Continuous-Time Selection Dynamics and Their Economic Applications

by

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ABSTRACT

In the recent years, there has been an impressive advance in the study of models in which (boundedly rational) economic agents adjust their behavior over time, in reaction to the additional information they acquire as time progresses.

Some of these models involve the use of autonomous continuous-time dynamical systems, known in the literature as *selection dynamics,* in which the adjustment process is directly linked to the relative performance of each strategy at each given point in time.

The aim of the thesis is to contribute to this line of research exploring the formal properties of these dynamics, as well as modeling suitable economic environments in which these dynamics can be applied.

In particular, the thesis explores the behavior of selection dynamics in three different game-theoretic frameworks: (game-form) mechanisms, extensive form games and games with pre-play communication.

A (game-form) mechanism is a game whose equilibria satisfies certain desirable properties but which does not necessitate vast amount of knowledge by the authorities *("the planner")* to put it in place. Instead, this social arrangement should basically self police itself, and the planner should only make sure that the rules of the game are correctly followed by the agents.

An extensive form game is a game in which players move sequentially, and make use of the information they acquire as the game proceeds to improve their performance.

A game with pre-play communication is a game which is preceded by a stage in which the agents are allowed to send costless signals to their opponent, in order to influence their behavior.

In all these situations, the analysis of the learning dynamics we described leads to conclusions which contradict traditional game theoretic analysis, but seem to suit more closely the empirical and experimental evidence in the field.

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Quoting an Italian pop song, I thank Rob for *explaining my own ideas in a way I can understand.* He also co-authored the paper on which chapter 4 is based.

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CONTENTS

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CHAPTER 1
INTRODUCTION

1.1. EVOLUTIONARY DYNAMICS AND EQUILIBRIUM ANALYSIS

Since Adam Smith [1776] introduced the notion of *natural price,* as "...*the central price, to which the prices of all commodities are continually gravitating*..." it is difficult to think of an idea that has been more widely applied in economic theory than the concept of an *equilibrium.* Yet, it is even more difficult to name another concept about whose interpretation has evolved so sharply along the years. Consider, for example, the alternative equilibrium notions proposed by Walras [1874], Cournot [1838], Marshall [1916], Keynes [1936], Arrow and Debreu [1954], Hahn [1973], Lucas [1972], Cass and Shell [1983] ...

Despite its pervasive use in economic modeling, serious questions concerning the foundational aspects of equilibrium analysis remain, questions that are too serious to be dismissed as mere academic puzzles. Market fluctuations and imperfections are endemic in real-life economics. From the *unemployment equilibria* of Keynesian memory, to the various *real business cycle* and *disequilibrium* theories proposed in later literature, the focus on equilibrium analysis has been continuously challenged, on various grounds and from different perspectives.[1]

Clearly, game theory cannot avoid considering the foundational aspects of equilibrium analysis, since it is an equilibrium concept, namely *Nash equilibrium,* which has made its fortune. In this respect, we shall follow Binmore [1987-8] in distinguishing between two alternative justifications of equilibrium analysis which have been maintained by the game-theoretic tradition:

- an *eductive* justification, which relies on the agents' ability to reach equilibrium through careful reasoning. Since agents are *fully rational,* they can always correctly *predict* their opponents' behavior and maximise against it;

[1]For a synthetic survey on *disequilibrium* theories, see Benassy [1987].

- an *evolutive* justification, which relies on the possibility that (*partially* rational) agents reach equilibrium by means on some adjustment processes.

The aim of this dissertation is to contribute to the latter methodological approach with the aid of *evolutionary dynamics,* which are meant to describe how imperfectly rational economic agents may adjust their behavior over time, in reaction to the additional information they acquire as time progresses. In other words, the evolutive approach we pursue tries to answer the following (apparently simple) question:

how do we learn to play games?

We shall break this grand question into smaller pieces. By doing so, we will introduce the main methodological assumptions on which the evolutionary paradigm (or, at least, the evolutionary paradigm we follow in this dissertation) is based.

- *Where do we learn?* That is: which are the institutional (strategic) features of the environment in which our agents are assumed to operate? In this respect, the literature to which we refer makes the following assumption: in formalising how people's behavior changes over time it considers models in which the interaction structure is exogenously given and *fixed*, in the form of an (infinitely) repeated game.[2] Moreover, and perhaps more crucially, the strategy space (i.e. the set of possible behaviors our agents may adopt) coincides with the strategy space of the stage game. In this respect, evolutionary games differ from other strategic frameworks like *differential games* (in which the current payoff is a function of time) or *supergames* (in which strategies are defined over time-paths).

  It may be worthwhile noting that any justification of this (drastic) assumption, apart from mathematical tractability, contains serious weaknesses. It is impossible to consider two situations as being absolutely *identical,* or in which we totally neglect the future consequences of our actions. However, when interaction is *anonymous* (i.e. takes place among a large population of agents who have no prior knowledge of the identity, the history, or any other relevant characteristics of their opponents), this framework appears to be more reasonable (and this is the reason why the literature has focused almost entirely on this case).
- *What do we learn (from)?* In this respect, we follow Selten [1991] by distinguishing three class of learning models:

(*i*) *rote* (individual) learning models, i.e. models in which success and failure directly influence choice probabilities;

(*ii*) *imitation* (social) learning models, in which success and failure *of others* directly influence choice probabilities;

[2]Exceptions that develop models of learning when games are *similar* are those of Li Calzi [1993] and Romaldo [1995].

(iii) belief learning models, in which experience has only a direct effect on players' *beliefs,* usually characterised as probability distributions over the possible states of play.[3]

In the dissertation, we will mainly deal with models of the first two categories, in which players need not know (or care) a great deal about the game they play, other than the payoff they (or other agents in the population) obtain.[4]

A related question along these lines could be the following:

- *Do we learn only from current states of events, or do we remember, making constant use of the information acquired in the past?* In formalising how people's behavior changes over time, we shall consider models in which *there is no memory.* The quantitative features of the adjustment process (i.e. the vector field) are in fact completely determined by the behavior of the system at each point in time.[5]

Numerous other questions could be asked[6], but we shall stop here. To address these problems correctly, we need a *theory of the mind* which accounts for the cognitive processes by which agents make their decisions, in games as well as in other circumstances. Unfortunately, no such a theory exists, and perhaps, never will. All we have is a set of conflicting paradigms, which one can use to shed some partial light on one of aspect of the problem.

The behavioral paradigm we shall follow is that of *evolutionary game theory.* In their standard formulation, evolutionary models are based on the assumption that agents' behavior is *genetically programmed,* i.e. encoded in the genes which characterize each agent's type. If such a representation is to be applied to study economic environments, it has been argued, this can only be at a *metaphorical* level. Börgers [1996] provides three good reasons why it should be so:

- it is not practically feasible, given the state-of-the art knowledge in genetics, to derive predictions of human behavior by appealing to its genetic determination;
- the way in which genetic codes determine behavior seems to be very complicated;
- the adaptation of human genes appears to occur so slowly that predictions which rely on this mechanism appear to be problematic.

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[3]A typical example of a belief learning model is the classic *fictitious play dynamics,* first introduced by Brown [1951]. According to fictitious play, the agent select, at each point in time, the pure strategy which maximises utility against the mixed strategy in which probabilities equal the relative frequency with which each pure strategy has been observed in the past. Recent papers in his research field include Jordan [1991], Fudenberg and Kreps [1988], Kalai and Lehrer [1191], Milgrom and Roberts [1990]. See Battigalli *et al.* [1992] and Fudenberg and Levine [1997] provide comprehensive surveys on this research field.

[4]A noticeable exception is the model contained in Chapter 4, in which belief learning will also be considered.

[5]Some might regard also this assumption as unreasonable. Take for example, the case in which the dynamics exhibit limit cycles like, for example, the model of chapter 3 of this dissertation. This would imply that the agents are not able to recognise this cyclic pattern and modify their response in reaction to it. As Fudenberg and Levine [1997] argue: *"we suspect that if cycles persisted long enough the agent would eventually use more sophisticated inference rules that detected them; for this reason we are not convinced that models of cycles in learning are useful descriptions of actual behavior..."* (p. 3). The fact that the system is autonomous simply discards this possibility: whenever the agents find themselves in the same *state* (i.e. play a mixed strategy in the limit cycle set), the adjustment process (i.e. the vector field of f) is the same, and therefore, leads to an infinite repetition of that cyclic behavior.

[6]Like, for example, *when* (or *how fast) do we learn* ?

Such criticisms take for granted that evolution is to be understood in a strictly ***biological*** sense. However, *social* evolution is also worthy of study, and evolutionary dynamic models have been justified by some recent literature which derives dynamic adjustment processes similar to those studied here starting from concrete models of social interaction.[7] But before we look at these models in more detail, it will be instructive to examine some simple examples from a biological perspective.

1.2 Some Simple Examples

Consider the following 2x2 game:

		C	D
x	C	3,3	2,4
$1-x$	D	4,2	1,1

Figure 1.1
The game *Chicken*

known in the literature as the game *Chicken*. In a biological context (see Maynard Smith [1982]), this payoff structure can be used to model the interaction between two animals contesting a resource of fixed value (6, in our example), which is efficiently distributed only if at least one of the two contestants "cooperates" (i.e. does not fight).[8] Alternatively, following Schelling [1960], one can think of *Chicken* as a *tacit bargaining* situation: given that the players differ in their ranking of the various equilibrium outcomes, the choice of a particular equilibrium has clear distributional effects, since a better bargain for a player always corresponds to less for the opponent.

Let "Anna" (A) denote the row player, and "Beppe" (B) the column player. First notice that the game has two asymmetric Nash equilibria in pure strategies, namely (C,D) and (D,C), and a symmetric Nash equilibrium in mixed strategies in which both strategies are played with equal probability. We start by assuming that Anna and Beppe are drawn from a single (large) population of agents whose behavior is genetically encoded. Let x (resp. 1-x) denote the relative frequency of agents programmed to play strategy C (D). Suppose that the game's payoffs represent the *fitness*, measured as the number of offspring per time unit. Each individual offspring is absolutely identical to its single parent. Under the above assumptions, it can be shown[9] that the fraction x of the population playing strategy C can be approximated by the following differential equation:

$$\dot{x} = x\big((2+x) - ((2+x)x + (1+3x)(1-x))\big) = x(1-x)(1-2x) \tag{1.2.1}$$

[7] See § 1.4.

[8] Following a well-established tradition, throughout the dissertation we shall label the two strategies C (for *cooperate)* and D (for *defect*), the latter identifying the *minmax* strategy of the game.

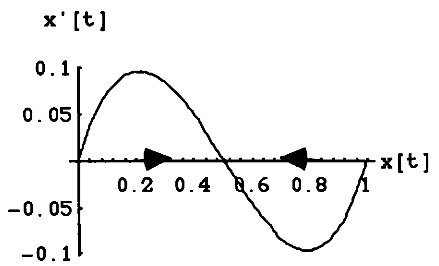
[9] See, for example, Binmore [1992], Chapter 9.

in which the growth rate $\frac{\dot{x}(t)}{x(t)}$ equals the difference in payoffs between the pure strategy *C* and the mixed strategy in which the pure strategy *C (D)* is played with probability *x* (1-*x*). We call the latter the *population strategy.* We will make constant reference, throughout the dissertation, to dynamic processes of this kind, known in the literature as *Replicator Dynamics,* which have been regarded as stereotypical *natural selection* processes.

We shall start by looking at the dynamics (1.2.1) in more detail. First note that (1.2.1) has three *restpoints,* i.e. solutions of the equation $\dot{x} = 0$. These restpoints are $x = 0$, $x = 1$ and $x = \frac{1}{2}$. If the system starts at any of these points, it will remain there forever. We shall be interested in the evolution of (1.2.1) when $x(0) \in (0,1)$, that is, when both strategies are present in the initial population:

- PROPOSITION 1.1. Any interior solution $x(x(0),t)$ of (1.2.1) converges to $1/2$.

To see why, look at the following figure 1.2. In figure 1.2*a)* we plot $x'(t)$ as a function of $x(t)$, while in figure 1.2*b)* we trace some interior trajectories of (1.2.1).



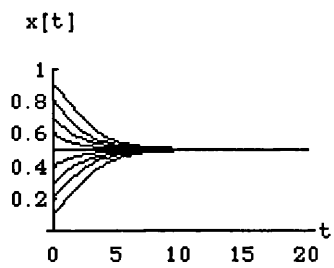


FIGURE 1.2
The Replicator Dynamics and the game *Chicken* in a single-population environment.

As figure 1.2*a)* shows, $x'(t)$ is positive (negative) for any $x(t)$ below (above) 1/2. Therefore, any interior trajectory must converge to the point which corresponds to the (symmetric) mixed strategy equilibrium.

Consider now an alternative scenario. Assume instead that the agents are drawn from *two* distinct populations; one population of Annas and one population of Beppes. matched at random to play the game of figure 1.1. [10] Since there are now two reference populations, it is no longer true that the relative frequency of Annas who cooperate should equalise (in expected terms) the relative frequency of Beppes who cooperate. Other things being equal (i.e. we shall maintain

[10] A more complex framework, which allows for *intra-specific* interaction, is analysed in the works of Selten [1983] and Cressman [1992, 1995].

the assumptions about the reproductive process, which now takes place in the two separate populations) we look at the dynamic properties of this new setting:

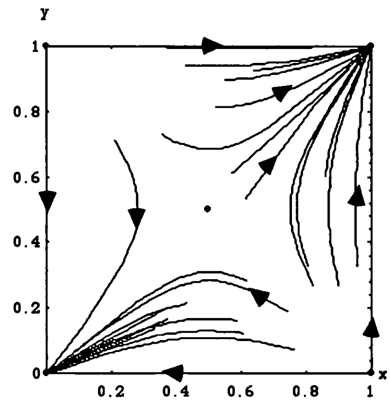


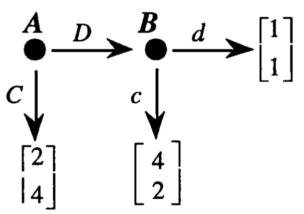
FIGURE 1.3
Replicator Dynamics and *Chicken* in a 2-population environment.

Let x be the relative frequency of Beppes who defect, and y be the relative frequency of Annas who cooperate. Figure 1.3 displays a phase diagram of the *Chicken* game given this new two-population setting. The picture is now quite different: the two asymmetric (strict) Nash equilibria in pure strategies attract any interior trajectory other than the one which corresponds to the Nash equilibrium in mixed strategies at (1/2, 1/2).

- PROPOSITION 1.2. Any generic interior trajectory of the Replicator Dynamics, in the two-population case, converges either to (C,D) or to (D,C).

PROOF. See Weibull [1995], Chapter 5. ◆

Consider now another modification of the strategic interaction proposed above. Assume that now players move *sequentially:* Anna moves first, deciding whether to cooperate or to defect. If Anna cooperates, the games ends; if Anna defects, then Beppe has to decide whether to reciprocate defection or to cooperate in return. Under this new assumption, the strategic scheme can be represented by means of the following extensive-form game of perfect information:



		$1-x$	x
		C	D
y	C	2,4	2,4
$1-y$	D	4,2	1,1

FIGURE 1.4
The *Entry* game

known in the literature as the *Entry* game[11]. The latter can be interpreted as a situation in which Anna (the potential entrant) has to decide whether to challenge Beppe (i.e. to play D) under the threat that Beppe (the incumbent) may fight back (and this would lead to an inefficient outcome for both players). On the other hand, she also knows that she can credibly *commit* herself to defection, since her action is perfectly observed by Beppe *before* he is asked to move.

We will analyse this game in more detail later in this chapter;[12] at this point we simply note that the game of figure 1.4 has a Nash (subgame-perfect) equilibrium in pure strategies, namely (D,C), and a *component* (that is, a closed and connected set) of Nash equilibria with the common property that Anna opts out with probability 1 and Beppe plays his (weakly dominated) strategy D with some positive probability . Let the symbol $\mathcal{NE}$ denote this component. In the following figure 1.5 we trace some interior trajectories of the Replicator dynamics of the *Entry* game:

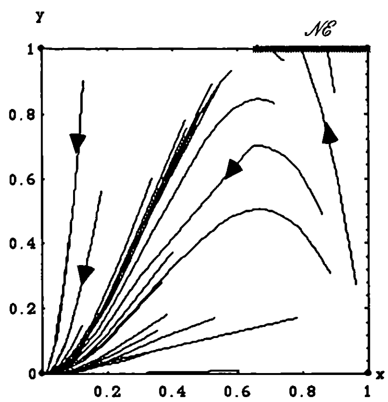


FIGURE 1.5
The Replicator Dynamics and the *Entry* game

Note that there are interior trajectories which lead to the Nash equilibrium component $\mathcal{NE}$. In other words, the players' limiting behavior may fail to eliminate strategies which are weakly dominated:

- PROPOSITION 1.3. Any interior trajectory of the Replicator Dynamics converges to a Nash equilibrium of the *Entry* game, with each $(x,y) \in \mathcal{NE}$ being the limit point of some interior solution.

PROOF. See Binmore *et al.* [1995], Proposition 1.

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[11]See Selten [1978].
[12]See § 1.7

We conclude this brief overview by returning to coming back to our original example, the game *Chicken* played by a single population. One might argue that the reason why, in equilibrium, the agents get (on average) less than the full share of the cake[13] is that the game is *symmetric,* but has with asymmetric (efficient) equilibria. Given that players are drawn from *a single* population, only symmetric mixed strategy profile are possible, therefore the possibility of a convergence toward one of the two efficient asymmetric equilbria is excluded.

However, even if payoffs are distributed in a symmetric fashion, it is difficult to think of a real-life situation in which the position of the two players is absolutely interchangeable. As Maynard Smith [1982] points out: "*...Most actual contests, however are asymmetric. They might be between a male and a female, between an old and a young, or a small and large individual, or between the owner of a resource and a non-owner. An asymmetry may be perceived beforehand by the contestant; if so, it can and usually will influence the choice of action...*"[14]

One might then argue, along these lines, that if the agents' possibilities were somehow enlarged, to allow some kind of "communication" before the game is played (or simply to condition their choice to some signal they receive, without any explicit communication actually taking place), this new feature of the game might solve their (anti)coordination problem, providing the (anti)coordination device they need to play efficiently in the stage game. In the terminology of game theory, such costless pre-play communication is often termed as *cheap talk.*

Following Maynard Smith [1982], assume that the (unmodeled) matching process is such that the stage game is *always* played between a female (Anna) and a male (Beppe), with Annas and Beppes present in the population in equal number.[15]Now introduce a new type *B* to the cooperators and the defectors labeled *C* and *D.* The *bourgeois* type which can condition its behavior on its own gender. In particular, any *bourgeois* will play *D* if Anna, and *C* if Beppe.

x	C	3,3	2,4	$\frac{5}{2},\frac{7}{2}$
y	D	4,2	1,1	$\frac{5}{2},\frac{3}{2}$
$1-x-y$	B	$\frac{7}{2},\frac{5}{2}$	$\frac{3}{2},\frac{5}{2}$	3,3

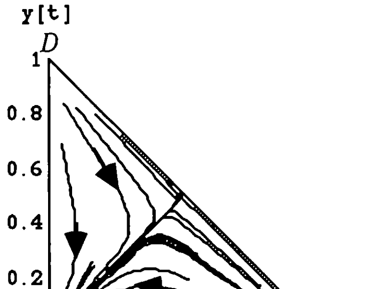

FIGURE 1.6
Maynard Smith's "Hawk-Dove-Bourgeois" Game

[13]Since the expected payoff, for each player, in the case of the mixed strategy equilibrium is equal to 2.5.

[14]See Maynard Smith [1982], p. 22.

[15]This assumption can be easily justified if such asymmetry comes naturally from the strategic setting (buyer *vs.* seller, incumbent *vs* entrant, and so on).

- PROPOSITION 1.4. Any interior solution of the Replicator Dynamics in the case of the "Hawk-Dove-Bourgeois" Game evolves to B.

PROOF. See the Appendix. ♠

In other words, only the agents who are able to condition their strategy on their player position in the stage game will eventually survive.[16] We have here an evolutionary explanation for the emergence of a *social norm of chivalry!*

To summarise: we are now in possession of new techniques which allow us to develop more sophisticated evolutionary models (both with respect to the stage game and the dynamics) in which asymmetric and sequential structures of various forms can be analysed.[17] As our brief overview suggests, these specifications matter, as they do indeed in traditional game-theoretic analysis. However, evolutionary models seem to provide answers to these problems which are closer to our intuition, and to the experimental evidence. The aim of this dissertation is to contribute to this line of research, both on the theoretical side (exploring the formal properties of these dynamics), and by providing interesting economic problems to which this methodology can be fruitfully applied.

1.3. CONTINUOUS-TIME EVOLUTIONARY DYNAMICS. HISTORY AND MAIN RESULTS

This section provides a summary of the theoretical results which will be used in subsequent analysis.

Before doing so, we choose the notation. Let $\mathfrak{I}$ denote a (finite) set of players, whose generic element will be denoted by i. The players repeatedly interact, playing a given (finite) normal form game $\Gamma \equiv \{\mathfrak{I}, S_i, u_i\}$. A player's behavior is formalised as the mixed strategy she adopts at each point in time. Denote with $r_i^k(t),\ i \in \mathfrak{I}$, the probability at time t with which player i, selects her pure strategy $s_i^k \in S_i$, with $r_i(t) \equiv (r_i^k(t))$ denoting the vector collecting such probabilities. Thus, we have $r_i(t) \in \Delta_i$, with Δ_i denoting the $|S_i - 1|$-dimensional simplex which describes player i's mixed strategy space. We also consider the vector $r(t) \equiv \{r_i(t)\}, i \in \mathfrak{I}$ as the *state of the system*, defined over the state space $\Delta \equiv \times_{i \in \mathfrak{I}} \Delta_i$, with Δ^0 denoting the relative interior of Δ, that is, the set of completely mixed strategy profiles.[18]

- ASSUMPTION 1.1. The evolution of $r(t)$ is given by the following system of continuous-time differential equations:

[16] A similar approach has been proposed in the analysis of coordination games in which pure strategy equilibria are Pareto rankable. In this case cheap talk acts as a coordination device for those who can condition their actions on such costless signal. See Robson [1990], Blume, Kim and Sobel [1993], Schlag [1993].

[17] A clear signal is the growing literature which reviews and organise, in the form of monographs if not textbooks, the state-of-the-art research in the field. Among others, see Fudenberg and Kreps [????], Fudenberg and Levine [1997], Samuelson [1997], Weibull [1995], together with the classic Hofbauer and Sigmund [1988].

[18] By analogy, Δ_i^0 will denote the set of completely mixed strategies of i.

$$\dot{r}_i^k(t) = f_i^k(r(t)) \tag{1.3.1}$$

We refer to the autonomous system $f \equiv (f_i) \equiv \left((f_1^1,...,f_1^{|S_1|}),...,(f_I^1,...,f_I^{|S_I|})\right)$ as the *selection dynamics,* i. e. the term that captures the relevant forces that govern the players' strategy revisions..[19] Some terminology is needed to specify the set of assumptions on f:

- DEFINITION 1.1. f is said to yield a (continuous-time) ***regular*** dynamic if the following conditions are satisfied:

- Lipschitz continuity[20]
- $\sum_{k \in S_i} f_i^k(r(t)) = 0; i \in \Im$
- $\lim_{r_i^k \to 0} \frac{f_i^k(r(t))}{r_i^k(t)}$ exists and is finite.

This regularity assumption has the implication of making the growth rates $\frac{f_i^k(r(t))}{r_i^k(t)}$ continuous over the state space Δ, and that (1.3.1) has a unique solution through any initial state which leaves Δ, as well as Δ^0, invariant: any solution path starting from (the relative interior of) Δ does not leave (the relative interior of) Δ:

$$r_i^k(0) > 0 \Leftrightarrow r_i^k(t) > 0; \forall t > 0$$

We have here a "no creation/no extinction" property: any pure strategy which is played with positive probability at time zero will also be played in any finite time interval. On the other hand, if a strategy is not played at time zero, it will never be used.

To complete the description of the dynamics we need to establish a formal link between the selection dynamics and the game payoffs. We do so by introducing what are probably the most popular evolutionary dynamics, that is, the *Replicator Dynamics* (Taylor and Jonker [1978]). According to the Replicator Dynamics the growth rate of strategy s_i^k, $\gamma_i^k(r(t)) \equiv \frac{\dot{r}_i^k(t)}{r_i^k(t)}$, equals the (expected) payoff difference between strategy s_i^k and the mixed strategy $r_i(t)$:

$$\dot{r}_i^k(t) = r_i^k(t)\left(u_i\left(s_i^k, r_{-i}(t)\right) - u_i\left(r_i(t), r_{-i}(t)\right)\right) \tag{1.3.2}$$

Taylor and Jonker [1978] propose two alternative interpretations for (1.3.2):

[19]Note that, as we noted earlier, there is an implicit "Markov" assumption in (1.3.1): the strategy revision process undertaken by the players is only affected by the current state $r(t)$.

[20]Sometime, (see, e. g. Cressman [1997]) the condition of Lipschitz continuity is replaced by the stronger requirement of continuous differentiability. In this latter case the dynamics are defined *smooth regular.*

- there is a single agent for each player's position *i*, who selects her pure strategy at time *t* starting from a probability distribution which follows the dynamics (1.3.2) as a result of some (unmodeled) learning adjustment process;
- there is a *population of agents* who are genetically programmed to play a particular strategy. An (unmodeled) *natural selection* mechanism governs the relative shares of each type in the population, whose law of motion evolves according with (1.3.2).

While the latter interpretation seems to suit more closely the biological *evolutionary metaphor* we used earlier in this chapter, the former considers the evolutionary dynamics as a proxy of an individual *learning process.* In the following § 1.4 both these interpretations will be derived formally.[21]

Since the introduction of the Replicator Dynamics by Taylor and Jonker [1978], a plethora of alternative definition has been proposed, in order to capture within a more general framework the "*success breeds success*" intuition implicit in the dynamics (1.3.2). The first noticeable attempt in this direction leads to the notion of *selection dynamics* due to Nachbar [1990]. In his paper, he defines a *selection dynamics* as a regular dynamics which satisfies the condition of *monotonicity:*

• DEFINITION 1.2. A regular selection dynamics is said to satisfy the property of monotonicity (MS hereafter) if:

$$\forall s_i^k, s_i^{k'} \in S_i \; ; \; u_i(s_i^k, r_{-i}(t)) \geq u_i(s_i^{k'}, r_{-i}(t)) \Leftrightarrow \gamma_i^k(r(t)) \geq \gamma_i^{k'}(r(t)) \qquad (1.3.3)$$

The condition of payoff monotonicity tries to capture an appealing property of the Replicator Dynamics: given the mixed strategy profile played at each point in time, more successful pure strategies grow at a higher rate, compared to poorly performing ones. Samuelson and Zhang [1992] extend the condition of monotonicity to *mixed strategies,* introducing the notion of *aggregate monotonicity:*

• DEFINITION 1.3. A regular selection dynamics is said to satisfy the property of aggregate monotonicity (AMS) if:

$$\forall \sigma_i, \sigma_i' \in \Delta_i \; ; \; u_i(\sigma_i, r_{-i}(t)) \geq u_i(\sigma_i', r_{-i}(t)) \Leftrightarrow \sum_{k \in S_i} \gamma_i^k(r(t)) \sigma_i^k \geq \sum_{k \in S_i} \gamma_i^k(r(t)) \sigma_i'^k \qquad (1.3.4)$$

that is, if a mixed strategy σ_i yields a higher expected payoff than σ_i' against the mixed strategy played by i's opponents at time t, the vector field induced by $f(r(t))$ should point "more" in

[21] One of the aims of Taylor and Jonker's [1978] article was indeed to establish a link between the equilibrium concept of Evolutionary Stable Strategy (Maynard Smith and Price [1973]) and an evolutionary dynamics which did converge to it. Cressman [1992] provides an excellent survey of the theoretical findings in the field.

the direction of the former. The following condition of *payoff positivity,* also due to Nachbar [1990][22], generalises the Replicator Dynamics in a different direction:

- DEFINITION 1.4. A regular selection dynamics is said to satisfy the property of payoff positivity (PPS) if:

$$\forall s_i^k \in S_i \; ; \; u_i(s_i^k, r_{-i}(t)) \geq u_i(r(t)) \Leftrightarrow \gamma_i^k(r(t)) \geq 0 \tag{1.3.5}$$

that is, only pure strategies which perform "better than the average" have a positive growth rate. The condition of payoff positivity can be relaxed by requiring instead that there is *at least* one pure strategy, among those which yield more than the average payoff, which has a positive growth rate.

- DEFINITION 2.2. A regular selection dynamics is said to satisfy the property of weak payoff positivity (WPPS) if for any $r \in \Delta$ and $i \in \mathfrak{I}$, there exists a $s_i^k \in S_i$ such that:

$$u_i(s_i^k, r_{-i}(t)) \geq u_i(r(t)) \Leftrightarrow \gamma_i^k(r(t)) \geq 0 \tag{1.3.6}$$

It follows from the definitions that $AMS \subset MS$ and $WPPS \subset (MS \cup PPS)$.[23] It also turns out that WPPS is the more general class of continuous-time selection dynamics which satisfies the following property, to which we will make constant use throughout the dissertation:

- PROPOSITION 1.5. Suppose f is a WPPS dynamics. Then, if $r \in \Delta$ is the limit to some interior solution, then r is a Nash equilibrium of Γ.

PROOF. See Weibull [1995], Theorem 5.2 *(c).* ♠

1.4 ECONOMIC MICROFOUNDATIONS

In this section, we shall review the literature which derives the evolutionary dynamics starting from an explicit model of social interaction. As we noticed earlier, the extent to which these (or similar) models capture the essence of concrete economic environments provides the limits within which the use of such dynamics can be applied in economic modeling.

There are basic two alternative "stories" which have been proposed to justify the use of selection dynamics in the context of economic learning. Not surprisingly, each story is based on one of the "informal" justifications provided by Taylor and Jonker [1978] for the case of the Replicator Dynamics. We shall look at each justification in more detail in the remainder of the section.

1.4.1. THE "INDIVIDUAL LEARNING" STORY: LEARNING BY REINFORCEMENT

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[22]The terminology of "payoff positivity" is due to Weibull [1995], while Nachbar [1990] refers to this dynamics as *sign-preserving.*
[23]See Weibull [1995], chapter 5.

To present this first class of models, we shall use as a reference the paper by Börgers and Sarin [1993]. In their model, *two* agents, Anna and Beppe, repeatedly interact to play a fixed normal form game. Their behavior is not fixed (as we assumed in § 1.2), but instead changes over time according to a very simple updating rule, which links the outcome of the stage game with the mixed strategy which will be used in the subsequent round. In particular, pure strategies which perform well against the opponents' actions are *reinforced,* and the probability with which they are selected grows accordingly.

As one may notice, there is a clear shift in the reference framework from § 1.2. In Börgers and Sarin [1993] alternative strategies compete *within* the mind of the agents, as *populations* of ideas. As the authors observe «*..Decision makers are usually not completely committed to just one set of ideas, or just one way of behaving. Rather, several system of ideas, or several possible ways of behaving are present in their mind simultaneously. Which of these predominate, and which are given less attention, depends on the experience which an individual makes...*».[24]

This approach is not new, as it follows the tradition of Estes' [1950] "stimulus sampling theory" of behavioral psychology, subsequently formalised in Bush and Mosteller's [1951], [1955] stochastic learning theory, and in the theory of "adaptive economic behavior" proposed by Cross [1973], [1983].[25]

Before summarising Börgers and Sarin findings on the relationship between the (stochastic) adjustment process they analyse and the (deterministic) dynamics studied in this dissertation, we shall describe the reference model more in detail. While the authors are mainly concerned with the relationship between their learning process and the Replicator Dynamics, we will modify slightly their assumptions to allow a higher degree of flexibility in defining the evolutionary dynamics.

Anna and Beppe are two *individual players,* who are engaged in a (possibly infinitely) repeated game $\Gamma = \{\Im, S_i, u_i\}$ interaction. Both players select, at each point in (discrete) time $n \in (0,1,2,...)$, an action $s_i^k \in S_i, i = A, B,$ starting from a probability distribution (mixed strategy) $r_i(n)$. It is assumed that, at each round, Anna/Beppe observes only the action s/he plays, and the payoff s/he obtains. Suppose that, at round n, Anna has played her pure strategy s_A^j and Beppe has played his pure strategy s_B^k; then Anna will update her mixed strategy as follows:

$$r_A^j(n+1) = v_A(s_A^j, s_B^k) + (1 - v_A(s_A^j, s_B^k)) r_A^j(n) \tag{1.4.1}$$

$$r_A^{j'}(n+1) = (1 - v_A(s_A^j, s_B^k)) r_A^{j'}(n), \text{ for any } j' \neq j \tag{1.4.2}$$

where $v_i{:}\,\alpha_i(r) + \beta_i(r) u_i$, with $\alpha_i{:}\,\Delta \to \Re$ and $\beta_i{:}\,\Delta \to \Re^+$ both Lipschitz continuous functions satisfying the following condition: $v_i(.) \in (0,1)$. In words: the change in probability $\Delta r_A^h(n) \equiv (r_A^h(n+1) - r_A^h(n))$ is proportional to a given affine transformation of the payoff Anna received in the stage game, with coefficients which may depend on the state variable $r(n)$, up to a rescaling which constrains $r_A(n+1)$ to be in the unit simplex.[26]

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[24] See Börgers and Sarin [1993], p. 1.

[25] Models of reinforcement learning are also those by Bendor *et al.* [1991], Börgers and Sarin [1996], Sarin [1995] and the experimental studies conducted by Mookerjee and Sopher [1994] and Roth and Erev [1983], [1995].

[26] An analogous expression holds for Beppe also.

If the dynamics lie in the relative interior of the state space, (1.4.1-2) imply that, no matter how s_A^j performed against s_B^k compared to other strategies in Anna's support, $r_A^j(n+1)$ will always be *bigger* than $r_A^j(n)$ since $\Delta r_A^j(n) = u_A(s_A^j, s_B^k)(1 - r_A^j(n)) > 0$, although the strength of the reinforcement is sensitive to the absolute magnitude of $v_A(s_A^j, s_B^k)$. From (1.4.1-2) we derive the following expression

$$\begin{aligned} E\left[\Delta r_A^j(n)\right] &= \left(E\left[v_A(s_A^j, s_B^k)\right]\left(1 - r_A^j(n)\right)\right) r_A^j(n) - r_A^j(n) \sum_{j' \neq j} E\left[v_A(s_A^{j'}, s_B^k)\right] r_A^{j'}(n) \\ &= r_A^j(n)\left(v_A(s_A^j, r_B(n)) - v_A(r(n))\right) \end{aligned} \quad (1.4.3)$$

which implies the following:

- PROPOSITION 1.6. For each player $i \in \mathfrak{I}$, the expected motion of the discrete-time dynamics is aggregate monotonic.

1.4.2. THE "CULTURAL EVOLUTION" STORY: LEARNING BY IMITATION.

The models which fall into this category will be said to represent ***cultural*** (or ***social)*** evolution, since they rely on the ability of agents to observe (and successfully imitate) behaviors from other agents in the population. In other words, these model follow the "biological metaphor" more closely, since they look at the *aggregate behavior* of a population of agents. To present this second class of models, we shall use as a reference the model proposed by Schlag [1994]. Suppose there are two (large) populations of agents, one population of Annas and one population of Beppes, involved in a repeated situation (either a game, or simply a decision problem). In each round each agent selects an action, and is then randomly matched with an agent of the opposite role, obtaining a payoff as a result of the encounter (in the absence of knowledge of both the action and the payoff of the opponent). Between rounds each agent receives information about another individual *of the same* population, via a symmetric random sampling procedure.[27] Each player then adjusts her current strategy according to an ***updating rule,*** which is a function that maps from current payoffs and actions of both sampling and sampled agents to the action which is to be played in the following round. Following the stationary assumption typical of this literature, only updating rules which condition on the current period information are considered.

In his analysis, Schlag [1994] first proposes a class of updating rules that agents might eventually select, if allowed to choose among those rules, once and for all, before entering into the matching and sampling scenario. He defend this class on the basis of a set of axioms intended to characterise a particular type of bounded rationality Such rules exhibit the following common properties:

[27]The information is the payoff obtained, together with the action played. By "symmetric" sampling the author means a sampling scheme in which the probability of agent x of sampling agent y equals the probability with which x may be sampled by y.

- they are *imitative,* in the sense that an agent will never switch to an action which has not been observed in the current period;
- the probability of switching to an action which performed better is *proportional* to the difference in payoffs, between the action of the sampled agent compared with the action currently used by the sampling agent.

We will refer to these rules as *proportional imitation rules.* Under these rules, at each round *n,* agent *i,* after having played strategy s_i^k and sampled agent *i'* who played strategy $s_i^{k'}$ will revise her strategy from s_i^k to $s_i^{k'}$ with probability $\beta_i\left(u_i(s_i^k,\cdot)-u_i(s_i^{k'},\cdot)\right)$ if $u_i(s_i^k,\cdot)\leq u_i(s_i^{k'},\cdot)$, and will not revise her strategy otherwise. By analogy with § 1.4.1, we shall assume $\beta_i:\Delta\rightarrow\Re^+$ Lipschitz continuous, with $\beta_i(.)\in(0,1)$.

The next step will be to consider an environment in which agents of the same population use the same (proportional imitation) updating rule and to look at the expected motion of the frequencies with which the various actions are played. Denote by $r_i^k(n)$ the frequency with which s_i^k is played at time $n\in(0,1,2,\ldots)$. Straightforward calculations lead to the following:

$$E\left(\Delta r_i^k(n)\right)=\beta_i\left[u_i\left(s_i^k,r_{-i}(n)\right)-u_i\left(r(n)\right)\right];i=A,B. \qquad (1.4.3)$$

which in turn implies

- PROPOSITION 1.7. For each population $i\in\Im$, the expected motion of the frequencies with which each pure strategy is played is aggregate monotonic.

As for the related literature on social evolution, Binmore and Samuelson [1993] set up another model in which agents may switch their strategy as a result of imitative behavior. However, their model has also some similarities with the "individual learning" approach proposed in § 1.4.1, since agents change their pure strategy only if the current payoff is *lower* than the payoff they received in the previous round (in this respect, their updating rule is governed by an endogenous *aspiration level,* which is set equal to the previous round payoff).[28] Under these assumptions, they show that the expected motion the expected motion of the frequencies with which each pure strategy is played over time follows, by analogy with Proposition 1.7, the Replicator Dynamics. Bjornerstedt and Weibull [1995] consider a model in which the agents receives a signal, which summarises the realised payoff of a sample of other agents in the population. However, this signal is perturbed and might distort the information delivered. Under these assumptions, they show that if the support of the noise is sufficiently large, the resulting dynamics satisfies the monotonicity condition (1.3.1). Cabrales [1993] also derives the Replicator Dynamics starting from a model which is similar to those presented in this section. Experimental evidence on games in which agents can imitate others can be found in Malawski [1989].

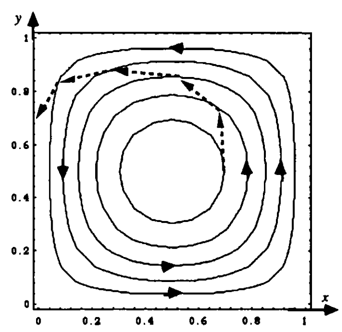
[28]A similar use of the notion of *aspiration level* is proposed by Bjornerstedt [1995]. See also Banerjee and Fudenberg [1995] and the model contained in Chapter 4 of this dissertation.

1.4.3. EXPECTED MOTION *VS.* ASYMPTOTIC BEHAVIOR

We conclude this section with a remark. We derived evolutionary dynamics as *expected motion* of two alternative stochastic processes, based on quite different models of social interaction. Moreover, it can be shown that, in both cases, the same evolutionary dynamics well approximates the stochastic process as the time scale gets to its continuous limit, since both stochastic processes converge in probability to the corresponding (aggregate monotonic) deterministic dynamics.[29] However, it turns out that the above results hold only for any *finite time,* i.e. they do not refer to asymptotics. Which may well differ whether we look at the stochastic process or at its (either continuous or discrete) deterministic approximation. Börgers and Sarin [1993] provides an useful example to clarify this point, considering the asymptotic behavior of their learning dynamics in the case of zero-sum games.

	$1-x$	x
	L	R
y T	1,0	0,1
$1-y$ B	0,1	1,0

a)



b)

FIGURE 1.7
Discrete-vs. Continuous -Time Replicator Dynamics and Zero -Sum-Games

Figure 1.7*b)* traces some orbits of the continuous-time Replicator Dynamics in the case of the (constant-sum) game of figure 1.7*a).* The dotted arrows of figure 1.7*b)* represent the (expected) jump of the discrete time dynamics (1.4.1-2). In other words, while the continuous time process cycles around the (unique) equilibrium in mixed strategies, the discrete time (deterministic) process does not converge, approaching the boundaries of the state space. Moreover, we also know that the stochastic dynamics will eventually end up in one of the four absorbing states of the system, i.e. one of the four pure strategy profiles. As a result of that, even if the (stochastic) learning dynamics is well approximated by the continuous-time deterministic process within any finite time interval, the asymptotic properties of the two processes may differ in a substantial way.[30]

1.5. SELECTION DYNAMICS AND DOMINATED STRATEGIES

[29] See Börgers and Sarin [1993], Proposition 4, and Schlag [1994], Theorem 6.

[30] However, this does not mean that the asymptotics of the two processes are *necessarily* so different. In the following chapter 4 we will provide conditions under which the continuous-time limit of the deterministic dynamics can be considered a good approximation of the asymptotic behavior of the stochastic dynamics (see also Boylan [1991], [1992]).

We have already found an example (the *Entry* game) in which the Replicator Dynamics fails to eliminate weakly dominated strategies. This result, which contrasts a well-established tradition in standard game-theoretic analysis[31], is also somehow counterintuitive if observed from an evolutionary perspective. In fact, if initial conditions lie in the relative interior of the (mixed strategy) state space, as it is commonly assumed by this literature, weakly dominated strategies will *always* yield strictly lower payoffs than those which dominate them, at least within any finite time, and this is essentially because, by forward invariance, the system will never reach (if not in the limit) one of the faces of the state space in which the dominant and the dominated strategy yield the same payoff.

Given this preliminary consideration, it may be interesting to investigate how general the above result is, and if it can also be extended to the case of strategies which are even *strictly* dominated. This is the aim of this section, in which we summarise some theoretical results concerning the relation between evolutionary dynamics and dominated strategies.

We shall consider strict dominance first. In this respect, it turns out to be crucial how the concept is formally defined. Conventionally, we say that a pure strategy s_i is strictly dominated if it exists another (pure or mixed) strategy $\sigma_i' \in \Delta_i$ which yields a (strictly) higher payoff against any mixed strategy in the support of the opponent(s):

$$u_i(s_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i}); \forall \sigma_{-i} \in \Delta_{-i} \tag{1.5.1}$$

Otherwise, to consider s_i strictly dominated, one might ask for the stronger requirement of $\sigma_i' \in \Delta_i$ being a *pure* strategy itself. If strict dominance is interpreted in this more restrictive sense, we then know that, under any monotonic selection dynamics, not only strategies which are strictly dominated[32], but also strategies which do not survive the iterated deletion of strictly dominated strategies, will eventually vanish:

- PROPOSITION 1.8. Let $r(r(0),t)$ be the interior solution of a regular MS dynamics (1.3.1). If $s_i^{k'}$ does not survive the iterated deletion of pure strategies strictly dominated by pure strategies, then $\lim_{t \to \infty} r_i^k(t) = 0$.

PROOF. See Samuelson and Zhang [1992], Theorem 1.

Things are different if we also consider *mixed* strategies. To obtain the same result as in Proposition 1.8, we then need to impose some more stringent conditions on the dynamics than monotonicity alone.[33] To clarify this point, we provide an example, adapted from Dekel and Scotchmer [1992], who first addressed the problem of elimination of strictly dominated strategies (by mixed strategies) in the context of (discrete-time) selection dynamics:

---

[31]Theoretical justifications of the (iterated) deletion of weakly dominated strategies, have been provided (among others) by Kohlberg and Mertens [1986], Moulin [1986], Dekel and Fudenberg [1991].

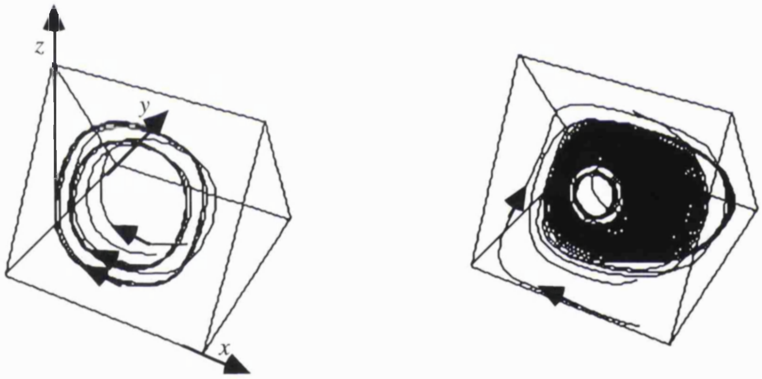
[32]This result is due to Nachbar [1990].

[33]For example, Akin [1980] had shown that strictly dominated strategies vanish along any interior solution of the single-population Replicator Dynamics.

		$1-x$	x
		L	R
y	T	1,0	0,1
z	M	.4,.4	.4,.4
$1-y-z$	B	0,1	1,0

FIGURE 1.8
An adaptation of Dekel and Scotchmer's [1992] counterexample

Note that the game of figure 1.8 differs from the game of figure 1.7 only by the fact that Anna has now an additional strategy *(M)* which yields a payoff equal to .4 to both players, regardless of what Beppe does. Moreover, strategy *B,* although it is not strictly dominated by any other pure strategy in Anna's support, is strictly dominated by any mixed strategy "sufficiently close" to the (unique Nash equilibrium) strategy, which attaches probability .5 to both strategies *T* and *B.*



a) *b)*

FIGURE 1.9
Monotonic Dynamics and Strictly Dominated Strategies

Figure 1.9*a)* traces some trajectories of the standard Replicator Dynamics for the game of figure 1.8. The orbits of figure 1.7*b)* are now ***limit cycles*** for the trajectories of figure 1.9*a)*., once the strictly dominated strategy *B* has been eliminated. Figure 1.9*b)* shows instead trajectories of the following MS dynamics:

$$\dot{r}_i^k(t) = r_i^k(t)\left(\sqrt{u_i\left(s_i^k, r_{-i}(t)\right)} - \sqrt{u_i(r(t))}\right) \tag{1.5.2}$$

in which the growth rate $\gamma_i^k(r(t))$ equals the difference between the ***square roots*** of $u_i\left(s_i^k, r_{-i}(t)\right)$ and $u_i(r(t))$. In this latter case, the system cycles ***in the interior*** of Δ. In consequence, the strictly dominated strategy *B* fails to be eliminated.

