To my parents

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On the Picard group of compact complex projective flat manifolds

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Abstract

We consider the problem of describing the projective imbeddings of a compact, complex, projective, flat manifold M .

L.S. Charlap has studied the algebraic classification of flat manifolds and recently F.E.A. Johnson using Albert's classification of rational positively involuted, finite dimensional algebras, succeeded in giving a necessary and sufficient condition for M to admit at least one complex, algebraic structure. This was done purely and solely in terms of the rational holonomy representation of M .

We carry this further and using extension theory we give a description of Pic(M) by proving a generalized equivariant version of the well known Appell-Humbert theorem.

We finally classify the projective flat manifolds whose holonomy group is either cyclic C_p or dihedral D_{2p} , where p is a prime. We use our results to provide an estimate for the size of the set of all positive line bundles and construct interesting imbeddings directly in terms of the integral holonomy representation of M .

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Chapter I

Flat Riemannian Manifolds

§ 1.1 Bieberbach's Three Theorems.

The aim of this section is to present the algebraic Bieberbach Theorems. For further details, we refer to [B1], [B2] and [C2] chapter I.

The *n-dimensional group of rigid motions* $\mathcal{M}_n$ is by definition the semidirect product (definition in §1.2) $\mathcal{M}_n := O_n \rhd\!< \mathbb{R}^n$ of the n-dimensional orthogonal group O_n and $\mathbb{R}^n$ (i.e. $(r_1,t_1)*(r_2,t_2) = (r_1 \cdot r_2 , r_1 \cdot t_2 + t_1)$). $\mathcal{M}_n$ is acting on $\mathbb{R}^n$, $\circ : \mathcal{M}_n \circ \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $(r,t) \circ x = r \cdot x + t$, where $x \in \mathbb{R}^n$ and $(r,t) \in \mathcal{M}_n$. If instead of the n-dimensional orthogonal group above we consider the n-dimensional general linear group, we get the *n-dimensional group of affine motions* $\mathcal{A}_n$.

A group is said to be *torsion free* if no element of it has finite order. A subgroup $G \subset \mathcal{M}_n$ is said to be *discontinuous* if all the orbits of G acting on $\mathbb{R}^n$ are discrete and *cocompact* if the orbit space $\mathbb{R}^n/G$ is compact. The action of $G \subset \mathcal{M}_n$ on $\mathbb{R}^n$ is *free* if the identity is the only element of G that leaves all elements of $\mathbb{R}^n$ fixed. If G acts freely then G is torsion free and if it is discontinuous and torsion free then it acts freely. If G is a discontinuous subgroup of the group of rigid motions whose action is free the orbit space $\mathbb{R}^n/G$ is a n-dimensional manifold. It is not known whether the same is true if simply $G \subset \mathcal{A}_n$. We call G *isotropic* if $G \cap \mathbb{R}^n$ spans $\mathbb{R}^n$ and G *crystallographic* if it is both discrete and cocompact. A torsion free, crystallographic subgroup of the group of rigid motions is called a *Bieberbach* subgroup of $\mathcal{M}_n$. Notice that if G is Bieberbach then $\mathbb{R}^n/G$ is a n-dimensional manifold.

We can now state Bieberbach's three theorems:

First Bieberbach Theorem (1.1.1): If G is a crystallographic subgroup of $\mathcal{M}_n$, then
(i) the rotational part r(G) of G is finite [where $r : \mathcal{M}_n \longrightarrow O_n$; $r(m,t) = m$]
(ii) G is isotropic.

In terms of the language we are going to use the First Bieberbach Theorem says that if G is crystallographic then G satisfies an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow G \longrightarrow \Phi \longrightarrow 1$$

where $\Lambda = G \cap \mathbb{R}^n$ is a lattice (finitely generated abelian group) of $\text{rank}_{\mathbb{Z}}$ n , and $\Phi = r(G)$ a finite group. Φ , for reasons we shall see later is called the *holonomy group* of G . We now have the following proposition whose proof can be found in [C] page 18 and Bieberbach's Second and Third Theorems.

Proposition (1.1.2): If G is a crystallographic subgroup of $\mathcal{M}_n$ then $G \cap \mathbb{R}^n$ is the unique normal, maximal abelian subgroup of G .

Second Bieberbach Theorem (1.1.3): Let G , G′ be two isomorphic crystallographic subgroups of $\mathcal{M}_n$, via $f : G \xrightarrow{\sim} G'$. Then, there exists $a \in \mathcal{A}_n$ such that $f(b) = aba^{-1}$, for all $b \in G$; in other words any isomorphism between crystallographic groups can be realized by an affine change of coordinates.

Third Bieberbach Theorem (1.1.4): Up to affine change of coordinates there are only finitely many crystallographic subgroups of $\mathcal{M}_n$.

Theorem (1.1.3) is really a corollary of Theorem (1.1.1) and Theorem (1.1.4) follows from (1.1.1), (1.1.3) and standard material. Except from Bieberbach's original proof [B1] and a recent one by P. Buser [B3], all the proofs are variations of the proof of Frobenius [F] given shortly after Bieberbach. P. Buser's proof arose from a study of the techniques involved in Gromov's work on almost flat manifolds [G]. For attempts to generalize Bieberbach's First Theorem to the affine group see [M2] and [F-G].

One has the following partial inverse to Theorem (1.1.1)

Theorem (Auslander-Kuranishi) (1.1.5): Let Φ be a finite group. Then, there exists a Bieberbach group G with $r(G) = \Phi$.

A.T. Vasquez's proof of this Theorem (1.1.5) which avoids R. Lyndon's technical results is reproduced on page 103 of [C2].

§ 1.2 Cohomology of groups.

The object of this subsection is to provide references rather than to attempt any introduction to the subject. We refer to Rotman [R3] or McLane, in particular chapter IV of [M], or even to the more recent publications by Evens [E] or Benson [B].

Let R be any ring with identity, a *free* left R-module is one that can be written as a sum of copies of R and a *projective* left R-module is one that is a summand of a free R-module. By a *resolution* $\mathfrak{X}$ of a left R-module Y , we mean a long exact sequence of left R-modules $\{X_n\}_{n \geq 0}$ and homomorphisms $\{d_n\}_{n \geq 0}$ with $d_n : X_n \longrightarrow X_{n-1}$; $d_0 \equiv \epsilon$ is usually called the augmentation of the complex $\{X_n, d_n\}$ onto Y . The resolution $\mathfrak{X}$ is *free (projective)* if and only if each the $\{X_n\}_{n \geq 0}$ is a *free (projective)* left R-module.

To define the cohomology of a finite group Φ one needs a free $\mathbb{Z}[\Phi]$-resolution of $\mathbb{Z}$ (regarded as a trivial $\mathbb{Z}[\Phi]$-module), where $\mathbb{Z}[\Phi]$ is the integral group ring. Theorem 6.1, Lemma 6.2, Corollary 6.3 on pages 87-88 of [M], assert that the definition of the cohomology is independent of the choice of the particular free Φ-resolution of $\mathbb{Z}$. On a bare-hands approach the choice of resolution can facilitate immensely the computation. For resolutions suitable for special classes of groups like cyclic or free abelian groups, we refer to [M] pages 121-123. The definition of the cohomology of groups is done by means of a standard free Φ-resolution of $\mathbb{Z}$, the so called *bar resolution* $\mathfrak{B}$ *for* Φ , see relatively chapter IV of [M]. For alternative general resolutions, see [G] .

As an immediate consequence of Theorem 6.2, page 23 of [M], we notice that

$$H^0(\Phi;A) = A^{\Phi}$$

where A^{Φ} is the maximal Φ-trivial submodule of A .

In what follows, we are mainly interested in the low dimensional cohomology groups. It is hence worth giving a geometric interpretation of $H^1(\Phi;A)$ and $H^2(\Phi;A)$.

Let A be an abelian $\mathbb{Z}$-module with a predefined Φ-action on it, precisely as above. A *group extension* is a short exact sequence of groups and homomorphisms of the form

$$\mathcal{E} := (0 \longrightarrow A \longrightarrow G \longrightarrow \Phi \longrightarrow 1) \ .$$

Let $\mathcal{E}xt(\Phi,A)$ denote the class of group extensions of the above form; two such extensions $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}xt(\Phi,A)$ are *congruent* if there exists an isomorphism $\alpha : G_1 \longrightarrow G_2$ which induces the identity on A and Φ. Let $\mathcal{EXT}(\Phi,A)$ be the set of all *congruence classes of extensions of A by* Φ.

Applying the definition and calculating the differentials on the bar resolution, see Theorem 10.24 on page 288 of [R3], we see that

$$H^2(\Phi;A) = \frac{Z^2(\Phi,A)}{B^2(\Phi,A)}$$

where $Z^2(\Phi,A)$ is the group of *factor sets* of $\mathcal{E}$ and $B^2(\Phi,A)$ the subgroup of all *special factor sets.* For precise definitions, we refer to [M], chapter IV. It is a theorem, see Theorem 4.1 page 112 of [M], that $\mathcal{EXT}(\Phi,A) \simeq Z^2(\Phi,A)/B^2(\Phi,A)$ and subsequently the set of congruence classes of extensions of A by Φ is classified by $H^2(\Phi;A)$. In fact, Eilenberg's and Maclane's theorem interprets geometrically the second cohomology group in even if A is not abelian. The respective congruence classes of extensions are then classified by $H^2(\Phi;\mathcal{Z}(A))$, where $\mathcal{Z}(A)$ is the center of A ; see further Theorem 8.8 on page 128 of [M].

Back to our case, where A is abelian as a group, it is useful to point out at least the one direction of Theorem 4.1 on page 112 of [M]. So given a factor set $f : \Phi \times \Phi \longrightarrow A$, one defines the extension corresponding to it by taking G to be the group with elements in $A \times \Phi$ and addition given by

$$(a_1, x_1) + (a_2, x_2) = (a_1 + x \cdot a_2 + f(x_1,x_2), x_1x_2) \ , \quad x_1,x_2 \in \Phi.$$

The extension corresponding to the zero element of $H^2(\Phi;A)$ is called the *semi-direct product* $A \rtimes \Phi$ of A and Φ .

We notice here that since $H^2(\Phi;A)$ is a group, there must be a corresponding way to add extensions. There is, and it is called *Baer* *addition.* For more on this, we refer to page 69 of [M].

A similar geometric interpretation of $H^1(\Phi;A)$ can be given. Following the similar lines as above one has

$$H^1(\Phi;A) = \frac{Z^1(\Phi,A)}{B^1(\Phi,A)}$$

where $Z^1(\Phi,A)$ is the group of all *crossed homomorphism* of Φ to A and $B^1(\Phi,A)$ is the subgroup of all *principal crossed homomorphisms* of A. Notice that if A is a trivial Φ-module a crossed homomorphism is just an ordinary homomorphism and $B^1(\Phi,A)$ is the trivial subgroup. It then follows immediately if A is a trivial Φ-module

$$H^1(\Phi;A) = \mathrm{Hom}(\Phi,A) .$$

For a geometric interpretation of the groups $Z^1(\Phi,A)$ and $B^1(\Phi,A)$ see Proposition 2.1 on page 106 of [M].

The calculation of $H^2(\Phi;A)$ is in general a rather tricky and technical question. The Lyndon-Hochschild-Serre spectral sequence, see page 355 of [R3] for example, is often a convenient tool. An easy but often useful remark is to notice that if $A = A_1 \oplus A_2$ then

(1.2.1) $$H^2(\Phi;A) \simeq H^2(\Phi;A_1) \oplus H^2(\Phi;A_2) .$$

At this point, to finish this quick journey over the essentials, we would like to notice that if Φ is a finite group and A is a finitely generated free Φ-module then $H^2(\Phi;A)$ is a finitely generated abelian group. For a proof of both, it suffices to notice that under these conditions $\mathbb{Z}[\Phi]$ is Noetherian and subsequently $\mathrm{Hom}_\Phi(\mathfrak{B},A)$ is a finitely generated $\mathbb{Z}[\Phi]$-module, thus a finitely generated abelian group.

§ 1.3 Flat Manifolds - Geometric Bieberbach's Theorems.

In this section we aim to explain the geometric meaning of Bieberbach's Theorems culminating to a presentation of Charlap's classification of compact flat manifolds.

A. Some Differential Topology.

Let M be a n-dimensional $\mathbb{R}$-manifold. Denote by M_x its tangent space at $x \in M$ and by ${}_M\nabla$ a connection on M . Precise definitions for those terms can be found in any book on differential geometry, for example in [K-N]. A diffeomorphism $F : M \longrightarrow N$, between two n-dimensional $\mathbb{R}$-manifolds is said to be an *affine equivalence* if in addition the induced connection on M , $F^*({}_N\nabla) = {}_M\nabla$, where

$$[F^*({}_N\nabla)]_U(V) = {}_M\nabla_{dF(U)}\,(dF(V))$$

with U , V vector fields on M and dF the linear differential of F .

By writing everything in local coordinates one can see that the *derivative of V* (a vector field on M) *along a curve* c ($c : [0,1] \longrightarrow M$, differentiable) exists and is the unique mapping $\frac{D}{dt}$ of vector fields along c to vector fields along c which satisfies

$$1)\quad \frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$$

$$2)\quad \frac{D}{dt}(g \cdot V) = \dot{c}(g)\, V + g\,\frac{DV}{dt}$$

and if V′ vector field on M such that its restriction $V'|_c = V$ then

$$\frac{DV}{dt} = \nabla_{\dot{c}}\, V' \ .$$

See [M1] pages 46-47 for a proof of this and of the following lemma.

Lemma (1.3.1): Given a n-dimensional $\mathbb{R}$-manifold, a connection ∇ , a curve c on it and a vector $V_{c(0)} \in M_{c(0)}$ there is a unique vector field V along c such that $V_0 = V_{c(0)}$ and V is *parallel along* c, i.e. $\frac{DV}{dt} \equiv 0$ along c.

A *parallel translation along* c *from* α *to* β is a linear transformation from $M_{c(\alpha)}$ to $M_{c(\beta)}$ defined, in the obvious way, by means of the above lemma. It is an easy exercise to show that the set of all those linear transformations given by parallel translation around loops at $x \in M$ forms a subgroup of $GL_n(M_x)$ called the *holonomy group of* M *at* x and denoted by $\Phi(M,x)$. The dependence of the holonomy group $\Phi(M,x)$ on the base point $x \in M$ is similar to that of the fundamental group $\pi_1(M,x)$. Here, the concept of two (curves) loops being homotopic to each other is being replaced by two (curves) loops being *holonomous* to each other if parallel translation around the one is the same as parallel translation around the other. We can therefore speak of $\Phi(M)$ and the Borel-Lichnerowitz theorem asserts that the identity component $\Phi_0(M)$ of the holonomy group $\Phi(M)$ consists precisely of those holonomy classes of loops which homotopic to the constant loop at $x \in M$. A proof of this can be found in [N] page 33. As an immediate corollary, one has

Corollary (1.3.2): There is a surjective homomorphism

$$r : \pi_1(M) \longrightarrow \Phi(M)/\Phi_0(M) .$$

If G a subgroup of $\mathcal{A}_n$ acting on $\mathbb{R}^n$ in such way that the orbit space $\mathbb{R}^n/G$ is a n-manifold with connection (the one naturally inherited by $\mathbb{R}^n$), for example G could be a Bieberbach subgroup of $\mathcal{M}_n \subset \mathcal{A}_n$ (see relatively §1.1), one can show that parallel translation around loops in M is given by the elements of $r(G)$. Further $\Phi(M)$ is isomorphic to $r(G)$ and this justifies the name holonomy group for $r(G)$ when G is a Bieberbach group; for more details we refer to example 3.1 page 50 of [C]. In the particular case where G is a Bieberbach subgroup of $\mathcal{M}_n \subset \mathcal{A}_n$ we can actually do more. $\mathbb{R}^n$ is the universal covering of M and by means of standard arguments in covering space theory, see [W3] pages 31-42, we can deduce that G is the group of deck transformations isomorphic to $\pi_1(M)$. Bieberbach's first theorem says that $\Phi(M) \simeq r(G)$ is finite therefore $\Phi_0(M)$ is trivial and the map $r : \pi_1(M) \longrightarrow \Phi(M)/\Phi_0(M)$ defined in Corollary (1.3.2) is nothing else than the rotational part $r : G \longrightarrow r(G)$.

The *curvature* of M with connection ∇ is the transformation R(U,V) , U,V vector fields on M , to vector fields on M given by

$$[R(U,V)](W) = -\nabla_U(\nabla_V(W)) - \nabla_V(\nabla_U(W)) + \nabla_{[U,V]}(W)$$

where W is a vector field on M and [U,V] the vector field defined by

$$[U,V](f) = U(V(f)) - V(U(f)) \quad ; \quad f \in C^\infty(M).$$

As a consequence of the Ambrose-Singer Holonomy Theorem, see either [A1] or [N] page 39, one gets the following corollary

Corollary (1.3.3): The curvature R of M is identically zero if and only if $\Phi_0(M) = \{I\}$, forcing the map $r : \pi_1(M) \longrightarrow \Phi(M)$ to be surjective.

We say that $(M,{}_M\nabla)$ is *flat* if the curvature R is identically zero. One can show that this is precisely the condition that needs to be satisfied for M to be locally not only topologically (even differentiably) similar to $\mathbb{R}^n$ but also to have a connection similar to the usual one on $\mathbb{R}^n$. For a discussion of this, see pages 301-303 in [S3].

For any two vector fields U,V on $(M,{}_M\nabla)$ one defines their *torsion* T by

$$T(U,V) = \nabla_U V - \nabla_V U - [U,V] \ .$$

A *geodesic* c is a curve on $(M,{}_M\nabla)$ so that $\nabla_{\dot c}\, \dot c = 0$. One can show that for every $x \in M$ and every $V_x \in M_x$ there is a unique geodesic on M , defined on $[0,\epsilon)$ for some $\epsilon > 0$, such that $c(0) = x$ and $\dot c_0 = V_x$.

$(M,{}_M\nabla)$ is said to be *symmetric* or *torsion free* if $T \equiv 0$; and $(M,{}_M\nabla)$ is said to be *complete* if for every $x \in M$ and every $V_x \in M_x$ the unique geodesic c with $c(0) = x$ and $\dot c_0 = Vx$ can be defined on all of [0,1]. The following theorem is basic

Theorem (1.3.4): Let M be a connected, simply connected n-manifold with a connection ${}_M\nabla$ such that $(M, {}_M\nabla)$ is flat, symmetric and complete, then M is affinely equivalent to $\mathbb{R}^n$ with the usual connection.

A proof of this theorem can be found on page 211 of [K-N]. As an immediate corollary by use of the theory of covering spaces we get

Corollary (1.3.5): Let M be a connected n-manifold with connection ${}_M\nabla$ such that $(M, {}_M\nabla)$ is flat, symmetric and complete. Then there is a subgroup G of $\mathcal{A}_n$ such that M is affinely equivalent to $\mathbb{R}^n/G$.

B. Flat Riemannian Manifolds .

A *Riemannian manifold* is an n-manifold M with a Riemannian structure i.e. a collection $\{ <,>_x \}_{x \in M}$ of positive definite inner products defined on every $x \in M$ such that if U , V two vector fields on M the function $x \mapsto < U_x, V_x >_x$ is a smooth function. It is an easy exercise to show that every manifold has a Riemannian structure. We have the following basic proposition

Proposition (1.3.6): Given a Riemannian manifold M , there is an unique symmetric connection on M such that parallel translation from M_x to M_y , c a curve on M with $c(0) = x$ and $c(1) = y$, is an isometry between inner product spaces.

See [M1] page 48 for a proof of Proposition (1.3.6). We call this connection the *Levi-Civita connection* and denote it by ∇ . It is a consequence of the Hopf-Rinow Theorem, see relatively page 62 of [M1] that the Levi-Civita connection of a compact Riemannian manifold is complete.

A *flat manifold* is a Riemannian manifold whose Levi-Civita connection is flat. Part of Clifford-Klein's [H] asserts the following stronger version of Theorem (1.3.4):

Theorem (1.3.7): If M is a connected, simply connected, Riemannian n-manifold such that (M,∇) is complete, flat then the differential of the affine equivalence between M and $\mathbb{R}^n$ is for every $x \in M$ an isometry of inner product spaces.

Following similar arguments as in Corollary (1.3.5) before one gets :

Corollary (1.3.8): Let M be a connected, Riemannian n-manifold such that (M,∇) is complete and flat. Then there is a discrete torsion free subgroup $G \in \mathcal{A}b_n$ such that the differential of the affine equivalence between M and $\mathbb{R}^n/G$ is an isometry. Furthermore, if M is compact G is a Bieberbach subgroup.

We can now state the geometric translation Bieberbach's theorems. Their proofs are immediate consequences of Corollary (1.3.8) above, the discussion following Corollary (1.3.2) and Theorems (1.1.2,3,4)

Theorem (Bieberbach) (1.3.9): Let M , N be compact, connected, Riemannian n-manifolds such that (M,∇) , (N,∇) are (complete) and flat. Then

(First) (1.3.9.I): There is a flat torus ($\mathbb{R}^n/G \cap \mathbb{R}^n$, where $G = \pi_1(M)$) covering M , and the covering map is a local isometry. Moreover, the holonomy group $\Phi(M)$ is isomorphic to the rotational part of G , r(G) and finite. The same, of course, is true for N .

(Second) (1.3.9.II): If $\pi_1(M) \simeq \pi_1(N)$, then M and N are affinely equivalent.

(Third) (1.3.9.III): The affine equivalence classes of compact, connected, flat, Riemannian n-manifolds are finite.

As a corollary to Theorem (1.1.5) one has immediately

Corollary (1.3.10): Let Φ be an arbitrary finite group. Then, there is a flat manifold whose holonomy group is precisely Φ.

C. Classification of Compact Flat Riemannian Manifolds.

The meaning of this discussion is that one can classify the compact flat manifolds up to diffeomorphisms preserving their connection, by classifying the torsion-free Bieberbach groups up to isomorphism. The term Φ-module will be used to denote a finitely generated free abelian group N which is a faithful left module over $\mathbb{Z}[\Phi]$, Φ is a finite group.

The problem in consideration now, comes down to the following classification scheme:

1) Find all Φ-modules N.
2) Find all extensions of N by Φ, i.e., compute $H^2(\Phi; N)$.
3) Find amongst the above those extensions that are torsion-free.
4) Check which of these extensions are isomorphic.

The answer to step 3) is given by

Theorem (Charlap) (1.3.10): Let N be a Φ-module. The extension of Φ by N corresponding to $\gamma \neq 0 \in H^2(\Phi; N)$ is torsion-free if and only if $(I_p)^*(\gamma) \neq 0$, for all C_p cyclic subgroups of G of prime order and $I_p : C_p \mapsto G$ the respective inclusion.

Summarizing the results in [C1], a compact flat manifold M is classified up to connection preserving diffeomorphism by a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Phi \longrightarrow 1$$

in which Φ is the holonomy group of M (a finite group), $\Gamma \simeq G_1(M)$ is a torsion free discrete cocompact subgroup of the group $\mathcal{M}_n$ of the rigid motions of $\mathbb{R}^n$ and Λ is its translation subgroup of Γ , free abelian of maximal rank. Conversely, given any such torsion free extension, Γ imbeds as a discrete cocompact subgroup of $\mathcal{M}_n$ and

$$M_\Gamma = \Gamma\backslash\mathcal{M}_n/O(n)$$

is a compact flat manifold ; O(n) is the isotropy group of the origin.

Chapter II

Algebraic Riemann Matrices and Projective Manifolds

§ 2.1 Positively Involuted Semisimple Algebras.

The purpose of this section is to provide Albert's classification of positively involuted division algebras, [A]. The first subsection attempts a concise introduction to the essentials of classical representation theory needed for the development of the second subsection, where we focus on involuted algebras.

All algebras in this section are to be finite-dimensional over a field of characteristic zero.

A. Rudiments of Classical Representation Theory.

In what follows all algebras are considered to be finite dimensional over $\mathbb{Q}$, or more generally over a field of characteristic zero but not necessarily algebraically closed. As general references for this section we point out [L2] chapter XVIII, or [C-R] chapter IV, or even [S] part II.

The notion of two-sided ideals is crucial throughout the structure theorems of simple and semi-simple algebras. A finite dimensional algebra $\mathcal{A}$ over a field K as above is said to be *simple* if the only two-sided ideals it contains are $\mathcal{A}$ and $\{0\}$. An $\mathcal{A}$-module M is said to be *simple* or *irreducible* if and only if it has no other $\mathcal{A}$-submodules except itself and the trivial one. The following proposition helps us understand how the notions of a simple algebra, a division algebra and that of a simple module are connected:

Proposition (2.1.1): If $\mathcal{A}$ is a finite dimensional simple K-algebra then the following are equivalent:

(i) $\mathcal{A}$ is a simple K-algebra;

(ii) $\mathcal{A} \simeq M_n(D)$ for some finite dimensional division K-algebra and some integer $n \geq 1$;

(iii) there is a K-module isomorphism $\mathcal{A} \simeq (S)^m$ where S is *any* simple $\mathcal{A}$-module and the division algebra in (ii) is recoverable since D $\simeq \mathrm{End}_{\mathcal{A}}(S)$.

