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**Abstract**

We show that for any set  $S \subseteq \mathbb{Z}$  with  $|S| = 4$  there exists a 3-coloring of  $\mathbb{Z}$  in which every translate of  $S$  receives all three colors. This implies that  $S$  has a codensity of at most  $1/3$ , proving a conjecture of Newman. We also consider related questions in  $\mathbb{Z}^d$ ,  $d \geq 2$ .

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### 1. Introduction

Throughout the paper, let  $G$  denote an arbitrary abelian group. Given  $S, T \subseteq G$ ,  $n \in G$ , define  $S + T = \{s + t : s \in S, t \in T\}$  and  $n + S = \{n\} + S$ . Any set of the form  $n + S$  is called a *translate* of  $S$ . Given a subset  $S$  of  $G$ , a coloring of the elements of  $G$  is  *$S$ -polychromatic* if every translate of  $S$  contains an element of each color. Define the *polychromatic number* of  $S$ , denoted  $p_G(S)$ , to be the largest number of colors allowing an  $S$ -polychromatic coloring of the elements of  $G$ . We just write  $p(S)$  when the choice of  $G$  is clear from context.

We begin with some elementary observations that will be used repeatedly. First, if  $S'$  is a subset of  $S$ , then any  $S'$ -polychromatic coloring is also an  $S$ -polychromatic coloring, and thus  $p(S') \leq p(S)$ . Also, if  $S' = n + S$  is a translate of  $S$ , then  $S'$  and  $S$  have the same set of translates, so  $p(S') = p(S)$ .

We are primarily concerned with the setting where  $G = \mathbb{Z}$  and  $S$  is finite. If  $S \subseteq \mathbb{Z}$  has cardinality 1 or 2,  $p(S) = |S|$ . For  $|S| = 3$ ,  $p(S)$  can be 2 or 3. For example, if  $S = \{0, 1, 5\}$  then every translate of  $S$  contains three elements which are each in different congruence classes (mod 3). Thus a 3-coloring of the integers where each congruence class (mod 3) is colored with a different color is  $S$ -polychromatic, and  $p(\{0, 1, 5\}) = 3$ . However  $p(\{0, 1, 3\}) = 2$ . To see that  $p(\{0, 1, 3\}) \neq 3$ , let  $\chi$  be a 3-coloring of  $\mathbb{Z}$  with  $\chi(0)$ ,  $\chi(1)$ , and  $\chi(3)$  all different. Some element  $s \in \{0, 1, 3\}$  has  $\chi(s) = \chi(2)$ , and there is a translate of  $\{0, 1, 3\}$  that contains both  $s$  and 2, so the coloring is not polychromatic. Our main result concerns the polychromatic numbers of sets with cardinality 4.

**Theorem 1.** *If  $S \subseteq \mathbb{Z}$  and  $|S| = 4$ , then  $p(S) \geq 3$ .*

The proof of Theorem 1 is given in Section 2. For larger sets  $S$ , Alon, Kríž, and Nešetřil [2] proved that  $p(S) \geq \frac{(1+o(1))|S|}{3 \ln |S|}$ , while there exists some set  $S$  where  $p(S) \leq \frac{(1+o(1))|S|}{\ln |S|}$ . Subsequently, Harris and Srinivasan [6] established a tight asymptotic lower bound on polychromatic numbers.

**Theorem 2 ([2], [6]).** *For a finite set  $S \subseteq \mathbb{Z}$ ,  $p(S) \geq \frac{(1+o(1))|S|}{\ln |S|}$ . Moreover, there exists some set  $S$  where  $p(S) \leq \frac{(1+o(1))|S|}{\ln |S|}$ .*

One motivation for studying polychromatic numbers is that they provide bounds for Turán type problems (see for example [1], [10], [11]). Call  $T \subseteq G$  a *blocking set* for  $S$  if  $G \setminus T$  contains no translate of  $S$ , i.e., if for all  $n \in G$ ,  $n + S \not\subseteq G \setminus T$ . A Turán type problem asks for the smallest blocking set for a given set  $S$ . In the case where  $S$  is finite and  $G = \mathbb{Z}$ , any blocking set is countably infinite, so we ask how small the density of a blocking set can be. Following the notation of Newman [8], (he worked in the setting of the natural numbers, but the definitions are equivalent),

define for any set  $T \subseteq \mathbb{Z}$  its *upper density*  $\bar{d}(T)$  and *lower density*  $\underline{d}(T)$  as

$$\bar{d}(T) = \limsup_{n \rightarrow \infty} \frac{|T \cap [-n, n]|}{2n + 1} \quad \text{and} \quad \underline{d}(T) = \liminf_{n \rightarrow \infty} \frac{|T \cap [-n, n]|}{2n + 1}.$$

If  $\bar{d}(T) = \underline{d}(T)$ , we call this quantity the *density* of  $T$  and denote it by  $d(T)$ . Define  $\alpha(S)$  to be a measure of how small the density of a blocking set for  $S$  can be. Let

$$\alpha(S) = \inf\{d(T) : T \text{ is a blocking set for } S \text{ and } d(T) \text{ exists}\}.$$

In Section 3, we describe the relationship between polychromatic colorings and blocking sets, and prove Lemma 1.

**Lemma 1.** *For any finite set  $S \subseteq \mathbb{Z}$ ,  $\alpha(S) \leq 1/p(S)$ .*

One of the main consequences of Theorem 1 concerns covering densities of sets of integers. Given a set  $S \subseteq G$ , we say  $T \subseteq G$  is a *complement set* for  $S$  if  $S + T = G$ . We say  $S$  *tiles  $G$  by translation* if it has a complement set  $T$  such that if  $s_1, s_2 \in S$ ,  $t_1, t_2 \in T$ , then  $s_1 + t_1 = s_2 + t_2$  implies  $s_1 = s_2$  and  $t_1 = t_2$ . We call such a complement set  $T$  a *tiling complement set* for  $S$ . Note that the set  $S$  tiles  $G$  by translation if all the translates  $S + t$ ,  $t \in T$ , are disjoint and every  $n \in G$  is an element of some translate  $S + t$ . In this paper we only consider tilings by translation, so if  $S$  tiles  $G$  by translation with tiling complement set  $T$  we will simply say  $S$  tiles  $G$  and write  $G = S \oplus T$ .

Again, our primary interest will be the case where  $G = \mathbb{Z}$  and  $S$  is finite. For example, if  $S = \{0, 1, 5\}$ , then  $S$  tiles  $\mathbb{Z}$  with complement set  $T = \{3n : n \in \mathbb{Z}\}$ . However  $S = \{0, 1, 3\}$  does not tile  $\mathbb{Z}$ . Newman [9] proved necessary and sufficient conditions for a finite set  $S$  to tile  $\mathbb{Z}$  if  $|S|$  is a power of a prime.

**Theorem 3 (Newman [9]).** *Let  $S = \{s_1, \dots, s_k\}$  be distinct integers with  $|S| = p^\alpha$  where  $p$  is prime and  $\alpha$  is a positive integer. For  $1 \leq i < j \leq k$  let  $p^{e_{ij}}$  be the highest power of  $p$  that divides  $s_i - s_j$ . Then  $S$  tiles  $\mathbb{Z}$  if and only if  $|\{e_{ij} : 1 \leq i < j \leq k\}| \leq \alpha$ .*

Later Coven and Meyerowitz [5] gave necessary and sufficient conditions for  $S$  to tile  $\mathbb{Z}$  when  $|S| = p_1^{\alpha_1} p_2^{\alpha_2}$ , where  $p_1$  and  $p_2$  are primes. The general question is still open. Kolountzakis and Matolcsi [7] and Amiot [3] have published recent work motivated by what are called rhythmic tilings in music.