In a recent paper, Hofbauer and Weibull [1996] show how this behavior is not peculiar to the functional form (1.5.2). In their paper, they consider a class of regular selection dynamics (which they call *functional selection dynamics)* in which all growth rate functions $\gamma_i^k(r(t))$ are as follows:

$$\gamma_i^k(r(t)) = \alpha_i(r(t)) + \beta_i(r(t))\varphi\left[u_i\left(s_i^k, r_{-i}(t)\right)\right] \tag{1.5.3}$$

with α_i and β_i as in (1.4.1-2), and φ Lipschitz continuous. In (1.5.3) the evolutionary dynamics are described by means of a function in which the utility of each pure strategy $u_i\left(s_i^k,\cdot\right)$ enters explicitly, instead of appearing as a side constraint, as in § 1.3. Although the class of evolutionary dynamics described by (1.5.3) is more restrictive than the class we introduced in § 1.3, all the conditions defined in § 1.3 have a natural counterpart as special cases of (1.5.3).[34]

The relation between the asymptotic behavior of strictly dominated strategies and functional selection dynamics of the form (1.5.3) is stated in the following proposition:

- PROPOSITION 1.9. Let $r(r(0),\cdot)$ be the interior solution of a functional selection dynamics (1.5.3). If s_i^k does not survive the iterated deletion of pure strategies strictly dominated by mixed strategies and φ is *strictly increasing and convex*, then $\lim_{t\to\infty} r_i^k(t) = 0$.

PROOF. See Hofbauer and Weibull [1996], Theorem 1.

The authors note that convex functional selection dynamics heuristically represent agents whose reaction to higher payoff is *at least* proportional (in other words, their utility exhibits *risk aversion with respect to fitness).*[35]

We now move to weak dominance. We will only consider here the case of pure strategies which are weakly dominated by other *pure* strategies. We already know, from the dynamic analysis of the *Entry* game„ that weakly dominated strategies may fail to be eliminated by the Replicator Dynamics, that is, by an evolutionary dynamics which satisfies all the conditions (1.3.1-6). This result might suggest that weak dominance considerations have no role in determining the limiting play under any selection dynamics. However, it turns out this not to be the case. In particular, if a weakly dominated strategy does not vanish, this implies the extinction of *all* the pure strategies in the support of the opponents against which the dominated strategy yields the same payoff as the dominant strategy. This result, first proved by Nachbar [1990] in the case of monotonic dynamics which converge to a Nash equilibrium, has now been also proved by Cressman [1996] for the case of the Replicator Dynamics without the need to assume that the process converges:

[34]For example, it is sufficient to assume φ strictly increasing to obtain a monotonic dynamics, or to assume $\alpha_i = -\beta_i(r(t))\varphi\left[u_i(r(t))\right]$, with φ still strictly increasing, to have a payoff positive dynamics, and so on.

[35]If the difference in growth rates is *exactly* proportional to the difference in payoffs the dynamic is then *aggregate monotonic.* In this respect, the result in Proposition 1.9 extends the result of Samuelson and Zhang [1992], who provided a sufficient condition (aggregate monotonicity) for the iterated deletion of pure strategies strictly dominated by mixed strategies.

- PROPOSITION 1.10. Let $r(r(0),\cdot)$ be the interior solution of the Replicator Dynamics (1.3.2). If a pure strategy s_i^k is weakly dominated by another pure strategy $s_i^{k'}$ then at least one of the following statements are true:

$$\lim_{t\to\infty} r_i^k(t) = 0 \tag{1.5.4}$$

$$\lim_{t\to\infty} r_{-i}^h(t) = 0, \forall s_{-i}^h \text{ s.t. } u_i(s_i^k, s_{-i}^h) < u_i(s_i^{k'}, s_{-i}^h) \tag{1.5.5}$$

- PROOF. See Cressman [1996] Proposition 3.1.♠

The above result will play a key role in our analysis and will be generalised to the class of monotonic selection dynamics (1.3.3). [36]

1.6. SELECTION DYNAMICS AND GAMES OF PERFECT INFORMATION

One of the standard rationality principles in the game-theoretic analysis of games with perfect information appeals to the notion of *backward induction,* first introduced by Zermelo [1913]. According to this principle, Anna should defect in the *Entry* game, because she *can rely* on the fact that Beppe will cooperate in return. This is because, if Anna plays *D,* Beppe is strictly better off if he cooperates, as cooperating yields him a higher payoff. In other words, any threat of defecting by Beppe is rendered *incredible* by the assumption that he is rational (and his, as well as Anna's rationality are common knowledge).

This conclusion has been repeatedly challenged on the grounds that, as with any solution concept based on an equilibrium notion, the backward induction principle leads is often ill-defined when applied *off* the equilibrium path *(what if Anna cooperates?)* , as it may well happen that it would be advantageous for *all* the players to violate the backward induction prescription. Far from proposing an alternative view on such a controversial matter,[37] we merely acknowledge that dynamic processes analogous to those proposed here have been used to analyse games of perfect information, precisely because the player behavior is completely specified under *all* contingencies. The study of the asymptotic behavior of some evolutionary dynamic may therefore provide a further argument in favour (or against) the backward induction principle (or any other rationality principle, for what matters).

In this respect, we mention here a recent contribution which has solved some unanswered questions on the dynamic properties of these games. Cressman and Schlag [1995] apply Cressman's [1996] technique in the analysis of two-player games of perfect information with no relevant payoff ties. In this class of games, the use of backward-induction (or alternatively, the iterative deletion of weakly dominated strategies) selects a unique Nash equilibrium, which is the subgame-perfect equilibrium of the game. They restrict their analysis to the Replicator Dynamics, for which they prove (among other properties) the following:

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[36] See Proposition 3.4.1.

[37] Among the recent papers which explore the logic and the rationale behind the backward induction procedure, we make reference to Aumann [1995], Battigalli [1995], Ben Porath [1994], Binmore [1995] and Reny [1993].

- every interior path converges to a Nash equilibrium (Theorem 2);
- for "simple" games, (games of perfect information in which at most three consecutive decisions are made), the Nash equilibrium component which contains (i.e. is outcome-equivalent to) the backward induction solution is the unique interior asymptotically stable set (theorem 5).

The first result is relevant, as it identifies a class of games for which an equilibrium notion (namely, Nash equilibrium) accurately describes the asymptotic play of a particular selection dynamics (namely, the Replicator Dynamics). However, this result does not support more stringent equilibrium requirements like, for example, subgame-perfection, since Nash equilibria which are not subgame-perfect may well be limit points of (a non-zero measure set of) interior trajectories, as we know from our analysis of the *Entry* game.

In this respect, theorem 5 tells us that, although the Nash equilibrium component $\mathcal{NE}$ of figure 1.5 is interior-attracting under the Replicator Dynamics, it cannot be ***asymptotically stable:*** trajectories starting arbitrarily close to $\mathcal{NE}$ will move away from it and never come back (whereas the same phenomenon cannot occur when we consider the subgame-perfect equilibrium of the game, namely the strategy profile *(D,C))*.

However, by theorem 5, the asymptotic stability of the backward induction solution is not guaranteed if games are not ***simple*** in Cressman and Schlag's terminology; for more ***complex*** games such an asymptotically stable set may even fail to exist.

As the authors comment, the predictive power of backward induction is supported only partially by the results contained in the paper, which they entitle *The Dynamic (In)stability of Backward Induction.*

1.7. Selection Dynamics with Drift

Binmore *et al.* [1995] explore the dynamic properties of the Replicator Dynamics in the case of another game of perfect information with no relevant ties, that is, the ***Ultimatum Game***. In this game Anna proposes to Beppe a particular division of a cake; subsequently, Beppe has to decide whether to accept Anna's proposal or not. If Beppe accepts, the pie is shared as agreed; if Beppe rejects the offer, nobody gets anything. This game has a unique subgame-perfect equilibrium in which Anna offers (an ε more than) nothing and Beppe accepts. The intuition is the same as in the *Entry* game: if Beppe's rationality is common knowledge, Anna can rely on the fact that Beppe will accept ***anything,*** no matter how little it is. As a matter of fact, there is a clear analogy between the two games: if we restrict the possible offers to two ***(high or low),*** assuming that a high offer is automatically accepted by Beppe, the *Ultimatum Game* collapses to a much simpler form which is absolutely analogous to the *Entry* game of figure 1.4 (and therefore has the same asymptotics under the Replicator Dynamics)[38].

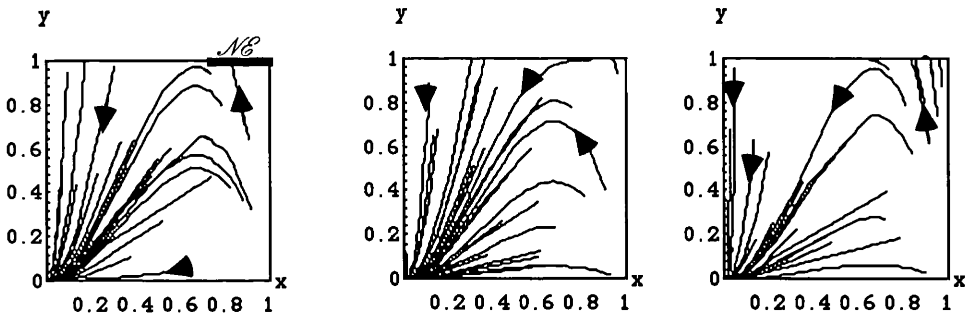
[38] This is the reason why Binmore *et al.* refer to the game of figure 1.4 as the *Ultimatum Minigame.*

The *Ultimatum Game* is an example in which the backward induction hypothesis is universally rejected by the experimental evidence, although the various experimental results provide no clear alternative hypothesis.[39]

To explain the fundamental weaknesses of the backward induction hypothesis in the context of the *Ultimatum Game,* Binmore *et al.* [1995] propose the following dynamics:

$$\dot{r}_i^k(t) = r_i^k(t)\left(u_i\left(s_i^k, r_{-i}(t)\right) - u_i(r(t))\right) + \lambda_i\left(\beta_i^k - r_i^k(t)\right); \lambda_i \geq 0, \beta_i^k \in (0,1) \qquad (1.7.1)$$

We will refer to the dynamics (1.7.1) as the *Replicator Dynamics with drift.* In Binmore *et al.* [1995] the dynamic (1.7.1) is derived from a population game in which agents die (or leave the game, or experiment new ways of playing) at a fixed rate equal to $\lambda_i dt$. Those who die are replaced by *novices* (or *experimenters)* who play each strategy s_i^k with probability β_i^k., while the rest of the population aggregate behavior follows the Replicator Dynamics. The relative importance of the drift is measured by λ, which we refer to as the *drift level.* We assume λ to be "very small", reflecting the fact that all the major forces which govern the dynamics should be captured by the evolutionary dynamics (1.3.1), which here takes the form of the Replicator Dynamics. In figure 1.10 we trace some trajectories of (1.7.1) for the *Entry* game, under different realizations of λ_i (setting $\beta_i^k = 1/2, \forall i, k$):



a) $\lambda_i = 0$ *b)* $\lambda_A = \lambda_B = 0.001$ *c)* $\lambda_A = 0.005$; $\lambda_B = 0.05$

FIGURE 1.10
Phase diagrams of the 1-stage Ultimatum Game

Figure 1.10*a)* shows trajectories of the unperturbed Replicator Dynamics that mimic the behavior already shown in the phase diagram of figure 1.5. Figure 1.10*b)* shows trajectories of (1.7.1) when both λ_A and λ_B are "negligible". In this case, the drift against Beppe's (weakly dominated) strategy *D* is sufficient to push the system away from the $\mathcal{NE}$ component. In figure 1.10*c)* λ_B is substantially higher than λ_A: in this case the system (1.7.1) has two restpoints close to the $\mathcal{NE}$ component, one of which is asymptotically stable. In other words, although the drift points toward the relative interior of the state space (since $\beta_i^k \in (0,1)$) it may not be

[39]Among the experimental studies on the Ultimatum Game, we make reference to Guth *et al.* [1982] and Binmore, *et al* [1985], together with the surveys by Bolton and Zwick [1995], Guth and Tietz [1990] and Roth [1995] and Thaler [1988]

sufficient to destabilize the $\mathcal{NE}$ component in which a suboptimal action is played with positive probability.

In the following proposition, we shall replicate Binmore *et al.* [1995] results fixing $\beta_A^C = 1/2, \beta_B^C = \beta, \beta_B^D = 1-\beta$, $\lambda_A = \lambda_B = \lambda$ and letting $\lambda \to 0$ (we do so to be consistent with the rest of the dissertation)[40]:

- PROPOSITION 1.11. Let $\hat{RE}(\beta)$ be the set of restpoints of (1.7.1) for λ sufficiently close to 0. The following properties hold:

a) for all $\beta \in (0,1)$, $\hat{RE}(\beta)$ contains the subgame-perfect equilibrium *(D,C)* which is also asymptotically stable.
b) when β is sufficiently large, $\hat{RE}(\beta)$ contains also two additional restpoints, both belonging to $\mathcal{NE}$, one of which is asymptotically stable.

PROOF. See the Appendix.

The authors comment: "...the question whether the subgame-perfect equilibrium should be regarded as the one and only game-theoretic prediction for the Ultimatum Game ... [has] a convincing and firmly negative answer...".[41]

We postpone any comment on such controversial matters to chapter 3, in which we provide a similar analysis for another game of perfect information with no relevant ties, that is Rosenthal's [1981] *Centipede Game*. What remains to be said at this stage is only that the analysis of (1.7.1) (or similar dynamics) may be used to test the robustness of any theoretical prediction based on the asymptotic properties of our continuous-time evolutionary dynamics. This is the reason why in chapters 2 and 3 of this dissertation the analysis of the Replicator Dynamics with drift (1.7.1) will complement the analysis of the pure evolutionary dynamics studied elsewhere.

1.8. PLAN OF THE THESIS

This dissertation consists of four chapters (including this introductory survey), followed by the reference index. In what follows, we shall briefly summarise the content of the other three main chapters.

1.8.1. EVOLUTIONARY DYNAMICS AND "THE IMPLEMENTATION PROBLEM"

In chapter 2, we try to understand the effect of taking an evolutionary approach to the study of *implementation theory*. The theory of implementation is that branch of game theory which addresses the problem of designing games, or *mechanisms,* whose equilibria satisfy certain socially desirable properties, but which do not necessitate vast amounts of knowledge by the

[40]See § 2.5 for a more detailed account of this parameter choice.
[41]See Binmore *et al.* [1995], p. 88.

authorities to put them in place. As the theory applies equilibrium concepts from non-cooperative game theory, it is implicitly assumed that these social arrangements should police themselves, and the authority should only make sure that the rules of the game are followed by the players.

In the last few years there has been impressive progress in the theory of implementation. As Sjöström [1994] points out "With enough ingenuity the planner can implement "anything"". This ingenuity often involves the design of complicated games and/or the use of highly "refined" equilibrium notions.

However, little attention has been paid to the issue of how equilibrium is reached, and whether it is stable. The only exceptions of which we know are the papers of Muench and Walker [1984] and De Trenqualye [1988] who study local stability in the Groves and Ledyard [1977] mechanism. This situation is worrisome. Given the fact that the theory makes normative recommendations, it does not seem sensible to apply such social engineering devices without thinking about whether real people will achieve the desired outcomes.

To approach these issues we study, as an example, the convergence and stability properties of Sjöström's [1994] mechanism relative to two alternative adjustment processes. In particular, we study the convergence and stability properties of the mechanism on the assumption that boundedly rational players find their way to equilibrium either using the monotonic learning dynamics (1.3.3) or with fictitious play.[42] The (conflicting) results we obtain from this comparison suggest that *a)* convergence to the solution of mechanisms (even of the simplest structure) should not be taken for granted and *b)* it may be necessary to do further empirical and experimental studies that reveal how people adjust their play in mechanisms if we really want to provide an empirical content to implementation theory.

1.8.2 Evolutionary Dynamics and Games of Perfect Information

As we pointed out earlier in this chapter, the backward-induction procedure (Zermelo [1913]) is one of the fundamental solution concepts of traditional game-theoretic analysis of games in extensive form. However, its applicability has been repeatedly challenged, especially for games with long chains of decision nodes off the equilibrium path. Such controversy is confirmed by the experimental evidence in the field: for example, in Rosenthal's [1981] Centipede Game, experiments show that subjects' behavior significantly differs from the theoretical prediction.

The Centipede Game is the object of chapter 3, in which we study its dynamic properties when the evolutionary dynamic satisfies the monotonicity condition (1.3.3).

Consistently with Crossman and Schlag [1996], our analysis of the Centipede Game does *not* provide a full justification for the backward induction procedure. An evolutionary argument in favour of backward induction surely comes from our Theorem 3.4.1, in which we prove convergence to the subgame-perfect equilibrium outcome for any monotonic continuous-time selection dynamics. However, in § 3.5, by introducing perturbations as in (1.7.1), we also show that these adjustment processes are intrinsically unstable, as the perturbed dynamics may easily

[42]See footnote 3 in this chapter.

give rise to limit cycles. Moreover, we show that this instability is also positively related to the length of the game.

1.8.3. Evolutionary Dynamics and Social Norms: Conventions and Social Mobility.

The analysis of Maynard Smith's [1982] "Hawk-Dove-Bourgeois" game shows that when interaction is anonymous and repeated over time people may resolve their conflicts by conditioning their behavior on some (public) signal, which reflects some asymmetry in the players' position. One might identify such a signal with the players' *reputation* (as in Rosenthal and Landau [1979]), or *status* (as in Okuno-Fujiwara and Postlewaite [1995]); it may be the players' *gender* (as in our simple example), or in their *race* (as in Kaneko and Kimura [1992]).

Whatever their source, or nature, the existence of such signals may help the agents to coordinate their actions on an equilibrium of the game they are playing. We are now accustomed to call these equilibria *conventions*. .[43]

There is a distinction to be made at this stage. In the two examples just presented, the signal is essentially *exogenous:* we do not determine (up to a certain extent!) our own gender or race. In the first two examples, however, the signal is instead *endogenous,* as it conveys information on how the agents reacted in the past when facing the same strategic situation.

Following the seminal contribution of Rosenthal [1979], the literature we just referred to explores the relationship between these signals and the convention which may be eventually selected by the population as a whole. The strategic framework employed by this literature is that of a *supergame:* in deciding whether to follow -or to challenge- the prevailing convention (i.e. the signal-extracting device which everybody uses in the population) each agent evaluates *ex ante* the expected payoff streams associated with the *superstrategy* (that is, the full contingent plan of actions under any possible history) associated with the use of such a convention.

In the model chapter 4 we follow an alternative route, tackling the problem with the aid of evolutionary techniques. In particular, we study the evolution of a population whose members use their *social class* to coordinate their actions in the *Chicken* game of figure 1.1. Essentially, the class of a player is the payoff she obtained the last time she was called to play; therefore, social promotions and failures depend entirely on the outcome of the stage game.

Following Rosenthal and Landau [1979], we interpret the equilibrium behaviours that the players may adopt, as a function of their class, as *customs*. However, in contrast with the literature we just referred to, we let a custom determine only the players' actions in the *current period.* Moreover (and more crucially), our analysis differs from that literature in that we allow people to *change* their custom during the course of the repeated game, as a result of some learning process.

Two alternative (and complementary) learning protocols are considered. First we consider a *coordination learning* procedure, which leads the players to revise their custom as a result of a disequilibrium play. We also consider an *aspiration learning* procedure, which leads the players

[43] Recent papers include Anderlini and Ianni [1996], Binmore *et al.* [1994], Crawford [1992], Ellison [1993], Kandori *et al.* [1993], Oechssler [1993], Vega Redondo [1993] and Young [1993],.

to revise their custom if the outcome of the stage game is below a certain threshold value which we take to be a proxy of a "satisfactory" outcome of the strategic interaction.

Given that the signal the players receive is directly linked with the outcome of the game (and therefore, with the evolutionary success of each custom), we can evaluate all the alternative customs in terms of their *efficiency,* i.e. their ability to minimise the probability of an inefficient resolution of the contest. Moreover, since we prove that, when it operates alone, each custom induces a unique (ergodic) *class distribution,* we can compare the (equilibrium) distributions associated with each custom in terms of their (social) *mobility* properties. As we will show, social mobility plays in fact a major role in shaping the evolutionary selection process among the alternative conventions.

APPENDIX

PROOF OF PROPOSITION 1.2.2. Denote by $x^k(t)$ the probability with which strategy k is played at time t. We evaluate $\frac{\dot{x}^B}{x^B} = \frac{1}{2}\left(x^C + x^D - 4x^C x^D\right)$, which is positive in any interior point, and this implies $\lim_{t\to\infty} x^B(t) = \hat{x}^B > 0$. We also know that:[44]

$$\frac{d}{dt}\log\left[\frac{x^D(t)}{x^C(t)}\right] = \frac{\dot{x}^D(t)}{x^D(t)} - \frac{\dot{x}^C(t)}{x^C(t)} = (x^C(t) - x^D(t)) \tag{A1.1}$$

which in turn implies $\lim_{t\to\infty} x^C(t) = \lim_{t\to\infty} x^D(t) = \hat{x} < \frac{1}{2}$. Thus

$$\begin{gathered} \frac{d}{dt}\log\left[\frac{x^D(t)}{x^B(t)}\right] = \frac{1}{2}(2x^C(t) - 1) \Rightarrow \\ \lim_{t\to\infty}\log\left[\frac{x^D(t)}{x^B(t)}\right] = \int_0^\infty \frac{1}{2}(2x^C(t) - 1) = -\infty \Rightarrow \\ \lim_{t\to\infty} x^D(t) = 0 \end{gathered} \tag{A1.2}$$

By a symmetric argument, $\lim_{t\to\infty} x^C(t) = 0$, which implies the result■

PROOF OF PROPOSITION 1.2.2. The proof follows the same technique used to prove Proposition 2.6 in this dissertation, to which we refer for a more comprehensive account of the methodology. The Replicator Dynamics with drift (1.7.1) in the case of the game of Figure 1.4 is as follows.

$$\dot{y} = y(1-y)(3x-2) + \lambda(\frac{1}{2} - y) \tag{A1.3}$$

$$\dot{x} = x(1-x)(y-1) + \lambda(\beta - x) \tag{A1.4}$$

Denote by $RE(\Gamma)$ the set of restpoints of (A1.3-4) when $\lambda = 0$, that is, the set of respoints of the unperturbed Replicator Dynamics. It is straightforward to show that $RE(\Gamma)$ contains (together with all the pure strategy profiles) only the following component: $RE^1 = \{(x,y) \in \Delta | y = 1, x \in [0,1]\}$

We know, from Binmore and Samuelson [1996], Proposition 1, that every limiting rest point of (A1.3-4) as $\lambda \to 0$ must lie in $RE(\Gamma)$. Only two cases have to be discussed:

- CASE 0: $\lambda \to 0$ and $y \to 0$. This yields (0,0) and (1,0) as possible candidates for the limit points in $\hat{RE}(\beta)$. The first (second) point is (not) a limiting restpoint of (A1.3-4) since it is a sink (source) of the unperturbed dynamics.. We also know, from Binmore and Samuelson

[44] See the proof of Proposition 3.4.1.

[1996], Proposition 2, that (0,0) is asymptotically stable, since it is a sink of the perturbed dynamics. This completes part *a)* of the proof.

- CASE 1. $\lambda \to 0$ and $y \to 1$, (that is $(1-y) \to 0$). Setting $\dot{y}=0$ yields the following equation:

$$\frac{1-y}{\lambda} = \frac{y-1/2}{y(3x-2)} \tag{A1.5}$$

Denote by x^1 a limiting value for x in a rest point, if a limit exists, when $y \to 1$. It must be

$$\lim_{\substack{y \to 1 \\ \lambda \to 0}} \frac{1-y}{\lambda} = \frac{1}{2(3x^0-2)} \tag{A1.6}$$

Setting $\frac{\dot{x}}{\lambda}=0$, substituting $\frac{(1-y)}{\lambda}$ with the expression in (A1.6) and taking limits leads to the following solutions for x^{1}0:

$$\widehat{x}^1 = \frac{3+6\beta+\sqrt{9-44\beta+36\beta^2}}{10} \quad \text{and} \quad \breve{x}^1 = \frac{3+6\beta-\sqrt{9-44\beta+36\beta^2}}{10}$$

Note that, from (A1.4), we know that x^1 must be a real, positive number, with $2/3 < x^1 < \beta$. For the expression under the square root at the numerator to be nonnegative, it must be that $\beta \in [(11-2\sqrt{10})/18, 1]$, which determines the feasible range for both roots..

We now move to establish the stability properties of these two limiting restpoints. The Jacobian matrix for the dynamic system is as follows:

$$J(x,y,\lambda) = \begin{bmatrix} (3x-2)(1-2y)-\lambda & 3y(1-y) \\ x(1-x) & (1-2x)(y-1)-\lambda \end{bmatrix}$$

We evaluate the limiting values of trace and determinant of $J(x,y,\lambda)$, factorising for λ and substituting λ, y, $\frac{(1-y)}{\lambda}$ with their limiting values. The limiting trace of $J(x,y,\lambda)$ equals to $2x^1-3$ which is negative for any feasible x^1. The sign of the limiting determinant of $J(x,y,\lambda)$ coincides with the sign of the following:

$$\psi(x^0) = (3x^0-2)(4x^0-3) - 3x^0(1-x^0) \tag{A1.7}$$

which is positive only in the domain of $\widehat{x}^1$. As a result of that, $\widehat{x}^1$ is asymptotically stable whereas $\breve{x}^1$ is not. This completes part *b)* of the proof.■

Chapter 2

Implementation, Elimination of Weakly Dominated Strategies and Evolutionary Dynamics

2.0. Abstract

This paper is concerned with the realism of mechanisms that implement social choice functions in the traditional sense. Will agents actually play the equilibrium assumed by the analysis? As an example, we study the convergence and stability properties of Sjöström's [1994] mechanism, on the assumption that boundedly rational players find their way to equilibrium using monotonic learning dynamics and also with fictitious play. This mechanism implements most social choice functions in economic environments using as a solution concept the iterated elimination of weakly dominated strategies (only one round of deletion of weakly dominated strategies is needed). There are, however, many sets of Nash equilibria whose payoffs may be very different from those desired by the social choice function. With monotonic dynamics we show that many equilibria in all the sets of equilibria we describe are the limit points of trajectories that have completely mixed initial conditions. The initial conditions that lead to these equilibria need not be very close to the limiting point. Furthermore, even if the dynamics converge to the "right" set of equilibria, it still can converge to quite a poor outcome in welfare terms. With fictitious play, if the agents have completely mixed prior beliefs, beliefs and play converge to the outcome the planner wants to implement.

2.1. Introduction

The theory of implementation studies the problem of designing decentralized institutions through which certain socially desirable objectives can be achieved. These social arrangements should be able to operate without extensive knowledge by the principal about the agents, and in a variety of environments. The principal should ensure that the rules of the game are respected by the agents, and such rules should be designed so that it is in the best interest of the agents to take actions that lead to the socially desirable outcome, given the environment.

More precisely, *a social choice rule* is *implemented* by a *mechanism* (game-form) if for every possible environment (preference profile) the solution (set of equilibrium outcomes) of the mechanism coincides with the set of outcomes of the social choice rule for every possible environment.

One of the problems that are faced by the implementation theorist is the choice of a solution concept. This is no trivial matter because for different solution concepts the range of social choice rules that can be implemented varies dramatically. As Moore [1990] points out "choice rules are unlikely to be implementable in dominant strategy equilibrium if the domain is very rich and/or the choice rule is efficient". In the case of single-valued choice rules he also notes that "the move from dominant strategy to Nash may not help at all: only the restricted class of strategy-proof choice may be Nash implementable". If the solution concept is more refined, then the domain of the social choice rule can be much larger. In fact, the social choice rule domains are considerably enlarged for subgame-perfect implementation (Moore and Repullo [1988]), and even more so when the solution concept is the iterative deletion of weakly dominated strategies (Abreu and Matsushima [1994], Jackson *et al.* [1994], Sjöström [1994]). In fact, as Sjöström [1994] says: "With enough ingenuity the planner can implement "anything"".

The question that arises then is whether the equilibrium concept chosen is a good one for the game in object. One way to answer the question is to assume that agents are boundedly rational and that they adjust their actions over time through some trial and error learning procedure. One can then analyze under which conditions the actions that lead to the socially desirable outcomes are played asymptotically, if at all. Research in implementation theory has paid little attention to the problem of how an equilibrium is reached. Some exceptions are the papers of Muench and Walker [1984] and De Trenqualye [1988] who study the conditions for local stability of the Groves and Ledyard [1977] mechanism. Walker [1984] proposes a stable mechanism yielding nearly Walrasian allocations in large economies. Jordan [1985] shows that for any mechanism which implements the Walrasian correspondence in Nash equilibria with agents that are uninformed about other agents characteristics and any dynamic adjustment process there is an environment for which the equilibria are unstable with respect to the dynamics. Vega-Redondo [1989] proposes a mechanism for which a best-response dynamic adjustment process is globally convergent to the Lindahl equilibrium outcome in an economy which has one private good, one public good and a linear production technology for the public good. De Trenqualye [1989] proposes a mechanism that is locally stable for the implementation of Lindahl equilibria in an economy with multiple private goods one public good, a linear production technology for the public good and quasi-linear preferences. Cabrales [1995] studies the global convergence of the canonical mechanism (Maskin [1977], Repullo [1987]) of Nash implementation and the mechanisms of Abreu and Matsushima [1992, 1994]. In this paper we study the convergence and stability properties of Sjöström's [1994] mechanism[1] when one assumes that the players are boundedly rational.

[1] Sjöström's [1994] mechanism and the one that Jackson, Palfrey and Srivastava [1994] study for separable environments are very similar and most of our results would generalize easily for that mechanism as well.

Our approach is twofold. First we study dynamics that are monotonic (Nachbar [1990]). One particularly well known member of the family of monotonic dynamics is the Replicator Dynamics of evolutionary game theory (Taylor and Jonker, [1978]). These dynamics have been given a learning theoretic foundation by Börgers and Sarin [1993], and they can also be interpreted as a model of imitation (Schlag, [1994]).

We also study the dynamic properties of this mechanism under fictitious play. Under these dynamics agents are assumed to understand the game well enough to be able to choose a strategy which is a best response to their beliefs. Beliefs are then assumed to be a time average of their past observations and their prior beliefs.

We concentrate on Sjöström's mechanism for several reasons. One is that the conditions for implementation are quite weak. Although the environments that are permitted are not universal, they are rich enough for most economic problems. Furthermore, this reduction in the domain permits the author to implement the social choice rule with a "bounded" game and thus makes it immune to the criticisms of Jackson [1992]. Finally, although the solution concept is the iterated elimination of (weakly) dominated strategies (it also implements in undominated Nash equilibria), it only needs one round of deletion of weakly dominated strategies (the first). This last feature of the mechanism makes it particularly attractive since under some assumptions of imperfect knowledge of agents (either because of payoff uncertainty as in Dekel and Fudenberg, [1990], or through lack of perfect common knowledge of rationality as in Börgers, [1994]) the appropriate solution concept implies one round of deletion of weakly dominated strategies and then the iterated deletion of strictly dominated strategies.

In Sjöström's [1994] mechanism the agents are arranged to announce their preferences and those of their two closest neighbors. The mechanism is designed in such a way that a truthful report of one's own preferences is weakly dominant (it does not affect one's payoff, except in a set of states which is called *totally inconsistent,* and in those states it is preferable to report them truthfully). Since in this mechanism it is advantageous to report the same preferences about your neighbors that they are reporting about themselves, it is clear that the only equilibrium that survives the first round of deletion of weakly dominated strategies is the truth-telling one.

There are, however, many other Nash equilibria. For every preference profile R, there is a component (i.e. a closed and connected set) of equilibria in which all agents report the preferences for their neighbors indicated in R and they report the preferences about themselves indicated in R with high enough (this need not be very high) probability. The reason for this is that the mechanism makes it important that all agents match their neighbors' announcements about themselves, but the report about oneself is only important in some unlikely *(totally inconsistent)* state.

For the monotonic dynamics we show that many equilibria in all the components of equilibria we have described are the limit points of trajectories of the learning dynamics that have completely mixed initial conditions (that is, initial conditions that give strictly positive weights to all possible messages). Although the general results are local, we can show by example (the game in Figure 1, Sjöström, [1994]) that the initial conditions that lead to these equilibria need not be close to the limiting point. Furthermore, and perhaps more worrying, the equilibria which belong to the same component as the completely truthful report are not outcome equivalent to such equilibrium, as they yield payoffs that are significantly different (lower) to the payoffs of the social choice functions outcome. Therefore, even if the dynamics converge to the "right" component of equilibria, it still can converge to quite a poor outcome in welfare terms.

One could naively expect that evolution would eliminate weakly dominated strategies. The reason why this doesn't happen is that the weakly dominant strategy grows faster than the dominated one only if the *totally inconsistent* states are met often enough by the players. But the weight of the *totally inconsistent* states is also decreasing over time since people are learning to avoid such states. It may be that they decrease fast enough so that the push towards the weakly dominant strategy is not enough to make the dominated strategy disappear.

The fact that evolution does not eliminate weakly dominated strategies has been known since at least Nachbar [1990]. Samuelson [1993] discusses the issue of elimination of weakly dominated strategies in evolutionary games. Binmore *et al.* [1995] have shown the implications of these findings for the *Ultimatum* game. In particular, they provide a numerical example, based on the classic *Entry* game, in which *a)* there are trajectories of the Replicator Dynamics which converge to the Nash equilibrium component in which the players choose a weakly dominated strategy with positive probability and *b)* in the presence of mutations, such component may even exhibit asymptotic stability properties.[2] These results are more than a theoretical curiosity. Binmore and Samuelson [1995] note that: "the experimental evidence is now strong that one cannot rely on predictions that depend on deleting weakly dominated strategies".

In the context of implementation theory, Cabrales [1996] studies the mechanism of Abreu and Matsushima [1994], which also uses as a solution concept the iterated deletion of weakly dominated strategies. He shows that, although convergence to the undominated solution of these games can be achieved, these solutions are not stable. The problem is that drift between strategies that have the same payoff as the equilibrium payoff can destabilize the equilibrium outcome. This result has an additional interest because it allows him to discuss the mechanism of Abreu and Matsushima [1992]. This mechanism virtually implements the social choice function (that is, it implements the social choice function with arbitrarily high probability) in strategies that survive the iterative deletion of strictly dominated strategies. This would seem to be a good mechanism from an evolutionary perspective, given that iteratively strictly dominated strategies are asymptotically eliminated for most adaptive dynamics.[3] The problem is that if the mechanism implements with very high probability the social choice function, then it will do so in iteratively strictly ε-undominated strategies, for ε very small. This implies that as the mechanism becomes more effective in doing its job, it becomes closer to the one in Abreu and Matsushima [1994] and thus it becomes open to the sort of instability problems which that mechanism has. More precisely, if the dynamics are not sensitive to ε differences (just as the planner is not sensitive to them) they converge to the right solution but they are not stable.

For fictitious play we show that as long as agents have completely mixed prior beliefs, their actions and beliefs converge to the unique equilibrium whose outcome is the one that the

[2]See § 1.7.
[3]See § 1.5

planner wants to implement. This is so because completely mixed initial beliefs make the weakly dominated strategies in which the agent lies about her own type suboptimal. Furthermore, because the initial beliefs are completely mixed, beliefs will always be completely mixed, so the weakly dominated strategies will always remain suboptimal, will never be played and their weight in beliefs will eventually vanish. Once their weight is sufficiently small the other nonequilibrium strategies become suboptimal as well and so they will never be played and their weight in beliefs will vanish asymptotically.

It is important to understand the limitations of this result. For example, it is crucial that beliefs remain totally mixed for all time. If an agent became convinced that the other agents would tell the truth about her type then she would not be hurt by lying about her own type. This is the same logic that underlies the instability result in Cabrales [1996]. Also notice that strategies that are not exact best responses get played with probability zero. If small differences in payoffs did not lead to such a large effect on probabilities of play (or proportions of the population playing them), then the results about monotonic dynamics could still hold.

Despite these theoretical considerations, the question about which of the dynamics assumptions is correct should have mainly an empirical content. In this sense, there is already some evidence on mechanism design and learning algorithms. Chen and Tang [1996] have done experiments with the Basic Quadratic mechanism by Groves and Ledyard [1977] and the Paired-Difference mechanism by Walker [1981]. They estimate different dynamic learning models using the experimental data and they show that variants of stimulus response learning algorithms (whose expected law of motion is the Replicator Dynamics[4]) outperform the generalized fictitious play model. This is consistent with the good performance that Roth and Erev [1995] show for stimulus response learning algorithms in mimicking the behavior of a range of experimental data which includes other weak dominance solvable games, like the *Ultimatum* game. By comparison, fictitious play would predict, contrary to the experimental evidence, convergence to the subgame perfect equilibrium of the ultimatum game.

The remainder of the paper is arranged as follows. In section 2.2 we introduce some notation, we describe the mechanism and we make the assumptions about the dynamics. In section 2.3 we fully characterize (for all interior initial conditions) the set of limit points of the dynamics for the game in Figure 1, Sjöström 1994, to be considered a simplified version of the mechanism. In section 2.4 we give local results (for some interior initial conditions) for the set of limit points of the dynamics for the general game. In section 2.5 we describe the asymptotic stability properties of the sets of limit points in the presence of mutations. Section 2.6 analyzes fictitious play in this game. Finally, section 2.7 concludes, together with an appendix containing the proofs of the relevant propositions.

2.2 THE MODEL AND THE DYNAMICS.

The only important change we make in Sjöström's [1994] approach is to employ a Von Neumann-Morgenstern utility function instead of a preference relation. We do this because we

[4]See Roth and Erev [1995] for the use of these dynamics with experimental data and Börgers and Sarin [1996] for their connection with the Replicator dynamics.

need to specify the payoff functions for mixed strategies, since the dynamics are defined on the mixed strategy simplex.

There is a set $\Im \equiv \{1,..,n\}$, $n \geq 3$, of agents and a set $A \subseteq \Re_+^m$ of feasible consumption plans. The preferences of agent $i \in \Im$ are represented with a (Von Neumann-Morgenstern) utility function $v_i : A \times \Phi_i \to \Re$, where Φ_i specifies a finite set of possible preference parameters. An element $R_i \in \Phi_i$ represents the preferences of agent *i* over *A*. A *preference profile* is a vector $R = (R_1, ..., R_I)$, where $R_i \in \Phi_i$. The preference profiles will be common knowledge among the agents. The following assumptions are made concerning preferences and feasible consumption profiles.

- ASSUMPTION P.1. The set of feasible consumption profiles is convex. For all $a, a' \in A$ and for all $\lambda \in [0,1]$ then $\lambda a + (1-\lambda)a' \in A$.
- ASSUMPTION P.2. The preferences represented by $R_i \in \Phi_i$ are complete and transitive.
- ASSUMPTION P.3. The preferences represented by $R_i \in \Phi_i$ are strictly convex. That is, for any $a, a' \in A$ and for all $\lambda \in (0,1)$ if $a \neq a'$ and $v_i(a, R_i) \geq v_i(a', R_i)$ then $v_i(\lambda a + (1-\lambda)a', R_i) > v_i(a', R_i)$.
- ASSUMPTION P.4. For any $R_i \in \Phi_i$ if $a \geq 0$ and $a \neq 0$ then $v_i(a, R_i) \geq v_i(0, R_i)$.
- ASSUMPTION P.5. Preference reversal. For any $R_i, R_i' \in \Phi_i$ if $R_i \neq R_i'$ then there are $a, a' \in A$ such that $v_i(a, R_i) > v_i(a', R_i)$ and $v_i(a', R_i') > v_i(a, R_i')$.

For any set $B \subseteq \Re_+^m$ and any $R_i \in \Phi_i$, a *choice correspondence* is defined as follows: $c(B, R_i) \equiv \{a \in B | \forall b \in B, v_i(a, R_i) \geq v_i(b, R_i)\}$.

A *social choice function* is a mapping $f : \Phi \to A$ where $f(R) = \{f_i(R)\}; i \in I$. A *mechanism* is a pair $\Gamma \equiv (M, \alpha)$, where $M = \times_{i \in \Im} M_i$ $\alpha : M \to A$, where $\alpha(M) = \{\alpha_i(m)\} i \in \Im$. M_i is the *message* space of agent *i* and α is the *outcome function*. A *mechanism* and a *preference profile* define a game.

We now construct a mechanism.

Let $M_i = \Phi_{i-1} \times \Phi_{i-1} \times \Phi_{i+1}$, so that each individual announces her own preferences as well as the preferences of her two neighbors, and let members of M_i and M be denoted m_i and m respectively. A generic strategy is therefore $m_i = (R_{i-1}^i, R_i^i, R_{i+1}^i)$. A *K*-tuple of messages $\{m_{j1}, ..., m_{jK}\}$ is *totally consistent* if whenever agents $i,k \in \{j_1, ..., j_K\}$ both announce the preference of player $j \in \Im$, then $R_j^i = R_j^k$. On the other hand, a *K*-tuple of messages $\{m_{j1}, ..., m_{jK}\}$ is *totally inconsistent* if whenever agents $i,k \in \{j_1, ..., j_K\}$ both announce the preference of player $j \in \Im$, then $R_j^i \neq R_j^k$.

Consider $R_i, R_i' \in \Phi_i$, where $R_i \neq R_i'$. By assumption p.6 there are $\tilde{a}, a \in A$, such that $v_i(a, R_i) > v_i(\tilde{a}, R_i)$ and $v_i(\tilde{a}, R_i') > v_i(a, R_i')$. We can choose a and $\tilde{a}$ so that $v_i(a, R_i) > v_i(a', R_i)$ for all a' in the line segment between a and $\tilde{a}$. Given this pair $(\tilde{a}, a)$ let $\beta_i(R_i, R_i') = \{b \in \Re_m^+ \,|\, b = \lambda a + (1-\lambda)\tilde{a}$, for $\lambda \in [0,1]\}$. By construction, for all $R_i, R_i' \in \Phi_i$

$c(\beta_i(R_i R_i'),R_i) \neq c(\beta_i(R_i R_i'),R_i')$. Let $\varphi(i,m)=(R_1^n,R_2^1,\ldots,R_i^{i-1},R_{i+1}^i,R_{i+2}^{i+1},\ldots,R_n^{n-1})$ and for every i and m_{-i}, define

$$B_i(m_{-i}) = \begin{cases} f_i(\varphi(i,m)) \text{ if } m_{-i} \text{ is totally consistent} \\ \beta_i(R_i^{i-1},R_i^{i+1}) \text{ if } m_{-i} \text{ is totally inconsistent} \\ \frac{1}{n} f_i(\phi(i,m)) \quad \text{otherwise} \end{cases}$$

Now we can define α :

$$\alpha_i(m) = \begin{cases} c(B_i(m_{-i}),R_i^i) R_{i-1}^i = R_{i-1}^{i-1} \,\&\, R_{i+1}^i = R_{i+1}^{i+1} \\ 0 \text{ otherwise} \end{cases}$$

Let $\hat{R}$ be the true preference profile and R^* an arbitrary preference profile. To understand the mechanism notice that the only time when the choice of an announcement R_i^i has any effect on payoffs is when m_{-i} is totally inconsistent. In that case, the outcome is the optimal choice within the set $\beta_i(R_i^{i-1},R_i^{i+1}))$ according to the announced R_i^i. For this reason announcing the true preference $\hat{R}_i$ can never hurt. Furthermore, for every alternative announcement $R_i^i = R_i^*$, there is some totally inconsistent m_{-i} with $R_i^{i-1} = R_i^*$ and $R_i^{i+1} = \hat{R}_i$ and the set $\beta_i(.,.)$ is constructed in such a way that $c(\beta_i(R_i^*,\hat{R}_i),\hat{R}_i)$ is strictly preferred to $c(\beta_i(R_i^*,\hat{R}_i),R_i^*)$. Therefore, a message $m_i=(R_{i-1}^i,R_i^*,R_{i+1}^i)$ is weakly dominated by a message $m_i=(R_{i-1}^i,\hat{R}_i,R_{i+1}^i)$, that is, untruthful announcements about oneself are weakly dominated.

Once these weakly dominated strategies are eliminated and all agents announce the true preferences about themselves, $R_j^j = \hat{R}_j^*$, it is strictly dominated to announce untruthful preferences about the neighbors, $R_{i+1}^i \neq \hat{R}_{i+1}=R_{i+1}^{i+1}$ or $R_{i-1}^i \neq \hat{R}_{i-1}=R_{i-1}^{i-1}$, since disagreeing with the neighbors is punished with the 0 consumption bundle.

These two facts establish the main theorem in Sjöström [1994].

- PROPOSITION 2.0. Let f be an arbitrary social choice function. The mechanism described above implements f in undominated Nash equilibrium and in iterated deletion of weakly dominated strategies.

It is important to notice, for the discussion we will undertake below, that the set of states for which not announcing the true preferences about oneself is weakly dominated are themselves states that typically produce very bad outcomes for the opponents (at least one of them will have 0 consumption, and probably many). If agents learn to avoid totally inconsistent states very fast, there is no incentive to tell the truth about oneself. The mechanism we have described puts a lot of emphasis in consensus announcements, since disagreement is punished with 0 consumption, and truth-telling is only rewarded in a set of states which need not be very prominent in the minds of the players. That is precisely the reason why convergence to outcomes of the social choice function may fail to occur. This conflict is typical of other mechanisms that implement in the iterated deletion of weakly dominated strategies, like Abreu and Matsushima [1994].

We now move on to the characterization of the evolutionary dynamics we analyze.

