The equivalence of two tax processes

Dalal Al Ghanim Ronnie Loeffen Alexander R. Watson

3rd September 2019

We introduce two models of taxation, the latent and natural tax processes, which have both been used to represent loss-carry-forward taxation on the capital of an insurance company. In the natural tax process, the tax rate is a function of the current level of capital, whereas in the latent tax process, the tax rate is a function of the capital that would have resulted if no tax had been paid. Whereas up to now these two types of tax processes have been treated separately, we show that, in fact, they are essentially equivalent. This allows a unified treatment, translating results from one model to the other. Significantly, we solve the question of existence and uniqueness for the natural tax process, which is defined via an integral equation. Our results clarify the existing literature on processes with tax.

Key words and phrases. Risk process, tax process, tax rate, spectrally negative Lévy process, ruin probability, tax identity, optimal control.

MSC2010 classification. 60G51, 91B30, 93E20, 91G80.

1 Introduction and main results

Risk processes are a model for the evolution in time of the (economic) capital or surplus of an insurance company. Suppose that we have some model $X = (X_t)_{t\geq 0}$ for the risk process, in which X_t represents the capital of the company at time t; for instance, a common choice is for X to be a Lévy process with negative jumps. Any such model can be modified in order to incorporate desirable features. For instance, reflecting the path at a given barrier models the situation where the insurance company pays out any capital in excess of the barrier as dividends to shareholders. Similarly, 'refracting' the path at a given level and with a given angle corresponds to the case where dividends are paid out at a certain fixed rate whenever the capital is above the level or, equivalently, corresponds to a two-step premium rate. These modifications are described in more detail in Chapter 10 of Kyprianou [8], in the Lévy process case.

Between the reflected and refracted processes are a class of processes where partial reflection occurs whenever the process reaches a new maximum. The motivation in risk theory for these processes is that the times of partial reflection can be understood to correspond to tax payments associated with a so-called loss-carry-forward regime in which taxes are paid only when the insurance company is in a profitable situation. In this paper we study *tax processes* of this kind.

Before we define rigorously the type of tax processes that we are interested in, we make some assumptions on X that are in place throughout the paper. We assume that X is a stochastic process with càdlàg paths (i.e., right-continuous paths with left-limits) and without upward jumps (that is, $X_t - \lim_{s \uparrow t} X_s \leq 0$ for all $t \geq 0$). We also assume $X_0 = x$ for some fixed $x \in \mathbb{R}$.

For example, these conditions are satisfied if X is a Lévy process without upward jumps. In fact, the main results presented in this work hold pathwise, in the sense that they apply to each individual path of the stochastic process. A random model is strictly only required for the study of specific examples; however, given the applications we have in mind, it seems appropriate to phrase everything in terms of stochastic processes.

One way to incorporate a loss-carry-forward taxation regime for the risk process X is to introduce the tax process $U^\gamma := (U_t^\gamma)_{t \geq 0}$ with

$$U_t^\gamma = X_t - \int_{0+}^t \gamma(\overline{X}_s)\, \mathrm{d}\overline{X}_s, \qquad t \geq 0, \tag{1}$$

where $\gamma \colon [x, \infty) \to [0, 1)$ is a measurable function and $\overline{X}_t = \sup_{s \leq t} X_s$ is the running maximum of X. Note that, here and later, $\int_{0+}^t = \int_{(0,t]}$ denotes the integral over $(0, t]$. Since every path $t \mapsto \overline{X}_t$ is increasing (in the weak sense), and is further continuous due to the assumptions on X, the integral in (1) is a well-defined Lebesgue-Stieltjes integral. We call U^γ a *latent tax process* or the tax process with *latent tax rate* γ. For this latent tax process we have that, roughly speaking, in the time interval $[t, t+h]$ with $h > 0$ small, a fraction $\gamma(\overline{X}_t)$ of the increment $\overline{X}_{t+h} - \overline{X}_t$ is paid as tax. In particular, tax contributions are made whenever X reaches a new maximum (which is whenever U^γ reaches a new maximum; see Lemma 4 below), which is why the taxation structure in (1) can be seen to be of the loss-carry-forward type. Since $\gamma < 1$, this can be seen as partial reflection; setting $\gamma = \mathbf{1}_{[b,\infty)}$ would correspond to fully reflecting the path at the barrier b.

A great deal of literature has emerged in the study of this tax process. It was introduced by Albrecher and Hipp [1] in the case where X is a Cramér–Lundberg process and γ is a constant, and in that work the authors studied the ruin probabilities, proving a strikingly simple relation between ruin probabilities with and without tax, the so-called tax identity. This work was extended by Albrecher, Renaud and Zhou [2], using excursion theory, to the case where X is a general spectrally negative Lévy process, with γ still constant. In [10], Kyprianou and Zhou took γ to be a function, and studied problems related to the two-sided exit problem and the net present value of the taxes paid before ruin. In the same setting Renaud [13] provided results on the distribution of the (present) value of the taxes paid before ruin. Wang and Hu [15] studied a problem of optimal control of latent tax processes in which one seeks to maximise the net present value of the taxes paid before ruin. A variation of the latent tax process in which the tax rate exceeds the value 1 can be found in Kyprianou and Ott [9].