If a finite set  $S$  tiles  $\mathbb{Z}$ , it has a complement set of density  $1/|S|$ . Following Newman [8], we define the *codensity* of a set  $S$ , denoted  $c(S)$ , as a measure of how small the density of a complement set can be. Let

$$c(S) = \inf\{d(T) : S + T = \mathbb{Z} \text{ and } d(T) \text{ exists}\}.$$

We are interested in the largest codensities for sets of a given cardinality. Define

$$c_k = \sup_{\{S : |S|=k\}} c(S).$$

An example of a complement set for  $\{0, 1, 3\}$  is  $\{t \in \mathbb{Z} : t \equiv 0 \text{ or } 1 \pmod{5}\}$ , so  $c(\{0, 1, 3\}) \leq 2/5$ . The following theorem and conjecture on  $c_4$  are due to Newman.

**Theorem 4 (Newman [8]).**

- $c(\{0, 1, 3\}) = 2/5$ .
- $c_3 = 2/5$ .
- $c(\{0, 1, 2, 4\}) = 1/3$ .

**Conjecture 1 (Newman [8]).**  $c_4 = 1/3$ .

Conjecture 1 is stated and attributed to Newman by Weinstein [16], who proved that  $c_4 < .339934$ . Based on a computer search, Bollobás, Janson, and Riordan [4] confirmed Newman’s conjecture for sets with diameter at most 22, where the *diameter* of a nonempty finite set of integers is defined to be the difference between the largest and smallest elements in the set. They also conjectured that  $c_5 = 3/11$  and  $c_6 = 1/4$  (See Remark 5.6 and Question 5.7 in [4]. Note they use different notation).

In Section 3 we prove the following lemma relating blocking sets and complement sets.

**Lemma 2.** *For any finite set  $S \subseteq \mathbb{Z}$ ,  $c(S) = \alpha(S)$ .*

Theorem 1, along with Lemmas 1 and 2, suffice to resolve Conjecture 1.

**Theorem 5.**  $c_4 = 1/3$ .

*Proof.* Theorem 4 implies  $c(\{0, 1, 2, 4\}) = 1/3$ , so it remains to show that for any other set  $S$  with cardinality four,  $c(S) \leq 1/3$ . Let  $S \subseteq \mathbb{Z}$  have four elements. Then Theorem 1 implies that  $p(S) \geq 3$ , and by Lemmas 1 and 2,

$$c(S) = \alpha(S) \leq 1/p(S) \leq 1/3.$$

□

In Subsection 3.1 we consider the relationship between polychromatic colorings and tilings. The main result is Theorem 6, which states that a set  $S$  tiles an abelian group  $G$  by translation if and only if  $p(S) = |S|$ .

Finally, in Section 4 we turn our attention to polychromatic numbers and tilings for finite sets in  $\mathbb{Z}^d$ . We begin by proving in Theorem 7 that the bound of Theorem 2 applies to subsets of  $\mathbb{Z}^d$ . We then show that if a set of points in  $\mathbb{Z}^d$  is collinear, determining its polychromatic number is equivalent to determining the polychromatic number of a specific projection of this set into  $\mathbb{Z}$ . Theorem 6 implies that a set  $S$  tiles  $\mathbb{Z}^d$  if and only if  $p_{\mathbb{Z}_d}(S) = |S|$ , so we use this to restate some well-known results on tilings of  $\mathbb{Z}^d$  by finite sets in the language of polychromatic colorings. We conclude by applying these results to determine polychromatic numbers of sets with cardinality 3 and 4 in  $\mathbb{Z}^d$ .

**2. Sets of Cardinality Four**

In this section we prove that every set of four integers has polychromatic number at least 3. We begin by stating some general lemmas that reduce the problem of finding an  $S$ -polychromatic coloring of  $\mathbb{Z}$  to finding an  $S$ -polychromatic coloring of  $\mathbb{Z}_m = \{0, 1, \dots, m - 1\}$  for a specific choice of  $m$ .

**Lemma 3.** *If  $G$  and  $H$  are abelian groups and  $\phi : G \rightarrow H$  is a homomorphism, then for all  $S \subseteq G$ ,*

$$p_G(S) \geq p_H(\phi(S)).$$

*Proof.* Let  $S \subseteq G$  and let  $\chi'$  be a  $\phi(S)$ -polychromatic coloring of  $H$  with  $p_H(\phi(S))$  colors. Define the coloring  $\chi$  on  $G$  such that  $\chi(g) = \chi'(\phi(g))$ . Consider a translate  $g + S$  of the set  $S$ . Since

$$\chi(g + S) = \chi'(\phi(g + S)) = \chi'(\phi(g) + \phi(S)),$$

and  $\phi(g) + \phi(S) \subseteq H$  is a translate of  $\phi(S)$ ,  $\chi(g + S)$  contains all  $p_H(\phi(S))$  colors, and  $\chi$  is  $S$ -polychromatic. □

**Corollary 1.** *If  $G$  and  $H$  are abelian groups and  $\phi : G \rightarrow H$  is an isomorphism, then for all  $S \subseteq G$ ,*

$$p_G(S) = p_H(\phi(S)).$$

**Lemma 4.** *If  $H$  is a subgroup of an abelian group  $G$ , and  $S \subseteq H$ , then  $p_H(S) = p_G(S)$ .*

*Proof.* Since  $H$  is a subset of  $G$ ,  $p_H(S) \geq p_G(S)$ . To prove the other inequality, suppose  $\chi'$  is an  $S$ -polychromatic coloring of  $H$  with  $p_H(S)$  colors. Let  $V \subseteq G$  be a set containing exactly one element from each coset of  $H$ . For every  $g \in G$ , there is a unique  $h \in H$  and  $v \in V$  such that  $g = h + v$ . Define a coloring  $\chi$  of  $G$  such that  $\chi(g) = \chi'(h)$ , i.e., the summand  $v$  is ignored. We show that  $\chi$  is  $S$ -polychromatic. Given  $g \in G$  with  $g = h + v$  for some  $h \in H$ ,  $v \in V$ , consider the translate  $g + S$ , and note

$$\chi(g + S) = \chi(h + v + S) = \chi'(h + S).$$

Since  $h + S \subseteq H$  is a translate of  $S$ ,  $\chi(g + S)$  contains all  $p_H(\phi(S))$  colors, and  $\chi$  is  $S$ -polychromatic. □

**Lemma 5.** *Suppose  $a, b, c, k \in \mathbb{Z}$  with  $0 < a < b < c$  and  $k \geq 1$ . Let  $S = \{0, ka, kb, kc\}$ ,  $S_1 = \{0, a, b, c\}$ , and  $S_2 = \{0, b - a, b, 2b - a\}$ . Then*

- (i)  $p_{\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$ .
- (ii) If  $q \in \mathbb{N}$ , then  $p_{\mathbb{Z}}(S_1) \geq p_{\mathbb{Z}_q}(S_1)$ .

(iii) If  $m = c - a + b$ , then  $p_{\mathbb{Z}_m}(S_1) = p_{\mathbb{Z}_m}(S_2)$ .

(iv) If  $q \in \mathbb{N}$ , with  $\gcd(k, q) = 1$ , then  $p_{\mathbb{Z}_q}(S) = p_{\mathbb{Z}_q}(S_1)$ .

