ON NON-DIFFRACTIVE CONES

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Abstract

A subject of recent interest in inverse problems is whether a corner must diffract fixed frequency waves. We study the related question of which cones $[0, \infty) \times Y$ which do not diffract high frequency waves. We prove that if Y is analytic and does not diffract waves at high frequency then every geodesic on Y is closed with period 2π. Moreover, we show that if $\dim Y = 2$, then Y is isometric to either the sphere of radius 1 or its $\mathbb{Z}^2$ quotient, $\mathbb{RP}^2$.

1. Introduction

A subject of recent interest in the study of inverse problems has been the question of whether corners must diffract fixed-frequency solutions of the Helmholtz equation with potential in $\mathbb{R}^2$; here a corner is the location of a singularity of the potential, which is of the form of a smooth function times the indicator function of a sector. Affirmative answers to this question have been obtained under various conditions by Blåsten–Päivärinta–Sylvester [2] and Päivärinta–Salo–Vesalainen [19] (who treat certain kinds of conic singularities in $\mathbb{R}^3$ as well). More recently, results on diffraction by partially transparent polygons and polyhedra have been obtained by Elschner–Hu [8].

In this note, we introduce a related problem that seems fundamental to the theory of diffraction. On a cone, perhaps the simplest setting in which diffraction is known to occur, must there be nontrivial diffraction at high frequency? In posing the problem as a high-frequency one, we restate it as a question about singularities of solutions to the wave equation. If we study the half-wave propagator $e^{-it\sqrt{\Delta}}$, we ask: must there be singularities to the solution other than those along (the closure of) the geodesics missing the cone tip, i.e., those predicted by geometric optics in its naïvest form? Passing to the frequency-domain via Fourier transform, the existence of these singularities implies nontrivial asymptotics, as the frequency parameter tends to infinity, in regions not predicted by geometric optics away from the cone tip. Hence the question under consideration here is equivalent to one of high-frequency asymptotics in stationary scattering.

Our main theorem, admittedly a very partial result in the desired direction, is that if a real-analytic cone exhibits no diffraction in this sense, then its link must have the property that every geodesic is 2π-periodic. In the special case when the link has dimension 2 (and is still analytic) we are further able to show that the link must be S^2 equipped with its standard round metric of circumference 2π, or else $\mathbb{RP}^2$, its $\mathbb{Z}_2$-quotient. Some remarks on conjectured stronger results may be found below.

We now state our results more precisely.

Definition 1. A cone $C(Y)$ over a Riemannian manifold (Y, h) of dimension $d-1$ is the d-manifold

$$C(Y) = [0, \infty)_x \times Y$$

whose interior is equipped with the metric

$$g = dx^2 + x^2 h.$$

Thus from the point of view of metric geometry, in the cone $C(Y)$, all points $(0, y),\ y \in Y$ are identified.

Remark 1. For brevity, the results below will all be stated for cones $C(Y)$, which are sometimes referred to as *product cones*. We remark, however, that in view of [9, Theorem 3.2], our results apply equally to diffraction by more general conic metrics. These are nondegenerate metrics on the interior of a manifold with boundary which near the boundary take the form

$$g = dx^2 + x^2 h(x, y, dx, dy),$$

where x is a boundary defining function and h is a smooth symmetric 2-cotensor that restricts to be a metric on the boundary. Boundary components thus become cone points, and the results of [9] show that the leading order contribution to the diffracted wave can be determined from the case of a model product cone obtained by freezing coefficients at the boundary after making an appropriate choice of boundary defining function.

Definition 2. We say that $C(Y)$ is *non-diffractive* if

$$\operatorname{singsupp} \kappa(e^{-it\sqrt{\Delta}}) = \overline{\{p, p' \colon p, p' \text{ are endpoints of a geodesic of length } |t| \text{ in } C(Y)^\circ\}},$$

where $\kappa(A)$ denotes the Schwartz kernel of the operator A and B° denotes the interior of B. Otherwise, we say $C(Y)$ is *diffractive*. (Here Δ denotes the Friedrichs extension of the nonnegative Laplace-Beltrami operator from $\mathcal{C}_c^\infty(C(Y)^\circ)$.)