An unusual property of the process U^γ is that the taxation at time t depends, not on the running maximum $\overline{U}_t^\gamma = \sup_{s \leq t} U_s^\gamma$ of the process U^γ itself, but on the running maximum of X, i.e., $\overline{X}_t$. In other words, the amount of tax the company pays out at time t is not determined by the amount of capital the company has at that time but it depends on a latent capital level, namely $\overline{X}_t$, which is the amount of capital that the company would have at time t if no taxes were paid out at all. Besides being somewhat unnatural, this also means that in the common case where X is modelled by a Markov process, the process $(U^\gamma, \overline{U}^\gamma)$ is not Markov (see the first paragraph of Section 2). In order to maintain the Markov property, one would need to consider to the three-dimensional process $(U^\gamma, \overline{U}^\gamma, \overline{X})$.

For these reasons, it may be more suitable to use another tax process $V^\delta = (V_t^\delta)_{t\geq 0}$, satisfying the equation

$$V_t^\delta = X_t - \int_{0+}^t \delta(\overline{V}_s^\delta)\, \mathrm{d}\overline{X}_s, \tag{2}$$

where $\overline{V}_t^\delta = \sup_{s\leq t} V_s^\delta$ and $\delta\colon [x, \infty) \to [0, 1)$ is a measurable function. We call V^δ a *natural tax process* or a tax process with *natural tax rate* δ. Since (2) is an integral equation, it is not immediately clear whether such a process V^δ exists and if so if it is uniquely defined. We will shortly give a simple condition for existence and uniqueness. Assuming that existence and uniqueness holds and that X is a Markov process, the natural tax process V^δ has the advantage that the two-dimensional process $(V^\delta, \overline{V}^\delta)$ is Markov. For the reason that we retain $\overline{X}$ as the integrator in (2), instead of using $\overline{V}^\delta$, see the second paragraph of Section 2.

Albrecher, Borst, Boxma and Resing [3] looked at tax processes with a natural tax rate in the case where X is a Cramér-Lundberg risk process and studied the ruin probability, though they do not provide a definition of the tax process in terms of an integral equation and in particular do not discuss existence and uniqueness. In the setting where X is a Cramér-Lundberg risk process, Wei [16] and Cheung and Landriault [6] considered a more general class of natural tax processes than ours in which the associated premium rate is allowed to be surplus-dependent. Although [16] and [6] do contain the definition (2) for the natural tax process in the case where X is a Cramér-Lundberg risk process (see [16, Section 1] with $\delta = 0$ and [6, Equation (1.2)] with the function $c(\cdot)$ being constant), neither paper addresses the question of existence and uniqueness.

The purpose of this work is to clarify the relationship between these two tax processes. Whereas latent and natural tax processes appear quite different when considering their definitions, it emerges that these two classes of tax processes are essentially equivalent, an observation which has seemingly gone unnoticed in the literature. This equivalence allows us to deal in a rather straightforward way with the existence and uniqueness of the natural tax process, which is something that has not been dealt with before.

Before presenting our main theorem, we emphasise that our results hold true for a large class of stochastic processes for X that includes, amongst others, spectrally negative Lévy processes, spectrally negative Markov additive processes (see [4]), diffusion processes (see [11]) and fractional Brownian motion. However, practically, (1) and (2) may not in all cases be the right way to define a taxed process. For instance, when one considers a Cramér-Lundberg risk process where the company earns interest on its capital as well as pays tax according to a loss-carry-forward scheme, then one should not work with a process of the form (1) or (2), but define the tax process differently, as in [16]. Our definitions (1) and (2) are practically suitable for modelling tax processes when the underlying risk process without tax X has a spatial homogeneity property, which is the case for, for instance, spectrally negative Lévy processes, spectrally negative Markov additive processes or Sparre

Andersen risk processes.

In order to present our main result, we will need to consider the following ordinary differential equation, for a given measurable function $\delta\colon [x,\infty) \to [0,1)$:

$$\begin{aligned} \frac{\mathrm{d}y_x^\delta(t)}{\mathrm{d}t} &= 1 - \delta\left(y_x^\delta(t)\right), \qquad t \geq 0, \\ y_x^\delta(0) &= x. \end{aligned} \tag{3}$$

We say that $y_x^\delta\colon [0,\infty) \to \mathbb{R}$ is a *solution* of this ODE if it is an absolutely continuous function and satisfies (3) for almost every t.

Theorem 1. *Recall that $X_0 = x$.*

(i) Let U^γ be the tax process with latent tax rate γ, where $\gamma\colon [x,\infty) \to [0,1)$ is a measurable function. Define $\bar{\gamma}_x\colon [x,\infty) \to \mathbb{R}$ by

$$\bar{\gamma}_x(s) = x + \int_x^s (1 - \gamma(y))\mathrm{d}y, \qquad s \geq x, \tag{4}$$

and consider its inverse $\bar{\gamma}_x^{-1}\colon [x,\infty] \to [x,\infty]$, with the convention that $\bar{\gamma}_x^{-1}(s) = \infty$ when $s \geq \bar{\gamma}_x(\infty)$. Define $\delta_x^\gamma\colon [x, \bar{\gamma}_x(\infty)) \to [0,1)$ by $\delta_x^\gamma(s) = \gamma(\bar{\gamma}_x^{-1}(s))$. Then,

$$\overline{U}_t^\gamma = \bar{\gamma}_x(\overline{X}_t), \qquad t \geq 0, \tag{5}$$

and U^γ is a natural tax process with natural tax rate δ_x^γ.