A left $\mathcal{A}$-module M is *(finitely) semisimple* if it can be written as a (finite) direct sum of simple $\mathcal{A}$-modules. If $M \simeq (N)^d$, where N is a simple $\mathcal{A}$-module and d a positive integer, we say that M is *isotypic of type N and multiplicity d*. A *block decomposition* for M is a finite expansion of the form $M \simeq \oplus (N_i)^{d_i}$ where the N_i's are non-isomorphic simple $\mathcal{A}$-modules and the d_i's psoitive integers. It can be seen that the block decomposiiton for semisimple modules is essentially unique. A fundamental result on the structure of semisimple $\mathcal{A}$-modules asserts that:

Proposition (2.1.2): If M is a finitely semisimple $\mathcal{A}$-module with block decomposition $(N_i, d_i)_{1 \leq i \leq n}$ then there is an isomorphism of algebras

$$\mathrm{End}_{\mathcal{A}}(M) \xrightarrow{\simeq} \prod_{1}^{n} M_{d_i}(D_i)$$

where $D_i = \mathrm{End}_{\mathcal{A}}(N_i)$ is a division algebra.

A finite dimensional K-algebra $\mathcal{A}$ is said to be *semisimple* if it is semisimple as a left K-module. Notice that the evaluation map ev : $\mathrm{End}_{\mathcal{A}}(\mathcal{A}) \longrightarrow \mathcal{A}$ defined by $ev(f) = f(1_{\mathcal{A}})$ is an isomorpism, and taken together with the above proposition yields:

Corollary (2.1.3): If $\mathcal{A}$ is a finite dimensional semisimple K-algebra there is an isomorphism of K-algebras

$$\mathcal{A} \simeq \prod_{1}^{n} M_{d_i}(D_i)$$

where D_i is a finite dimensional K-division algebra and n and each d_i is a positive integer.

This decomposition of a semisimple algebra is known as the *Wedderburn Decomposition* and it can be seen that it is essentially unique up to K-algebra isomorphisms and rearrangement.

The notions of semisiplicity for an algebra and its modules are reciprocal. Theorem (2.1.4) throws more light in the structure of such algebras and displays the connection between semisimplicity and projectivity. We recall that an $\mathcal{A}$-module M is called *projective* if there exists an $\mathcal{A}$-module N such that $M \oplus N$ is a free $\mathcal{A}$-module.

Theorem (2.1.4): If $\mathcal{A}$ is a finite dimensional K-algebra, then the following are equivalent:

(i) $\mathcal{A}$ is semisimple;

(ii) every finitely generated left (right) $\mathcal{A}$-module is semisimple;

(iii) every finitely generated left (right) $\mathcal{A}$-module is projective;

(iv) $\mathcal{A}$ is isomorphic to a product $\mathcal{A} \simeq \prod M_{d_i}(D_i)$ where each D_i is a K-division algebra and d_i a positive integer.

To shed further light in the structure of semisimple algebras and modules we remark that every irreducible $\mathcal{A}$-module is isomorphic to some minimal left ideal of $\mathcal{A}$. Furthermore, the sum of all isomorphic minimal left ideals of $\mathcal{A}$ is a two-sided ideal of $\mathcal{A}$. It turns out that the isotypic components of $\mathcal{A}$, when $\mathcal{A}$ is considered as a left module over itself, are isomorphic to precisely the simple two-sided ideals of $\mathcal{A}$. Moreover, if $\{A_i\}_{1 \leq i \leq n}$ is the set of all simple two-sided ideals of $\mathcal{A}$, it can be seen that $\mathcal{A}$ is the internal direct sum of all the A_i's. For more details and complete proofs of all these remarks, the interested reader is referred to Propositions 25.10, 25.15, 25.22 of [C-R].

B. Positive Division Algebras.

Let A be a finite dimensional semisimple over a field K of characteristic zero.

A K-homomorphism $\tau : A \longrightarrow A$ such that $\tau(xy) = \tau(y)\tau(x)$ for all x, $y \in A$ and $\tau^2 = 1_A$ is called an *involution* on A. If K is a real field, the involution τ is said to be *positive* if $Tr_K(x\tau(y)) > 0$, for all $x \in A - \{0\}$. (Here, by $Tr_K(z)$ for an element z of a K-algebra A, we mean the trace of the matrix representing the map $f : A \longrightarrow A$ with $f(x) = zx$, $x \in A$, with respect to a (and therefore any) fixed K-basis of A).

Pairs (A,τ) where A is a finite dimensional algebra over k and τ an involution on A form a category with morphisms $h : (A,\tau) \longrightarrow$

(B,σ), K-algebra homorhisms that commute with the respective involutions, $h \circ \tau = \sigma \circ h$. This category contains products of the form

$$\prod_1^n (A_i,\tau_i) = (\prod_i A_i, \prod_i \tau_i) .$$

According to the discussion following Theorem (2.1.4) one can express A as a direct sum of simple two-sided ideals $\{A_i\}$ say $A = \oplus_1^n A_i$. We can easily see that a positive involution on A restricts to a positive involution on A_i, for all i =1, ..., n. Therefore, it suffices to consider the case where A is simple, and by Proposition (2.1.1) above of the form $A = M_n(D)$ for some finite dimensional division K-algebra D. One can easily see that an involution τ on D extends to an involution $\hat{\tau}$ on A by letting

$$\hat{\tau}((x_{ij})) = (\tau(x_{ji})) .$$

Also, by means of the Skolem-Noether Theorem, one can show that every involution on $A = M_n(D)$ has precisely this form. Furthermore, $\hat{\tau}$ is positive if and only if τ is positive. One has proved thus:

Proposition (2.1.5): If (A,τ) is a positively involuted finite dimensional semisimple K-algebra, K a real field of zero characteristic, then one has the following isomorphism of positively involuted K-algebras

$$(A,\tau) = (M_{n_1}(D_1),\hat{\tau}_1) \times \cdots \times (M_{n_r}(D_r),\hat{\tau}_r)$$

where (D_i,τ_i) is a positively involuted division algebra of finite dimension over K.

τ is said to be *of the first kind* if it restricts to the identity on the center of a simple algebra A , otherwise it is said to be *of the second kind.*

Let a, b $\in$ K , denote by $(\frac{a,b}{K})$ the *quaternion algebra* over K with basis {1, i, j, k} and multiplication obeying the relations

$$i^2 = a \cdot 1 , \qquad j^2 = b \cdot 1 , \qquad i \cdot j = -j \cdot i = k .$$

$(\frac{a,b}{K})$ is a central simple algebra over K which admits essentially two distinct involutions of the first kind, namely *conjugation* c , and *reversion* r , defined as follows:

$$c(x_0 + x_1 i + x_2 j + x_3 k) = x_0 - x_1 i - x_2 j - x_3 k$$

$$r(x_0 + x_1 i + x_2 j + x_3 k) = x_0 + x_1 i - x_2 j + x_3 k \ .$$

More information on quaternion algebras can one find in [Po] chapter X, or alternatively in chapter IX of [L].

The following standard proposition relates quaternion algebras to quadratic forms. Its proof can be found either in [Pi] page 15.

Proposition (2.1.6): If $Q = (\frac{a,b}{K})$ is as above, then the following are equivalent:

(i) Q is a division algebra;

(ii) $x \cdot c(x) \neq 0$ for all non-trivial $x \in Q$;

(iii) the quadratic form $\Omega : K^3 \longrightarrow K$, given by $\Omega(x,y,z) = x^2 - a \cdot y^2 - b \cdot z^2$, is anisotropic over K.

The classification of finitely generated, positively involuted division algebras over $\mathbb{R}$ is quite simple. One gets only three distinct kinds, namely $(\mathbb{R},1_{\mathbb{R}})$, ($\mathbb{C}$,"-") and ($\mathbb{H}$,c), where $\mathbb{H} = (\frac{-1,-1}{\mathbb{R}})$, the classical quaternions. "-" is the complex conjugation and c the quaternionic conjugation defined as above. For more detail, we refer to [Pi] page 206 Proposition 11.23.

C. Albert's Classification of Finite Dimensional Positively Involuted $\mathbb{Q}$-Algebras.

Before we proceed with Albert's result, we need a definition. Let K be a field, f a field automorphism of K and a a non-zero element of K that stays fixed under f. The *cyclic algebra* (K,f,a) is composed of a two-sided vector space (K,f,a) of dimension n and basis $\{[x^r]\}_{0 \leq r \leq n-1}$

over K , subject to relations

$$[x^r]\, y = f^r(y)\, [x^r] , \quad y \in K .$$

It is an algebra with center $\mathfrak{Z}((K,f,a)) = \{ x \in K : f(x) = x \}$ and multiplication

$$[x] \cdot [x^r] = \begin{cases} [x^{r+1}] , & 0 \le r < n-1 \\ a\,[x^0] , & r = n-1 . \end{cases}$$

Albert's result [A] essentially asserts that there are only four distinct kinds of positively involuted division algebras (D,τ) of finite dimension over $\mathbb{Q}$; F and K here are algebraic number fields :

(I) $D = F$ is totally real and $\tau = 1_F$;

(II) $D = (\frac{a,b}{F})$, with F totally real, a totally positive, b totally negative, and $\tau = r$;

(III) $D = (\frac{a,b}{F})$, with F totally real, a and b both totally negative, and $\tau = c$;

(IV) $D = (K,f,a)$, where f is a field automorphism of K whose fixed point field F $(a \in F)$ is an imaginary quadratic extension, $F = F_0(\sqrt{b})$, of a totally real field F_0 ; [furthermore, if L is a maximal totally real subfield of K , there exists a totally positive element $d \in L$ such that $N_{F/F_0}(a) = N_{L/F_0}(d)$].

Notice that Albert's Theorem asserts that the first three types of algebras are precisely those finite dimensional $\mathbb{Q}$-division algebras which admit a positive involution of the first kind, while type IV is the only one admitting a positive involution of the second kind.

D. The Group Algebra and Representations of Finite Groups.

Let now Φ be a finite group and $K[\Phi]$ denote the group algebra of Φ over K; this is an algebra whose basis is indexed by all elements of Φ. Each element of $K[\Phi]$ can then be written uniquely in the form $x = \sum_{g \in G} x_g\, g$, where $x \in K[\Phi]$ $x_g \in K$ and multiplication in $K[\Phi]$ extends that in K.

Maschke's theorem, see for example Theorem 1.2 on page 641 of [L2], asserts that for a finite group Φ and a field K whose characteristic is coprime to the order of Φ every $K[\Phi]$-module is projective. It then follows from Theorem (2.1.4) that $K[\Phi]$ is semisimple and therefore it accepts a Wedderburn Decomposition. The uniqueness of this decomposition along with the natural one-to-one correspondence between K-representations of Φ and left modules over the group algebra $K[\Phi]$ (this correspondence preserves direct sums, and irreducible representations correspond to simple modules) give us a handy tool to tackle the representation theory of groups of small order. It is worth noting here the following particular characteristics of this decomposition. If $K[\Phi] \simeq \prod_1^n M_{d_i}(D_i)$ then

(i) at least for one i , $D_i = K$ and $d_i = 1$.

If further K is algebraically closed one has the following additional properties:

(ii) each division algebra $D_i = K$ and n is equal to the number of conjugacy classes of Φ

(iii) $\sum_1^n d_i^2 = |\Phi|$

(iv) each d_i divides $|\Phi|$

(v) there exists an irreducible K-representation of Φ of dimension r if and only if $r = d_i$ for some i ; moreover, the number of isomorphically distinct K-representations of dimension r is equal to the number of indices i for which $d_i = r$.

One can easily see that in view of the information available the representation theory over an algebraically closed field is relatively easy. Questions arise when one attempts to restrict the field to a non-algebraically closed one, like $\mathbb{Q}$ for example. The main theorems do descend although major modifications are needed. For more details on this see [S] chapters 12-13 of Part II. The situation worsens rapidly if instead of a field we consider a ring, even if that ring is a principal ideal domain like $\mathbb{Z}$ for example. Crucial theorems like the Jordan-Hölder Theorem and the Krull-Schmidt [R1] and also Maschke's Theorem fail to hold. Furthermore, indecomposable and irreducible modules are no longer synonymous concepts. Although a $\mathbb{Z}[\Phi]$-module M is reducible if and only if its extension over $\mathbb{Q}$, $\mathbb{Q} \otimes_{\mathbb{Z}} M$, is reducible, Theorem (73.9) on page 497 of [C-R], one needs to find how the irreducible ones may be combined into idecomposable modules. A classification would also require to determine when two direct sums of indecomposable $\mathbb{Z}[\Phi]$-modules are $\mathbb{Z}[\Phi]$-isomorphic since the Krull-Schmidt Theorem fails. Also, it is by no means predictable whether the number $n(\mathbb{Z}[\Phi])$ of indecomposable $\mathbb{Z}[\Phi]$-modules, where we chose one module from each isomorphism class, is finite or not for the general group Φ.

The simplest problem of this kind, the case where Φ is a group of prime order, was settled by Diederichsen [D] and Reiner [R]. Heller and Reiner [H-R, 1] have shown that $n(\mathbb{Z}[\Phi])$ is finite for Φ cyclic of order p^2, where p is any prime. (This result has been obtained independently by Knee [K], and for the special case where $p = 2$, by Roiter [R2] and Troy [T]). On the other hand, Heller and Reiner [H-R, 2] have proved that $n(\mathbb{Z}[\Phi])$ is infinite if Φ is cyclic of order p^n, where $n \geq 3$, p prime, and also that $n(\mathbb{Z}[\Phi])$ is infinite if Φ is a non-cyclic p-group. A. Jones [J1] studied the representation theory for direct products of groups. This approach enabled him to provide a necessary and sufficient condition for $n(\mathbb{Z}[\Phi])$ to be finite. Thus

Theorem (2.1.7): Let Φ be a finite group. Then $n(\mathbb{Z}[\Phi])$ is finite if and only if each p-Sylow subgroup of Φ is cyclic of order at most p^2, for all p primes dividing the order of Φ.

J. Oppenheim [O1] studied the case where Φ is a cyclic group of square free order considering in particular *regular* $\mathbb{Z}[\Phi]$-modules, i.e. finitely generated torsion free $\mathbb{Z}$-modules. M.P. Lee [L3] studied the case Φ is the dihedral group of order $2p$, p a prime number.

If K is a subfield of $\mathbb{R}$, we saw above that the group algebra $K[\Phi]$ is semisimple. Furthermore, it is naturally endowed with a positive involution τ given by

$$\tau(x) = \sum_{g \in \Phi} x_g \, g^{-1}$$

where $x = \sum_{g \in \Phi} x_g \, g$, $x_g \in K$. By Proposition (2.1.5) there is an isomorphism

$$\textit{(2.1.8)} \qquad (K[\Phi],\tau) = (M_{n_1}(D_1),\hat{\tau}_1) \times \cdots \times (M_{n_r}(D_r),\hat{\tau}_r)$$

where each D_i is a finite dimensional division algebra over K , admitting the positive involution τ_i . By the structure theorem for simple algebras, Proposition (2.1.1), and the remarks that follow the structure theorem of semisimple algebras, Proposition (2.1.2), we let S_i be a simple left $K[\Phi]$-module isomorphic to a simple left ideal of $M_{n_i}(D_i)$ and $\mathrm{End}_{K[\Phi]}(S_i) = D_i$. We shall be particularly interested in the cases $K = \mathbb{R}$ and $K = \mathbb{Q}$. If V is an isotypic $K[\Phi]$-module, i.e. $V \simeq S^d$, where S a simple $K[\Phi]$-module and d a positive integer, then we say V is of type D if and only if $D = \mathrm{End}_{K[\Phi]}(S)$. Thus, if $K = \mathbb{R}$ then, according to our comments following Proposition (2.1.6), an isotypic $\mathbb{R}[\Phi]$-module can be of type $\mathbb{R}$, $\mathbb{H}$ or $\mathbb{C}$. If $K = \mathbb{Q}$ then Albert's classification asserts that an isotypic $\mathbb{Q}[\Phi]$-module can be of type I, II, III or IV.

§ 2.2 Construction of Algebraic Riemann Matrices.

We work in the category of Riemann matrices over $\mathbb{Z}$, denoted by $\mathcal{RM}_{\mathbb{Z}}$. In the following definitions let K be a subring of $\mathbb{R}$.

The objects of $\mathcal{RM}_K$ are triples (N, Λ, t), where N is a K[Φ]-module of finite rank, Λ is a lattice in N and $t \in \mathrm{End}_{\mathbb{R}}(\mathbb{R} \otimes_K N)$ such that $t^2 = -1$. The morphisms are maps of Riemann matrices $\phi : (V, \Lambda, t) \mapsto (U, \Omega, t)$ where $\phi : V \mapsto U$ is a K-linear map such that $\phi(\Lambda) \subseteq \Omega$ and $(1 \otimes \phi) \circ t = s \circ (1 \otimes \phi)$ where $1 \otimes \phi$ is the iduced map on the respective realizations.

For a Riemann matrix (V, Λ, t), we let $(V_{\mathbb{R}})^t$ denote the complex vector space whose underlying real space is $V_{\mathbb{R}}$ and the complex multiplication is given through t. An object (N, Λ, t) in $\mathcal{RM}_K$ is called *simple* over K if and only if it contains no non-trivial K-submodule M such that $t(M \otimes_K \mathbb{R}) = M \otimes_K \mathbb{R}$. $\mathcal{RM}_K$ has finite products given by $(V_1, \Lambda_1, t_1) \times (V_2, \Lambda_2, t_2) = (V_1 \times V_2, \Lambda_1 \times \Lambda_2, t_1 \times t_2)$.

A *Riemann form* for (V, Λ, t) is a nondegenerate alternating bilinear form $B : V \times V \mapsto K$ such that $B(\Lambda,\Lambda) \in \mathbb{Z}$ and, if $\mathfrak{B} : V_{\mathbb{R}} \times V_{\mathbb{R}} \mapsto \mathbb{R}$ denotes the realisation of B, then

(i) $\mathfrak{B}$ is invariant under the complex structure, i.e.

$$\mathfrak{B}(x, y) = \mathfrak{B}(t(x), t(y)) , \quad \forall\, x, y \text{ , and}$$

(ii) $\hat{\mathfrak{B}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \mapsto \mathbb{R}$ defined by

$$\hat{\mathfrak{B}}(x, y) \equiv \mathfrak{B}(t(x), y), \quad \forall\, x, y$$

is symmetric and positive definite.

A Riemann matrix is said to be *algebraic* when it admits a Riemann form.

We shall see now how one can construct algebraic Riemann matrices when one has at his disposal a positively involuted division algebra.

Let K be a subfield of $\mathbb{R}$. If V is a finite dimensional K-vector space over K then one can easily check that the K-linear map $t : V \oplus V$

$\longrightarrow V \oplus V$ defined by $t(x_1,x_2) = (-x_2,x_1)$, extends to a complex structure for $V \oplus V$. The Riemann matrix $D(V) := (V \oplus V,\ t \otimes 1)$, thus defined is called the *double* of V . If further $\{f_i\}_{1 \le i \le n}$ is a basis for the K-dual of V , we can define a form $\beta : D(V) \times D(V) \longrightarrow K$ as follows

$$\beta\big((x_1,x_2)\ ,\ (y_1,y_2)\big) = \sum_1^n \big(f_j(x_2)\ f_j(y_1) - f_j(x_1)\ f_j(y_2)\big)\ .$$

It is not difficult to check that all three conditions of the definition of a Riemann form hold true for β and $D(V)$ is an algebraic Riemann matrix. This construction is a standard tric and is called *the doubling construction.* It will be of particular interest to us when trying to produce an algebraic Riemann matrix out of those division algebras that are of type I in Albert's classification. The other three types are uniformly treated through the notion of the *complex multiplication-algebra.*

A *CM-algebra* is a quadruple $(\mathcal{A},\mathcal{E},\tau,\alpha)$ with the following properties:

(i) $\mathcal{E}$ is a totally real algebraic number field of finite degree over $\mathbb{Q}$;

(ii) $(\mathcal{A},\tau)$ is a finite dimensional positively involuted $\mathcal{E}$-algebra such that $\tau_{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$;

(iii) $\alpha \in \mathcal{A}$ such that α^2 is a totally negative element of $\mathcal{E}$;

(iv) $\mathcal{F} := \mathcal{E}(\alpha)$ is a purely imaginary extension of $\mathcal{E}$; and

(v) $\tau(\mathcal{F}) = \mathcal{F}$ and τ restricts to the non-trivial element of $\mathrm{Gal}(\mathcal{F}/\mathcal{E})$.

There is a *canonical complex structure* $t : \mathcal{A} \otimes_{\mathcal{E}} \mathbb{R} \longrightarrow \mathcal{A} \otimes_{\mathcal{E}} \mathbb{R}$ one can associate with a CM-algebra $(\mathcal{A},\mathcal{E},\tau,\alpha)$ defined by

$$t\ (x \otimes 1) = \alpha\ x \otimes \frac{1}{\sqrt{-\alpha^2}}\ ,$$

for $x \in \mathcal{A}$ and extended $\mathbb{R}$-linearly.

On a CM-algebra $(\mathcal{A},\mathcal{E},\tau,\alpha)$ one can further define the form $\beta : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{E}$ by

$$\beta\ (x\ ,\ y) = \mathrm{Tr}_{\mathcal{E}}\big(\alpha\ y\ \tau(x) - \alpha\ x\ \tau(y)\big)\,.$$

From the definition of a CM-algebra $\alpha^2 \in \mathcal{E}$, and the observation $\tau(\alpha)$

$= -\alpha$ easily follows. Also recall that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$. These two remarks along with the positivity of τ suffice to show straightforwardly that both conditions of the definition of a Riemann form hold true for β . Thus,

Proposition (2.2.1): If $(\mathcal{A},\mathcal{E},\tau,\alpha)$ is a CM-algebra and t its canonical complex structure, then $(\mathcal{A},\tau)$ is an algebraic Riemann matrix over $\mathcal{E}$.

The following proposition, for finite dimensional division algebras of type II in Albert's classification, is a result of an observation of Shimura [S1] Proposition 2 on page 153, and a tautology for algebras of types III and IV.

Proposition (2.2.2): If $(\mathcal{A},\tau)$ is a positively involuted finitely dimensional division algebra of type II, III or IV over $\mathbb{Q}$ then there exists a subfield $\mathcal{E} \subseteq \mathcal{A}$, and $\alpha \in \mathcal{A}$ such that $(\mathcal{A},\mathcal{E},\tau,\alpha)$ is a CM-algebra.

To be able to extend our results to semisimple algebras we need the following *complete reducibility* theorem of Poincaré. For a proof in terms of tori, see [L] chapter VII, §4 on page 117.

Theorem (2.2.3): If K is a subfield of $\mathbb{R}$ and $(\mathcal{A},\Lambda,t)$ an algebraic Riemann matrix over K , then $(\mathcal{A},\Lambda,t)$ is isomorphic in $\mathfrak{RM}_K$ to a product

$$(\mathcal{A},\Lambda,t) \simeq (\mathcal{A}_1,\Lambda_1,t_1)^{d_1} \times \cdots \times (\mathcal{A}_n,\Lambda_n,t_n)^{d_n}$$

where $\{(\mathcal{A}_i,\Lambda_i,t_i)\}_{1 \le i \le n}$ are simple Riemann matrices over K , and the isomorphism types and multiplicities d_i are unique up to order.