It is known that in general there are additional "diffracted" singularities of this Schwartz kernel, at

$$\mathsf{D}_t \equiv \{p, p' \colon x(p) + x(p') = |t|\};$$

indeed there is a conormal singularity along this set, degenerating near its intersection with the set of endpoints of geodesics in $C(Y)^\circ$, which always carries singularities. We remark that this intersection occurs exactly at the set

$$\{p, p' \colon x(p) + x(p') = |t|,\ y(p), y(p') \text{ endpoints of a geodesic of length } \pi \text{ in } Y\}.$$

It follows from the work of Cheeger–Taylor [3], [4] (see [9, Corollary 2.3]) that the *principal symbol* of the diffracted wave on D_t is a nonvanishing multiple of the Schwartz kernel of the operator

$$\exp\left(-i\pi\sqrt{\Delta_Y + \frac{(d-2)^2}{4}}\right),$$

where Δ_Y is the (positive definite) Laplacian on the link Y of the cone, with respect to the metric h. Setting

$$\nu = \sqrt{\Delta_Y + \frac{(d-2)^2}{4}},$$

we thus find that *a sufficient condition for $C(Y)$ to be diffractive* is that for some $y \in Y$, $\kappa(e^{-i\pi\nu}\delta_y)$ should have support outside the distance sphere of radius π centered at y. It is this condition that we exploit in proving the following.

Theorem 1. *Let $C(Y)$ be non-diffractive, and Y real analytic. Then every geodesic on Y must be periodic with (not necessarily minimal) period 2π.*

Remark 2. Many manifolds exist on which all geodesics are periodic with the same period: in addition to the compact rank one symmetric spaces and their quotients, there is a menagerie of so-called Zoll manifolds which enjoy this property—see [1] for detailed discussion.

Conversely, we remark that if Y is a spherical space form, i.e., the quotient of S^{d-1} with the standard metric on the *unit* sphere by the fixed-point-free action of a finite subgroup of $G \subset O(d)$, then $C(Y)$ is the quotient of $\mathbb{R}^d$ by the action of G, blown-up at the origin (i.e., viewed in polar coordinates). The method of images then shows that $C(Y)$ is non-diffractive, since the Schwartz kernel of $e^{-it\nu}$ on $C(Y)$ may be obtained by averaging over the action of G the corresponding Schwartz kernel on $\mathbb{R}^d$, where ordinary propagation of singularities along geodesics holds true. In the case of $d = 2$, i.e., $\dim Y = 1$, $e^{-i\pi\nu}$ can be calculated explicitly as in the work of Hillairet [15] and it is easy to verify that these are the only non-diffractive links. In fact *we conjecture that these are the only examples,* even in the smooth category: if $C(Y)$ is non-diffractive and Y merely $\mathcal{C}^\infty$, then we conjecture that Y must be a spherical space form. This conjecture seems out of reach for the moment.

Returning to the analytic case, we have been able to verify our conjecture in the case of dimension 2, ruling out Zoll manifolds that are not spherical space forms.

Theorem 2. *Let $C(Y)$ be non-diffractive with Y analytic and $\dim Y = 2$. Then Y is either S^2 or $\mathbb{RP}^2$ equipped with its standard metric.*

We emphasize that by "standard metric" on S^2 or $\mathbb{RP}^2$ we do *not* mean "standard metric up to scale," but rather the metric on the *unit* sphere in $\mathbb{R}^3$ and its $\mathbb{Z}_2$-quotient respectively; spheres and projective spaces of other sizes *do* diffract (as our proof shows). Unlike in many other geometric situations, the scaling of the metric plays a role since it corresponds to the size of the "opening" of the cone.

In order to clarify these distinctions, we will use the notation S^2_a and $\mathbb{RP}^2_a$ for the sphere equipped with the round metric of *circumference* a and its $\mathbb{Z}_2$-quotient, respectively. Hence $S^2_{2\pi}$ and $\mathbb{RP}^2_{2\pi}$ are the standard sphere and projective space. We introduce the non-standard terminology that a $\widetilde{P}_a$ manifold is one on which all geodesics are periodic with common period a, while we follow [1] in letting P_a manifolds denote those $\widetilde{P}_a$ manifolds on which a is the *minimal* common period. Thus, S^2_a is a P_a surface while $\mathbb{RP}^2_a$ is a $P_{a/2}$ surface.

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2. Proof of Theorem 1

Let Φ_t denote geodesic flow for time t on S^*Y, i.e., the time-t flow generated by H_p, the Hamilton vector field of $(1/2)|\xi|^2_g$, restricted to the unit cotangent bundle. Let π_Y denote the projection $S^*Y \to Y$.

Let
$$K \equiv \kappa(e^{-i\pi\nu}).$$
Recall that a necessary condition for Y to be non-diffractive is
$$\operatorname{supp} K \subset \{y, y' \colon y, y' \text{ are endpoints of a geodesic of length } \pi\}.$$
(Standard propagation of singularities results [7] show that the *singular support* of K lies in the latter set.)

