly in } L^1(\Omega\times(0,T)\times D)^d,\\ \xi^\delta &\longrightarrow \xi \quad && \text{weakly in } L^1(\Omega\times(0,T)\times D), \end{aligned}$$

from which it follows that (u,η,ξ) solves the original equation. Moreover, the strong-weak closure of β readily implies that $\xi \in \beta(u)$ a.e. in $\Omega\times(0,T)\times D$. Finally, arguing as in the previous subsection, by weak lower semicontinuity of convex integrals and Itô's formula one can show that

$$\limsup_{\lambda\to 0} \mathbb{E}\int_0^T\int_D \eta_\lambda\cdot\nabla u_\lambda \le \mathbb{E}\int_0^T\int_D \eta\cdot\nabla u,$$

so that $\eta\in\gamma(\nabla u)$ a.e. in $\Omega\times(0,T)\times D$ as well.

Continuous dependence on the initial datum is a consequence of Itô's formula and the monotonicity of γ and β. Finally, the generalization to the case of multiplicative noise follows using the Lipschitz continuity of B and a classical fixed point argument. A detailed exposition of the arguments needed to prove these claims can be found in [9, 10, 11, 13].

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