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Choking and hydraulic jumps in laminar flow

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The viscous hydraulic jump still represents research in progress rather than a finalised edifice. The existing rigorous approaches show how this phenomenon is tied in with a bifurcation of the upstream flow adjacent to the guiding rigid plate of finite length, aligned perpendicularly to the direction of gravity. Here, this together with the upstream influence by the detached flow triggers transition from super- to subcritical flow (sensing its susceptibility to the upstream propagation of small disturbances). We present recent advances in the self-consistent theory of single-layer jumps continued as free shear layers.

1 Moderately large Froude numbers: strong jump terminated by choked flow

Consider the planar steady flow of a Newtonian liquid layer exhibiting uniform properties over a finite rigid horizontal plate. Constant gravity acts in negative vertical direction onto its wetted side. Typically, the ambient gaseous environment is at rest.

Let a reciprocal Froude number or non-dimensional gravitational acceleration g be of $O(1)$ for asymptotically large values the Reynolds number Re; both are formed with a flow speed $\tilde{U}$ and a vertical height $\tilde{H}$ characteristic of the slender layer described in the shallow-water limit. We furthermore identify $1/Re$ with the shallow-water parameter $\epsilon := \tilde{H}/\tilde{L}$ so that the layer becomes a developed one over the horizontal adjustment length $\tilde{L}$. According to the pioneering study [1], the phenomenon of a correspondingly smooth "jump" occurring in a developed flow appears self-evidently just as being the result of the upstream influence present across the entire layer due to the hydrostatic pressure gradient, parametrised by g. Consequently, an appropriate downstream conditions in the form of a generic singularity prescribed at the plate edge, accounting for the shortening of the streamwise scale as the flow approaches the trailing edge of the plate to undergo transcritical conditions, i.e. choking, there, closes the then weakly elliptic problem. Otherwise, the specific generation of the flow a horizontal distance $\tilde{L}$ upstream of the plate edge provides a source of ill-posedness of the underlying marching problem: as a typical scenario, the impingement of a free jet onto the plate provokes a very thin and therefore nearly supercritical, fully developed wall jet, first discussed in [2], asymptotically far downstream of jet impact but still asymptotically close to it when its streamwise extent is measured by $\tilde{L}$. Hence, the match with this flow allows for a one-parametric family of eigensolutions of the linearised shallow-water operator emerging in the virtual origin of the developed flow, cf. [3]. Most importantly, we find that the singularity terminates numerical downstream marching: placed as the downstream condition, it selects a specific member of that class to render the problem a well-posed one. This first completes the rational description of the fully viscous hydraulic jump.

The included limit of an underdeveloped flow and accordingly weak jump as $Re^{-1} \ll \epsilon \ll 1$ was already appreciated by viscous–inviscid interaction in [4] as long as $(0 <)\ 1 - g = O(1)$. Notably, the study [5] of the according transcritical situation in the least-degenerate distinguished limit $\epsilon = O(Re^{-1/10})$, $(0 <)\ 1 - g = O(\epsilon^2)$ introduces the impact of streamline curvature and surface tension. By the localised interaction, choking is fully absent in the first case. However, the flow almost chokes in the second given the $O(\epsilon^2)$-detuning of an upstream propagating wave from coming to rest.

We settle an open point crucial for understanding the detailed flow structure for $g \gg 1$ in an appealing manner. To this end, we introduce the distance s from the virtual origin in streamwise direction and the local vertical height h and kinematic horizontal momentum flux I of the layer, non-dimensional with $\tilde{L}$, $\tilde{H}$, and $\tilde{H}\tilde{U}^2$ respectively. Then $M_{u,d} := I_{u,d} + g h_{u,d}^2/2$ denotes its total momentum evaluated sufficiently far up- and downstream (subscripts u, d) of the limiting discontinuous jump at the sought position $s = s_J$, say. The aforementioned self-preserving near-supercritical flow, cf. [2], yields $I_u \sim \pi C/h_u$, C is a universal constant, and $h_u \sim \pi s/\sqrt{3}$. On the contrary, the large-g limit entails a subcritical flow governed by a lubrication approximation valid not too close to the plate edge ($s = 1$), hence $I_d \sim 5/(6h_d)$ and $h_d^4 - h_e^4 \sim 12(1 - s)/g$ where h_e is a virtual layer height at the edge. Here we take $h_e \ll h_d$ as matching with the terminal flow regime where fluid inertia is reinstated in full to finally allow for the edge singularity requires also h_d given above to behave singular as $s \to 1_-$. On the other hand, the apparent impossibility to extend the lubrication limit to arbitrarily large values of s hampers the flow from attaining a steady state over an infinitely long plate. This confirms its intrinsic downstream termination by a singularity. Eventually, from the jump conditions stating conservation of the volumetric flow rate and total momentum or

$$M_u \sim M_d, \quad s_J \sim C/\sqrt{g} \quad \text{with} \quad C = [\Gamma(1/3)/\Gamma(5/6)]^3/(6\pi^{3/2}) \simeq 0.4001 \tag{1}$$