Since Y is analytic, we note that in order to show that all geodesics are periodic with period 2π, it suffices to show that on a nonempty open set in S^*Y,
$$\Phi_{2\pi} = \operatorname{Id}.$$
Hence our strategy is to show that the support condition for $e^{-i\pi\nu}$ implies the existence of these closed geodesics.

Consider first the manifold
$$\Lambda \equiv \operatorname{graph}(\Phi_\pi) \subset S^*Y \times S^*Y$$
A key observation is now that $\operatorname{WF} K = \Lambda'$ (see, e.g., [6, Theorem 1]) where
$$\Lambda' := \{(x, \xi, y, \eta) \mid (x, -\xi, y, \eta) \in \Lambda\}.$$

Setting

$$\begin{aligned}\Psi, \Psi' : S^*Y &\to S^*Y \times S^*Y \\ \Psi(y,\eta) &= (y, \eta, \Phi_\pi(y,\eta)) \\ \Psi'(y,\eta) &= (y, -\eta, \Phi_\pi(y,\eta))\end{aligned}$$

we thus obtain diffeomorphisms

$$\begin{aligned}\Psi : S^*Y &\to \Lambda, \\ \Psi' : S^*Y &\to \Lambda'.\end{aligned}$$

We now consider the projection,

$$\pi_{Y\times Y}\Lambda \subset Y \times Y,$$

which is where $\operatorname{supp} K$ lives, by hypothesis. We remark that $\pi_{Y\times Y}\Lambda$ is certainly not guaranteed to be a smooth manifold. However, since our hypotheses imply that Λ is analytic, certainly $\pi_{Y\times Y}\Lambda$ is subanalytic, by definition. A theorem of Gabrielov [10], later rediscovered by Hironaka [16] and Hardt [14] then implies that $\pi_{Y\times Y}\Lambda$ is a *stratified* space, and in particular, contains as an open subset, F, a maximal-dimensional embedded submanifold (note that the "semianalytic shadows" in [14] are synonymous with subanalytic sets). We will employ a slight strengthening of this statement, also following from the results of [14].

Lemma 2.1. *There is an open subset $F \subset \pi_{Y\times Y}\Lambda$ such that*

1) F is a maximal-dimensional embedded submanifold.
2) The set $\tilde F = \Psi^{-1}\pi_{Y\times Y}^{-1}(F) \subset S^*Y$ is open
3) For $\rho \in \tilde F$, $\operatorname{rank} d\pi_{Y\times Y} d\Psi(\rho) = \dim F$.

Proof. Recall that a *stratification* of a manifold M is a locally finite collection $\mathcal{S}$ of connected, embedded open submanifolds such that $\sqcup_{S\in\mathcal{S}} S = M$ and

$$\text{if } S, T \in \mathcal{S} \text{ and } T \cap \partial S \neq \emptyset, \text{ then } \dim T < \dim S,\ T \subset \partial S.$$

Let $\Theta := \pi_{Y\times Y} \circ \Psi$. By [14, Corollary 4.4], since $\Theta : S^*Y \to Y \times Y$ is an analytic mapping of real analytic manifolds, there is a stratification, $\mathcal{S}$, of S^*Y and $\mathcal{T}$ of $Y \times Y$ such that for $S \in \mathcal{S}$, $\Theta(S) \in \mathcal{T}$, with

$$\operatorname{rank} d\Theta|_S = \dim \Theta(S).$$

Define $k := \sup_{S^*Y} \operatorname{rank} d\Theta$ and let $\tilde F \in \mathcal{S}$ such that

$$\operatorname{rank} d\Theta|_{\tilde F} = k, \qquad \dim \tilde F = 2n-1$$

Then $F := \Theta(\tilde F) \in \mathcal{T}$ is an embedded submanifold of dimension k. Moreover, $\pi_{Y\times Y}\Lambda$ is contained in a finite union of submanifolds of dimension $\leq k$ and hence F has maximal dimension.

Now, since $\mathcal{T}$ is a stratification, and $\Theta(S) \in \mathcal{T}$ for $S \in \mathcal{S}$,

$$\bigcup_{\substack{S\in\mathcal{S}\\ S\neq \tilde F}} \Theta(S) \cap F = \emptyset.$$

In particular, $\Theta^{-1}(F) = \tilde{F}$ and the proof is complete. q.e.d.