We can improve on the above theorem precisely as we did with the decomposition theorem of finitely dimensional semisimple algebras. For a Riemann matrix $(\mathcal{A},\Lambda,t)$ over K , write $(\mathcal{A},\Lambda,t)^* := (\mathcal{A}^*,\Lambda^*,t^*)$ where $\mathcal{A}^* \equiv \mathrm{Hom}_K(\mathcal{A},K)$ the K-dual of $\mathcal{A}$, $\Lambda^* \equiv \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})$ the $\mathbb{Z}$-dual of Λ , t^* is the $\mathbb{R}$-dual of t . A Riemann form $\beta : \mathcal{A} \times \mathcal{A} \longrightarrow K$ on $(\mathcal{A},\Lambda,t)$ gives rise to an injective K-linear map $\breve{\beta} : \mathcal{A} \longrightarrow \mathcal{A}^*$ by letting $[\breve{\beta}(x)](y)$

$= \beta(x,y)$ for all $x, y \in \mathcal{A}$. Moreover, if K is a subfield of $\mathbb{R}$, $\breve{\beta}$ is bijective, and after identifying $\mathcal{A} \equiv \mathcal{A}^{**}$, one can define a Riemann form $\breve{\beta}$ on $(\mathcal{A},\Lambda,t)^*$ by letting for all $f, g \in \mathcal{A}^*$

$$\beta^* (f , g) = [(\breve{\beta})^{-1}(f)] (g) \ .$$

It turns out that $(\mathcal{A},\Lambda,t)$ is algebraic if and only if $(\mathcal{A},\Lambda,t)^*$ is algebraic. Furthermore, a Riemann form β on $(\mathcal{A},\Lambda,t)$ induces an isomorphism $\breve{\beta}$: $(\mathcal{A},\Lambda,t) \longrightarrow (\mathcal{A},\Lambda,t)^*$ in $\mathfrak{R}\mathcal{M}_K$. Rosati [R4], see also [M3] page 189, observed that the K-algebra $\mathrm{End}_K(\mathcal{A},\Lambda,t)$ can be given a positive involution, the so called *Rosati involution* denoted by " σ " ; thus for $f \in \mathrm{End}_K(\mathcal{A},\Lambda,t)$ we have

$$\sigma(f) = (\breve{\beta})^{-1} \circ f^* \circ \breve{\beta}$$

where $f^* : (\mathcal{A},\Lambda,t)^* \longrightarrow (\mathcal{A},\Lambda,t)^*$ the obvious map induced by f . Using earlier results, Propositions (2.1.3) and (2.1.5), it can be shown that

Corollary (2.2.4): The algebra $\mathrm{End}_K(\mathcal{A},\Lambda,t)$, of K-endomorphisms of $(\mathcal{A},\Lambda,t)$, is isomorphic to a product

$$\left(\mathrm{End}_K(\mathcal{A},\Lambda,t) \ , \sigma\right) \simeq M_{d_1}(D_1 \ , \sigma_1) \times \cdots \times M_{d_n}(D_n \ , \sigma_n)$$

where $D_i = \mathrm{End}_K(\mathcal{A}_i,\Lambda_i,t_i)$ is a positively involuted division algebra over K and σ_i is the Rosati involution on D_i . In particular, $\left(\mathrm{End}_K(\mathcal{A},\Lambda,t) \ , \sigma\right)$ is a positively involuted semisimple algebra over K . The d_i's and $\mathcal{A}_i$'s are kept as in the theorem above.

Remark (2.2.5): The formal similarity between Corollary (2.2.4) and the decomposition (2.1.8) of the group ring is behind the algebraic description of complex projective flat manifolds, Theorem (2.4.1).

§ 2.3 Projective Structures on Representation Spaces and Abelian Varieties.

Throughout this section, we fix the following notation:

Λ : a free abelian group of even rank ;

V : the real vector space $V \equiv \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$; Λ is considered to be imbedded in V via the map $\lambda \mapsto \lambda \otimes 1$, for all $\lambda \in \Lambda$;

t : a complex structure on V , i.e. a real linear map $t : V \longrightarrow V$ such that $t^2 \equiv -(\mathrm{id})_V$;

V^t : the complex vector space whose underlying real vector space is V , and complex multiplication is defined by means of t , i.e. $(x+iy) \cdot v = x \cdot v + y \cdot t(v)$ where $x, y \in \mathbb{R}$, and $v \in V$;

Φ : a finite group ;

ρ : a faithful integral representation $\rho : \Phi \longrightarrow GL_{\mathbb{Z}}(\Lambda)$;

${}_{\rho}\Lambda$: the $\mathbb{Z}[\Phi]$-module whose additive group is given by Λ and Φ acts on it by means of ρ ;

${}_{\rho_{\mathbb{R}}}V$: the $\mathbb{R}[\Phi]$-module whose additive group is V and the action of Φ on it is given by the extension $\rho_{\mathbb{R}}$ of ρ to $\mathbb{R}$, i.e. $\rho_{\mathbb{R}} : \Phi \longrightarrow GL_{\mathbb{Z}}(\Lambda) \hookrightarrow GL_{\mathbb{R}}(V)$.

When there is no fear of confusion we may write Λ instead of ${}_{\rho}\Lambda$, and V instead of ${}_{\rho_{\mathbb{R}}}V$. We shall say Λ admits a *complex structure* if there exists a map $t \in \mathrm{End}_{\mathbb{R}[\Phi]}({}_{\rho_{\mathbb{R}}}V)$ such that $t^2 = -1$. In view of the well known correspondence between Φ-representations over a ring K and $K[\Phi]$-modules, we can, in the language of representation theory, equivalently define t to be a *complex structure* for (Λ,ρ) when $\mathrm{Im}(\rho) \subset GL_{\mathbb{C}}(V^t)$. $GL_{\mathbb{C}}(V^t)$ denotes the following set:

$$GL_{\mathbb{C}}(V^t) := \{ m \in GL_{\mathbb{R}}(V) \text{ such that } mt = tm \} .$$

One says that Λ or (Λ,ρ) admits a *projective structure* if, in addition to its complex structure, the complex torus V^t/Λ is projective as a complex manifold. A complex torus is said to be *abelian, algebraic* or *projective* if it is projective as a complex manifold; i.e. if it can be holomorphically embedded in the complex projective space. Its image regarded as an algebraic subvariety of the projective space is then called an *abelian variety.* Often, the two notions are being confused.

The following key theorem helps us study the holomorphic embeddings of flat projective varieties by means of algebraic Riemann matrices. For a complete proof see either [L] chapters VI and VIII .

Theorem (2.3.1): If (V,Λ,t) is a Riemann matrix over $\mathbb{Z}$, then the following statements are equivalent:

(i) V^t/Λ is an abelian variety ;

(ii) (V,Λ,t) is an algebraic Riemann matrix over $\mathbb{Z}$;

(iii) $(V,\Lambda_{\mathbb{Q}},t)$ is an algebraic Riemann matrix over $\mathbb{Q}$, where $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

We recall that two subgroups H_1 , H_2 , of the same group are said to be *commensurable* if and only if $H_1 \cap H_2$ has finite index in both H_1 and H_2 . We have

Lemma (2.3.2): If Λ_1 , Λ_2 are two commensurable lattices in V^t , then V^t/Λ_1 is algebraic if and only if V^t/Λ_2 is algebraic.

proof: Clearly and without loss of generality, we can assume that (V, Λ_1, t) admits a Riemann form $\beta_1 : V \times V \longrightarrow \mathbb{R}$ and that $\Lambda_2 \subset \Lambda_1$. It is then obvious that β_1 is a Riemann form for (V, Λ_2, t) . Conversely, assume that (V,Λ_2,t) admits a Riemann form $\beta_2 : V \times V \longrightarrow \mathbb{R}$. Let n be the exponent of the finite abelian group Λ_1/Λ_2 . It follows that $n \cdot \Lambda_1 \subseteq \Lambda_2$. Now if we let $\beta_1 := n^2 \cdot \beta_2$, all we need to show is that $\beta_1(\Lambda_1, \Lambda_1) \subset \mathbb{Z}$; but this is obvious since $\beta_1(x, y) = n^2 \cdot \beta_2(x, y) = \beta_2(nx, ny) \in \mathbb{Z}$, where $x, y \in \Lambda_1$.

An *isogeny* f between two complex tori $T_1 = W_1/\Lambda_1$ and $T_2 = W_2/\Lambda_2$, where W_1 , W_2 finite dimensional $\mathbb{C}$-vector spaces and Λ_1 , Λ_2 lattices in W_1 , W_2 respectively, is a complex analytic homomorphism $f : T_1 \longrightarrow T_2$ which is surjective and of finite kernel. If further, n is a common multiple of the orders of all elements in Ker(f) , then there exists a homomorphism $g : T_2 \longrightarrow T_1$ such that $g \circ f = n \cdot (id)$. Thus, f admits an inverse over $\mathbb{Q}$, namely $n^{-1} \cdot g$. T_1 and T_2 , will be called isogenous if and only if there is an isogeny from either one of the two on the other. It is straightforward that this defines an equivalence relationship. From the above lemma, one immediately sees that if Λ_1 , Λ_2 are two commensurable lattices in V^t , then V^t/Λ_1 and V^t/Λ_2 are isogenous. Also notice that by extending scalars from $\mathbb{Z}$ to $\mathbb{Q}$, we get $(V,(\Lambda_1)_{\mathbb{Q}},t) \simeq (V,(\Lambda_2)_{\mathbb{Q}},t)$ in $\mathcal{RM}_{\mathbb{Q}}$. The following bijection is clear:

$$\left\{\begin{array}{c}\text{isogeny classes}\\ \text{of}\\ \text{abelian varieties}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\mathbb{Q}\text{-isomorphism classes}\\ \text{of}\\ \text{algebraic Riemann matrices over } \mathbb{Q}\end{array}\right\}.$$

The proof of the following fundamental proposition utilizes ideas from representation theory that resemble to those used in the proof of Theorem (2.3.6). For details, we refer, for example, to [J] Proposition 3.1.

Proposition (2.3.3): If V is a finitely generated $\mathbb{R}[\Phi]$-module, then the following statements are equivalent:
(i) V admits a complex structure;
(ii) each isotypic component of V admits a complex structure;
(iii) each simple summand of type $\mathbb{R}$ occurs with even multiplicity in V.

We now proceed with the following lemma:

Lemma (2.3.4): If S is a simple $\mathbb{Q}[\Phi]$-module of type I, then $S \oplus S$ admits a projective structure. If S is a simple $\mathbb{Q}[\Phi]$-module of type II, III or IV then S admits a projective structure.

proof: From the doubling construction, §2.2, we know that the Riemann matrix $D(S) := (S \oplus S,\ t \otimes 1)$, where t is the $\mathbb{Q}$-linear map $t : S \oplus S \longrightarrow S \oplus S$ defined by $t(x_1,x_2) = (-x_2,x_1)$ is algebraic and by Theorem (2.3.1) the first part of the statement follows immediately. To prove the second part of the statement, it is clear that we shall use the notion of a CM-algebra and also a way of descending from the division algebra $D = \mathrm{End}_{\mathbb{Q}[\Phi]}(S)$ to S . By Proposition (2.2.2), and since (D,τ) is a positively involuted, τ being the canonical involution on D inherited from $\mathbb{Q}[\Phi]$, finitely dimensional division algebra of type II, III or IV over $\mathbb{Q}$ then there exists a subfield $\mathcal{S} \subseteq D$, and $\alpha \in D$ such that $(D,\mathcal{S},\tau,\alpha)$ is a CM-algebra. We have shown that we can associate a canonical complex structure t to $(D,\mathcal{S},\tau,\alpha)$ such that (D,t) is an algebraic Riemann matrix. One only needs to notice now that S is naturally a vector space over D of dimension, say n. Immediately, $t_S : S \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow S \otimes_{\mathbb{Q}} \mathbb{R}$ defined by

$$t_S(s \otimes 1) = \alpha \cdot s \otimes \frac{1}{\sqrt{-\alpha^2}}$$

is a complex structure on S. Moreover, it is easy to check that t_S commutes with the Φ-action on S, because the D-action on S commutes with Φ-action on S. Therefore, we get an isomorphism of Riemann matrices $(S,t_S) \simeq (D,t)^n$ making (S,t_S) an algebraic Riemann matrix and the result follows from Theorem (2.3.1).

Next we need the following

<u>*Lemma (2.3.5):*</u> If V_i, V_j $(i < j)$ are simple left ideals of $\mathbb{Q}[\Phi]$ no $\mathbb{R}[\Phi]$-simple summand of $V_i \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to any $\mathbb{R}[\Phi]$-simple sumand of $V_j \otimes_{\mathbb{Q}} \mathbb{R}$.

<u>*proof:*</u> The remarks following the structure theorem for semisimple algebras in §2.1 tell us that V_i, V_j appear in the isotypic decomposition of

$$\mathbb{Q}[\Phi] \simeq (V_1)^{d_1} \oplus \cdots \oplus (V_i)^{d_i} \oplus \cdots \oplus (V_j)^{d_j} \oplus \cdots \oplus (V_n)^{d_n}$$

where V_k's are non-isomorphic, simple left ideals of $\mathbb{Q}[\Phi]$. As a result of Corollary (2.1.3) one gets the decomposition

$$\mathbb{Q}[\Phi] \simeq M_{d_1}(D_1) \times \cdots \times M_{d_n}(D_n)$$

where $(V_k)^{d_k} \simeq M_{d_k}(D_k)$ with $D_k = \mathrm{End}_{\mathbb{Q}[\Phi]}(V_k)$ is a module isomorphism. The identity element in $\mathbb{Q}[\Phi]$ therefore can be written as a sum of primitive central idempotents

$$1 = \epsilon_1 + \cdots + \epsilon_n .$$

It then follows that ϵ_j induces the identity on $(V_j)^{d_j}$ and thus on V_j as well, and also that it annihilates all elements in $(V_i)^{d_i}$ and consequently V_i too. We then have for the map induced by $\epsilon_j \otimes 1 \in \mathbb{R}[\Phi]$

$$(\epsilon_j \otimes 1)\,|_{V_k \otimes \mathbb{R}} = \begin{cases} 0 \quad , & \text{for } k \neq j \\ 1\,|_{V_j \otimes \mathbb{R}} \quad , & \text{for } k = j \end{cases}$$

Let the $\mathbb{R}$-extensions of V_i and V_j admit the following isotypic decompositions over $\mathbb{R}[\Phi]$

$$V_i \otimes_{\mathbb{Q}} \mathbb{R} \simeq (T_1)^{\mu_1} \oplus \cdots \oplus (T_\rho)^{\mu_\rho}$$

$$V_j \otimes_{\mathbb{Q}} \mathbb{R} \simeq (S_1)^{\nu_1} \oplus \cdots \oplus (S_\lambda)^{\nu_\lambda} .$$

If $T_n \simeq S_m$ as $\mathbb{R}[\Phi]$-modules then $(\epsilon_j \otimes 1)\, T_n = 0$ while $(\epsilon_j \otimes 1)\, S_m = S_m$, a contradiction.

We can now give the main theorem of this section

Theorem (2.3.6): If W is a finitely generated $\mathbb{Q}[\Phi]$-module, then the following statements are equivalent:

(i) W admits a projective structure :

(ii) W admits a complex structure ;

(iii) every $\mathbb{Q}[\Phi]$-simple summand of type I has even multiplicity in W .

proof: By Theorem (2.1.4) W is semisimple, therefore

$$W \simeq (S_1)^{e_1} \oplus \cdots \oplus (S_n)^{e_n} \simeq W_1 \oplus W_2 \oplus W_3 \oplus W_4$$

where $e_1, \ldots, e_n$ are non-negative integers, S_i is a simple left ideal of $M_{d_i}(D_i)$, $D_i = \mathrm{End}_{\mathbb{Q}[\Phi]}(S_i)$, and positive integers $d_1, \ldots, d_n$ as in

$$\mathbb{Q}[\Phi] \simeq M_{d_1}(D_1) \times \cdots \times M_{d_n}(D_n) .$$

This is clear from the discussion of semisimple algebras in §2.1. W_j is the direct sum of all S_i's of type j , with j = 1, 2, 3 or 4 for types I, II, III or IV respectively.

(i) ⇒ (ii) : It is straightforward from the definitions.

(ii) ⇒ (iii) : Now, suppose S_i is of type I, i.e. D_i is totally real algebraic number field with $r_i = \dim_{\mathbb{Q}}(D_i)$. If

$$W \simeq (S_1)^{e_1} \oplus \cdots \oplus (S_i)^{e_i} \oplus \cdots \oplus (S_n)^{e_n}$$

is the isotypic decomposition of W as above, we need to prove that e_i is even. If W admits a complex structure, then tautologically $W \otimes_{\mathbb{Q}} \mathbb{R}$ admits a complex structure. When we extend scalars we get

$$W \otimes_{\mathbb{Q}} \mathbb{R} \simeq \left((S_1)^{e_1} \otimes_{\mathbb{Q}} \mathbb{R}\right) \oplus \cdots \oplus \left((S_i)^{e_i} \otimes_{\mathbb{Q}} \mathbb{R}\right) \oplus \cdots$$
$$\oplus \left((S_n)^{e_n} \otimes_{\mathbb{Q}} \mathbb{R}\right)$$

where if

$$S_i \otimes_{\mathbb{Q}} \mathbb{R} \simeq T_1 \oplus \cdots \oplus Tr_i$$

breaks down to r_i non-isomorphic simple $\mathbb{R}[\Phi]$-modules since $D_i \otimes_{\mathbb{Q}} \mathbb{R} \simeq (\mathbb{R})^{r_i}$,

$$\left((S_i)^{e_i} \otimes_{\mathbb{Q}} \mathbb{R}\right) \simeq (S_i \otimes_{\mathbb{Q}} \mathbb{R})^{e_i} \simeq (T_1 \oplus \cdots \oplus Tr_i)^{e_i}$$
$$\simeq (T_1)^{e_i} \oplus \cdots \oplus (Tr_i)^{e_i}$$

The T_i's in the above decomposition are not $\mathbb{R}[\Phi]$-isomorphic by default. Also, by Lemma (2.3.5) we know that no $\mathbb{R}[\Phi]$-simple summand of $S_i \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to any $\mathbb{R}[\Phi]$-simple summand of $S_j \otimes_{\mathbb{Q}} \mathbb{R}$ for $j \neq i$, thus the multiplicity of each T_k , k = 1, ..., r_i is equal to the multiplicity of S_i , i.e. it is exactly e_i which has to be even by Proposition (2.3.3).

(iii) ⇒ (i) : It is immediate from Lemma (2.3.4) and the fact that each simple summand of W_1 has even multiplicity by hypothesis.

§ 2.4 The holonomy group of a flat projective variety.

The following theorem characterizing the class of fundamental groups of complex, projective, smooth varieties in terms of the representation space of their holonomy group. The proof makes use of Theorem (2.3.6) and the theory of covering spaces, see [J] Theorem 4.1.

Theorem (F.E.A. Johnson)(2.4.1): A compact, algebraic, flat manifold is classified by an extension

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Phi \longrightarrow 1$$

as above, in which the operator homomorphism $\rho : \Phi \longrightarrow GL_{\mathbb{Z}}(\Lambda)$, the notation is being kept as in §2.3, admits a projective structure. Conversely, given such an extension, there exists an algebraic, compact, flat manifold that corresponds to it.

Before we give the equivalent of the Auslander-Kuranishi Theorem in this context we need a lemma. We keep the notation as in §2.3

Lemma (2.4.2): $D(V)/D(\Lambda)$ is a complex algebraic manifold, where $D(\Lambda) = \Lambda \times \Lambda$, the double of Λ .

proof: In §2.2 , we have shown that the double of a real vector space admits a natural complex structure. Given this structure, we need to show that $\big(D(V),\ D(\Lambda)\big)$ is algebraic, i.e. it admits a Riemann form. Consider the following $\mathbb{Z}$-basis for $D(\Lambda)$

$$\mathbb{U} := \Big\{\, \mu_i = (\lambda_i, 0)\ ,\ \nu_i = (0, \lambda_i) \ \text{ where } \ 1 \le i \le \operatorname{rank}_{\mathbb{Z}}(\Lambda) \,\Big\}$$

where $\{\lambda_i\}_i$ is a $\mathbb{Z}$-basis for Λ . Define $\beta : D(V) \times D(V) \longrightarrow \mathbb{R}$ by

$$\beta(\mu_i, \nu_j) = \delta_{ij} = -\,\beta(\nu_j, \mu_i) \quad \text{and} \quad \beta(\mu_i, \mu_j) = 0 = \beta(\nu_i, \nu_j) \quad .$$

Then β is $\mathbb{R}$-bilinear, skew-symmetric and $\mathrm{Im}\big(\beta(D(\Lambda))\big) \subset \mathbb{Z}$. Moreover $\mathbb{U}$ is an orthonormal basis for the realization $\tilde{\beta}$ and $\tilde{\beta}$ is positive definite.

We now have the corollary

<u>*Corollary (2.4.3):*</u> Any finite group Φ can occur as the holonomy group of a smooth flat projective variety.

<u>*proof:*</u> By Theorem (1.3.10) there exists a compact flat manifold M such that Φ is the holonomy group of M and we have

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \xrightarrow{p} \Phi \longrightarrow 1$$

where $\Gamma \simeq \pi_1(M)$ and $\Lambda \simeq \mathbb{Z}^{2n}$ can be regarded as a discrete cocompact subgroup of $V \simeq \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\Delta := \{ (x,y) \in \Gamma\times\Gamma$ such that $p(x)=p(y) \}$. Then Δ occurs in an extension

$$0 \longrightarrow \Lambda\times\Lambda \longrightarrow \Delta \xrightarrow{p'} \Phi \longrightarrow 1$$

where $p'\big((x,y)\big) = p(x) = p(y)$. Since Δ is a subgroup of finite index in $\Gamma\times\Gamma$, Δ itself is the fundamental group of a compact flat complex manifold of dimension n. Let ρ and ρ' be the faithful integral representations $\rho : \Phi \longrightarrow GL_{\mathbb{Z}}(\Lambda)$, $\rho' : \Phi \longrightarrow GL_{\mathbb{Z}}(\Lambda\times\Lambda)$ associated with the first and second extension respectively. Then, for every $g \in \Phi$

$$\rho'(g) = \begin{bmatrix} \rho(g) & 0 \\ 0 & \rho(g) \end{bmatrix} .$$

Since $D(\Lambda)$ is invariant under ρ' and by Lemma (2.4.2) $D(V)/D(\Lambda)$ is algebraic, it follows that ρ' admits a projective structure and the proof is complete.

Chapter III

Projective Embeddings of Complex Tori

§ 3.1 Vector Bundles - Sheaves - Sheaf Cohomology.

A. Vector Bundles.

Let M be a complex manifold . A differentiable manifold E is called an *n-dimensional complex holomorphic vector bundle* over M and denoted by (E,M,π) if E is locally isomorphic through π to $U_i \times \mathbb{C}^n$, where U_i open neighbourhood of M . These neighbourhoods are glued together by means of *holomorphic transition functions* for (E,M,π)

$$f_{jk} : U_j \cap U_k \longrightarrow GL_n(\mathbb{C})$$

It turns out that the transition functions completely determine the vector bundle since E . For complete definitions, we refer to [Ko] pages 97-102. If n = 1 then the vector bundle is called *line bundle.* Two holomorphic vector bundles (E,M,π_E) and (G,M,π_G) over M are said to be *equivalent* if there is a biholomorphism h of E onto G such that h maps each *fiber* $\pi_E^{-1}(x)$ linearly onto $\pi_G^{-1}(x)$. One has several constructions on vector bundles, corresponding essentially to the respective constructions of complex vector spaces.

The *Picard group* over M is defined as the quotient

$$\mathrm{Pic}(M) := \{\text{holomorphic line bundles over } M\} / \approx$$

where " $\approx$ " is the equivalence between holomorphic vector bundles defined above. It is indeed a group as it can be easily verified with "multiplication" the tensor product of line bundles.

Let (E,M,π) be a holomorphic line bundle over M . A holomorphic map $\sigma : x \mapsto \sigma(x)$ of a domain $U \subset M$ into E is called a *holomorphic section of E over U* provided $\sigma(x) \in \pi^{-1}(x)$ for every x $\in$ U .

B. Sheaves.

Let M be a complex manifold. Two holomorphic functions f , g on M are said to be *equivalent at* $x \in M$ if there is a neighbourhood U of x such that $f|_U = g|_U$. The equivalence class to which f belongs is called *the germ of f at x* and is denoted f_x . Let now $\mathcal{O}_x$ be the set of all germs of holomorphic functions on a neighbourhood of $x \in M$. By defining termwise addition and multiplication $\mathcal{O}_x$ acquires the structure of a ring. $\mathcal{O} = \bigcup \mathcal{O}_x$, $x \in M$, is turned into a Hausdorff topological space by a suitable introduction of a topology on it; see further [Ko] page 109.

A topological space $\mathcal{S}$ is said to be a *sheaf* over M if the following conditions are satisfied

(i) a local homeomorphism $p : \mathcal{S} \longrightarrow M$, onto M is defined

(ii) for any $x \in M$, $p^{-1}(x)$ is a K-module, where K is a ring, usually one of $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$

(iii) for any $a_1, a_2 \in K$ and $\phi , \psi \in \mathcal{S}$ with $p(\phi) = p(\psi)$, $a_1\phi + a_2\psi \in \mathcal{S}$ and depends continuously on ϕ and ψ .

A *section* σ of $\mathcal{S}$ over $W \subset M$ is a continuous map $\sigma : x \mapsto \sigma(x)$ into $\mathcal{S}$ such that $\sigma(x) \in \mathcal{S}_x = p^{-1}(x)$ the *stalk* over x . For sections σ , τ of $\mathcal{S}$ over W , one defines their linear combination with coefficients in K by

$$c_1\sigma + c_2\tau \ : \quad x \mapsto c_1\,\sigma(x) + c_2\,\tau(x) \quad , \qquad c_1 , c_2 \in K .$$

Then the set of all sections of $\mathcal{S}$ over W naturally forms a K-module denoted by $\Gamma(W,\mathcal{S})$.