*Proof.*

- (i) Define  $\phi : \mathbb{Z} \rightarrow k\mathbb{Z}$  such that  $\phi(n) = kn$ . Then  $\phi$  is an isomorphism where  $\phi(S_1) = S$ , so Corollary 1 implies  $p_{k\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$ . Since  $k\mathbb{Z}$  is a subgroup of  $\mathbb{Z}$  and  $S \subseteq k\mathbb{Z}$ , Lemma 4 implies  $p_{k\mathbb{Z}}(S) = p_{\mathbb{Z}}(S)$ . By combining these equations, we conclude  $p_{\mathbb{Z}}(S) = p_{k\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$ .
- (ii) This part follows from Lemma 3 using the homomorphism  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_q$  where  $\phi(n) = n \pmod{q}$ .
- (iii) In  $\mathbb{Z}_m$ , with addition  $\pmod{m}$ ,  $S_2 = S_1 + (b - a)$ . Thus in  $\mathbb{Z}_m$ ,  $S_1$  and  $S_2$  are translates of each other and have the same polychromatic number.
- (iv) Define  $\phi : \mathbb{Z}_q \rightarrow \mathbb{Z}_q$  so that  $\phi(n) = kn$ . Since  $\gcd(k, q) = 1$ ,  $\phi$  is an isomorphism. Since  $\phi(S_1) = S$ , Corollary 10 implies  $p_{\mathbb{Z}_q}(S) = p_{\mathbb{Z}_q}(S_1)$ .

□

We now prove Theorem 1.

*Proof.* Let  $S \subseteq \mathbb{Z}$  have cardinality four. Since all translates of  $S$  have the same polychromatic number, we may assume that 0 is the smallest element of  $S$ , and by Lemma 5, Part (i), it suffices to prove the theorem in the case that  $S = \{0, a, b, c\}$  with  $0 < a < b < c$  and  $\gcd(a, b, c) = 1$ .

It is possible, though tedious, to prove the entire theorem by hand. Thus in the interest of simplifying the exposition, we verified using a computer search that for every  $S$  with diameter at most 288 there exists an  $S$ -polychromatic 3-coloring of  $\mathbb{Z}_q$  for some  $q$  depending on  $S$ . The code for this search has been included as an ancillary file with the preprint of this paper at [arxiv.org/abs/1704.00042](https://arxiv.org/abs/1704.00042). By Lemma 5, Part (ii), this gives a periodic  $S$ -polychromatic 3-coloring of  $\mathbb{Z}$ . Hence we suppose that  $c \geq 289$ .

For the remainder of the proof, let  $m = c - a + b$ . By Lemma 5, Parts (ii) and (iii), it suffices to show that we can 3-color  $\mathbb{Z}_m = \{0, 1, \dots, m - 1\}$  so that the translates of  $\{0, b - a, b, 2b - a\}$  are polychromatic. So for the remainder of the proof we assume  $S = \{0, b - a, b, 2b - a\}$  and seek an  $S$ -polychromatic 3-coloring of  $\mathbb{Z}_m$ . The key observation regarding  $S$  is that it contains two repeated differences:  $b - a$  and  $b$ .

Define  $d_1 = \gcd(b, m)$  and  $d_2 = \gcd(b - a, m)$ . Since  $1 = \gcd(a, b, c) = \gcd(b - a, b, c - a + b) = \gcd(b - a, b, m)$ , we know  $\gcd(d_1, d_2) = 1$ . We distinguish two main cases. In the first case, which we call “single cycle,” we assume  $\min\{d_1, d_2\} = 1$  and give a coloring of  $\mathbb{Z}_m$ . In the second case, which we call “multiple cycle,” we

assume  $\min\{d_1, d_2\} > 1$  and partition  $\mathbb{Z}_m$  into multiple cycles of length  $m/d_i$  for one of the choices of  $i$ . We then give a rule for coloring each cycle.

**Main Case 1 (Single cycle):** Suppose  $\min\{d_1, d_2\} = 1$ . Without loss of generality, assume  $d_1 = 1$  (if not, then simply switch all occurrences of  $b$  and  $b - a$  in the argument below). Let  $2 \leq g \leq m - 2$  satisfy  $gb \equiv b - a \pmod{m}$ , so that  $S = \{0, bg, b, b(g + 1)\}$ . Applying Lemma 5, Part (iv), with  $q = m$  and  $k = b$ , we can instead work with  $S = \{0, g, 1, g + 1\} = \{0, 1, g, g + 1\}$ .

We may assume that  $g \leq m/2$ , as otherwise we could work with the translate  $(m - g) + S = \{0, 1, m - g, m - g + 1\}$ . Let  $s$  be the smallest multiple of 3 such that  $g > \lceil m/s \rceil$ . We consider four subcases: The first two are (1a)  $g = 2, 3$ , or 4 and (1b)  $5 \leq g < 2\lfloor m/s \rfloor$ . In the remaining subcases (1c) and (1d),  $2\lfloor m/s \rfloor \leq g \leq \lfloor m/(s - 3) \rfloor$ . For  $m > 8$ , if  $2\lfloor m/s \rfloor \leq g \leq m/2$  then  $s > 3$ , and for  $m > 44$ , if  $2\lfloor m/s \rfloor \leq g \leq \lfloor m/(s - 3) \rfloor$  then  $s < 9$ . Since  $m > c \geq 289 > 44$ , we can assume  $s = 6$ , so  $2\lfloor m/6 \rfloor \leq g \leq \lfloor m/3 \rfloor$ . This implies  $m = 3g + k$  where  $-2 \leq k \leq 5$  and there are two further subcases to consider, depending on the residue class of  $m$  modulo 6: (1c)  $m = 3g - 2, 3g - 1, 3g + 1, 3g + 2, 3g + 4$ , or  $3g + 5$ , and (1d)  $m = 3g$  or  $3g + 3$ .

**Subcase (1a):** Suppose  $g = 2, 3$ , or 4. Then  $S = \{0, 1, 2, 3\}, \{0, 1, 3, 4\}$ , or  $\{0, 1, 4, 5\}$ , respectively. In Subcase (1c) we will construct  $S$ -polychromatic 3-colorings of  $\mathbb{Z}_m$  for each of these sets.

**Subcase (1b):** Suppose  $5 \leq g < 2\lfloor m/s \rfloor$ . Then split  $\mathbb{Z}_m$  into  $s$  intervals as equally as possible (i.e., of lengths  $\lfloor m/s \rfloor$  and  $\lceil m/s \rceil$ ) and color these intervals 010101..., followed by 121212..., then 202020..., repeating  $s/3$  times. Since  $\lceil m/s \rceil < g < 2\lfloor m/s \rfloor$ , any translate of  $S'$  where the pairs  $\{0, 1\}$  and  $\{g, g + 1\}$  lie in different intervals gets all three colors. If one of the pairs  $\{0, 1\}$  or  $\{g, g + 1\}$  straddles two consecutive intervals, this pair may get only the single color common to these two intervals, but then the other pair lies fully inside a third interval which is colored with the remaining two colors.

**Subcase (1c):** Suppose  $m = 3g - 2, 3g - 1, 3g + 1, 3g + 2, 3g + 4$ , or  $3g + 5$ . In this case we know that  $m \not\equiv 0 \pmod{3}$  so we can apply Lemma 5, Part (iv), with  $q = m$  and  $k = 3$ , and instead work with one of the sets in  $\mathcal{S} = \{\{0, 2, 3, 5\}, \{0, 1, 3, 4\}, \{0, 1, 2, 3\}, \{0, 3, 4, 7\}, \{0, 3, 5, 8\}\}$ . For example, if  $m = 3g - 2$ , then multiplying by 3,  $S$  is transformed into  $\{0, 3, 3g, 3g + 3\} \equiv \{0, 2, 3, 5\}$ , while if  $m = 3g + 4$ , then multiplying by 3,  $S$  is transformed into  $\{0, 3, 3g, 3g + 3\} \equiv \{0, 3, -4, -1\}$ , which is a translate of  $\{0, 3, 4, 7\}$ .