Let F, $\tilde{F}$ as in Lemma 2.1. Then, by construction, for $\rho \in \tilde{F}$, $d\pi_{Y\times Y} d\Psi : T_\rho S^*Y \to T_{\pi_{Y\times Y}\circ\Psi(\rho)}F$ is surjective.

Lemma 2.2. *Suppose that* $d\pi_{Y\times Y} d\Psi : T_\rho S^*Y \to T_{\pi_{Y\times Y}\circ\Psi(\rho)}F$ *is surjective. Then* $\Psi'(\rho) \in SN^*F$.

Proof. Fix any $\rho_0 \equiv (y, \eta) \in S^*Y$ satisfying the hypotheses. We need to show that all vectors in $T_{\pi_{Y\times Y}\circ\Psi(\rho_0)}F$ are annihilated by pairing with $\Psi'(\rho_0)$.

By hypothesis,

$$\begin{aligned} T_{\pi_{Y\times Y}\circ\Psi(\rho_0)}F &= \{d\pi_{Y\times Y} \circ d\Psi(V) \colon V \in T_{\rho_0}S^*Y\} \\ &= \{(d\pi(V), d\pi \circ d\Phi_\pi(V)) \colon V \in T_{\rho_0}S^*Y\}, \end{aligned}$$

hence we need to show that the pairing of a vector of this form with $\Psi'(\rho_0)$ vanishes, i.e., (letting square bracket denote the pairing of a covector with a vector) that

$$-\rho_0[d\pi(V)] + \Phi_\pi(\rho_0)[d\pi \circ d\Phi_\pi(V)] = 0, \qquad \text{for all } V \in T_{\rho_0}S^*Y. \tag{1}$$

Now we investigate the quantity $\Phi_\pi(\rho_0)[d\pi \circ d\Phi_\pi(V)]$. For any $V \in T_{\rho_0}S^*Y$, choose $\rho : (-\epsilon, \epsilon) \to S^*Y$ with $\rho(0) = \rho_0$ and $\partial_s\rho|_{s=0} = V$. Next, define $\Gamma(s,t) = \pi(\Phi_t(\rho(s)))$. Then $J(t) := \partial_s\Gamma(s,t)|_{s=0} = d\pi \circ d\Phi_t V$ is a Jacobi field along $\gamma(t) := \Gamma(0,t)$ and

$$\Phi_t(\rho(0))[J(t)] = \langle \dot\gamma(t), J(t)\rangle_g.$$

Since J is a Jacobi field, $\partial_t^2\langle\dot\gamma(t), J(t)\rangle_g = 0$ (see [17, p.288]). Recalling that H_p denotes the Hamilton vector field of $(1/2)|\xi|_g^2$, hence the generator of unit speed geodesic flow in the cosphere bundle, we compute (using symmetry of the connection—[17, Lemma 6.2])

$$\begin{aligned} \partial_t\langle\dot\gamma(t), J(t)\rangle_g|_{t=0} &= \langle\dot\gamma(0), D_tJ(0)\rangle_g \\ &= \langle\dot\gamma(0), D_s d\pi H_p(\rho(s))|_{s=0}\rangle_g \\ &= \langle d\pi H_p(\rho(0)), D_s d\pi H_p(\rho(s))|_{s=0}\rangle_g \\ &= \frac{1}{2}\partial_s\langle d\pi H_p(\rho(s)), d\pi H_p(\rho(s))\rangle_g|_{s=0} \end{aligned}$$

Now, in coordinates, we have $d\pi H_p = g^{ij}\xi_i\partial_{x_j}$ and therefore, since $\rho(s) \in S^*Y$,

$$\langle d\pi H_p(\rho(s)), d\pi H_p(\rho(s))\rangle_g = g^{ij}\xi_i(s)\xi_j(s) \equiv 1.$$

Therefore, $\partial_t\langle\dot\gamma(t), J(t)\rangle_g|_{t=0} = 0$. We have now shown that for any $V \in T_{\rho_0}S^*Y$, $\Phi_t(\rho_0)[d\pi \circ d\Phi_t V]$ is constant and in particular,

$$\rho_0[d\pi \circ d\Phi_t V] - \Phi_\pi(\rho_0)[d\pi \circ d\Phi_\pi V] = 0,$$

thereby establishing (1). q.e.d.