(ii) Let $\delta\colon [x,\infty) \to [0,1)$ be a measurable function and assume that there exists a unique solution $y_x^\delta(t)$ of (3). Define $\gamma_x^\delta\colon [x,\infty) \to [0,1)$ by $\gamma_x^\delta(s) = \delta\left(y_x^\delta(s-x)\right)$.

Then, the integral equation (2) defining the natural tax process has a unique solution $V^\delta = (V_t^\delta)_{t\geq 0}$. Moreover,

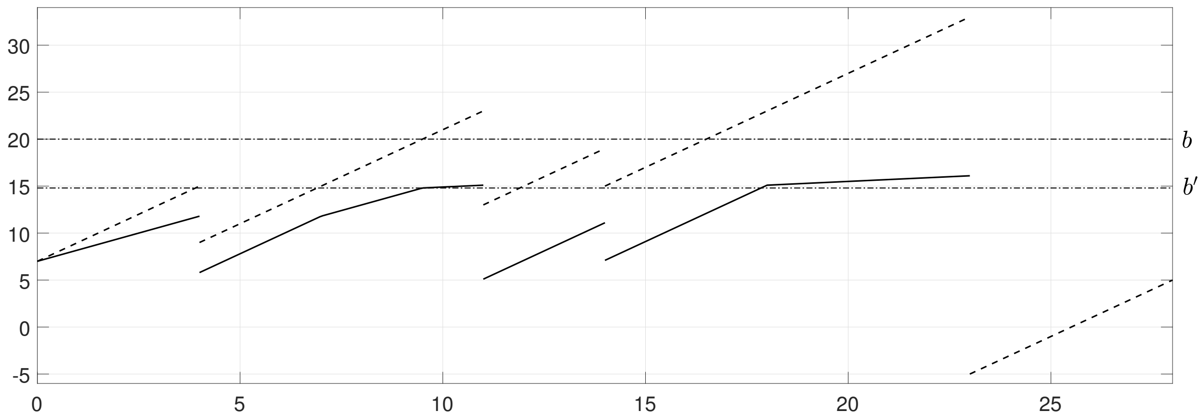
$$\overline{V}_t^\delta = y_x^\delta(\overline{X}_t - x), \qquad t \geq 0, \tag{6}$$

and so the solution V^δ to (2) is a latent tax process with latent tax rate given by γ_x^δ.

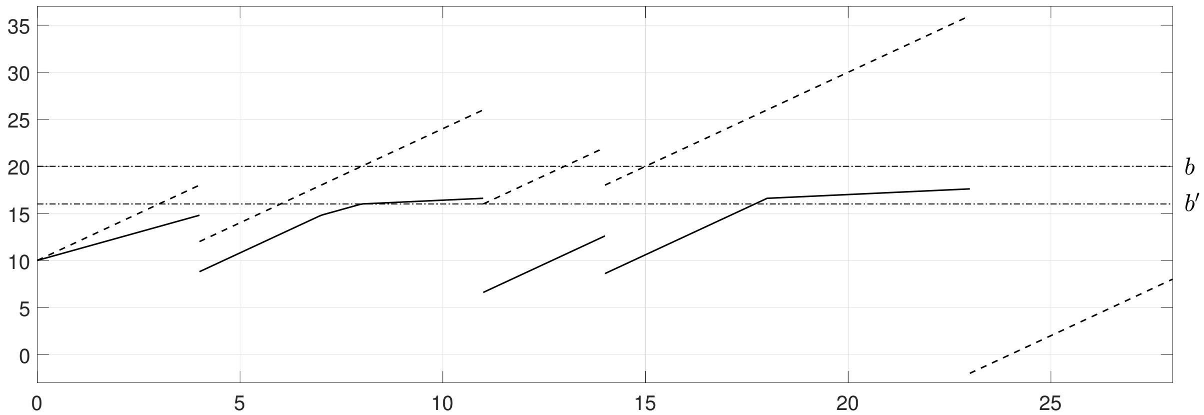
This theorem is the main contribution of the article. It states that a sufficient condition for existence and uniqueness of solutions to (2) can be given in terms of a simple ODE. From the proofs given in Section 3 below, it is not difficult to see that the existence and uniqueness of the ODE (3) is also a necessary condition for existence and uniqueness of a solution to (2). Theorem 1 also gives a precise relationship between the two types of tax processes. In particular, every latent tax process is a natural tax process, though the corresponding latent and natural tax rates may differ. Conversely, every well-defined natural tax process is also a latent tax process. The next example illustrates this equivalence for piecewise constant tax rates.