Thus we have reduced the problem to finding an  $S$ -polychromatic 3-coloring of  $\mathbb{Z}_m$  for each of the sets  $S \in \mathcal{S}$ . For each  $S \in \mathcal{S}$ , in Table 1 we list one interval of length  $r$  and one of length  $r + 1$  obtained by adding an initial 0 to the other interval. We also include an interval for  $\{0, 1, 4, 5\}$  to cover Subcase (1a). Each of the intervals has the property that concatenating the intervals of length  $r$  and  $r + 1$  in any way results in an  $S$ -polychromatic coloring for the corresponding set. One can

| $S$              | $r$ | period $r$ | period $r + 1$ |
|------------------|-----|------------|----------------|
| $\{0, 2, 3, 5\}$ | 6   | 001122     | 0001122        |
| $\{0, 1, 3, 4\}$ | 6   | 001212     | 0001212        |
| $\{0, 1, 2, 3\}$ | 3   | 012        | 0012           |
| $\{0, 3, 4, 7\}$ | 9   | 000111222  | 0000111222     |
| $\{0, 3, 5, 8\}$ | 9   | 000111222  | 0000111222     |
| $\{0, 1, 4, 5\}$ | 7   | 0001212    | 00001212       |

Table 1: One interval of a periodic coloring for sets in Subcases (1a) and (1c).

check this by hand, using the fact that, in each case, a translate of  $S \in \mathcal{S}$  intersects at most two consecutive intervals. Hence if  $m$  can be expressed as a positive integer combination of  $r$  and  $r + 1$ ,  $m = hr + k(r + 1)$ , we can obtain an  $S$ -polychromatic coloring with period  $m$ . For  $r = 3, 6, 7, 9$ , by the 2-coin Frobenius problem,  $m$  can be expressed as a positive integer combination of  $r$  and  $r + 1$  for any  $m$  greater than  $r^2 - r - 1 \leq 71 < 289$ .

**Subcase (1d):** Suppose  $m = 3g$  or  $3g + 3$ . If  $g \not\equiv 0 \pmod{3}$  then simply color  $\mathbb{Z}_m$  with the pattern 0120120...012. If  $g \equiv 0 \pmod{3}$  and  $m = 3g$ , color  $\mathbb{Z}_m$  in 3 equal intervals, each of length  $g$ : 012012...012 followed by 120120...120 followed by 201201...201. Finally, if  $g \equiv 0 \pmod{3}$  and  $m = 3g + 3$  we color  $\mathbb{Z}_m$  in 3 equal intervals, each of length  $g + 1$ : 012012...0120 followed by 201201...2012 followed by 120120...1201.

**Main case 2 (Multiple cycles):** Suppose  $\min\{d_1, d_2\} > 1$ . Since  $d_1$  and  $d_2$  are relatively prime, at most one of them can be a multiple of 3. Choose the smallest of these numbers that is not a multiple of 3, and as in the single cycle case, without loss of generality assume it is  $d_1$ .

Let  $e_1 = m/d_1$  and  $e_2 = m/d_2$ . For  $0 \leq i < d_1$ , let

$$C_i = \{(b - a)i + bj \pmod{m} : 0 \leq j < e_1\}.$$

Since

$$\mathbb{Z}_m = \{(b - a)i + bj \pmod{m} : 0 \leq i < d_1, 0 \leq j < e_1\},$$

the  $C_i$ 's form a partition of  $\mathbb{Z}_m$  into  $d_1$  cycles, each with  $e_1$  elements.

Let  $c_{i,j}$  denote the  $j$ th element of  $C_i$ , i.e.,  $c_{i,j} = i(b - a) + jb \pmod{m}$ . Note that any translate of  $S$  contains two consecutive elements of two consecutive cycles, i.e., any translate of  $S$  has the form  $\{c_{i,j}, c_{i,j+1}, c_{i+1,j}, c_{i+1,j+1}\}$ , where the first entry in the subscript is taken  $\pmod{d_1}$  and the second entry is taken  $\pmod{e_1}$ . We describe an  $S$ -polychromatic 3-coloring for each of four subcases: (2a)  $e_1$  is even, (2b)  $d_1$  is even and  $e_1$  is odd, (2c)  $d_1$  and  $e_1$  are both odd, with  $e_1 \leq 17$ , and (2d)  $d_1$  and  $e_1$  are both odd, with  $e_1 \geq 19$ .

**Subcase (2a):** Suppose  $e_1$  is even. For  $i = 0, \dots, \lfloor d_1/2 \rfloor - 1$ , color each  $C_{2i}$  by



01010...01 and each  $C_{2i+1}$  by 02020...02. Finally, if  $d_1$  is odd, color  $C_{d_1-1}$  by 1212...12.

**Subcase (2b):** Suppose  $d_1$  is even and  $e_1$  is odd. For  $i = 0, \dots, d_1/2 - 1$ , color each  $C_{2i}$  by 01010...011 and each  $C_{2i+1}$  by 22020...02.

**Subcase (2c):** Suppose  $d_1$  and  $e_1$  are both odd, with  $e_1 \leq 17$ . Since  $e_1 e_2 \geq m > c \geq 289$ , one of  $e_1$  and  $e_2$  is larger than 17, so  $e_2 > e_1$  and hence  $d_1 > d_2$ . Since  $d_1$  is the smaller of  $d_1$  and  $d_2$  that is not a multiple of 3,  $d_2$  must be a multiple of 3, and thus so is  $e_1$ .

We color each  $C_i$  with one of three patterns: 012012...012, 120120...120, or 201201...201. Such a coloring is  $S$ -polychromatic so long as for all  $i$ ,  $C_i$  and  $C_{i+1}$  are colored with different patterns. For  $0 \leq i \leq (d_1 - 3)/2$ , color  $C_{2i}$  with the first pattern and color  $C_{2i+1}$  with the second pattern. Finally, color  $C_{d_1-1}$  with the third pattern.

**Subcase (2d):** Suppose  $d_1$  and  $e_1$  are both odd, with  $e_1 \geq 19$ . Since  $d_1$  is not divisible by 3 and  $\min\{d_1, d_2\} > 1$ ,  $d_1 \geq 5$ . Let  $e_1 = u + v + w$  be a sum of odd integers  $u, v, w$  with  $u \geq v \geq w \geq u - 2$ . Color  $C_0$  in intervals of size  $u, v, w$ , using the patterns 0101...010 then 1212...121 and then 2020...202. For each  $i \geq 1$ , color  $C_i$  by taking a “counterclockwise rotation” of length  $r_i$  of the coloring of  $C_{i-1}$ , so that the color of  $c_{i,j+r}$  is the same as the color of  $c_{i-1,j}$ . For  $1 \leq i \leq d_1 - 1$ , if  $u \leq r_i \leq v + w = e_1 - u$ , then each translate of  $S$  meeting  $C_{i-1}$  and  $C_i$  receives all 3 colors.