Our hypotheses are that $\operatorname{supp} K \subset \pi_{Y\times Y}\Lambda$, hence in a neighborhood V of any point in F, $\operatorname{supp} K \subset F$. Since F is a smooth embedded submanifold, we thus know that on V, we may express

$$K = \sum \delta^{\alpha}(u)\phi_{\alpha}(y)$$

where $u = (u_1, \dots, u_k)$ are defining functions for F, y complete u to a local coordinate system, and $\phi_\alpha \in \mathcal{D}'(F)$. Moreover, by Lemma 2.2, for $\rho \in \tilde{F} = \Psi^{-1}\pi_{Y\times Y}^{-1}(F)$, $\Psi'(\rho) \in SN^*F$. In particular,

$$\mathrm{WF}(K) \cap \pi_{Y\times Y}^{-1}(F) = \Lambda' \cap \pi_{Y\times Y}^{-1}(F) \subset SN^*F$$

which implies $\phi_\alpha \in C^\infty(F)$.

Such a distribution has the property that its wavefront set is invariant under the negation map on fibers:

$$(y, \eta, y', \eta') \in \mathrm{WF}\, K \cap \pi_{Y\times Y}^{-1}(F \cap V) \Longrightarrow (y, -\eta, y', -\eta') \in \mathrm{WF}\, K \cap \pi_{Y\times Y}^{-1}(F \cap V).$$

Thus, since $\mathrm{WF}\, K = \Lambda'$,

$$(y, \eta, y', \eta') \in \Lambda \cap \pi_{Y\times Y}^{-1}(F \cap V) \Longrightarrow (y, -\eta, y', -\eta') \in \Lambda \cap \pi_{Y\times Y}^{-1}(F \cap V).$$

This precisely means that for $(y, \eta) \in \pi_L(\Lambda \cap \pi_{Y\times Y}^{-1}(F \cap V))$,

$$\Phi_\pi(y, \eta) = -\Phi_\pi(y, -\eta) = \Phi_{-\pi}(y, \eta)$$

(with negation interpreted as acting on the fibers). Hence

$$\Phi_{2\pi}(y, \eta) = (y, \eta).$$

Now set $U = \pi_L(\Lambda \cap \pi_{Y\times Y}^{-1}(F \cap V))$. Since $\pi_L : \Lambda \to S^*Y$ is bijective, and $\Lambda \cap \pi_{Y\times Y}^{-1}(F \cap V)$ is open we have proved the desired periodicity of geodesics on a nonempty open set in S^*Y. q.e.d.

3. Proof of Theorem 2

By Theorem 1, Y is a $\widetilde{P}_{2\pi}$ surface. Thus, it is diffeomorphic to either S^2 or $\mathbb{RP}^2$—see [**1**, Section 4.3].

We begin with the case where Y is diffeomorphic to S^2. As in the proof of Theorem 1, we consider $\pi_{S^2\times S^2}\Lambda \subset S^2 \times S^2$, the projection of the graph of time-π geodesic flowout in $S^*(S^2)$; we again use crucially that this is a stratified space. Since the dimension of Λ itself is 3 and since projections onto the left and right factor of Y of $\pi_{Y\times Y}\Lambda$ are surjective, the dimension of the maximal stratum of $\pi_{S^2\times S^2}\Lambda$ may only be 2 or 3. If it is 3, then there is an open set, F in $\pi_{S^2\times S^2}\Lambda$ that is a submanifold of $S^2 \times S^2$ of codimension-1, so that the Schwartz kernel of the propagator K is locally given by

$$K = \sum_{|\alpha|\leq M} \delta^{(\alpha)}(u)\phi_\alpha(y) \tag{2}$$

where now $u \in \mathbb{R}$ is locally a defining function for $\pi_{S^2\times S^2}\Lambda$, $M < \infty$ and $\phi_\alpha \in C^\infty(F)$.

We will need a slightly stronger consequence of [6, Theorem 1] than that $(\mathrm{WF}(K))' = \Lambda$. In particular, we need that

$$\begin{aligned}(\mathrm{WF}^{-1}(K))' = \Lambda,\\ \mathrm{WF}^{-1-\epsilon}(K) = \emptyset \text{ for all } \epsilon > 0,\end{aligned} \tag{3}$$

where WF^s denotes the s-wavefront set, i.e., $\rho \notin \mathrm{WF}^s(u)$ if and only if there exists $A \in \Psi^0$ so that $Au \in H^s$ and $\sigma(A)(\rho) \neq 0$. Now, let $x_0 \in F$ and V a neighborhood of x_0 so that (2) is valid on V. Let $\chi \in C_c^\infty(V)$ with $\chi(x_0) = 1$. Then, by (2)

$$\chi K \in \bigcup_{\epsilon>0} H^{-1/2-j-\epsilon} \setminus H^{-1/2-j}, \tag{4}$$

where j is the largest $|\alpha|$ such that the coefficient ϕ_α in (2) is nonvanishing on $\operatorname{supp}\chi$. On the other hand,

$$\begin{aligned}T^*_{x_0}(S^2\times S^2)\cap \mathrm{WF}^{-1}(K) \neq \emptyset,\\ T^*_{x_0}(S^2\times S^2)\cap \mathrm{WF}^{-1-\epsilon}(K) = \emptyset.\end{aligned}$$

Either the first or the second of these statements contradicts (4) depending on whether $j = 0$ or $j \geq 1$.