Remark (3.1.1): By a *locally constant function* we mean a local C^∞ function f with $f \equiv c \in \mathbb{C}$ on D(f). If one identifies f with $D(f) \times \{c\} \subset M \times \mathbb{C}$, then its germ f_x at $x \in D(f)$ is identified with $(x,c) \in M \times \mathbb{C}$ and $\mathcal{U}(f_x;\ f,\ U)$ with $U \times \{c\} \subset M \times \mathbb{C}$, where U open. Therefore, the sheaf of germs of locally constant functions over M is identified with $M \times \mathbb{C}$ where the topology is defined by means of the following system of open sets: $\{U \times \{c\}$; where U open $\subset M$, $c \in M\}$. For reasons that will become obvious later, see Remark (3.1.5), one denotes the sheaf of germs of locally constant $\mathbb{C}$-valued functions, simply by $\mathbf{C}$. Similarly, $\mathbf{Z}$ is the sheaf of germs of $\mathbb{Z}$-valued locally constant functions.

Remark (3.1.2): It is an easy exercise to show that $\mathcal{O}$ is indeed a sheaf and that a section σ of $\mathcal{O}$ over an open subset W of M corresponds in a one-to-one manner to a holomorphic function f defined on W by $\sigma(x) = f_x$. In this sense, we can identify a section of $\mathcal{O}$ over an open set with a holomorphic function defined there, (see further, page 113 of [Ko]).

C. Cohomology with Coefficients in a Sheaf.

The cohomology of M with coefficients in $\mathcal{S}$ is defined at two steps. One first defines a cochain complex for M that depends on an arbitrary locally finite open covering $\mathbb{U}$ of M . A 0-cochain with respect to $\mathbb{U}$ of M is a set $\{\sigma_j\}$ of sections where $\sigma_j \in \Gamma(U_j, \mathcal{S})$, a 1-cochain a set $\{\sigma_{jk}\}$ where $\sigma_{jk} \in \Gamma(U_j \cap U_k, \mathcal{S})$ for all indices j, k with $U_j \cap U_k \neq \emptyset$ and one continues similarly for higher order cochains. The boundary map δ is given, for example, for a 0-cochain $\{\sigma_j\}$ by

$$\delta(\{\sigma_j\}) = \{\tau_{jk}\} \quad , \quad \tau_{jk} = \sigma_k - \sigma_j$$

where we assume that $U_j \cap U_k \neq \emptyset$, etc. One shows that this is indead a cochain complex and its cohomology $H^p(\mathbb{U},\mathcal{S})$ is defined in the usual way. Taking the direct limit of $H^p(\mathbb{U},\mathcal{S})$ over sebsequent refinments of $\mathbb{U}$ one defines

$$H^p(M,\mathcal{S}) :\equiv \varinjlim_{\mathbb{U}} H^p(\mathbb{U},\mathcal{S}) \ .$$

For further particulars on this we refer to [Ko] chapter 3, §3.3 .

Chasing the definitions one easily verifies

Remark (3.1.3): $H^0(M,\mathcal{S}) = \Gamma(M,\mathcal{S})$.

We know that a vector bundle is completely determined once its transition functions are known. One observes that the transition functions of the tensor product of two line bundles over M is the product of their transition functions. Remark (3.1.2) and the definiton above make then clear the following identification

$$\mathrm{Pic}(M) \simeq H^1(M; \mathcal{O}^*)$$

where $\mathcal{O}^*$ is the sheaf of non-vanishing holomorphic sections of M .

One needs a theory of cohomology where most of the known algebraic tools can be applied. So one defines *subsheaves, (iso)-homomorphisms* between sheaves, *quotient sheaves. Short exact sequences* of sheaves follow. The definitions of cohomology are such that a short exact sequence of sheaves induces a long exact sequence of cohomology groups. We shall be particularly interested in the long cohomology sequence induced by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0 \ ,$$

where $[j(f)](x) = 2\pi i f(x)$ and $[e(g)](x) = \exp\{g(x)\}$, with $x \in M$ and f , g $\in \mathbb{Z}$, $\mathcal{O}$ respectively.

A sheaf over M is said to be *fine* if $H^p(M,\mathcal{S}) = 0$ for all $p \geq 1$. An exact sequence of sheaves is called a *fine resolution* if and only every sheaf involved in it is fine. One has the following important

Theorem (3.1.4): If

$$0 \longrightarrow \mathcal{S} \xrightarrow{i} \mathcal{S}_0 \xrightarrow{d} \mathcal{S}_1 \xrightarrow{d} \cdots$$

is a fine resolution of $\mathcal{S}$ over M , then

$$H^p(M,\mathcal{S}) \equiv \Gamma(M,d\mathcal{S}_{p-1}) / d\Gamma(M,\mathcal{S}_{p-1}) \ , \quad p \geq 1 \ .$$

Before we state the theorems of de Rham and Dolbeault, we make the following

Remark (3.1.5): For K a simplicial complex with underlying topological space M

$$H^*(K;\mathbb{Z}) \simeq H^*(M;\mathbf{Z}) \ .$$

The same is true if $\mathbb{Z}$ is replaced by $\mathbb{R}$ or $\mathbb{C}$ and $\mathbf{Z}$ by $\mathbf{R}$ or $\mathbf{C}$ respectively. Also this means that for any "good" topological space, one can use indiscriminately sheaf, simplicial, singular (since any differentiable manifold M can be realized as the underlying topological space of a simplicial complex K) or even group cohomology with coefficients in $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$. To see this identity, associate to every vertex v_k in K , the *star* of v_k , which is the interior of the union of all simplices in K having v_k as a vertex and consider the open covering $\mathbb{U} = \{ U_k \mid \text{where } U_k = \text{star}(v_k;K) \}$. A p-cochain σ with respect to $\mathbf{Z}$ sends $\sigma : (k_0,\ldots,k_p) \mapsto \sigma_{k_0\cdots k_p}$ an element

$$\sigma_{k_0\cdots k_p} \in \mathbb{Z}(\cap \text{star}(v_{k_i};K)) = \begin{cases} \mathbb{Z} \ , & \text{if } \{v_{k_i}\}_i \text{ span a p-simplex in K} \ , \\ 0 \ , & \text{otherwise} \ . \end{cases}$$

Define $i : C^p(U;\mathbf{Z}) \longrightarrow C^p(K;\mathbb{Z})$ by letting $i(\sigma)$ be the simplicial p-cochain defined by

$$[i(\sigma)](\Delta) = \sigma_{k_0\cdots k_p}$$

where $\Delta = <k_0,\ldots,k_p>$ a p-simplex with $\{v_{k_i}\}_i$ vertices. It is not difficult to see that i is an isomorphism. Chasing the definitions, we see further that it commutes with the respective boundary homomorphisms inducing thus an isomorphism

$$i : H^p(U;\mathbf{Z}) \xrightarrow{\ \simeq\ } H^p(K;\mathbb{Z}) \ .$$

It now suffices to consider subdivisions of K to produce refinements of $\mathbb{U}$ and the result immediately follows.

If $\mathcal{A}_r$ is the sheaf of germs of C^∞ r-forms and $\mathcal{A}_{i,j}$ the sheaf of germs of C^∞ (i.j)-forms on M then we have, essentially as corollaries of Theorem (3.1.4)

Theorem (3.1.6) (de Rham): The exact sequence

$$0 \longrightarrow \mathbb{C} \xrightarrow{i} \mathcal{A}_0 \xrightarrow{d} \mathcal{A}_1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_n \longrightarrow 0$$

gives a fine resolution of M , where n=dimM and

$$\mathrm{H}^p(\mathrm{M},\mathbb{C}) \equiv \Gamma(\mathrm{M},\mathrm{d}\mathcal{A}_{p-1}) \,/\mathrm{d}\Gamma(\mathrm{M},\mathcal{A}_{p-1}) \quad , \quad \mathrm{p} \geq 1 \ .$$

De Rham's Theorem, in combination with Remark (3.1.4), really says that the *de Rham cohomology groups*

$$\mathrm{H}^p_{dR}(\mathrm{M}) :\equiv \Gamma(\mathrm{M},\mathrm{d}\mathcal{A}_{p-1}) \,/\mathrm{d}\Gamma(\mathrm{M},\mathcal{A}_{p-1}) \quad , \quad \mathrm{p} \geq 1$$

which are defined with respect to the differential structure of M , actually depend only on the topology of M , since

$$\mathrm{H}^p_{dR}(\mathrm{M}) = \mathrm{H}^p(\mathrm{M},\mathbb{C}) \quad , \quad \mathrm{p} \geq 1 \ .$$

Theorem (3.1.7) (Dolbeault): The exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{i} \mathcal{A}_{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_{0,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}_{0,n} \longrightarrow 0$$

gives a fine resolution of M , where n=dimM and

$$\mathrm{H}^p(\mathrm{M},\mathcal{O}) \equiv \Gamma(\mathrm{M},\mathrm{d}\mathcal{A}_{0,p-1}) \,/\mathrm{d}\Gamma(\mathrm{M},\mathcal{A}_{0,p-1}) \quad , \quad \mathrm{p} \geq 1 \ .$$

§ 3.2 The Long Cohomology Sequence for a Complex Torus.

Let $M = V/\Lambda$ be a complex torus, V a m-dimensional complex vector space and Λ a free abelian subgroup of V of maximal dimension. We fix the following notation

$\mathbb{Z}$: the sheaf of germs of locally constant $\mathbb{Z}$-valued functions on M ;
$\mathcal{O}$: the sheaf of germs of holomorphic functions on M ;
$\mathcal{O}^*$: the sheaf of germs of non-vanishing holomorphic functions on M ;

The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0 ,$$

where $[j(f)](x) = 2\pi i f(x)$ and $[e(g)](x) = \exp\{g(x)\}$, with $x \in M$ and f , g $\in \mathbb{Z}$, $\mathcal{O}$ respectively, induces the long cohomology sequence

$$0 \longrightarrow H^0(M;\mathbb{Z}) \longrightarrow H^0(M;\mathcal{O}) \longrightarrow H^0(M;\mathcal{O}^*) \longrightarrow$$

$$H^1(M;\mathbb{Z}) \longrightarrow H^1(M;\mathcal{O}) \longrightarrow H^1(M;\mathcal{O}^*) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow \cdots .$$

Remark (3.2.1): Let $M = X/G$ where G is a discrete group acting freely and discontinuously on a "good" topological space X . Let $\pi : X \longrightarrow M$ be the projection, then for any sheaf $\mathcal{S}$ on M , there is a natural map

$$i : H^p(G;\Gamma(X,\pi^*\mathcal{S})) \longrightarrow H^p(M;\mathcal{S})$$

with the following properties:

(i) For every two sheaves $\mathcal{S}_1$, $\mathcal{S}_2$ on M that fit in a short exact sequence

$$0 \longrightarrow \mathcal{S}_1 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_2 \longrightarrow 0$$

such that

$$0 \longrightarrow \Gamma(X,\pi^*\mathcal{S}_1) \longrightarrow \Gamma(X,\pi^*\mathcal{S}) \longrightarrow \Gamma(X,\pi^*\mathcal{S}_2) \longrightarrow 0$$

is exact, then i is a homomorphism;

(ii) i is compatible with the cup product, and

(iii) if

$$H^p(X,\pi^*\mathcal{S}) = 0 \ , \quad \text{for all } p \geq 1$$

then i is an isomorphism.

For more details, we refer to [Gr] chapter V, page 195.

Let $\mathcal{O}$ be the $\mathbb{C}$-algebra

$$\mathcal{O} := \{ f \mid f: V \longrightarrow \mathbb{C} \ , \ f \text{ holomorphic} \} \qquad \text{and}$$

$$\mathcal{O}^* := \{ f \mid f: V \longrightarrow \mathbb{C}^* \ , \ f \text{ holomorphic} \}$$

its group of units. Notice that in the case where $M = V/\Lambda$ is a complex torus, $\pi^*\mathcal{O} = \mathcal{O}$ [resp. $\pi^*\mathcal{O}^* = \mathcal{O}^*$] , thus $\mathcal{O} = \Gamma(V,\mathbb{C})$ [resp. $\mathcal{O}^* = \Gamma(V,\mathbb{C}^*)$] . Therefore, according to this discussion and because $\mathcal{O}$ [resp. $\mathcal{O}^*$] is a "good" sheaf, in as much as the cohomology is concerned we can indiscriminately use, according to our convenience, either sheaf cohomology $H^*(M,\mathcal{O})$ [resp. $H^*(M,\mathcal{O}^*)$] or group cohomology $H^*(\pi_1(M),\mathcal{O})$ [resp. $H^*(\pi_1(M),\mathcal{O}^*)$] or even singular cohomology $H^*(M,\mathcal{O})$ [resp. $H^*(M,\mathcal{O}^*)$] . Moreover, since $\mathcal{O}$ is a complex vector space, so are $H^n(M;\mathcal{O})$.

We have the following

Proposition (3.2.2): $H^0(M;\mathbb{Z}) = \mathbb{Z}$, $H^0(M;\mathcal{O}) = \mathbb{C}$, $H^0(M;\mathcal{O}^*) = \mathbb{C}^*$ and $\delta : H^0(M;\mathcal{O}^*) \longrightarrow H^1(M;\mathbb{Z})$ is the zero map.

proof: Immediate from the fact that global holomorphic functions on M are constant and from Remarks (3.1.3) and (3.1.2) ; $\delta(\text{constant}) = 0$.

Therefore the long cohomology sequence above splits into two parts

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 1$$

(3.2.3)

$$0 \longrightarrow H^1(M;\mathcal{O})/H^1(M;\mathbb{Z}) \longrightarrow H^1(M;\mathcal{O}^*) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow \cdots .$$

By use of the known identification $\mathrm{Pic}(M) = H^1(M;\mathcal{O}^*)$ and by letting

$$\mathrm{Pic}^0(M) := H^1(M;\mathcal{O})/H^1(M;\mathbb{Z})$$

(3.2.3) becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 1$$

(3.2.4)

$$0 \longrightarrow \mathrm{Pic}^0(M) \longrightarrow \mathrm{Pic}(M) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow \cdots .$$

We now want to take a look at $H^n(M;\mathbb{Z})$. As topological groups, they are all discrete because their topology comes from $\mathbb{Z}$. It turns out that they all can be computed out of $H^1(M;\mathbb{Z})$. First

$$\pi_1(M) = \Lambda = \pi_1^{ab}(M) = H_1(M;\mathbb{Z})$$

so by the universal coefficient theorem there is a natural isomorphism s.t.

$$H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z}) = \mathrm{Hom}(\Lambda,\mathbb{Z})$$

Lemma (3.2.5): If Λ is a free abelian group finitely generated, and A any Λ-trivial module, then one has

$$H^i(\Lambda; A) \simeq \wedge^i H^1(\Lambda; A).$$

proof: Use induction on the number of generators, say n, Λ has. To verify the truth of the statement for the case $n = 1$, observe that

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z}[\mathbb{Z}] \xrightarrow{d} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where i is the inclusion map, d is defined by $d(x) = x - (\sum_{n\in\mathbb{Z}} x_n)1$

for $x = \sum_{n\in\mathbb{Z}} x_n n \in \mathbb{Z}[\mathbb{Z}]$ and ϵ is the augmantation map $\epsilon : x \mapsto \sum_{n\in\mathbb{Z}} x_n$

is a free resolution for $\mathbb{Z}$. One follows the definitions to see that

$$H^i(\mathbb{Z};\,A) = \begin{cases} \mathbb{Z}\,, & \text{if } i = 0,1 \\ 0\,, & \text{otherwise} \end{cases}$$

which verifies the statement in this case. To complete the proof assume it is true for n , and use the Künneth formula to show it is true for n+1 ,

$$\begin{aligned} H^i = (\Lambda\times\mathbb{Z};\,A) &= \bigoplus_{p+q=i}\left(H^p(\mathbb{Z};\,A)\otimes H^q(\mathbb{Z};\,A)\right) \\ &= \bigoplus_{p+q=i}\left(\overset{p}{\wedge} H^1(\mathbb{Z};\,A)\,\otimes\,\overset{q}{\wedge} H^1(\mathbb{Z};\,A)\right) \\ &= \overset{i}{\wedge}\ H^1(\mathbb{Z}\times\mathbb{Z};\,A) \quad . \end{aligned}$$

Remark (3.2.6): The isomorphism of Lemma (3.2.5) is an algebra isomorphism sending the cup product to the exterior product.

$H^1(\Lambda;\,\mathbb{Z}) = \mathrm{Hom}(\Lambda;\,\mathbb{Z})$ and denoting by $\mathrm{Alt}^n(\Lambda;\,\mathbb{Z})$ the group of $\mathbb{Z}$-valued alternating n-forms on Λ we get

Corollary (3.2.7): There is a canonical isomorphism

$$H^n(M;\mathbb{Z}) \simeq \mathrm{Alt}^n(\Lambda;\mathbb{Z}) \text{ for every } n \geq 1.$$

$$H^n(M;\mathbb{Z}) \overset{(3.1.5)}{\simeq} H^n(\Lambda;\mathbb{Z}) \simeq \overset{n}{\wedge} H^1(\Lambda;\mathbb{Z}) = \overset{n}{\wedge} \mathrm{Hom}(\Lambda,\mathbb{Z}) \simeq \overset{n}{\wedge} H^1(M;\mathbb{Z}) \,.$$

For an explicit such isomorphism, at level n=2 , see lemma (3.3.7). Also

Corollary (3.2.8): $H_n(M;\,\mathbb{Z})$ and $H^n(M;\,\mathbb{Z})$ are free $\mathbb{Z}$-modules of rank $\binom{2m}{n}$ for all $n \geq 1$.

Applying the universal coefficient theorem we get

$$H^*(M; \mathbb{Z}) \otimes \mathbb{C} = H^*(M; \mathbb{C}) \ .$$

Thus,

$$H^n(M; \mathbb{C}) = H^n(M; \mathbb{Z}) \otimes \mathbb{C} = (\overset{n}{\wedge} H^1(M; \mathbb{Z})) \otimes \mathbb{C}$$

$$= \overset{n}{\wedge} (H^1(M; \mathbb{Z}) \otimes \mathbb{C}) = \overset{n}{\wedge} H^1(M; \mathbb{C}) \ .$$

Furthermore if $\mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C})$ denotes the group of $\mathbb{R}$-bilinear alternating n-forms on $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with values on $\mathbb{C}$, then since $\mathrm{Alt}^n(\Lambda, \mathbb{Z}) \otimes \mathbb{C} = \mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C})$ we get

Corollary (3.2.9): For every $n \geq 1$ there are canonical isomorphisms

$$H^n(M; \mathbb{C}) \simeq \mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C}) \simeq \overset{n}{\wedge} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \overset{n}{\wedge} H^1(M; \mathbb{C}).$$

A. The Hodge Decomposition

Let, Ω^p be the sheaf of germs of holomorphic p-forms on M. The cohomology groups $H^*(M; \Omega^*)$ are one of the most important invariants of any compact complex manifolds. The following theorem is of great importance and is based amongst other things on the well known Dolbeault Theorem (3.1.7). The interested reader can find a proof in [G-H] page 116.

Theorem (3.2.10): If $V^* = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and $\overline{V^*} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C})$ then

$$H^q(M; \mathcal{O}) \simeq \wedge^q(\overline{V^*}) \quad \text{and} \quad H^q(M; \Omega^p) \simeq \wedge^p(V^*) \otimes \wedge^q(\overline{V^*}) \ .$$

Remark (3.2.11): By use of arguments of general sheaf theory [Go] it follows that in the above isomorphism $H^q(M; \mathcal{O}) \xrightarrow{\sim} \wedge^q(\overline{V^*})$ the cup product in $H^q(M; \mathcal{O})$ corresponds to the exterior product in $\wedge^q(\overline{V^*})$.

Corollary (3.2.12): The natural map $\wedge^q H^1(M; \mathcal{O}) \longrightarrow H^q(M; \mathcal{O})$ induced by the cup product is an isomorphism.

Remark (3.2.13): Notice that $V^{\bullet} \equiv \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \simeq V^* \oplus \overline{V^*}$

This along with corollary (3.2.8) yield

$$H^n(M; \mathbb{C}) \simeq \overset{n}{\wedge}(V^* \oplus \overline{V^*}) \simeq \bigoplus_{p+q=n} \left(\wedge^p(V^*) \otimes \wedge^q(\overline{V^*})\right)$$

$$\simeq \bigoplus_{p+q=n} H^q(M; \Omega^p) \ .$$

This is the famous Hodge decomposition. We have the following

Lemma (3.2.14): $\dim_{\mathbb{C}} H^1(M; \mathcal{O}) = \frac{1}{2} \operatorname{rank}_{\mathbb{Z}} H^1(M; \mathbb{Z})$ and

$$\mathrm{Pic}^0(M) = H^1(M; \mathcal{O}) \,/\, H^1(M; \mathbb{Z})$$

is a complex torus.

proof: By the Hodge Decomposition

$$H^1(M; \mathbb{C}) \simeq V^* \oplus \overline{V^*} \simeq H^1(M; \mathcal{O}) \oplus H^0(M; \Omega^1)$$

because $$\dim_{\mathbb{C}} V^* = \dim_{\mathbb{C}} \overline{V^*}$$

$$\dim_{\mathbb{C}} H^1(M; \mathbb{C}) = 2\dim_{\mathbb{C}} H^1(M; \mathcal{O}) \qquad (*)$$

By the universal coefficient theorem

$$H^1(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^1(M; \mathbb{C})$$

$$\Rightarrow \qquad \operatorname{rank}_{\mathbb{Z}} H^1(M; \mathbb{Z}) \simeq \dim_{\mathbb{C}} H^1(M; \mathbb{C}) \qquad (**)$$

By $(*)$ and $(**)$ now

$$\dim_{\mathbb{C}} H^1(M; \mathcal{O}) = \frac{1}{2} \operatorname{rank}_{\mathbb{Z}} H^1(M;\mathbb{Z})$$

Because $H^1(M:\mathbb{Z})$ inherits the topology of $\mathbb{Z}$, it is discrete and the statement of the Lemma follows.

§ 3.3 Line Bundles over Complex Tori.

In this section we show that the Chern class $\mathcal{F} \equiv \mathcal{F}(L)$ of a line bundle L over $M = V/\Lambda$ is an integral skew-symmetric form on Λ whose realization $F : V \times V \longrightarrow \mathbb{R}$ is t-invariant, where t is the complex structure on V . Conversely, for any such form $\mathcal{F}$, one can construct a line bundle $L_{\mathcal{F}}$ whose Chern class is presisely $\mathcal{F}$.

We recall the following well-known fact whose proof can be found in [G-R] for example

Proposition (3.3.1): For all integers $n > 0$, $H^n(\mathbb{C}^m; \mathcal{O})$ is trivial.

By use of the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 1$$

and the fact that $H^n(\mathbb{C}^m; \mathbb{Z})$ is trivial for positive n , because $\mathbb{C}^n$ is connected, we have the

Corollary (3.3.2): $H^n(\mathbb{C}^m; \mathcal{O}^*)$ is trivial for all positive n.

If now L is any line bundle over M, then because of the lemma the pull-back bundle $\pi^*(L)$ is trivial and we can choose a global trivialization

$$t : \pi^*(L) \xrightarrow{\simeq} V \times \mathbb{C}.$$

For any $z \in V$ and any $u \in \Lambda$, the fibers of $\pi^*(L)$ over z and z+u are identical with the fiber of L over $\pi(z)$. Then, the choice of trivialization t gives us a linear automorphism of $\mathbb{C}$ for the pair (u,z)

$$\mathbb{C} \xleftarrow{t_z} [\pi^*(L)]_z = L_{\pi(z)} = [\pi^*(L)]_{z+u} \xrightarrow{t_{z+u}} \mathbb{C}.$$

Such an automorphism amounts essentially to multiplication by a non-zero complex number which we denote $e_u(z)$. We thus get a function

$$e : \Lambda \longrightarrow \mathcal{O}^*,$$

where $\mathcal{O}^* \equiv \{ f\colon V \longrightarrow \mathbb{C}^* = \mathbb{C}-\{0\},\ f \text{ holomorphic} \}$, which is easily seen to satisfy the cocycle condition

$$e_u(z+v)\, e_v(z) = e_v(z+u)\, e_u(z) = e_{u+v}(z),$$

for all $u, v \in \Lambda$, and $z \in V$. So, clearly $e \in Z^1(\Lambda; \mathcal{O}^*)$.

Furthermore, if the trivialization is being altered by multiplication by a nowhere vanishing holomorphic function f on V, e is being replaced by the cohomologous cocycle

$$e_u'(z) = e_u(z)\, f(z+u)\, f(z)^{-1}.$$

Therefore we have defined a map from $H^1(M; \mathcal{O}^*)$ to $H^1(\Lambda; \mathcal{O}^*)$, where the different trivializations essentially correspond to 1-coboundaries.