It remains to show that there are choices of  $r_1, \dots, r_{d_1-1}$  with  $u \leq r_i \leq v + w = e_1 - u$  so that of the translates of  $S$  meeting  $C_{d_1-1}$  and  $C_0$  receive all three colors. The coloring of  $C_0$  is a “clockwise rotation” of length  $R = -r_1 - r_2 - \dots - r_{d_1-1}$  of the coloring of  $C_{d_1-1}$ , i.e., the color of  $c_{0,j-R}$  is the same as the color of  $c_{d_1-1,j}$ . Since for each  $i$ ,  $u \leq r_i \leq v + w = e_1 - u$ , it suffices to show that there is a multiple of  $e_1$  in the interval  $[d_1 u, d_1(e_1 - u)]$ , ensuring there are choices for the  $r_i$ ’s such that  $R$  is congruent to a number between  $u$  and  $e_1 - u \pmod{e_1}$ . This certainly holds if  $d_1(e_1 - 2u) \geq e_1 - 1$  which, since  $d_1 \geq 5$ , holds if  $4e_1 \geq 10u - 1$ . This inequality is true for  $e_1 \geq 19$ .

This completes the multiple cycles case and the proof. □

### 3. Colorings, Blocking Sets, Coverings, and Tilings

In this section we prove the results necessary to resolve Newman’s conjecture. The key insight in proving Lemma 1 is that the elements of a given color in an  $S$ -polychromatic coloring form a blocking set for  $S$ . While it is possible for  $\alpha(S)$  to be equal to  $1/p(S)$  (e.g. if  $|S| = 2$  then  $\alpha(S) = 1/2 = 1/p(S)$ ), in general these two quantities are not equal. For example,  $p(\{0, 1, 3\}) = 2$ , but by Lemma 2 and Theorem 4,  $\alpha(\{0, 1, 3\}) = 2/5 < 1/2$ .

We now prove Lemma 1.

*Proof.* Let  $\chi$  be an  $S$ -polychromatic coloring of  $\mathbb{Z}$  with  $p(S)$  colors. Suppose  $d \in \mathbb{Z}$  is greater than the diameter of  $S$  and let  $I_j = \{n \in \mathbb{Z} : jd \leq n < (j+1)d\}$ . By the pigeonhole principle, for some  $0 \leq j_1 < j_2 \leq (p(S))^d$  the coloring of the intervals  $I_{j_1}$  and  $I_{j_2}$  are identical, i.e., for  $0 \leq k < d$ ,  $\chi(j_1d + k) = \chi(j_2d + k)$ . Let  $m = (j_2 - j_1)d$ . For any  $n \in \mathbb{Z}$ , denote by  $r$  the remainder when  $n$  is divided by  $m$ , so  $0 \leq r < m$ . Let  $\chi'$  be the coloring of  $\mathbb{Z}$  where  $\chi'(n) = \chi(j_1d + r)$ . Note that  $\chi'$  uses  $p(S)$  colors and is periodic with period  $m$ , i.e., for all  $n \in \mathbb{Z}$ ,  $\chi(n) = \chi(n + m)$ . Furthermore, the coloring under  $\chi'$  of any  $d$  consecutive integers is identical to the coloring under  $\chi$  of some  $d$  consecutive integers, so  $\chi'$  is  $S$ -polychromatic. Let  $T_i = \{n \in \mathbb{Z} : \chi'(n) = i\}$ . Since any periodic set has a defined density,  $d(T_i)$  is defined for each  $i$ , and  $\sum_{i=1}^{p(S)} d(T_i) = 1$ . Since  $\chi'$  is  $S$ -polychromatic, for each  $i$ , each translate of  $S$  contains an element of  $T_i$ , i.e.,  $T_i$  is also a blocking set for  $S$ . Thus for some  $i$ ,  $T_i$  is a blocking set for  $S$  with density at most  $1/p(S)$ , which implies that  $\alpha(S) \leq 1/p(S)$ .  $\square$

For any subset  $T$  of an abelian group  $G$ , let  $-T$  denote the set  $\{-t : t \in T\}$ . Lemma 6 is well-known (see e.g. [14]) but for completeness we present a proof.

**Lemma 6.** *Let  $G$  be an abelian group, and  $S \subseteq G$ . Then  $T \subseteq G$  is a complement set for  $S$  if and only if  $-T$  is a blocking set for  $S$ .*

*Proof.* Suppose  $T$  is a complement set for  $S$ . For any  $n \in G$ , we have  $-n \in S + T$ , so there must be some  $t \in T, s \in S$  such that  $t + s = -n$ . This implies  $t = -n - s$ , so  $-n - s \in T$ , and  $n + s \in -T$ . Thus for every  $n$ , some element of  $n + S$  is in  $-T$ , and  $-T$  is a blocking set for  $S$ .

Conversely, suppose  $-T$  is a blocking set for  $S$ . For the sake of contradiction, assume  $T$  is not a complement set for  $S$ , i.e., there is some  $-n \in G$  such that  $-n \notin S + T$ . This implies that for all  $s \in S$ ,  $-n - s \notin T$ , which means for all  $s \in S$ ,  $n + s \notin -T$ . Thus  $n + S \subseteq G \setminus -T$ , and so  $-T$  is not a blocking set for  $S$ , a contradiction.  $\square$

We now prove Lemma 2.

*Proof.* Lemma 6 implies that  $T$  is a complement set for  $S$  if and only if  $-T$  is a blocking set for  $S$ . If they exist, the densities of  $T$  and  $-T$  are the same.  $\square$

### 3.1. Polychromatic Colorings and Tilings

We now describe some relationships between polychromatic colorings and tilings.

**Theorem 6.** *Let  $G$  be any abelian group. A finite set  $S \subseteq G$  tiles  $G$  by translation if and only if  $p(S) = |S|$ . Moreover, if  $\chi$  is an  $S$ -polychromatic coloring of  $G$  with*

$|S|$  colors and  $T$  is the set of elements of  $G$  colored by  $\chi$  with any given color, then  $S \oplus T = G$ .

*Proof.* Let  $S = \{s_1, s_2, \dots, s_k\}$ , and suppose  $S$  tiles  $G$  with complement set  $T \subseteq G$ . For each  $n \in G$ , define a coloring  $\chi$  on  $G$  so that  $\chi(n) = i$  if  $n = s_i + t$  for some  $t \in T$ . By the definition of tiling, this coloring is well-defined. For the sake of contradiction, assume  $\chi$  is not  $S$ -polychromatic. Then for some  $l$  where  $1 \leq l \leq k$ , there exists  $n \in G$  and  $s_i, s_j \in S$  with  $i \neq j$  such that  $\chi(n + s_i) = \chi(n + s_j) = l$ . Then there exist  $t_1, t_2 \in T$ ,  $t_1 \neq t_2$ , such that  $n + s_i = t_1 + s_l$  and  $n + s_j = t_2 + s_l$ . Subtracting these equations, we find that  $s_i - s_j = t_1 - t_2$ . Thus  $t_2 + s_i = t_1 + s_j$ , which is a contradiction.

Conversely, let  $S = \{s_1, s_2, \dots, s_k\}$ , suppose  $p(S) = |S|$ , and let  $\chi$  be an  $S$ -polychromatic coloring of  $G$  with  $|S|$  colors. Then for all  $n \in G$ , if  $i \neq j$  then  $\chi(n + s_i) \neq \chi(n + s_j)$ . Let  $T \subseteq G$  be the set of elements colored with a given color. We show that  $S \oplus T = G$ . First assume for the sake of contradiction that two translates of  $S$  share an element, i.e., there exist  $s_i, s_j \in S$ ,  $i \neq j$ ,  $t_1, t_2 \in T$ ,  $t_1 \neq t_2$ , such that  $s_i + t_1 = s_j + t_2$ . Let  $n = t_1 - s_j = t_2 - s_i$ , so  $t_1 = n + s_j$  and  $t_2 = n + s_i$ . Since  $\chi(t_1) = \chi(t_2)$  we get  $\chi(n + s_j) = \chi(n + s_i)$ , so two elements of  $n + S$  are colored identically, which is a contradiction.