- ASSUMPTION D.1. The evolution of $x(t)$ is given by a system of continuous-time differential equations:

$$\dot{x} = D(x(t)) \tag{2.1}$$

We require that the autonomous system (2.1) satisfies the standard regularity condition as in definition 1.1. Furthermore, D must also satisfy the following requirements:

- ASSUMPTION D.2. D is a monotonic selection dynamic.[5]
- ASSUMPTION D.3. Let $Y_i(m_i, m_i') = \{x_{-i} \mid u_i(m_i, x_{-i}(t)) - u_i(m_i', x_{-i}(t)) = 0\}$. Then, for all $\delta > 1$:

$$\frac{\lim\limits_{|x_{-i} - Y_i(m_i, m_i')| \to 0} \sup\left[g_i(m_i, x_{-i}(t)) - g_i(m_i', x_{-i}(t))\right]}{\mathrm{sign}\left[u_i(m_i', x_{-i}(t)) - u_i(m_i, x_{-i}(t))\right]\left(\ln\left[\left|u_i(m_i', x_{-i}(t)) - u_i(m_i, x_{-i}(t))\right|\right]\right)^{-\delta}} > -1$$

Assumption d.3 is less standard in the evolutionary literature and we will expand on it when we discuss Proposition 2.4 because it will be helpful to understand why weakly dominated strategies need not disappear in the limit. What assumption d.3 says is that if the difference in payoffs between two strategies is going to zero at rate exp[$-n$], the difference in growth rates has to go to zero at a rate of at least $1/n^{\delta}$.

Continuity and assumption d.2 demand that strategies that have the same payoff grow at the same rate, but they impose no requirements on the speed at which the difference in growth rates goes to zero as the difference in payoff go to zero. Assumption d.3 can be satisfied even if the sensitivity of growth rates to payoffs is much higher than linear around zero (as would be implied, for example, by the Replicator Dynamics and other aggregate monotonic dynamics).

- ASSUMPTION D.4. $x(0) \in \Delta^0$.

Finally, Assumption d.4, which is also standard in the evolutionary literature, is necessary because it excludes the possibility that the selection dynamic acts only on a subset of the strategy space. This possibility arises because the system is forward invariant, and therefore a strategy that has zero weight at time zero would also have zero weight at all subsequent times. We want to avoid this possibility because the selection dynamics would be operating on a game which might be qualitatively different from the game we are trying to analyze.

2.3 An Example

[5]See Definition 1.2.

We prefix the analysis of the dynamics of the mechanism with the following example, taken from Sjöström [1994], p. 504, which is intended to convey the essence of our results. A unit of a good has to be divided among three players: 1, 2 and 3. The (Von-Neumann Morgenstern) utility functions of players 1 and 2 are linear in the amount of good consumed, and this is common knowledge among the players and the planner. The utility function of player 3 may have one of two possible types: either it is linear in the amount of good consumed (we index these preferences by the number 1) or linear until the amount of good consumed is 1/3, for consumptions larger than 1/3 the utility decreases, (in other words, preferences peak at consumption 1/4). The index for these latter preferences is 0; the true preferences of player 3 are common knowledge among the players, but the planner does not know them.

The social choice function recommends the consumption vector (1/3,1/3,1/3) for preferences of type 1 and (1/4,1/4,1/2) for preferences of type 0. Notice that this social choice function is such that agent 3 would like to conceal her preferences, and therefore the planner needs a nontrivial mechanism to elicit the true preferences.

The mechanism proposed by Sjöström requires the three players to make a simultaneous statement about the preferences of player 3. Let $m_i^1, i \in \Im$ represent the message in which preferences of type 1 are announced, with m_i^0 denoting the announcement of type 0 preferences. Figure 2.1 illustrates the outcome function. We will assume for the analysis that the true preferences are of type 1 and therefore Figure 2.1 is also the payoff function of a game, which we call Γ:

$m_3 = m_3^0$

	m_2^0	m_2^1
m_1^0	$\frac{1}{4},\frac{1}{4},\frac{1}{2}$	$\frac{1}{3},0,\frac{1}{3}$
m_1^1	$0,\frac{1}{3},\frac{1}{3}$	$0,0,\frac{1}{3}$

$m_3 = m_3^1$

	m_2^0	m_2^1
m_1^0	$0,0,\frac{1}{2}$	$0,\frac{1}{3},\frac{1}{2}$
m_1^1	$\frac{1}{3},0,\frac{1}{2}$	$\frac{1}{3},\frac{1}{3},\frac{1}{3}$

FIGURE 2.1
Sjöström's Example: game Γ.

Player 1 picks a row, player 2 a column, and player 3 picks a matrix. We first notice that the mechanism leads to a game which is weakly dominance solvable, in the sense that it can be reduced to a single cell, corresponding to the truth-telling equilibrium outcome, by the iterated deletion of weakly dominated strategies. Unlike other weakly solvable games, this procedure yields, when applied to Γ, a unique outcome, independently of the order of removal of strategies. Player 3 first deletes her (weakly) dominated strategy m_3^0 (the other agents have no dominated strategies at this stage). Once m_3^0 is removed, the strategies m_1^0 and m_2^0 for players 1 and 2 become strictly dominated. The unique strategy profile selected in this way is (m_1^1, m_2^1, m_3^1). Notice, however, that the strategy profile (m_1^0, m_2^0, m_3^0) is also an equilibrium, and that this equilibrium yields a higher payoff for agent 3 than (m_1^1, m_2^1, m_3^1).

Given that each player has only two strategies in her support, with an abuse of notation we set $x_i \equiv x_i^{m_i^1}$.[6] We first characterize the set of Nash equilibria of the game:

- PROPOSITION 2.1. The set $\mathcal{NE}$ of Nash equilibria of Γ is the union of precisely two disjoint components NE^0 and NE^1, where:

$$NE^0 = \{x \in \Delta | x_1 = x_2 = 0, x_3 \leq 3/7\}$$
$$NE^1 = \{x \in \Delta | x_1 = x_2 = 1, x_3 \geq 1/2\}$$

PROOF. See the Appendix.

Denote the set of restpoints of Γ under some monotonic dynamic by $RE(\Gamma)$. It is straightforward to show that $RE(\Gamma)$ contains (together with all the pure strategy profiles) only the following two components: $RE^h = \{x \in \Delta | x_1 = x_2 = h, x_3 \in [0,1]\}, h = 0,1.$ Our task is to study the asymptotics of a monotonic selection dynamic whose initial state lies in the relative interior of the state space:

- PROPOSITION 2.2. Any interior solution $x(t,x(0))$ of a monotonic selection dynamics $\dot{x} = D(x)$ converges to $\mathcal{NE}$.

PROOF. See the Appendix.

If initial conditions are completely mixed, we then know that the evolutionary dynamics will eventually converge to a Nash equilibrium of the game. In the following section we extend the result to the more general setting of Sjöström's mechanism.

2.4 LOCAL RESULTS FOR THE GENERAL GAME

In this section we show that the results of the previous section generalize locally. Proposition 2.3 characterizes some components of Nash equilibria for the game induced by the mechanism in Sjöström [1994], which we described in section 2. Any message profile in which the agents are unanimous in the (arbitrary) preference profile they announce, R^*, (or, more appropriately, the preferences they announce about their neighbors and themselves are taken from the profile R^*) is an equilibrium. Furthermore, a mixed strategy profile in which every agent mixes between messages consistent with R^* and other preference profiles that only differ in the announcement they make about their own preferences is also an equilibrium, as long as R^* is given a high enough weight. As we showed in the example, the weight given to R^* need not be very high. The equilibria in a component are not payoff equivalent, since disagreeing with a neighbor (an event with nonzero probability in the mixed strategy equilibria) results in a

[6] The fact that each player has only two available options will also allow us to express the dynamics in terms of the payoff difference between player i's truthful and untruthful strategy, which we call $\Delta\Pi_i(x(t))$ (i.e. $\Delta\Pi_i(x(t)) = u_i(m_i^1, x_{-i}(t)) - u_i(m_i^0, x_{-i}(t))$).

punishment. Proposition 2.4 shows that any of the previous equilibria that gives enough weight to R^* is the limit point of some interior path for a monotonic selection dynamic. Figure 2.2 shows that the initial condition need not be very close to the limit point.

Let $m_i^* = (R_{i-1}^*, R_i^*, R_{i+1}^*)$, $u_i = \max_R \ v_i(f_i(R), \hat{R}_i)$ and $U_{in} = \max_R \ v_i(\frac{1}{n} f_i(R), \hat{R}_i)$. The message m_i^* is a consensus announcement by agent i, u_i is the utility associated to the most preferred outcome from the social choice function for agent i with true preferences $\hat{R}_i$ and U_{in} is the utility associated to the most preferred consumption bundle among those that result from dividing the bundles assigned by the social choice function by n. Let

$$S_i = \{ m_i \in M_i | \ R_{i-1}^i = R_{i-1}^*, R_{i+1}^i = R_{i+1}^* \}$$

and $\overline{S}_i = \{ m_i \in M_i | \ m_i \notin S_i \}$ The set S_i is the set of all mixed strategies in which announcements about the neighbors agrees with R^*, and $\overline{S}_i$ is the complement of S_i with respect to M_i. Also let

$$S_i^{k_i} = \{ x_i | \ x_i^{m_i} = 0, \text{ for all } m_i \notin S_i \text{ and } x_i^{m_i^*} > k_i \},$$

where we assume

$$(k_i)^n \geq \frac{U_{jn} - v_j(0, \hat{R}_j)}{v_j(f_i(\phi(i, R^*)), \hat{R}_j) - 2v_j(0, \hat{R}_j) + U_{jn}}$$

for all i and all $j \neq i$. The set $S_i^{k_i}$ is the set of all mixed strategies in which announcements about the neighbors agrees with R^*, and the probability of announcing R_i^* is higher than k_i.

- PROPOSITION 2.3. For all $\hat{R}, R^* \in \Phi$ and $x_i \in S_i^{k_i}$, x is a Nash equilibrium of $(\alpha, \hat{R})$.

PROOF. See the Appendix.♠

Now we prove that not only are there other Nash equilibria, but that elements in those components can be reached by paths starting in the interior to the simplex. By assumption d.2 we know that for all $h_v > 0$ with $u_i(m_i, x_{-i}(t))$-$u_i(m_i', x_{-i}(t)) < -h_v$, there is $h_g > 0$, such that $g_i(m_i, x_{-i}(t))$-$g_i(m_i', x_{-i}(t)) < -h_g$.

Let h_v be a constant such that $0 \leq h_v \leq \min_{i,R} v_i(f_i(\varphi(i, R^*)), \hat{R}_i)$-$v_i(0, \hat{R}_i)$. Let the corresponding h_g and

$$H = \max_i \left\{ \left(\frac{U_i - v_i(0, \hat{R}_i) + h_v}{v_i(f_i(\phi(i, R^*)), \hat{R}_i) - 2v_i(0, \hat{R}_i) + U_i} \right)^{\frac{1}{n-1}} \right\}$$

Note that by definition $H \in [0,1]$. By Assumption d.3 we know that there exists $\theta_i(m_i, m_i') > 0$ such that if $| u_i(m_i, x_{-i}(t)) - u_i(m_i', x_{-i}(t)) | < \theta_i(m_i, m_i')$:

$$\frac{\lim\limits_{|x_{-i}(t)-Y_i(m_i,m_i')|\to 0} \sup\left|g_i(m_i,x_{-i}(t)) - g_i(m_i',x_{-i}(t))\right|}{\left|\ln\left[u_i(m_i',x_{-i}(t)) - u_i(m_i,x_{-i}(t))\right]\right|^{-\delta}} > 1$$

Let $0 < \theta < \min\limits_{i.m_i,m_i'} [\theta_i(m_i, m_i')]$. For any set $\Phi_i \in M_i$, let $x_{\Phi_i} = \sum\limits_{m_i \in \Phi_i} x_i^{m_i}$ and

$$L = \exp\left[\frac{1}{h_g(\delta-1)\left(-\ln\left[U_i \prod\limits_{j\neq i} x_j^{\overline{S}_j}(0) \frac{H}{x_i^{m_i^*}(0)}\right]\right)^{\delta-1}}\right]$$

The following proposition holds:

- PROPOSITION 2.4. Assume that for all i, $x_i^{m_i^*}(0)$ is big enough so that, $x_i^{m_i^*}(0)L > H$ and $U_i \prod\limits_{j\neq i} x_j^{\overline{S}_j}(0) U_i \prod\limits_{j\neq i} x_j^{\overline{S}_j}(0)\ (H/x_i^{m_i^*}(0)) < \theta$. Under these conditions we have that

a) For all $m_i \in \overline{S}_i$, $x_i^{m_i}(t)/x_i^{m_i}(0) < \exp[-h_g t](H/x_i^{m_i^*}(0))$ for all t and all i;
b) $x_i^{m_i^*}(t) > H$ for all t;
c) $(x_i^{m_i^*}(t)/x_i^{m_i}(t)) < (x_i^{m_i^*}(0)/x_i^{m_i}(0))L$ for all t and all $m_i \in S_i$

PROOF. See the Appendix.

Part *a)* of the Proposition says that the weight of any strategy in $\overline{S}_i$ decreases over time at a rate higher than h_g. This is important because the strategies for which not telling the truth about oneself is dominated are all in $\overline{S}_i$, so if the weight of these strategies decrease over time, the payoff advantage of the dominating strategy disappears over time, and makes it possible for a dominated strategy to have positive limiting weight.

Part *b)* ensures that the weight of m_i^* is always high enough. If the weight of m_i^* is high enough, then the strategies in the sets $\overline{S}_j$ have a lower payoff than strategies in S_j since an announcement about your neighbor that does not coincide with her announcement about herself is punished.

In fact parts *a)* and *b)* reinforce each other. While m_i^* keeps having a high enough weight, the weight of strategies in $\overline{S}_j$ decreases, and if strategies in $\overline{S}_j$ decrease fast enough the weight of m_i^* does not go below a certain bound. All of this provided that m_i^* started with high enough weight, which as Figure 2.2 demonstrates, need not be very high.

Notice that part *b)* guarantees that pure strategy equilibria in the "wrong" component are attractors of interior paths. Part *c)* says that the weight of m_i^* in the limit is less than 1, and therefore some mixed strategy equilibria are attractors as well if the initial conditions give sufficiently little weight to strategies in $\overline{S}_j$. This guarantees that even if there is convergence to the "right" component it need not be to the pure strategy equilibrium, and remember that the equilibria are not payoff equivalent (the mixed strategy equilibria have lower expected payoff because agents are punished for announcing discordant preferences).

The convergence to the mixed equilibria can happen because the payoffs to all strategies in S_i are similar if the weight of strategies in $\overline{S}_j$ is small, and by *a)* the weight of strategies in $\overline{S}_j$ is decreasing. So even though m_i^* has a payoff advantage, the advantage goes to zero over time, and assumption d.3 guarantee that it does not accumulate fast enough.

If d.3 didn't hold, equilibria which do not implement the social choice function may fail to be a limit point for the dynamics. Convergence to the "wrong" equilibrium obtains only if a weakly dominated strategy for player i (call it m, and call m' the strategy that weakly dominates m) gets positive weight in the limit. But along the way to the limit the strategy against which m and m' differ (call it y) has also positive weight (since the system is regular, and therefore forward invariant). So by assumption d.2 the growth rate of m' is larger than the growth rate of m. The weight of m'/m is the integral of the difference in growth rates of m' and m. If the limiting value of this integral is infinite the limiting value of $x_i^m(t)$ would be zero. But the weight of y (and thus the difference in payoffs) may be going to zero, thus the weight of m'/m is an integral of a function that goes to zero, which may be finite.

Assumption d.3 describes how the growth rates have to relate to payoffs (when differences in payoffs are small) so that the limiting value of a dominated strategy is not zero. Assumption d.3 would hold, for example, if the growth rates were linear in the payoffs, as it happens with the Replicator Dynamics. But the requirement is much weaker than that, because it is only a local requirement around zero, and because the rate at which growth rates go to zero with payoffs can be much higher than linear. In other words, even if the growth rates were much more sensitive to payoff differences (around zero) than the Replicator Dynamics allow, assumption d.3 could still be satisfied.

The elimination of a weakly dominated strategy in an evolutionary context requires that the strategy against which the dominated strategy gives a lower payoff than the dominating strategy has to appear sufficiently often or that its appearance has to provoke a dramatic enough reduction in the dominated strategy.

2.5. More on the example (stability with/out drift)

In the previous section, we have extended the convergence result contained in Proposition 2.2, and we have shown that the limit points of the dynamics for interior initial conditions are generally different from the outcomes intended by the planner. We now go back to the example in order to test the stability properties of $\mathscr{NE}$. To do so, some further terminology is needed:

- DEFINITION 2.1. Let $x(t,x(0))$ be the solution of a differential equation on state space Δ given initial condition $x(0)$. Let also C denote a closed set of restpoints in Δ of the same differential equation. Then:
  (i) C is (interior) ***stable*** if, for every neighbourhood O of C, there is another neighbourhood U of C, with $U \subset O$, such that, for any $x(0) \in U \cap \Delta$ ($U \cap \Delta^0$), we have $x(t,x(0)) \in O$;
  (ii) C is (interior) ***attracting*** if is contained in an open set O such for any $x(0) \in O \cap \Delta$ ($O \cap \Delta^0$) we have $\lim_{t\to\infty} x(t,x(0)) \in C$.;
  (iii) C is ***globally*** (interior) attracting if for any $x(0) \in \Delta$ (Δ^0) we have $\lim_{t\to\infty} x(t,x(0)) \in C$;
  (iv) C (interior) ***asymptotically stable*** if it is (interior) attracting and (interior) stable.

To simplify the analysis, we set additional conditions on the dynamics, which is the purpose of the following assumption, (which replaces assumptions d.1-5):

- ASSUMPTION D.5. The evolution of $x(t)$ is given by the following system of continuous-time differential equations:

$$\dot{x}_i \equiv \tilde{D}_i(x(t),\lambda) = x_i(t)(1-x_i(t))\Delta\Pi_i(.) + \lambda(\beta_i - x_i(t)) \tag{2.2}$$

with $\lambda > 0$, $\beta_1 = \beta_2 = 1/2$ and $\beta_3 = \beta \in (0,1)$. We will make reference to the Dynamic (2.2) as the Replicator Dynamic with drift.[7]

We check how the model reacts to the introduction of such a perturbation. The stability analysis of the Replicator Dynamics with drift will give us information about the effects of small changes in the vector field on the equilibria of the system defined by the Replicator Dynamic (in other words, it will test the *structural stability* of such equilibria). To simplify the exposition, we have fixed $\beta_1 = \beta_2 = \frac{1}{2}$, since only the value of β_3 turns out to be genuinely significant.

We start our analysis on the stability properties of $\mathcal{NE}$ looking at the case of the Replicator Dynamic without drift (i.e. when $\lambda = 0$). We know from Proposition 2.2, that $\mathcal{NE}$ is globally interior attracting, since it attracts every interior path under any monotonic selection dynamic (of which the Replicator Dynamic is a special case). We now take a closer look at the stability properties of each component of Nash equilibria separately (i.e. NE^0 and NE^1):

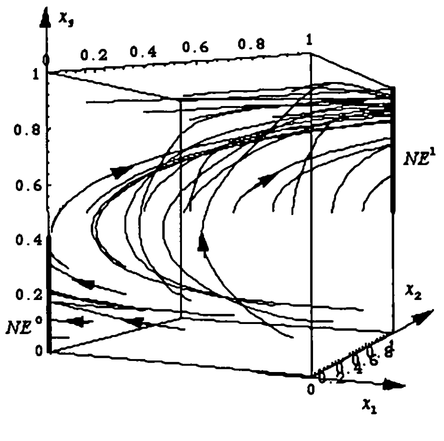


FIGURE 2.2
The Replicator Dynamic and game Γ

Figure 2.2 shows a phase diagram describing trajectories of the Replicator Dynamic starting from some interior initial conditions. The Nash equilibrium component NE^0 (NE^1) is

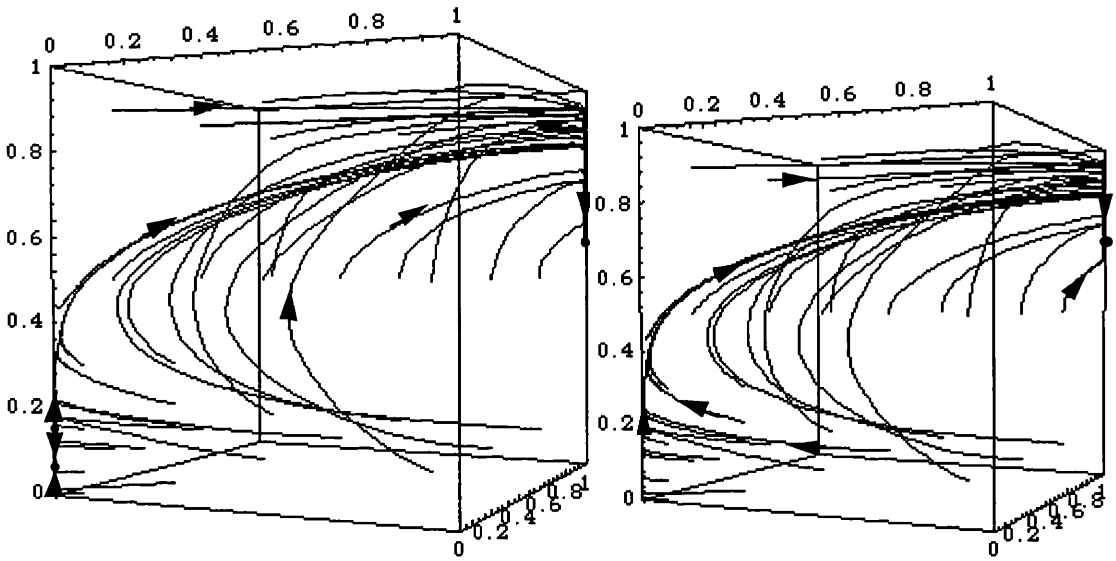
[7]See § 1.7.

represented by a bold segment in the bottom-left (top-right) corner of the state space Δ. First notice that, as we know from Proposition 2.2, all trajectories converge to a Nash equilibrium of the game. Moreover, the diagram shows (consistently with Proposition 2.4) that there are some trajectories of the Replicator Dynamics which converge to NE^0, the Nash equilibrium component in which both players 1 and 2 deliver the false message with probability 1. However, this latter component is not asymptotically stable, as can be easily spotted from the diagram. Trajectories starting arbitrarily close to NE^0, provided $x_3 > \frac{3}{7}$ will eventually converge to the truth-telling component NE^1. We summarize the key properties of these trajectories in the following proposition:

- PROPOSITION 2.5. Under the unperturbed Replicator Dynamic *(i)* NE^1 is interior asymptotically stable, whereas *(ii)* NE^0 is not.

PROOF. See the Appendix.♠

We now move to the analysis of the Replicator Dynamic with drift when $\lambda > 0$:



a) $\beta = .001$ *b)* $\beta = .5$

FIGURE 2.3

The dynamic with drift and game Γ

Let $\beta \in (0,1)$ be a generic element of the space of the feasible perturbations. Figure 2.3 shows the trajectories of (2.2) when $\lambda = .001$ under two different specifications of β. Diagram 3 represents a situation in which, in the proximity of NE^0, the drift against m_i^0 is uniform across players, where in diagram 2.3*a)* the drift against m_3^0 is significantly lower. As the figures show, there is a local attractor close to NE^1 in both cases. Moreover, none of the elements of NE^0 is a restpoint of the dynamic with drift in Figure 2.3*b)*, while Figure 2.3*a)* shows that there is an additional local attractor which belongs to NE^0: trajectories starting close to NE^0: converge to it, as it happens in the case of the unperturbed Replicator Dynamics.

We are interested in the convergence and stability properties of (2.2) when $\lambda \to 0$, considering two different configurations of the drift parameter β:

$$\text{CASE A: } \beta \in \left(0, \frac{23-4\sqrt{30}}{49}\right)$$
$$\text{CASE B: } \beta \in \left(\frac{23-4\sqrt{30}}{49}, 1\right)$$

Given $\frac{23-4\sqrt{30}}{49} \approx 0222673$, CASE A depicts a situation in which, for small values of x_i, the drift against the untruth-telling strategy is significantly lower for player 3 than for her opponents. In the following proposition we characterize the set of restpoints of the dynamic with drift, together with their stability properties:

- PROPOSITION 2.6. Let $\hat{RE}(\beta)$ be the set of restpoints of (2.2) for λ sufficiently close to 0. The following properties hold:

a) for all $\beta \in (0,1)$ $\hat{RE}(\beta)$ contains an element of NE^1 which is also asymptotically stable.
b) under CASE A $\hat{RE}(\beta)$ contains also two additional restpoints, both belonging to NE^0, one of which is asymptotically stable.

PROOF. See the Appendix.♠

There is a striking similarity between the content of Proposition 2.6 and the findings of Binmore *et al.* [1995], as we pointed out in the introduction. They analyze the *Entry* game, in one of whose equilibrium components a player selects a weakly dominated strategy with positive probability. This component is interior attracting. Moreover, like our NE^0, such component fails to be interior asymptotically stable, but for certain parameter values it may be asymptotically stable when the system is slightly perturbed.[8] Given the failure of asymptotic stability without perturbations, one would expect any perturbation to move the system away from the unstable component and the weakly dominated strategy to become extinct. Proposition 2.6 tells us that evolutionary game theory does not provide a ground for such claim. The intuition is similar to the one in Binmore *et al.* [1995]. When there is drift, the strategies against which the weakly dominated strategy does poorly will have positive weight at all times and therefore the part of the dynamics that depend on payoffs pushes against the dominated strategy. But the drift may provide a direct push in favour of the dominated strategy (and more crucially, in favour of those strategies of the other players which do well against such dominated

[8]See § 1.7.

strategy). When the balance between these two forces is right, one gets a stable equilibrium with positive weight for the dominated strategy, as it happens in our example.

2.6. FICTITIOUS PLAY AND SJÖSTRÖM'S MECHANISM

In this section we shall consider an alternative scenario. Suppose that the players are now endowed with some *belief* about their opponents' strategies, which are constantly updated along the sequence of plays $\zeta(t) = (m(1), m(2), \ldots, m(t))$ which defines the (discrete-time) *history* of the game. In particular, we shall assume that each player *i*, after having put initial (arbitrary) weights $\chi_i(0)$: $M_{-i} \rightarrow (0, \infty)$ to any pure strategy profile of the opponents (which constitutes her initial non-normalized beliefs), will update these beliefs as follows:

$$\xi_i^{m_{-i}}(t) = \frac{\chi_i^{m_{-i}}(0) + \kappa^{m_{-i}}(\zeta(t))}{\sum_{M_{-i}} \chi_i^{m_{-i}}(0) + \kappa^{m_{-i}}(\zeta(t))}$$

with $\kappa^{m_{-i}}(\zeta(t))$ denoting the number of times the pure strategy profile m_{-i} has been observed for a given history $\zeta(t)$. In other words, no matter how these initial beliefs are set, they tend asymptotically toward the empirical frequencies with which each pure strategy profile has been played (and perfectly observed) in the past.

Furthermore, we shall assume that each player selects, at each point in time, the pure strategy which maximizes her expected payoff, given her current beliefs (with ties broken at random), that is $m_i(t) \in \arg\max \sum_{M_{-i}} v_i(m_i, m_{-i})$

This alternative set of assumptions specifies a (discrete-time) version of the classic *fictitious play* dynamics, often proposed as an approximation of a learning model when agents are boundedly rational alternative to the evolutionary dynamics studied hereto.[9] The aim of this section is to study the asymptotics of this alternative dynamics, where now $\xi_i(t) \in \Delta_{-i}$ is to be interpreted as the vector collecting, at each point in time, player *i*'s beliefs about their opponents' strategies. The analysis of fictitious play will be restricted, to be consistent with the rest of the paper, to the case of only completely mixed initial beliefs (since $\xi_i^{m_{-i}}(0) > 0$, for all $m_{-i} \in M_{-i}$). We shall begin by characterizing the asymptotics of fictitious play in the case of Sjöström's example, that is, game Γ:

- PROPOSITION 2.7. For any possible history $\zeta(t)$ of Γ, if players behave according with fictitious play and initial beliefs are completely mixed, there will be a time T after which $m(t) = (m_1^1, m_2^1, m_3^1)$ for all $t > T$.

PROOF. See the Appendix.

[9] The so-called fictitious-play dynamics, firstly introduced by Brown [1951] as an algorithm to compute Nash equilibria, has been recently re-interpreted as a learning model in the works of Fudenberg and Kreps [1992], . Milgorm and Roberts [1990] extend some of the properties of fictitious play to the more general class of adaptive learning dynamics. The results contained in this section come as straightforward applications of the findings of this literature [see, e.g., Fudenberg and Kreps [1992], Proposition 2.2]. We prefer the standard version in discrete-time (as opposed to the rest of the paper) for the sake of simplicity, although it can be shown that the same results proved here still hold if the dynamics is defined in continuous-time.

The argument used to prove Proposition 2.7 can be extended, in a straightforward way, to the case of the general mechanism. Let $\hat{S}_i = \{ m_i \in M_i | R^i_i = \hat{R}_i \}$ denote the set of pure strategies in which each player reveals her true preferences, with $\hat{s} = \{ \hat{s}_i \}$, $i \in \mathfrak{I}$, being the Nash equilibrium in which the true preferences of all players are consistently revealed (i.e. the "solution"). The following proposition holds:

- PROPOSITION 2.8. For any possible history $\zeta(t)$ of the game $(\alpha, \hat{R})$, if players behave according with fictitious play and initial beliefs are completely mixed, there will be a time T after which $m(t) = \{ \hat{s}_i \}$,$i \in \mathfrak{I}$, for all $t > T$.

PROOF. See the Appendix.

Since the results obtained here are so different from the ones in the previous sections it would be interesting to know what are the main reasons behind this difference. In this respect it is interesting to note that fictitious play (a suitably modified version for continuous time) does not satisfy assumption D.3. The reason is that the growth rate of a strategy reacts very strongly to changes in payoffs. The only strategies whose weight in beliefs increase are the best responses, while the weight of the remaining strategies decreases. This implies that there is an infinite response of growth rates to changes in the sign of the differences in payoffs, which is precisely what assumption D.3 rules out. Another key difference is that fictitious play rules out the possibility that the weight of a strategy which has initial positive weight becomes zero. This is important because what keeps an agent from lying about herself is the possibility that the neighbors also lie about her type. But suppose an agent never sees (or has only seen very long ago) her neighbors lying about her type and becomes convinced that they will always tell the truth about her. Then she could start lying about her type with positive probability, which doesn't hurt her (but hurts the other players)[10] and this would be stable, or at least it can go on for a very long time, provided that the neighbors don't lie about her type, which is optimal for them as long as she doesn't lie too often about herself. This situation is very negative from the point of view of the planner, who will be punishing, possibly quite harshly, the truth-tellers a non-negligible portion of the time. The final answer to which dynamic system (and therefore which implied limit outcomes and a verdict on the usefulness of the mechanism in the presence of boundedly rational players) is a better model for the situation is empirical. Chen and Tang [1996] have done experiments with the Basic Quadratic mechanism by Groves and Ledyard [1977] and the Paired-Difference mechanism by Walker [1981]. They evaluate the performance of different dynamic learning models in explaining the experimental data by computing the quadratic deviation measure and some other scoring rules, like the absolute deviation measure and the proportion of inaccuracy scores. They try variants of stimulus response learning algorithms, which make the probability with which an individual chooses a strategy a function of the relative payoffs obtained by that strategy and the other strategies in the past.[11] They try a

[10]Anyone who has lived in a condominium knows that residents often derive pleasure from actions that hurt their neighbors, which would make this mechanism somewhat dangerous in such an environment. There is also some evidence that "spite" is a factor in experimental findings about ultimatum bargaining (see Camerer and Thaler [1995], and Abbink, Bolton, Sadrieh and Tang [1996])

[11]See Roth and Erev [1995] for the use of these dynamics with experimental data and Börgers and Sarin [1996] for their relationship with the Replicator dynamics.

linear version similar to the one of Roth and Erev [1995] but modified to accommodate for negative payoffs and some non-linear versions like the ones proposed by Tang [1996], Chen, Friedman and Thisse [1996] and Mookherjee and Sopher [1996]. They show that these dynamics explain the data significantly better, according to quadratic deviation measures and others, than a generalized fictitious play model which can accommodate behaviors ranging from fictitious play to best response dynamics by the estimation of a "forgetfulness parameter" which weights past information.

2.7. Conclusions

We have argued that there is room for doubt about the practicability of one the of the leading examples of implementation with iterated deletion of weakly dominated strategies when agents are boundedly rational. This result complements that obtained by Cabrales [1996] for the Abreu and Matsushima [1994] mechanism. Since Cabrales [1996] uses dynamics that are different from those used here, it would be interesting to check if the results we obtain here extend to Abreu and Matsushima [1994] games. More generally, a deeper study with evolutionary tools of other mechanisms studied in the literature would enhance our understanding of the performance of these mechanisms with boundedly rational agents, a necessary step before mechanisms are used in real life.

Ideally one would like to design a game for which convergence to the preferred social outcome could be guaranteed for the learning protocols that agents use. To achieve this goal, it is necessary to do further empirical and experimental studies that reveal how people adjust their play in games like that studied in detail in this paper. We intend to perform such experimental studies in the future. The history of actual social arrangements may also give clues as to how people learn in such environments. Different mechanisms for public good provision have existed for centuries in many countries. These considerations imply the need for a substantial program of future research.

Appendix

PROOF OF PROPOSITION 2.1. We have already noticed that agent 3 has a weakly dominated strategy (namely, m_3^0). In particular, m_3^1 (truth-telling) makes agent 3 (strictly) better off than m_3^0 (lying), unless agents 1 and 2 coordinate their actions completely, that is, unless they play m_i^0 $i=1,2$ with probability 1 or they play m_i^1 $i=1,2$ with probability 1, (in which case, 3 is completely indifferent). This leads to the following lemma:

- LEMMA 2.1. No strategy profile in which $x_3 \in (0,1)$ can be a Nash equilibrium unless $x_1=x_2=1$ or $x_1=x_2=0$, that is, unless agents 1 and 2 play the same strategy with probability 1.

With this consideration in mind, we construct the proof as follows: we fix the mixed strategy of player 3 and check which mixed strategies for player 1 and 2 can sustain a Nash equilibrium. Noting that

$$\Delta\Pi_1 = u_1(m_1^1,x) - u_1(m_1^0,x) = \frac{1}{12}(x_2(x_3-1)+7x_3-3) \tag{2.3}$$

$$\Delta\Pi_2 = u_2(m_2^1,x) - u_2(m_2^0,x) = \frac{1}{12}(x_1(x_3-1)+7x_3-3) \tag{2.4}$$

we can make the following observations:

a) when $x_3 3/7$, m_i^0 (lying) yields a strictly higher payoff than m_i^1 for $i=1,2$, independently of what the other player does. Therefore, the strategy profiles in NE^0 (and only those) will be Nash equilibria;
b) when $x_3=3/7$, m_1^0 yields a strictly higher payoff than m_1^1 unless $x_2=0$, and $x_2=0$ makes player 1 indifferent between m_1^0 and m_1^1 (a symmetric argument holds for player 2). This excludes the possibility of (1,1,3/7) being a Nash equilibrium of the game, leaving $(0,0,3/7) \in NE^0$ as the unique Nash equilibrium when $x_3=3/7$;
c) when $x_3 \in (3/7,1/2)$ there are no Nash equilibria. This happens because in this case if $x_1=1$, the best response of player 2 is $x_2=0$ and if $x_1=0$, the best response for player 2 is $x_2=1$. However, neither $(0,1,x_3)$ nor $(1,0,x_3)$ can be Nash equilibria when $x_3 \in (3/7,1/2)$ by Lemma 2.1;
d) $x_3=1/2$. By analogy with the case $x_3=3/7$, it is an implication of Lemma 2.1 that (1,1,1/2) $\in NE^1$ is the unique Nash equilibrium when $x_3=1/2$;
e) when $x_3>1/2$ announcing m_i^1 (truth-telling) is optimal for $i=1$ and 2, independently of what the other player does. Thus, the strategy profiles in NE^1 (and only those) will be Nash equilibria.
Since this exhausts all cases the result follows. ■

PROOF OF PROPOSITION 2.2. To prove the proposition, it is enough to show that any interior trajectory converges. This is because, once convergence has been proved, we can apply the standard result "convergence implies Nash under any monotonic selection dynamics" (see, e.g. Weibull, 1995, Theorem 5.2 *(iii)*) to obtain the result.

We start by observing that the fact that the dynamic is forward invariant implies that $x_i(t)$ is always defined and positive, for any nonnegative t. By monotonicity, $x_3(t)$ is also a positive, increasing function of t and bounded above by 1 (since m_3^1 is a weakly dominant strategy), therefore it must converge. This already implies convergence of player 3's mixed strategy. Let us denote $x_i^* = \lim_{t\to\infty} x_i(t)$, when such a limit exists. Three alternative cases have to be discussed:

a) $x_3^*=0$. If $x_3^*=0$ there must be a time t' such that $x_3(t)3/7$ for $t > t'$. This implies that there is a $k>0$ such that for all $t'>$t, $\Delta\Pi_i(x(t))<-k$ for i=1,2. This implies, by monotonicity, $\lim_{t\to\infty} x_i(t)=0$ for $i=1,2$, thus $x^*=(0,0,0)$.

b) $x_3^*=1$. By a similar argument, monotonicity implies $x^*=(1,1,1)$.

c) $x_3^*\in$,(0,1) . We want to prove that x_3^* cannot converge to a value within this range unless the system converges to a Nash equilibrium. To do so, given the special features of our example, it is enough to show that, if $x_3^*\in(0,1)$ it then must be that both players 1 and 2 select, in the limit, the same pure strategy. Given that this result implies convergence of the entire mixed strategy profile, the result follows. More formally, what we have to prove is contained in the following lemma:

- LEMMA 2.2. If $x_3^*\in(0,1)$ then:

either

$x_i^*=0$,i=1,2 (CASE 0 hereafter)

or

$x_i^*=1$,i=1,2. (CASE 1)

PROOF. Let's assume, for the purpose of contradiction, that neither of the above statements is true. In that case, there must exist a sequence $\{t_k\}_{k=1}^{\infty}$ and a positive constant $\varepsilon>0$ such that either $x_i(t_k)>\varepsilon$,i=1,2 or $x_i(t_k)<1-\varepsilon$,i=1,2 for all k (in other words, the system must stay infinitely often an ε away from the faces of Δ in which player 1 and 2 play the same pure strategy). We already noticed that these are the only faces of Δ in which both pure strategies for player 3 yield the same payoff: if the system stays away from them infinitely often along the solution path, it then must be that the cumulative payoff difference will grow unbounded as time goes to infinity. As we will see, this in turn implies (by monotonicity) that $x_3(t)$ will also reach, in the limit, its highest value, that is, $x_3^*=1$, as a result of the extinction of the weakly dominated strategy m_3^0, which is a contradiction.

To show this, we first notice that the payoff difference $\Delta\Pi_i(x(t))$ is a continuous function of $x(t)$ defined over a compact set (Δ). In the case of player 3, such function takes the following form:

$$\Delta\Pi_3(x(t))=\frac{1}{6}((x_1(t)-x_2(t))^2+x_1(t)(1-x_1(t))+x_2(t)(1-x_2(t))) \quad (2.5)$$

Take $g_M=\max_{i\in I,\in x_{-i}\Delta_{-i}}[\,|\,g_i(m_i,x_{-i}(t))\,|\,]$, i.e. the highest possible growth rate (in absolute value) over all strategies and players (we know a max exists, since also $g_i(.)$ is continuous in Δ). Then define τ_1,τ_2,τ_3 and τ_4 as follows:

τ_1 solves $\varepsilon\exp[-g_M\tau_1] = (\varepsilon/2)$ [i.e. $\tau_1=(\ln[2]/g_M]$
τ_2 solves $(1-\varepsilon)\exp[-g_M\tau_2] = (\varepsilon/2)$ [i.e. $\tau_2=\ln[-2+(2/\varepsilon)]/g_M]$
τ_3 solves $\varepsilon\exp[g_M\tau_3] = 1-(\varepsilon/2)$ [i.e. $\tau_3=\ln[-(1/2)+(1/\varepsilon)]/g_M]$
τ_4 solves $(1-\varepsilon)\exp[g_M\tau_4] = 1-(\varepsilon/2)$ [i.e. $\tau_4=\ln[(2-\varepsilon)/(2-2\varepsilon)]/g_M]$

and take $\partial\tau=\min[\tau_1,\tau_2,\tau_3,\tau_4]$, that is, set a lower bound for the time interval in which, after each t_K, $(\varepsilon/2)<x_i<1-(\varepsilon/2)$, $i=1,2$ and therefore $\Delta\Pi_3(x(t))$ still remains bounded away from 0 (i.e $\Delta\Pi_3(x(t))>(\varepsilon(1-\varepsilon/2))/3>0$, for all $t\in[t_k, t_k+\partial\tau]$). Denote by

$$G_\varepsilon=\left\{x\in\Delta\left|\Delta\Pi_3(x)\geq\frac{\varepsilon(1-\frac{\varepsilon}{2})}{3}\right.\right\}.$$

Now define:

$$\gamma_i(x(t))=\frac{d}{dt}\ln\left[\frac{x_i(t)}{1-x_i(t)}\right]=\frac{\dot{x}_i(t)}{x_i(t)-(x_i(t))^2}$$

i.e. the time derivative of the log of the ratio between the probabilities with which each of player i's pure strategies are played, which can be expressed in terms of the difference in the growth rates. Notice that also $\gamma_3(x(t))$ will be a positive number bounded away from 0 infinitely often since, by assumption d.1, the difference in growth rates is a continuous function of $x(t)$ defined on a compact set, which preserves the same sign of $\Delta\Pi_3(x(t))$. This implies that we can always define a constant $g_\varepsilon=\min_{x\in G_\varepsilon}\gamma_3(x(t))$, with $g_\varepsilon>0$ by assumption d.2. Also by assumption d.2, $\gamma_3(x(t))>g_\varepsilon\Leftrightarrow\Delta\Pi_3(x(t))>\frac{\varepsilon(1-\frac{\varepsilon}{2})}{3}$. If we integrate the value of $\gamma_3(x(t))$ over time we then obtain:

$$\lim_{t\to\infty}\int_0^t\gamma_3(x(t))dt\geq\sum_{k=1}^{\infty}\int_{t_k}^{t_k+\partial\tau}\gamma_3(x(t))dt>g_\varepsilon\sum_{k=1}^{\infty}\int_{t_k}^{t_k+\partial\tau}dt=\infty$$

which implies that $x_3^*=1$, which leads to a contradiction.♦

To summarize, Lemma 2.2 shows that, if $x_3{}^* \in (0,1)$, $x_1(t)$ and $x_2(t)$ must converge (and therefore $x(t)$ must converge to a Nash equilibrium). Since this exhausts all cases the result follows.

PROOF OF PROPOSITION 2.3. For all $\hat{x}_i$, such that $\hat{x}_i^{m_i} > 0$ only if $m_i \in S_i$ we have,

$$u_i(\hat{x}_i, x_{-i}) \geq \prod_{j \neq i} x_j^{m_j^*} v_i(f_i(\varphi(i, R^*)), \hat{R}_i) + (1 - \prod_{j \neq i} x_j^{m_j^*}) v_i(0, \hat{R}_i).$$

For all $\bar{x}_i \neq \hat{x}_i$,

$$u_i(\bar{x}_i, x_{-i}) \leq (1 - \sum_{m_i \in S_i} \bar{x}_i) u_i(x_i, x_{-i}) + \sum_{m_i \in S_i} \bar{x}_i [\prod_{j \neq i} x_j^{m_j^*} v_i(0, \hat{R}_i) + (1 - \prod_{j \neq i} x_j^{m_j} U_{in}] .$$

Then

$$u_i(\hat{x}_i, x_{-i}) - u_i(\bar{x}_i, x_{-i}) \geq \sum_{m_i \in S_i} \bar{x}_i [\prod_{j \neq i} x_j^{m_j^*} (v_i(f_i(\varphi(i, R^*)), \hat{R}_i) - v_i(0, \hat{R}_i))$$
$$+ (1 - \prod_{j \neq i} x_j^{m_j^*})(v_i(0, \hat{R}_i) - U_{in})]$$

which is greater than zero since by the definition of k_j,

$$\prod_{j \neq i} x_j^{m_j^*} \geq \prod_{j \neq i} k_j \geq (U_{in} - v_i(0, \hat{R}_i)) / (v_i(f_i(\varphi(i, R^*)), \hat{R}_i) - v_i(0, \hat{R}_i) + U_{in} - v_i(0, \hat{R}_i)).$$

PROOF OF PROPOSITION 2.4. By contradiction.
Suppose that *a)* is the statement that stops being true earliest, that it does it for agent *i* and strategy $m_i \in \bar{S}_i$ and that the boundary time is t'. Then it must be true that

$$\frac{x_i^{m_i}(t')}{x_i^{m_i}(0)} = \exp[-h_g t'] \frac{H}{x_i^{m_i^*}(0)}$$

Notice that for all *t*,

$$u_i(x_i^{m_i}(t), x_{-i}(t)) - u_i(x_i^{m_i^*}(t), x_{-i}(t)) \leq v_i(0, \hat{R}_i) \prod_{j \neq i} x_j^{m_j^*}(t) + U_i(1 - \prod_{j \neq i} x_j^{m_j^*}(t))$$
$$-(v_i(f_i(\varphi(i, R^*)), \hat{R}_i) \prod_{j \neq i} x_j^{m_j^*}(t) + v_i(0, \hat{R}_i)(1 - \prod_{j \neq i} x_j^{m_j^*}(t)))$$
$$= [U_i - v_i(0, \hat{R}_i) - \prod_{j \neq i} x_j^{m_j^*}(t)(v_i(f_i(\varphi(i, R^*)), \hat{R}_i) + U_i - 2 v_i(0, \hat{R}_i))]$$

But since *b)* is true for $t < t'$

$$u_i(x_i^{m_i}(t),x_{-i}(t))\text{-}u_i(x_i^{m_i^*}(t),x_{-i}(t))<[\mathrm{U}_i\text{-}v_i(0,\hat{R}_i)\text{-}H^{n\text{-}1}(v_i(f_i(\varphi(\mathrm{i},R^*)),\hat{R}_i)+U_i\text{-}2v_i(0,\hat{R}_i))]$$

So we have that $u_i(x_i^{m_i}(t),x_{-i}(t))\text{-}u_i(x_i^{m_i^*}(t),x_{-i}(t))<\text{-}h_v$ which by assumption d.2 and the definition of h_v and h_g implies that $g_i(m_i,x_{-i}(t))\text{-}g_i(m_i^*,x_{-i}(t))<\text{-}h_g$, which integrating from 0 to t' and given that $x_i^{m_i^*}(t')\leq H$ implies that

$$\frac{x_i^{m_i}(t')}{x_i^{m_i}(0)}<\exp\left[-h_g t'\right]\frac{H}{x_i^{m_i^*}(0)}$$

This is a contradiction.