We conclude from this contradiction that in fact the dimension of the maximal stratum is 2. On the other hand, the rank of the projection from $\Lambda \subset S^*S^2\times S^*S^2$ to S^2 in the first factor already has rank 2. Hence in order for the stratum dimension not to exceed 2, it must be the case that $d_\xi \pi_{S^2}\Phi_\pi(x,\xi) = 0$ for all $x, \xi \in S^*(S^2)$. This means that $\pi_{S^2}\Phi_\pi(S^*_x(S^2))$ is a *single point* for each $x \in S^2$; for brevity we denote this point $\Phi_\pi(x)$. Since Y is a $\widetilde{P}_{2\pi}$ manifold, by positivity of the injectivity radius of a compact manifold, there is some positive minimal common period, hence Y is a $P_{2\pi/k}$ manifold for some positive integer k. We now consider separately the cases where k odd and even.

Case 1: k odd. It has been shown by Gromoll–Grove [12] that on Y diffeomorphic to S^2, the P_a condition implies that Y is an SC_a manifold (again in the terminology of [1, Section 7.8]), which is to say, all geodesics have minimal period exactly a and are without self-intersection (“simple”). If Y is a $P_{2\pi/k}$ manifold for k *odd*, we thus conclude from [12] that $\Phi_\pi(x) \neq x$ for all x, as otherwise this would contradict simplicity of the geodesics.

Lemma 3.1. *Suppose that Y is as above and Y is a $P_{2\pi/k}$ surface for some k odd. Then, Y is a Blaschke surface.*

Proof. We recall from [1, Theorem 5.43] that among several equivalent definitions of a Blaschke surface is that *the cut locus is spherical*, which is to say the distance to the first cut point is independent of direction at each point. For Y a $P_{2\pi/k}$ surface with k odd, $\Phi_{\pi/k}(x)$ has distance π/k from x for all x, since otherwise a geodesic from x would pass through $\Phi_{\pi/k}(x)$ at time $t_0 \in (0, \pi/k)$ and then would self-intersect at time π/k, contradicting the simplicity of the geodesics from [12]. But then every geodesic must in fact be minimizing up to time-π/k, as a failure to be minimizing would allow us to construct a continuous, piecewise smooth curve from x to $\Phi_{\pi/k}(x)$ of length shorter than π/k. Hence the cut-radius is exactly π/k, at every point in every direction, and our surface is indeed Blaschke. q.e.d.

By Lemma 3.1 together with the resolution of the Blaschke Conjecture [11] (cf. [1, Theorem 5.59]), we conclude that $Y = S^2_{2\pi/k}$.

We now rule out nonstandard spheres, with $k \neq 1$. We will identify spheres of all radii with one another using standard polar coordinates, and note that for all k, on $S^2_{2\pi/k}$, $\Phi_{\pi/k}(x) = -x$, the antipode. Now observe that since $e^{-i\pi\nu}\delta_x$ is supported at $\Phi_\pi(x) = -x$ and K takes the form (2),

$$e^{-i\pi\nu}\delta_x = c_1\delta_{-x}$$

for some $c_1 \neq 0$. Moreover, c is independent of x since isometries act transitively on $S^2_{2\pi/k}$. Now, since $S^2_{2\pi}$ is non-diffractive,

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-i\pi(\ell-\frac{1}{2})}\overline{Y_\ell^m(y)}Y_\ell^m(x) = c_2\delta_{-x}(y)$$

for some $c_2 \neq 0$. Thus, there is $c \neq 0$ such that for all x, y

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-i\pi(\ell-\frac{1}{2})}\overline{Y_\ell^m(y)}Y_\ell^m(x) = c\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-i\pi\sqrt{k^2\ell(\ell-1)+\frac{1}{4}}}\overline{Y_\ell^m(y)}Y_\ell^m(x).$$

Since the Y_ℓ^m form an orthonormal basis for L^2, this implies that for all $\ell \in \mathbb{N}$,

$$\sqrt{k^2\ell(\ell-1)+\frac{1}{4}} - \ell + \frac{1}{2} \in \beta + 2\mathbb{Z}$$

for some fixed real number $\beta \in [0,2)$. We note, though, that

$$\sqrt{k^2\ell(\ell-1)+\frac{1}{4}} - \ell + \frac{1}{2} = k\ell - \frac{1}{2}k - \frac{k^2-1}{8k\ell} - \ell + \frac{1}{2} + O(\ell^{-2}),$$

and, by Lemma 3.2 (below), this quantity cannot have constant fractional parts as $\ell \to \infty$ unless $k = 1$.