Example 2. Define the piecewise constant function f^b by

$$f^b(z) = \begin{cases} \alpha, & z \leq b, \\ \beta, & z > b, \end{cases} \tag{7}$$



(a) $x = 7$, $b = 20$ and $b' = 14.8$.



(b) $x = 10$, $b = 20$ and $b' = 16$.

Figure 1: Plots of the risk process X (dashed line) and the associated latent tax process U^{f^b} or equivalently natural tax process $V^{f^{b'}}$ (solid line), where f^b is the piecewise constant function defined by (7) with $\alpha = 0.4$ and $\beta = 0.9$. The dashed-dot lines mark the values of b and b'.

where $b > x = X_0$ and $0 \leq \alpha \leq \beta < 1$. Note that the ODE (3) with $\delta = f^b$ has a unique solution, see e.g. Section 2. It is clear that the tax process with latent tax rate f^b differs from the tax process with natural tax rate f^b, unless $\alpha = \beta$ or $\alpha = 0$. However, from Theorem 1 we deduce that the tax process with latent tax rate f^b is equal to the tax process with natural tax rate $f^{b'}$ for

$$b' = (1 - \alpha)b + \alpha x.$$

Note that b' depends on the starting point x, unless $\alpha = 0$. Figure 1 contains two plots in which an example of X and the corresponding tax process U^{f^b}, or equivalently $V^{f^{b'}}$, are drawn. From this figure we see that indeed the first time X reaches the level b is equal to the first time the tax process reaches the level b'.

The theorem allows us to very easily translate results derived for the latent tax process to results on the natural tax process, or vice versa. As an example, by using the corresponding result derived in [10] for the latent tax processes, we provide below an analytical expression of the so-called two-sided exit problem of the natural tax process in the case where X is a spectrally negative Lévy process. For an introduction to spectrally negative Lévy processes and their scale functions we refer to Chapter 8 in [8].

Corollary 3. *Let X be a spectrally negative Lévy process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$ such that $\mathbb{P}_x(X_0 = x) = 1$. Let $\delta\colon [x,\infty) \to [0,1)$ be a measurable function such that there exists a unique solution y_x^δ to (3). Let V^δ be the tax process with natural rate δ associated with the spectrally negative Lévy process X. Define the first passage times*

$$\tau_a^- = \inf\{t \geq 0 : V_t^\delta < a\} \quad \text{and} \quad \tau_a^+ = \inf\{t \geq 0 : V_t^\delta > a\},$$

*where $a \in \mathbb{R}$. Let $q \geq 0$ and let $W^{(q)}\colon \mathbb{R} \to [0,\infty)$ be the q-*scale function *of X, defined by $W^{(q)}(z) = 0$ for $z < 0$ and characterised on $[0,\infty)$ as the continuous function whose Laplace transform is given by*

$$\int_0^\infty e^{-\lambda y} W^{(q)}(z)\,\mathrm{d}z = \left(\log\left(\mathbb{E}\left[e^{\lambda X_1}\right]\right) - q\right)^{-1}, \qquad \textit{for } \lambda > 0 \textit{ sufficiently large.}$$

Then, for $0 \leq x < a < y_x^\delta(\infty)$, we have

$$\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}\right] = \exp\left\{-\int_x^a \frac{W^{(q)\prime}(y)}{W^{(q)}(y)(1-\delta(y))}\,\mathrm{d}y\right\}, \tag{8}$$

where $W^{(q)\prime}$ denotes a density of $W^{(q)}$ on $(0,\infty)$. On the other hand, if $a \geq y_x^\delta(\infty)$, then $\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}\right] = 0$.

The rest of this article is organised as follows. In Section 2, we explain the consequences of our results and make further connections with the literature. Section 3 is devoted to the proofs of Theorem 1 and Corollary 3.

2 Related work and further applications

Markov property Assume X is a Markov process. As we have already commented in the previous section, it follows from the integral equation (2) for the natural tax process V^δ that the process $(V^\delta, \overline{V}^\delta)$ is Markov. One might expect that the equivalence between the two types of tax processes should imply the same for $(U^\gamma, \overline{U}^\gamma)$ where U^γ is an arbitrary latent tax process, since we know by Theorem 1(i) that U^γ is also a natural tax process. However, the corresponding natural tax rate is $\delta_x^\gamma = \delta_{X_0}^\gamma$, which depends upon the initial value of X. Looked at another way, although one can recover $\overline{X}$ from the formula $\overline{X}_t = \bar{\gamma}_x^{-1}(\overline{U}_t^\gamma)$, this too depends on knowledge of the initial value X_0. For this reason, we do not obtain the Markov property for $(U^\gamma, \overline{U}^\gamma)$ in general.

An alternative definition of the natural tax process It would also appear to be reasonable to define a natural type of tax process as a solution to the SDE

$$W_t = X_t - \int_{0+}^{t} \kappa(\overline{W}_s)\,\mathrm{d}\overline{W}_s, \tag{9}$$

where $\kappa\colon [x,\infty) \to [0,\infty)$. Define $\delta = \frac{\kappa}{1+\kappa}$. The process V^δ, when it exists, is a solution to (9), as can be shown using Lemma 4. The natural tax rate δ describes the tax rate as a proportion of the increments of capital prior to taxation rather than after taxation, and therefore appears to us to be preferable to κ as a parameter.