Suppose that *b)* is the statement that stops being true earliest, that it does it for agent i and that the boundary time is t'. Then it must be true that $x_i^{m_i^*}(t')=H$.
First notice that for all $m_i\in \mathrm{S}_i\backslash\{m_i^*\}$ since the payoffs of strategy m_i^* and other strategies in S_i differ only when playing against strategies not in S_{-i}

$$u_i(x_i^{m_i^*}(t),x_{-i}(t))\text{-}u_i(x_i^{m_i}(t),x_{-i}(t))>\text{-}\mathrm{U}_i\prod_{j\neq i}x_j^{\bar{S}_j}(t)$$

since *a)* holds for $t<t'$

$$u_i(x_i^{m_i^*}(t),x_{-i}(t))\text{-}u_i(x_i^{m_i}(t),x_{-i}(t))>\text{-}U_i(\exp[\text{-}h_g t]\frac{H}{x_i^{m_i^*}(0)}\prod_{j\neq i}x_j^{\bar{S}_j}(0)$$

Since $U_i(\frac{H}{x_i^{m_i^*}(0)}\prod_{j\neq i}x_j^{\bar{S}_j}(0)<\theta$, this implies by assumption d.3 that

$$(\,g_i(m_i^*,x_{-i}(t))\text{-}g_i(m_i,x_{-i}(t)))>\text{-}(\text{-}\ln(U_i\frac{H}{x_i^{m_i^*}(0)}\prod_{j\neq i}x_j^{\bar{S}_j}(0))+h_g t)^{-\delta}$$

So by integration we have that

$$\frac{x_i^{m_i^*}(t')}{x_i^{m_i}(t')}\frac{x_i^{m_i}(0)}{x_i^{m_i^*}(0)}>\exp\left[\frac{1}{h_g(\delta-1)\left(-\ln\left[U_i\frac{H}{x_i^{m_i^*}(0)}\prod_{j\neq i}x_j^{\bar{S}_j}(0)\right]\right)^{\delta-1}}\right]=L$$

Adding over all strategies in S_i we have

$$\frac{x_i^{m_i^*}(t')}{x_i^{m_i^*}(0)}>\frac{x_i^{S_i}(t')}{x_i^{S_i}(0)}L=\frac{1-x_i^{\bar{S}_i}(t')}{1-x_i^{\bar{S}_i}(0)}L\geq L$$

But this implies that $x_i^{m_i^*}(t') > H$ (using the assumption that $x_i^{m_i^*}(0)\, L > H$), which is a contradiction.

Suppose that *c)* is the statement that stops being true earliest, that it does it for agent *i* and that the boundary time is t'. Then it must be true that $\frac{x_i^{m_i^*}(t')}{x_i^{m_i}(t')} = L\frac{x_i^{m_i}(0)}{x_i^{m_i^*}(0)}$

As before, notice that for all $m_i \in S_i \backslash \{m_i^*\}$ the payoffs of strategy m_i^* and m_i differ only when playing against strategies not in S_{-i}, so

$$u_i(x_i^{m_i^*}(t), x_{-i}(t)) - u_i(x_i^{m_i}(t), x_{-i}(t)) \leq U_i \prod_{j \neq i} x_j^{\bar{S}_j}(t)$$

which by part *a)* of the proposition implies that for all $t < t'$

$$u_i(x_i^{m_i^*}(t), x_{-i}(t)) - u_i(x_i^{m_i}(t), x_{-i}(t)) \leq U_i(\exp[-b_g t] \frac{H}{x_i^{m_i^*}(0)} \prod_{j \neq i} x_j^{\bar{S}_j}(0))$$

Since $U_i(\frac{H}{x_i^{m_i^*}(0)} \prod_{j \neq i} x_j^{\bar{S}_j}(0)) < \theta$, this implies by assumption d.3 that

$$(g_i(m_i^*, x_{-i}(t)) - g_i(m_i, x_{-i}(t))) > -(-\ln[U_i \frac{H}{x_i^{m_i^*}(0)} \prod_{j \neq i} x_j^{\bar{S}_j}(0)] + b_g t)^{-\delta}$$

So by integration we have that

$$\frac{x_i^{m_i^*}(t')}{x_i^{m_i}(t')} < L\frac{x_i^{m_i^*}(0)}{x_i^{m_i}(0)}$$

which is a contradiction.
Since this exhausts all cases the result follows.♦

PROOF OF PROPOSITION 2.5. *(i)* We know, from Proposition 2.2, that $\dot{x}_3 > 0$ in any interior point. This implies that if there is a time *t* such that $x_3(t) > 1/2$, then $x_3(t') > 1/2$ for all $t' > t$. From equations (2.3-4) we have that, whenever $x_3(t) > 1/2$, $\Delta\Pi_i(x) > 0$ for $i = 1,2$. This implies that if there is a time t such that $x_3(t) > 1/2$, then $\dot{x}_i > 0$ for all $t' >$ t for $i = 1,2$ and therefore $x(t)$ converges. Since convergence must be to a Nash equilibrium and x_1 and x_2 have been increasing, *x* converges to NE^1. To show the stability of NE^1 it suffices to show that there is a neighbourhood of NE^1 such that, for all $x(0)$ in this neighbourhood, there is a time *t* such that $x_3(t) > 1/2$. Let $x_i(0) = 1 - \varepsilon_i$ for i=1,2 and $x_3(0) = 1/2 - \delta$, with $\varepsilon_i > 0$, $\delta > 0$. From (2.3-4) we also have that $-1 < \Delta\Pi_i(x) < 1$ for $i = 1,2$, thus

$$\exp[-t](1-\varepsilon_i) < x_i(t) < \exp[t](1-\varepsilon_i) \tag{2.6}$$

Since $\Delta\Pi_3(x) \leq x_1(1-x_1)/6$ we have by equation (2.6):

$$\frac{\dot{x}_3(t)}{x_3(t)} > \frac{\exp[-t](1-\varepsilon_i)(1-\exp[t](1-\varepsilon_i))}{6}$$

thus

$$\frac{\dot{x}_3(t)}{x_3(t)} > \frac{(1-\varepsilon_1)(\exp[-t]-(1-\varepsilon_1))}{6} > \frac{(1-\varepsilon_1)(-t+\varepsilon_1)}{6}$$

This implies that

$$x_3(t) > \exp\left[\frac{(1-\varepsilon_1)(-\frac{t^2}{2}+\varepsilon_1 t)}{6}\right]\left(\frac{1}{2}-\delta\right)$$

Note that for $t=\varepsilon_1$:

$$\exp\left[\frac{(1-\varepsilon_1)(-\frac{t^2}{2}+\varepsilon_i t)}{6}\right]\exp\left[\frac{(1-\varepsilon_1)\frac{\varepsilon_1^2}{2}}{6}\right]>1$$

and therefore $x_3(t) > 1/2$ for δ small enough, which is what we wanted to show.
(ii). Assume that $x_3(0) > 3/7$. Since $\dot{x}_3 > 0$ for all t, $x_3(t)$ is an increasing function of t, therefore it must converge. Since the initial condition $x_3(0)$ is larger than 3/7 it must converge to a number larger than 3/7. We know that $x(t)$ converges to a Nash equilibrium by Proposition 2.2. Since there is no equilibrium in NE^0 with $x_3 > 3/7$, $x(t)$ cannot converge to a point in NE^0. Since $x_3(0)$ can be arbitrarily close to 3/7 and therefore to the set NE^0, this set must be unstable.♠

PROOF OF PROPOSITION 2.6. The proof is constructed as follows. We first characterize the limit of the set of rest points $\hat{RE}(\beta)$, and then analyze the stability properties of each of its elements.

We start by observing that, given $\beta \in (0,1)$, any rest point must be completely mixed, and it also must be $x_3 > \beta$, as $\Delta\Pi_3(.)$ is always positive in the interior of the state space Δ (because m_3^0 is a weakly dominated strategy). We also know, by the continuity of the vectorfield with respect to λ, that every limiting rest point of the dynamic, as λ goes to zero, must lie in the set of restpoints of the unperturbed dynamic $RE(\Gamma)$.
We analyze first the limit set of rest points under CASE 0. In this case, both players 1 and 2 play their strategy m_i^0 with probability 1, that is $x_i^0=0$, for $i=1,2$. Setting $\dot{x}_1=0$ yields the following equation:

$$x_1/\lambda = 12(\,1/2\text{-}x_1)/(\,1\text{-}x_1)\,(\,3+x_1\text{-}x_3(7\text{-}x_2)) \tag{2.7}$$

and an analogous expression can be obtained for x_2/λ. Denote by $x_3{}^0$ a limiting value in a rest point, if a limit exists, for x_3. When the limiting values for x_1 and x_2 are zero we have:

$$\lim_{\substack{x_i \to 0 \\ \lambda \to 0}} x_i/\lambda = 1/(2(3\text{-}7x_3{}^0)) \tag{2.8}$$

Notice that in this case if a rest point exists it must be $x_3{}^0<3/7$, since $x_i/\lambda>0$. We then set $\frac{\dot{x}_3}{\lambda}=0$, substitute $\frac{x_i}{\lambda}$ with the expression in (2.8), solve for x_3, and substitute x_i, $i=1,2$ and λ by their limiting value of zero. The solutions for $x_3{}^0$ take the following form:

$$\hat{x}_3^0 = \frac{3+4\beta+\sqrt{9-16\beta(1-\beta)}}{10} \quad \text{and} \quad \check{x}_3^0 = \frac{1+7\beta-\sqrt{1-\beta(46\beta)}}{10}$$

Remember that $x_3{}^0$ must be a real, positive number, with $\beta< x_3{}^0<3/7$. For the expression under the square root at the numerator to be nonnegative, it must be that $\beta \in [0,(23-4\sqrt{30})/49]$, which determines the feasible range for both roots. Within this interval of values for β, $\hat{x}_3^0$ ($\check{x}_3^0$) is a strictly decreasing (increasing) function of β, which has a minimum and a maximum, whose values are $(15-2\sqrt{30})/35$ (0) and 2/10 ($(15-2\sqrt{30})/35$) respectively. As $\beta \to (23-4\sqrt{30})/49$ both solutions converge to $(15-2\sqrt{30})/35$.

We now deal with the subset of limiting rest points under CASE 1, i.e. with limiting values for $x_i=1$ for i=1,2. The equations corresponding to (2.7-8) are now the following:

$$(1\text{-}x_1)/\lambda = (\,1/2\text{-}x_1)/(x_1(1/3+(1\text{-}x_3)((1/12)(1\text{-}x_2)\text{-}(2/3))) \tag{2.9}$$

$$\lim_{\substack{x_i \to 0 \\ \lambda \to 0}} (1\text{-}x_1)/\lambda = 1/2(1/3\text{-}(2/3)(1\text{-}x_3)) \tag{2.10}$$

Denote by $x_3{}^1$ a limiting value in a rest point for x_3 in this latter case. By analogy with CASE 0, we know from (2.9) that, if a rest point exists, it must be $x_3{}^1>1/2$. There is a unique feasible solution for $x_3{}^1$, for all $\beta\in(\,0,1)$ with the following form:

$$\hat{x}_3^1 = \frac{3+4\beta+\sqrt{9-16\beta(1-\beta)}}{10}$$

Following the same procedure for the remaining rest points of the unperturbed dynamics (i.e. the pure strategy profiles which belong to $RE(\,\Gamma)$ and do not satisfy either CASE 0 or CASE 1) does not add any element to the limiting set of rest points of the perturbed dynamics. This should not be surprising, as any other rest point of the unperturbed Replicator Dynamics is unstable with respect to the interior. Since this exhausts all cases, the result follows.

We now move to establish the stability properties of each limiting restpoint separately. The Jacobian matrix for the dynamic system is as follows:

$$J(x,\lambda)=\begin{pmatrix} (1-2x_1)\Delta\Pi_1-\lambda & \frac{-(1-x_1)x_1(1-x_3)}{12} & \frac{(1-x_1)x_1(7+x_2)}{12} \\ \frac{-(1-x_2)x_2(1-x_3)}{12} & (1-2x_2)\Delta\Pi_2-\lambda & \frac{(1-x_2)x_2(7+x_1)}{12} \\ \frac{(1-2x_2)x_3(1-x_3)}{6} & \frac{(1-2x_1)x_3(1-x_3)}{6} & (1-2x_3)\Delta\Pi_3-\lambda \end{pmatrix}$$

We analyze CASE 0 first. We know that, in this case, we have two restpoints, which we call $\hat{x}^0=(0,0,\hat{x}_3^0)$ and $\breve{x}^0=(0,0,\breve{x}_3^0)$. We evaluate the Jacobian when x_1, x_2 and λ are equal to their limiting value (i.e. 0). The corresponding eigenvalues are: $\{0,(-3+7x_3^0)/12,(-3+7x_3^0)/12\}$. There are then two (identical) negative eigenvalues (since any limiting $x_3^0<3/7$ for CASE 0), while the third eigenvalue is equal to zero. To determine the stability properties of the perturbed system, the sign of the eigenvalue whose limit is zero becomes crucial given that the continuity of $J(.)$ ensures that the other two will be negative, for any λ sufficiently small. We now linearize the rest points (as a function of λ) around NE^0. We set $\tilde{x}(\lambda)=(\delta_1\lambda\ ,\delta_2\lambda\ ,x_3^0+\delta_3\lambda\ )$, where $\delta=(\delta_1,\delta_2,\delta_3)$ denotes the vector collecting the coefficients of the linearised system. We then evaluate the following expression:

$$\phi^0(x_3^0,\delta)=\lim_{\lambda\to 0}\frac{\partial \det(J(x,\lambda))}{\partial\lambda}$$

We do so because $\det(J(x,\lambda))$, which is equal to zero for all $x\in NE^0$, will preserve the sign of the third eigenvalue, given that the sign of the other two will stay constant (and negative) when x is sufficiently close to NE^0 and λ is sufficiently small. For CASE 0 we get the following result:

$$\phi^0(x_3^0,\delta)=(-54+x_3^0(252+294x_3^0)+(\delta_1+\delta_2)(9-39x_3^0+63(x_3^0)^2-49(x_3^0)^3)/864 \qquad (2.11)$$

We first notice that (2.11) does *not* depend on δ_3. To evaluate $\text{sign}(\phi^0(x_3^0,\delta))$ we only need to get estimates of δ_1 and δ_2, the linear coefficients which measure the responsiveness of the equilibrium values of x_i, $i=1,2$ to small changes in λ . We do so setting $\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\tilde{D}(x,\lambda)\big|_{\tilde{x}(\lambda,\delta)}=0$ and solving for $\{\delta_1,\delta_2,x_3^0\}$. There are two alternative set of solutions, each of them corresponds to each of the restpoints. In particular:

$$\breve{\delta}_1^0=\breve{\delta}_2^0=\frac{23-49\beta-7\sqrt{1-\beta(46-49\beta)}}{8}$$

$$\hat{\delta}_1^0=\hat{\delta}_2^0=\frac{23-49\beta+7\sqrt{1-\beta(46-49\beta)}}{8}$$

We evaluate the numerator of (2.11) for both sets of solutions, and we get the following expressions::

$$\breve{\phi}(\beta)=\frac{3\left(-7+322\beta-343\beta^2+(49\beta-23)\sqrt{1-46\beta-49\beta^2}\right)}{10}$$
$$\widehat{\phi}(\beta)=\frac{3\left(-7+322\beta-343\beta^2-(49\beta-23)\sqrt{1-46\beta-49\beta^2}\right)}{10} \tag{2.12}$$

Both $\breve{\phi}(\beta)$ and $\widehat{\phi}(\beta)$ are plotted in Figure 2.4. As the diagram shows, $\breve{\phi}(\beta)$ is always negative in the domain $[0,(23-4\sqrt{30})/49]$, whereas $\widehat{\phi}(\beta)$ is not. As a result of that, $\breve{\phi}(\beta)$ is asymptotically stable whereas $\widehat{\phi}(\beta)$ is not.

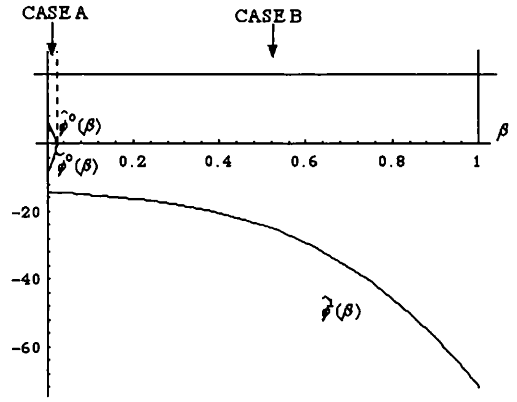


FIGURE 24
Asymptotic stability of the dynamic with drift

We now move on to CASE 1. Here we have a unique rest point, which we call $\hat{x}^1=(1,1,\hat{x}_3^{\,1})$.The eigenvalues of the unperturbed dynamics are as follows: $\{0,(1\text{-}2x_3)/3,(1\text{-}2x_3)/3\}$. As in CASE 0, there are two (identical) negative eigenvalues (given that $x_3>1/2$), and the remaining eigenvalue equal to zero. By analogy with CASE 0, we now define $\tilde{x}(\lambda)=(1\text{-}\delta_1\lambda,1\text{-}\delta_2\lambda\ ,x_3^{\,0}+\delta_3\lambda\ )$ and solve $\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\tilde{D}(x,\lambda)\big|_{\tilde{x}(\lambda,\delta)}=0$ to get estimates of δ. The unique feasible solution (corresponding to the unique limiting equilibrium), takes the following form:

$$\hat{\delta}_1^0=\hat{\delta}_2^0=\frac{3\left(2-4\beta+\sqrt{9-16\beta+16\beta^2}\right)}{2}$$

The function corresponding to (2.12) takes now the following form:

$$\hat{\phi}^1(\beta)=\frac{24(-\alpha+(2-4\beta)\sqrt{\alpha}}{5}$$

with $\alpha = 9 - 16\beta$. The function $\hat{\phi}^1(\beta)$ is also plotted in Figure 2.4. As the diagram shows, $\hat{\phi}^1(\beta)$ is negative for all $\beta \in (0,1)$. As a result of that, $\hat{x}^1$ is asymptotically stable under any drift configuration.

PROOF OF PROPOSITION 2.7. Given m_3^0 is weakly dominated by m_3^1, it will never be played if player 3's initial beliefs are completely mixed. As a consequence of that, its weight will decrease monotonically in player 3 's opponents beliefs as $t \to \infty$. Take the role of player 1 and define:

$$T_1 = \min_{t \in (1,2,\dots)} \left[\xi_1^{(m_2^0, m_3^1)}(t) + \xi_1^{(m_2^1, m_3^1)}(t) > \frac{1}{2} \right]$$

We know, from (2.3), that for all $t > T_1$, player 1 has an optimal strategy (namely, m_1^1), regardless of what player 2 does (since (2.3) will stay positive from T_1 on). Therefore, for all $t > T_1$, player 1 will follow player 3 in delivering the true message, independently of player 2's choice. A symmetric argument for player 2 completes the proof (setting $T = \max[T_1, T_2]$)

PROOF OF PROPOSITION 2.8. By analogy with Proposition 2.7, if beliefs are completely mixed, only strategies in $\hat{S}_i$ will be selected, since this set corresponds, for each player, to the set of undominated strategies. Let $T_1 = \min_{t \in (1,2,\dots)} \left[\sum_{M_{-i} \in \hat{S}_{-i}} \xi_1^{m_{-i}}(t) > k_i \right]$, that is, the point in time in which the set $\hat{S}_{-i}$ accumulates enough weight in player i's belief to make $\hat{s}_i$ the unique best response. We know that $m_i(t) = \{\hat{s}_i\}$, for all $t > T_i$ and this completes the proof (setting $T = \max_{i \in I}[T_i]$).

CHAPTER 3
CYCLES OF LEARNING
IN THE CENTIPEDE GAME

3.0. ABSTRACT

Traditional game theoretic analysis proposes backward induction as a model of rational behavior in games with perfect information. However, counterintuitive results have cast doubt on the predictive power of the theory. For example, in the Centipede Game, experimental evidence shows that subjects' behavior significatively differs from what the theory expects.

In our paper, we construct a dynamic model based on the Centipede Game. Our claim is that the source of these discrepancies between theory and experimental evidence can be explained by appealing to some form of bounded rationality. Traditional game theoretical analysis could then still accurately predict the players' behavior, provided that they are given time enough to appreciate the strategic environment in which they operate. We prove convergence to the subgame-perfect equilibrium outcome for any monotonic continuous-time selection dynamics (Nachbar [1990]). By introducing perturbations, we also show that such adjustment processes are intrinsically unstable, and study how this instability is positively related with the length of the game.

3.1. INTRODUCTION

Inspired by Nachbar [1990] and Cressman [1996] works on the Prisoner's Dilemma, this paper explores the properties of an evolutionary model based on the *Centipede Game,* first introduced by Rosenthal [1981]. Its extensive form is shown in Figure 3.1.1.

In the class of games we investigate, the use of backward induction (or alternatively, the iterative deletion of weakly dominated strategies) selects a unique Nash equilibrium, which is the subgame perfect equilibrium of the game. This equilibrium requires the two players to adopt the strategy of opting out at each information set. Since the game is characterized by a unique subgame perfect equilibrium, the latter is also trembling-hand perfect (Selten [1975]) and proper (Myerson [1978]) when you consider the appropriate normal forms. Moreover, every other Nash equilibrium of the game is outcome-equivalent to the subgame perfect strategy profile. All the most popular solution concepts therefore appear (for once!) to agree on a unique outcome. However, there are clear benefits to the players if, for some reason, they deviate from this prediction: both are always better off if the game continues for at least two more stages.

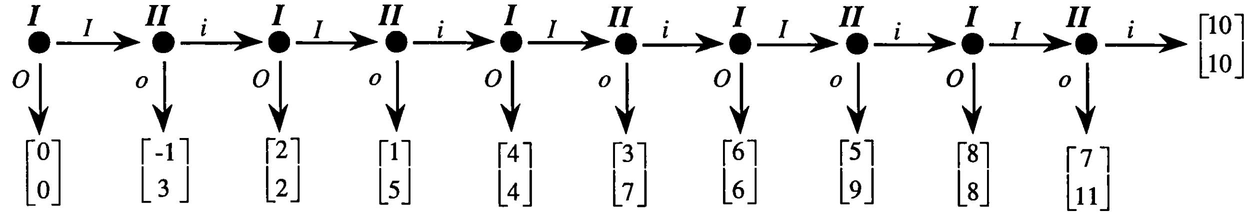


FIGURE 3.1.1
Rosenthal's Centipede Game

This *backward induction paradox* has been the focus of a debate about the logic and the rationale of backward induction. Recent papers include Aumann [1995], Battigalli [1997], Ben Porath [1994], Binmore [1996] and Reny [1993]. The urgency of this debate is underlined by the experimental literature which confirms that subjects deviate substantially from subgame-perfection when the number of stages is sufficiently large. Recent papers include, among others, McKelvey and Palfrey [1992], Nagel and Tang [1995] and Roth and Erev [1995].[1]

How to justify such discrepancy between theory and the more natural intuition? Rosenthal's [1981] original approach to the problem can be summarized as follows: finite, non-cooperative games with both complete and perfect information should be treated, from the players' viewpoint, as stylized single-person decision problems. The rationality criterion he proposes is, indeed, backward induction, which, nevertheless, should be subject to some modification by the players whenever they recognize that the game actually played is characterized by some degree of incomplete information, the exact characterization of which would be to render the game unmanageably complex for them. Instead of modeling such incomplete information directly, they modify their equilibrium calculations in a way they feel is intuitively appealing. In Rosenthal's example, the players assign a given probability that, at each information set, a suboptimal action is chosen, this probability being negatively related to the payoff difference between the optimal and the suboptimal action. If so, the players may well decide to select a strategy which does *not* require to opt always out, at least for the first stages.

The aim of this paper is to explore the backward induction paradox in the Centipede Game with evolutionary techniques. In the model, the two players are assumed to adjust their (mixed)

[1]McKelvey and Palfrey [1992] analyse their data with a static model based on the "incomplete information" approached of Kreps *et al.* [1982], but their experiment results strongly support the learning hypothesis we pursue in the paper. Commenting their results, the authors notice in fact that: " ***there are some differences between the earlier and later plays of the game in a given treatment which are supportive of the proposition that as subjects gain more experience with the game, their behavior appears "more rational."*** " (p. 809). The experimental studies of Nagel and Tang [1995] and Roth and Erev [1995] also propose, analysing their results, adaptive learning models which has some similarities with the approach pursued here.

strategies according to a continuous-time *Monotonic Selection dynamics* (Nachbar [1990]). This condition requires that the relative frequency of fitter strategies should increase at the expenses of less fit competitors.[2]

The remainder of the paper is arranged as follows. Section 3.2 provides a formal description of the *N-legged* Centipede Game. Section 3.3 sets up the dynamical system as a continuous-time Monotonic Selection dynamics. Section 3.4 explores the asymptotic properties of such dynamics for the *N*-legged Centipede Game. If the initial conditions lie in the relative interior of the state space, any Monotonic Selection dynamics converges to a Nash equilibrium (Theorem 3.4.1). Since all Nash equilibria of this game are outcome-equivalent to the unique subgame perfect equilibrium, players adjusting their behavior according to any Monotonic Selection dynamics will therefore eventually behave as though using backward induction, regardless of their initial behavior.

The proof of Theorem 3.4.1 is an application of Nachbar [1990] and Cressman's [1996] results on the finitely-repeated Prisoner's Dilemma, which show that the Replicator Dynamics (probably the most commonly known and studied Monotonic Selection dynamics) converge to one of the Nash equilibria of the game from any interior initial condition. Like the finitely-repeated Prisoner's Dilemma, the Centipede Game is *weakly* dominance solvable. Samuelson and Zhang [1992] have shown that any Monotonic Selection dynamics converges to the solution of *strictly* dominance solvable games from any interior initial condition, whereas we know of counterexamples which show that same property does not generally hold for games which are only weakly dominance solvable.[3] Our paper describes a class of games in which the iterated deletion of weakly dominated strategies leads to a solution which is outcome equivalent to the strategy profile selected by a commonly popular class of evolutionary selection dynamics.

Section 3.5 is devoted to simulations based on the type of dynamics studied here. In the 3-legged Centipede Game, we see orbits that start close to the Nash equilibrium component, then move away from it and eventually come back. In other words, even though Theorem 3.4.1 guarantees convergence, we show cases in which it is not monotonic. Borrowing the term from Binmore *et al* [1989] we interpret this phenomenon in terms of *unlearning.* Although the players seem to understand backward induction initially, because they mostly opt out immediately, they gradually learn that they can earn more by opting in at the first move, and it is only after they have learned to opt out at the third and then the second stage, that they return opting out at the first stage.

Is such mimicking of the backward induction procedure more likely to occur the longer is the Centipede? This is one of the questions one would address, given these first simulation results. The problem is clearly related to the local stability properties of the subgame perfect equilibrium outcome, that is, the properties of the vector field characterizing the dynamic process sufficiently "close" to the subgame perfect strategy profile. In an independently conducted study, Cressman and Schlag [1995] analyse conditions for convergence and stability for the subgame perfect equilibrium outcome of games with perfect information without

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[2]There is a growing literature which explores the conditions under which dynamics analogous to the one studied here can approximate a learning adjustment process. See, among others, Börgers and Sarin [1993], Cabrales [1993], and Schlag [1994].

[3]As it will be explained later -see (3.4.2)- we only consider dominance relations between *pure* strategies. On the behavior of pure strategies strictly dominated by *mixed* strategies, see Hofbauer and Weibull [1996].

relevant ties, of which the Centipede is a special case. In the case of the Replicator Dynamics they provide a sufficient condition for local (asymptotic) stability of the backward induction outcome which they call *simplicity*. In their terminology, a simple Centipede Game must have at most three legs. The intuition they provide is the following: if the Centipede is longer, then learning how to select the subgame perfect outcome ***might be difficult***, and boundedly rational players might persist in playing strategies that are not justified if the backward induction procedure is applied correctly, even if this sort of behavior disappears in the long run.

Their analysis leaves open the qualitative features of the adjustment process *(i)* when the learning dynamics are slightly "perturbed" and *(ii)* in the case of longer Centipede Games. We tackle this problem in the following way. We run simulations of a modified version of the dynamics analysed in Section 4 using a perturbed version of the Replicator Dynamics which "forces" the players to adopt a completely mixed strategy, regardless of their initial behavior, and no matter how each strategy performs against the current opponent's profile. Following Binmore and Samuelson [1995], this perturbation is called ***drift***. Its role is to open the model to the possibility of a ***heterogeneity*** of behaviors, which we think reasonable in every social environment populated by boundedly rational agents.[4] The source of this heterogeneity is left unmodeled here; following the standard literature in the field, we attribute the drift to unexplained *mutations*, and simply check how the model reacts to the introduction of such a perturbation.

In the three-legged Centipede Game, when drift is reinforced the ***unlearning*** phenomenon is enhanced, as the dynamics exhibit limit cycles. Moreover, even with a relatively small amount of drift, we find that the cyclic behavior described for the three-legged Centipede Game gets repeated over time, and the longer the game, the more it gets repeated. Increasing the length of the game also has the effect of increasing the average payoff of the players. This effect, which is inconsistent with the backward induction prediction, is again supported by the experimental evidence in the field. We interpret such trajectories as *cycles of learning*. The existence of such cycles would seem to support Rosenthal's intuition that backward induction will not predict the play of agents who do not reason perfectly. Finally, Section 3.6 concludes.

3.2. The Centipede Game.

The aim of this section is to provide a simple characterization of the N-legged Centipede Game. It is a 2-player game with perfect information and N moves that alternate between players. The formal condition associated with the so-called "centipede structure" is that there exists an information set such that the set of its predecessors (including itself) coincides with the set of decision nodes of the game. This particular feature allows us to adopt the following notation:

$i \in \Im \equiv \{I, II\}$ denotes a generic player, with $-i$ indicating her opponent;
N is the number of 'legs' of the centipede (we assume $N > 1$);

[4]The notion of stability we appeal is therefore stability with respect of small perturbation of the vector field, conventionally defined as *structural stability*.

Θ is the set of *stages* of the game, i. e. $\Theta \equiv \{1,...,N+1\}$[5];
$u_i : \Theta \to \Re$ is the payoff function for player i, that is the reward she receives when the game ends at stage θ; $\theta \in \Theta$.

We now look at the payoff ranking. We require that, at any stage, both players are better off if the game continues for more than one stage:

$$u_i(\theta + n) > u_i(\theta - 1);\ n \geq 1 \tag{3.2.1}$$

Condition (3.2.1) formalizes a natural property of the Centipede Game: the player who is entitled to move has always an incentive to opt in, conditional on the opponent doing the same in the following round. However, opting out is always optimal if the opponent is doing the same in the following stage:

$$\begin{aligned} u_i(\theta) > u_i(\theta+1) \text{ when } \delta_i^\theta = 1 \\ u_i(\theta) < u_i(\theta+1) \text{ when } \delta_i^\theta = 0 \end{aligned} \tag{3.2.2}$$

where δ_i^θ is a Kronecker delta function that takes the value 1 when i and θ have the same parity, and 0 otherwise. To complete the description of the game, we need to define the strategy set for both players. We introduce the following restriction: we group together all the equivalent pure strategies, i.e. the strategies that lead, for each player, to the same probability distribution over the terminal nodes for all the pure strategies of the opponent. In other words, we shall consider only the strategies of the reduced normal form, i. e. the following:

$$\begin{aligned} S_I \equiv \{s_I^1, s_I^3, ..., s_I^{N+1}\} \\ S_{II} = \{s_{II}^2, s_{II}^4, ..., s_{II}^{N+1}\} \end{aligned}$$

with $S = S_I \times S_{II}$. For any $i = I, II$, $s_i^\theta \in S_i$ denotes the pure strategy that player i adopts when she opts out at stage θ. With an abuse of notation, we identify, for both players, the strategy s_i^{N+1} as the «*always opt in*» behavior. Analogously, the symbol σ_i identifies, for player i, a mixed strategy, with σ_i^θ denoting the probability attached to the pure strategy s_i^θ under σ_i.

We also need to specify the relation between a generic (pure) strategy profile, and the corresponding outcome. This is formalized by means of the outcome function $v : S \to \Theta$ which has the following properties:

$$v\left(s_I^\theta, s_{II}^{\theta'}\right) = \begin{cases} \theta \text{ when } \theta \leq \theta' \\ \theta' \text{ otherwise} \end{cases}$$

[5]We prefer the terminology of *stages* to indicate the terminal nodes simply to stress the natural ordering provided by the sequential structure of the game.

in fact, as can be seen by looking at the extensive form of Figure 3.2.1, the outcome of the game is determined by the player who decides to opt out first.[6]

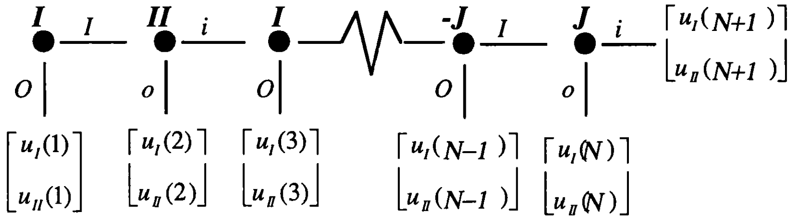


FIGURE 3.2.1
The N-legged Centipede Game

We quote, for the sake of reference, the following standard results, the proofs of which are omitted:

- PROPOSITION 3.2.1. The N-legged Centipede Game exhibits the following properties:

- A unique subgame perfect equilibrium, namely (s_I^1, s_{II}^2);
- A (unique) *component*, that is a closed and connected set, of Nash equilibria with the common property that player I plays strategy s_I^1 with probability 1.

Throughout the paper, the symbol $\mathcal{NE}$ will denote such a component, the exact characterization of which clearly depends on the length of the Centipede, N.

The game has another property of interest which can be summarized as follows. If J is the player who is asked to move last (i. e. $J \equiv \{i \in \{I, II\} | \delta_i^N = 1\}$), then J has an incentive to opt out at stage N. Hence, opting in at stage N, i.e. s_J^{N+1}, is a weakly dominated strategy. A clear implication of this property is that the game is (weakly) *dominance solvable,* in the sense that it can be reduced to a single cell (the subgame perfect outcome) by the iterated deletion of weakly dominated strategies. Unlike other weakly solvable games, this latter procedure is unique here and correctly reproduces the backward induction argument: at each stage, only the player which is asked to move last has a weakly dominated strategy that can be deleted, actually reducing the game to the $N-1$ Centipede Game, where the same argument can be re-applied, in a recursive fashion.

3.3. THE DYNAMICS

[6]With another abuse of notation, we will use the symbol $u_i(\cdot)$ not only for the function $u_i : \Theta \to \Re$, but also the compound function $v \circ u_i$. In this latter case, the symbol $u_i(s_I, s_{II})$ will indicate the payoff received by player i given the strategy profile (s_I, s_{II}).

We now move on to the characterization of the evolutionary dynamics we analyze. We formalize players' behavior in terms of the mixed strategy they adopt at each point in time. Denote with $r_i^\theta(t)$ the probability with which player i selects her pure strategy s_i^θ at time t, with $r_i(t) \equiv \left(r_i^{1+\delta_i^{II}}(t), r_i^{3+\delta_i^{II}}(t), \dots, r_i^{N+1}(t)\right)$ denoting the vector collecting such probabilities. We then have $r_i(t) \in \Delta_i$, with Δ_i denoting $|S_i - 1|$-dimensional simplex which describes player i's mixed strategy space. We also interpret the vector $r(t) \equiv (r_I(t), r_{II}(t))$ as the *state of the system* at time t, defined over the state space $\Delta \equiv \Delta_I \times \Delta_{II}$, with Δ^0 denoting the relative interior of Δ. By analogy, Δ_i^0 will denote the set of completely mixed strategies of i.

- ASSUMPTION 3.3.1. The evolution of $r(t)$ is given by the following system of continuous-time differential equations:

$$\dot{r}_i^\theta(t) = f_i^\theta(r(t)) \tag{3.3.1}$$

We refer to the autonomous system $f \equiv (f_I, f_{II}) \equiv ((f_I^1, \dots, f_I^{N+1}), (f_{II}^2, \dots, f_{II}^{N+1}))$ as the *selection dynamics,* i. e. the term that captures the relevant forces that govern the players' strategy revisions. As in chapter 2, we will require that (3.3.1) satisfies the monotonicity condition (1.3.3):

- ASSUMPTION 3.3.1. f is a regular Monotonic Selection dynamics (MS hereafter).

In the following section, we shall explore the asymptotic properties of any MS in the case of the N-legged Centipede Game.

3.4. THE MS DYNAMICS AND THE CENTIPEDE GAME

In this section, we examine the asymptotic properties of any MS when initial conditions lie in the relative interior of the state space Δ:

- ASSUMPTION 3.4.1. $r(0) \in \Delta^0$

Assumption 3.4.1 excludes the possibility that the selection dynamics, given that the system is forward invariant, act only on a subset of the strategy set S (i.e. acts on a game which might be qualitatively different from the game object of study).

As noted previously, the Centipede Game is weakly dominance-solvable. We shall therefore begin by specifying the relation between weak dominance and the asymptotic behavior of any MS dynamics. Before we proceed, some further terminology is needed. Consider a finite normalform game $\Gamma = \{\Im, S_i, u_i\}$. We say that a pure strategy $s_i^\theta \in S_i$ is said to be *strictly* dominated by some pure strategy $s_i^{\theta'}$ if it yields a smaller payoff against any mixed strategy in the support of the opponent:

$$u_i(s_i^\theta, \sigma_{-i}) < u_i(s_i^{\theta'}, \sigma_{-i}); \forall \sigma_{-i} \in \Delta_{-i} \tag{3.4.1}$$

Analogously, s_i^θ is *weakly* dominated by $s_i^{\theta'}$ if $u_i(s_i^\theta,\sigma_{-i}) \le u_i(s_i^{\theta'},\sigma_{-i});\forall\sigma_{-i}\in\Delta_{-i}$ with strict inequality holding for some $\sigma_{-i}\in\Delta_{-i}$ and, *a fortiori,* for any completely mixed strategy of the opponent:

$$u_i(s_i^\theta,\sigma_{-i}) < u_i(s_i^{\theta'},\sigma_{-i});\forall\sigma_{-i}\in\Delta_{-i}^0 \qquad (3.4.2)$$

The following definition makes use of this property and applies the notion of dominance under weaker conditions, that is, *along the solution path:*

• DEFINITION 3.4.1. Let $r(r(0),t)$ be the interior solution of a MS dynamics $\dot r = f(r(t))$. A pure strategy s_i^θ is said to be strictly τ-dominated by some pure strategy $s_i^{\theta'}$ ($s_i^\theta <_\tau s_i^{\theta'}$ hereafter) if we can identify a time τ and a non-empty compact set $C_{-i}\subseteq\Delta_{-i}$ s. t.:

$$r_{-i}(t)\in C_{-i};\forall t>\tau; \qquad (3.4.3)$$
$$u_i(s_i^\theta,\sigma_{-i}) < u_i(s_i^{\theta'},\sigma_{-i});\forall\sigma_{-i}\in C_{-i}\,; \qquad (3.4.4)$$

Moreover, s_i^θ is said to be weakly τ-dominated by $s_i^{\theta'}$ ($s_i^\theta \le_\tau s_i^{\theta'}$ hereafter), if (3.4.3) holds and we substitute (3.4.4) with the following:

$$u_i(s_i^\theta,\sigma_{-i}) \le u_i(s_i^{\theta'},\sigma_{-i});\forall\sigma_{-i}\in C_{-i} \qquad (3.4.5)$$
$$u_i(s_i^\theta,\sigma_{-i}) < u_i(s_i^{\theta'},\sigma_{-i});\forall\sigma_{-i}\in C_{-i}^0 \qquad (3.4.6)$$

where, by analogy, C_{-i}^0 denotes the relative interior of C_{-i}.

Let $\omega_i(r(0))$ the ω-limit set for player i of an interior solution $r(r(0),t)$; i.e. $\omega_i = \{\sigma_i\in\Delta_i | r_i(r(0),t)\to\sigma_i \text{ for some sequence } t_k\to\infty\}$. The following proposition holds:

• PROPOSITION 3.4.1. Let $r(r(0),t)$ be the interior solution of a MS dynamics $\dot r = f(r(t))$. If $s_i^\theta \le_\tau s_i^{\theta'}$ then:

- $$\frac{d}{dt}\left(\frac{r_i^\theta(t)}{r_i^{\theta'}(t)}\right)<0;\forall t>\tau \qquad (3.4.7)$$

- $$\lim_{t\to\infty}\frac{r_i^\theta(t)}{r_i^{\theta'}(t)} = L\ge 0 \qquad (3.4.8)$$

- If $L>0$ then:

$$u_i(s_i^\theta,\sigma_{-i}) = u_i(s_i^{\theta'},\sigma_{-i});\forall\sigma_{-i}\in\omega_{-i}(r(0)). \qquad (3.4.9)$$

• if $s_i^{\theta''} \le_\tau s_i^\theta$, then $s_i^{\theta''} \le_\tau s_i^{\theta'}$ (3.4.10)

- PROOF. We start by observing that, since $r(0) \in \Delta^0$, forward invariance implies that $\frac{r_i^\theta(\tau)}{r_i^{\theta'}(\tau)} > 0$.

- (3.4.7). $\forall t > \tau$, $\frac{d}{dt}\log\left(\frac{r_i^\theta(t)}{r_i^{\theta'}(t)}\right) = \frac{f_i^\theta(r(t))}{r_i^{\theta'}(t)} - \frac{f_i^{\theta'}(r(t))}{r_i^{\theta'}(t)} < 0$ by monotonicity.
- (3.4.8). $\forall t > \tau$, $\frac{r_i^\theta(t)}{r_i^{\theta'}(t)}$ is a positive, continuous, decreasing function of t and bounded below by 0, so it must converge.
- (3.4.9). First notice that (3.4.3) implies that $\omega_{-i}(r(0)) \subset C_{-i}$; thus $u_i(s_i^\theta, \sigma_{-i}) \le u_i(s_i^{\theta'}, \sigma_{-i})$, $\forall \sigma_{-i} \in \omega_{-i}(r(0))$. Assume, for the purpose of contradiction, that there exists some $\sigma_{-i} \in \omega_{-i}(r(0))$ such that $u_i(s_i^\theta, \sigma_{-i}) < u_i(s_i^{\theta'}, \sigma_{-i})$. If so, by regularity of $f(.)$ and (absolute) continuity of $(u_i(s_i^\theta, r_{-i}(t)) - u_i(s_i^{\theta'}, r_{-i}(t)))$ in C_{-i}, there must be a sequence $\{t_k\}_{k=1}^\infty$, and some positive constants ε and Δt such that $(u_i(s_i^\theta, r_{-i}(t)) - u_i(s_i^{\theta'}, r_{-i}(t))) \ge \varepsilon$ within each interval $[t_k, t_k + \Delta t]$.[7] Now recall that, $\forall t > \tau$, also $\frac{d}{dt}\log\left(\frac{r_i^\theta(t)}{r_i^{\theta'}(t)}\right) = \left(\frac{f_i^\theta(r(t))}{r_i^\theta(t)} - \frac{f_i^{\theta'}(r(t))}{r_i^{\theta'}(t)}\right)$ will be a negative number bounded away from 0 infinitely often, since also the difference of growth rates is a Lipschitz continuous function of $r(t)$ defined over a compact set (C_{-i}), which preserves the same sign of $(u_i(s_i^\theta, r_{-i}(t)) - u_i(s_i^{\theta'}, r_{-i}(t)))$. This implies that there must be another positive constant g_ε such that:

$$\left(\frac{f_i^\theta(r(t))}{r_i^\theta(t)} - \frac{f_i^{\theta'}(r(t))}{r_i^{\theta'}(t)}\right) < g_\varepsilon \Leftrightarrow (u_i(s_i^\theta, r_{-i}(t)) - u_i(s_i^{\theta'}, r_{-i}(t))) < \varepsilon$$

which in turn implies

$$\begin{aligned} \lim_{t\to\infty}\log\left(\frac{r_i^\theta(t)}{r_i^{\theta'}(t)}\right) &= \int_0^\infty\left[\frac{f_i^\theta(r(t))}{r_i^\theta(t)} - \frac{f_i^{\theta'}(r(t))}{r_i^{\theta'}(t)}\right]dt \\ &< g_\varepsilon \sum_{k=1}^\infty \int_{t_k}^{t_k+\Delta t} dt = -\infty \end{aligned} \tag{3.4.11}$$

which leads to a contradiction, as the improper integral (3.4.11) does not exist.[8]

- (3.4.10). Let τ' (τ'') and C'_{-i} (C''_{-i}) be the parameters which describe, following Definition 3.4.1, the τ-dominance relation between s_i^θ and $s_i^{\theta'}$ ($s_i^{\theta''}$ and s_i^θ). Take $\hat{\tau} = \max[\tau', \tau'']$ and $\hat{C}_{-i} = C'_{-i} \cap C''_{-i}$. Then, the result follows, since, by (3.4.3-5):

$$r_{-i}(t) \in \hat{C}_{-i}; \forall t > \hat{\tau}; \tag{3.4.12}$$
$$u_i(s_i^{\theta''}, \sigma_{-i}) \le u_i(s_i^\theta, \sigma_{-i}) \le u_i(s_i^{\theta'}, \sigma_{-i}); \forall \sigma_{-i} \in \hat{C}_{-i} \tag{3.4.13}$$

[7] The argument follows, by analogy, the proof of Lemma 2.2.