We can also go the converse way too. Given a 1-cocycle e, we can define the line bundle L_e on M as the orbit space of $V \times \mathbb{C}$ by the action of Λ given by

$$(z, \xi) \mapsto (z+u,\, e_u(z) \cdot \xi).$$

We have thus established an isomorphism

(3.3.3) $$i : H^1(M; \mathcal{O}^*) \xrightarrow{\ \simeq\ } H^1(\Lambda; \mathcal{O}^*).$$

The short exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

induce respectively the following long cohomology sequences

(3.3.4)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^1(\Lambda;\, \mathcal{O}^*) & \longrightarrow & H^2(\Lambda;\, \mathbb{Z}) & \longrightarrow & \cdots \\ & & \simeq \big\downarrow & & \simeq \big\downarrow & & \\ \cdots & \longrightarrow & H^1(M;\, \mathcal{O}^*) & \longrightarrow & H^2(M;\, \mathbb{Z}) & \longrightarrow & \cdots \end{array}$$

The vertical isomorphisms are immediate from (3.3.3) and Remark (3.1.5). The horizontal homomorphisms are the connecting homomorphisms in group cohomology and sheaf cohomology repectively. By chasing the definitions in the more general setting of Remark (3.2.1), one can further show that the square commutes (see also the appendix on page 22 of [M3]. Therefore, using the definition of the connecting homomorphisms in group cohomology one immediately gets :

Proposition (3.3.5): The Chern class of a line bundle L over M with *multipliers* $\{e_u\}_{u \in \Lambda}$ is given by the 2-cocycle in Λ with coefficients in $\mathbb{Z}$

(3.3.6) $$[\delta([L])](u,v) = f_v(z+u) - f_{u+v}(z) - f_u(z) \in \mathbb{Z}$$

where $f_u(z)$ such that $e_u(z) = \exp\{2\pi i f_u(z)\}$.

The following lemma will be useful to us

Lemma (3.3.7): There is an isomorphism

$$A : H^2(\Lambda;\, \mathbb{Z}) \xrightarrow{\simeq} \mathrm{Hom}(\overset{2}{\wedge} \Lambda;\, \mathbb{Z}) \simeq \overset{2}{\wedge} \mathrm{Hom}(\Lambda, \mathbb{Z})$$

proof: Let $C \in Z^2(\Lambda;\, \mathbb{Z})$ and define A by setting $[A([C])](u,v) := C(u,v) - C(v,u)$.
To show first that the above defined map is well defined let $C = \delta(B)$ for

a 1-cochain B. Then

$$[A([C])](u,v) = [\delta(B)](u,v) - [\delta(B)](v,u)$$

$$= [B(v) - B(u+v) + B(u)] - [B(u) - B(u+v) + B(v)] = 0.$$

Now by the definition of C for $u,v,w \in \Lambda$

$$C(v,w) - C(u+v,w) + C(u,v+w) - C(u,v) = 0$$

$$C(u,v) - C(u+w,v) + C(w,v+u) - C(w,u) = 0$$

$$C(w,v) - C(u+w,v) + C(u,w+v) - C(u,w) = 0$$

By adding the first two and subtracting the third equation we get

$$[A([C])](w,u+v) = [A([C])](w,u) + [A([C])](w,v)$$

and since $[A([C])](u,u) = 0$ and $[A([C])](u,v) = -[A([C])](v,u)$, it follows that $A([C])$ is alternating, bilinear.

So far we have a homomorphism

$$A : H^2(\Lambda;\, \mathbb{Z}) \xrightarrow{\ \sim\ } \mathrm{Hom}(\overset{2}{\wedge}\Lambda;\, \mathbb{Z}) \simeq \overset{2}{\wedge}\mathrm{Hom}(\Lambda;\, \mathbb{Z})$$

To prove that A is an isomorphism, one can show something stronger. By Lemma (3.2.5) $H^*(\Lambda;\mathbb{Z})$ is the exterior algebra of $H^1(\Lambda;\mathbb{Z}) = \mathrm{Hom}(\Lambda;\mathbb{Z})$. Let $f, g \in \mathrm{Hom}(\Lambda,\mathbb{Z}) = H^1(\Lambda;\mathbb{Z})$. Their cup product $f \cup g$ is given by the 2-cocycle $(f\cup g)(\lambda_1,\lambda_2) = f(\lambda_1) \cdot g(\lambda_2)$ $\lambda_1, \lambda_2 \in \Lambda$. So

$$[\, A\, (f\cup g)\,]\, (\lambda_1,\lambda_2) = f(\lambda_1)g(\lambda_2) - f(\lambda_2)g(\lambda_1) = (f\wedge g)(\lambda_1,\lambda_2)$$

which proves the stronger result, $A(f\cup g) = f\wedge g$, that A maps the cup product into the exterior product. This is just a special case of Remark (3.2.6).

In view of this lemma and by abuse of terminology, we are now at a position to formulate

Proposition (3.3.8): The Chern class of a line bundle L over M corresponding to the multipliers $\{e_u\}_{u \in \Lambda} \in Z^1(\Lambda; \mathcal{O}^*)$ is the integral skew-symmetric form $\mathcal{F}$ on Λ given by

$$(3.3.9) \qquad \mathcal{F}(u,v) = f_v(z+u) + f_u(z) - f_u(z+v) - f_v(z) \ ,$$

where $z \in V$, $u,v \in \Lambda$, and $f_u(z)$ such that $e_u(z) = \exp\{2\pi f_u(z)\}$.

Furthermore if F is the realization of $\mathcal{F}$ from $V \times V \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}$, F is invariant under t , the complex structure in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ i.e. $F(tx,ty) = F(x,y)$ for all $x,y \in V$.

proof: By (3.3.6) and Lemma (3.3.7) above, we have for all $z \in V$ $u,v \in \Lambda$

$$\begin{aligned}\mathcal{F}(u,v) &= [\delta([L])](u,v) - [\delta([L])](v,u) \\ &= f_v(z+u) - f_{u+v}(z) + f_u(z) - f_u(z+v) + f_{v+u}(z) - f_v(z) \\ &= f_v(z+u) + f_u(z) - f_u(z+v) - f_v(z) \ .\end{aligned}$$

For the second part of the statement, notice that F represents an element in $H^2(M; \mathbb{Z})$. The map

$$j^* : H^2(M; \mathbb{Z}) \longrightarrow H^2(M; \mathcal{O})$$

factors through $H^2(M; \mathbb{C})$ as follows

$$H^2(M; \mathbb{Z}) \xrightarrow{i^*} H^2(M; \mathbb{C}) \xrightarrow{p^*} H^2(M; \mathcal{O}) \ .$$

By the Hodge Decomposition

$$H^2(M; \mathbb{C}) \simeq (\wedge^2(V^*)) \oplus (V^* \otimes \overline{V^*}) \oplus (\wedge^2(\overline{V^*}))$$

By theorem (3.2.10) we have the isomorphism

$$H^2(M; \mathcal{O}) \simeq \wedge^2 (\overline{V^*})$$

where p^* is the projection

$$p^* : (\wedge^2(V^*)) \oplus (V^* \otimes \overline{V^*}) \oplus (\wedge^2(\overline{V^*})) \longrightarrow \wedge^2(\overline{V^*})$$

Write $$F = F_{2,0} + F_{1,1} + F_{0,2}$$

with $F_{2,0} \in \wedge^2(V^*)$, $F_{1,1} \in V^* \otimes \overline{V^*}$, $F_{0,2} \in \wedge^2(\overline{V^*})$.

Since $F = i(\mathcal{F})$ is real $F_{2,0} = \overline{F}_{0,2} = 0$ because $F_{0,2} = p(F) = (p \circ i)(\mathcal{F}) = j(\mathcal{F}) = 0$. Thus

$$F = F_{1,1} \in V^* \otimes \overline{V^*}$$

and $$F(tx,ty) = F_{1,1}(tx,ty) = i(-i)\, F_{1,1}(x,y)$$

$$= F_{1,1}(x,y) = F(x,y) \quad \text{for all } x,y \in V .$$

The converse is proven constructively in §3.4 and we can summarize the results in this and the previous sections in the following concise way.

Proposition (3.3.10): If $M = V/\Lambda$ a complex torus with t the complex structure on V , then there is a short exact sequence as follows

$$(3.3.11) \quad 0 \longrightarrow \mathrm{Pic}^0(M) \longrightarrow \mathrm{Pic}(M) \longrightarrow \left\{ \begin{array}{c} \text{t-invariant skew-symmetric} \\ \text{forms } F : V \times V \longrightarrow \mathbb{R} \\ \text{such that } F(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right\} \longrightarrow 0 .$$

§ 3.4 The Appell-Humbert Theorem

The Appell-Humbert Theorem essentially asserts that the short exact sequence (3.3.11) splits up to semicharacters. The notation is being kept the same as above. So let L be a line bundle over a complex torus $M = V/\Lambda$ represented by multipliers $\{e_u\}_{u \in \Lambda}$. To L one associates the hermitian form H defined by

$$H(x,y) = F(tx,y) + iF(x,y) .$$

It is elementary to check that there is a 1-1 correspondence between non-degenerate hermitian forms H on V and non-degenerate, real skew-symmetric forms F . See for example Lemma (3.5.7).

We start with the following lemma:

Lemma (3.4.1): Let H be a hermitian form on V such that $F = \mathrm{Im}(H)$ is a real, skew-symmetric, bilinear form, invariant under the complex structure t on V and integral on $\Lambda \times \Lambda$. Let

$$\psi : \Lambda \longrightarrow \mathbb{C}_1^* = \{ z \in \mathbb{C}^* \mid \|z\| = 1 \}$$

be a map with

$$\psi(u+v) = \exp\{i\pi F(u,v)\} \, \psi(u) \, \psi(v) \; , \quad u,v \in \Lambda$$

Such maps exist for any given H and will be called *semicharacters of* Λ *, associated with H or F .* Now let

$$(3.4.2) \qquad e_u(z) := \psi(u) \exp\{\pi H(z,u) + \pi/2 \, H(u,u)\}$$

Then $e \in H^1(\Lambda; \mathcal{O}^*)$ and the Chern class of the associated line bundle is precisely $F = \mathrm{Im}(H)$.

proof: For the first part, given H , we need to show the existence of a semicharacter of Λ associated with H . One can check that the functions

$$f_u(z) = \frac{1}{2i} H(z,u) + l_u$$

satisfy equation (3.3.9) for any contants l_u . Furthermore, by substitution in (3.3.6) , we get

$$il_u + il_v - il_{u+v} + \frac{1}{2} H(u,v) \in i\mathbb{Z}$$

for all $u,v \in \Lambda$. Choosing

$$l_u = r_u - i \frac{1}{4} F(iu,u) \qquad r_u \in \mathbb{R}$$

and writing $il_u = k_u + 1/4 \, H(u,u)$, this reduces to

$$k_u + k_v - k_{u+v} + \frac{1}{2} iF(u,v) \in i\mathbb{Z}$$

with k pure imaginary.

Let $\psi(u) = e^{2\pi k_u}$. It is immediate to verify that $\| \psi(u) \| = 1$ and

$$\frac{\psi(u+v)}{\psi(u)\,\psi(v)} = e^{i\pi F(u,v)}$$

making, thus ψ a semicharacter of Λ associated with H . The rest is a straightforward verification and the proof of the lemma is complete.

The line bundle L over M is said to be *of type (H,ψ)* , L $\equiv L(H,\psi)$, where H a hermitian form on V and ψ a semicharacter of Λ associated with it, if it is given by the orbit space of $\mathbb{C} \times V$ for the action

$$\phi : \Lambda * (\mathbb{C} \times V) \longrightarrow \mathbb{C} \times V$$

defined by

$$\phi_u(c,z) = (c \cdot e_u\ ,\ z+u) = (c \cdot \psi(u) \exp\{\pi H(z,u)+\pi/2H(u,u)\}\ ,\ z+u)$$

Notice that for the map $(H,\psi) \longrightarrow \{e_u\}$ defined in (3.4.2) we have

Remark (3.4.3): Let H_1 , H_2 be two hermitian forms and ψ_1 , ψ_2 two semicharacters of Λ associated with H_1 and H_2 respectively. Then by means of (3.4.2)

$$(H_1,\psi_1) \longrightarrow \{e_{1u}\} \text{ and } (H_2,\psi_2) \longrightarrow \{e_{2u}\} \quad .$$

One can check that

$$(H_1+H_2\ ,\ \psi_1\psi_2) \longrightarrow \{e_{1u} \cdot e_{2u}\}$$

and therefore we have an isomorphism of line bundles

$$L(H_1,\psi_1) \otimes L(H_2,\psi_2) \ \simeq\ L(H_1+H_2\ ,\ \psi_1\psi_2)$$

So the map $(H,\psi) \longrightarrow \{e_u\}$ constitutes a "splitting" up to semicharacter for the short exact sequence (3.3.11).

Next we have

Lemma (3.4.4): There is an isomorphism

$$\mathrm{Hom}(\Lambda, \mathbb{C}_1^*) \xrightarrow{\ \sim\ } \mathrm{Pic}^0(M)$$

proof: By the Hodge Decomposition

$$H^1(M; \mathbb{C}) \ \simeq\ V^* \oplus \overline{V^*} \ \simeq\ H^1(M; \mathcal{O}) \oplus H^0(M; \Omega^1)$$

and the $H^1(M; \mathbb{C}) \longrightarrow H^1(M; \mathcal{O})$ in the long cohomology sequence is the natural projection

$$p:\ H^1(M; \mathcal{O}) \oplus H^0(M; \Omega^1) \longrightarrow H^1(M; \mathcal{O})\ ,$$

[M3] page 13, in particular, p is surjective. Now if $L \in \mathrm{Pic}^0(M)$ then its Chern class $\delta([L]) = 0$ and L is presentable as the orbit space of the action

$$\phi : \Lambda * (\mathbb{C} \times V) \longrightarrow \mathbb{C} \times V$$

defined by

$$\phi_u(c,z) = (c \cdot e_u(z) \ , \ z+u)$$

where $e_u(z)$ is a coboundary in $H^1(\Lambda; \mathcal{O}^*)$ thus $e_u(z) = \psi'(u) : \Lambda \longrightarrow \mathbb{C}^*$ a homomorphism. But then we can normalize to get $\psi(u) = \psi'(u)/|\psi(u)| : \Lambda \longrightarrow \mathbb{C}_1^*$ and $\mathrm{Hom}(\Lambda,\mathbb{C}_1^*) \longrightarrow \mathrm{Pic}^0(M)$ is surjective.

To prove injectivity suppose that there is a line bundle $L \equiv L(0,\psi_L) \in \mathrm{Pic}^0(M)$ such that there is another homomorphism $\psi \in \mathrm{Hom}(\Lambda,\mathbb{C}_1^*)$ with $L(0,\psi) \simeq L(0,\psi_L)$. Let $\{e_u(z) = \psi_L(u)\}_{u \in \Lambda}$ and $\{e'_u(z) = \psi(u)\}_{u \in \Lambda}$ be the representing multipliers. It follows that ψ_1 , ψ_2 are equivalent cocycles in $Z^1(\Lambda; \mathcal{O}^*)$ and $\psi_L(u) = \psi(u)\, f(z+u)/f(z)$ for all $z \in V$ and $u \in \Lambda$, where f is a nowhere vanishing holomorphic function on V . We have

$$\|\psi_L\| = \|\psi\| = 1 \quad \Rightarrow \quad \|f(z+u)\| = \|f(z)\|$$

for all $z \in V$, $u \in \Lambda$ and by Liouville's theorem f is a constant and $\psi = \psi_L$. In conjunction with the Remark (3.4.3) above the proof of the lemma is now complete.

Putting it all together we get the Appell-Humbert Theorem

Theorem (3.4.5): If $M = V/\Lambda$ is a complex torus then the following diagram commutes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\Lambda,\mathbb{C}_1^*) & \longrightarrow & P & \longrightarrow & \left[\begin{array}{c}\text{Group of hermitian} \\ H : V\times V \longrightarrow \mathbb{C} \ \text{s.t.} \\ (\mathrm{Im}H)(\Lambda\times\Lambda)\subset\mathbb{Z}\end{array}\right] & \longrightarrow & 0 \\
 & & \simeq\Big\downarrow & & \simeq\Big\downarrow s & & \simeq\Big\downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}^0(M) & \longrightarrow & \mathrm{Pic}(M) & \longrightarrow & \left[\begin{array}{c}\text{t-invariant skew-symmetric} \\ \text{forms } F : V\times V \longrightarrow \mathbb{R} \\ \text{such that } F(\Lambda\times\Lambda)\subset\mathbb{Z}\end{array}\right] & \longrightarrow & 0
\end{array}
$$

where the dotted map $s : \{ (H,\psi)\ldots\} \longrightarrow \mathrm{Pic}(M)$ is the isomorphism $s(H,\psi) = L(H,\psi)$ and P is the group of pairs (H,ψ) with $(H_1,\psi_1)*(H_2,\psi_2)=(H_1+H_2,\psi_1\psi_2)$, H hermitian s.t. $(\mathrm{Im}H)(\Lambda\times\Lambda) \subset \mathbb{Z}$ and ψ a semicharacter for H .

§ 3.5 Sections of Line Bundles over Complex Tori - - Connection to the classical theory of Theta Functions

According to the Appell-Humbert Theorem, see Theorem (3.4.5) every line bundle L over a complex torus $M = V/\Lambda$ is of the form $L(H,\psi)$ where H a hermitian form on V such that $(\mathrm{Im}\ H)(\Lambda \times \Lambda) \subset \mathbb{Z}$ and ψ a semicharacter for H .

This means that $L = L(H,\psi)$ is taken from $\mathbb{C} \times V$ as the orbit space of the action

$$\phi : \Lambda * (\mathbb{C} \times V) \longrightarrow (\mathbb{C} \times V)$$

$$\phi_u(c,z) = (c \cdot \psi(u) \cdot \exp\{\pi H(z,u) + \tfrac{\pi}{2} H(u,u)\} \ , z+u)$$

In other words, $L = (E, M, p_L)$, with $E \simeq \mathbb{C} \times V/\Lambda$ and p_L : $E \longrightarrow M$ the projection map given by $p_L([c,z]) = [z]$.

Consider now the trivial line bundle over V , $\widetilde{L} = (\mathbb{C} \times V, V, p_{\widetilde{L}})$ with $p_{\widetilde{L}} : \mathbb{C} \times V \longrightarrow V$ defined by $p_{\widetilde{L}}(c,z) = z$. If $\pi : V \longrightarrow V/\Lambda$ the quotient map then one easily verifies that the following diagramm

$$\begin{array}{ccc} \mathbb{C} \times V & \xrightarrow{\ \pi\ } & \mathbb{C} \times V/\Lambda \\ \downarrow{\scriptstyle p_{\widetilde{L}}} & & \downarrow{\scriptstyle p_L} \\ V & \xrightarrow{\ \pi\ } & V/\Lambda \end{array}$$

commutes. (Λ acts on $\mathbb{C} \times V$ as above and on V by translation, i.e. $(u,z) \mapsto u+z$ for $u \in \Lambda$, $z \in V$), ($\widetilde{L}$ is really the universal covering of L , which justifies the notation also.)

We are interested in the holomorphic sections of $L(H,\psi)$, denoted $\Gamma(L)$. As already mentioned, $\Gamma(L)$ is naturally a complex vector space.

We have the following

Proposition (3.5.1): There is a natural imbedding

$$\pi^* : \Gamma(L) \longrightarrow \Gamma(\widetilde{L})$$

which induces an isomorphism (we call this π^* again)

$$\pi^* : \Gamma(L) \xrightarrow{\simeq} \left\{ \begin{array}{l} \text{The space of equivariant sections of} \\ \widetilde{L} \text{ with respect to the } \Lambda \text{ actions} \\ \text{on } V \text{ and } \mathbb{C}\times V \text{ respectively.} \end{array} \right\}$$

proof: First, notice that $\Gamma(\widetilde{L}) \simeq \mathcal{O}$, for $s \in \mathcal{O}$ is sent to $\mathbf{s} \in \Gamma(\widetilde{L})$ with $\mathbf{s}(z) = (s(z), z)$, $z \in V$ and $s \mapsto \mathbf{s}$ gives the isomorphism.
Now if $\sigma \in \Gamma(L)$ define $\pi^*(\sigma) \in \Gamma(\widetilde{L})$ by $(\pi^*(\sigma))(z) = (\eta(z), z)$ with $[\eta(z), z] = \sigma([z])$ and $\eta(z) \in \mathbb{C}$. Obviously $\eta \in \mathcal{O}$ and $\eta = \pi^*(\sigma) \in \Gamma(\widetilde{L})$. Further, it is not difficult to check that because π is surjective π^* is injective and the first part of the Proposition is proved. For the second part $[z] = [z+u]$, $\forall u \in \Lambda$. Thus

$$[\ \eta(z+u), z+u\ ] = \sigma([z+u]) = \sigma([z]) = [\ \eta(z), z\ ] = [\ e_u(z)\ \eta(z), z+u\ ]\ .$$

Thus,

$$(3.5.2)\ \eta(z+u) = e_u(z)\ \eta(z) \ \text{ with } \ e_u(z) = \psi(u) \exp\{\pi H(z,u) + \tfrac{\pi}{2}H(u,u)\}\ .$$

Definition (3.5.3): Any such $\eta \in \mathcal{O}$ satisfying (3.5.2) will be called a *Theta function for H and ψ* . Denote the complex vector space of all such theta functions by $\mathbf{Th}(H,\psi)$.

In what follows we shall give the classical definition of a theta function and show that the two definitions can be thought as identical, at least for our purposes.

Definition (3.5.4): Classically, a theta function of type (L,J) on V with respect to the lattice Λ is defined to be an entire function θ , not identically zero, that satisfies the following relation:

$$\theta(x+u) = \theta(x) \exp\{ 2\pi i(L(x,u) + J(u)) \} \tag{3.5.5}$$

for all $z \in V$ and $u \in \Lambda$, where L is $\mathbb{C}$-linear on x and no other specifications are made for the dependence of either L or J on u .

One immediately sees that the theta functions of type (L,J) form a complex vector space under pointwise addition and scalar multiplication. We denote this vector space by $Th(L,J)$. Furthermore, the theta functions (in general, not of a particular (L,J)-type) form a multiplicative group under pointwise multiplication and so we can speak of the algebra of theta functions.

Computing $\theta(x+u+v)/\theta(x)$ in the two obvious ways, we obtain

$$L(x,u+v)+J(u+v) \equiv L(x,u)+L(x+u,v)+J(u)+J(v) , \pmod{\mathbb{Z}}.$$

By letting $x = 0$, we get

$$J(u+v) - J(u) - J(v) \equiv L(u,v), \quad \pmod{\mathbb{Z}}. \tag{3.5.6}$$

Because left hand side of the above is commutative in u and v

$$L(u,v) \equiv L(v,u), \qquad \pmod{\mathbb{Z}}. \tag{3.5.7}$$

Also $$L(x,u+v) \equiv L(x,u) + L(x,v), \quad \pmod{\mathbb{Z}}.$$

The difference between the two sides of the last relation is an integer, but L is linear in the first variable and therefore this integer has to vanish forcing L to be $\mathbb{R}$-linear in the second variable.

Proposition (3.5.8): If we set

$$F(z,w) := L(z,w) - L(w,z)$$

then F is a skew-symmetric, $\mathbb{R}$-bilinear form

$$F : V \times V \longrightarrow \mathbb{R}$$

such that $F(\Lambda \times \Lambda) \subset \mathbb{Z}$ and F is invariant under the complex structure t on V.
Furthermore if

$$\widetilde{F} : V \times V \longrightarrow \mathbb{R}$$

is defined by $\widetilde{F}(z,w) := F(tz,w)$

$\widetilde{F}$ is symmetric.

proof: We have seen that L is $\mathbb{R}$-bilinear. Thus F is $\mathbb{R}$-bilinear, alternating. F takes integral values on $\Lambda \times \Lambda$ since $L(u,v) \equiv L(v,u)$ (mod $\mathbb{Z}$) and because F $\mathbb{R}$-bilinear and Λ generates V over $\mathbb{R}$, F is also real valued. For the rest of the statement notice that

$$\widetilde{F}(z,w) - \widetilde{F}(w,z) = L(tz,w) - L(w,tz) - L(tw,z) + L(z,tw)$$

$$= i\, L(z,w) - i\, L(w,z) - i\, (L(tz,tw) - L(tw,tz))$$

$$= i\, [\, F(z,w) - F(tz,tw)\,] \ .$$

But $\widetilde{F}(z,w) - \widetilde{F}(w,z) \in \mathbb{R}$, thus $\widetilde{F}(z,w) = \widetilde{F}(w,z)$ which proves $\widetilde{F}$ symmetric and $F(z,w) - F(tz,tw) = 0$ which proves that F is t-invariant and the proof is now complete.

Lemma (3.5.9): Let L be a $\mathbb{C}$-valued, $\mathbb{R}$-bilinear form on $V \times V$, $\mathbb{C}$-linear in the first variable, such that

$$F(z,w) := L(z,w) - L(w,z)$$

is $\mathbb{R}$-valued, antisymmetric. Then $2iL = H+G$, where $H(z,w) = F(tz,w)$

$+ i\, F(z,w)$ is hermitian, and G is a $\mathbb{C}$-bilinear , symmetric form. The correspondence between F and (H,G) is one-to-one, H is non-degenerate if and only if F is non-degenerate and H is positive definite if and only if $\widetilde{F}(z,w) := F(tz,w)$ is positive definite.