Existence of the tax process with progressive natural tax rates When the tax rate increases with the amount of capital one has, the taxation regime is typically called progressive. We will show that, when δ is an increasing (in the weak sense) measurable function $\delta\colon [x,\infty) \to [0,1)$, then the ODE (3) has a unique solution, which implies the existence and uniqueness of the natural tax process with tax rate δ.

For existence, since δ is an increasing function, we have that

$$g(z) \coloneqq \frac{1}{1-\delta(z)}, \qquad z \geq x,$$

is a strictly positive, increasing measurable function, and hence integrable, so

$$G(y) \coloneqq \int_x^y g(z)\,\mathrm{d}z, \qquad y \geq x,$$

is absolutely continuous. Moreover, since G is continuous and strictly increasing, G^{-1} exists and, as $G' > 0$ a.e., G^{-1} is absolutely continuous [5, Vol. I, p. 389]. Thus, $(G^{-1})'(t)$ exists for almost every t, and it follows that a solution to (3) is given by $y_x(t) = G^{-1}(t)$. This is because, by the inverse function theorem [12, Theorem 31.1], it holds that

$$\frac{\mathrm{d}G^{-1}(t)}{\mathrm{d}t} = \frac{1}{g(G^{-1}(t))} = 1 - \delta(G^{-1}(t)), \qquad \text{for a.e. } t > 0,$$

and since $G(x) = 0$, we have $G^{-1}(0) = x$.

For uniqueness, since δ is increasing, the right hand side of (3) is decreasing. This guarantees uniqueness, as can be proved using, for instance, [7, Theorem 1.3.8].

Optimal control In the case where X is a spectrally negative Lévy process, Wang and Hu [15] studied a very interesting optimal control problem for the latent tax process U^γ, given by

$$\sup_{\gamma\in\Pi} \mathbb{E}_x\left[\int_0^{\sigma_0^-} e^{-qt}\gamma(\overline{X}_t)\,\mathrm{d}\overline{X}_t\right], \tag{10}$$

where $\sigma_0^- = \inf\{t \geq 0 : U_t^\gamma < 0\}$, and Π is the set of measurable functions $\gamma\colon [0,\infty) \to [\alpha,\beta]$, where $0 \leq \alpha \leq \beta < 1$ are fixed. Denote by γ^* the function $\gamma \in \Pi$ which maximises (10), if it exists. A remarkable feature of Wang and Hu's work is that they obtain a *natural* tax process as the optimal solution to the problem of controlling a latent tax process, as we will now explain.

In their Theorem 3.1, Wang and Hu state that γ^* should satisfy the equation

$$\gamma^*(\overline{X}_t) = \eta\left(x + \int_x^{\overline{X}_t} (1-\gamma^*(y))\,\mathrm{d}y\right) = \eta(\overline{U}_t^{\gamma^*}),$$

for some function η which they call the *optimal decision rule*. On the other hand, let δ be a function satisfying the assumptions of Theorem 1(ii), and define γ_x^δ as in that result. If we write $\xi = \gamma_x^\delta$, then, by the definition of γ_x^δ together with (5) and (6), we have the relation

$$\xi(\overline{X}_t) = \delta\left(x + \int_x^{\overline{X}_t} (1-\xi(y))\,\mathrm{d}y\right) = \delta(\overline{U}_t^{\xi}).$$

It follows from Theorem 1 that the relationship between Wang and Hu's optimal decision rule η and optimal tax rate γ^* is nothing other than the relationship between a particular natural tax rate δ and the equivalent latent tax rate γ_x^δ. Our results clarify that this connection is a sensible one even outside of the optimal control context, and make clear under exactly which conditions this connection is valid.

Wang and Hu go on to show that η must be piecewise constant, and in particular $\eta = f^b$, as defined in (7), where b is specified in terms of scale functions of the Lévy process but is independent of x; see section 4 and equation (5.15) in their work (in which b is denoted u_0). Combining this with our result, we see that Wang and Hu's solution of the optimal control problem (10) is actually a tax process with the piecewise constant natural tax rate f^b, or equivalently the piecewise constant latent tax rate $f^{\tilde{b}(x)}$, where $\tilde{b}(x)$ depends on x as in Example 2.

Tax identity Assume we are in the setting of Corollary 3 where in particular X is a spectrally negative Lévy process. We are interested here in the tax identity: a relationship between the survival probability of the natural tax process V^δ and the one of the risk process with out tax X. To this end, let

$$\phi_\delta(x) = \mathbb{P}_x\left(\inf_{t\geq 0} V_t^\delta \geq 0\right) \quad \text{and} \quad \phi_0(x) = \mathbb{P}_x\left(\inf_{t\geq 0} X_t \geq 0\right)$$

be the survival probability in the risk model with and without taxation, respectively.

If $y_x^\delta(\infty) < \infty$, the process V^δ cannot exceed the level $y_x^\delta(\infty)$. Since from every starting level (and thus in particular from $y_x^\delta(\infty)$), there is a strictly positive probability of V going below zero, a standard renewal argument shows that the survival probability $\phi_\delta(x)$ is zero in this case.