[8] We omit to consider the constant of integration in the improper integral (3.4.11), as the integral does *not* exist.

$$u_i(s_i^{\theta''},\sigma_{-i}) < u_i(s_i^{\theta},\sigma_{-i}) < u_i(s_i^{\theta'},\sigma_{-i}); \forall \sigma_{-i} \in \hat{C}_{-i}^0 \quad (3.4.14)$$

Proposition 3.4.1, which generalises similar results in the literature,[9] can be rephrased as follows: given interior initial conditions, if s_i^k is weakly τ-dominated by $s_i^{k'}$, its relative weight will eventually fall, no matter how the weakly τ-dominant strategy performs compared with other strategies in player i's support. Moreover, according to (3.4.8), the ratio $\frac{r_i^{\theta}(t)}{r_i^{\theta'}(t)}$ must converge, whether player i's mixed strategy converges or not. By (3.4.9), if $\frac{r_i^{\theta}(t)}{r_i^{\theta'}(t)}$ converges to a positive constant, this implies that both pure strategies must yield the same payoff against any mixed strategy in the ω-limit set of the opponent. Finally, (3.4.10) ensures that the weak τ-dominance relation is transitive, as it is the "classic" one.

Proposition 3.4.1 tells us that the extent to which the intuition "domination implies extinction" holds is related to the relative performance of the two strategies *in the limit*, that is, when $t \to \infty$.[10] In particular, to ensure extinction, we need *strict* dominance, even if in the weaker form of τ-dominance:

- PROPOSITION 3.4.2. Let $r(r(0),t)$ be the interior solution of a MS dynamics $\dot{r}=f(r(t))$. If $s_i^{\theta} <_{\tau} s_i^{\theta'}$ then:

$$\lim_{t\to\infty} \frac{r_i^{\theta}(t)}{r_i^{\theta'}(t)} = \lim_{t\to\infty} r_i^{\theta}(t) = 0 \quad (3.4.15)$$

- PROOF. Since C_{-i} is a compact set, there exists some $g_{\varepsilon} > 0$ such that $\left[\frac{f_i^{\theta}(r(t))}{r_i^{\theta}(t)} - \frac{f_i^{\theta'}(r(t))}{r_i^{\theta'}(t)}\right] < -g_{\varepsilon}; \forall t > \tau$. We can therefore apply, by analogy, (3.4.11) to show that $\lim_{t\to\infty} \frac{r_i^{\theta}(t)}{r_i^{\theta'}(t)} = \lim_{t\to\infty} r_i^{\theta}(t) = 0$.

Proposition 3.4.2 generalizes the standard result of extinction of strictly dominated strategies to the case of pure strategies which are only strictly τ-dominated. The intuition behind the two results is exactly the same: if the relative performance of a pure strategy is uniformly worse than another, and this property still holds in the limit, this implies the extinction of the dominated strategy, regardless of any further consideration. We apply Propositions 3.4.1-2 in the case of the N-legged Centipede Game:

[9] We make reference to Cressman [1996], Theorem 3.1, and the Lemma in Nachbar [1990].

[10] It is important to notice that $\lim_{t\to\infty} \frac{r_i^{\theta}(t)}{r_i^{\theta'}(t)} = L > 0$ is only a *necessary* condition for the weakly dominated strategy to be in the support of the limiting distribution, whilst $\lim_{t\to\infty} \frac{r_i^{\theta}(t)}{r_i^{\theta'}(t)} = 0$ is sufficient for extinction.

- THEOREM 3.4.1. Any interior solution $r(r(0),t)$ of a MS dynamic $\dot{r}=f(r(t))$ converges to $\mathcal{NE}$.

Before proving the theorem, we need an additional lemma. We know from (3.4.10) that the relation $\leq_\tau$ is transitive. This result, as well as any other result proved so far, is not peculiar of the Centipede Game, as it holds for any finite normal form game. However, if we restrict our attention to the Centipede Game, we can can prove the following:

- LEMMA 3.4.1. For any interior solution, $r(r(0),t)$, of a MS dynamics $\dot{r}=f(r(t))$ the relation $\leq_\tau$ strictly orders the sets S_I and S_{II}.

PROOF. The proof is by induction on θ, as we will show that, for any $i \in \mathfrak{I}$ and $\theta \in \{1,\dots,N\}$ $\leq_\tau$ strictly orders the sets $\mathscr{S}_i(\theta) \equiv \{s_i^{\theta'}\}_{\theta' \geq \theta}$.

Let J be the player who is required to move last. When $\theta = N$, $\mathscr{S}_J(\theta) \equiv \{s_J^N, s_J^{N+1}\}$ and $\mathscr{S}_{-J}(\theta) \equiv \{s_{-J}^{N+1}\}$, and the lemma is obviously true, since s_J^{N+1} is weakly dominated by s_J^N.

- STEP 1. $\theta = N-1$. In this case, $\mathscr{S}_J(\theta) = \{s_J^N, s_J^{N+1}\}$ and $\mathscr{S}_{-J}(\theta) = \{s_{-J}^{N-1}, s_{-J}^{N+1}\}$. To prove the lemma, it is sufficient to show that $\mathscr{S}_{-J}(N-1)$ can be ordered by $\leq_\tau$, since $\mathscr{S}_J(N-1) = \mathscr{S}_J(N)$. We evaluate the payoff difference $\left(u_{-J}(s_{-J}^{N+1}, r_J(t)) - u_{-J}(s_{-J}^{N-1}, r_J(t))\right)$ explicitly, factorising $r_J^N(t)$ out :

$$\begin{aligned} & u_{-J}(s_{-J}^{N+1}, r_J(t)) - u_{-J}(s_{-J}^{N-1}, r_J(t)) \\ &= r_J^N(t)\left(\left(u_{-J}(N+1) - u_{-J}(N-1)\right)\frac{r_J^{N+1}(t)}{r_J^N(t)} - \left(u_{-J}(N-1) - u_{-J}(N)\right)\right) \end{aligned} \tag{3.4.16}$$

with, by (3.3.2.1-2), both $\left(u_{-J}(N-1) - u_{-J}(N)\right)$ and $\left(u_{-J}(N+1) - u_{-J}(N-1)\right)$ positive. Only the payoffs $u_{-J}(N-1)$, $u_{-J}(N)$ and $u_{-J}(N+1)$ enter in the evaluation of (3.4.16), since s_{-J}^{N+1} and s_{-J}^{N-1} yield the same payoff against any pure strategy in which J opts out before stage $N-1$. Moreover, the sign of (3.4.16) depends only on the sign of the term into round brackets of the right hand side of (3.4.16), since, by forward invariance, $r_J^N(t) > 0, \forall t \geq 0$. Define $\kappa > 0$ by $\kappa \equiv \frac{\left(u_{-J}(N-1) - u_{-J}(N)\right)}{\left(u_{-J}(N+1) - u_{-J}(N-1)\right)}$, that is, the threshold value of $\frac{r_J^{N+1}(t)}{r_J^N(t)}$ that makes player *-J* indifferent between s_{-J}^{N-1} and s_{-J}^{N+1}. Taking limits of (3.4.16) we obtain the following:

$$\begin{aligned} & \lim_{t\to\infty} \operatorname{sign}\left[u_{-J}(s_{-J}^{N+1}, r_J(t)) - u_{-J}(s_{-J}^{N-1}, r_J(t))\right] \\ &= \operatorname{sign}\left[\left(\left(u_{-J}(N+1) - u_{-J}(N-1)\right)L^{N+1} - \left(u_{-J}(N-1) - u_{-J}(N)\right)\right)\lim_{t\to\infty} r_J^N(t)\right] \end{aligned} \tag{3.4.17}$$

with $L_J^{N+1} \equiv \lim_{t\to\infty} \frac{r_J^{N+1}(t)}{r_J^N(t)}$. There are only two possible alternatives:

- CASE A. $L_J^{N+1} \geq \kappa$. Then $s_{-J}^{N-1} \leq_\tau s_{-J}^{N+1}$ (with $\tau = 0$ and $C_J = \left\{ \sigma_J \in \Delta_J \middle| \frac{\sigma_J^{N+1}}{\sigma_J^N} \geq L_J^{N+1} \right\}$, since, by (3.4.7), $\frac{r_J^{N+1}(t)}{r_J^N(t)}$ is continuously decreasing in t;.
- CASE B. $L_J^{N+1} < \kappa$. Then $s_{-J}^{N+1} \leq_\tau s_{-J}^{N-1}$ (with τ solving $\frac{r_J^{N+1}(t)}{r_J^N(t)} = L^{N+1}$ and $C_J = \left\{ \sigma_J \in \Delta_J \middle| \frac{\sigma_J^{N+1}}{\sigma_J^N} \leq L_J^{N+1} \right\}$).

Since this exhausts all cases, the result follows.[11]

- STEP 2. $1 \leq \theta < N$. Assume that the lemma is true for $1 \leq \theta < N$. Let i be the player who is required to move at stage θ. When $1 \leq \theta < N$, $\mathscr{S}_i(\theta) \equiv \{s_i^\theta, s_i^{\theta+2}, \ldots, s_i^{N+1}\}$ and $\mathscr{S}_{-i}(\theta) = \{s_{-i}^{\theta+1}, s_{-i}^{\theta+3}, \ldots, s_{-i}^{N+1}\}$. To prove the lemma, is sufficient to show that $\mathscr{S}_{-i}(\theta-1)$ is an ordered set, since $\mathscr{S}_i(\theta-1) = \mathscr{S}_i(\theta)$. For a fixed θ, let $\hat{\vartheta}(\theta)$ index the pure strategy of player i which weakly τ-dominates all other strategies in $\mathscr{S}_i(\theta)$ (i.e. $s_i^{\theta'} \leq_\tau s_i^{\hat{\vartheta}}$ for any $s_i^{\theta'} \in \mathscr{S}_i(\theta)$, with $\theta' \neq \hat{\vartheta}$). When $1 \leq \theta < N$, for any $s_{-i}^{\theta'} \in \mathscr{S}_{-i}(\theta)$, the payoff difference $u_{-i}(s_{-i}^{\theta'}, r_i(t)) - u_{-i}(s_{-i}^{\theta-1}, r_i(t))$ takes the following form:

$$u_{-i}(s_{-i}^{\theta'}, r_i(t)) - u_{-i}(s_{-i}^{\theta-1}, r_i(t)) = r_i^{\hat{\vartheta}}(t) \sum_{\theta'' \in \mathscr{S}_i(\theta)} \frac{r_i^{\theta''}(t)}{r_i^{\hat{\vartheta}}(t)} \left(u_{-i}(k) - u_{-i}(\theta-1)\right),\ k = \begin{cases} \theta' \text{ when } \theta'' > \theta' \\ \theta'' \text{when } \theta'' \leq \theta' \end{cases} \tag{3.4.18}$$

with all $(u_{-i}(k) - u_{-i}(\theta-1))$ strictly positive, except for $(u_{-i}(\theta) - u_{-i}(\theta-1)) < 0$. If $\mathscr{S}_i(\theta)$ is an ordered set, then (3.4.8) implies that $\xi_i^\theta(t) \equiv \sum_{\theta'' \in \mathscr{S}_i(\theta)} \frac{r_i^{\theta''}(t)}{r_i^{\hat{\vartheta}}(t)} (u_{-i}(k) - u_{-i}(\theta-1))$ must converge to some constant L_i^θ. There are two possible cases:

- $L_i^\theta > (<) 0$. Then $s_{-i}^{\theta-1} \leq_\tau s_{-i}^{\theta'}$ ($s_{-i}^{\theta'} \leq_\tau s_{-i}^{\theta-1}$), by anakogy with CASE B.
- $L_i^\theta = 0$. Then $s_{-i}^{\theta-1} \leq_\tau s_{-i}^{\theta'}$. To show this, notice that the result is obviously true if $\hat{\vartheta} = \theta$, since this would make $\xi_i^\theta(t)$ strictly decreasing in t.

Assume instead $\hat{\vartheta} \neq \theta$. Then, it must be:

$$\lim_{t\to\infty} \left[\frac{r_i^\theta(t)}{r_i^{\hat{\vartheta}}(t)} \right] = \frac{\lim_{t\to\infty} \left[\sum_{\theta'' \in \mathscr{S}_i(\theta) \setminus \theta} \frac{r_i^{\theta''}(t)}{r_i^{\hat{\vartheta}}(t)} (u_{-i}(k) - u_{-i}(\theta-1)) \right]}{(u_{-i}(\theta-1) - u_{-i}(\theta))} > 0. \tag{3.4.19}$$

[11] It might be argued that we are not allowed to determine the limiting sign of (3.4.17) looking only at the term into round brackets of the right hand side, since $r_J^N(t)$ might go to zero faster than $\frac{r_J^{N+1}(t)}{r_J^N(t)} \to L_J^{N+1}$. However, this possibility is ruled out by the fact that, by (3.4.7), L_J^{N+1} is finite.

Given that $\frac{r_i^\theta(t)}{r_i^{\hat{\vartheta}}(t)}$ converges to a positive constant, (3.4.10) implies that:

$$u_i(s_i^\theta, \sigma_{-i}) = u_i(s_i^{\hat{\vartheta}}, \sigma_{-i}) \geq u_i(s_i^{\theta'}, \sigma_{-i}), \text{ for any } \sigma_{-i} \in \omega_{-i}(r(0)). \tag{3.4.20}$$

In other words, the relative performance of s_i^θ must eventually improve, compared with any other strategy in $\mathcal{A}_i(\theta)$. Thus, for t sufficiently large, also $s_{-i}^{\theta-1}$ must improve compared to $s_{-i}^{\theta'}$, since s_i^θ is the only strategy in $\mathcal{A}_i(\theta)$ against which $s_{-i}^{\theta-1}$ does better than $s_{-i}^{\theta'}$. From the above consideration we have that $\xi_i^\theta(t)$ must converge to 0 from above, which in turn implies, by analogy with CASE A, $s_{-i}^{\theta-1} \leq_\tau s_{-i}^{\theta'}$.

Since this exhausts all cases, the result follows. ♠

We are now in the position to prove Theorem 3.4.1.

- PROOF OF THEOREM 3.4.1. Since S_I and S_{II} are strictly ordered by $\leq_\tau$, (3.4.8) implies that $\lim_{t\to\infty} \frac{r_I^\theta(t)}{r^{\theta'}(t)}$ exists for any i, θ and θ'. If all the ratios converge, then also the mixed strategy profile must converge. We can therefore apply the standard result "convergence implies Nash" in the case of MS dynamics (see, e. g. Weibull [1995], Theorem 5.2 *(c))* to complete the proof. ♠

Another way to rephrase the content of Theorem 3.4.1 and its corollary could be the following. The theorem shows that the Nash equilibrium component denoted by $\mathcal{NE}$ is ***globally interior attracting***, that is, it attracts every interior path under any MS dynamics. Note that the above result is not directly linked with any local stability property of the set $\mathcal{NE}$: it may well happen that trajectories starting close to $\mathcal{NE}$ move away and then, eventually, come back. This is exactly what happens in the Centipede Game. In the next section, we shall explore this phenomenon through simulations.

3.5. «CYCLES OF LEARNING»: SOME SIMULATION RESULTS

This section is devoted to simulations of the dynamics studied hitherto. To perform this task, we shall begin by specifying the payoff structure, as well as the dynamics:

- ASSUMPTION 3.5.1. The payoff function is as follows:

$$u_I(\theta) = \frac{\alpha^\theta}{\alpha^{2(1-\delta_I^\theta)}}, \text{ with } \alpha > 1; \tag{3.5.1}$$

$$u_{II}(\theta) = u_I(\theta + 1) \tag{3.5.2}$$

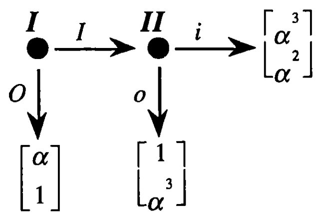
In words: the payoff of player i is multiplied by some positive constant α after every other round. We can therefore interpret α as a measure of the *increasing returns to cooperation* in the Centipede Game, since it reflects how much the payoffs increase as the game ends further away from the beginning of the tree. The assumption for the dynamics (which replaces Assumption 3.3.1) is the following:

- ASSUMPTION 3.5.2. The evolution of $r(t)$ is given by the following system of continuous-time differential equations:

$$\dot{r}_i^\theta(t) = r_i^\theta(t)\left(u_i\left(s_i^\theta, r_{-i}(t)\right) - u_i(r(t))\right) + \lambda_i\left(\beta_i^\theta - r_i^\theta(t)\right) \tag{3.5.3}$$

with $\lambda \geq 0$ and $\beta_i^\theta \in (0,1)$. These dynamics are a linear combination between the standard Replicator Dynamics and a perturbation term which ensures that, at each point in time, every pure strategy is played with positive probability, no matter how it performs against the opponent's mixed strategy. Following Binmore and Samuelson [1995] we call the latter ***drift***. Such a deterministic perturbation term can serve as a high probability approximation to a stochastic ***noise*** term in a model in which time is discrete and the population size is finite[12], as we approach the limiting case of continuous-time and infinite population suitably. The relative importance of the drift is measured by λ_i, which we refer to as the *drift level.* We assume λ_i to be "small", since the major forces which govern the adjustment process should be captured by the unperturbed dynamics.

We shall analyze the 2-legged Centipede Game of Figure 3.5.1 first.



		$(1-x)$ 2	x 3
y	1	$\alpha,1$	$\alpha,1$
$(1-y)$	3	$1,\alpha^3$	α^3,α^2

FIGURE 3.5.1
A 2-legged Centipede Game.

In Figure 3.5.1 x and y denote the probabilities with which the corresponding pure strategies are played and $\mathcal{NE} \equiv \left\{(x,y) \in \Delta \middle| y = 1, x \in \left[0, \frac{1}{\alpha^2 + \alpha + 1}\right]\right\}$.

In the phase diagrams of Figure 3.5.2 we trace some interior solutions of the unperturbed Replicator Dynamics (i.e. when $\lambda_I = \lambda_{II} = 0$) under two different realizations of the payoff parameter α.

[12]Model which fall in this category are, for example, those of Kandori *et al*. [1993] and Young [1993]. In a biological context, this noise may be interpreted as a ***mutation,*** i. e. a random alteration of the agents' genetic code. In a learning context, it can be interpreted as a ***mistake,*** i. e. a random alteration of the agents' behavior, or as an effect of the players' ***experimentation.*** We prefer the terminology of drift (as opposed to noise) because the latter is usually modeled as a genuine random variable, whereas the former takes the form of a purely deterministic dynamics. For a general discussion on motivations and general properties of evolutionary dynamics with drift, see Samuelson [1997], Chapter 6.

Figure 3.5.2 gives a good description of the content of Propositions 3.4.1-2. First note that weak domination of s_{II}^3 implies that $\frac{r_{II}^3(t)}{r_{II}^2(t)}$, and therefore x, is strictly decreasing in t, for any $t \geq 0$. Whenever x falls below $\frac{1}{\alpha^2+\alpha+1}$, the threshold level of x which makes player I indifferent between her two pure strategies, $r_I^3(t)$ starts to fall, until it vanishes in the limit. For any interior solution, s_I^3 is in fact a strictly τ-dominated under the Replicator Dynamics (even if it is not strictly dominated in the conventional sense) and this implies, as we know from Proposition 3.4.2, $\lim_{t \to \infty} r_I^3(t) = 0$.

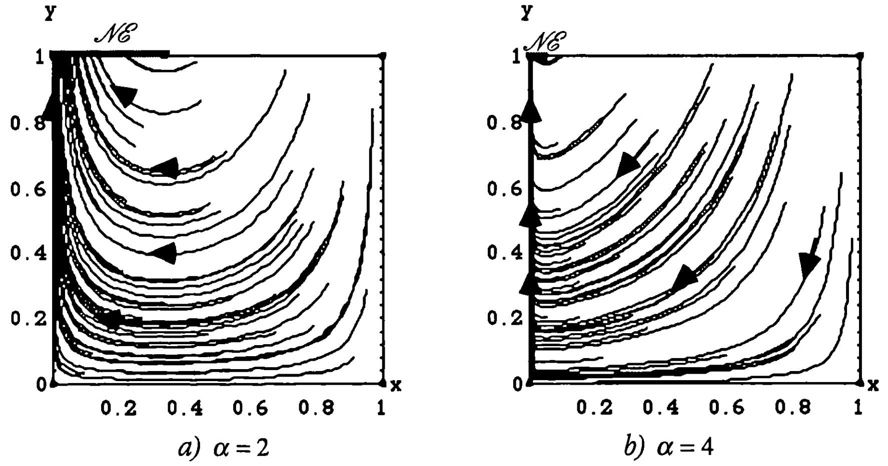


FIGURE 3.5.2
The 2-legged Centipede Game and the Replicator Dynamics.

As $r_I^1(t) \to 1$ the (expected) payoff difference between s_{II}^2 and s_{II}^3 tends to zero, as both strategies yield the same payoff against s_I^1. In consequence, the evolutionary pressure against the weakly dominated strategy s_{II}^3 vanishes, and this is why s_{II}^3 remains in the support of the limiting play.

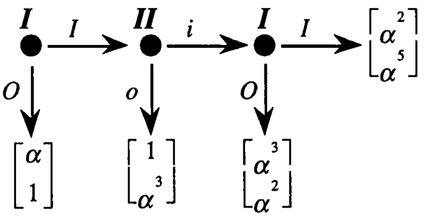
Consistently with Theorem 3.4.1 and its corollary, every interior trajectory converges to the corresponding Nash-equilibrium component $\mathcal{NE}$, highlighted by a bold segment in the upper-left corner of the two diagrams. Increasing the payoff parameter α has the following effects:

- $\mathcal{NE}$ shrinks, i.e. the measure of states compatible with the Nash prediction is reduced;
- the dynamics speed up (this is because in the Replicator Dynamics, as well as in any MS, growth rates are increasing functions of payoff differences).

It is interesting to note that both effects are qualitatively consistent with McKelvey and Palfrey [1992]'s experimental results on the Centipede Game, for which our model may provide a theoretical account.[13]

[13]We compare the two sessions of comparable length and number of observations characterized by a "LOW" payoff treatment (Sessions 1 and 3), with the unique session characterized by a "HIGH" payoff treatment (Session 4) of McKelvey and Palfrey's

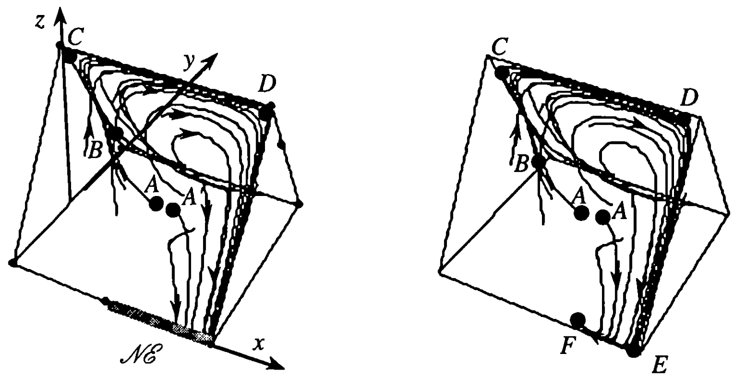
We move on to the 3-legged Centipede Game of Figure 3.5.3.



		x	$(1-x)$
		2	4
$(1-y-z)$	1	$\alpha,1$	$\alpha,1$
z	3	$1,\alpha^3$	α^3,α^2
y	4	$1,\alpha^3$	α^2,α^5

FIGURE 3.5.3
A 3-legged Centipede Game

In Figure 3.5.4 we show the phase diagrams for the dynamics (3.5.3) in this game under two different realizations of the drift parameters $(\lambda_i, \beta_i^\theta)$:[14]



a) $\lambda = 0$ *b)* $\lambda = .001$

FIGURE 3.5.4
Unlearning in the 3-legged Centipede Game

Figure 3.5.4*(a)* refers to the unperturbed Replicator Dynamics. As we know from Theorem 3.4.1 and its corollary, any interior path converges to $\mathcal{NE}$ (the bold segment in the bottom-

experiments on the 4-legged Centipede Game. In their Table IIA, p. 808, they call f_θ the observed frequency of games ended at stage θ. If F_θ is the corresponding cumulative distribution over the terminal nodes, then they show that such distribution, in the case of the HIGH treatment, stochastically dominates the distribution derived from the LOW treatments. In other words, when payoffs are higher, the game ends, on average, "closer" to the subgame perfect outcome. However, if we look at the data in more detail, we discover that this evidence is due to the fact that in the last five repetitions (out of a series of ten) the cumulative distributions of the HIGH treatment consistently dominate the corresponding distributions derived from both the LOW payoff sessions, whereas this never happens if we look at any of the first five repetitions. In other words, not only higher payoffs produce *more* learning, but also at a *faster* rate. We obtain a similar result if we estimate the mixed strategies, for both players, consistent with the observed frequencies at each node. In the last five repetitions, the cumulative distributions of the HIGH treatment stochastically dominate the relative distributions of both the LOW payoff treatments, showing a less dispersed behavior compared with the subgame perfect strategy profile.

[14]Hereafter all the simulations are characterized by $\alpha = 2$.

right corner of the diagram). Whenever x is sufficiently high (see, for example, the point A' in the diagram) the system converges to $\mathcal{NE}$ monotonically . This happens whenever player *II* adopts an initial behavior which is sufficiently *close* to the subgame-perfect prediction (that is, when $r_{II}^2(0)$ is sufficiently high). Otherwise, as with the trajectory starting from the point labelled A in the diagram, we can observe the following pattern:

- $A \rightarrow B$: player *II* opts in often enough to induce player *I* to increase the probability of playing both strategies s_I^3 and s_I^4 which yield a higher payoff. Player *II* then has a clear incentive to play s_{II}^4, which is the reason why x decreases significantly. At the point labelled B, the system is sufficiently close to the pure strategy profile (s_I^4, s_{II}^4) to consider this as the *cooperative phase;*
- $B \rightarrow C$: this is the beginning of what we may consider as the ***backward induction phase.*** Now x is too small for player *I* not to discriminate between s_I^3 and s_I^4. The probability of playing the latter increases gradually at the expenses of the alternative options (and the system moves toward the point C in the diagram, corresponding to the pure strategy profile (s_I^3, s_{II}^4));
- $C \rightarrow D$: now it is player *II* who modifies her behavior significantly, given the fact that, whenever the last node is reached, player *I* is now opting out with a sufficiently high probability. The system moves gradually toward a position characterized by the strategy profile (s_I^3, s_{II}^2) (point D in the diagram);
- $D \rightarrow E$: x is sufficiently high to make s_I^1 optimal compared with any alternative option: $r_I^1(t)$ is therefore bound to increase until the process eventually converges to $\mathcal{NE}$.

The behavior of the dynamics with drift is reported in Figure 3.5.4*(b)*. In this first example we set $\lambda_i = .001$ and $\beta_i^\theta = \frac{1}{|S_i|}$, for any i, θ. In other words, drift is "negligible" and uniformly distributed across players and strategies. The only significant difference between Figures 3.5.4 *(a)* and 3.5.4*(b)* is that, whenever the system gets sufficiently close to $\mathcal{NE}$, the drift component overcomes the selection dynamics, pushing the system toward the (unique) restpoint denoted by F.[15] In other words, the dynamics with drift of Figure 3.5.4 *(b)* is characterized by an additional phase:

- $E \rightarrow F$: the system eventually reaches the unique restpoint, in which I plays s_I^1 with (almost) probability 1 and II mixes (although not sufficiently to induce I to come back into a new cooperative phase). In our interpretation, this latter phase is driven by a *pure drift effect,* which does not have, in this first example, a significant impact on the play in the limit (in fact, whenever I opts out at the first stage almost with sufficiently high probability, II 's behavior is completely irrelevant to determine the outcome of the play).

In Figure 3.5.5 we perturb the system (3.5.3) differently. In particular, we keep the same parameter setting as in Figure 3.5.4*(b)* for player *II,* enhancing the drift in favour of player *I*'s "more cooperative" strategies, setting $\lambda_I = .1$, $\beta_I^1 = 1/12$, $\beta_I^3 = 1/6$ and $\beta_I^4 = 2/3$. Under

[15] In a subsequent paper (see Ponti [1996] we prove in fact uniqueness of the restpoint of the dynamics (3.5.3] in the case of the 2 and 3-legged Centipede Games when $\lambda_i > 0$.

these conditions, the cyclic pattern exhibited by Figure 3.5.4 gets repeated over time, as the dynamics settle down to a limit cycle around the (unique) restpoint, F.

For this phenomenon to appear, we do not need the special parameter setting of Figure 3.5.5, although we need a drift against subgame-perfect equilibrium strategy that is stronger for player I than for player II. Why should it be so? If drift reflects players' experimentation, it is then reasonable to assume this effect to be stronger in the case of player I, whose subgame-perfect equilibrium strategy precludes any observation of the opponent's reaction. If drifts reflects the fact that players make ***mistakes*** (or misperceive the game) a similar argument applies: as in Rosenthal [1981], it may well be that the probability of a mistake is higher for actions which stop the game further away from the end of the tree.

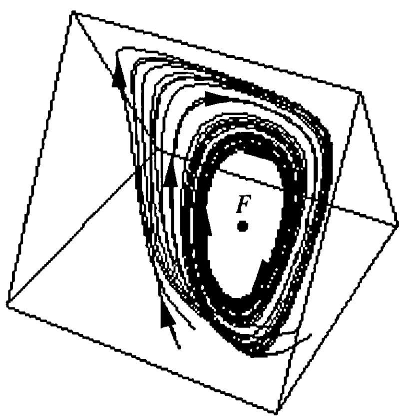


FIGURE 3.5.5
Cycles of learning in the 3-legged Centipede Game

What happens when N (the length of the Centipede) increases? Figure 3.5.6 shows different diagrams summarizing the behavior of (3.5.3) under different specifications of the parameters N and λ_i.[16] As can be easily spotted in the diagrams, raising the drift level λ has the following effects:

- cycles persist, and are more frequent, the longer is the Centipede;
- the average length of play (and therefore, the average payoff) increases.

When the drift is no longer to be considered "negligible" (though "small") for both players, the cyclic behavior observed in the three-leg case persists over time. Not being able to provide

[16] All the diagrams in figure 3.5.6 are characterised by $\lambda_I = \lambda_{II} = \lambda$ and $\beta_i^\theta = \frac{1}{|S_i|}$. However, a much wider range of simulations for Centipede Games of various length has been carried out, varying payoffs, drift parameters and initial conditions. Results are analogous and available on request.

a formal analysis of these dynamics, our simulations leave open the question whether such qualitative behavior is bound to disappear in the long run. If it does not disappear, we may be observing limit cycle behavior equivalent to the one of Figure 3.5.5.

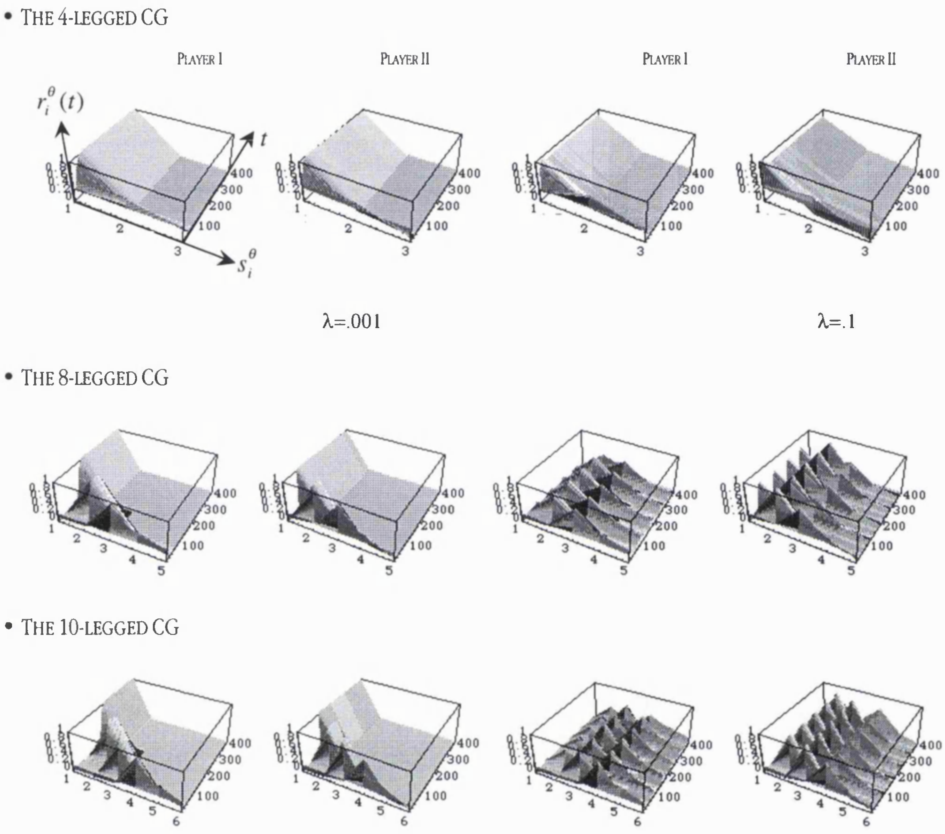


FIGURE 3.5.6
Cycles of learning in longer Centipede Games

3.6. CONCLUSIONS

The results contained in the paper suggest two different (and possibly antithetical) conclusions. Theorem 3.4.1 establishes a strong link between the dynamic outcome of a popular class of evolutionary dynamics and the traditional game-theoretical analysis of games with perfect information. Moreover, in the special case of the Replicator Dynamics, our simulations suggest that the *actual* learning path might resemble the backward induction procedure in a closer way. If the initial conditions are sufficiently "mixed" (i.e. if the players' initial behavior gives sufficient weight to the pure strategies which require to opt out at the latter stages) the adjustment process replicates closely the iterative deletion of weakly dominated strategies prescribed by the backward induction procedure, and the players act *as if* they experience the procedure step by step, until equilibrium is achieved. On the other hand, as the simulations on longer Centipede Games suggest, you need only a small perturbation of the adjustment process to produce a cyclic behavior (this latter effect being stronger the longer is the centipede).

As pointed out in the introduction, these results are to be compared with to the approach proposed by Cressman and Schlag [1995], to which our paper relates in several ways. In their Theorem 2, Cressman's [1996] technique is applied to show that, in the case of games with perfect information and no relevant ties (of which the Centipede Game is a special case), every interior path of the Replicator Dynamics asymptotically converges to a Nash equilibrium. The key argument is contained in their Theorem 1, which our Proposition 3.4.1 generalizes to any MS dynamics, as both results rely only on weak-dominance considerations. Moreover, their result on the (in)stability of the subgame perfect outcome in "complex" games is consistent with our simulation results of the Replicator Dynamics with drift.

In addition, our simulations may provide an evolutionary account of Rosenthal's [1981] original analysis of the Centipede Game. The drift modeled in our simulations has similar effects of the "tremble" each player considers in her calculations: it is negligible when the selection dynamics are in action, but it becomes crucial in shaping the adjustment process as the latter approaches the $\mathscr{NE}$ component. A (more remote) analogy could be also established with the treatment of the finitely-repeated Prisoner's Dilemma proposed by Kreps *et al.* [1982]. In their model, the players consider an enriched model, by means of a game with imperfect information, where some other view of what is rational is taken into account, including possible scenarios off the equilibrium path. In our case, the assumption of a completely mixed drift term makes Δ^0 forward invariant: at any point in time (as well as in the limit) every pure strategy must be played with positive probability.

CHAPTER 4

CONVENTIONS AND SOCIAL MOBILITY IN BARGAINING SITUATIONS

> The force of many rules of etiquette and social restraint ... seems to depend on their having become "solutions" to a coordination game.
>
> Schelling [1963]

4.0. ABSTRACT

This paper studies the evolution of a population whose members use their social class to coordinate their actions in a simple tacit bargaining game. In the spirit of Rosenthal and Landau [1979], we interpret the equilibrium behaviours that the players may adopt, as a function of their class, as *customs*. Players may change their class depending on the outcome of the game, and may also change their custom, as a result of some learning process. We are interested in the characterization of the fixed points of the adjustment process over the space of classes and customs from a distributional point of view. We find that, although any custom (when it operates alone) generates the same limiting class distribution as any other, these limiting distributions can be ranked with respect of their *mobility*. If players are allowed to change their custom when they find it unsatisfactory, then social mobility appears to be the key variable to predict the type of custom which will predominate in the long run even though, in general, no one custom is dominant. In particular, customs which promote social mobility appear to exhibit, in all the cases we have analysed, stronger stability properties.

4.1. INTRODUCTION

There are many economic situations in which informal means are employed to execute mutually beneficial agreements. In such cases, some social variable (call it *class,* or *reputation*) may help the agents to coordinate their actions on an equilibrium of the game they are playing. The notion of *convention,* often used to describe these equilibria, may then involve some sociological background: a particular behaviour may have no intrinsic merit, but is selected on the basis of some social or cultural link among the players. The role of these social variables may be even more important in those situations where, for such an equilibrium to be implemented, different agents are required to adopt different behaviours (and receive, in return, different rewards). In this case, the social context may in fact determine who is supposed to do what (and, consequently, who deserves the lion's share).

The society we have in mind is modelled by a constant utility flow which is to be allocated, in each time period, by means of a simple bargaining scheme between two players, randomly selected from the population. Each player has to choose, simultaneously, whether to *defect* (requiring the biggest share for herself) or to *cooperate* (accepting the division proposed by the opponent). If both players cooperate, then the pie is equally divided; if both defect, then the size of the pie will be substantially reduced, as a result of the negotiation breakdown. The only information available to each player is the opponent's *class*, that is, a signal from which it can be partially deduced the opponent's past behaviour in the stage game. We shall assume that the strategic choice of the two players is conditioned only on this information. The outcome of the stage game may modify the class of the players, who are then placed back in the original population. In the following time period, other two players will be paired at random, and so on.

Rosenthal and Landau [1979] (R&L hereafter) explore, under similar conditions,[1] how some behavioural patterns, which they call *customs*, may influence the long-run distribution of plays of the population game. In their paper, these customs are described as "*...possible decision rules which members of the society might unanimously employ to determine their moves in the game...*"[2]. Two properties characterize a custom under their perspective:

- it uniquely determines the players' behaviour in the stage game;
- such behaviour must be self-enforcing, in the sense that it must be justified, from the players' viewpoint, on the ground of some rationality assumption.

This definition clearly recalls what economists are now accustomed to call *conventions*, with reference to the flourishing stream of research in the recent game-theoretic literature which studies coordination games.[3] Behind this analogy stands the fact that each player faces a symmetric situation characterized by multiple equilibria. However, unlike a pure coordination setting, in the stage game we have just described, the players rank the various equilibrium outcomes differently: the selection of a particular custom can then be observed from a *distributional* point of view, since a better bargain for a player implies less for the opponent.

In R&L's model, the social variable upon which players condition their choice is termed *reputation* (higher reputation signifying tendency to defect). Moreover, they assume that each individual in the population follows the same custom: different customs generate different limiting distributions, which are then compared in terms of their efficiency properties. Intuition suggests, the authors claim, that customs which prescribe cooperating against a player with higher reputation (seemingly more prevalent in real-life bargaining situations) might also be justified on efficiency grounds, once they minimised the social loss generated in equilibrium.

[1]See Rosenthal [1979] for a formal description of the general framework.
[2]See Rosenthal and Landau [1979], p. 234.
[3]See, among others, Kandori *et al.* [1993] and Young [1993]

However, commenting on their results, R&L admit that, in their model "...*this has proved not to be the case*..." (p. 234), since the social ranking of the equilibrium customs depends crucially on how reputation is formally defined.

Our model differs from R&L's original formulation in (at least) two respects. First, we assume that the social variable (we call it *class*) is directly linked to the payoffs received during the past history of the game. In particular, we shall assume that the class of a player is simply the payoff she received the last time she was called to play. We justify this assumption by interpreting the class as a signal of each individual's wealth.

Moreover (and more crucially), we do not necessarily assume that a unique custom is commonly shared in the society. Instead, we allow the possibility that the agents hold different customs. This feature of our model opens the possibility of modelling a learning process: players may in fact change their custom if they find it somehow unsatisfactory.

We design the learning process at two different, and somehow complementary, levels. We consider first what we call *coordination learning:* players holding different customs may fail to coordinate their actions. We therefore model a procedure which leads the players to revise their custom as a result of a disequilibrium play. In addition, we introduce a further type of learning, which we call *aspiration learning.* After the stage game has been played, each player compares her own payoff to some threshold value by which we take to be an estimate of a "satisfactory" outcome of the strategic interaction. Whenever this aspiration level is not reached, a player is assumed to modify her custom with positive probability.

Our coordination and aspiration learning schemes allow some individual feed-back to the social outcome induced by each custom; one of the aims of the paper is to explore how this feed-back interacts with the social pressures generated by our custom society.[4]

The remainder of the paper is arranged as follows. Section 2 describes the main features of the model. Section 3 develops the formal theory on which our analysis is based. Following R&L, section 4 assumes that only one custom is adopted by the entire population, and explores the asymptotic properties of the limiting class distribution, under different customs. In this respect, we find (consistently with R&L) that the limiting class distribution under any particular custom *is exactly the same.* We interpret this result as follows. If a custom allows the players to coordinate on one of the Nash equilibria of the game, and the class of a player is the payoff received, the limiting class distribution will concentrate most of its mass on the classes which correspond to the payoffs that the players get when a pure strategy Nash equilibrium is played, no matter how this coordination takes place (i.e. regardless of the custom which is actually established).

It is important to notice that it does not follow from the above result that, once the equilibrium distribution has been reached, the same players will stay in the same class forever after. On the contrary, each custom is characterised in equilibrium by a complex, but balanced, network of flows among classes. Section 5 explores the properties of a society in which only on custom is available from this perspective, interpreting these flows as measures of *social mobility.*

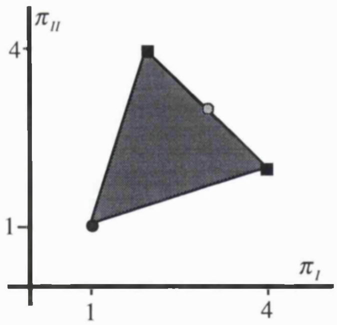
---

[4]There are many references, in the macroeconomic litterature on income distribution, which stress the role of social variables in the determination of the income distribution in society. Becker and Tomes [1979], for example, point out that: "...The concept of endowment is also a fundamental part of our analysis. Children are assumed to receive endowments of capital that are determined by the reputations and "connections" of their families, the contribution to the abilities, race, and other characteristics of children from the genetic constitutions of their families, and the learning, skills, goals, and other "family commodities" acquired through belonging to a particular family culture..." (p. 1158).

We then move to a situation where different customs are present at the same time within the population. Section 6 explore the simplest possible case (that is, a two-custom society); section 7 considers the case of a society in which all possible customs may be present. If players are allowed to change their custom through learning processes in the way we described, then social mobility appears to be the key variable for predicting the type of custom which will predominate in the long run. In particular, even though no custom is dominant, customs which promote social mobility appear to exhibit stronger stability properties in all the cases we have analysed. A final section devoted to additional remarks concludes, followed by four sections of appendix containing the most elaborate proofs.

4.2 THE BASIC MODEL

We deal with a market economy characterized by a constant utility flow which is to be allocated in each time period within a large, but finite population of N players. At each point in time two individuals are drawn at random and sequentially from the population to play the symmetric normalform game of figure 2.1, known in the literature as *Chicken,* which tries to capture the intuition of a simple tacit bargaining situation. The game is characterized by two asymmetric Nash equilibria in pure strategies, namely *(C,D)* and *(D,C),* and a symmetric Nash equilibrium in which each pure strategy is played with equal probability:



	C	D
C	3,3	2,4
D	4,2	1,1

FIGURE 4.1.
Payoff matrix and cooperative payoff region of the *Class Game*

At any given time, the *type* of each player is characterized by a class} and a custom} as explained in the following two definitions.

- DEFINITION 4.1. The class of a player is simply the payoff received in the last round she was called to play.

We interpret the class as a measure of the stock of wealth inherited from the past history of the game. Let $\mathcal{C}=\{1,2,3,4\}$ be the set of classes and $S=\{C,D\}$ the strategy set in the Class Game. The custom of a player determines her behaviour in the Class Game:

• DEFINITION 4.2. A generic *custom* $k \in K$ is a function $k: \mathscr{C} \times \mathscr{C} \Rightarrow \Delta(S)$ which satisfies the following conditions:

$$\begin{aligned} k(c,c') &= 0,1 \text{ when } c \neq c' \\ k(c,c) &= 1/2 \\ k(c,c') &= 1 - k(c',c) = 0,1 \end{aligned} \tag{4.2.1}$$

In words: if two players follow the same custom, they are able to coordinate their actions on one of the Nash equilibria of the Class Game.[5] In particular, if they belong to different classes, the custom tells them who is supposed to cooperate and who is supposed to defect. In the case of a play between two players of the same class, given the fact that they are absolutely indistinguishable for each other, the custom still assures that an optimal behaviour, even if only *ex ante,* is selected; namely the symmetric mixed-strategy Nash equilibrium. The interpretation is the following: the players aim to maximize their class (and therefore their share of the utility pie), and use their current class as a signal for their opponents, who condition (via the custom they follow) their behaviour on that signal. Each player can observe the class of her opponent (but not his custom), and reacts according to the dictate of her own custom, which acts as a signal extracting device.[6]

Definition 4.2.2 allows for the possibility of 64 different customs, since there are 6 possible encounters between players of a different class, and two choices for each player (and therefore there are $2^6 = 64$ different customs). From Definition 4.2.2, it is clear that each of the 64 possible customs is completely specified by the list of six numbers, either zero or one,

$$\kappa = \{k(1,2), k(1,3), k(1,4), k(2,3), k(2,4), k(3,4)\}$$

indicating the pure strategy selected by the row player in the event of being matched with an opponent belonging to a different class. Taking as alphabet the pair $\{0,1\}$, we may therefore number the customs in their lexicographic ordering, as shown in Table 4.2.1.

[5]Given each player can choose only between two pure strategies in the Class Game, Definition 2.2 interprets the mixed strategy *k(c,c')* as the probability of defecting. The probability of cooperating is then uniquely determined as 1-*k(c,c')*.

[6]We confine our attention to the range of possible behaviours represented by the set of customs, as described in Definition 2.2. This restriction is not innocent: we do not consider here a wide range of alternative behaviours which may affect the dynamics of the system. We justify this focus by arguing that a behaviour which is internally coherent (represented by a custom) may not be *consistent*, given the fact that the custom followed by each agent is not publicly known, and the customs used by different players may be lead to a non-Nash outcome. In this way we introduce non-equilibrium behaviour in the model, while keeping its complexity under control.