Thus, $Y = S^2_{2\pi}$. This finishes the case where k is odd.

Lemma 3.2. *Let $p, q \in \mathbb{Z}$, $q > 0$, $b, c \in \mathbb{R}$ and $c \neq 0$ and define*

$$\alpha_\ell := \frac{p}{q}\ell + b + c\ell^{-1} + O(\ell^{-2}), \qquad \ell = 1, 2, \dots$$

Suppose that $\{j_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ with $j_k \underset{k\to\infty}{\longrightarrow} \infty$. Then there are ℓ, m such that

$$\alpha_{j_\ell} - \alpha_{j_m} \notin \mathbb{Z}.$$

Proof. Let
$$f(\ell) = \frac{p}{q}\ell + b + \frac{c}{\ell}.$$
Then, there is $L > 0$ such for $j \geq L$, α_j satisfies
$$|\alpha_j - f(j)| \leq \frac{|c|}{3j}.$$
Hence, for $\ell, m \geq L$, letting $e_\ell = \alpha_\ell - f(\ell)$, $e_m = \alpha_m - f(m)$, we obtain
$$\begin{aligned} \alpha_m - \alpha_\ell &= \frac{p}{q}m + b + \frac{c}{m} + e_m - (\frac{p}{q}\ell + b + \frac{c}{\ell}) - e_\ell \\ &= \frac{p}{q}(m-\ell) + \frac{c}{m} - \frac{c}{\ell} + e_m - e_\ell. \end{aligned} \tag{5}$$
Now, our estimates on e_m, e_ℓ easily give
$$|e_m - e_\ell| \leq \frac{|c|}{3m} + \frac{|c|}{3\ell}.$$
Now, since $j_k \to \infty$, there is $M > 0$ such that for $m \geq M$, $j_m > \max(L, 4|c|q)$. Fix $m \geq M$. Let $\ell \geq M$ such that $j_\ell \geq 4j_m$. Then, we have
$$\alpha_{j_m} - \alpha_{j_\ell} \bmod \frac{1}{q}\mathbb{Z} = \frac{c}{j_m} - \frac{c}{j_\ell} + e_{j_m} - e_{j_\ell} \bmod \frac{1}{q}\mathbb{Z},$$
and
$$0 < \frac{|c|}{3j_m} \leq \Big|\frac{c}{j_m} - \frac{c}{j_\ell} + e_{j_m} - e_{j_\ell}\Big| \leq \frac{3|c|}{2j_m} \leq \frac{3}{4q}.$$
Hence the fractional part of $\alpha_{j_m} - \alpha_{j_\ell}$ is nonzero. q.e.d.

CASE 2: k EVEN. We now assume that Y is a $P_{2\pi/k}$ surface for some *even* integer k, hence that Y is a $\widetilde{P}_\pi$ surface; we will then derive a contradiction.

For each $y \in S^2$, $e^{-i\pi\nu}\delta_y$ is by hypothesis supported at the flowout of $S^*_y(S^2)$, which we now know to be the point y itself. Thus $e^{-i\pi\nu}\delta_y$ must equal $\psi(y)\delta_y$ for some function ψ; more generally this tells us that
$$e^{-i\pi\nu}f = \psi f \tag{6}$$
for every $f \in L^2$. Applying (6) to $f = \phi_j$, an eigenfunction of Δ_Y with eigenvalue λ^2, tells us that
$$e^{-i\pi\sqrt{\lambda_j^2+1/4}} = \psi(y). \tag{7}$$
The left side is of course constant, so ψ is in fact constant, and all of these values must agree, i.e., there exists $\beta \in [0,2)$ such that for all λ_j^2 in the spectrum of Δ_Y,
$$\sqrt{\lambda_j^2 + 1/4} \equiv \beta \bmod 2\mathbb{Z}; \tag{8}$$
equivalently this is just the statement that the spectrum of ν lies in $\beta + 2\mathbb{Z}$.