On the other hand, if $y_x(\infty) = \infty$, then we can apply Corollary 3 to get a relation between the two survival probabilities. Namely, by letting $q \to 0$ and $a \to \infty$ in (8) and using the well-known expression for $\phi_0(x)$ (see, e.g., [8, equation (8.10)]), we have that

$$\phi_\delta(x) = \exp\left\{-\int_x^\infty \frac{W'(y)}{W(y)(1-\delta(y))}\mathrm{d}y\right\} = \exp\left\{-\int_x^\infty \frac{\mathrm{d}\ln(\phi_0(y))}{\mathrm{d}y} \cdot \frac{1}{(1-\delta(y))}\mathrm{d}y\right\}.$$

This agrees with [3, Proposition 3.1] for the special case where X is a Cramér-Lundberg risk process, which confirms that in [3] natural tax processes are considered.

3 Proofs

We start with a lemma generalising a result from [10].

Lemma 4. *Let $K = (K_t)_{t\geq 0}$ be a stochastic process for which every path is measurable as a function of time and such that $K_t < 1$ for every $t \geq 0$. Define*

$$H_t = X_t - \int_{0+}^{t} K_s \mathrm{d}\overline{X}_s, \qquad t \geq 0.$$

Then,

$$\overline{H}_t = \overline{X}_t - \int_{0+}^{t} K_s \mathrm{d}\overline{X}_s,$$

where $\overline{H}_t = \sup_{s\leq t} H_s$. Moreover, $\{t \geq 0 : H_t = \overline{H}_t\} = \{t \geq 0 : X_t = \overline{X}_t\}$.

Proof. Since $K_t < 1$ for all $t \geq 0$, the proof in [10, Lemma 2.1] works without alteration. □

Next we prove part (ii) of Theorem 1 except for the existence of the integral equation.

Lemma 5. *Let $\delta\colon [x,\infty) \to [0,1)$ be a measurable function and assume that there exists a unique solution y_x^δ of (3). Define $\gamma_x^\delta\colon [x,\infty) \to [0,1)$ by $\gamma_x^\delta(s) = \delta\left(y_x^\delta(s-x)\right)$. If there exists a solution $V^\delta = (V_t^\delta)_{t\geq 0}$ to the integral equation*

$$V_t^\delta = X_t - \int_{0+}^{t} \delta(\overline{V}_r^\delta)\mathrm{d}\overline{X}_r, \qquad t \geq 0, \tag{11}$$

then $\overline{V}_t^\delta = y_x^\delta(\overline{X}_t - x)$ and hence V^δ is a latent tax process with latent tax rate given by γ_x^δ.

Proof. Suppose that V^δ solves (11). By Lemma 4,

$$\overline{V}_t^\delta = \overline{X}_t - \int_{0+}^{t} \delta(\overline{V}_r^\delta)\,\mathrm{d}\overline{X}_r, \qquad t \geq 0. \tag{12}$$

We define $L_t = \overline{X}_t - x$ and we let L_a^{-1} be its right-inverse, i.e.

$$L_a^{-1} := \begin{cases} \inf\{t > 0 : L_t > a\} = \inf\{t > 0 : \overline{X}_t > a + x\}, & \text{if } 0 \leq a < L_\infty, \\ \infty, & \text{if } a \geq L_\infty. \end{cases}$$

As X does not have upward jumps, $t \mapsto \overline{X}_t$ is continuous, which implies

$$\overline{X}_{L_a^{-1}} = x + (a \wedge L_\infty). \tag{13}$$

Using respectively (12) for $t = L_a^{-1} = L_{a\wedge L_\infty}^{-1}$, (13) and the change of variables formula with $r = L_b^{-1}$ (see, for instance, [14, footer of p. 8]), we have for $a \geq 0$,

$$\begin{aligned}
\overline{V}^\delta_{L^{-1}_{a\wedge L_\infty}} &= \overline{X}_{L^{-1}_{a\wedge L_\infty}} - \int_{0+}^{L^{-1}_{a\wedge L_\infty}} \delta(\overline{V}_r^\delta)\,\mathrm{d}\overline{X}_r \\
&= x + (a \wedge L_\infty) - \int_{0+}^{\infty} \mathbf{1}_{\left\{r \leq L^{-1}_{a\wedge L_\infty}\right\}} \delta(\overline{V}_r^\delta)\,\mathrm{d}\overline{X}_r \\
&= x + (a \wedge L_\infty) - \int_{0}^{\infty} \mathbf{1}_{\left\{0 < L_b^{-1} \leq L^{-1}_{a\wedge L_\infty}\right\}} \delta(\overline{V}^\delta_{L_b^{-1}})\,\mathrm{d}b \\
&= x + \int_0^{a\wedge L_\infty} \left(1 - \delta\left(\overline{V}^\delta_{L_b^{-1}}\right)\right)\,\mathrm{d}b,
\end{aligned}$$

where for the last equality we used that L_b^{-1} is strictly increasing on $[0, L_\infty]$, which follows because $t \mapsto \overline{X}_t$ is continuous. By the hypothesis that (3) has a unique solution y_x^δ, we deduce,