#	κ	#	κ	#	κ	#	κ
1	{0, 0, 0, 0, 0, 0}	17	{0, 1, 0, 0, 0, 0}	33	{1, 0, 0, 0, 0, 0}	49	{1, 1, 0, 0, 0, 0}
2	{0, 0, 0, 0, 0, 1}	18	{0, 1, 0, 0, 0, 1}	34	{1, 0, 0, 0, 0, 1}	50	{1, 1, 0, 0, 0, 1}
3	{0, 0, 0, 0, 1, 0}	19	{0, 1, 0, 0, 1, 0}	35	{1, 0, 0, 0, 1, 0}	51	{1, 1, 0, 0, 1, 0}
4	{0, 0, 0, 0, 1, 1}	20	{0, 1, 0, 0, 1, 1}	36	{1, 0, 0, 0, 1, 1}	52	{1, 1, 0, 0, 1, 1}
5	{0, 0, 0, 1, 0, 0}	21	{0, 1, 0, 1, 0, 0}	37	{1, 0, 0, 1, 0, 0}	53	{1, 1, 0, 1, 0, 0}
6	{0, 0, 0, 1, 0, 1}	22	{0, 1, 0, 1, 0, 1}	38	{1, 0, 0, 1, 0, 1}	54	{1, 1, 0, 1, 0, 1}
7	{0, 0, 0, 1, 1, 0}	23	{0, 1, 0, 1, 1, 0}	39	{1, 0, 0, 1, 1, 0}	55	{1, 1, 0, 1, 1, 0}
8	{0, 0, 0, 1, 1, 1}	24	{0, 1, 0, 1, 1, 1}	40	{1, 0, 0, 1, 1, 1}	56	{1, 1, 0, 1, 1, 1}
9	{0, 0, 1, 0, 0, 0}	25	{0, 1, 1, 0, 0, 0}	41	{1, 0, 1, 0, 0, 0}	57	{1, 1, 1, 0, 0, 0}
10	{0, 0, 1, 0, 0, 1}	26	{0, 1, 1, 0, 0, 1}	42	{1, 0, 1, 0, 0, 1}	58	{1, 1, 1, 0, 0, 1}
11	{0, 0, 1, 0, 1, 0}	27	{0, 1, 1, 0, 1, 0}	43	{1, 0, 1, 0, 1, 0}	59	{1, 1, 1, 0, 1, 0}
12	{0, 0, 1, 0, 1, 1}	28	{0, 1, 1, 0, 1, 1}	44	{1, 0, 1, 0, 1, 1}	60	{1, 1, 1, 0, 1, 1}
13	{0, 0, 1, 1, 0, 0}	29	{0, 1, 1, 1, 0, 0}	45	{1, 0, 1, 1, 0, 0}	61	{1, 1, 1, 1, 0, 0}
14	{0, 0, 1, 1, 0, 1}	30	{0, 1, 1, 1, 0, 1}	46	{1, 0, 1, 1, 0, 1}	62	{1, 1, 1, 1, 0, 1}
15	{0, 0, 1, 1, 1, 0}	31	{0, 1, 1, 1, 1, 0}	47	{1, 0, 1, 1, 1, 0}	63	{1, 1, 1, 1, 1, 0}
16	{0, 0, 1, 1, 1, 1}	32	{0, 1, 1, 1, 1, 1}	48	{1, 0, 1, 1, 1, 1}	64	{1, 1, 1, 1, 1, 1}

TABLE 4.2.1.
Numbering customs.

Table 4.2.1 lists every possible behaviour allowed by Definition 4.2.2: from k_1, which always prescribes the lower class player to cooperate against a higher class opponent, to the opposite extreme k_{64}, in which the lower class player always defect, together with every possible combination between the two.

We assume that all the players follow a custom (not necessarily the same) that completely characterizes their strategic behaviour in the Class Game, which in turn determines their current payoff and, therefore, their new class when they are then placed back in the original population. At the beginning of the following round, two new players will be paired at random, and so on. Loosely speaking, the above mechanism generates a dynamic over the set of classes $\mathscr{C}$; ie for each agent a in the population, there is generated a *class history*, in the form of a sequence $c_0(a), c_1(a), \ldots, c_n(a),$, with $c_i(a) \in C$, and $c_n(a)$ agent a's class in round n. In the remainder of the paper, we will refer to this as the *Class Dynamic*.

Given our assumptions, at each point in time, the state of the system is identified by the vector $x(t) = \{x_{(c,k)}(t)\}$ of proportions of players characterized by the class c and the custom k at time t. Denote by Ω_N the set of such states, i. e. the state space of the system. Notice that Ω_N is a finite set: the underlying dynamic is therefore a stochastic process defined over a finite state space, the properties of which will be formally explored in the following sections.

4.3 Some General Theory

Our analysis of the system described above will be based on the general theory developed in Seymour (1994). In this section we give a brief synopsis of those features of the theory we require. We consider a (large) population of N agents, each of whom can be any one of m possible "types", $\{1,2,...,m\}$, at any given time.[7]

Let $x=(x_i), i=1,...,m$ be the vector of proportions of the population in each type. We assume that during each small time interval of length Δt, two individuals are chosen at random (without replacement) from the population. These individuals (and no others) then interact in some way (eg by playing the 2-player game described in section 2), the effect of which is to change their type.

Thus, if the agents have types (i,j) before the interaction, then the interaction results in a transition $(i,j)\rightarrow(i',j')$ with some specified probability, $p(i',j'|i,j)$. After the interaction, the agents return to the population, and the process is repeated in the next time interval. The transition probabilities are assumed to satisfy

$$\sum_{i',j'} p(i',j'|i,j)=1, \forall i,j \in \mathfrak{I} \qquad (4.3.1a)$$

$$p(i',j'|i,j)=p(j',i'|j,i) \qquad \text{(symmetry)} \qquad (4.3.1b)$$

The symmetry condition simply means that the interaction outcome is unaffected by whichever of the two participants is chosen first. Condition (4.3.1a) also implies that the repeated process is a discrete-time Markov process on the rational lattice

$$\Omega_N \subset \Delta^{m-1} = \{x \in \Re^m \mid 0 \le x_i \le 1, \text{and } \sum_i x_i = 1\} \qquad (4.3.2)$$

consisting of those points x for which $N_i = x_i N$ is an integer. Here, Δ^{m-1} is the $(m-1)$-dimensional simplex in its standard embedding in $\Re^m$. In fact, each interaction results in a state transition, $x \rightarrow x'$, between points in Ω_N, which, if the participants have initial types (i,j), has the form

$$x' = x + \varepsilon^{ij}\Delta x, \qquad (4.3.3)$$

where $\Delta x = 1/N$, and $\varepsilon^{ij}=(\varepsilon^{ij}_1, \varepsilon^{ij}_2, ..., \varepsilon^{ij}_m)$, is a vector-valued random variable, with each $\varepsilon^{ij}_a \in \{0, \pm 1 \pm 2\}$, such that ε^{ij}_a is the change in the ***number of individuals*** of type a which results from an interaction between agents of types (i,j). Thus, the possible values of ε^{ij}_a are given by:

$$\varepsilon^{ij}_a = \varepsilon^{ij}_a(i',j') := (\delta^{i'}_a + \delta^{j'}_a) - (\delta^i_a + \delta^j_a) \text{ with probability } p(i',j'|i,j) \qquad (4.3.4)$$

[7] As already mentioned, in the framework of section 2, a *type* is simply a pair (c,k), denoting the class and the custom of an individual.

where δ^b_a is the Kronecker-delta function: $\delta^b_a=1$ if $a=b$, and $\delta^b_a=0$ otherwise. From (4.3.4) we can easily compute the expected value of ε^{ij}_a,

$$\bar{\varepsilon}^{ij}_a = \sum_s \left[p(a,s|i,j) + p(s,a|i,j)\right] - \left(\delta^i_a + \delta^j_a\right) \tag{4.3.5}$$

We shall be interested in the limiting, continuous-time process as $N \to \infty$ and $\Delta t \to 0$. We may think of $\Delta x = 1/N$ as the probability that a particular individual will be picked at random from the population, so that, for large N, the probability that a particular individual will participate in an interaction is $2\Delta x$ (to first order in Δx).[8] It follows that $\Delta x/\Delta t$ is the (number) frequency with which a specified individual participates in an interaction. We shall take the above limits while keeping this frequency constant; ie keeping $\Delta x/\Delta t = c$, constant. For convenience we assume that the time scale is chosen so that $c=1$. The result is a deterministic system on Δ^{m-1} given by the system of differential equations:

$$\frac{dx_a}{dt} = \sum_{i,j} \bar{\varepsilon}^{ij}_a x_i x_j \tag{4.3.6}$$

Equations (4.3.6) are derived formally in Seymour (1994), but the intuition is clear: the rate of change in x_a is the sum, for all possible type pairs, of the expected changes to type a resulting from interactions between players of types (i,j), the probability with which such an interaction occurs being $x_i x_j$

Now suppose that $P_N(x)$ is a probability distribution on the finite lattice Ω_N. In general, this distribution will change under the discrete-time Markov process on Ω_N. If the Markov process is ergodic, then there is a unique stationary (ergodic) distribution, $\hat{P}_N(x)$, such that $P_N(x, n\Delta t) \to \hat{P}_N(x)$ as $n \to \infty$. Now, as explained in Seymour (1994), if $\lim_{t\to\infty} P_N$ is represented by a density[9], $\pi(x,t)$, on Δ^{m-1}, then π satisfies the *continuity equation*

$$\frac{\partial \pi(x,t)}{\partial t} + div[\bar{\varepsilon}(x)\pi(x,t)] = 0 \tag{4.3.7}$$

where $\bar{\varepsilon}(x)$ is the vector field on Δ^{m-1} given by the right hand side of (4.3.6). In particular, if the limiting ergodic distribution is represented by the density $\hat{\pi}$, then

$$div[\bar{\varepsilon}(x)\pi(x,t)] = 0 \tag{4.3.8}$$

From this we can prove

[8]An individual has two chances of being picked, one as player-I, with probability Δx and the other as player-II, with probability $\Delta x/(1-\Delta x)$.

[9]The limit keeps $\Delta x/\Delta t=1$, so that Δt goes to 0 as N goes to infinity, yielding a continuous-time model. Also, it is not strictly necessary to assume that the limit of densities is a density, we can work with measures instead. We assume densities here only to avoid uninteresting technicalities (see Seymour (1994)).

- PROPOSITION 4.3.1[10] Suppose the Markov process on Ω_N is ergodic for each $N \geq N_0$, and that the system (4.3.6) has a unique, globally asymptotically attracting equilibrium, $\hat{x}$. Then the limiting ergodic distribution on Δ^{m-1}, as $N \to \infty$, is represented by the mass-point density, $\hat{\pi}(x) = \delta[x-\hat{x}]$.

PROOF. It suffices to show that the mass-point density is the unique solution of (4.3.8). By Proposition 4.52 of Seymour (1994), the hypotheses on $\bar{\varepsilon}(x)$ imply that any solution of (4.3.7) satisfies, $\pi(x,t) \to \delta[x-\hat{x}]$ as $t \to \infty$. But, $\hat{\pi}(x)$ is a stationary solution of (4.3.7), and so $\hat{\pi}(x) = \delta[x-\hat{x}]$. ◆

4.4. ONE-CUSTOM SOCIETY

The aim of this (and the next) section is to analyze the asymptotic properties of the model described in Section 2 when only one generic custom k is followed by the entire society. In this section, therefore, the dynamic will act only on a subset of states $\Omega^k \subset \Omega^N$ with the following properties: $x_c^{k'}(t) = 0$ when $k' \neq k$ and $\sum_{c \in C} x_{(c,k)}(t) = 1$, for all t.

Let $k \in K$ be the custom used by everyone in the population. If player-I and player-II have classes i and j, respectively, then $k(i,j)$ is the probability (either 0,1 or 1/2, see (4.2.1)) that player-I will defect, and $k(i,j)$ is the probability that player-II will defect. As discussed in section 4.3, the game results in class transitions $(i,j) \to (i',j')$, and we denote by $p(i',j' \mid i,j)$ the probability for such a pairwise transition. These transition probabilities are easy to specify in this single custom case, and are

$$p(i',j' \mid i,i) = \frac{1}{4}(\delta^1_{i'}\delta^1_{j'} + \delta^2_{i'}\delta^4_{j'} + \delta^3_{i'}\delta^3_{i'} + \delta^4_{i'}\delta^2_{i'}) \tag{4.4.1a}$$

$$p(i',j' \mid i,j) = k(j,i)\delta^2_{i'}\delta^4_{j'} + k(i,j)\delta^4_{i'}\delta^2_{j'},\ (i \neq j) \tag{4.4.1b}$$

$$p(i',j' \mid i,j) = 0 \text{ otherwise} \tag{4.4.1c}$$

Thus, the only possible transitions are: $(i,i) \to (i',j') \in \{(1,1),(2,4),(3,3),(4,2)\}$, each with probability $\frac{1}{4}$, and, if $i \neq j$, $(i,j) \to (2,4)$ if $k(j,i) = 1$ (player-I Cooperates and player-II Defects), or $(i,j) \to (4,2)$ if $k(i,j) = 1$ (player-I Defects and player-II Cooperates). As explained in the previous section, these probabilities determine a Markov process on the rational lattice $\Omega_N{}^m$ contained in the 3-dimensional simplex $\Delta^3 = \{x = (x_1, x_2, x_3, x_4) \mid 0 \leq x_i \leq 1 \text{ and } \sum_i x_i = 1\}$. Here, $x_i = N_i/N$ is the proportion of the total population (of size N) in class i.

- PROPOSITION 4.4.1. For $N \geq 4$ the one-custom Markov process defined on $\Omega_N{}^k$ is ergodic.

PROOF. See Appendix D. ◆

[10] A similar result, expressed in the language of sample paths, is obtained by Boylan, [1991], corollary 2.3.

It now follows from Proposition 4.3.1 and Proposition 4.4.1 that if the derived continuous-time, deterministic system (4.3.6) admits a globally stable equilibrium, $\hat{x}$, then the limiting ergodic distribution is represented by the mass-point density, $\delta[x-\hat{x}]$. We are interested in the asymptotic properties of the class dynamic when all the individuals follow the same custom:

- PROPOSITION 4.4.2. The system (4.3.6) is independent of the custom k and has a unique equilibrium,

$$\hat{x} = \left\{ \frac{3-\sqrt{7}}{4}, \frac{\sqrt{7}-1}{4}, \frac{3-\sqrt{7}}{4}, \frac{\sqrt{7}-1}{4} \right\}$$

which is globally asymptotically stable.

PROOF. The coefficients in equations (4.3.6) are given by (4.3.5), and, using equations (4.4.1), we have

$$\bar{\varepsilon}^{i,i}{}_{i} = -3/2 \qquad (4.4.2a)$$

$$\bar{\varepsilon}^{i,i}{}_{r} = \frac{1}{2} \ (i \neq r) \qquad (4.4.2b)$$

$$\bar{\varepsilon}^{i,j}{}_{r} = (\delta^2{}_r + \delta^4{}_r) - (\delta^i{}_r + \delta^j{}_r) \ (i \neq j) \qquad (4.4.2c)$$

Note in particular, that these coefficients are *independent of the custom, k* [this is true in equation (4.4.2c) because $k(i,j)+k(j,i)=1$]. Thus, so is the deterministic dynamic (4.3.6), and hence, so is the equilibrium, $\hat{x}$. In fact, we shall show in Appendix D that the Markov process on $\Omega^k{}_N$ is independent of k. We can now obtain an explicit form for the equations (4.3.6). Thus,

$$\frac{dx_a}{dt} = \bar{\varepsilon}^{a,a}{}_{a} x_a{}^2 + \sum_{i \neq a} \bar{\varepsilon}^{i,i} x_i^2 + \sum_{i \neq j} \bar{\varepsilon}^{i,j} x_i x_j$$

$$= -(3/2) x_a{}^2 + \frac{1}{2}(||x||^2 - x_a{}^2) + (\delta^2{}_a + \delta^4{}_a)(1 - ||x||^2) - (2x_a - 2x_a{}^2)$$

$$= -2x_a + \frac{1}{2} ||x\}||^2 + (\delta^2{}_a + \delta^4{}_a)(1 - ||x||^2)$$

where, as usual, $||x||^2 = x.x = \sum_i x_i^2$. Explicitely,

$$\frac{dx_1}{dt} = -2x_1 + \frac{1}{2} ||x||^2 \qquad (4.4.3a)$$

$$\frac{dx_2}{dt} = 1 - 2x_2 - \frac{1}{2} ||x||^2 \qquad (4.4.3b)$$

$$\frac{dx_3}{dt} = -2x_3 + \frac{1}{2} ||x||^2 \qquad (4.4.3c)$$

$$\frac{dx_4}{dt} = 1 - 2x_4 - \frac{1}{2} ||x||^2 \qquad (4.4.3d)$$

We are now in a position to prove Proposition 4.4.2. Let $\xi = x_1 - x_3$ and $\eta = x_2 - x_4$. From equations (4.4.3),

$$\frac{d\xi}{dt} = -2\xi \qquad \text{and} \qquad \frac{d\eta}{dt} = -2\eta$$

Thus, $(\xi(t),\eta(t)) = (\xi_o,\eta_o)e^{-2t}$, so that $(\xi(t),\eta(t)) \to (0,0)$ as $t \to \infty$. It follows that, for any equilibrium $\hat{x}$, we have $\hat{x}_1 = \hat{x}_3$ and $\hat{x}_2 = \hat{x}_4$. Furthermore, the subspace, $(x_1,x_2) = (x_3,x_4)$, is invariant under the dynamic (4.3), and is globally attracting. It remains to show that there is a unique attracting equilibrium inside this subspace.

Under the above constraints we have, $||x||^2 = 2(x_1^2 + x_2^2)$. Also, $\sum_i x_i = 1$, reduces to $x_1 + x_2 = \frac{1}{2}$. Thus, the dynamic inside the invariant subspace is 1-dimensional, and is determined by equation (4.4.3a),

$$\frac{dx_1}{dt} = -2x_1 + (x_1^2 + (\frac{1}{2} - x_1)^2) = \frac{1}{4} - 3x_1 + 2x_1^2 \qquad (4.4.4)$$

The equilibria of (4.4.4) are, $\hat{x}_1(\pm) = \frac{1}{4}(3 \pm \sqrt{7})$. However, only the minus sign lies in the interval $[0,1]$, and is therefore the only allowable solution. Clearly then, $\hat{x}_1 = \hat{x}_3 = \hat{x}_1(-)$ and $\hat{x}_2 = \hat{x}_4 = \frac{1}{2} - \hat{x}_1 = \frac{1}{4}(\sqrt{7} - 1)$. Also note that (4.4.4) may be written

$$\frac{dx_1}{dt} = 2\ [\hat{x}_1(-) - x_1]\ [\hat{x}_1(+) - x_1]$$

The second bracket is always positive for $x_1 \in [0,1]$, and the first bracket is positive if $x_1 < \hat{x}_1$, and negative if $x_1 > \hat{x}_1$. This shows that $\hat{x}$ is globally asymptotically attracting, and therefore completes the proof of Proposition 4.4.2. ∎

Proposition 4.4.2 tells us that the limiting class distributions of the 64 customs coincide, and concentrate most of their mass between classes 2 and 4, the payoffs of the pure strategy Nash equilibrium. We provide, for illustrative purposes, the following histogram:

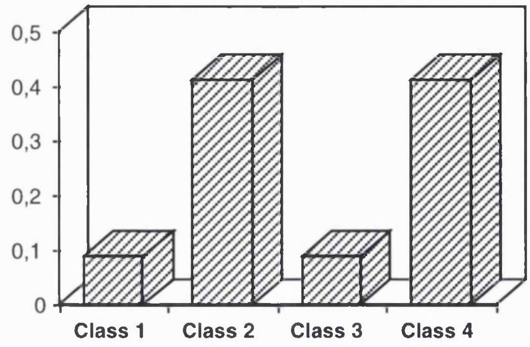


FIGURE 4.4.3.
The limiting class distribution under a generic custom k.

From Proposition 4.4.2, we can easily calculate the expected payoff $\hat{u}$ of a single Class Game, given the equilibrium distribution $\hat{x}$:

$$\overline{u} \equiv \sum_{c \in \mathscr{C}} c\hat{x}_c = (\sqrt{7} + 3)/2 = 2.82288 \qquad (4.4.5)$$

4.5. SOCIAL MOBILITY.

We can read the results of the previous section in the following way. In the spirit of Schotter [1981], we can interpret the game *Chicken* as an *inequality preserving* social institution, given the distributional effects associated with any self-enforcing class profile.[11] From this standpoint, it would be surprising if the limiting class distribution did not reflect the strategic features of the stage game which generates it. On the other hand, little is lost in equilibrium (since the proportion of plays in which the utility pie is not allocated in full is relatively small); from this perspective, the (expected) equilibrium payoff $\overline{u}$ can hence be considered as a measure of the efficiency generated by the adoption of a custom (whatever it is).

However, it is important to note that, even if the limiting distribution is the same under each custom, it does not follow that, at each point in time, the *same* individuals belong to the same class. On the contrary, the equilibrium flows between classes may well differ in magnitude from custom to custom, with only the overall proportion remaining, on average, constant. In fact, each custom is characterised, in equilibrium, by a complex, but balanced, network of flows among classes. In this section, we shall interpret these flows in terms of *social mobility*.

A first distinction has to be made at this stage. Different mobility structures may, first of all, determine different equilibrium class distributions: this is what sociologists label as *structural* mobility. This notion refers to the idea that, via the equilibrium distribution they produce, different mobility structures imply different availability of positions in higher or lower social classes. This is not, however, the only way to look at mobility: different mobility structures also influence the intertemporal movement of individuals among the social classes, for a given

[11] Here we restrict our attention to pure strategy Nash equilibria, for which the coordinating role of a custom is fully effective.

equilibrium distribution. This latter effect, known as *exchange* (or *pure*) mobility can be regarded as the dynamic counterpart of the comparative statics on different income distributions characterized by the same average income. It is this effect which we examine in this paper.

First some notation. Let $q(i'|i)$ denote the (not necessarily equilibrium) transition probability for an individual initially in class i to move to class i' after participation in a game. If the population is using a custom, k, and the prior state of the system is $x \in \Omega^k$, then it follows from (4.3.1a) that

$$q(i'\,|\,i) = \sum_{i',j'} p(i',j'|i,j) \tag{4.5.1}$$

Note that $q(i'|i)$ is independent of whether the player is player-I or player-II, and is the same in either case by the symmetry condition (4.3.1b). Using the formulae in equations (4.1), we therefore obtain

$$q(i'\,|\,i) = \frac{1}{4}x_i + \sum_{i \neq j} \; [k(j,i)\delta_{i'^2} + k(i,j)\delta_{i'^4}]x_j. \tag{4.5.2}$$

Denote by $Q(k)$ the (state dependent) matrix of transition probabilities, $q(i'|i)$, with i labelling rows and i' columns. Then, for example, the transition matrices for customs k_1 and k_{64} (that is, the two extremes in the spectrum of customs shown in Table 2.1):

$$Q(1) = \begin{array}{|c|c|c|c|} \hline \frac{1}{4}x_1 & 1-\frac{3}{4}x_1 & \frac{1}{4}x_1 & \frac{1}{4}x_1 \\ \hline \frac{1}{4}x_2 & 1-x_1-\frac{3}{4}x_2 & \frac{1}{4}x_2 & x_1+\frac{1}{4}x_2 \\ \hline \frac{1}{4}x_3 & x_4+\frac{1}{4}x_3 & \frac{1}{4}x_3 & 1-x_4-\frac{3}{4}x_3 \\ \hline \frac{1}{4}x_4 & \frac{1}{4}x_4 & \frac{1}{4}x_4 & 1-\frac{3}{4}x_4 \\ \hline \end{array} \tag{4.5.3a}$$

$$Q(64) = \begin{array}{|c|c|c|c|} \hline \frac{1}{4}x_1 & \frac{1}{4}x_1 & \frac{1}{4}x_1 & 1-\frac{3}{4}x_1 \\ \hline \frac{1}{4}x_2 & x_1+\frac{1}{4}x_2 & \frac{1}{4}x_2 & 1-x_1-\frac{3}{4}x_2 \\ \hline \frac{1}{4}x_3 & 1-x_4-\frac{3}{4}x_3 & \frac{1}{4}x_3 & x_4+\frac{1}{4}x_3 \\ \hline \frac{1}{4}x_4 & 1-\frac{3}{4}x_4 & \frac{1}{4}x_4 & \frac{1}{4}x_4 \\ \hline \end{array} \tag{4.5.3b}$$

Notice that Columns 1 and 3 are equal, and that $Q(64)$ is obtained from $Q(1)$ by interchanging columns 2 and 4. Of course, $Q(k)$ is the (conditional) transition matrix which characterises the stochastic process faced by a single player in a society operating custom k, conditional on the class distribution being x. From this *individual* perspective, x_i must be interpreted as the probability that an individual chosen at random from the population, belongs to class i. We call this the *individual* process. Note that these transition probabilities are the same for each player belonging to the same class. If we consider these probabilities as *proportions*, the same transition matrix $Q(k)$ expresses, at a *population* level, the expected motion of a population in which the proportion in class i is represented by x_i.[12] The (expected) class distribution in the next time period will then be:

$$x(t+1) = x(t)Q(k). \tag{4.5.4}$$

A caveat here. In analysing the mobility structure of our society, we will make constant reference to these transition matrixes $Q(k)$, as is customary practise in the literature on social mobility. The transition matrix $Q(k)$ describes a well-defined stochastic process over the set of classes which is *not*, however, the one described in section 2. In particular, if we describe the transition from class to class by means of the individual process $Q(k)$, we notwithstand the fact that the actual transition process comes as a result of *a game* being played between two agents randomly paired (as a matter of fact, we have always considered transition probabilities of the form $(i,j) \rightarrow (i',j')$). Nonetheless, given that $Q(k)$ is, by construction, the transition matrix which characterises the stochastic process faced by a single player *before* she has been paired, and there is no correlation between the random matching and the (possibly mixed) strategy profile that may be played, it would be surprising if the population process described by $Q(k)$ produced a different limiting class distribution than the one we constructed in the previous section.

Therefore, for consistency with the development in section 4, we expect (and obtain) the following:

- PROPOSITION 4.51. For the equilibrium distribution, $\hat{x}$, given by Proposition 4.2, we have $\hat{x} = \hat{x}\,\hat{Q}(k)$, where $\hat{Q}(k)$ is the transition matrix at $\hat{x}$. That is, $\hat{x}$ is an equilibrium of the (non-linear) discrete-time dynamic (4.54). In fact, $\hat{x}$ is the unique global attractor for this dynamic.

PROOF. See Appendix A.♠

We now move to social mobility, and compare the universal equilibrium $\hat{x}$ with respect to its exchange structure under different customs. Following a well established tradition, we do this with the aid of a ***mobility index***. Among the various alternative indices proposed in the literature, we choose the Bartholomew [1973] index, defined as follows:

- DEFINITION 4.5.1. The Bartholomew mobility index for custom k is defined by

$$B(k) = \sum_{i,i'} |i-i'|\, \hat{q}(i'|i)\,\hat{x}_i, \tag{4.5.5}$$

[12] See Kemeny and Snell [1976], sec. 6.

where $\hat{q}(i' \mid i)$ is the *(i,i')*-th entry in the equilibrium transition matrix $\hat{Q}(k)$.

This index is, of course, just the expected value in equilibrium of the possible (non-directional) class changes, $|i-i'|$, representing the possible changes in individual class resulting from a play of the game. The mobility index is easily calculated for each of the 64 possible customs, and we obtain:

- PROPOSITION 4.5.2. The mobility index B induces a linear ordering on the set of customs. This ordering identifies k_1 as having the minimum mobility, and k_{64} as having the maximum mobility.

PROOF. See Appendix A.

The intuition behind the result is not difficult to understand. What follows is the transition matrix $Q(1)$ evaluated when the society is at equilibrium, $\hat{x}$, given by Proposition 4.2:

0.0221405	0.933578	0.0221405	0.0221405
0.102859	0.602859	0.102859	0.191422
0.0221405	0.433578	0.0221405	0.522141
0.102859	0.102859	0.102859	0.691422

(4.5.6)

Note that $\hat{q}(4 \mid 4)=0.69$ and $\hat{q}(2 \mid 2)=0.6$. This means that an agent who belongs to one of the most represented classes (in equilibrium) at time *t*, will stay in the same class at time $t+1$ with a fairly high probability. The reason is that a player of class 2 under custom k_1 always cooperates against higher-class opponents (while a player of class 4 will defect in return). Therefore, after the encounter, each will find herself in the same class as before the play. Consider instead what happens under custom k_{64}:

0.0221405	0.0221405	0.0221405	0.933578
0.102859	0.191422	0.102859	0.602859
0.0221405	0.522141	0.0221405	0.433578
0.102859	0.691422	0.102859	0.102859

(4.5.7)

Now $\hat{q}(4 \mid 4)$ has gone down to 0.1 and $\hat{q}(2 \mid 2)=0.19$ (while $\hat{q}(2 \mid 4)$ and $\hat{q}(4 \mid 2)$ have moved up from 0.19 to 0.6 and from 0.1 to 0.69 respectively). This is because, under custom k_{64}, the social ranking is always reversed after the play, enhancing the overall mobility of the society.

It might be worth noting that the intuitive appeal of Proposition 4.52 is not to be taken for granted. Alternative indices do not produce the same clear-cut result, as is well known in the literature which focuses on social mobility.[13] The results of this section specify, in a formal way, why we think of k_{64} (i.e. a code of behaviour which consistently favors the lower class player in the division of the pie) as a code of conduct which promotes social mobility, and custom k_1 (where the opposite holds) as a code which discourages it. For this reason, we will hereafter refer to k_1 as the *Immobile Custom* and to k_{64} as the *Mobile Custom.*

4.6. TWO-CUSTOM SOCIETY

Up to this point, we have studied the case of a society which unanimously agrees on a unique custom. We now move to a setting in which we allow the possibility of an heterogeneous society. In this section we analyse the simplest possible case, in which agents use one of two possible customs. In the following section the analysis will be extended to a society in which all 64 customs may be present.

Different codes of behaviour are followed by different players who, occasionally, interact. The first, intuitive, implication is that coordination on one of the possible Nash equilibria of the Class Game is no longer guaranteed when agents belonging to different classes meet.[14] It may happen that both customs prescribe the same pure strategy; so that people fail to play optimally. If guaranteeing an optimal play is what a custom is for, the simple coexistence of multiple customs creates, within the constraints of our simple model, a clear inefficiency, due to the fact that players now miscoordinate much more often. The extent to which this problem can arise depends, of course, upon the relative frequencies of the different customs. Think, for example, of a custom which is comparatively rare: people who follow it are more likely to mismatch their behaviour compared to those who follow "more popular" customs (this is, essentially, because the "rare" custom fails to act as a coordinating device). If so, it is reasonable to assume that, when multiple customs coexist, some kind of *coordination learning* might take place in the population; i.e. agents modify their custom in the light of experience, with a view to finding better coordination devices.

In addition, we consider a further source of learning, which we label *aspiration learning.* According to this, an agent will change her custom with positive probability only if her realised payoff lies *below* a threshold value, which partially depends on her class (and is therefore *endogenous),* and partly on some *exogenous* constant, which is fixed and common for all the individuals in the population. In both cases, we shall assume that the probability with which an agent may switch her custom will be *proportional* to the difference between these threshold values and the game payoffs, as will be specified explicitly shortly.

We now give a formal description of this two-custom society. There are two customs, k_A and k_B. A player's (instantaneous) state is represented by a pair (i,α), with $i \in \{1,2,3,4\}$ and $\alpha \in \{A,B\}$. We denote by $\overline{\alpha}$ the complementary custom to α in $\{A,B\}$. Thus, $\overline{A} = B$ and $\overline{B} = A$.

[13]See, for example, Dardanoni [1993].

[14]Remember that when two players belonging to the same class meet, any custom prescribes the same mixed stratey (i.e. $k(i,i) = 1/2$, for any k in K.

If player-I has state (i,α) and player-II has state (j,β), then the game results in state transitions, $(i,\alpha)\to(i',\alpha')$ and $(j,\beta)\to(j',\beta')$. This transition occurs with probability

$$p(i',j';\alpha',\beta' \mid i,j;\alpha,\beta). \tag{4.6.1}$$

Our first aim is to compute all the possible non-zero transition probabilities (4.6.1). First note that the possible transitions are not completely determined by the game alone, because now each player may change her custom by applying either a *coordination test,* with probability (1-λ), or an *aspiration test,* with probability λ, where $\lambda\in(0,1)$ is some exogenous constant, after the Class Game has been played:

- THE COORDINATION TEST. We say that the customs used by the two players *coordinate* at (i,j) if

$$k_\alpha(i,j)=k_\beta(i,j). \tag{4.6.2}$$

The customs therefore fail to coordinate at (i,j) if $k_\alpha(i,j)\neq k_\beta(i,j)$. Of course, when there are only two possible customs, this can only happen if $\beta=\overline{\alpha}$. Note also that, if $i=j$ the two customs always coordinate at (i,j).

When player-I applies a coordination test, then $\alpha'=\alpha$ if α and β coordinate at (i,j), and $\alpha'=\overline{\alpha}$ otherwise. Note that a failure to coordinate is detected by player-I from her subsequent class i'. The public information available to both players prior to the game is the pair of class numbers (i,j); information about the other player's custom is not available. Thus, if the customs coordinate at (i,j) with $i\neq j$, then $(i',j')=(2,4)$ if $k_\alpha(i,j)=0$ (player-I Cooperates), and $(i',j')=(4,2)$ if $k_\alpha(i,j)=1$ (player-I Defects). However, if there is a failure of coordination, then $(i',j')=(3,3)$ if $k_\alpha(i,j)=0$, and $(i',j')=(1,1)$ if $k_\alpha(i,j)=1$.

The intuition is the following. If the main function of a custom is to lead to coordination, this is what one should check first. In this respect, we should expect each custom to work as well as any other. We confine our attention to pure strategy outcomes for the following reason. As already noticed, a custom operates effectively only when two players from different classes meet, exactly the situation where a player would expect to be guided by the custom toward an optimal play. If this does not happen, then it is reasonable to assume that players may cast doubt on the validity of the custom they follow. We interpret this process as taking place on an individual level. Thus, there is a positive probability (1-λ), which we assume to be the same for each player, that, after the game has been played, each agent applies a test of this kind, and updates her custom accordingly. As mentioned previously, with the remaining probability λ, each agent will judge the performance of her custom from a different perspective, as follows.

- THE ASPIRATION TEST. If player-I applies an aspiration test, then she will change her custom with a probability $\gamma_{ii'}$, which depends only on her class change $i\to i'$.

The particular form of the aspiration test probabilities we shall consider is an amalgam of two complementary tests, whose relative weight is measured by an exogenous constant $\eta \in (0,1)$, which is assumed to be the same for all individuals in the population. These are defined as follows:

- THE ENDOGENOUS ASPIRATION TEST. Under this test, each individual compares her relative position before and after the encounter. We assume that this part of the test will lead to a change in the custom only when there is a status loss, i.e. when $(i-i')>0$, and that the probability of such a change is proportional to this loss.

- THE EXOGENOUS ASPIRATION TEST. Under this test, each individual compares the game outcome with some exogenous constant $\sigma \in [1,4]$, here meant to represent a commonly shared "social standard" of what should be considered a *fair* split of the cake. Given that this comparison could be performed equally with the prior class i, or with the posterior class i', and we have no definite criterion for preferring one over the other (given that each alternative has its pros and cons), we assume that each individual will average out the class transition, comparing the social standard σ with $(i+i')/2$. As for the case of the endogenous aspiration test, we assume that this part of the test will lead to a change in the custom only when a player's averaged position is still below what is considered socially fair, i.e. when $(\sigma-(i+i')/2)>0$. Moreover, we will assume that this probability also will be proportional to the difference $(\sigma-(i+i')/2)$. The exact form in which the two parts of the aspiration test, exogenous and endogenous, are combined together to determine the transition probabilities $\gamma_{ii'}$ is given by

$$\gamma_{ii'}=(1/3)\eta\, h[\sigma-(i+i')/2] + (1/3)(1-\eta)\, h[i-i'], \qquad (4.6.3)$$

where $\eta \in (0,1)$ measures the relative weight of the exogenous aspiration test and $h[x]=x$ if $x>0$, $h[x]=0$ otherwise. [The factor 1/3 is for normalization purposes].

We provide a justification for this structure of the updating process, which is driven by two (rather different) forces. While coordination learning implicitly assumes that the agents are well aware of the fact that the outcome of the Class Game is the product of some interactive decision, those who update their custom according to the aspiration test need not know anything about the strategic features of situation in which they are involved (apart from the share of the pie they obtain). Otherwise, they simply do not care. While coordination learning recalls the classic "best-reply dynamics" over the space of customs (given that such a learning protocol is active only if an agent has not played a best response against the opponent's move), with our aspiration test we try to model some form of "learning through reinforcement", the object of recent interest both in the learning and experimental literature. These are the two learning models which have been given most attention by economists, as both learning schemes seem to provide a suitable framework in the economic modelling of boundedly rational agents, and their predictions have both empirical and experimental support.[15]

While coordination learning suits environments in which the agents play strategically (though not in a very sophisticated way), our aspiration learning seems to be more appropriate in situations in which people know or care very little of the strategic aspects of the environment in which they act. We do not have, in principle, any reason to favor one learning protocol over the other, and we are actually interested in testing the predictions of our model in the presence of both these effects, using λ, that is, the probability with which an agent will choose one test or the other, as a control variable in our simulations. As people hold different customs, they can also react in different ways, and we allow some degree of freedom in modelling the updating process, appealing to the two most influential candidates learning theory has provided so far.

To summarise, in a two-custom society, if player-I applies an aspiration test (with probability λ), then $\alpha' = \alpha$ in (4.6.1) with probability $(1-\gamma_{ii'})$ (i.e. there is no change of custom), and $\alpha' = \overline{\alpha}$ with probability $\gamma_{ii'}$ (i.e. there is a change of custom).

In appendix B we compute the transition probabilities (4.6.1) for general $\gamma_{ii'}$. We also prove the following.

- PROPOSITION 4.6.1. For $0<\lambda, \eta<1$ and $N \geq 2\times 4=8$, the two-custom Markov process defined on the lattice $\Omega_N^{(k_A,k_B)} \subset \Delta^7$ is ergodic.

Proof. See Appendix D.

For custom *A*, let $x_A = (x_{(1,A)}, x_{(2,A)}, x_{(3,A)}, x_{(4,A)})$ be the vector of proportions in each of the four classes, and let $x_B = (x_{(1,B)}, x_{(2,B)}, x_{(3,B)}, x_{(4,A)})$ be the vector of proportions for custom B. Thus, the total vector, $x = (x_A, x_B) \in \Delta^7 \subset \Re^8$. The deterministic equations (4.3.6) now have the form

$$\frac{dx_{(r,\rho)}}{dt} \sum_{(i,\alpha)} \sum_{(j,\beta)} \overline{\varepsilon}_{(r,\rho)}^{(i,\alpha),(j,\beta)} x_{(i,\alpha)} x_{(j,\beta)} \tag{4.6.4}$$

where the coefficients (4.3.5) are given by

$$\overline{\varepsilon}_{(r,\rho)}^{(i,\alpha),(j,\beta)} = \sum_{(s,\sigma)} p(i',j';\rho,\sigma \mid i,j;\alpha,\beta) + p(s,r;\sigma,\rho \mid i,j;\alpha,\beta)\} - \{\delta_{(r,\rho)}^{(i,\alpha)} + \delta_{(r,\rho)}^{(j,\beta)}\} \tag{4.6.5}$$

Substituting from (4.6.5) into (4.6.4) and using the symmetry condition (4.3.1b), we obtain

$$\frac{dx_{(r,\rho)}}{dt} = -2x_{(r,\rho)} + 2\sum_{(i,\alpha)} \sum_{(j,\beta)} \{\sum_{(s,\sigma)} p(i',j';\rho,\sigma \mid i,j;\alpha,\beta)\} x_{(i,\alpha)} x_{(j,\beta)}$$

[15] Here we consider best-reply dynamics as a special case of a broader class of adjustment process, namely *adaptive learning* dynamics, following the terminology of Milgrom and Roberts [1991]. Learning procedures similar to our aspiration test have been studied recently by, among others, Bendor et al. [1991] and Börgers and Sarin[1994]. For the experimental evidence, see Mookerjee and Sopher [1994] and Roth and Erev [1983]. Proportional learning rules have been proposed by Cabrales [1993], and Schlag [1994], who also provides conditions under which a similar adjustment process can be justified on normative grounds.

(4.6.6)

An explicit form for these equations is computed in Appendix B. We cannot provide a formal analysis of the solutions of the system (4.6.6), whose properties will be derived by simulation. Nonetheless, before we proceed, it may be interesting to analyse how the learning protocols we have designed would operate if they were applied to the exchange structure defined by $\hat{Q}(k)$ considered in the previous section. In other words, as an exercise, we try to gain intuition about the selection process over the custom space looking at the probabilities with which, given that a one-custom society has reached the equilibrium distribution $\hat{x}$, our coordination and aspiration tests *would* lead to an individual changing her custom.

It is already obvious that such an exercise can be carried out only for the aspiration test, since our coordination test, by construction, will never fail when everybody follows the same custom. In what follows, we will therefore calculate the effects of the various parts of the aspiration test if it were to be applied in a one-custom society. This should be seen as a measure of the ease with which a one-custom society at equilibrium could be invaded by mutants who apply an aspiration test with some probability. This is therefore a measure of *social stability*.

Suppose we are given a matrix of real numbers, $\alpha=\{\alpha_{ii'} \mid 1\leq i,i'\leq 4\}$. Then we can define an α-mobility index on the set of customs, by

$$\hat{I}_{\alpha}(k)=\sum_{i,i'} \alpha_{ii'}\hat{q}(i'\mid i)\hat{x}_{i}, \tag{4.6.7}$$

where, as usual, the `hat' refers to evaluation at the equilibrium $\hat{x}$. An α-mobility index of the form (1) induces an ordering on the set of customs by:

$$k_1\leq_{\alpha} k_2\text{q if and only if } \hat{I}_{\alpha}(k_1)\leq \hat{I}_{\alpha}(k_2). \tag{4.6.8}$$

We shall be concerned with ordering customs according to various indices of this type. If we think of $\alpha_{ii'}$ as a "reward" (if $\alpha_{ii'}>0$), or a "penalty" (if $\alpha_{ii'}<0$), payable on an agent's transition from class i to class i' after playing the stage game, then $\hat{I}_{\alpha}(k)$ is just the expected reward, at equilibrium, when everyone uses custom k. A custom which has a high α-mobility index, therefore, has a high expected α-reward. For example, when $\alpha_{ii'}=\mid i\text{-}i'\mid$, then $\hat{I}_{\alpha}(k)=B(k)$ is just the Bartholomew mobility index, and a custom with a high `expected reward' corresponds to a more mobile society. In this section, we shall be mainly interested in the case in which the $\alpha_{ii'}=\gamma_{ii'}$, the aspiration test probabilities for a custom change. Thus, $\hat{I}_{\gamma}(k)$ is just the expected probability, at equilibrium, that an application of the aspiration test will lead to a change of custom.

Now recall that $\gamma_{ii'}=\eta\gamma_{ii'}^{ex}+(1\text{-}\eta)\gamma_{ii'}^{en}$, splits into two components, an exogenous part, $\gamma_{ii'}^{ex}=(1/3)b[\sigma\text{-}(i+i')/2]$, and an endogenous part, $\gamma_{ii'}^{en}=(1/3)b[i\text{-}i']$. Hence, we may write,

$$\hat{I}_{\gamma}(k)=\eta\hat{I}_{ex}(k)+(1\text{-}\eta)\hat{I}_{en}(k), \tag{4.6.9}$$

where $\hat{I}_{ex}(k)$ is the γ^{ex}-mobility index, and $\hat{I}_{en}(k)$ is the γ^{en}-mobility index. Clearly, $\eta\,\hat{I}_{ex}(k)$ is just the expected probability that the exogenous part of the aspiration test will lead to a change of custom, and similarly, $(1\text{-}\eta)\,\hat{I}_{en}(k)$, is the expected probability for the endogenous part of the test. We aim to prove:

- PROPOSITION 4.6.2. *(i)*, $\hat{I}_{en}(\cdot)$ induces the same ordering on the set of customs as the Bartholomew index.

(ii) When $\sigma=3$, $\hat{I}_{ex}(\cdot)$ induces the reverse of the Bartholomew index ordering on the set of customs.

(iii) When $\sigma=4$, $\hat{I}_{ex}(k)$ is independent of k, and so induces the uniform ordering on the set of customs.

Before proving the proposition, we first need some lemmas.

- LEMMA 4.6.1. For a matrix α, define an index $\hat{J}_\alpha(k)$, by

$$\hat{J}_\alpha(k)=\sum_{i,j}(\alpha_{j2}\text{-}\alpha_{j4})k(i,j)\,\hat{x}_i\hat{x}_j. \tag{4.6.10}$$

Then $\hat{J}_\alpha(\cdot)$ induces the same ordering on the set of customs as $\hat{I}_\alpha(\bullet)$.

PROOF. Recall that

$$q(i'\,|\,i)=\frac{1}{4}x_i+\sum_{i\neq j}[k(j,i)\delta_{i'}{}^2+k(i,j)\delta_{i'}{}^4]x_j=\frac{1}{4}(1\text{-}\delta_{i'}{}^2\text{-}\delta_{i'}{}^4)x_i+\sum_j[k(j,i)\delta_{i'}{}^2+k(i,j)\delta_{i'}{}^4]x_j.$$

Thus,

$$\begin{aligned}\hat{I}_\alpha(k)&=\frac{1}{4}\sum_{i,i'}\alpha_{ii'}\hat{x}_i{}^2\text{-}\frac{1}{4}\sum_i(\alpha_{i2}+\alpha_{i4})\,\hat{x}_i{}^2+\sum_{i,j}[\alpha_{i2}k(j,i)+\alpha_{i,4}k(i,j)]\,\hat{x}_j\hat{x}_i\\&=\{\frac{1}{4}\sum_i(\alpha_{i1}+\alpha_{i3})\,\hat{x}_i{}^2+\sum_i\alpha_{i4}\hat{x}_i\}+\sum_{i,j}(\alpha_{i2}\text{-}\alpha_{i4})k(j,i)\,\hat{x}_j\hat{x}_i\\&=\hat{A}_\alpha+\hat{J}_\alpha(k)\end{aligned}$$

where $\hat{A}_\alpha$ is independent of the custom k. It therefore follows from the definition (2) (and the corresponding definition for $\hat{J}_\alpha$), that $\hat{I}_\alpha$ and $\hat{J}_\alpha$ induce the same ordering on the set of customs.