It remains to show that  $S + T = G$ . Suppose there is some  $n \in G$  such that  $n \notin S + T$ . Then for all  $i$ ,  $n - s_i \notin T$ , which implies that the  $|S|$  elements of  $n - S$  are colored with at most  $|S| - 1$  colors, i.e., two are colored identically. Suppose  $\chi(n - s_i) = \chi(n - s_j)$ , where  $i \neq j$ . Let  $m = n - s_j - s_i$ . Then  $m + S$  contains both  $m + s_i = n - s_j$  and  $m + s_j = n - s_i$ . Since these integers are colored identically,  $m + S$  is a translate of  $S$  that does not contain all colors, which is a contradiction.  $\square$

Sets of integers with cardinality  $n = 3$  or  $4$  always have polychromatic number  $n$  or  $n - 1$ , and a corollary of Theorem 6 is that they have polychromatic number  $n - 1$  if and only if they do not tile  $\mathbb{Z}$ . According to Remark 5.6 in [4],  $c(\{0, 1, 3, 4, 8\}) = 3/11 > 1/4$ . Thus by Lemma 1,  $\{0, 1, 3, 4, 8\}$  is an example of a set with cardinality 5 and polychromatic number 3. The results of [2] and [6] imply that for sets  $S$  with large cardinality  $n$  the cardinality and polychromatic number of  $S$  can differ by a factor of  $1/\ln n$ .

We now state some other corollaries of Theorem 6.

**Corollary 2.** *If a finite set  $S$  tiles an abelian group  $G$  by translation, then any  $S$ -polychromatic coloring of  $G$  with  $|S|$  colors is also a  $(-S)$ -polychromatic coloring.*

*Proof.* Suppose  $S$  tiles  $G$ . By Theorem 6, there exists an  $S$ -polychromatic coloring  $\chi$  of  $G$  with  $|S|$  colors. Let  $T \subseteq G$  be the set of all elements of a given color. Again by Theorem 6,  $S + T = G$ . Therefore by Lemma 6,  $-T$  is a blocking set for  $S$ , i.e., for all  $n \in G$ ,  $n + S \not\subseteq G \setminus (-T)$ . This implies that for all  $n \in G$ ,  $-n - S \not\subseteq G \setminus T$ , i.e.,  $T$  is a blocking set for  $-S$ . Since  $T$  is a blocking set for  $-S$

for every color choice, every translate of  $-S$  contains every color, i.e., the coloring  $\chi$  is  $(-S)$ -polychromatic.  $\square$

Define  $t(S)$  to be the cardinality of the largest subset of  $S$  that tiles  $G$ .

**Corollary 3.** *For any finite subset  $S$  of an abelian group  $G$ ,  $p(S) \geq t(S)$ .*

If  $S \subseteq \mathbb{Z}$ ,  $|S| \leq 3$ , then  $p(S) = t(S)$ . But these parameters can be different for sets of integers with at least four elements. For example,  $S = \{0, 1, 3, 7\}$  is an example of a set where  $t(S) = 2$ , but  $p(S) = 3$ .

**Question 1.** For sets  $S$  of a given cardinality, how large can the gap between  $t(S)$  and  $p(S)$  be?

#### 4. Polychromatic Colorings in $\mathbb{Z}^d$

In this section we consider polychromatic numbers in the case where  $G = \mathbb{Z}^d$ ,  $d \geq 2$ . We will frequently “project” a set  $S \subseteq \mathbb{Z}^d$  to another set  $S' \subseteq \mathbb{Z}^{d-1}$  as follows. Let  $d \geq 2$ , and  $\mathbf{w} = (w_1, \dots, w_{d-1}, 1) \in \mathbb{Z}^d$ . Define  $f_{\mathbf{w}} : \mathbb{Z}^d \rightarrow \mathbb{Z}^{d-1}$  so that if  $\mathbf{s} = (v_1, \dots, v_d) \in \mathbb{Z}^d$ ,

$$f_{\mathbf{w}}(\mathbf{s}) = (v_1, \dots, v_{d-1}) - v_d(w_1, \dots, w_{d-1}).$$

We call  $f_{\mathbf{w}}(\mathbf{s})$  the *projection* of  $\mathbf{s}$  along  $\mathbf{w}$ . Given a set  $S \subseteq \mathbb{Z}^d$ , we call the set  $f_{\mathbf{w}}(S) \subseteq \mathbb{Z}^{d-1}$  the *projection* of  $S$  along  $\mathbf{w}$ .

For example, if  $\mathbf{s} = (2, 7, 4)$  and  $\mathbf{w} = (3, 1, 1)$ , the projection of  $\mathbf{s}$  along  $\mathbf{w}$  is  $f_{\mathbf{w}}(\mathbf{s}) = (2, 7) - 4(3, 1) = (-10, 3)$ . As another example, note that if  $\mathbf{s} = (v_1, \dots, v_d) \in \mathbb{Z}^d$ , the vector  $\mathbf{s}' = (v_1, \dots, v_{d-1}) \in \mathbb{Z}^{d-1}$  is the projection of  $\mathbf{s}$  along  $\mathbf{w} = (0, \dots, 0, 1)$ .

**Lemma 7.** *Let  $d \geq 2$ , and  $\mathbf{w} = (w_1, \dots, w_{d-1}, 1) \in \mathbb{Z}^d$ . Let  $S \subseteq \mathbb{Z}^d$ , and suppose  $S' \subseteq \mathbb{Z}^{d-1}$  is the projection of  $S$  along  $\mathbf{w}$ . Then  $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}^{d-1}}(S')$ .*

*Proof.* Since  $f_{\mathbf{w}} : \mathbb{Z}^d \rightarrow \mathbb{Z}^{d-1}$  is a homomorphism, the result follows from Lemma 3.  $\square$

**Proposition 1.** *Let  $d \geq 2$ . For any  $S \subseteq \mathbb{Z}^d$ , there is a projection  $S' \subseteq \mathbb{Z}^{d-1}$  where  $|S| = |S'|$ .*

*Proof.* Let  $S = \{\mathbf{s}_1, \dots, \mathbf{s}_k\} \subseteq \mathbb{Z}^d$  and suppose  $\mathbf{w} = (w_1, \dots, w_{d-1}, 1) \in \mathbb{Z}^d$ . For  $1 \leq i \leq k$  let  $\mathbf{s}'_i = f_{\mathbf{w}}(\mathbf{s}_i)$ . For  $1 \leq i \leq k$ , let  $s_{id}$  denote the last coordinate of  $\mathbf{s}_i$  and note that if  $i \neq j$ ,  $\mathbf{s}'_i = \mathbf{s}'_j$  if and only if

$$\mathbf{w} = \frac{1}{s_{id} - s_{jd}}(\mathbf{s}_i - \mathbf{s}_j).$$

In other words  $\mathbf{s}'_i = \mathbf{s}'_j$  if and only if  $\mathbf{w}$  is parallel to  $\mathbf{s}_i - \mathbf{s}_j$ . Since the number of differences  $\mathbf{s}_i - \mathbf{s}_j$  is finite, we can choose  $\mathbf{w}$  so that it is not parallel to any of these. For this choice of  $\mathbf{w}$ , for all  $1 \leq i \neq j \leq k$ ,  $\mathbf{s}'_i \neq \mathbf{s}'_j$ . Then  $S' = \{\mathbf{s}'_1, \dots, \mathbf{s}'_k\}$ , is a projection  $S$  with  $|S'| = |S|$ .  $\square$

**Theorem 7.** Fix  $d \geq 2$ . For a finite set  $S \subseteq \mathbb{Z}^d$ ,  $p(S) \geq \frac{(1+o(1))|S|}{\ln |S|}$ .