$$\overline{V}^\delta_{L^{-1}_{a\wedge L_\infty}} = y_x^\delta\left(a \wedge L_\infty\right) = y_x^\delta\left(\overline{X}_{L^{-1}_{a\wedge L_\infty}} - x\right), \qquad a \geq 0, \tag{14}$$

where the last equality follows by (13). As $t \mapsto \overline{X}_t$ does not jump upwards, $\overline{X}_{L^{-1}_{L_t}} = \overline{X}_t$ for all $t \geq 0$, which implies via (12) that $\overline{V}^\delta_{L^{-1}_{L_t}} = \overline{V}^\delta_t$ for all $t \geq 0$. So by invoking (14) for $a = L_t$, we conclude that $\overline{V}^\delta_t = y_x^\delta(\overline{X}_t - x)$ for all $t \geq 0$. □

We are now ready to prove Theorem 1 and Corollary 3.

Proof of Theorem 1. (i) Fix $t \geq 0$. By Lemma 4, we have

$$\overline{U}^\gamma_t = \overline{X}_t - \int_{0^+}^t \gamma(\overline{X}_r)\mathrm{d}\overline{X}_r.$$

By applying the change of variable $y = \overline{X}_r$, we obtain

$$\overline{U}^\gamma_t = \overline{X}_t - \int_x^{\overline{X}_t} \gamma(y)\mathrm{d}y = \bar{\gamma}_x(\overline{X}_t),$$

where we recall that $\bar{\gamma}_x(s) = x + \int_x^s (1 - \gamma(y))\,\mathrm{d}y$. Hence, $\bar{\gamma}_x^{-1}(\overline{U}^\gamma_t) = \overline{X}_t$, and so $\gamma(\overline{X}_t) = \gamma(\bar{\gamma}_x^{-1}(\overline{U}^\gamma_t)) = \delta_x^\gamma(\overline{U}^\gamma_t)$. It follows that U^γ is a natural tax process with natural tax rate δ_x^γ.

(ii) The uniqueness of a solution to (2), and the equality (6), follow directly from Lemma 5. So it remains to prove the existence of a solution to (2). By the hypothesis there exists a unique solution y_x^δ to (3). With $\gamma_x^\delta(z) = \delta\left(y_x^\delta(z - x)\right)$, we define $\bar{\delta}\colon [x, \infty) \to [0, 1)$ by

$$\bar{\delta}(z) = \gamma_x^\delta\Big((\bar{\gamma}_x^\delta)^{-1}(z)\Big) = \delta\left(y_x^\delta\left(\left(\bar{\gamma}_x^\delta\right)^{-1}(z) - x\right)\right),$$

where $(\bar{\gamma}_x^\delta)^{-1}$ is the inverse function of

$$\bar{\gamma}_x^\delta(z) = x + \int_x^z (1 - \gamma_x^\delta(y))\mathrm{d}y. \tag{15}$$

By part (i) of Theorem 1, the tax process with latent tax rate γ_x^δ is a natural tax process with natural tax rate $\bar{\delta}$. Thus, it remains to show that $\bar{\delta}(z) = \delta(z)$ for $z \geq x$.

Note that $\bar{\gamma}_x^\delta$ is an absolutely continuous function and hence $(\bar{\gamma}_x^\delta)'$ exists almost everywhere. By (3) we have that, for z such that $(\bar{\gamma}_x^\delta)'(z)$ exists,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\left(y_x^\delta((\bar{\gamma}_x^\delta)^{-1}(z) - x)\right) &= \left[1 - \delta\left(y_x^\delta\left((\bar{\gamma}_x^\delta)^{-1}(z) - x\right)\right)\right]\frac{\mathrm{d}}{\mathrm{d}z}\left((\bar{\gamma}_x^\delta)^{-1}(z)\right)\\ &= \left[1 - \gamma_x^\delta\left((\bar{\gamma}_x^\delta)^{-1}(z)\right)\right]\frac{\mathrm{d}}{\mathrm{d}z}\left((\bar{\gamma}_x^\delta)^{-1}(z)\right).\end{aligned}$$

Since by the inverse function theorem [12, Theorem 31.1],

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((\bar{\gamma}_x^\delta)^{-1}(z)\right) = \frac{1}{(\bar{\gamma}_x^\delta)'\left((\bar{\gamma}_x^\delta)^{-1}(z)\right)} = \frac{1}{1-\gamma_x^\delta\left((\bar{\gamma}_x^\delta)^{-1}(z)\right)},$$

we see that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(y_x^\delta((\bar{\gamma}_x^\delta)^{-1}(z)-x)\right) = 1 \qquad \text{a.e.},$$

and therefore, by the absolute continuity, for some constant c, we have that

$$y_x^\delta((\bar{\gamma}_x^\delta)^{-1}(z)-x) = z + c, \qquad z \geq x.$$

Since $(\bar{\gamma}_x^\delta(x))^{-1} = x = y_x^\delta(0)$, we get that $c = 0$. We conclude that $\bar{\delta}(z) = \delta(z)$ for $z \geq x$, and this completes the proof.