- LEMMA 4.6.2. For any custom k, $\sum_{i,j}k(i,j)x_ix_j=\frac{1}{2}$.

PROOF. $\sum_{i,j} k(i,j)x_i x_j = \sum_{i,j} [1\text{-}k(j,i)]x_i x_j = 1\text{-}\sum_{i,j} k(j,i)x_i x_j$. Now interchange dummy indices in the right hand sum to obtain the result.

- LEMMA 4.6.3. Let $\alpha=\{\alpha_{ii'}\}$ and $\beta=\{\beta_{ii'}\}$ be matrixes. Suppose there are real numbers, a and b, such that $a(\alpha_{i2}\text{-}\alpha_{i4})+b(\beta_{i2}\text{-}\beta_{i4})=1$ for each i. Then $\hat{I}_\alpha(\cdot)$ and $\hat{I}_\beta(\cdot)$ induce the same (resp. reverse) ordering on the set of customs if $ab<0$ (resp. $ab>0$).

PROOF By Lemma 4.6.1, it suffices to prove the result for $\hat{J}_\alpha(\cdot)$ and $\hat{J}_\beta(\cdot)$. But, $a(\alpha_{i2}\text{-}\alpha_{i4})+b(\beta_{i2}\text{-}\beta_{i4})=1$, together with Lemma 4.6.2, implies that $a\hat{J}_\alpha(k)+b\hat{J}_\beta(k)=\frac{1}{2}$.

Suppose $ab\neq 0$. Then $k_1\leq_\alpha k_2$ if and only if $\hat{J}_\alpha(k_2)\text{-}\hat{J}_\alpha(k_1)\geq 0$; if and only if $a^2\ (\hat{J}_\alpha(k_2)\text{-}\hat{J}_\alpha(k_1))\geq 0$; if and only if $ab\ (\hat{J}_\alpha(k_2)\text{-}\hat{J}_\beta(k_1))\leq 0$; if and only if, either $ab<0$ and $k_1\leq_\beta k_2$, or $ab>0$ and $k_1\geq_\beta k_2$.

PROOF OF PROPOSITION 4.6.2 *(i)*Let $\alpha_{ii'}=|i\text{-}i'|$, so that $\hat{I}_\alpha(k)=B(k)$ is the Bartholomew index, and $\beta_{ii'}=3\gamma_{ii'}^{en}=h[i\text{-}i']$. Then,

$$\{\alpha_{i2}\text{-}\alpha_{i4}\}=\{-2,-2,0,2\},$$
$$\{\beta_{i2}\text{-}\beta_{i4}\}=\{0,0,1,2\}.$$

Hence, $-\frac{1}{2}(\alpha_{i2}\text{-}\alpha_{i4})+(\beta_{i2}\text{-}\beta_{i4})=1$ for each i. Since $-\frac{1}{2}\times 1=-\frac{1}{2}<0$, it follows from Lemma 4.6.3 that α and β induce the same ordering on the set of customs. This proves *(i)*.

(ii). Let $\alpha_{ii'}=|i\text{-}i'|$, as above, and $\beta_{ii'}=3\gamma_{ii'}^{ex}=b[3\text{-}(i+i')/2]$. Then, $\{\beta_{i2}\text{-}\beta_{i4}\}=\{1,1,\frac{1}{2},0\}$. Hence, $\frac{1}{2}(\alpha_{i2}\text{-}\alpha_{i4})+2(\beta_{i2}\text{-}\beta_{i4})=1$ for each i. Since $\frac{1}{2}\times 2=1>0$, it follows from Lemma 4.6.3 that α and β induce the reverse ordering on the set of customs. This proves *(ii)*.

(iii). When $\sigma=4$, set $\beta_{ii'}=3\gamma_{ii'}^{ex}=(4\text{-}(i+i')/2)$. Then, $\{\beta_{i2}\text{-}\beta_{i4}\}=\{1,1,1,1\}$. Thus, $\hat{J}_\beta(k)=\frac{1}{2}$ by Lemma 4.6.2, which is independent of the custom k. It follows that $\hat{J}_\beta(\cdot)$, and hence $\hat{I}_{ex}(\cdot)$, induces the uniform ordering on the set of customs. This proves *(iii)*.

For a given matrix α, the α-ordering is determined by the J_α-index (4.6.10). We therefore attempt to give an interpretation of this index.. First note that, given that player-II has class j, $\hat{P}_j(\mathrm{D})=\sum_i k(i,j)\hat{x}_i$ is the probability (at equilibrium) that player-I will Defect. If, on this event, and after having played his own strategy (dictated by $k(i,j)$), player-II moves into class 2, then he receives a "reward" α_{j2}. On the other hand, if he moves into class 4 he receives a reward α_{j4}. If he does 'better' (in terms of the α-reward scheme) in the former case, then

$\alpha_{j2} > \alpha_{j4}$, whereas the reverse is true if he does better in the latter case. Thus, if moving into class 2 is more advantageous to player-II, as measured by the reward scheme α, than moving into class 4, then it is advantageous to player-II that player-I should Defect (i.e. $\hat{P}_j$(D) should be high). Conversely, if it is more advantageous to move into class 4, then it is advantageous to player-II that player-I should Cooperate. In terms of the reward scheme for the endogenous part of the aspiration learning rule, $\alpha_{ii'} = 3\gamma_{ii'}{}^{en} = h[i-i']$, we have

$$\{\alpha_{j2}\} = \{0,0,1,2\}, \qquad \{\alpha_{j4}\} = \{0,0,0,0\}.$$

Thus, for $j = 3,4$ it is advantageous for player-II that player-I should Defect, and for $j = 1,2$, player-II is indifferent to player-I's strategy. Thus, $\hat{P}_3$(D) and $\hat{P}_4$(D) should be high. The former has its maximum when $k(1,3) = k(2,3) = k(4,3) = 1$, and the latter when $k(1,4) = k(2,4) = k(3,4) = 1$. These requirements are incompatible at (3,4). However, the relevant terms in $\hat{J}_\alpha(k) = \sum_j (\alpha_{j2} - \alpha_{j4})\hat{x}_j P_j(D)$, are $\hat{x}_3 k(4,3)\hat{x}_4$ and $2\hat{x}_4 k(3,4)\hat{x}_3$. Thus, since $2\hat{x}_4\hat{x}_3 > \hat{x}_3\hat{x}_4$it is more advantageous for the overall index that $k(3,4) = 1$ rather than $k(4,3) = 1$. If we write

$$\kappa = \{k(1,2), k(1,3), k(1,4), k(2,3), k(2,4), k(3,4)\}$$

we therefore find that customs with the highest $\hat{I}_{en}(k)$-mobility index, satisfy

$$\kappa(k) = \{*,1,1,1,1,1\}, \qquad (4.6.11)$$

where * can be either 0 or 1. In particular, the Mobile Custom, $\kappa(k_{64}) = \{1,1,1,1,1,1\}$, has the highest possible $\hat{I}_{en}(k)$-mobility index, as does $\kappa(k_{32}) = \{0,1,1,1,1,1\}$. A similar analysis shows that the Immobile custom, $\kappa(k_1) = \{0,0,0,0,0,0\}$, has the lowest possible $\hat{I}_{en}(k)$-mobility index, as does $\kappa(k_{33}) = \{1,0,0,0,0,0\}$.

We can play the same game with the exogenous part of the aspiration test learning rule, $\alpha_{ii'} = 3\gamma_{ii'}{}^{ex} = h[3-(i+i')/2]$, when $\sigma = 3$. In this case,

$$\{\alpha_{j2}\} = \{3/2, 1, \tfrac{1}{2}, 0\}, \qquad \{\alpha_{j4}\} = \{\tfrac{1}{2}, 0,0,0\}.$$

Thus, to ensure a high expectation of custom change from the exogenous part of the aspiration test, we require P_j(D) to be large for $j = 1,2,3$, which in turn requires that $k(2,1) = k(3,1) = k(4,1) = 1$, and $k(1,2) = k(3,2) = k(4,2) = 1$, and $k(1,3) = k(2,3) = k(4,3) = 1$.

The incompatibilities here occur at (1,2), (1,3), (2,3), and the relevant choices are between the pairs of terms in the J_α-index, $\{\hat{x}_2 k(1,2)\hat{x}_1, \hat{x}_1 k(2,1)\hat{x}_2\}$, $\{\frac{1}{2}\hat{x}_3 k(1,3)\hat{x}_1, \hat{x}_1 k(3,1)\hat{x}_3\}$ and $\{\frac{1}{2}\hat{x}_3 k(2,3)\hat{x}_2, \hat{x}_2 k(3,2)\hat{x}_3\}$. For the first pair, it is a matter of indifference whether we take $k(1,2) = 1$ or $k(2,1) = 1$; for the second pair, it is more advantageous to take $k(3,1) = 1$; and for the third pair, it is more advantageous to take $k(3,2)=1$. Thus, to achieve a maximum of $\hat{I}_{ex}(k)$, we require $\kappa(k)$ to have the form,

$$\kappa(k)=\{*,0,0,0,0,0\}.$$

In particular, this occurs when $k=k_1$. Conversely, $\hat{I}_{ex}(k)$ is a minimum when $\kappa(k)=\{*,1,1,1,1,1\}$; eg when $k=k_{64}$.

The content of Proposition 4.6.2 can be rephrased as follows. If people mainly look at their class change to determine whether to keep their custom or not (Proposition 4.6.2*(i)),* then less mobile customs exhibit stronger stability properties (in the sense that, given they are already established, they minimize the probability of a failure in the aspiration test). On the other hand, if people mainly care about fairness considerations (Proposition 4.6.2*(ii)*), the opposite will occur, and we can expect more mobile customs to be predominant.[16] We test our (preliminary) conclusions by simulations, evaluating numerical solutions of the system (4.6.6). It seems somehow natural to start looking at the case in which people follow the two "extreme" customs, i.e. when $k_A=1$ and $k_B=64$. In this case, the society is split into two subgroups which follow, respectively, the Immobile and the Mobile Custom.

FIGURE 4.6.1.
Mobile vs. Immobile Custom.

The diagrams of Figure 4.6.1 show the limiting class distributions under both customs, as well as population shares and average payoffs under four different configurations of the parameter pair $\{\lambda,\eta\}$.[17] In the last column of the matrix associated with each diagram, the Bartholomew indexes for the one-custom case are to be compared with the one exhibited by the (equilibrium) two-custom society.

First notice that the population share of those who follow the Immobile Custom ***never*** exceeds 1/2, while it is substantially smaller than this value when η is high (i.e. when the exogenous aspiration test is performed with sufficiently high probability). When λ is also high (i.e. when the aspiration test is applied much more frequently than the coordination test) the proportion of the population which follows the Mobile Custom is almost as twice as much as the proportion which follows the Immobile Custom.

While this latter result is consistent with the content of Proposition 4.6.2(ii) (since the conjunction of λ and η high, with $\sigma=3$ provides in principle the "best of the possible environments" for the Mobile Custom), the overall poor performance (in terms of limiting population share) of the Immobile Custom under any parameter configuration is puzzling.

One reason for this outcome might be the somehow arbitrary choice of the two contestants, placed at the opposite extremes of the custom space. To test this, we stage a Round Robin

[16]Even if the "social standard" σ could take, in principle, any value within the interval [1,4], we restricted our analysis to the cases $\sigma=3$ and $\sigma=4$ for the following reasons. The choice of $\sigma=4$ describes a situation in which an individual aims to reach the top of the social ranking, regardless of her current status (since 4 is the maximum payoff she can achieve in the Class Game). The choice $\sigma=3$ can be justified on fairness grounds, since, when $\sigma=3$ the utility pie is equally divided.

[17]In the simulation displayed in Figures 6.1-2, we fixed $\sigma=3$, although (somewhat surprisingly) this choice does not seem to affect substantially the essence of the results (see Table 6.1 below).

tournament among all the customs, in which each custom is paired against each of the other 63 to form a two-custom society. Table 4.6.1 summarises the relevant summary statistics of the performance of the Mobile and Immobile customs in the tournament described above, under the same parameter settings as the previous example:

TABLE 4.6.1.
A Round Robin tournament.

Once again, while it seems to be true that an environment characterised by both λ and η small favours the Immobile Custom[18], it is also true that the Mobile Custom obtains a (much) larger share of the population in all the other cases, especially in the case when $\{\lambda,\eta\} = \{.999,.001\}$, where we should still expect a good performance of the Immobile Custom against any possible contestant (given that η is relatively "small").

Our tournament suggests that there is something missing if we look at the data relying only on the conclusions of Proposition 4.6.2. Moreover, intuition suggests that the missing factor is to be found in the effects of the aspiration test, since, in a two-custom society, whenever coordination does not take place, there is an equal push against both customs (and therefore these opposite pushes should in principle cancel out). To proceed with the analysis, we display a detailed summary of the encounter between the two least mobile customs, i.e. k_1 and k_{33}:

FIGURE 4.6.2.
The two least mobile customs (1 vs. 33)

Once again, the limiting population share of the Immobile Custom never exceeds 1/2, with the gap increasing with η and λ, as it happened in k_1 *vs.* k_{64} case.

Remember that both customs, k_1 and k_{33} exhibit the same mobility in the one-custom society, at least when mobility is measured with the aid of the equilibrium Bartholomew index. In other words, one cannot appeal to mobility alone to explain why our dynamic seems to work against the Immobile Custom, as this also happens when k_1 is paired with a custom which exhibits the same mobility.

Note that the only difference between k_1 and k_{33} is that $k_1(1,2)=0$ while $k_{33}(1,2)=1$. In other words, k_{33} prescribes the same behaviour as the Immobile Custom, under all contingencies except when an individual of class 1 meets an opponent of class 2.

We shall look at this encounter in more detail. To do so, some further terminology is needed. For $\alpha \in \{A,B\}$, let $\phi_{ii'}(\alpha,x)$ denote the conditional probability that player-I changes her custom by application of the aspiration test, given that *(i)* she has class *i*, *(ii)* her prior custom is k_α, *(iii)* player-II has class *j*, and *(iv)* the (not necessarily equilibrium) state of society is $x = (x_{(1,A)}, x_{(2,A)}, \ldots, x_{(4,A)})$.

Let $X_j^\alpha = \dfrac{x_{(j,\alpha)}}{x_{(j,\alpha)} + x_{(j,\bar{\alpha})}}$. Thus, given that a player has class j, X_j^a is the probability that he uses custom k_α. We then have

---

[18]Notice, however, that when λ and n are both small, the relative performance of the various customs tends to converge, as in the simulations shown in Figure 6.1.

$$f_{ii'}(a,x) = \sum_{i',j'} [p(i',j';\bar{\alpha},a \mid i,j;a,a) + p(i',j';\bar{\alpha},\bar{\alpha} \mid i,j;a,a)]X^a_j$$
$$+ [p(i',j';\bar{\alpha},a \mid i,j;a,\bar{\alpha}) + p(i',j';\bar{\alpha},\bar{\alpha} \mid i,j;a,\bar{\alpha})]\, X^{\bar{\alpha}}_j$$

where the transition probabilities, p(• | •), are taken as conditional on player-I using an aspiration test. The relevant formulae for these conditional probabilities are given in Appendix B, (B1.4*b,d*), and (B1.5,*b*, $\bar{b}$,*d*, $\bar{d}$). Using these formulae, together with (B1.2) and (B1.8), we obtain

$$\phi_{ii}(\alpha,x) = \frac{1}{4}\sum_{i} \cdot\gamma_{ii'}$$
$$\phi_{ii'}(\alpha,x) = [\gamma_{i2}k_\alpha(j,i) + \gamma_{i4}k_\alpha(i,j)]X^a_j + [\gamma_{i1}k_\alpha(i,j)\,k_{\bar{\alpha}}(j,i) + \gamma_{i2}k_\alpha(j,i)\,k_{\bar{\alpha}}(j,i) +$$
$$\gamma_{i3}k_\alpha(j,i)\,k_{\bar{\alpha}}(i,j) + \gamma_{i4}k_\alpha(i,j)\,k_{\bar{\alpha}}(i,j)]\, X^{\bar{\alpha}}{}_j;\ (i \neq j).$$

We shall compute the net flow of these custom transition probabilities $\Delta\phi_{ij'}(x) = \phi_{ii'}(A,x) - \phi_{ii'}(B,x))$ in the context of our example, that is, when $(i,j)=(1,2)$ and $\{A,B\}=\{1,33\}$. Remember that, when $\{A,B\}=\{1,33\}$, we have $k_A(1,2)=0$ and $k_B(1,2)=1$. Thus,

$$\phi_{12}(1,x) = \gamma_{12}X^1{}_2 + \gamma_{13}X^{33}{}_2;$$
$$\phi_{12}(33,x) = \gamma_{14}X^{33}{}_2 + \gamma_{11}X^1{}_2;$$
$$\phi_{21}(1,x) = \gamma_{24}X^1{}_1 + \gamma_{21}X^{33}{}_1;$$
$$\phi_{21}(33,x) = \gamma_{22}X^{33}{}_1 + \gamma_{23}X^1{}_1.$$

from which we obtain:

$$\Delta\phi_{12}(x) = \frac{1}{6}\eta\ [X^{33}{}_2 - X^1{}_2],$$
$$\Delta\phi_{21}(x) = \frac{1}{6}[2X_1{}^{33} - \eta(X_1{}^1 + X_1{}^{33})].$$

The above analysis refers to the *out of equilibrium* behaviour of the two-custom society, which has been completely neglected in our considerations so far. Note that if $X^{33}{}_2 = X^1{}_2$ then $\Delta\phi_{12}(x)=0$. On the other hand, if $X_1{}^1 = X_1{}^{33}$ then $\Delta\phi_{21}(x)>0$, for any $\eta \in (0,1)$. We have here a way to discriminate between the Immobile Custom and k_{33}. Although k_{33} exhibits the same mobility in equilibrium in the one-custom case, (and prescribes the same behaviour in five out of six cases), out of equilibrium (i.e. when a player of class 2 meets an opponent of class 1), it prescribes a more efficient behaviour for the higher-class player. The latter is in fact the one who is more likely to change her custom, since she has more to loose in the encounter: if she cooperates against a lower-class opponent, she can avoid the inefficient outcome (1,1). This in turn will reduce the probability of changing her custom (measured by $\phi_{21}(k,x)$), producing the slight preference for custom k_{33} exhibited by our simulations.

4.7. The full 64-custom society

In this final section we consider the case of a society in which any of the 64 possible customs may be present. As in the two-custom case, we suppose that players apply a coordination test or an aspiration test. However, whereas in the two custom case, an agent changing her custom must change to the only other alternative available, in the full system there are many possible choices. We shall assume that custom changes (for whatever reason) are effected only by *local* modification. Thus, if player-I's state prior to the game is (i,k_I), and player-II's is (j,k_{II}), then player-I's strategy in the game is to Defect with probability $k_I(i,j)$. If player-I has cause to change her custom as a result of this experience, then *she only modifies her* (i,j)*-response and nothing else*; ie the change of custom is $k_I \to k_I'$, where

$$k_I'(r,s) = \begin{cases} k_I(s,r) \text{ if } (r,s) = (i,j) \text{ or } (j,i) \\ \quad k_I(r,s) \text{ otherwise} \end{cases} \tag{4.7.1}$$

Notice that $k_{I'}=k_I$ if $i=j$, so that two players of the same class never modify their customs in response to the game outcome. This is in contrast to the two-custom case discussed previously.

We call such a change a *local* modification because it depends only on the information available to player-I in the particular game, namely the prior class types of the players, (i,j), and the posterior class type of player-I. For example, if there is a failure of coordination at (i,j) (see (4.6.2)), then, necessarily $i \neq j$, and player-II will make an unexpected move (Cooperate instead of Defect, or vice-versa). If player-I applies a coordination test, then she will attempt to coordinate with player-II in future by coordinating with whatever strategy player-II played when the same (i,j) situation arises again. Similarly, if the custom change is the result of an aspiration test, then player-I reasons that, in order to do better in a similar situation next time, she should play the alternative strategy. This leads to the rule (4.7.1). In doing this, player-I does not make any assumption about what player-II will do in response to a coordination failure. This is because she only ever knows player-II's class, and not what custom he might be using. Her prior working assumption is always that player-II's custom is the same as hers. If she didn't make this assumption, then the notion of a custom as a coordinating device would lose its force; she might just as well pick a strategy at random.

The result of a game, together with the application of coordination or aspiration tests and local modification with per-player probabilities $(1-\lambda)$ and λ, respectively, is a transition of the form, $((i,k_I),(j,k_{II})) \to ((i',k_I'),(j',k_{II}'))$. We denote by

$$p(i',j';k_{I'},k_{II}' \mid i,j;k_I,k_{II}) \tag{4.7.2}$$

the probability with which such a transition occurs. We compute the transition probabilities (4.7.2) in Appendix C. Again we have

- Proposition 4.7.1. For $0<\lambda, \eta<1$ and $N \geq 64\times 4=256$, the 64-custom Markov process defined on the lattice $\Omega_N \subset \Delta^{255}$, is ergodic.

PROOF. See Appendix D.♠

For simulation purposes in this section, we use the form (4.6.3) for $\gamma_{ii'}$. We are mainly interested in checking whether the push in favour of "more mobile" customs we observed in the two-custom case still operates in the full 64-custom society. Figure 4.7.1 displays graphically the summary statistics of a set of 400 simulations, 100 for each of the parameter settings $\{\lambda,\eta\}$, which we used for the two-custom society:

FIGURE 4.7.1.
The full 64-custom society

Customs are ordered with respect of their mobility index *B(k)* evaluated at the estimated equilibrium class distribution of the corresponding 64-custom simulation. As can be spotted from the graphs, the trend toward more mobile customs is evident in all cases. Moreover, this preference increases with both λ and η as in the two-custom case.

4.8. CONCLUSIONS

Our research program is clearly at a preliminary stage, and the reader may feel uncomfortable finding *ad hoc* assumptions every now and then. Our first (and cheap-talk) justification invokes simplicity and mathematical tractability: the model appears to be complicated enough, even with all these (some would argue) quite implausible short-cuts.

Still, we claim to have some ground for further defence. This is why we tried to justify each assumption on the basis of some plausible intuition (this is, at least, the authors' hope). However, we devote these concluding remarks to point out some critical points, which should be interpreted as guidelines for future research.

Non-equilibrium behaviours. We restrict our attention to the set of behaviours specified in definition 2.2 as "customs". This assumptions is, of course, not innocent, and we have no reason to conjecture that, once we allowed a larger set of possible behaviours, our conclusions would not differ in a substantial way. Think, for example, of a sub-population of 'die-harders', prone to defect regardless of the identity of their opponents: to what extent would their presence affect the dynamics of the system? Or, alternatively, think of sub-populations of 'mixers', playing a mixed strategy all the time, or 'doves', cooperating with anybody, etc... All these possibilities are ruled out by Definition 4.2.2, and we simply do not attempt to predict what would happen if the set of possible behaviours were enlarged substantially.

"Smoother" class ranking In our model, the class of a player is simply the payoff received the last round she was called to play. In other words, we allow the possibility that a player moves from the top of the social ranking to the bottom (or vice versa) within a single period. This feature of the model is indeed unrealistic, and it could be modified, for example, if we defined the class as some weighted average of the last *n* payoffs received in the Class Game.

The role of memory. A similar remark can be addressed to the structure of the learning process. Our players have no memory, and every comparison is made with respect to current payoffs: a grain of 'bad luck' could completely upset the *weltansbaung* of a player, regardless any other consideration. Alternative sets of assumptions could design the learning process in a more realistic way; for instance, we might let players apply our coordination and aspiration tests on a longer string of outcomes ("don't let your choices be driven by your last impression!"). Our conjecture is that a modification in this direction should not change our conclusions in a substantial way, but we are not able, at this stage, to provide a formal justification of this claim.

More complex Class Games. One could argue that some of our results crucially depend upon the particular features of the Class Game we have chosen, namely: *Chicken*. It would be interesting to apply the same analysis to more complex strategic frameworks. For example, a natural extension of the model would be to the classic *Nash Demand Game,* where the strategic framework of the *Chicken* game is extended to a much richer strategy space for each player.

APPENDIX A
ONE-CUSTOM SOCIETY

A1. PROOF OF PROPOSITION 4.5.1

From equations (4.5.1), we obtain the explicit form for the discrete dynamic (4.5.4):

$$x_1'=x_3'=\frac{1}{4}||x||^2 \qquad \text{(A1.1}a\text{)}$$

$$x_2'=x_4'=\frac{1}{2}-\frac{1}{4}||x||^2 \qquad \text{(A1.1}b\text{)}$$

where we have written $x=x(t)$ and $x'=x(t+1)$, and $||x||=\sum_i x_i^2$. Note that this dynamic is independent of the custom *k*, and so any equilibria will be universal.

Clearly, after at most one time step, the dynamic is confined to the invariant subspace, $x_1=x_3, x_2=x_4$. We also have the constraint, $\sum_i x_i=1$. Thus, in this constarined space, $||x||^2=2\,(x_1^2+(\frac{1}{2}-x_1)^2)=\frac{1}{2}-2x_1+4x_1^2$. The resulting constrained dynamic is therefore 1-dimensional, and can be written in the form

$$x'-x=1/8-(3/2)x+x^2, \qquad \text{(A1.2)}$$

where $x=x_1$. The quadratic factorizes as, $(x-\hat{x}_+)\,(x-\hat{x}_-)$, where $\hat{x}\pm=\frac{1}{4}(3\pm\sqrt{7})$. The first factor is always negative since $\hat{x}_+>1$, and the second factor is negative if $0\leq x<\hat{x}_-$, and positive if $\hat{x}_-<x\leq 1$. It follows that $\hat{x}_-$ is the unique global attractor for this dynamic, and hence that $\hat{x}=(\frac{1}{4}(3-\sqrt{7}),\frac{1}{4}(\sqrt{7}-1),\frac{1}{4}(3-\sqrt{7}),\frac{1}{4}(\sqrt{7}-1))$ is the unique global attractor for the dynamic (A1.1).

A2 PROOF OF PROPOSITION 4.5.2. Using the formula (4.5.1) for the transition probabilities, we can compute the Bartholemew mobility index for each custom, and order the customs by increasing mobility. The resulting order is shown in Table A.1. Note that the custom k_1 (together with custom k_{33} has mobility index which is strictly less than any other custom, and the Liberal custom has an index which is strictly greater than any other. Table A.1 therefore constitutes a proof of Proposition 4.5.2

κ	B(κ)	κ	B(κ)	κ	B(κ)	κ	B(κ)
	0.677124	22	0.838562	3	1.35425	24	1.51569
	0.677124	25	0.838562	35	1.35425	27	1.51569
	0.692811	54	0.838562	19	1.36994	56	1.51569
	0.692811	57	0.838562	51	1.36994	59	1.51569
	0.75	10	0.895751	4	1.42712	12	1.57288
	0.75	13	0.895751	7	1.42712	15	1.57288
	0.75	42	0.895751	36	1.42712	44	1.57288
	0.75	45	0.895751	39	1.42712	47	1.57288
	0.765687	26	0.911438	20	1.44281	28	1.58856
	0.765687	29	0.911438	23	1.44281	31	1.58856
	0.765687	58	0.911438	51	1.44281	60	1.58856
	0.765687	61	0.911438	55	1.44281	63	1.58856
	0.822876	14	0.968627	8	1.5	16	1.64575
	0.822876	46	0.968627	11	1.5	48	1.64575
	0.822876	30	0.984313	40	1.5	32	1.66144
	0.822876	62	0.984313	43	1.5	64	1.66144

TABLE A.1.
Customs ordered by increasing Bartholomew mobility index.

Appendix B. Two-Custom Society

B1. Explicit forms for transition probabilities and deterministic equations.

To compute the transition probabilities (4.6.1), we must consider four cases.

- Case 1. Both players apply a coordination test. This occurs with probability $(1-\lambda)^2$. In this case the possible non-zero transition probabilities are:

$$p(i',j';\alpha,\beta \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)p_\alpha(i',j' \mid i,j) \tag{B1.1a}$$
$$p(i',j';\bar{\alpha},\bar{\beta} \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j' \mid i,j) \tag{B1.1b}$$

Here, $\delta_{\alpha,\beta}$ is the *coordination index* function: $\delta_{\alpha,\beta}(i,j)=1$ if α and β coordinate at (i,j), and $\delta_{\alpha,\beta}(i,j)=0$ otherwise. Also, $\bar{\delta}_{\alpha,\beta}(i,j)=1-\delta_{\alpha,\beta}(i,j)$, so that α and β fail to coordinate at (i,j) if and only if $\bar{\delta}_{\alpha,\beta}(i,j)=1$. Clearly, $\delta_{\alpha,\beta}(i,j)=\delta_{\alpha,\beta}(j,i)$. The probabilities $p_\alpha(i',j' \mid i,j)$ and $\bar{p}_\alpha(i',j' \mid i,j)$, are the coordinated and uncoordinated transition prbabilities, respectively; i.e.

$$p_\alpha(i',j' \mid i,i)=(1/4)(\delta^1{}_{i'}\delta^1{}_{j'}+\delta^2{}_{i'}\delta^4{}_{j'}+\delta^3{}_{i'}\delta^3{}_{j'}+\delta^4{}_{i'}\delta^2{}_{j'}) \tag{B1.2a}$$
$$p_\alpha(i',j' \mid i,j)=k_\alpha(j,i)\delta^2{}_{i'}\delta^4{}_{j'}+k_\alpha(i,j)\delta^2{}_{j'}\delta^4{}_{i'} \quad (i\neq j) \tag{B1.2b}$$
$$\bar{p}_\alpha(i',j' \mid i.i)=0 \tag{B1.2c}$$
$$\bar{p}_\alpha(i',j' \mid i,j)=k_\alpha(i,j)\delta^1{}_{i'}\delta^1{}_{j'}+k_\alpha(j,i)\delta^3{}_{i'}\delta^3{}_{j'} \quad (i\neq j) \tag{B1.2d}$$

- Case 2. Player-I uses a coordination test and player-II uses an aspiration test. This occurs with probability $(1-\lambda)\lambda$. The possible non-zero transition probabilities are

$$p(i',j';\alpha,\beta \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)(1-\gamma_{jj'})p_\alpha(i',j' \mid i,j) \tag{B1.3a}$$
$$p(i',j';\alpha,\bar{\beta} \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)\gamma_{jj'}p_\alpha(i',j' \mid i,j) \tag{B1.3b}$$
$$p(i',j';\bar{\alpha},\beta \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)(1-\gamma_{jj'})\bar{p}_\alpha(i',j' \mid i,j) \tag{B1.3c}$$
$$p(i',j';\bar{\alpha},\bar{\beta} \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)\gamma_{jj'}\bar{p}_\alpha(i',j' \mid i,j) \tag{B1.3d}$$

- Case 3. Player-I uses an aspiration test and player-II uses a coordination test. This occurs with probability $\lambda(1-\lambda)$. The possible non-zero transition probabilities are

$$p(i',j';\alpha,\beta \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)(1-\gamma_{ii'})p_\alpha(i',j' \mid i,j) \tag{B1.4a}$$
$$p(i',j';\bar{\alpha},\beta \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)\gamma_{ii'}p_\alpha(i',j' \mid i,j) \tag{B1.4b}$$
$$p(i',j';\alpha,\bar{\beta} \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)(1-\gamma_{ii'})\bar{p}_\alpha(i',j' \mid i,j) \tag{B1.4c}$$
$$p(i',j';\bar{\alpha},\bar{\beta} \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)\gamma_{ii'}\bar{p}_\alpha(i',j' \mid i,j) \tag{B1.4d}$$

- CASE 4. Both players use an aspiration test. This occurs with probability λ^2. The possible non-zero transition probabilities are

$$p(i',j';\alpha,\beta \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)(1-\gamma_{ii'})(1-\gamma_{jj'})p_\alpha(i',j' \mid i,j) \quad \text{(B1.5a)}$$
$$p(i',j';\alpha,\beta \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)(1-\gamma_{ii'})(1-\gamma_{jj'})\bar{p}_\alpha(i',j' \mid i,j) \quad \text{(B1.5}\bar{a}\text{)}$$
$$p(i',j';\bar{\alpha},\beta \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)\gamma_{ii'}(1-\gamma_{jj'})p_\alpha(i',j' \mid i,j) \quad \text{(B1.5b)}$$
$$p(i',j';\bar{\alpha},\beta \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)\gamma_{ii'}(1-\gamma_{jj'})\bar{p}_\alpha(i',j' \mid i,j) \quad \text{(B1.5}\bar{b}\text{)}$$
$$p(i',j';\alpha,\bar{\beta} \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)(1-\gamma_{ii'})\gamma_{jj'}p_\alpha(i',j' \mid i,j) \quad \text{(B1.5c)}$$
$$p(i',j';\alpha,\bar{\beta} \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)(1-\gamma_{ii'})\gamma_{jj'}\bar{p}_\alpha(i',j' \mid i,j) \quad \text{(B1.5}\bar{c}\text{)}$$
$$p(i',j';\bar{\alpha},\bar{\beta} \mid i,j;\alpha,\beta)=\delta_{\alpha,\beta}(i,j)\gamma_{ii'}\gamma_{jj'}p_\alpha(i',j' \mid i,j) \quad \text{(B1.5d)}$$
$$p(i',j';\bar{\alpha},\bar{\beta} \mid i,j;\alpha,\beta)=\bar{\delta}_{\alpha,\beta}(i,j)\gamma_{ii'}\gamma_{jj'}\bar{p}_\alpha(i',j' \mid i,j) \quad \text{(B1.5}\bar{d}\text{)}$$

We can now compute the unconditional transition probabilities (4.6.1). For example,

$$p(i',j';\alpha,\beta \mid i,j;\alpha,\beta)=(1-\lambda)^2\text{(B1.1a)}+(1-\lambda)\lambda\text{(B1.3a)}+\lambda(1-\lambda)\text{(B1.4a)}+\lambda^2 \quad \text{(B1.5a)}.$$

We obtain thus,

$$p(i',j';\alpha,\beta \mid i,j;\alpha,\beta)=$$
$$(1-\lambda\gamma_{ii'})(1-\lambda\gamma_{jj'})\delta_{\alpha,\beta}(i,j)p_\alpha(i',j' \mid i,j)+\lambda^2(1-\gamma_{ii'})(1-\gamma_{jj'})\bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j' \mid i,j) \quad \text{(B1.6a)}$$

$$p(i',j';\alpha,\bar{\beta} \mid i,j;\alpha,\beta)=$$
$$(1-\lambda\gamma_{ii'})\lambda\gamma_{jj'}\delta_{\alpha,\beta}(i,j)p_\alpha(i',j' \mid i,j)+\lambda(1-\gamma_{ii'})(1-\lambda(1-\gamma_{jj'}))\bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j' \mid i,j) \quad \text{(B1.6b)}$$

$$p(i',j';\bar{\alpha},\beta \mid i,j;\alpha,\beta)=$$
$$\lambda\gamma_{ii'}(1-\lambda\gamma_{jj'})\delta_{\alpha,\beta}(i,j)p_\alpha(i',j' \mid i,j)+(1-\lambda(1-\gamma_{ii'}))\lambda(1-\gamma_{jj'})\bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j' \mid i,j) \quad \text{(B1.6c)}$$

$$p(i',j';\bar{\alpha},\bar{\beta} \mid i,j;\alpha,\beta)=$$
$$\lambda^2\gamma_{ii'}\gamma_{jj'}\delta_{\alpha,\beta}(i,j)p_\alpha(i',j' \mid i,j)+(1-\lambda(1-\gamma_{ii'}))(1-\lambda(1-\gamma_{jj'}))\bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j' \mid i,j) \quad \text{(B1.6d)}$$

We now compute an explicit form for the deterministic equations (4.6.4). Note from equations (B1.2) that

$$\sum_s p_\alpha(i',j' \mid i,i)=\frac{1}{4} \quad \text{(B1.7a)}$$
$$\sum_s p_\alpha(i',j' \mid i,j)=k_\alpha(j,i)\delta^2{}_r+k_\alpha(i,j)\delta^4{}_r \quad (i\neq j) \quad \text{(B1.7b)}$$
$$\sum_s \bar{p}_\alpha(i',j' \mid i,i)=0 \quad \text{(B1.7c)}$$
$$\sum_s \bar{p}_\alpha(i',j' \mid i,j)=k_\alpha(i,j)\delta^1{}_r+k_\alpha(j,i)\delta^3{}_r \quad (i\neq j) \quad \text{(B1.7d)}$$

Also, noting that $k_\alpha(i,j)^2=k_\alpha(i,j)$ and $k_\alpha(i,j)k_\alpha(j,i)=0$, for $i\neq j$, it is easy to check that, for $i\neq j$,

$$\delta_{\alpha,\beta}(i,j)k_\alpha(i,j)=\delta_{\alpha,\beta}(i,j)k_\beta(i,j)=k_\alpha(i,j)k_\beta(i,j) \quad \text{(B1.8a)}$$
$$\bar{\delta}_{\alpha,\beta}(i,j)k_\alpha(i,j)=\bar{\delta}_{\alpha,\beta}(i,j)k_\beta(j,i)=k_\alpha(i,j)k_\beta(j,i) \quad \text{(B1.8b)}$$

It therefore follows from equations (B1.7) and (B1.8) that

$$\sum_s \delta_{\alpha,\beta}(i,i)p_\alpha(i',j' \mid i,i)=\frac{1}{4} \quad \text{(B1.9a)}$$
$$\sum_s \delta_{\alpha,\beta}(i,j)p_\alpha(i',j' \mid i,j)=k_\alpha(j,i)k_\beta(j,i)\delta^2{}_r+k_\alpha(i,j)k_\beta(i,j)\delta^4{}_r \quad (i\neq j) \quad \text{(B1.9b)}$$
$$\sum_s \bar{\delta}_{\alpha,\beta}(i,i)\bar{p}_\alpha(i',j' \mid i,i)=0 \quad \text{(B1.9c)}$$
$$\sum_s \bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j' \mid i,j)=k_\alpha(i,j)k_\beta(j,i)\delta^1{}_r+k_\alpha(j,i)k_\beta(i,j)\delta^3{}_r \quad (i\neq j) \quad \text{(B1.9d)}$$

Using equations (B1.6) and (B1.9), we first obtain an explicit form for equations (4.6.4) with ρ=A. Thus, equations (4.6.4) can be written

$$\begin{aligned}\frac{dx_{(r,A)}}{dt}\,\frac{dx_{(r,A)}}{dt} &= -2x_{(r,A)} \\ &+2\sum_{i,j}\{\sum_s [p(i',j';A,A \mid i,j;A,A)+p(i',j';A,B \mid i,j;A,A)]\backslash\}x_{(i,A)}x_{(j,A)} \\ &+2\sum_{i,j}\{\sum_s [p(i',j';A,A \mid i,j;A,B)+p(i',j';A,B \mid i,j;A,B)]\}x_{(i,A)}x_{(j,B)} \\ &+2\sum_{i,j}\{\sum_s [p(i',j';A,A \mid i,j;B,A)+p(i',j';A,B \mid i,j;B,A)\ ]\}x_{(i,B)}x_{(j,A)} \\ &+2\sum_{i,j}\{\sum_s [p(i',j';A,A \mid i,j;B,B)+p(i',j';A,B \mid i,j;B,B)\ ]\}x_{(i,B)}x_{(j,B)}\end{aligned} \quad \text{(B1.10)}$$

Now, from (B1.6a,b) and (B1.3c) we have

$$\begin{aligned}p(i',j';A,A \mid i,i;A,A)&+p(i',j';A,B \mid i,i;A,A) \\ &= [(1-\lambda\gamma_{ir})(1-\lambda\gamma_{is})+(1-\lambda\gamma_{ir})\lambda\gamma_{is}\ ]\delta_{(A,A)}(i,j)p_A(i',j' \mid i,i) \\ &=(1-\lambda\gamma_{ir})\delta_{(A,A)}(i,j)p_A(i',j' \mid i,i)\end{aligned}$$

Thus, from (B1.9a) we obtain

$$\sum_s [p(i',j';A,A \mid i,i;A,A)+p(i',j';A,B \mid i,i;A,A)\ ]=\frac{1}{4}(1-\lambda\gamma_{ir}) \quad \text{(B1.11a)}$$

Similar calculations yield

$$\sum_s [p(i',j';A,A \mid i,i;A,B)+p(i',j';A,B \mid i,i;A,B)\ ]=\frac{1}{4}(1-\lambda\gamma_{ir}) \quad \text{(B1.11b)}$$
$$\sum_s [p(i',j';A,A \mid i,i;B,A)+p(i',j';A,B \mid i,i;B,A)\ ]=\frac{1}{4}\lambda\gamma_{ir} \quad \text{(B1.11c)}$$

$$\sum_s \ [p(i',j';A,A\,|\,i,i;B,B)+p(i',j';A,B\,|\,i,i;B,B)\ ]=\frac{1}{4}\lambda\gamma_{ir} \qquad \text{(B1.11d)}$$

From equations (B1.11) we can now pull out the term in the summations on the right hand side of (B1.10) for which $i=j$, to obtain

$$\frac{1}{2}||\,x_A\,||^2+\frac{1}{2}x_A\cdot x_B-\lambda\frac{1}{2}\sum_i \ \gamma_{ir}\ [x_{(i,A)}{}^2-x_{(i,B)}{}^2\ ] \qquad \text{(B1.12)}$$

where $u\cdot v=\sum_i \ u_i v_i$, and $||\,u\,||^2=u\cdot u$.