*Proof.* Given  $S \subseteq \mathbb{Z}^d$ , Proposition 1 implies we can project  $d-1$  times to ultimately obtain a set  $S' \subseteq \mathbb{Z}$ , with  $|S'| = |S|$ . Theorem 2, along with repeated application of Lemma 7, implies  $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}}(S') \geq \frac{(1+o(1))|S'|}{\ln |S'|} = \frac{(1+o(1))|S|}{\ln |S|}$ .  $\square$

We will be interested in the case where  $S = \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_k\} \subseteq \mathbb{Z}^d$  contains a set of collinear points. In the general case, we can assume for each  $i$  that  $\mathbf{s}_i = (b_1 + l_i a_1, b_2 + l_i a_2, \dots, b_d + l_i a_d)$ , where  $0 = l_0 < l_1 < l_2 < \dots < l_k$ ,  $a_i, b_i \in \mathbb{Z}$ , and  $\gcd(a_1, a_2, \dots, a_d) = 1$ . However since translation does not affect the polychromatic number, we will restrict our attention to the case where  $a_1 > 0$  and for all  $i$ ,  $b_i = 0$ .

**Theorem 8.** Let  $d \geq 2$ . Let  $S = \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_k\}$  be a set of  $k+1$  collinear points in  $\mathbb{Z}^d$  where for each  $i$ ,  $\mathbf{s}_i = (l_i a_1, l_i a_2, \dots, l_i a_d)$ , where  $0 = l_0 < l_1 < l_2 < \dots < l_k$ ,  $a_i \in \mathbb{Z}$ ,  $a_1 > 0$ , and  $\gcd(a_1, a_2, \dots, a_d) = 1$ . Let  $S' = \{0, l_1, l_2, \dots, l_k\} \subseteq \mathbb{Z}$ . Then  $p_{\mathbb{Z}^d}(S) = p_{\mathbb{Z}}(S')$ .

*Proof.* Let  $S'' = \{0, l_1 a_1, l_2 a_1, \dots, l_k a_1\} \subseteq \mathbb{Z}$ . By Lemma 5, Part (i),  $p_{\mathbb{Z}}(S') = p_{\mathbb{Z}}(S'')$ . Since  $S''$  can be obtained from  $S$  by a sequence of  $d-1$  projections, Lemma 7 implies  $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}}(S'') = p_{\mathbb{Z}}(S')$ . For the other direction, let  $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$  and note that the function  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}^d$  where  $\phi(n) = n\mathbf{a}$  is a homomorphism where  $\phi(S') = S$ . Thus by Lemma 3,  $p_{\mathbb{Z}}(S') \geq p_{\mathbb{Z}^d}(S)$   $\square$

Now we return to the subject of tilings. Lemma 8 and Theorems 9, 10, and 11 are well-known in the field of discrete geometry (see e.g. Section III of [14]) as simple examples of “splitting” groups. We restate them here using the language of polychromatic colorings.

**Lemma 8.** If a set  $S \subseteq G$  tiles a nontrivial subgroup  $H$  of  $G$ , then  $S$  tiles  $G$ .

*Proof.* Suppose  $S \subseteq G$  tiles a nontrivial subgroup  $H$  of  $G$ . Theorem 6 implies  $p_H(S) = |S|$ , so by Lemma 4,  $p_G(S) = |S|$ . By Theorem 6,  $S$  tiles  $G$ .  $\square$

For any  $d \geq 1$ , let  $\mathbf{0}$  denote the element  $(0, 0, \dots, 0) \in \mathbb{Z}^d$  and let  $\mathbf{e}_i$  denote the element  $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^d$  with all 0’s except for a 1 in the  $i$ th position. For  $\mathbf{s} = (v_1, \dots, v_d) \in \mathbb{Z}^d$ , let  $-\mathbf{s} = (-v_1, \dots, -v_d)$ . Define the  $d$ -semicross  $SC_d = \{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d\}$  and the  $d$ -cross  $C_d = \{\mathbf{0}, \mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2, \dots, \mathbf{e}_d, -\mathbf{e}_d\}$ . Theorem 6 implies that any finite set  $S \subseteq G$  with  $p(S) = |S|$  tiles  $G$ , and we use this insight to show that these sets tile  $\mathbb{Z}^d$ .

**Theorem 9.** For all  $d \geq 1$ , the  $d$ -semicross  $SC_d = \{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d\}$  tiles  $\mathbb{Z}^d$ .

*Proof.* Consider the coloring  $\chi : \mathbb{Z}^d \rightarrow [d + 1]$  where  $\chi(v_1, \dots, v_d) = v_1 + 2v_2 + 3v_3 + \dots + dv_d \pmod{d + 1}$ . On any translate  $\mathbf{n} + SC_d \subseteq \mathbb{Z}^d$ , the colors  $\chi(\mathbf{n} + \mathbf{0}), \chi(\mathbf{n} + \mathbf{e}_1), \chi(\mathbf{n} + \mathbf{e}_2), \dots, \chi(\mathbf{n} + \mathbf{e}_d)$  are  $\chi(\mathbf{n}), \chi(\mathbf{n}) + 1, \chi(\mathbf{n}) + 2, \dots, \chi(\mathbf{n}) + d \pmod{d + 1}$ . They are all different, so  $\chi$  is  $SC_d$ -polychromatic with  $|SC_d| = d + 1$  colors. By Theorem 6,  $SC_d$  tiles  $\mathbb{Z}^d$ .  $\square$

**Theorem 10.** For all  $d \geq 1$ , the  $d$ -cross  $C_d = \{\mathbf{0}, \mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2, \dots, \mathbf{e}_d, -\mathbf{e}_d\}$  tiles  $\mathbb{Z}^d$ .

*Proof.* The  $(2d + 1)$ -coloring  $\chi : \mathbb{Z}^d \rightarrow [2d + 1]$  where  $\chi(v_1, \dots, v_d) = v_1 + 2v_2 + 3v_3 + \dots + dv_d \pmod{2d + 1}$  is  $C_d$ -polychromatic: On any translate  $\mathbf{n} + C_d \subseteq \mathbb{Z}^d$ , the colors  $\chi(\mathbf{n} + \mathbf{0}), \chi(\mathbf{n} + \mathbf{e}_1), \chi(\mathbf{n} - \mathbf{e}_1), \chi(\mathbf{n} + \mathbf{e}_2), \chi(\mathbf{n} - \mathbf{e}_2), \dots, \chi(\mathbf{n} + \mathbf{e}_d), \chi(\mathbf{n} - \mathbf{e}_d)$  are  $\chi(\mathbf{n}), \chi(\mathbf{n}) + 1, \chi(\mathbf{n}) - 1, \chi(\mathbf{n}) + 2, \chi(\mathbf{n}) - 2, \dots, \chi(\mathbf{n}) + d, \chi(\mathbf{n}) - d \pmod{2d + 1}$ .  $\square$

**Theorem 11.** Let  $d \geq 2$ . Let  $S \subseteq \mathbb{Z}^d$  be a set that contains  $\mathbf{0}$  and  $j \leq d$  other elements  $\mathbf{s}_1, \dots, \mathbf{s}_j$ , where no nontrivial integer linear combination of  $\{\mathbf{s}_1, \dots, \mathbf{s}_j\}$  is  $\mathbf{0}$ . Then  $S$  tiles  $\mathbb{Z}^d$ .