□

Proof of Corollary 3. From (6) we see that $\tau_a^+ = \infty$ when $a \geq y_x(\infty)$. Hence we can assume without loss of generality that $a < y_x^\delta(\infty)$. By part (ii) of Theorem 1 we know that V^δ is a latent tax process with latent tax rate γ_x^δ. Hence we can use Theorem 1.1 in [10] to conclude that,

$$\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] = \exp\left\{-\int_x^a \frac{W^{(q)\prime}(y)}{W^{(q)}(y)\left(1-\gamma_x^\delta\left((\bar{\gamma}_x^\delta)^{-1}(y)\right)\right)}\mathrm{d}y\right\},$$

where $(\bar{\gamma}_x^\delta)^{-1}$ is the inverse of the function $\bar{\gamma}_x^\delta$ given by (15). Note that in [10] the additional assumption $\int_0^\infty(1-\gamma_x^\delta(z))\mathrm{d}z = \infty$ is made on the latent tax rate, but from the proof of Theorem 1.1 in [10] it is clear that this assumption is unnecessary when $a < y_x^\delta(\infty)$. In the proof of Theorem 1(ii) we showed that $\gamma_x^\delta\left((\bar{\gamma}_x^\delta)^{-1}(y)\right) = \delta(y)$ for all $y \geq x$, which finishes the proof. □

Acknowledgement We thank two anonymous referees for their helpful comments.

References

[1] H. Albrecher and C. Hipp. Lundberg's risk process with tax. *Bl. DGVFM*, 28(1): 13–28, 2007. ISSN 1864-0281. doi:10.1007/s11857-007-0004-4.

[2] H. Albrecher, J.-F. Renaud, and X. Zhou. A Lévy insurance risk process with tax. *J. Appl. Probab.*, 45(2):363–375, 2008. ISSN 0021-9002. doi:10.1239/jap/1214950353.

[3] H. Albrecher, S. Borst, O. Boxma, and J. Resing. The tax identity in risk theory—a simple proof and an extension. *Insurance Math. Econom.*, 44(2):304–306, 2009. ISSN 0167-6687. doi:10.1016/j.insmatheco.2008.05.001.

[4] H. Albrecher, F. Avram, C. Constantinescu, and J. Ivanovs. The tax identity for Markov additive risk processes. *Methodol. Comput. Appl. Probab.*, 16(1):245–258, 2014. ISSN 1387-5841. doi:10.1007/s11009-012-9310-y.

[5] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007. ISBN 978-3-540-34513-8; 3-540-34513-2. doi:10.1007/978-3-540-34514-5.

[6] E. C. K. Cheung and D. Landriault. On a risk model with surplus-dependent premium and tax rates. *Methodol. Comput. Appl. Probab.*, 14(2):233–251, 2012. ISSN 1387-5841. doi:10.1007/s11009-010-9197-4.

[7] Z. Jackiewicz. *General linear methods for ordinary differential equations.* John Wiley & Sons, Inc., Hoboken, NJ, 2009. ISBN 978-0-470-40855-1. doi:10.1002/9780470522165.

[8] A. E. Kyprianou. *Fluctuations of Lévy processes with applications.* Universitext. Springer, Heidelberg, second edition, 2014. ISBN 978-3-642-37631-3; 978-3-642-37632-0. doi:10.1007/978-3-642-37632-0. Introductory lectures.

[9] A. E. Kyprianou and C. Ott. Spectrally negative Lévy processes perturbed by functionals of their running supremum. *J. Appl. Probab.*, 49(4):1005–1014, 2012. ISSN 0021-9002.

[10] A. E. Kyprianou and X. Zhou. General tax structures and the Lévy insurance risk model. *J. Appl. Probab.*, 46(4):1146–1156, 2009. ISSN 0021-9002. doi:10.1239/jap/1261670694.

[11] B. Li, Q. Tang, and X. Zhou. A time-homogeneous diffusion model with tax. *J. Appl. Probab.*, 50(1):195–207, 2013. ISSN 0021-9002. doi:10.1239/jap/1363784433.

[12] G. B. Price. *Multivariable analysis.* Springer-Verlag, New York, 1984. ISBN 0-387-90934-6. doi:10.1007/978-1-4612-5228-3.

[13] J.-F. Renaud. The distribution of tax payments in a Lévy insurance risk model with a surplus-dependent taxation structure. *Insurance Math. Econom.*, 45(2):242–246, 2009. ISSN 0167-6687. doi:10.1016/j.insmatheco.2009.07.004.

[14] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999. ISBN 3-540-64325-7. doi:10.1007/978-3-662-06400-9.

[15] W. Wang and Y. Hu. Optimal loss-carry-forward taxation for the Lévy risk model. *Insurance Math. Econom.*, 50(1):121–130, 2012. ISSN 0167-6687. doi:10.1016/j.insmatheco.2011.10.011.

[16] L. Wei. Ruin probability in the presence of interest earnings and tax payments. *Insurance Math. Econom.*, 45(1):133–138, 2009. ISSN 0167-6687. doi:10.1016/j.insmatheco.2009.05.004.