Now in order to derive a contradiction, we turn to the strong results known about spectral asymptotics of Zoll surfaces; this argument is based on the fact that the spectrum of a $P_{\pi/k}$ manifold must closely resemble that of $S^2_{\pi/k}$, which is indeed diffractive. Duistermaat–Guillemin [6], Weinstein [20], and Colin de Verdière [5] have obtained very precise estimates of the *clustering* of the eigenvalues of such a Zoll surface. Thus, e.g., [5, Corollaire 1.2] (see also [13], [21]) shows that there is $M > 0$ so that the spectrum of $\Delta_Y + \frac{1}{4}$ is entirely contained in a union of intervals

$$I_n = \left[4(n+\alpha/4)^2 - M, 4(n+\alpha/4)^2 + M\right],$$

where α is the Maslov index of all the π-periodic geodesics; the (crucial)[1] factors of 4 arise since we are dealing with a $\widetilde{P}_\pi$ surface rather than a $\widetilde{P}_{2\pi}$ surface as in [5]. Since the eigenvalues of $\Delta_Y + \frac{1}{4}$, $\lambda_\bullet^2 + \frac{1}{4}$, lie in I_n, the square roots $\sqrt{\lambda_\bullet^2 + \frac{1}{4}}$ of the eigenvalues lie in intervals

$$J_n = \left[2(n+\alpha/4) - C/n, 2(n+\alpha/4) + C/n\right].$$

On the other hand, for n large, the constraint (8) implies that *each interval I_n can contain at most one eigenvalue* (possibly with high multiplicity). Indeed, for n large enough, $2C/n < 2$ and hence the interval J_n has length less than 2 and contains at most one element of the form (8). We have thus reduced to the situation studied by Zelditch in [22] of a *maximally degenerate Laplacian*; Zelditch proves [22, Theorem C] that this places a yet stronger constraint on the locations of the eigenvalues and that there is an operator A with spectrum in $\mathbb{N}$ such that

$$\Delta_Y = 4\Big(A + \frac{1}{2}\Big)^2 - 1 + S$$

with S a smoothing operator; here again we have rescaled by a factor of 4 since we are dealing with a $\widetilde{P}_\pi$ surface. Hence the eigenvalues $\sqrt{\lambda_\bullet^2 + 1/4}$ of ν are all of the form

$$\sqrt{4(\ell+1/2)^2 - 3/4} + O(\ell^{-\infty}) = 2\ell + 1 - \frac{3}{16\ell} + O(\ell^{-2}). \tag{9}$$

Recall on the other hand that they are in $\beta + 2\mathbb{Z}$ by (8). By Lemma 3.2, these two constraints are incompatible, i.e. solutions to (9) cannot asymptotically differ by even integers.

Hence we have ruled out all $\widetilde{P}_\pi$ manifolds, and completed the case of Y diffeomorphic to S^2.

To finish the proof, we now turn to the (easier) case when Y is diffeomorphic to $\mathbb{RP}^2$. In this case, Lin–Schmidt [18, Theorem 2] shows that since all geodesics on Y are closed, $Y = \mathbb{RP}^2_a$ for some $a > 0$. (Cf. Green's proof of Blaschke's Conjecture [11].) Next, since Y is a $\widetilde{P}_{2\pi}$ manifold, we know that $Y = \mathbb{RP}^2_{4\pi/k}$ for some $k \in \mathbb{N}$. We now rule out all but $\mathbb{RP}^2_{2\pi}$. To start, we know by the same argument as in the sphere case that $\Phi_\pi(x)$ is a single point for each x. This does not happen unless k is even and at least 2. For k even, $\mathbb{RP}^2_{4\pi/k}$ is a $\widetilde{P}_\pi$-manifold so the

[1] These factors will later give rise to our contradiction and arise from the rescaling of a $\widetilde{P}_{2\pi}$ metric to a $\widetilde{P}_\pi$ metric.

same argument as in the sphere case tells us that the spectrum of $\nu = \sqrt{\Delta_Y + 1/4}$ lies in $\beta + 2\mathbb{Z}$ for β fixed. Next, the spectrum of $\mathbb{RP}^2_{4\pi/k}$ is the set

$$\frac{k^2}{4} 2\ell(2\ell+1), \quad \ell \in \mathbb{N};$$

thus the spectrum of ν is

$$\left(\frac{k^2}{4} 2\ell(2\ell+1) + \frac{1}{4}\right)^{\frac{1}{2}} = k\ell + \frac{k}{4} - \frac{k^2-4}{32k\ell} + O(\ell^{-2}).$$

By Lemma 3.2, these cannot have constant fractional part unless $k = 2$. q.e.d.

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