*Proof.* Let  $H \subseteq \mathbb{Z}^d$  be the set of all integer linear combinations of  $\{\mathbf{s}_1, \dots, \mathbf{s}_j\}$ . By Theorem 9, there is a set  $T \subseteq \mathbb{Z}^j$  such that  $\{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_j\} \oplus T = \mathbb{Z}^j$ . Let  $M : \mathbb{Z}^j \rightarrow \mathbb{Z}^d$  be the unique linear transformation which maps  $\mathbf{e}_i$  to  $\mathbf{s}_i$  for each  $i \leq j$ . Then  $\{\mathbf{0}, \mathbf{s}_1, \dots, \mathbf{s}_j\}$  tiles  $H$  with complement set  $\{M(t) : t \in T\}$ . Since  $H$  is a subgroup of  $\mathbb{Z}^d$ , by Lemma 8,  $S$  tiles  $\mathbb{Z}^d$ .  $\square$

We can now determine the polychromatic number of any set  $S$  of cardinality 3 or 4 in  $\mathbb{Z}^d$ ,  $d \geq 2$ . Since translation does not affect polychromatic numbers, in all cases we may assume  $\mathbf{0} \in S$ .

**Theorem 12.** Let  $d \geq 2$  and suppose  $S \subseteq \mathbb{Z}^d$  has cardinality 3, with  $\mathbf{0} \in S$ . Then  $p_{\mathbb{Z}^d}(S) = 3$  if the three points are in general position or if they are collinear and there exists  $S' \subseteq \mathbb{Z}$  with  $p_{\mathbb{Z}}(S') = 3$  such that  $S'$  is the image of  $S$  after  $d - 1$  projections. Otherwise  $p_{\mathbb{Z}^d}(S) = 2$ .

*Proof.* Theorem 11 implies that if  $d \geq 2$  and  $S \subseteq \mathbb{Z}^d$  consists of three points in general position, then  $S$  tiles  $\mathbb{Z}^d$ , and thus  $p(S) = 3$ . If  $S \subseteq \mathbb{Z}^d$  has three collinear points, then Theorem 8 implies the problem is equivalent to finding the polychromatic number of a set of three integers, which is either 2 or 3 and can be determined using Theorem 3.  $\square$

**Theorem 13.** Let  $d \geq 2$  and suppose  $S \subseteq \mathbb{Z}^d$  has cardinality 4, with  $\mathbf{0} \in S$ . Then

1. If all points of  $S$  are collinear,  $p_{\mathbb{Z}^d}(S)$  is 3 or 4.
2. If exactly three points of  $S$  are collinear,  $p_{\mathbb{Z}^d}(S) = 4$ .

- 3. If  $d \geq 3$  and  $S$  has four points in general position,  $p_{\mathbb{Z}^d}(S) = 4$ .
- 4. If  $d = 2$  and  $S$  has four points in general position,  $p_{\mathbb{Z}^2}(S)$  is 3 or 4.

*Proof.* For  $d \geq 2$  and a set  $S \subseteq \mathbb{Z}^d$  with  $|S| = 4$ , Proposition 1 implies that there is a set  $S' \subseteq \mathbb{Z}$  where  $|S'| = 4$  and  $S'$  can be obtained by  $d - 1$  projections of  $S$ . Thus Theorem 1 and Lemma 7 imply that  $p(S) \geq 3$ . Determining whether  $p(S)$  is 3 or 4 is equivalent to determining whether  $S$  tiles  $\mathbb{Z}^d$ . As with the  $|S| = 3$  case, we can examine cases depending on how many points of  $S$  are collinear.

If the four points of  $S$  are in general position, then if none is a nontrivial integer linear combination of the others,  $p(S) = 4$  by Theorem 11. Otherwise, we can assume  $S \subseteq \mathbb{Z}^2$ . In this case,  $p(S)$  can be 3, e.g., if  $S = \{(0, 0), (1, 0), (0, 1), (1, 2)\} \subseteq \mathbb{Z}^2$ . It can also be 4, for example if  $S = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \subseteq \mathbb{Z}^2$ . Szegedy [15] gave an algorithm to determine if a set of cardinality 4 tiles  $\mathbb{Z}^2$ .

If the four points of  $S$  are all collinear, then  $p(S)$  is determined by applying Theorems 8 and 3.

If exactly three of the four points are collinear, then without loss of generality assume  $S = \{\mathbf{0}, (a, 0, \dots, 0), (b, 0, \dots, 0), \mathbf{s}\}$ , with  $0 < a < b$  and  $\mathbf{s} = (s_1, \dots, s_d)$ . Then  $S$  is in the subgroup  $\{x \in \mathbb{Z}^d : x_i \in s_i\mathbb{Z} \text{ for } 2 \leq i \leq d\}$ . By Lemmas 3 and 4 we may assume that  $s_i = 1$  for some  $2 \leq i \leq d$ , and thus by a sequence of projections,  $S$  can be projected to the set  $\{0, a, b, c\}$ , for any  $c \in \mathbb{Z}$ . By Lemma 7 it suffices to show that there exists  $c \in \mathbb{Z}$  such that  $p_{\mathbb{Z}}(\{0, a, b, c\}) = 4$ . Without loss of generality, we may assume that  $a$  and  $b$  have different parity, and in this case Theorem 3 implies that  $S = \{0, a, b, a + b\}$  has polychromatic number 4.  $\square$

The fact that  $p_{\mathbb{Z}^d}(S) = 4$  if  $S$  contains exactly three collinear points implies that for any set  $S$  of three integers, there is a 4-coloring of  $\mathbb{Z}$  so that every translate of  $S$  gets three different colors. Here is an explicit example of one such coloring. Without loss of generality we need only consider sets of the following form: Let  $S = \{0, a, b\} \subseteq \mathbb{Z}$  where  $a$  and  $b$  are positive with  $a$  even and  $b$  odd (note that we do not specify which is larger). Define the *alternating block 4-coloring relative to  $S$*  as follows: Given any  $m \in \mathbb{Z}$ , let  $q_m$  and  $r_m$  be the unique integers such that  $m = 2aq_m + r_m$ , where  $-a \leq r_m < a$ . Let  $X(m) = 0$  if  $r_m \geq 0$ ,  $X(m) = 1$  otherwise. Let  $Y(m) = 0$  if  $m$  is even,  $Y(m) = 1$  otherwise. Define  $\chi$ , the alternating block 4-coloring relative to  $S$ , so that  $\chi(m) = (X(m), Y(m))$ .

**Theorem 14.** *Let  $S = \{0, a, b\} \subseteq \mathbb{Z}$  with  $a, b > 0$ ,  $a$  even, and  $b$  odd. If the integers are colored with the alternating block 4-coloring relative to  $S$  then every translate of  $S$  has elements of three different colors.*

*Proof.* For any translate  $n + S = \{n, n + a, n + b\}$  of  $S$ ,  $X(n) \neq X(n + a)$ , while  $Y(n) = Y(n + a) \neq Y(n + b)$ . Thus  $\chi$  has the property that any translate of  $S$  contains elements with three different colors.  $\square$

Given a set of three integers, the alternating block 4-coloring shows that there is a 4-coloring of the integers so that every translate gets three different colors. If  $S \subset \mathbb{Z}$ ,  $|S| = 4$ , is there a 5-coloring of  $\mathbb{Z}$  so that every translate of  $S$  has 4 colors? More generally, we ask the following question.

**Question 2.** Let  $d \geq 1$ . Given  $k, n \in \mathbb{Z}$  with  $k \leq n$ , let  $p(n, k)$  denote the minimum  $r$  so that any  $S \subseteq \mathbb{Z}$  with  $|S| = n$  has an  $r$ -coloring where every translate of  $S$  gets at least  $k$  colors. What is an asymptotic upper bound on  $p(n, k(n))$  for natural choices of  $k(n)$ ?

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