<u>*proof:*</u> We first check that H defined as above is indeed hermitian. The same arguments used in the proof of Proposition (3.5.8) are valid under the conditions of this lemma, and so we have ready that $F(tx,ty) = F(x,y)$ with F is $\mathbb{R}$-bilinear, antisymmetric, and $\tilde{F}$ symmetric. By direct computation one sees that

$$H(tx,y) = iH(x,y)\ ,\quad H(x,ty) = -iH(x,y)\ ,\quad H(x,y) = \overline{H(y,x)}\ ,$$

which proves what was to be shown. Going the other direction, it is immediate that the imaginary part of a hermitian form is an $\mathbb{R}$-bilinear, antisymmetric form, invariant under the complex structure, and that $Re(H(x,y)) = Im(H(tx,y))$ is a symmetric, $\mathbb{R}$-bilinear form. Moreover, it follows easily that the form

$$G(x,y) = 2iF(x,y) - H(x,y)$$

$$= Im(L(ty,tx)) - iIm(L(y,tx)) - iIm(L(ty,x)) + Im(L(y,x)),$$

is symmetric and $\mathbb{C}$-linear in x, therefore $\mathbb{C}$-bilinear. The uniqueness is clear. It is obvious that the non-degeneracy of H implies that of F and vice versus. Furthermore H is positive definite if and only if ReH is positive definite. But $ReH = \widetilde{F}$ and the proof is now complete.

The following proposition sheds more light into Definition (3.5.4).

<u>*Proposition (3.5.10):*</u> Let $f_u(z) := \exp\{\, 2\pi i(L(z,u)+J(u))\,\}$ where L and J are as in Definition (3.5.4). Then $f_u(z) \in H^1(\Lambda;\mathcal{O}^*)$, $u \in \Lambda$ and $z \in V$.
<u>*proof:*</u> All we really need to prove is the cocycle condition

$$f_{u+v}(z) = f_u(z+v)\, f_v(z).$$

But this is really immediate as a consequence of the identity (3.5.6), i.e.

$$J(u+v) \equiv L(v,u)+J(u)+J(v) \pmod{\mathbb{Z}} .$$

For $\quad f_{u+v}(z) = \exp\{\ 2\pi i(L(x,u+v)+J(u+v))\ \}$

$$= \exp\{\ 2\pi i(L(z,u)+J(x,v)+L(v,u)+J(u)+J(v))\ \}$$

$$= \exp\{\ 2\pi i(L(x+v,u)+J(u))\ \} \exp\{\ 2\pi i(L(x,v)+J(v))\ \}$$

$$= f_u(z,v)\ f_v(z) \quad .$$

We can now suspect that the difference between the two definitions (3.5.3) and (3.5.4) respectively lays in the choice of the cocycle. Cohomologous cocycles should give equivalent theta functions.

Indeed, a theta function of type (L,J) is called *trivial* if for $f_u(z) = \exp\{\ 2\pi i(L(z,u)+J(u))\ \}$ we have $[f_u(z)] = 0$, i.e. if $f_u(z) \in B^1(\Lambda;\mathcal{O})$.

Two theta functions are said to be *equivalent* if their quotient is a trivial theta function.

Lemma (3.5.11): There is an isomorphism of complex vector spaces

$$\mathrm{Th}(L,J) \xrightarrow{\ \sim\ } \mathrm{Th}(\tfrac{1}{2i}H,J)$$

proof: Consider $Q(z,w) := \frac{1}{2i} H(z,w) - L(z,w)$, then $Q(z,w) = \frac{1}{2i} G(z,w)$, where G is as in Lemma (3.5.9) and therefore Q is $\mathbb{C}$-bilinear , symmetric. We also have $Q(\Lambda \times \Lambda) \subset \mathbb{Z}$. Take

$$g(z) = \exp\{\ \pi i\ Q(z,z)\ \} \ \in \mathcal{O}^* ,$$

$$\frac{g(z+u)}{g(z)} = \exp\{\ \pi i(Q(z+u,\ z+u) - Q(z,z))\ \} = \exp\{\ 2\pi i\ (Q(z,u))\ \} \ \in B^1(\Lambda;\mathcal{O}).$$

Thus g is a trivial theta function. It is now immediate that the funcion

$$\phi_1 : \mathrm{Th(L,J)} \longrightarrow \mathrm{Th}(\tfrac{1}{2i}\mathrm{H,J})$$

defined by $\phi_1(\theta) = g\cdot\theta$ is an isomorphism. In fact we proved more than we needed, i.e. we proved that θ and $\phi_1(\theta)$ are equivalent.

Lemma (3.5.12): There is an isomorphism of complex vector spaces as follows:

$$\mathrm{Th}(\tfrac{1}{2i}\mathrm{H,J}) \xrightarrow{\ \sim\ } \mathbf{Th}(\mathrm{H},\psi) \ .$$

proof: For $K(u) := J(u) - \frac{1}{2} L(u,u)$ we have from (3.5.6) by letting $L = \frac{1}{2i}H$

$$K(u+v) \equiv K(u) + K(v) + 1/2\, F(u,v) \pmod{\mathbb{Z}} \qquad (3.5.13)$$

There is a possibility that K is not a real-valued function. In that case define

$$l(z) := (\mathrm{Im}K)(tz)+i(\mathrm{Im}K)(z)$$

(Notice that ImK is by (3.5.13) an additive function on Λ and can, thus, be extended to an $\mathbb{R}$-linear function on V) . Consider the function $K-l$ which is real-valued and (3.5.13) is still true for $K-l$. Moreover, remark that

$$\begin{aligned} l(tz) &= (\mathrm{Im}K)(-z) + i(\mathrm{Im}K)(tz) \\ &= -(\mathrm{Im}K)(z) + i(\mathrm{Im}K)(tz) = i\, l(z) \end{aligned}$$

i.e. l is $\mathbb{C}$-linear . Define

$$\psi_J(u) = \exp\{\, 2\pi i((K-l)(u))\, \}$$

then (3.5.13) for $K-l$ makes ψ_J a semicharacter for H . Since ψ is also a semicharacter for H

$$\frac{\frac{\psi}{\psi_J}(u+v)}{\frac{\psi}{\psi_J}(u)\,\frac{\psi}{\psi_J}(v)} = 1 \qquad \text{i.e. } \frac{\psi}{\psi_J} \in \mathrm{Hom}(\Lambda,\mathbb{C}_1^*)$$

Therefore it can be extended to $g_2 \in \mathcal{O}^*$. Notice that g_2 becomes automatically a trivial theta function since

$$\frac{g_2(z+u)}{g_2(z)} = \frac{\psi}{\psi_J}(u) \in B^1(\Lambda;\mathcal{O}^*) .$$

Let

$$g_1(z) = \exp\{ 2\pi i(-l(z)) \} \in \mathcal{O}^*$$

g_1 is again a trivial theta function since

$$\frac{g_1(z+u)}{g_1(u)} = \exp\{ 2\pi i(-l(u)) \} \in B^1(\Lambda;\mathcal{O}^*)$$

because l is linear. Now define

$$\phi_2 : \mathrm{Th}(\tfrac{1}{2i} H,J) \longrightarrow \mathbf{Th}(H,\psi)$$

by letting $$\phi_2(\theta) = g_1 g_2 \theta$$

It is easy now to verify that ϕ_2 is an isomorphism.

We have again proved more than we needed, since θ and $\phi_2(\theta)$ are equivalent. Also from the statement and the proof of lemma (3.5.12) it is clear that ψ is only required to be a semicharacter for H . Combining the two lemmata (3.5.11) and (3.5.12) we have shown

Corollary (3.5.13): There is an isomorphism of complex spaces

$$\mathrm{Th}(L,J) \xrightarrow{\sim} \mathbf{Th}(H,\psi)$$

where $$H(z,w) = [\, L(tz,w) - L(w,tz) \,] + i \{\, L(w) - L(w,z) \,]$$

and ψ is a semicharacter for H .

Definition 1 is thus equivalent to Definition 2. Moreover the dim Th(L,J) = dim $\mathbf{Th}(H,\psi)$ depends only on H and not on the choice of ψ .

§ 3.6 The space of Theta Functions - Algebraizability of Complex Tori.

Suppose $\mathcal{L}$ is a holomorphic line bundle over an analytic space M . Assume further, that this can be generated by a finite number of global sections, say $\phi_0, \ldots, \phi_n \in H^0(M;\mathcal{O}(\mathcal{L}))$. Then, by picking a trivialization of $\mathcal{L}$ around any point a , one gets functions $\phi_{0,U}, \ldots, \phi_{n,U}$ which do not vanish similtaneously, and from these one gets a holomorphic map U $\longrightarrow \mathbb{C}^{n+1} - \{0\}$ which in turn defines a holomorphic map U $\longrightarrow \mathbb{CP}(n)$ has been defined. It is important to know when this map is an embedding. According to the Inverse Function Theorem for holomorphic functions, see relatively [G-H] page 18-19, for ϕ to be embedding one needs ϕ to be one-to-one and its differential to be non-zero everywhere.

In the previous section, we saw that over a complex torus the space of the 'global sections of a holomorphic line bundle can be identified with the space of the classical theta functions of a certain type (L,J). An invariant was associated to this space, namely its hermitian form H or the skew-symmetric form Im(H) . Lemma (3.5.9) show that Im(H) is a Riemann form if and only if H is non-degenerate, positive definite.

In this section, we want to find the dimension of the space of global holomorphic sections of $\mathcal{L}$, $H^0(M;\mathcal{O}(\ell))$. Furthermore, we follow Shimura [S2] to exhibit a basis for this space.

First of all, if we select a basis for Λ over $\mathbb{Z}$, then this is also a basis of V over $\mathbb{R}$. The matrix representing a Riemann form with respect to such a basis has integer coefficients, and its determinant is a perfect square and called *the pfaffian of the Riemann form with respect to* Λ. Because a change of basis in this case ammounts to multiplication by a unimodular matrix and its transpose, the determinant is an invariant of any such change of basis.

Lemma (3.6.1): If $\mathcal{F}$ is an integral, skew-symmetric, bilinear form on $\Lambda = \mathbb{Z}^{2n}$, then there exists a basis $\{\lambda_1,\ldots,\lambda_n,l_1,\ldots,l_n\}$ for Λ in terms of which $\mathcal{F}$ is given by the matrix

$$\mathfrak{T} = \begin{bmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{bmatrix}, \qquad \Delta_\delta = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix}, \qquad \delta_i \in \mathbb{Z}.$$

<u>*proof:*</u> For each $\lambda \in \Lambda$, the set $\{\mathfrak{T}(\lambda,\mathrm{l}), \mathrm{l} \in \Lambda\}$ is a principal ideal in $\mathbb{Z}$ and thus of the form $\mathrm{d}_\lambda\mathbb{Z}$, where d_λ is a non-negative integer. Let $\delta_1 = \min(\ \mathrm{d}_\lambda\colon \lambda \in \Lambda,\ \mathrm{d}_\lambda \neq 0\ )$ and choose λ_1 and l_1 such that $\delta_1 = \mathfrak{T}(\lambda_1,\mathrm{l}_1)$. Now for each $\mu \in \Lambda$, $\delta_1 \mid \mathfrak{T}(\mu,\lambda_1)$ and $\delta_1 \mid \mathfrak{T}(\mu,\mathrm{l}_1)$ and thus

$$\mu + \frac{\mathfrak{T}(\mu,\lambda_1)}{\delta_1}\cdot\mathrm{l}_1 - \frac{\mathfrak{T}(\mu,\mathrm{l}_1)}{\delta_1}\cdot\lambda_1 \ \in\ \mathbb{Z}\{\lambda_1,\mathrm{l}_1\}^\perp$$

i.e.,

$$\Lambda = \mathbb{Z}\{\lambda_1,\mathrm{l}_1\} \ \oplus\ \mathbb{Z}\{\lambda_1,\mathrm{l}_1\}^\perp.$$

By setting $\Lambda' = \mathbb{Z}\{\lambda_1,\mathrm{l}_1\}^\perp$ and repeating the process above we can get elements λ_2, $\mathrm{l}_2 \in \Lambda'$ such that

$$\Lambda' = \mathbb{Z}\{\lambda_2,\mathrm{l}_2\} \ \oplus\ \mathbb{Z}\{\lambda_2,\mathrm{l}_2\}^\perp.$$

Continuing in this way, it becomes clear that we finally obtain a basis $\{\lambda_1,\ldots,\lambda_n,\mathrm{l}_1,\ldots,\mathrm{l}_n\}$ for Λ having the desired properties.

The above decomposition of Λ is called a *Frobenius decomposition of* Λ *with respect to the form* $\mathfrak{T}$.

<u>*Lemma (3.6.2):*</u> If $\{\lambda_1,\ldots,\lambda_n,\mathrm{l}_1,\ldots,\mathrm{l}_n\}$ is a Frobenius decomposition for Λ with respsect to a Riemann form, then $\{\epsilon_1,\ldots,\epsilon_n\}$ forms a $\mathbb{C}$-basis for $\mathrm{V} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, where $\epsilon_i = \delta_i^{-1}\lambda_i$.

<u>*proof:*</u> Let V' be the $\mathbb{R}$-space generated by $\epsilon_1,\ldots,\epsilon_n$, and $\mathrm{V}'' := \mathrm{t}(\mathrm{V}')$. The relation

$$\mathrm{x} + \mathrm{ty} = 0, \quad \mathrm{x}\ ,\mathrm{y} \in \mathrm{V}'$$

implies that $\mathrm{ty} \in \mathrm{V}'$, because it is equal to $-\mathrm{x}$, hence ty is H-orthogonal to V'. But

$$H(ty, y) > 0 \ , \ \text{provided} \ y \neq 0$$

forcing y to equal 0, since H is non-degenerate. We thus, get $x = 0$.

To compare with treatments where one starts with an algebraic, complex torus and therefore a non-degenerate, positive (1,1)-form (for example the one given by Kodaira's Embedding Theorem) and proceeds subsequently to finding the so called Riemann Conditions, we call the $n \times (2n)$-matrix $\mathfrak{Z} = (\Delta_\delta, T)$ expressing the elements $\{\lambda_1, \ldots, \lambda_n, l_1, \ldots, l_n\}$ of a basis of a Frobenius decomposition of Λ in terms of the $\mathbb{C}$-basis $\{\epsilon_1, \ldots, \epsilon_n\}$ as above, *the normalized period matrix of* (V, Λ) *with respect to the given Frobenius decomposition of* Λ.

Lemma (3.6.3): The matrix T above in the period matrix of $\Lambda \subseteq V$ is symmetric and its imaginary part is negative definite.

proof: Notice that $\mathfrak{B} \equiv \{\epsilon_1, \ldots, \epsilon_n, l_1, \ldots, l_n\}$ is an $\mathbb{R}$-basis for V with respect to which

$$\mathfrak{F}(\epsilon_i, \epsilon_j) = 0 = \mathfrak{F}(l_i, l_j) \quad \text{and} \quad \mathfrak{F}(\epsilon_i, l_j) = -\mathfrak{F}(l_j, \epsilon_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} . \end{cases}$$

Consider now $[t]$ the matrix representing the complex structure on V with respect to $\mathfrak{B}$, say

$$[t] = \begin{bmatrix} [t'_{i,j}] & [s'_{i,j}] \\ [t''_{i,j}] & [s''_{i,j}] \end{bmatrix} \quad , \quad 1 \leq i,j \leq n \ .$$

Now, consider

$$\mathfrak{F}(i\epsilon_i, \epsilon_j) = \mathfrak{F}(t(\epsilon_i), \epsilon_j) = \mathfrak{F}(\sum_k t'_{k,i}\epsilon_k + \sum_k t''_{k,i} l_k, \ \epsilon_j) = t''_{j,i}\mathfrak{F}(l_i, \epsilon_j) = -t''_{j,i}$$

$$\mathfrak{F}(i\epsilon_j, \epsilon_i) = \mathfrak{F}(t(\epsilon_j), \epsilon_i) = \mathfrak{F}(\sum_k t'_{k,j}\epsilon_k + \sum_k t''_{k,j} l_k, \ \epsilon_i) = t''_{i,j}\mathfrak{F}(l_i, \epsilon_i) = -t''_{ij} \ .$$

Because $\mathfrak{F}$ is a Riemann form and subsequently $\tilde{\mathfrak{F}}(x,y) \equiv \mathfrak{F}(t(x),y)$ is positive definite, we get

$$\mathcal{F}(\mathrm{i}\epsilon_i,\epsilon_j) = \mathcal{F}(\mathrm{i}\epsilon_j,\epsilon_i) \Rightarrow \mathrm{t}''_{i,j} = \mathrm{t}''_{j,i} \quad , \ \forall \text{ i,j}$$

and

$$\mathcal{F}(\mathrm{i}\epsilon_i,\epsilon_i) = -\mathrm{t}''_{i,i} \geq 0 \quad , \quad \forall \text{ i} \ ,$$

i.e. the matrix $[\mathrm{t}''_{i,j}]$ is symmetric and its main diagonal elements are non-positive.

Similarly, we evaluate

$$\mathcal{F}(\mathrm{il}_i,\mathrm{l}_j) = \mathcal{F}(\mathrm{t}(\mathrm{l}_i),\mathrm{l}_j) = \mathcal{F}(\sum_k \mathrm{s}'_{k,i}\epsilon_k + \sum_k \mathrm{s}''_{k,i}\mathrm{l}_k, \mathrm{l}_j) = \mathrm{s}'_{j,i}\mathcal{F}(\epsilon_j,\mathrm{l}_j) = \mathrm{s}'_{j,i}$$

$$\mathcal{F}(\mathrm{il}_j,\mathrm{l}_i) = \mathcal{F}(\mathrm{t}(\mathrm{l}_j),\mathrm{l}_i) = \mathcal{F}(\sum_k \mathrm{s}'_{k,j}\epsilon_k + \sum_k \mathrm{s}''_{k,j}\mathrm{l}_k, \mathrm{l}_i) = \mathrm{s}'_{i,j}\mathcal{F}(\epsilon_i,\mathrm{l}_i) = \mathrm{s}'_{ij} \ .$$

Reasoning as above, we get $[\mathrm{s}'_{ij}]$ and all the elements on its main diagonal are non-negative. Furthermore, following similar lines we have

$$\begin{aligned} \mathrm{t}'_{j,i} = \mathrm{t}'_{j,i}\mathcal{F}(\epsilon_i,\mathrm{l}_i) &= \mathcal{F}(\sum_k \mathrm{t}'_{k,i}\epsilon_k + \sum_k \mathrm{t}''_{k,i}\mathrm{l}_k, \mathrm{l}_j) = \mathcal{F}(\mathrm{t}(\epsilon_i),\mathrm{l}_j) \\ &= \mathcal{F}(\mathrm{t}(\mathrm{l}_j),\epsilon_i) = \mathcal{F}(\sum_k \mathrm{s}'_{k,j}\epsilon_k + \sum_k \mathrm{s}''_{k,j}\mathrm{l}_k, \epsilon_i) = \mathrm{s}''_{i,j}\mathcal{F}(\mathrm{l}_i,\epsilon_i) = -\mathrm{s}''_{i,j} \ . \end{aligned}$$

Thus, we get a further relation

(3.6.3.I) $$[\mathrm{t}'_{i,j}] = -{}^t[\mathrm{s}''_{i,j}] \ .$$

If we now recall the fact that t is a complex structure, i.e.

$$[\mathrm{t}^2] = \begin{bmatrix} [\mathrm{t}'_{i,j}]^2 + [\mathrm{s}'_{i,j}][\mathrm{t}''_{i,j}] & [\mathrm{t}'_{i,j}][\mathrm{s}'_{i,j}] + [\mathrm{s}'_{i,j}][\mathrm{s}''_{i,j}] \\ [\mathrm{t}''_{i,j}][\mathrm{t}'_{i,j}] + [\mathrm{s}''_{i,j}][\mathrm{t}''_{i,j}] & [\mathrm{t}''_{i,j}][\mathrm{s}'_{i,j}] + [\mathrm{s}''_{i,j}]^2 \end{bmatrix} = -\mathrm{id} \ ,$$

we have

$$[\mathrm{t}'_{i,j}][\mathrm{s}'_{i,j}] + [\mathrm{s}'_{i,j}][\mathrm{s}''_{i,j}] = 0$$

which combined with (3.6.3.I) gives

$$-{}^{t}[s''_{i,j}][s'_{i,j}] + [s'_{i,j}][s''_{i,j}] = 0 \quad , \text{ or}$$

(3.6.3.II)
$$[s'_{i,j}][s''_{i,j}] = {}^{t}[s''_{i,j}][s'_{i,j}] \ .$$

Set now $A = (A_{i,j}) \equiv \mathrm{Re}(T)$ and $B = (B_{i,j}) \equiv \mathrm{Im}(T)$.

$$t(l_i) = \sum_k s'_{k,i}\epsilon_k + \sum_k s''_{k,i}l_k = t(\sum_k \tau_{k,i}\epsilon_k) = \sum_k A_{k,i}t(\epsilon_k) - \sum_k B_{k,i}\epsilon_k \ .$$

We, thus, have

(3.6.3.III)
$$\sum_k (s'_{k,i}+B_{k,i})\epsilon_k + \sum_k s''_{k,i}l_k - \sum_k A_{k,i}t(\epsilon_k) = 0 \ .$$

We recall that V is generated as a real vector space by the elements $\{\epsilon_1,\ldots,\epsilon_n,l_1,\ldots,l_n\}$ and by the proof of the Lemma (3.6.2) the spaces generated by $\{\epsilon_1,\ldots,\epsilon_n\}$ and $\{t(\epsilon_1),\ldots,t(\epsilon_n)\}$ over $\mathbb{R}$ are disjoint, thus

(3.6.3.IV)
$$B_{k,i} = -s'_{k,i} \ .$$

This, along with the conditions on $[s'_{i,j}]$, immediately proves that $\mathrm{Im}(T)$ is symmetric negative definite. Equation (3.6.3.III) gives us further

$$\sum_k s''_{k,i}l_k - \sum_k A_{k,i}t(\epsilon_k) = 0 \ .$$

$$\sum_k s''_{k,i}(\sum_\rho A_{\rho,k}\epsilon_\rho + \sum_\rho B_{\rho,k}t(\epsilon_\rho)) - \sum_k A_{k,i}t(\epsilon_k) = 0$$

$$\sum_k (\sum_\rho s''_{k,i}A_{\rho,k}\epsilon_\rho + \sum_\rho s''_{k,i}B_{\rho,k}t(\epsilon_\rho)) - \sum_k A_{k,i}t(\epsilon_k) = 0$$

$$\sum_\rho (\sum_k s''_{k,i}A_{\rho,k})\epsilon_\rho + \sum_\rho (\sum_k s''_{k,i}B_{\rho,k})t(\epsilon_\rho) - \sum_\rho A_{\rho,i}t(\epsilon_\rho) = 0$$

and because of Lemma (3.6.2) $\{\epsilon_1,\ldots,\epsilon_n,t(\epsilon_1),\ldots,t(\epsilon_n)\}$ is an $\mathbb{R}$-basis of V

$$\sum_k s''_{k,i}B_{\rho,k} - A_{\rho,i} = 0 \quad , \quad \forall\, \rho$$

i.e. $$A_{\rho,i} = \sum_k s''_{k,i} B_{\rho,k} = -\sum_k s''_{k,i} s'_{\rho,k} = -\sum_k s'_{\rho,k} s''_{k,i}$$

$$= -\sum_\mu s''_{\mu,\rho} s'_{\mu,i} = -\sum_\mu s''_{\mu,\rho} s'_{i,\mu} = -\sum_\mu s'_{i,\mu} s''_{\mu,\rho} = A_{i,\rho} \quad ,$$

where the second equality stems from (3.6.3.IV) and the fourth from (3.6.3.II) while the fifth is justified because $[s'_{i,j}]$ is symmetric. It follows that Re(T) is symmetric as well and the proof is now complete.