We now turn to the off-diagonal cases, $i\neq j$. From (B1.6a,b), we have

$$p(i',j';A,A\,|\,i,j;A,A)+p(i',j';A,B\,|\,i,j;A,A)$$
$$=(1-\lambda\gamma_{ir})\delta_{(A,A)}(i,j)p_A(i',j'\,|\,i,j)+\lambda(1-\gamma_{ir})\,\bar{\delta}_{(A,A)}(i,j)\bar{p}_A(i',j'\,|\,i,j)$$

and using (B1.7b,d) gives

$$\sum_s \ [p(i',j';A,A\,|\,i,j;A,A)+p(i',j';A,B\,|\,i,j;A,A)]=(1-\lambda\gamma_{ir})[k_A(j,i)\delta^2{}_r+k_A(i,j)\delta^4{}_r] \quad \text{(B1.13a)}$$

Remember that $k_\alpha(i,j)^2=k_\alpha(i,j)$ and $k_\alpha(i,j)k_\alpha(j,i)=0$. Similar calculations yield

$$\sum_s \ [p(i',j';A,A\,|\,i,j;A,B)+p(i',j';A,B\,|\,i,j;A,B)]=$$
$$q(1-\lambda\gamma_{ir})[k_A(j,i)k_B(j,i)\delta^2{}_r+k_A(i,j)k_B(i,j)\delta^4{}_r\ ]$$
$$+\lambda(1-\gamma_{ir})\ [k_A(i,j)k_B(j,i)\delta^1{}_r+k_A(j,i)k_B(i,j)\delta^3{}_r\ ] \qquad \text{(B1.13b)}$$

$$\sum_s \ [p(i',j';A,A\,|\,i,j;B,A)+p(i',j';A,B\,|\,i,j;B,A)\ ]=$$
$$q\lambda\gamma_{ir}\ [k_B(j,i)k_A(j,i)\delta^2{}_r+k_B(i,j)k_A(i,j)\delta^4{}_r\ ]$$
$$+\ (1-\lambda(1-\gamma_{ir})\ )\ [k_B(i,j)k_A(j,i)\delta^1{}_r+k_B(j,i)k_A(i,j)\delta^3{}_r\ ] \qquad \text{(B1.13c)}$$

$$\sum_s \ [p(i',j';A,A\,|\,i,j;B,B)+p(i',j';A,B\,|\,i,j;B,B)\ ]=\lambda\gamma_{ir}\ [k_B(j,i)\delta^2{}_r+k_B(i,j)\delta^4{}_r\ ]$$

(B1.13d)

Now multiply (B1.13b) by $x_{(i,A)}x_{(j,B)}$ and sum over $i\neq j$, then multiply (B1.13c) by $x_{(i,B)}x_{(j,A)}$, sum over $i\neq j$ and interchange the dummy indices, $i\leftrightarrow j$, to obtain

$$\sum_{i\neq j} \ \{\sum_s \ [p(i',j';A,A\,|\,i,j;A,B)+p(i',j';A,B\,|\,i,j;A,B)]\}x_{(i,A)}x_{(j,B)}$$
$$+\sum_{i\neq j} \ \{\sum_s \ [p(i',j';A,A\,|\,i,j;B,A)+\ p(i',j';A,B\,|\,i,j;B,A)\ ]\}x_{(i,B)}x_{(j,A)}$$

$$=\sum_{i\neq j}\{k_A(i,j)k_B(j,i)\delta^1{}_r+k_A(j,i)k_B(j,i)\delta^2{}_r+k_A(j,i)k_B(i,j)\delta^3{}_r+k_A(i,j)k_B(i,j)\delta^4{}_r\}x_{(i,A)}x_{(j,B)}$$
$$-\lambda\sum_{i\neq j}\{(\gamma_{ir}-\gamma_{jr})[k_A(i,j)k_B(j,i)\delta^1{}_r+k_A(j,i)k_B(i,j)\delta^3{}_r]$$
$$+[\gamma_{ir}k_A(j,i)k_B(j,i)-\gamma_{jr}k_A(i,j)k_B(i,j)]\delta^2{}_r+[\gamma_{ir}k_A(i,j)k_B(i,j)-\gamma_{jr}k_A(j,i)k_B(j,i)]\delta^4{}_r\}x_{(i,A)}x_{(j,B)} \tag{B1.14}$$

Finally, multiplying (B1.13a) by $x_{(i,A)}x_{(j,A)}$, (B1.13d) by $x_{(i,B)}x_{(j,B)}$, and summing over $i\neq j$, we obtain terms

$$\sum_{i\neq j}\{\sum_{s}[p(i',j';A,A\,|\,i,j;A,A)+p(i',j';A,B\,|\,i,j;A,A)]\}x_{(i,A)}x_{(j,A)}$$
$$=\sum_{i\neq j}(1-\lambda\gamma_{ir})[k_A(j,i)\delta^2{}_r+k_A(i,j)\,\delta^4{}_r]\,x_{(i,A)}x_{(j,A)} \tag{B1.15}$$

$$\sum_{i\neq j}\{\sum_{s}[p(i',j';A,A\,|\,i,j;B,B)+p(i',j';A,B\,|\,i,j;B,B)]\}x_{(i,B)}x_{(j,B)}$$
$$=\lambda\sum_{i\neq j}\gamma_{ir}[k_B(j,i)\delta^2{}_r+k_B(i,j)\delta^4{}_r]x_{(i,B)}x_{(j,B)} \tag{B1.16}$$

By combining (B1.12), (B1.14), (B1.15) and (B1.16), we can now write down explicit forms for equations (B1.10), to obtain:

$$\frac{dx_{(1,A)}}{dt}=-2x_{(1,A)}+\frac{1}{2}\,||x_A||^2+\frac{1}{2}x_A\cdot x_B+2\sum_{i\neq j}k_A(i,j)k_B(j,i)x_{(i,A)}x_{(j,B)}$$
$$-\lambda\{\frac{1}{2}\sum_{i}\gamma_{i1}[x_{(i,A)}{}^2-x_{(i,B)}{}^2]+2\sum_{i\neq j}[\gamma_{i1}-\gamma_{j1}]k_A(i,j)k_B(j,i)x_{(i,A)}x_{(j,B)}\} \tag{B1.17a}$$

$$\frac{dx_{(2,A)}}{dt}=-2x_{(2,A)}+\frac{1}{2}\,||x_A||^2+\frac{1}{2}x_A\cdot x_B$$
$$+2\sum_{i\neq j}k_A(j,i)x_{(i,A)}x_{(j,A)}+2\sum_{i\neq j}k_A(j,i)k_B(j,i)x_{(i,A)}x_{(j,B)}$$
$$-\lambda\{\frac{1}{2}\sum_{i}\gamma_{i2}[x_{(i,A)}{}^2-x_{(i,B)}{}^2]+2\sum_{i\neq j}\gamma_{i2}[k_A(j,i)x_{(i,A)}x_{(j,A)}-k_B(j,i)x_{(i,B)}x_{(j,B)}]$$
$$+2\sum_{i\neq j}[\gamma_{i2}k_A(j,i)k_B(j,i)-\gamma_{j2}k_A(i,j)k_B(i,j)]x_{(i,A)}x_{(j,B)}\} \tag{B1.17b}$$

$$\frac{dx_{(3,A)}}{dt}=-2x_{(3,A)}+\frac{1}{2}\,||x_A||^2+\frac{1}{2}x_A\cdot x_B+2\sum_{i\neq j}k_A(j,i)k_B(i,j)x_{(i,A)}x_{(j,B)}$$
$$-\lambda\{\frac{1}{2}\sum_{i}\gamma_{i3}[x_{(i,A)}{}^2-x_{(i,B)}{}^2]+2\sum_{i\neq j}[\gamma_{i3}-\gamma_{j3}]k_A(j,i)k_B(i,j)x_{(i,A)}x_{(j,B)}\} \tag{B1.17c}$$

$$\frac{dx_{(4,A)}}{dt} = -2x_{(4,A)} + \frac{1}{2}||x_A||^2 + \frac{1}{2}x_A \cdot x_B$$
$$+2\sum_{i \neq j} k_A(i,j)x_{(i,A)}x_{(j,A)} + 2\sum_{i \neq j} k_A(i,j)k_B(i,j)x_{(i,A)}x_{(j,B)}$$
$$-\lambda\{\frac{1}{2}\sum_{i} \gamma_{i4}\ [x_{(i,A)}{}^2 - x_{(i,B)}{}^2\ ] + 2\sum_{i \neq j} \gamma_{i4}\ [k_A(i,j)x_{(i,A)}x_{(j,A)} - k_B(i,j)x_{(i,B)}x_{(j,B)}\ ]$$
$$+2\sum_{i \neq j} [\gamma_{i4}k_A(i,j)k_B(i,j) - \gamma_{j4}k_A(j,i)k_B(j,i)]x_{(i,A)}x_{(j,B)}\ \} \quad \text{(B1.17d)}$$

Similarly, the equations for x_B are:

$$\frac{dx_{(1,B)}}{dt} = -2x_{(1,B)} + \frac{1}{2}||x_B||^2 + \frac{1}{2}x_A \cdot x_B + 2\sum_{i \neq j} k_A(i,j)k_B(j,i)x_{(i,A)}x_{(j,B)}$$
$$+\lambda\{\frac{1}{2}\sum_{i} \gamma_{i1}\ [x_{(i,A)}{}^2 - x_{(i,B)}{}^2\ ] + 2\sum_{i \neq j} [\gamma_{i1} - \gamma_{j1}\ ]k_A(i,j)k_B(j,i)x_{(i,A)}x_{(j,B)}\} \quad \text{(B1.18a)}$$

$$\frac{dx_{(2,B)}}{dt} = -$$
$$2x_{(2,B)} + \frac{1}{2}||x_B||^2 + \frac{1}{2}x_A \cdot x_B + 2\sum_{i \neq j} k_B(j,i)x_{(i,B)}x_{(j,B)} + 2\sum_{i \neq j} k_A(i,j)k_B(i,j)x_{(i,A)}x_{(j,B)}$$
$$+\lambda\{\frac{1}{2}\sum_{i} \gamma_{i2}[x_{(i,A)}{}^2 - x_{(i,B)}{}^2] + 2\sum_{i \neq j} \gamma_{i2}[k_A(j,i)x_{(i,A)}x_{(j,A)} - k_B(j,i)x_{(i,B)}x_{(j,B)}]$$
$$+2\sum_{i \neq j} [\gamma_{i2}k_A(j,i)k_B(j,i) - \gamma_{j2}k_A(i,j)k_B(i,j)]x_{(i,A)}x_{(j,B)}\} \quad \text{(B1.18b)}$$

$$\frac{dx_{(3,B)}}{dt} = -2x_{(3,B)} + \frac{1}{2}||x_B||^2 + \frac{1}{2}x_A \cdot x_B + 2\sum_{i \neq j} k_A(j,i)k_B(i,j)x_{(i,A)}x_{(j,B)}$$
$$+\lambda\{\frac{1}{2}\sum_{i} \gamma_{i3}[x_{(i,A)}{}^2 - x_{(i,B)}{}^2] + 2\sum_{i \neq j} [\gamma_{i3} - \gamma_{j3}]k_A(j,i)k_B(i,j)x_{(i,A)}x_{(j,B)}\} \quad \text{(B1.18c)}$$

$$\frac{dx_{(4,B)}}{dt} = -2x_{(4,A)} + \frac{1}{2}||x_B||^2 + \frac{1}{2}x_A \cdot x_B$$
$$+2\sum_{i \neq j} k_B(i,j)x_{(i,B)}x_{(j,B)} + 2\sum_{i \neq j} k_A(j,i)k_B(j,i)x_{(i,A)}x_{(j,B)}$$
$$+\lambda\{\frac{1}{2}\sum_{i} \gamma_{i4}[x_{(i,A)}{}^2 - x_{(i,B)}{}^2] + 2\sum_{i \neq j} \gamma_{i4}[k_A(i,j)x_{(i,A)}x_{(j,A)} - k_B(i,j)x_{(i,B)}x_{(j,B)}]$$
$$+2\sum_{i \neq j} [\gamma_{i4}k_A(i,j)k_B(i,j) - \gamma_{j4}k_A(j,i)k_B(j,i)]x_{(i,A)}x_{(j,B)}\} \quad \text{(B1.18d)}$$

APPENDIX C - THE 64-CUSTOM SOCIETY

C1. TRANSITION PROBABILITIES FOR THE 64-CUSTOM CASE

We compute the transition probabilities (4.7.2). First note that, because the local modification rule (4.7.1) can have no effect on customs when $i=j$, we have

$$p(i',j';k'_{I},k'_{II}|i,i;k_{I},k_{II}) = \begin{cases} p(i',j'|i,i) \text{ when } (k'_{I},k'_{II}) = (k_{I},k_{II}) \\ 0 \text{ otherwise} \end{cases} \tag{C1.1}$$

where $p(i',j' \mid i, i)$ is the elementary (one custom) transition probability given by (4.3.1a). Now suppose $i \neq j$. Let $\alpha=k_I(i,j)$ and $\beta=k_{II}(j,i)$ be the probabilities with which player-I and player-II will Defect. The $\alpha,\beta \in \{0,1\}$. By the local modification rule (4.7.1), $k_{I'}$ and k_{II}' depend only on α and β, and not on the values of k_I or k_{II} at any other class-pair. Also, $k_{I'}$ or k_{II}' can differ from k_I or k_{II} only at (i,j) and (j,i). Denote by ${}^{(i,j)}k$ the custom obtained from k by local modification at (i,j); i.e.

$${}^{(i,j)}k'_{i}\ (r,s) = \begin{cases} k_{I}(s,r) \text{ if } (r,s) = (i,j) \text{ or } (j,i) \\ k_{I}(r,s) \text{ otherwise} \end{cases} \tag{C1.2}$$

Clearly ${}^{(i,j)}k={}^{(j,i)}k$. Then $k_{I'}=k_I$ or ${}^{(i,j)}k_I$ and $k_{II}'=k_{II}$ or ${}^{(j,i)}k_{II}'$. It follows that a possible transition $((i,k_I), (j,k_{II}))\rightarrow((i',k_{I'}),(j',k_{II}'))$ is completely specified by the local transition, $((i,\alpha),(j,\beta))\rightarrow((i',\alpha'),(j',\beta'))$, where $\alpha'=k_{I'}(i,j)$ and $\beta'=k_{II}'(j,i)$. In fact, if

$$p(i',j';\alpha',\beta' \mid i,j;\alpha,\beta) \tag{C1.3}$$

is the probability of this latter transition, then the possible non-zero probabilities (20) with $i \neq j$ are given by

$$p(i',j';k_I,k_{II} \mid i,j;k_I,k_{II})=p(i',j';\alpha,\beta \mid i,j;\alpha,\beta) \tag{C1.4a}$$
$$p(i',j';{}^{(i,j)}k_I,k_{II} \mid i,j;k_I,k_{II})=p(i',j';1-\alpha,\beta \mid i,j;\alpha,\beta) \tag{C1.4b}$$
$$p(i',j';k_I,{}^{(j,i)}k_{II} \mid i,j;k_I,k_{II})=p(i',j';\alpha,1-\beta \mid i,j;\alpha,\beta) \tag{C1.4c}$$
$$p(i',j';{}^{(i,j)}k_I,{}^{(j,i)}k_{II} \mid i,j;k_I,k_{II})=p(i',j';1-\alpha,1-\beta \mid i,j;\alpha,\beta) \tag{C1.4d}$$

It therefore remains to compute the probabilities (C1.3). There are four cases.

- CASE 1. Both players apply a coordination test. This occurs with probability $(1-\lambda)^2$. Note that a coordination failure can occur only if $\alpha=\beta$, and in this case $\alpha'=\beta'=1-\alpha$. Otherwise, $\alpha'=\alpha$ and $\beta'=\beta=1-\alpha$. For $\alpha \in \{0,1\}$, define elementary transition probabilities

$$p_\alpha(i',j' \mid i,j)=(1-\alpha)\delta^2_{i'}\delta^4_{j'}+\alpha\delta^4_{i'}\delta^2_{j'} \tag{C1.5a}$$

$$\bar{p}_\alpha(i',j' \mid i,j) = \alpha\delta^1{}_{i'}\delta^1{}_{j'} + (1\text{-}\alpha)\delta^3{}_{i'}\delta^3{}_{j'} \qquad \text{(C1.5b)}$$

We then have

$$p(i',j';1\text{-}\alpha,1\text{-}\alpha \mid i,j;\alpha,\alpha) = \bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.6a)}$$
$$p(i',j';\alpha,1\text{-}\alpha \mid i,j;\alpha,1\text{-}\alpha) = p_\alpha(i',j' \mid i,j) \qquad \text{(C1.6b)}$$

- CASE 2 Player-I applies a coordination test and player-II applies an aspiration test. This occurs with probability (1-λ)λ. The possible non-zero transition probabilities are

$$p(i',j';1\text{-}\alpha,\alpha \mid i,j;\alpha,\alpha) = (1\text{-}\gamma_{jj'})\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.7a)}$$
$$p(i',j';1\text{-}\alpha,1\text{-}\alpha \mid i,j;\alpha,\alpha) = \gamma_{jj'}\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.7b)}$$
$$p(i',j';\alpha,1\text{-}\alpha \mid i,j;\alpha,1\text{-}\alpha) = (1\text{-}\gamma_{jj'})p_\alpha(i',j' \mid i,j) \qquad \text{(C1.7c)}$$
$$p(i',j';\alpha,\alpha \mid i,j;\alpha,1\text{-}\alpha) = \gamma_{jj'}p_\alpha(i',j' \mid i,j) \qquad \text{(C1.7d)}$$

- CASE 3 Player-I applies an aspiration test and player-II applies a coordination test. This occurs with probability λ(1-λ). The possible non-zero transition probabilities are

$$p(i',j';\alpha,1\text{-}\alpha \mid i,j;\alpha,\alpha) = (1\text{-}\gamma_{ii'})\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.8a)}$$
$$p(i',j';1\text{-}\alpha,1\text{-}\alpha \mid i,j;\alpha,\alpha) = \gamma_{ii'}\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.8b)}$$
$$p(i',j';\alpha,1\text{-}\alpha \mid i,j;\alpha,1\text{-}\alpha) = (1\text{-}\gamma_{ii'})p_\alpha(i',j' \mid i,j) \qquad \text{(C1.8c)}$$
$$p(i',j';1\text{-}\alpha,1\text{-}\alpha \mid i,j;\alpha,1\text{-}\alpha) = \gamma_{ii'}p_\alpha(i',j' \mid i,j) \qquad \text{(C1.8d)}$$

- CASE 4 Both players apply an aspiration test. This occurs with probability λ^2. The possible non-zero transition probabilities are

$$p(i',j';\alpha,\alpha \mid i,j;\alpha,\alpha) = (1\text{-}\gamma_{ii'})(1\text{-}\gamma_{jj'})\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.9a1)}$$
$$p(i',j';\alpha,1\text{-}\alpha \mid i,j;\alpha,\alpha) = (1\text{-}\gamma_{ii'})\gamma_{jj'}\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.9a2)}$$
$$p(i',j';1\text{-}\alpha,\alpha \mid i,j;\alpha,\alpha) = \gamma_{ii'}(1\text{-}\gamma_{jj'})\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.9a3)}$$
$$p(i',j';1\text{-}\alpha,1\text{-}\alpha \mid i,j;\alpha,\alpha) = \gamma_{ii'}\gamma_{jj'}\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.9a4)}$$
$$p(i',j';\alpha,1\text{-}\alpha \mid i,j;\alpha,1\text{-}\alpha) = (1\text{-}\gamma_{ii'})(1\text{-}\gamma_{jj'})\, p_\alpha(i',j' \mid i,j) \qquad \text{(C1.9b1)}$$
$$p(i',j';\alpha,\alpha \mid i,j;\alpha,1\text{-}\alpha) = (1\text{-}\gamma_{ii'})\gamma_{jj'}p_\alpha(i',j' \mid i,j) \qquad \text{(C1.9b2)}$$
$$p(i',j';1\text{-}\alpha,1\text{-}\alpha \mid i,j;\alpha,1\text{-}\alpha) = \gamma_{ii'}(1\text{-}\gamma_{jj'})p_\alpha(i',j' \mid i,j) \qquad \text{(C1.9b3)}$$
$$p(i',j';1\text{-}\alpha,\alpha \mid i,j;\alpha,1\text{-}\alpha) = \gamma_{ii'}\gamma_{jj'}p_\alpha(i',j' \mid i,j) \qquad \text{(C1.9b4)}$$

We can now obtain the unconditional transition probabilities (C1.3). Thus,

$$p(i',j';\alpha,\alpha \mid i,j;\alpha,\alpha) = \lambda^2(1\text{-}\gamma_{ii'})(1\text{-}\gamma_{jj'})\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.10a1)}$$
$$p(i',j';\alpha,1\text{-}\alpha \mid i,j;\alpha,\alpha) = \lambda(1\text{-}\gamma_{ii'})\ (1\text{-}\lambda(1\text{-}\gamma_{jj'})\ )\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.10a2)}$$
$$p(i',j';1\text{-}\alpha,\alpha \mid i,j;\alpha,\alpha) = \ (1\text{-}\lambda(1\text{-}\gamma_{ii'})\ )\lambda(1\text{-}\gamma_{jj'})\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.10a3)}$$
$$p(i',j';1\text{-}\alpha,1\text{-}\alpha \mid i,j;\alpha,\alpha) = \ (1\text{-}\lambda(1\text{-}\gamma_{ii'})\ )\ (1\text{-}\lambda(1\text{-}\gamma_{jj'})\ )\bar{p}_\alpha(i',j' \mid i,j) \qquad \text{(C1.10a4)}$$

$$p(i',j';\alpha,\alpha \mid i,j;\alpha,1\text{-}\alpha) = (1\text{-}\lambda\gamma_{ii'})\lambda\gamma_{jj'}p_\alpha(i',j' \mid i,j) \qquad \text{(C1.10b1)}$$
$$p(i',j';\alpha,1\text{-}\alpha \mid i,j;\alpha,1\text{-}\alpha) = (1\text{-}\lambda\gamma_{ii'})(1\text{-}\lambda\gamma_{jj'})p_\alpha(i',j' \mid i,j) \qquad \text{(C1.10b2)}$$
$$p(i',j';1\text{-}\alpha,\alpha \mid i,j;\alpha,1\text{-}\alpha) = \lambda^2\gamma_{ii'}\gamma_{jj'}p_\alpha(i',j' \mid i,j) \qquad \text{(C1.10b3)}$$
$$p(i',j';1\text{-}\alpha,1\text{-}\alpha \mid i,j;\alpha,1\text{-}\alpha) = \lambda\gamma_{ii'}(1\text{-}\lambda\gamma_{jj'})p_\alpha(i',j' \mid i,j) \qquad \text{(C1.10b4)}$$

This completes the derivation of the transition probabilities.

APPENDIX D. ERGODICITY RESULTS

D1. Proof of Proposition 4.1. We begin by computing the possible coefficients,

$$\varepsilon^{ij}(i',j')=(\varepsilon^{ij}_1(i',j'),\varepsilon^{ij}_2(i',j'),\varepsilon^{ij}_3(i',j'),\varepsilon^{ij}_4(i',j')).$$

By considering all the possible transitions, $(i,j)\to(i',j')$, and using the transition probabilities (4.1), it is straightforward to show:

$$\varepsilon^{11}=\left\{\begin{matrix}\varepsilon^{11}(1,1)=(0,0,0,0)\\ \varepsilon^{11}(2,4)=(-2,1,0,1)\\ \varepsilon^{11}(3,3)=(-2,0,2,0)\\ \varepsilon^{11}(4,2)=(-2,1,0,1)\end{matrix}\right\}\text{each with probability }\frac{1}{4} \tag{D1.1a}$$

$$\varepsilon^{22}=\left\{\begin{matrix}\varepsilon^{22}(1,1)=(2,-2,0,0)\\ \varepsilon^{22}(2,4)=(0,-1,0,1)\\ \varepsilon^{22}(3,3)=(0,-2,2,0)\\ \varepsilon^{22}(4,2)=(0,-1,0,1)\end{matrix}\right\}\text{each with probability }\frac{1}{4} \tag{D1.1b}$$

$$\varepsilon^{33}=\left\{\begin{matrix}\varepsilon^{33}(1,1)=(2,0,-2,0)\\ \varepsilon^{33}(2,4)=(0,1,-2,1)\\ \varepsilon^{33}(3,3)=(0,0,0,0)\\ \varepsilon^{33}(4,2)=(0,1,-2,1)\end{matrix}\right\}\text{ each with probability }\frac{1}{4} \tag{D1.1c}$$

$$\varepsilon^{44}=\left\{\begin{matrix}\varepsilon^{44}(1,1)=(2,0,0,-2)\\ \varepsilon^{44}(2,4)=(0,1,0,-1)\\ \varepsilon^{44}(3,3)=(0,0,2,-2)\\ \varepsilon^{44}(4,2)=(0,1,0,-1)\end{matrix}\right\}\text{ each with probability }\frac{1}{4} \tag{D1.1d}$$

and, for $i\neq j$, $\varepsilon^{ij}=\varepsilon^{ij}(2,4)=\varepsilon^{ij}(4,2)$, with probability $p(2,4\,|\,i,j)+p(4,2\,|\,i,j)=1$, using (4.3.1b) and the fact that $k(i,j)+k(j,i)=1$. Thus,

$$\left.\begin{matrix}\varepsilon^{12}=\varepsilon^{21}=(-1,0,0,1)\\ \varepsilon^{12}=\varepsilon^{21}=(-1,0,0,1)\\ \varepsilon^{14}=\varepsilon^{41}=(-1,1,0,0)\\ \varepsilon^{23}=\varepsilon^{32}=(0,0,-1,1)\\ \varepsilon^{24}=\varepsilon^{42}=(0,0,0,0)\\ \varepsilon^{24}=\varepsilon^{42}=(0,0,0,0)\end{matrix}\right\}\text{ each with probability 1.} \tag{D1.2}$$

Note that, since each one-step transition on Ω^k_N is of the form $x \rightarrow x + \varepsilon^{ij}\Delta x$, it follows from the above calculations that the corresponding state transition probabilities are independent of the custom k. Thus, the Markov process is independent of k.

Let $e_1 = (1,0,0,-1), e_2 = (0,1,0,-1)$, and $e_3 = (0,0,1,-1)$ be vectors in $\Re^4$, and let $\Delta x = 1/N$. Then, for $x \in \Omega^k_N$, we may define the set of nearest neighbours of x in Ω^k_N to be

$$\mathcal{N}(x) = \{x \pm \Delta x e_r \mid 1 \leq r \leq 3\} \cap \Omega^k_N \qquad \text{(D1.3)}$$

Let $\text{int}\Delta^3$ be the interior of Δ^3, and $\text{int}\Omega^k_N = \text{int}\Delta^3 \cap \Omega^k_N$. Then, for $x \in \text{int}\Omega^k_N$, set

$$\text{int}\mathcal{N}(x) = \mathcal{N}(x) \cap \text{int}\Omega^k_N \qquad \text{(D1.4)}$$

Let $\partial\Delta^3$ be the boundary of Δ^3 and $\partial\Omega^k_N = \partial\Delta^3 \cap \Omega^k_N$. In order to prove Proposition A, it suffices to show that, for each $x \in \Omega^k_N$, there are paths of finite length and positive probability in Ω^k_N joining x to each point of $\text{int}\mathcal{N}(x)$. From this it follows that there is a finite, positive probability path joining any two points in $\text{int}\Omega^k_N$, and that any point on $\partial\Omega^k_N$ eventually moves into $\text{int}\Omega^k_N$.

Let $x \in \Omega^k_N \in$ en the single step transition, $x \rightarrow x + \varepsilon^{ij}(i',j')\Delta x$, occurs with probability $p(i',j' \mid i,j) f_{ij}(x,\Delta x)$, where

$$f_{ij}(x,\Delta x) = \frac{x_i(x_j - \delta_{ij}\Delta x)}{1 - \Delta x}$$

is the without replacement probability of choosing players of classes i and j. We first show that, for $x \in \text{int}\Omega^k_N$, there is a finite sequence of single step transitions, each of positive probability, linking x to $x \pm \Delta x e_r$, whenever this latter point is also in $\text{int}[\Omega^k_N.]$

Since $x \in \text{int}\Omega^k_N$, we have that $x_i \geq \Delta x$ for each i. Such points exist since $N \geq 4$. Then, using (D1.2) and (D1.1b), a possible path is

$$x \xrightarrow{1\times 2} x - \Delta x e_1 \xrightarrow{4\times 4} x + \Delta x e_1 \qquad \text{(D1.5)}$$

where $\xrightarrow{i\times j}$ means that the transition is due to a game between players of classes i and j. Here, the first transition occurs with positive probability Cx_1x_2, and the second with positive probability $\frac{1}{4}C(x_4 + \Delta x)x_4$, where $C = 1/(1-\Delta x)$. Similarly, a possible path is

$$x \xrightarrow{1\times 4} x - \Delta x e_1 + \Delta x e_2 \xrightarrow{2\times 2} x - \Delta x e_1 \qquad \text{(D1.6)}$$

with positive step probabilities Cx_1x_4 and $\frac{1}{2}C(x_2 + \Delta x)x_2$. The other nearest neighbour transitions may be effected as follows. If $x_4 \geq 2\Delta x$,

$$x \xrightarrow{4\times 4} x + \Delta x e_2 \qquad \text{(D1.7)}$$

with positive step probability $\frac{1}{2}Cx_4(x_4\text{-}\Delta x)$. If $x_2 \geq 2\Delta x$,

$$x \xrightarrow{2\times 2} x\text{-}\Delta x e_2 \tag{D1.8}$$

with positive step probability $\frac{1}{2}Cx_2(x_2\text{-}\Delta x)$. Also,

$$x \xrightarrow{2\times 3} x\text{-}\Delta x e_3 \xrightarrow{4\times 4} x+\Delta x e_3 \tag{D1.9}$$

with positive step probabilities Cx_2x_3 and $\frac{1}{4}C(x_4+\Delta x)x_4$, and

$$x \xrightarrow{3\times 4} x+\Delta x e_2\text{-}\Delta x e_3 \xrightarrow{2\times 2} x\text{-}\Delta x e_3 \tag{D1.10}$$

with positive step probabilities Cx_3x_4 and $\frac{1}{2}C(x_2+\Delta x)x_2$. Note that the assumption $x_4 \geq 2\Delta x$ in (D1.7) is without loss of generality. For, if $x_4 \leq \Delta x$, then $x+\Delta x e_2 \eta$otin intΩ^k_N, so we don't need to effect this transition. A similar remark applies to (D1.8).

It remains to show that there is a finite, positive probability path from any $x \in \partial\Omega^k_N$ into intΩ^k_N. To do this, first suppose that N≥7. Then, if $x_r=0$, there is at least one j≠r for which $x_j \geq 3\Delta x$. Thus, we can effect a transition, $(j,j) \rightarrow (r,j') \in \{(1,1),(2,4),(3,3),(4,2)\}$, with positive probability $\frac{1}{4}Cx_j(x_j\text{-}\Delta x)$. It follows that there is a positive probability transition after which x_r increases to $x_r' \geq \Delta x$, x_j decreases to $x_j' \geq x_j\text{-}2\Delta x \geq \Delta x$, and $x_s \geq x_s$ for $s \neq r$ or j. Then x can be moved into intΩ^k_N after at most 3 such transitions. In fact, by a careful consideration of the various possibilities, this result can also be shown to hold for $N=4$, 5 and 6. We omit the laborious detatils. This completes the proof of Proposition A. ♠

PROOF OF PROPOSITION 4.6.1 Let $\mathbf{0} \in \Re^4$ be the zero vector, and define vectors $e^A_r = e_r \times \mathbf{0}$, $e^B_r = \mathbf{0} \times e_r \in \Re^8$, for $1 \leq r \leq 3$, and $\mathbf{f} = (0,0,0,\text{-}1) \times (0,0,0,1) \in \Re^8$. Then any two points in the lattice $\Omega_N^{(k_A,k_B)} \subset \Delta^7 \subset \Re^8$ may be joined by a sequence of elementary transitions of the form

$$x \longrightarrow x \pm \Delta x e^A{}_r \tag{D2.1a}$$

$$x \longrightarrow x \pm \Delta x e^B{}_r \tag{D2.1b}$$

$$x \longrightarrow x \pm \Delta x \mathbf{f} \tag{D2.1c}$$

We first show that each of the transitions (D2.1) can be effected with positive probability whenever both x and the terminal point lie in int$\Omega_N^{(k_A,k_B)}$.

If $x = x_A \times x_B \in$ int$\Omega_N^{(k_A,k_B)}$, then $x_{(i,A)}, x_{(i,B)} \backslash \geq \Delta x$ for each i. Hence, the transitions (D2.1a) can be effected by interactions with players who also use custom A, as in (D1.5)-(D1.10). Similarly for the transitions (D2.1b). Now suppose player-I has state (4,A) and player-II has state (4,B). Then, from (B1.6c), a transition ((4,A),(4,B))→((2,B),(4,B)) occurs with probability

$p(2,4;B,B\,|\,4,4;A,B) = \frac{1}{4}\lambda\gamma_{42}(1\text{-}\lambda\gamma_{44}) = \frac{1}{6}\lambda$ using the form (4.6.3) for $\gamma_{ii'}$. This is positive when $\lambda > 0$. In the notation of (D2.1), this transition is

$$x \longrightarrow x' = x + \Delta x\mathbf{f} + \Delta x e^{B}{}_{2}. \tag{D2.2}$$

Now, since $x_{(2,B)}' \geq 2\Delta x$, the transition (D1.8) may be performed on x_B' with positive probability. Composing this with (D2.2) then effects the transition $x \rightarrow x + \Delta x f$ with positive probability. The move $x \rightarrow x\text{-}\Delta x\mathbf{f}$ may be constructed in a similar manner by considering the possible transitions ((4,A),(4,B))→ ((4,A),(2,A)).

Finally, if $x \in \partial\,\Omega_N^{(k_A,k_B)}$, then either x_A or x_B has a zero component. But, as discussed for the one custom case, provided $l_A = \sum_i x_{(i,A)} \geq 4\Delta x$, all the components of x_A may be made non-zero by a sequence of positive probability interactions with players who also use custom A. These interactions leave x_B unaffected. Similarly for the components of x_B. If, on the other hand, $l_A < 4\Delta x$, then $l_B > 4\Delta x$ (because $N \geq 8$), and there is at least one i for which $x_{(i,B)} \geq 2\Delta x$. We may therefore increase l_A and decrease l_B by Δx, without reducing any component of x_B to zero, via a transition of the form $((i,B),(i,B)) \rightarrow ((i',A),\, (j',B))$, with probability

$$\sum_{i',j'} p(i',j';A,B\,|\,i,i;B,B) = \frac{1}{4}\lambda\{\gamma_{i1}(1\text{-}\lambda\gamma_{i1}) + \gamma_{i2}(1\text{-}\lambda\gamma_{i4}) + \gamma_{i3}(1\text{-}\lambda\gamma_{i3}) + \gamma_{i4}(1\text{-}\lambda\gamma_{i2})\}$$

(see B1.6c). With the form (4.6.3) for $\gamma_{ii'}$, this is positive for $0 < \lambda \leq 1$ and $0 < \eta < 1$. Thus, by a finite sequence of such manouvers, we may ensure that both l_A and $l_B \geq 4\Delta x$.

We have now shown that any point in $\Omega_N^{(k_A,k_B)}$ may be joined to any point in $\mathrm{int}\,\Omega_N^{(k_A,k_B)}$ by a finite, positive probability path. This proves Proposition 4.6.1. ♠

D3. PROOF OF PROPOSITION 4.7.1 Represent a point $x \in \Omega_N \subset \Delta^{255} \subset \Re^{256}$ as a product, $x = \prod_k x_k$ with $x_k \in \Re^4$ the state vector for custom k. Here, k runs over the full set of the 64 possible customs. Denote by $e^k{}_r \in \Re^{256}$ the vector with components $x_{k'} = \mathbf{0}$ for $k' \neq k$, and $x_{\geq} = e_r$. Also, for a pair of customs, (k,k'), with $k \neq k'$, denote by $\mathbf{f}^{(k,k')}$, the vector with components

$$f_m^{(k,k')} = \begin{cases} (0,0,0,-1); m = k \\ (0,0,0,1); m = k' \\ 0 \text{ otherwise} \end{cases} \tag{D3.1}$$

Then, since $\mathbf{f}^{(k',k)} = \text{-}\mathbf{f}^{(k,k')}$, any two points in the lattice Ω_N may be joined by a sequence of elementary transitions of the form

$$x \longrightarrow x \pm \Delta x e^k{}_r \tag{D3.2a}$$
$$x \longrightarrow x + \Delta x\mathbf{f}^{(k,k')} \tag{D3.2b}$$

for which k' differs from k by a single local modification of the form (19). We first show that each of the transitions (D3.2) can be effected with positive probability whenever both x and the terminal point lie in $\mathrm{int}\Omega_N$.

If $x \in \mathrm{int}\Omega_N$, then $x_{(i,k)} \geq \Delta x$ for each (i,k). Hence, the transitions (D3.2a) can be effected by interactions with players who also use custom k, as in (D1.5)-(D1.10).

For (D3.2b), suppose that $k' = {}^{(i,j)}k$ for some $i<j$ (see (C1.2)), and set $\alpha = k(i,j)$. Consider a transition $((i,k'),(j,k)) \to ((i',k'),(j',k'))$. By (C1.4c), (C1.10a2) and (C1.5b), a transition of this form occurs with probability

$$P(k \to k') = \sum_{i',j'} p(i',j';1-\alpha,\alpha \mid i,j;1-\alpha,1-\alpha)\lambda\{(1-\alpha)(1-\gamma_{i1})[1-\lambda(1-\gamma_{j1})] + \alpha(1-\gamma_{i3})[1-\lambda(1-\gamma_{j3})]$$

Here, $\alpha \in \{0,1\}$ and $(i,j) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$. Using the form (4.6.3) for $\gamma_{ii'}$, one easily checks that $P(k \to k')$ is non-zero in all these cases, provided $0<\lambda, \eta<1$. Such a positive probability transition results in a state change of the form

$$x \longrightarrow x' = x + \Delta x \mathbf{f}^{(k,k')} + (1-\alpha)\Delta x(2e^{k'}{}_1 - e^{k'}{}_i) + \alpha \Delta x\,(2e^{k'}{}_3 - e^{k'}{}_i) \qquad \text{(D3.3)}$$

(note that $i \leq 3$). Providing $x'_{k'} \in \mathrm{int}\Omega^{k'}{}_N$ (e.g. if $x_{(i,k')} \geq 2\Delta x$) we may compose (D3.3) with transitions of the form (D3.2a) to effect (D3.2b) with positive probability. On the other hand, if $x_{k'}'$ has a zero component, then, providing $l_{k'} = \sum_j x\_\{(j,k')\} \geq 4\Delta x$, we may connect this point to any interior point by a positive probability path, as in the one custom case, without affecting any of the other components of x. In particular, we may connect x' to the interior point $x + \Delta x \mathbf{f}^{(k,k')}$.

It remains to show that we can arrange for any point $x \in \partial\Omega_N$ to satisfy $l_k \geq 4\Delta x$ for each k. Suppose $l_{k'} < 4\Delta x$ for some k'. Then, since $N \geq 256$, there exists $k \neq k'$ such that $l_{\geq} > 4\Delta x$. We may then choose a sequence, $k = k_0 \to k_1 \to \ldots \to k_{q-1} \to k_q = k'$, such that k_i is obtained from k_{i-1} by a single local modification. Furthermore, we may assume that $l_{k_i} = 4\Delta x$ for $1 \leq i < q$. For, if $l_{k_i} < 4\Delta x$, we may take $k' = k_i$ and if $k_i > 4\Delta x$, we may take $k = k_i$. It suffices to show therefore, that if k' is obtained from k by a single local modification, then there is a positive probability transition having the effect $l_{\geq} \to l_{\geq} - \Delta x$ and $l_{k'} \to l_{k'} + \Delta x$.

Suppose that $k' = {}^{(i,j)}k$ with $i<j$. We may assume that $x_{(i,k)}$ and $x_{(j,k)} \backslash \geq \Delta x$. For, if not, then $l_{\geq} > 4\Delta x$ means that, as described for the one custom case, a preliminary shuffling amongst the components of $x_{\geq}$ can effect this with positive probability, while leaving the other components of x unchanged. We now consider a transition of the form $((i,k),(j,k)) \to ((i',k),(j',k'))$. By (C1.4c), (C1.10b1) and (C1.5a), such a transition occurs with probability

$$\begin{aligned} P(k \to k') &= \sum_{i',j'} p(i',j';\alpha,\alpha \mid i,j;\alpha,1-\alpha) \\ &= \lambda\{(1-\alpha)(1-\lambda\gamma_{i2})\gamma_{j4} + \alpha(1-\lambda\gamma_{i4})\gamma_{j2}\} \\ &= \alpha\lambda\gamma_{j2} \end{aligned}$$

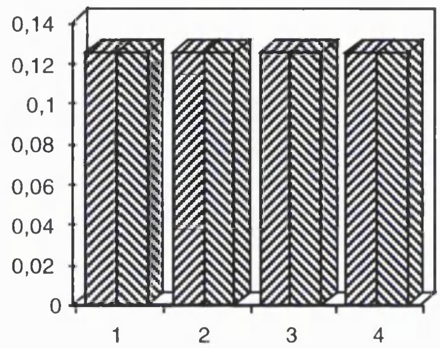
using the form (4.6.3) for $\gamma_{ii'}$. This is non-zero provided $\lambda, \eta > 0$ and $\alpha = k(i,j) = 1$. Similarly for transitions of the form $((i,k),(j,k)) \rightarrow ((i',k'),(j',k))$, we have, from (C1.4b), (C1.10b4) and (C1.5a),

$$\begin{aligned} P(k \rightarrow k') &= \sum_{i',j'} p(i',j';1-\alpha,1-\alpha \mid i,j;\alpha,1-\alpha) \\ &= \lambda\{(1-\alpha)\gamma_{i2}(1-\lambda\gamma_{j4}) + \alpha\gamma_{i4}(1-\lambda\gamma_{j2})\} \\ &= (1-\alpha)\lambda\gamma_{i2} \end{aligned}$$

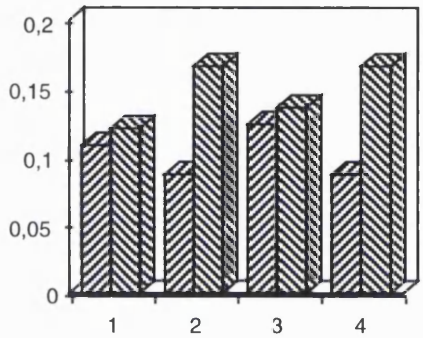
which is non-zero provided $\lambda, \eta > 0$ and $\alpha = 0$. We have therefore completed the proof of Proposition 4.7.1.

k_1 k_{64}

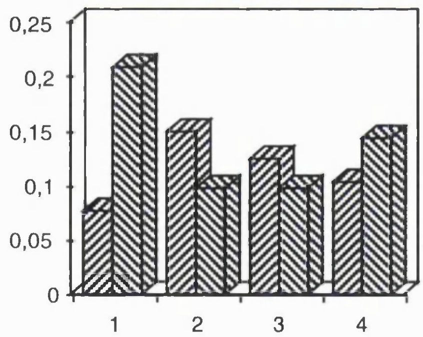
		$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
	k_1	.5	2.5	0.677
λ =**.001**	k_{64}	.5	2.5	1.661
η=**.001**	*TOT*	1	2.5	1.25



		$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
	k_1	.409	2.466	0.677
λ =**.001**	k_{64}	.591	2.591	1.661
η=**.999**	*TOT*	1	2.54	1.326



		$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
	k_1	.452	2.564	0.677
λ =**.999**	k_{64}	.548	2.319	1.661
η=**.001**	*TOT*	1	2.43	1,254



		$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
	k_1	.359	2.38	0.677
λ =**.999**	k_{64}	.641	2.73	1.661
η=**.999**	*TOT*	1	2.6	1.359

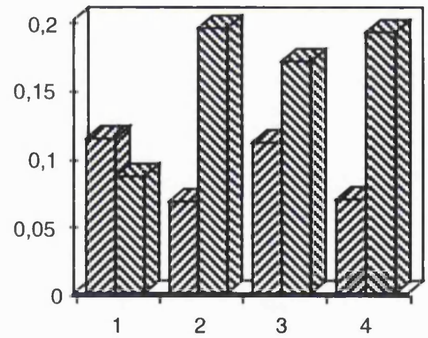
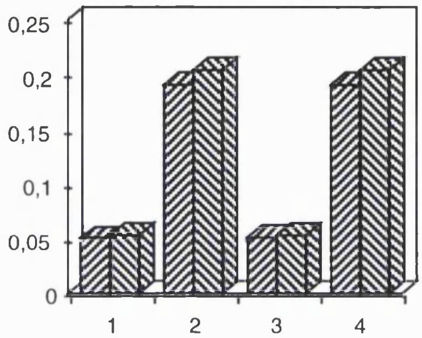


FIGURE 6.1

k_1 k_{33}

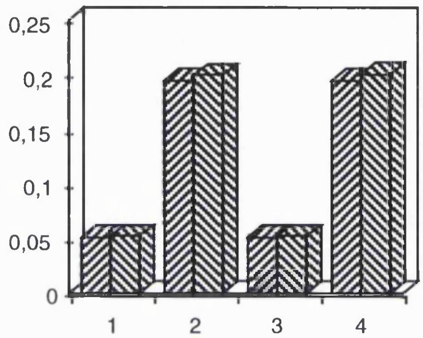
$\lambda = .001$, $\eta = .001$

	$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
k_1	.485	2.79	.667
k_{64}	.515	2.793	.667
TOT	1	2.792	.667



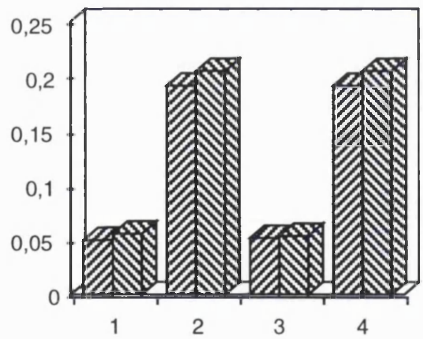
$\lambda = .001$, $\eta = .999$

	$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
k_1	.495	2.791	.667
k_{64}	.505	2.792	.667
TOT	1	2.791	.667



$\lambda = .999$, $\eta = .001$

	$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
k_1	.483	2.793	.667
k_{64}	.517	2.787	.667
TOT	1	2.79	.667



$\lambda = .999$, $\eta = .999$

	$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
k_1	.489	2.78	.667
k_{64}	.511	2.80	.667
TOT	1	2.79	.667

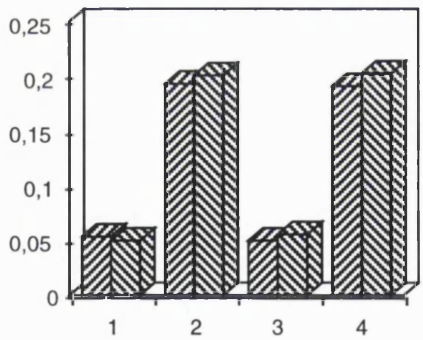


FIGURE 6.2

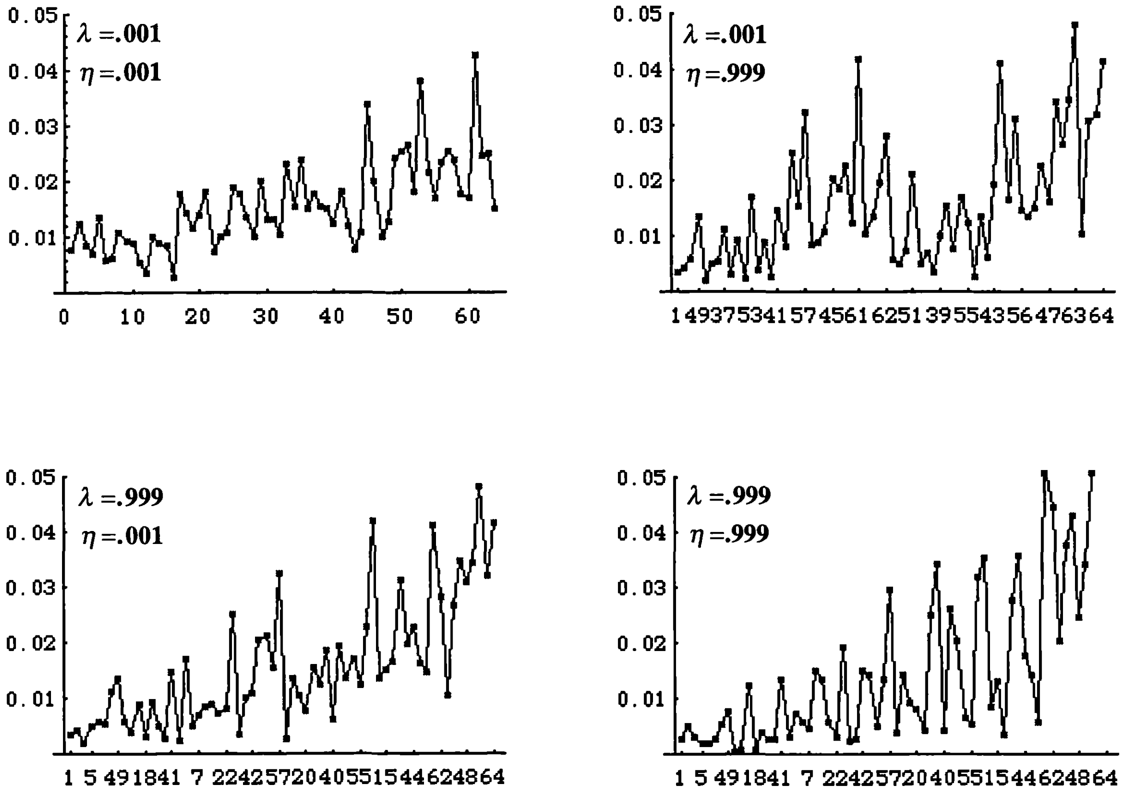

λ =.001
η =.001
0.05
0.04
0.03
0.02
0.01
0 10 20 30 40 50 60
λ =.001
η =.999
0.05
0.04
0.03
0.02
0.01
1 4937 5341 57 4561 6251 39 5543 56 4763 64
λ =.999
η =.001
0.05
0.04
0.03
0.02
0.01
1 5 491841 7 22425720405515446248 64
λ =.999
η =.999
0.05
0.04
0.03
0.02
0.01
1 5 491841 7 22425720405515446248 64

FIGURE 7.1

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