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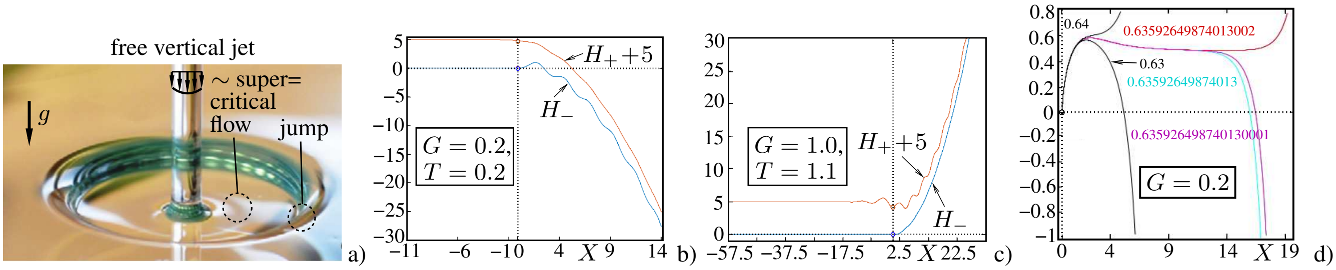

Fig. 1: **a** Concentric jump (image from [7], used with permission); **b**, **c** $H_\pm$ vs. X; **d** P vs. X parametrised by T (coloured), $T^* \simeq 0.6359$.

ensues. The scaling law (1) is at the basis for completing the regularisation of the jump by virtue of its internal structure at smaller distances $|s - s_J|$, commenced in [1]. Specifically, the invariance of the shallow-water momentum equations against a change of variables $(s, h, g) \mapsto (s_l, h_l, g_l) := (s/s_J, h/s_J, s_J^3 g)$ gives $g_l \sim C^3/\sqrt{g} \ll 1$ typical of the flow branching abruptly off its near-supercritical state close to the virtual origin to undergo the jump. Thus, the generic description of the onset of a strong jump in [6], also involving capillarity, applies on the smallest streamwise, interactive scale $s - s_J = O(g_l^3)$ at play.

The circular counterpart of this strong jump, undular by surface tension, is as ubiquitous in applications as clearly visible in experiments: see Fig. 1a. For the extension of the current theory coping with the jump over a rotating disc we refer to [8].

2 Very large Froude number: choking by detachment

As demonstrated in [9], the effect of streamline curvature suppresses hydrostatically-induced choking for sufficiently small values of g. In the very supercritical, extreme limit taking g as of $O(\epsilon^{4/7})$, classical choking at the plate edge is replaced by the upstream influence within the double-tiered interactive flow passing the edge over $s = 1 + O(\epsilon^{6/7})$ and predicting vertical variations of the free streamlines relative to the base flow upstream of $O(\epsilon^{2/7})$. Let us introduce appropriately scaled $O(1)$-quantities: reciprocal Froude and Weber numbers G and T (based on $\tilde{H}$) respectively, horizontal distance X from the plate edge, leading-order fluid pressure $P(X)$, related displacement function $A(X)$, vertical deflections $H_\pm(X)$ of the lower- $(-)$ and uppermost $(+)$ streamlines. We complete the typical P/A law as given in [9] by considering the detached flow $(X > 0)$:

$$\{P, H_-, H_+\} = \left\{ \frac{T}{2T-1} [G + \mathrm{sgn}(T-1)A''], \frac{T-1}{2T-1} \left[A - A(0) - A'(0)X + \mathrm{sgn}(T-1)\frac{GX^2}{2} \right], H_- - A \right\}. \quad (2)$$

Here $-A$ describes the variation of the layer height; the quadratic one of $H_\pm$ with X far downstream provides a match with the downfall $(T < 1/2)$ or, notably, uprising $(T > 1/2)$ parabola recovered on a larger scale. Accordingly, $P(X)$ and $H_\pm(X)$ result from post-processing the numerical solution of the interaction problem obtained by marching downstream: this is only entered by $P'(X)$ and $A(X)$ and parametrised by T, whereas the influence of gravity is condensed in the flow profile in the lower tier at the edge. Serving as an initial condition, it is determined by the preceding marching sweep towards the edge so as to meet the choking condition $P(0) = 0$. We finally discuss some preliminary analytical/numerical results obtained from (2).

For $0 < T < 1/2$, the free layer exhibits nearly symmetric capillary waves, thus reminiscent of and being an inertia-driven modification of neutral Rayleigh–Plateau modes: Fig. 1b. In contrast, these modes are replaced by "varicose" ones if $T > 1$: Fig. 1c. For $1/2 < T < 1$, however, waves are absent. Rather, for each $G > 0$ there seems to exist a threshold T^* such that the flow terminates in the form of an expansive singularity if $T < T^*$; this is shifted infinitely far downstream as $T \to T^*_-$ and superseded by a Goldstein wake adjacent to the lower free streamline as $P \to GT/(2T-1)$ for $T = T^*$; grossly reversed flow replaces this far wake if $T > T^*$: Fig. 1d. Current activities focus on an Euler stage regularising that singularity.

The exceptional, singular cases $T \sim 1/2$ and $T \sim 1$, not captured by (2), pose formidable challenges of our ongoing research. We finally remark that the first controls a sign change of capillary dispersion and a presumably nonlinear P/A law.

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