Theorem (Frobenius) (3.6.4): Let H be a positive definite, non-degenerate hermitian form on V such that $H(\Lambda \times \Lambda) \subset \mathbb{Z}$. Then, $\dim(\mathbf{Th}(H,\psi)) = \delta_1 \cdots \delta_n$, where ψ is any semicharacter for H and $\delta_1,\ldots,\delta_n$ the invariants resulting from a Frobenius decomposition of $F \equiv \mathrm{Im}(H)$ on Λ .

proof: First, if $\mathcal{F} \equiv F|_\Lambda$ it is immediate from Lemma (3.5.9) that $\mathcal{F}$ satisfies the hypothesis of Lemma (3.6.1) and thus the existence of $\delta_1,\ldots,\delta_n$ follows. By definition, the trivial theta functions correspond to 1-coboundaries $B^1(\Lambda;\mathcal{O}^*)$. Thus, the vector space of the theta functions defined by means of a 1-cocycle is isomorphic to the vector space of thete functions defined by means of a cohomologous cocycle. The isomorphism is explicitly realized by multiplication with a trivial theta function. In the proof of Lemma (3.4.1) we chose a typical cocycle

$$e_u(z) = \psi(u) \exp\{\pi H(z,u) + \tfrac{\pi}{2} H(u,u)\} = \exp\{\pi H(z,u) + \tfrac{\pi}{2} H(u,u) + 2\pi K(u)\}$$

where $K : \Lambda \longrightarrow i\mathbb{R}$ is associated with H . In the proofs of Lemmata (3.5.11) and (3.5.12) we saw that if $H_1(z,w)$ is a symmetric $\mathbb{C}$-bilinear form on V and K_1 a $\mathbb{C}$-linear function on V

$$\exp(2\pi i\{H_1(z,u) + \tfrac{1}{2} H_1(u,u) + K_1(u)\} \;\in\; B^1(\Lambda;\mathcal{O}^*) \;.$$

By Lemma (3.6.2) $\{\epsilon_1,\ldots,\epsilon_n\}$ is a $\mathbb{C}$-basis for V . Define $H_1 \equiv H|_{V'}$, where $V' \equiv \mathbb{R}[\epsilon_1,\ldots,\epsilon_n]$ and extend $\mathbb{C}$-linearly to all of V. Then Lemma (3.4.1) H_1 is a symmetric $\mathbb{C}$-bilinear form. Furthermore, $(H-H_1)|_{V'} = 0$,

$$(H-H_1)(\epsilon_i,l_j) = H(l_j,\epsilon_i) - H(\epsilon_i,l_j) = -2i\ \mathrm{Im}(H(l_j,\epsilon_i)) = 2i\ F(\epsilon_i,l_j)$$

and $$(H-H_1)(l_i,l_j) = 2i\ F(l_i,l_j) = 0\ .$$

Define further, $K_1 \equiv -iK|_{V'}$ and extend $\mathbb{C}$-linearly throughout V. Then consider the cohomologous cocycle $e'_u(z)$

$$e'_u(z) = e_u(z)\ \exp\{-2\pi i\{H_1(z,u) + \tfrac{1}{2}H_1(u,u) + K_1(u)\} \in H^1(\Lambda;\mathcal{O}^*)\ .$$

It follows that $\mathbf{Th}(H,\psi)$ is isomorphic to the vector space of all entire functions satisfying the conditions

$$(3.6.5) \qquad \theta(z+\epsilon_i) = \theta(z) \quad \text{and} \quad \theta(z+l_i) = \theta(z)\ \exp\{2\pi i(z_i+c_i)\}$$

where $\{z_i\}_{i=1,\ldots,n}$ are the coordinates of $z \in V$ with respect to $\mathbb{C}$-basis $\{\epsilon_1,\ldots,\epsilon_n\}$ and $c_i = (-iK-K_1)(l_i)$. Lemma (3.6.2) certifies that $\{\lambda_1,\ldots,\lambda_n\}$ is a $\mathbb{C}$-basis of V as well, thus if we consider a Frobenius decomposition $\{\lambda_1,\ldots,\lambda_n,l_1,\ldots,l_n\}$ for $\mathcal{F}$ (3.6.5) takes the form

$$(3.6.6) \quad \theta(z+\lambda_i) = \theta(z) \quad \text{and} \quad \theta(z+l_i) = \theta(z)\ \exp\{2\pi i(\delta_i z_i+c_i)\}$$

where $\{z_i\}_{i=1,\ldots,n}$ are now the new coordinates of $z \in V$ with respect to $\mathbb{C}$-basis $\{\lambda_1,\ldots,\lambda_n\}$. From the first equation (3.6.6) we have a Fourier expansion for θ

$$\theta(z) = \sum_{\mu \in \mathbb{Z}^n} a_\mu \exp\{2\pi i\mu\cdot z\}\ .$$

By use of the second of (3.6.3)

$$\theta(z+l_i) = \sum_{\mu \in \mathbb{Z}^n} a_\mu \exp\{2\pi i(\mu\cdot z+\mu\cdot l_i)\}$$

$$= \sum_{\mu \in \mathbb{Z}^n} a_\mu \exp\{2\pi i\mu\cdot z\}\ \exp\{2\pi i(\delta_i z_i+c_i)\}$$

and after equating coefficients, we get

$$a_\mu \exp\{2\pi i\mu\cdot l_i\} = a_{\mu-\delta_i} \exp\{2\pi i c_i\}\ .$$

Thus all the Fourier coefficients are determined by the a_μ with $\mu = (\mu_1,$

$\ldots,\mu_n)$, where $0 \leq \mu_i \leq \delta_i$ which explains the dimension of $\mathbf{Th}(H,\psi)$. Further, the choice of

$$a_\mu = \exp\left\{2\pi i\left(\tfrac{i}{4}H(t_\mu,t_\mu)+c_1 \cdot \mu+c_2\right)\right\},$$

with $t_\mu = \sum_i \frac{\mu_i}{\delta_i} l_i$ and $c_1 \in \mathbb{C}^n$, $c_2 \in \mathbb{C}$ two constants, produces a theta function in $\mathbf{Th}(H,\psi)$. Convergence follows the fact that the real part of the exponent comes out with minus a positive definite quadratic form. The proof is now complete.

Remark (3.6.7): Shimura in [S2] gives a basis for $\mathbf{Th}(H,\psi)$:

$$\phi(z,T,j) = \exp\{\tfrac{1}{2}\, {}^t z(T-\overline{T})^{-1}z\} \cdot \sum_{m \in \mathbb{Z}^n} \exp\{\tfrac{1}{2}\, {}^t(j+m)T(j+m) + {}^t(j+m)z\}$$

j ranges over a complete system of representatives for $\Delta_\delta{}^{-1}\mathbb{Z}^n \,/\, \mathbb{Z}^n$.

One can now consider explicit embeddings of complex algebraic tori into the $\mathbb{CP}(n)$.

To this end if $\{\phi_1,\ldots,\phi_m\}$ is a *linear system for the line bundle* L, *represented by multipliers* $\{e_u\}_{u \in \Lambda}$, i.e., a basis for the space of theta functions of type (H,ψ), one can define a map $I : V \longrightarrow \mathbb{CP}(n)$ by

$$I : \; z \mapsto (\phi_1(z),\ldots, \phi_m(z))$$

which induces a map from the torus V/Λ into the projective space $\mathbb{CP}(n)$.

The main theorem is

Theorem (Lefschetz) (3.6.8): Let F be a skew, $\mathbb{R}$-linear form on (V,t), integral on Λ and invariant under the complex structure t , and L be the line bundle $L(H,a)$ over V/Λ where $H(x,y) = F(tx,y)+iF(x,y)$ and a a semicharacter for H . Then, F is a Riemann form if and only if the map I above, defined by means of a linear system ϕ for $L^{\otimes n}$ is an analytic imbedding into the projective space, for all $n \geq 3$.

Chapter IV

Compact Complex Projective Flat Manifolds

§ 4.1 The Long Cohomology Sequence of a Compact Complex Flat Manifold.

In what follows, M denotes a connected, compact, complex, flat manifold. According to Theorem (1.3.9), there is a natural exact sequence associated with M

(4.1.1) $$0 \longrightarrow \Lambda \longrightarrow G \longrightarrow \Phi \longrightarrow 1 ,$$

in which Φ , the holonomy group of M is finite and $\Lambda \simeq \mathbb{Z}^{2n}$ is the translation subgroup of $G \simeq \pi_1(M)$. We maintain the same notation as in § 2.3 and let $V \simeq V^t$ denote the complex vector space of dimension n and complex structure t whose underlying real vector space is $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathbf{Z}$, $\mathcal{O}$, $\mathcal{O}^*$ be the sheaves of locally constant $\mathbb{Z}$-valued, holomorphic, and non-vanishing holomorphic functions on M . Following similar lines as in § 3.2, the short exact sequence of sheaves

$$0 \longrightarrow \mathbf{Z} \xrightarrow{j} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0 ,$$

where $[j(f)](x) = 2\pi i f(x)$ and $[e(g)](x) = \exp\{g(x)\}$, with $x \in M$ and f , $g \in \mathbf{Z}$, $\mathcal{O}$ respectively, gives rise to a long exact sequence in cohomology

$$0 \longrightarrow H^0(M;\mathbf{Z}) \longrightarrow H^0(M;\mathcal{O}) \longrightarrow H^0(M;\mathcal{O}^*) \longrightarrow$$

$$H^1(M;\mathbf{Z}) \longrightarrow H^1(M;\mathcal{O}) \xrightarrow{e^*} H^1(M;\mathcal{O}^*) \longrightarrow H^2(M;\mathbf{Z}) \longrightarrow \cdots .$$

Remark (4.1.2): Remark (3.1.5) is always true and because of Corollary (1.3.8) Remark (3.2.1) is true. Therefore, we can freely replace $\mathbf{Z}$ with $\mathbb{Z}$ and $\mathcal{O}$ [resp. $\mathcal{O}^*$] with $\mathcal{O}$ [resp. $\mathcal{O}^*$] , where $\mathcal{O}$ is the $\mathbb{C}$-algebra

$$\mathcal{O} := \{ f \mid f : V \longrightarrow \mathbb{C} , \ f \text{ holomorphic} \} \qquad \text{and}$$

$$\mathcal{O}^* := \{ f \mid f : V \longrightarrow \mathbb{C}^* , \ f \text{ holomorphic} \}$$

its group of units. So, in the case where M is a flat manifold, G is obviously the Bieberbach group $\pi_1(M)$ and $H^*(M;\mathbb{Z}) \simeq H^*(\pi_1(M);\mathbb{Z})$, $H^*(M;\mathcal{O}) \simeq H^*(\pi_1(M);\mathcal{O})$ [resp. $H^*(M;\mathcal{O}^*) \simeq H^*(\pi_1(M);\mathcal{O}^*)$. Furthermore, Proposition (3.2.2) holds true for precisely the same reasons as before.

Therefore the long cohomology sequence above splits again into two parts

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 1$$

(4.1.3)

$$0 \longrightarrow H^1(M;\mathcal{O})/H^1(M;\mathbb{Z}) \longrightarrow H^1(M;\mathcal{O}^*) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow \cdots .$$

By use of the known identification $\mathrm{Pic}(M) = H^1(M;\mathcal{O}^*)$ and by letting

$$\mathrm{Pic}^0(M) := H^1(M;\mathcal{O})/H^1(M;\mathbb{Z})$$

one gets

(4.1.4) $$0 \longrightarrow \mathrm{Pic}^0(M) \longrightarrow \mathrm{Pic}(M) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow \cdots .$$

We now want to take a look at $H^p(M;\mathbb{Z})$. As topological groups, they are all discrete because their topology comes from $\mathbb{Z}$. For $p \geq 2$, $H^p(M;\mathbb{Z})$ are no longer torsion free. However, one can easily compute the $\mathrm{rank}_{\mathbb{Z}}[H^p(M;\mathbb{Z})]$ by means of the following lemma. It also turns out that this depends only on the holonomy representation $\rho : \Phi \longrightarrow GL_{\mathbb{Z}}(\Lambda)$ and not on the cohomology class defining (4.1.1)

Lemma (4.1.5): If G is a Bieberbach group as above and $\rho : \Phi \longrightarrow GL_{\mathbb{Z}}(\Lambda)$ the holonomy representation associated with (4.1.1), then one has

$$H^i(G;\mathbb{Q}) = [\wedge^i H^1(\Lambda;\mathbb{Q})]^{\Phi} ,$$

the Φ-invariant elements of i-th exterior power of $H^1(\Lambda;\mathbb{Q})$.

proof: Look at the Lyndon-Hochshild-Serre spectral sequence associated with (4.1.1). The E_2-page is given by $E_2^{j,i} = H^j(\Phi;H^i(\Lambda;\mathbb{Q}))$. Because $\mathbb{Q}$ is divisible , everything collapses except the i-axis. Therefore

$$H^i(G;\mathbb{Q}) \simeq H^0(\Phi;H^i(\Lambda;\mathbb{Q})) \simeq [H^i(\Lambda;\mathbb{Q})]^\Phi$$

By Lemma (3.2.5) and the universal coefficient theorem $H^i(\Lambda;\mathbb{Q}) \simeq \wedge^i H^1(\Lambda;\mathbb{Q})$, where $H^1(\Lambda;\mathbb{Q}) \simeq \mathrm{Hom}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q},\mathbb{Q})$, because Φ acts trivially on $\mathbb{Q}$.

A. The Hodge Decomposition

Let, Ω^p denote again the sheaf of germs of holomorphic p-forms on M. The cohomology groups $H^*(M;\Omega^*)$ remain one of the most important invariants of any compact complex manifold. The Hodge Theorem remains true and one gets the following Hodge Decomposition Theorem for any Kähler manifold. A proof of the Theorem can be found, for example, in [G-H] page 116.

Theorem (4.1.6): If M is a compact Kähler manifold

$$H^i(M;\mathbb{C}) \simeq \bigoplus_{p+q=i} H^p(M;\Omega^q)$$

where

$$H^p(M;\Omega^q) = \overline{H^q(M;\Omega^p)} \quad .$$

Lemma (4.1.7): If M is a n-dimensional, complex, compact, connected, flat manifold, then we have

$$\dim_{\mathbb{C}} H^1(M;\mathcal{O}) = \tfrac{1}{2}\,\mathrm{rank}_{\mathbb{Z}}\, H^1(M;\mathbb{Z}) \qquad \text{and}$$

$$\mathrm{Pic}^0(M) = H^1(M;\mathcal{O}) \;/\; H^1(M;\mathbb{Z})$$

is a complex torus.

proof: By the Hodge Decomposition

$$H^1(M; \mathbb{C}) \simeq H^1(M; \mathcal{O}) \oplus H^0(M; \Omega^1)$$

where $H^0(M; \Omega^1)$ is the space of holomorphic 1-forms and

$$H^1(M; \mathcal{O}) = \overline{H^0(M; \Omega^1)} \quad .$$

Thus,

$$\dim_{\mathbb{C}} H^1(M; \mathbb{C}) = 2 \dim_{\mathbb{C}} H^1(M; \mathcal{O}) \qquad (*)$$

By the Poincaré Duality $H^n(M;\mathbb{Z}) \simeq H_0(M;\mathbb{Z})$ and because M is connected $H_0(M;\mathbb{Z}) \simeq \mathbb{Z}$. It follows that $H_{n-1}(M;\mathbb{Z})$ is torsion free. For

$$H^n(M;\mathbb{Z}) \simeq \mathrm{Hom}(H_n(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}) \ ,$$

and $\mathrm{Ext}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z})$ is isomorphic to the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$. By Poincaré Duality once more, one concludes that $H^1(M;\mathbb{Z})$ has no torsion. By the universal coefficient theorem

$$H^1(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^1(M; \mathbb{C})$$

$$\Rightarrow \qquad \mathrm{rank}_{\mathbb{Z}} H^1(M; \mathbb{Z}) \simeq \dim_{\mathbb{C}} H^1(M; \mathbb{C}) \qquad (**)$$

By $(*)$ and $(**)$ now

$$\dim_{\mathbb{C}} H^1(M; \mathcal{O}) = \tfrac{1}{2}\, \mathrm{rank}_{\mathbb{Z}} H^1(M;\mathbb{Z})$$

Because $H^1(M:\mathbb{Z})$ inherits the topology of $\mathbb{Z}$, it is discrete and the statement of the Lemma follows.

One can actually find the dimension of $\mathrm{Pic}^0(M)$,

Lemma (4.1.8): $\dim_{\mathbb{C}} \mathrm{Pic}^0(M) = \tfrac{1}{2}\, \mathrm{rank}_{\mathbb{Z}} [H^1(\Lambda;\mathbb{Z})]^{\Phi}$

where Λ , Φ are as in (4.1.1).

proof: V is the universal covering of M and $\hat{M} \equiv V/\Lambda$ is a complex torus which is a finite holomorphic covering of M . Combining (3.2.3) and (4.1.3), we get the following commutative diagram

$$
(4.1.9) \qquad
\begin{array}{ccccccccc}
0 \longrightarrow & H^1(M;\mathbb{Z}) & \longrightarrow & H^1(M;\mathcal{O}) & \longrightarrow & \mathrm{Pic}(M) & \longrightarrow & H^2(M;\mathbb{Z}) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H^1(\hat{M};\mathbb{Z}) & \longrightarrow & H^1(\hat{M};\mathcal{O}) & \longrightarrow & \mathrm{Pic}(\hat{M}) & \longrightarrow & H^2(\hat{M};\mathbb{Z}) & \longrightarrow \cdots
\end{array}
$$

M and $\hat{M}$ are $K(\pi_1,1)$-spaces and from the Lyndon-Hotschild-Serre spectral sequence, we have that

$$H^1(M;\mathbb{Z}) = E_\infty^{1,0} \oplus E_\infty^{0,1}$$

One easily sees that in this case

$$E_\infty^{1,0} = E_2^{1,0} = H^1(\Phi;H^0(\hat{M};\mathbb{Z})) = H^1(\Phi;\mathbb{Z})) = \mathrm{Tors}(H_0(\Phi;\mathbb{Z})) = 0$$

$$E_\infty^{0,1} = E_3^{0,1} = \mathrm{Ker}\Big([H^1(\hat{M};\mathbb{Z})]^\Phi \longrightarrow H^2(\Phi;\mathbb{Z})\Big) \ .$$

Because Φ is a finite group $H^2(\Phi;\mathbb{Z})$ is a finite group and thus $H^1(M;\mathbb{Z})$ is a subgroup of finite index of the torsion free group $[H^1(\hat{M};\mathbb{Z})]^\Phi$. Thus

$$\mathrm{rank}_{\mathbb{Z}}\, H^1(M;\mathbb{Z}) = \mathrm{rank}_{\mathbb{Z}}\, [H^1(\hat{M};\mathbb{Z})]^\Phi = \mathrm{rank}_{\mathbb{Z}}\, [H^1(\Lambda;\mathbb{Z})]^\Phi$$

and the proof of the Lemma follows now from Lemma (4.1.7).

If we let $H^2(M;\, \mathbb{Z}) = \Delta \oplus \Psi$, where Δ is the free abelian part and Ψ torsion group, we have

Proposition (4.1.10): The $\mathrm{Im}\{\, \delta : H^1(M;\, \mathcal{O}^*) \longrightarrow H^2(M;\, \mathbb{Z})\, \}$ is a direct summand of $H^2(M;\, \mathbb{Z})$ and contains all its torsion elements. In fact, more is true, $\mathrm{Pic}(M)/\mathrm{Pic}^0(M)$ is discrete and

$$\mathrm{Tors}\Big(\frac{\mathrm{Pic(M)}}{\mathrm{Pic^0(M)}}\Big) = \mathrm{Tors}(\mathrm{H}^2(\mathrm{M};\ \mathbb{Z}))\ .$$

proof: $\mathrm{H}^2(\mathrm{M};\ \mathbb{Z})/\mathrm{Im}\{\delta\} \longrightarrow \mathrm{H}^2(\mathrm{M};\ \mathcal{O})$ is injective by exactness, and $\mathrm{H}^2(\mathrm{M};\ \mathcal{O})$ being a vector space over $\mathbb{C}$ is torsion free. Thus all torsion elements of $\mathrm{H}^2(\mathrm{M};\ \mathbb{Z})$ are contained in $\mathrm{Im}\{\delta\}$. $\mathrm{H}^2(\mathrm{M};\ \mathbb{Z})/\mathrm{Im}\{\delta\}$ is finitely generated abelian and thus we can write

$$\mathrm{H}^2(\mathrm{M};\ \mathbb{Z}) \simeq \mathrm{H}^2(\mathrm{M};\ \mathbb{Z})/\mathrm{Im}\{\delta\} \oplus \mathrm{Im}\{\delta\}\ .$$

Since $\mathrm{H}^2(\mathrm{M};\ \mathbb{Z})/\mathrm{Im}\{\delta\}$ is torsion free, $\mathrm{Tors}(\mathrm{H}^2(\mathrm{M};\ \mathbb{Z})) \subset \mathrm{Im}\{\delta\}$. For the second part of the statement, it suffices to notice that from (4.1.4) we get an exact sequence

$$0 \longrightarrow \frac{\mathrm{Pic(M)}}{\mathrm{Pic^0(M)}} \longrightarrow \mathrm{H}^2(\mathrm{M};\mathbb{Z}) \longrightarrow \mathrm{H}^2(\mathrm{M};\mathcal{O}) \longrightarrow \cdots ,$$

where $\mathrm{H}^2(\mathrm{M};\ \mathcal{O})$ is a vector space.

Furthermore, exploiting the topological structure of the groups in (4.1.4), we have

Proposition (4.1.11): $\mathrm{Ker}\{\ \delta : \mathrm{H}^1(\mathrm{M};\ \mathcal{O}^*) \longrightarrow \mathrm{H}^2(\mathrm{M};\ \mathbb{Z})\ \}$ is precisely the identity component $(\mathrm{H}^1(\mathrm{M};\mathcal{O}^*))_0$ of $\mathrm{H}^1(\mathrm{M};\mathcal{O}^*)$.

proof: Clearly, because $\mathrm{H}^2(\mathrm{M};\mathbb{Z})$ is discrete $\delta\big((\mathrm{H}^1(\mathrm{M};\mathcal{O}^*))_0\big) = 0$. We now need to show the other direction. Since $\mathrm{H}^1(\mathrm{M};\mathcal{O})$ is a complex vector space, it is connected and so is $e_*\big(\mathrm{H}^1(\mathrm{M};\mathcal{O})\big) \subset \mathrm{H}^1(\mathrm{M};\mathcal{O}^*)$. But $\mathrm{Ker}\{\ \delta : \mathrm{H}^1(\mathrm{M};\ \mathcal{O}^*) \longrightarrow \mathrm{H}^2(\mathrm{M};\ \mathbb{Z})\ \} = e_*\big(\mathrm{H}^1(\mathrm{M};\mathcal{O})\big)$ is connected and contains $(\mathrm{H}^1(\mathrm{M};\mathcal{O}^*))_0$, thus

$$\mathrm{Ker}\{\ \delta : \mathrm{H}^1(\mathrm{M};\ \mathcal{O}^*) \longrightarrow \mathrm{H}^2(\mathrm{M};\ \mathbb{Z})\ \} = (\mathrm{H}^1(\mathrm{M};\mathcal{O}^*))_0\ .$$

Remark (4.1.12): Propositions (4.1.10) and (4.1.11) are still true even if δ is the connecting homomorphism $\mathrm{H}^n(\mathrm{M};\mathcal{O}^*) \longrightarrow \mathrm{H}^{n+1}(\mathrm{M};\mathbb{Z})$

We thus have, keeping the above notation, the following two

$$\Psi \oplus \Delta_1 \simeq \mathrm{Im}\{\delta\} , \qquad \Delta_1 \oplus \Delta_2 \simeq \Delta ,$$

where Δ_1 , Δ_2 are free abelian groups.

Proposition (4.1.13): $\mathrm{rank}_{\mathbb{Z}}\, H^2(M;\, \mathbb{Z}) = \mathrm{rank}_{\mathbb{Z}}\, [H^2(\hat{M};\, \mathbb{Z})]^{\Phi}$.

proof: From the Lyndon-Hochschild-Serre spectral sequence of (4.1.1), we have

$$H^2(M;\, \mathbb{Z}) = E_\infty^{2,0} \oplus E_\infty^{1,1} \oplus E_\infty^{0,2} .$$

$$E_\infty^{2,0} = E_3^{2,0} = E_2^{2,0} \,/\, \mathrm{Im}(E_2^{0,1} \longrightarrow E_2^{2,0})$$

$$E_\infty^{1,1} = E_3^{1,1} = \mathrm{Ker}(E_2^{1,1} \longrightarrow E_2^{3,0})$$

$$E_\infty^{0,2} = E_4^{0,2} = \mathrm{Ker}(E_3^{0,2} \longrightarrow E_3^{3,0})$$

Now, the groups $E_2^{2,0} = H^2(\Phi;H^0(\hat{M};\mathbb{Z})) = H^2(\Phi;\mathbb{Z})$ and $E_2^{1,1} = H^1(\Phi;H^1(\hat{M};\mathbb{Z}))$ are finite, thus

$$\mathrm{rank}_{\mathbb{Z}}\, H^2(M;\mathbb{Z}) = \mathrm{rank}_{\mathbb{Z}}\, E_\infty^{0,2} .$$

$$E_3^{3,0} = E_2^{3,0} \,/\, \mathrm{Im}(E_2^{1,1} \longrightarrow E_2^{3,0}) \qquad \text{with}$$

$E_2^{3,0} = H^3(\Phi;H^0(\hat{M};\mathbb{Z})) = H^3(\Phi;\mathbb{Z})$ a finite group. We thus have,

$$\mathrm{rank}_{\mathbb{Z}}\, H^2(M;\mathbb{Z}) = \mathrm{rank}_{\mathbb{Z}}\, E_\infty^{0,2} = \mathrm{rank}_{\mathbb{Z}}\, E_3^{0,2} ,$$

where $E_3^{0,2} = \mathrm{Ker}(E_2^{0,2} \longrightarrow E_2^{2,1})$ with $E_2^{2,1} = H^2(\Phi;H^1(\hat{M};\mathbb{Z}))$ a finite group. It follows that

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}}\, H^2(M;\mathbb{Z}) &= \mathrm{rank}_{\mathbb{Z}}\, E_\infty^{0,2} = \mathrm{rank}_{\mathbb{Z}}\, E_3^{0,2} \\ &= \mathrm{rank}_{\mathbb{Z}}\, E_2^{0,2} = \mathrm{rank}_{\mathbb{Z}}\, [H^2(\hat{M};\mathbb{Z})]^{\Phi} .\end{aligned}$$

§ 4.2 Line Bundles and Algebraizability of Flat Manifolds.

Let M denote a connected, compact, complex, flat Riemannian manifold. Corollary (1.3.8) says that M is isometric to one of the form $G\backslash \mathcal{M}_{2n}/O_{2n}$ where O_{2n} is the isotropy group of the origin, $\mathcal{M}_{2n}$ the group of rigid motions of $\mathbb{R}^{2n}$ and $G \simeq \pi_1(M)$ a torsion-free, discrete, cocompact subgroup of $\mathcal{M}_{2n}$. According to Theorem (1.3.9), there is a natural exact sequence associated with M

$$(4.2.1) \qquad 0 \longrightarrow \Lambda \longrightarrow G \longrightarrow \Phi \longrightarrow 1 ,$$

in which Φ, the holonomy group of M is finite and $\Lambda \simeq \mathbb{Z}^{2n}$ is the translation subgroup of G . In what follows we maintain the same notation as in § 2.3 and let $V \simeq V^t$ denote the complex vector space of dimension n and complex structure t whose underlying real vector space is $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

If V is the universal covering of M , $\hat{M} \equiv V/\Lambda$ is a complex torus which is a finite holomorphic covering of M . If further M is a flat algebraic manifold, then so is $\hat{M}$. The main result in this section is that we can understand Pic(M) as the Φ-invariant elements of Pic($\hat{M}$).

Associated with (4.2.1) one has the cohomology class $c \in H^2(\Phi;{}_\rho\Lambda)$ that classifies the extension and which we assume to be non-trivial. If $i : \Lambda \longrightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$ is the map defined by sending $\lambda \mapsto \lambda \otimes 1$, $\lambda \in \Lambda$, then the class $i_*(c) \in H^2(\Phi;V)$ is trivial for Φ is finite and V is divisible. We choose

$$\hat{c}(g) = \frac{1}{|\Phi|} \sum_{h \in \Phi} c(g, h) ,$$

since Φ is always finite, the above definition makes sence. It is not difficult to show that $\hat{c} \in H^1(\Phi, V)$ is a canonical choice, amongst several ones, of a 1-cocycle such that

$$(\delta\hat{c})(g_1, g_2) = \hat{c}\big(\theta((g_1, g_2))\big)$$

$$= \hat{c}(g_1(g_2)-(g_1g_2)+(g_1)) = g_1\hat{c}(g_2)-\hat{c}(g_1g_2)+\hat{c}(g_1)$$

(4.2.2)

$$= \frac{1}{|\Phi|}\sum_{h\in\Phi} g_1 c(g_2, h) - \frac{1}{|\Phi|}\sum_{l\in\Phi} c(g_1 g_2, l) + \frac{1}{|\Phi|}\sum_{k\in\Phi} c(g_1, k)$$

$$= \frac{1}{|\Phi|}\Big(\sum_{h\in\Phi} g_1 c(g_2, h) - \sum_{h\in\Phi} c(g_1 g_2, h) + \sum_{h\in\Phi} c(g_1, g_2 h)\Big)$$

$$= \frac{1}{|\Phi|}\sum_{h\in\Phi}\Big(g_1 c(g_2, h) - c(g_1 g_2, h) + c(g_1, g_2 h)\Big)$$

$$= \frac{1}{|\Phi|}\sum_{h\in\Phi} c(g_1, g_2) = c(g_1, g_2),$$

for $c \in H^2(\Phi, \Lambda)$ and thus

$$g_1 c(g_2, h) + c(g_1, g_2 h) = c(g_1, g_2) + c(g_1 g_2, h) \tag{4.2.3}$$

for all $g_1, g_2, h \in \Phi$; θ is the second differential in the bar resolution and δ the corresponding differential from the 1-cochains to the 2-cochains.

The elements of G are being represented as the elements of the cartesian product $\Lambda \times \Phi$, where the addition is being defined by

$$(\lambda, g) + (\lambda', g') = (\lambda + g\cdot\lambda' + c(g,g'),\ gg'),$$

for all $\lambda, \lambda' \in \Lambda$, $g, g' \in \Phi$.

The action of G on V, $G \times V \longrightarrow V$ is given by

$$(\lambda,g)(z) = g(z) + |\Phi|(\lambda + \hat{c}(g)),$$

which induces the action of Φ on $\hat{M}$, $\Phi \simeq G/\Lambda \times V/\Lambda \longrightarrow V/\Lambda$ given by

$$g\,[z] = [g(z) + |\Phi|(\lambda + \hat{c}(g))] = [g(z)],$$

since $|\Phi|\hat{c}(g) \in \Lambda$ (recall that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ inherits its G-module structure form Λ).

For any G-module A the action of Φ on $H^n(\Lambda;A)$ is defined by means of the action of G on $H^n(\Lambda;A)$

$$G \times H^n(\Lambda;A) \longrightarrow H^n(\Lambda;A)$$

given by

$$(\gamma * c)\,(\lambda_1, \ldots, \lambda_n) = \gamma \cdot c\,(-\gamma+\lambda_1+\gamma, \ldots, -\gamma+\lambda_n+\gamma)$$

for $\gamma \in G$ and $\lambda_i \in \Lambda$. For $n = 1$ and $A \equiv \mathcal{O}^*$, this gives us

$$\Phi \times H^1(\Lambda;\mathcal{O}^*) \longrightarrow H^1(\Lambda;\mathcal{O}^*)$$

as follows

$$(g * e_u)(z) = e_{g^{-1}u}(g^{-1}z)$$

where $e \in H^1(\Lambda;\mathcal{O}^*)$, $g \in \Phi$, $u \in \Lambda$, and $z \in V$.

Remark (4.2.4): One can easily see that the skew, $\mathbb{R}$-bilinear form F which is integral on Λ and invariant under the complex structure is Φ-invariant if and only if its associated hermitian form H, $H(x,y) = F(tx,y)+iF(x,y)$ is Φ-invariant, for

$$F(tx,y)+iF(x,y) = H(x,y) = (g*H)(x,y)$$

$$= H(g^{-1}x,g^{-1}y) = F(t(g^{-1}x),g^{-1}y)+iF(g^{-1}x,g^{-1}y)\ .$$

The imaginary parts have to be equal, thus

$$F(x,y) = F(g^{-1}x,g^{-1}y) = (g*F)(x,y)\ .$$

The other direction is obvious.

Lemma (4.2.5): If F as in the Remark (4.2.4) above is Φ-invariant then so is the 1-cocycle

$$e_u(z) = a(u)\,\exp\{\pi H(z,u)+\tfrac{\pi}{2}H(u,u)\}\ ,$$

where $u \in \Lambda$, $z \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, $H(x,y) = F(tx,y)+iF(x,y)$ and a is a semicharacter for H.

proof: Immediate from the Remark (4.2.4) above, (a and H are Φ-

invariant) .

Proposition (4.2.6): Let $H \in [NS(\hat{M})]^{\Phi}$, $F = \mathrm{Im}H$ and a be a semicharacter for H . Define

$$h_{(\lambda,g)}(z) :\equiv f_{|\Phi|(\lambda+\hat{c}(g))}(z)$$

where $f_u(z) = e_{-u}(z) = e_u^{-1}(z-u)$ and

$$e_u(z) = a(u)\exp\{\pi H(z,u)+\tfrac{\pi}{2}H(u,u)\}\ ,\ \ u,\lambda \in \Lambda\ ,\ z \in V\ ,\ g \in \Phi.$$

Then $h \in H^1(G;\mathcal{O}^*)$. Furthermore, the Chern class of the line bundle defined by h is F .

proof: First from the Remark (4.2.4) and Lemma (4.2.5) $a \in [\mathrm{Pic}^0(\hat{M})]^{\Phi}$ and $e \in [\mathrm{Pic}(\hat{M})]^{\Phi}$. One easily sees that

(4.2.7) $$f_{u+v}(z) = f_v(z-u)\, f_u(z)\ .$$

The cocycle condition for h is as follows

$$h_{(\lambda,g)+(\lambda',g')}(z) = (\lambda,g)\cdot h_{(\lambda',g')}(z)\, h_{(\lambda,g)}(z)\ .$$

We then have

$$\begin{aligned}
&h_{(\lambda,g)+(\lambda',g')}(z)\\
&= h_{(\lambda+g\lambda'+c(g,g'),\,gg')}(z)\\
&= f_{|\Phi|(\lambda+g\lambda'+c(g,g')+\hat{c}(gg'))}(z)\\
&\overset{(4.2.2)}{=} f_{|\Phi|(\lambda+\hat{c}(g)+g\lambda'+g\hat{c}(g'))}(z)\\
&\overset{(4.2.7)}{=} f_{|\Phi|(g\lambda'+g\hat{c}(g'))}(z-|\Phi|(\lambda+\hat{c}(g)))\ f_{|\Phi|(\lambda+\hat{c}(g))}(z)\\
&\overset{(4.2.2(3))}{=} f_{|\Phi|(g\lambda'+g\hat{c}(g'))}(z-|\Phi|(\lambda+gc(g^{-1},g)-g\hat{c}(g^{-1})))
\end{aligned}$$

$$\cdot f_{|\Phi|(\lambda+\hat{c}(g))}(z)$$

$$\overset{f\in[Pic(\hat{M})]^{\Phi}}{=} f_{|\Phi|(\lambda'+\hat{c}(g'))}(g^{-1}z-|\Phi|(g^{-1}\lambda+c(g^{-1},g)-\hat{c}(g^{-1})))$$

$$\cdot f_{|\Phi|(\lambda+\hat{c}(g))}(z)$$

$$= h_{(\lambda',g')}((\lambda,g)^{-1}\cdot z)\ h_{(\lambda,g)}(z) = (\lambda,g)\cdot h_{(\lambda',g')}(z)\ h_{(\lambda,g)}(z)$$

(where we recall that $(\lambda,g)^{-1} = (-g^{-1}\lambda - c(g^{-1},g)\ ,\ g^{-1})$).

For the second part of the proof, first notice that the line bundle $L(H,a)$ corresponding to H and a as above is obtained as the quotient of $\mathbb{C}\times V$ for the action of $\pi_1(M) = G$ given by

$$\phi_{(\lambda,g)}(u,z) = \left(f_{|\Phi|(\lambda+\hat{c}(g))}(z)\cdot u\ ,\ gz+|\Phi|(\lambda+\hat{c}(g)) \right)$$

where the notation is being kept as above. To find the Chern class of $L(H,a)$ we look at the commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & Pic(M) & \longrightarrow & H^2(G;\mathbb{Z}) & \longrightarrow \cdots \\ & \downarrow i & & \downarrow j & \\ \cdots \longrightarrow & Pic(\hat{M}) & \longrightarrow & H^2(\Lambda;\mathbb{Z}) & \longrightarrow \cdots \end{array}$$

The vertical maps are inclusions. In particular,

$$j(c_1(L(H,a))) = c_1(i(L(H,a))) = c_1(i(h)) = c_1(f) = F\ .$$

Also, $c_1(L(H,a))$ is a torsion free element of $H^2(G;\mathbb{Z})$, thus by the proof of Proposition (4.1.13)

$$c_1(L(H,a))\ \in\ E_\infty^{0,2} < E_2^{0,2} = [H^2(\Lambda;\mathbb{Z})]^{\Phi}\ .$$

Since j is the inclusion it follows $c_1(L(H,a))\ \in\ [NS(\hat{M})]^{\Phi}$ and this completes the proof.

One can have further

Lemma (4.2.8): Let L_1 and L_2 $\in$ Pic(M) such that $c_1(L_1)$ and $c_1(L_2)$ are torsion free. Then $c_1(L_1) = c_1(L_2)$ if and only if there exists a $(\lambda, g) \in G$ such that the one line bundle is the translate of the other, i.e. $L_1 = \tau^*_{(\lambda,g)} L_2$.

proof: Since for any $(\lambda, g) \in G$ the translation $\tau_{(\lambda,g)} : M \longrightarrow M$ is homotopic to the identity

$$c_1(\tau^*_{(\lambda,g)} L) = c_1(L) .$$

To prove the other direction, it suffices to show that any line bundle with chern class zero can be realized by constant multipliers. To this end, notice that the diagram of short exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{j} & \mathbb{C} & \xrightarrow{e} & \mathbb{C}^* & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{j} & \mathcal{O} & \xrightarrow{e} & \mathcal{O}^* & \longrightarrow & 0 \end{array}$$

induces for any compact Kähler manifold a commutative diagram as follows

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^1(M;\mathbb{C}) & \longrightarrow & H^1(M;\mathbb{C}^*) & \longrightarrow & H^2(M;\mathbb{Z}) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_2^*} & & \| & \\ \cdots \longrightarrow & H^1(M;\mathcal{O}) & \longrightarrow & H^1(M;\mathcal{O}^*) & \longrightarrow & H^2(M;\mathbb{Z}) & \longrightarrow \cdots \end{array}$$

The map i_1^* represents the projection of $H^1(M;\mathbb{C}) = H^{1,0}(M) \oplus H^{0,1}(M)$ on the second factor and so is surjective. It follows that any cocycle $h \in H^1(M;\mathcal{O}^*)$ in the kernel of c_1 is in the image of i_2^* , i.e. is cohomologous to a cocycle with constant coefficients and this proves the Lemma.

By Proposition (4.1.11) and the comments at the end of §1.2 we deduce that $\mathrm{Pic}^0(M)$ is precisely the identity component of $\mathrm{Pic}(M)$. Furthermore, $\mathrm{Pic}(M)/\mathrm{Pic}^0(M)$ is finitely generated abelian and therefore discrete as a Lie group. So, in conjunction with Proposition (4.1.10), we may write

(4.2.9) $$\frac{\mathrm{Pic}(M)}{\mathrm{Pic}^0(M)} \simeq \mathrm{Tors}(H^2(M;\mathbb{Z})) \oplus \mathbb{Z}^N .$$

Now, let $\phi : \mathrm{Pic}(M) \longrightarrow \dfrac{\mathrm{Pic}(M)}{\mathrm{Pic}^0(M)}$ be the identification map.

Define $\Gamma \equiv \phi^{-1}(\mathrm{Tors}(H^2(M;\mathbb{Z}))$, then Γ occurs in the following two extensions:

(4.2.10) $$0 \longrightarrow \mathrm{Pic}^0(M) \longrightarrow \Gamma \longrightarrow \mathrm{Tors}(H^2(M;\mathbb{Z})) \longrightarrow 0 \text{ , and}$$

(4.2.11) $$0 \longrightarrow \Gamma \longrightarrow \mathrm{Pic}(M) \longrightarrow \mathbb{Z}^N \longrightarrow 0 .$$

The 2-cocycle in $H^2(\mathrm{Tors}(H^2(M;\mathbb{Z}));\mathrm{Pic}^0(M))$ that classifies extension (4.2.10) is trivial. In the opposite case $\mathrm{Pic}^0(M)$ would be a subgroup of finite index in a connected Lie group and so contradicts Proposition (4.1.11). Since $\mathrm{Pic}(M)$ is an abelian topological group we can write

(4.2.12) $$\mathrm{Pic}(M) \simeq \mathrm{Pic}^0(M) \oplus \mathrm{Tors}(H^2(M;\mathbb{Z})) \oplus \mathbb{Z}^N .$$

This, for a compact abelian topological group, follows directly from a theorem of Pontrjagin, Theorem 55 on page 213 of [P]. The case of a general abelian topological group A with identity component a Lie group A_0 such that A/A_0 be finitely generated, can be reduced to that of a compact abelian topological group through the following reasoning:
Let $\phi : A \longrightarrow A/A^0 \simeq \Phi \oplus \mathbb{Z}^n$ be the identification map. Define $G_1 \equiv \phi^{-1}(\Phi)$. Then $G_1 \triangleleft A$, $A/G_1 \simeq \mathbb{Z}^n$ and $A \simeq G_1 \times \mathbb{Z}^n$. Now, G_1 is a Lie group with finitely many components, so it has a maximal compact subgroup, say K , see relatively [M4]. $K \triangleleft G_1$ with G_1 abelian, so $G_1/K \simeq \mathbb{R}^l$ and finaly $G_1 \simeq K \times \mathbb{R}^l$. The decomposition now follows from

Pontrjagin's theorem on K .

We have now proved, Proposition (4.2.6) and extension (4.2.11), the following version of the Appell-Humbert Theorem (3.4.5) for flat manifolds

Theorem (4.2.13): If M is a compact, Kähler, flat manifold, and Γ the extension of $\mathrm{Tors}(H^2(M;\mathbb{Z}))$ by $\mathrm{Pic}^0(M)$ as above. Then there is a short exact sequence as follows

$$0 \longrightarrow \Gamma \longrightarrow \mathrm{Pic}(M) \longrightarrow [\mathrm{NS}(\hat{M})]^{\Phi} \longrightarrow 0 \ .$$

Corollary (4.2.14): If Pic(M) is written as in (4.2.9), then N is the rank of $[\mathrm{NS}(\hat{M})]^{\Phi}$.

Corollary (4.2.15): Let H be a Φ-invariant hermitian form on V , integral on Λ and $L(H,\psi)$ the line bundle on $\hat{M}$ associated to (H,ψ) , ψ a semicharacter for H . Then H is positive definite if and only if the map induced by a linear system of Φ-invariant holomorphic sections of $L^{\otimes n}$ gives an imbedding of M as a closed submanifold in a projective space for each $n \geq 3$.

proof: By means of Grothendieck's spectral sequence, see page 202 of [Gr], we have

$$H^0(M;\mathcal{O}(L)) = [H^0(\hat{M};\mathcal{O}(\hat{L}))]^{\Phi} \ .$$

The proof now follows from (3.6.8) and (4.2.10).

Chapter V

Examples

§ 5.1 Projective Flat Manifolds of Prime Holonomy.

In this section, we classify the projective flat manifolds whose holonomy group is either a cyclic group C_p, or dihedral D_{2p}, with p prime, give an estimate for the size of the set their positive line bundles and constuct explicitly and directly in terms of the representation space of the holonomy group some interesting imbeddings.

We first recall the basic facts of the integral representation theory of the cyclic group of prime order.

A. Integral Representations of Cyclic Groups of Prime Order.

The main result here is that the arbitrary $\mathbb{Z}[C_p]$-module is essentially composed of three much simpler types of $\mathbb{Z}[C_p]$-modules.

Throughout this section, let ζ_p denote a pth primitive root of unity over $\mathbb{Q}$, and let $K \equiv \mathbb{Q}(\zeta_p)$. We also let R denote the ring of all algebraic integers in K which we know to be a Dedekind domain with $\mathbb{Z}$-basis $\{1, \zeta_p, \cdots, \zeta_p{}^{p-2}\}$, so that $(K:\mathbb{Q}) = (R:\mathbb{Z}) = p-1$.

Suppose A is any fractional ideal of K. Then we turn A, that has a $\mathbb{Z}$-rank $p-1$, into a $\mathbb{Z}[C_p]$-module by defining the scalar multiplication by

$$g \cdot a = \zeta_p a \,, \quad a \in A$$

and g a generator of C_p. From the above definition, it is obvious that the module structure on A is well defined and also that two such ideals A and B in K are the same as $\mathbb{Z}[C_p]$-modules if and only if A and B are isomorphic as R-modules; and this in turn is equivalent to A and B being in the same ideal class.

The second type of $\mathbb{Z}[C_p]$-module is constructed as follows:
Let again A be an ideal in K as previously and a_o a fixed element in A. Consider now the external direct sum $A \oplus \mathbb{Z}$ as $\mathbb{Z}$-modules and turn this into a $\mathbb{Z}[C_p]$-module by defining the scalar multiplication

$$g \cdot (a, n) = (\zeta_p a + n a_o,\ n)\ ,\quad a \in A\ ,\ n \in N.$$

Because ζ_p is a root of the cyclotomic polynomial of order p one can easily check that the above definition is indeed "good". The construction implies also that the $\mathbb{Z}$-rank of our newly constucted module is p. In what follows we shall denote this module by (A, a_o). Furthermore, one sees in [C-R] pg. 507, that if $m \in \mathbb{Z}$ is such that p does not divide m, then $(A, a_o) \simeq (A, ma_o)$.

Theorem (Diederichsen, Reiner)(5.1.1): Every $\mathbb{Z}[C_p]$-module, say Λ , that is finitely generated and torsion-free as abelian group is isomorphic to a direct sum

$$(A_1, a_1) \oplus \cdots \oplus (A_r, a_r) \oplus B_{r+1} \oplus \cdots \oplus B_n \oplus Y$$

where the $\{A_i\}$, $\{B_j\}$ are fractional ideals in K and the $\{a_i\}$ satisfy , $a_i \in A_i$ but $a_i \notin (\zeta_p - 1)A_i$. Y is a trivial $\mathbb{Z}$-module of $\mathrm{rank}_{\mathbb{Z}} Y = s$. The isomorphism class of Λ is completely determined by r, n, s and the ideal class of the product of $A_1 \cdots A_n$ in K.

(*proof:* For the details of the proof see [C-R] pgs. 508 - 514.)

According to the scheme described in §1.3.C, to classify the flat manifolds one first needs faithful C_p-representations. It is easily seen that this is the case if and only if $r+n > 0$. Further, the cohomology 2-classes that give rise to torsion-free extensions of $\mathbb{Z}_p$ by a module Λ are needed. To this end, one essentialy has to compute $H^2(C_p;\ \Lambda)$. This is greatly facilitated by Theorem (1.5.1) since, by use of the remark (2.1.1), we only need to do so for each of the previous three specific types of $\mathbb{Z}[C_p]$-modules. The exact computation is carried out in detail in [C2] pages 136–138 and the final outcome is:

Proposition (5.1.2) : If Λ is a C_p-module, then for $n \geq 0$

$$H^0(C_p;\ \Lambda) \simeq \mathbb{Z}_p{}^{s+r}\ ,\ H^{2n}(C^p;\ \Lambda) \simeq \mathbb{Z}_p{}^{s}\ ,\ H^{2n+1}(C_p;\ \Lambda) \simeq \mathbb{Z}_p{}^{n}.$$

B. The Classification Theorem.

To complete the classification of C_p-modules one needs to decide on conditions preventing us from receiving isomorphic extensions from different 2-cocycles and and different C_p-modules. This is done fully in [C2] pgs. 139–152, or alternatively in [C1]. We shall only state here his results. We turn our attention to the group of automorphisms of C_p which we denote by $\mathrm{Aut}(C_p)$. First, we note that $\mathrm{Aut}(C_p) \simeq \mathbb{Z}_{p-1}$ and thus there is a unique element ${}^2f \in \mathrm{Aut}(C_p)$ such that $({}^2f)^2 = 1$, the identity in $\mathrm{Aut}\,(C_p)$.

Let $\mathfrak{A} \equiv \mathrm{Aut}(C_p)$. It is easy to show that $\mathfrak{A} \equiv \mathrm{Gal}[\mathbb{Q}(\zeta_p){:}\mathbb{Q}]$. It is well known, see for example Theorem (21.13) on page 140 of [C-R], that the ring of algebraic integers of $\mathbb{Q}(\zeta_p)$ is precisely $\mathbb{Z}(\zeta_p)$. So by restriction to $\mathbb{Z}(\zeta_p) \subset \mathbb{Q}(\zeta_p)$, $\mathfrak{A}$ acts on the algebraic integers of $\mathbb{Q}(\zeta_p)$ and therefore on its fractional ideals. Moreover one can show that if A and B are ideals in the same class, then $f \cdot A$ and $f \cdot B$ are in the same ideal class, where $f \in \mathfrak{A}$; thus $\mathfrak{A}$ acts on $\mathfrak{C}_p$, the ideal class group of $\mathbb{Q}(\zeta_p)$.

We let $\mathfrak{E}_p \equiv \mathfrak{C}_p/\mathfrak{A}$ be the orbit set which is not a group. Further, we let $\mathfrak{E}_p^2$ denote the orbit set of the action of ${}^2f$ on $\mathfrak{C}_p$.

Putting everything together

Theorem (5.1.3): There is a one-one correspondence between affine equivalence classes of C_p-manifolds and quadruples (s, r, n; $[\alpha]$), where s, r, n are integers $s > 0$, $n \geq 0$, $r \geq 0$, $n + r > 0$ and α a fractional ideal whose class $[\alpha] \in \mathfrak{C}_p$ if $(s, r) \neq (1, 0)$ or $[\alpha] \in \mathfrak{E}_p^2$ if $(s, r) = (1, 0)$.

Consider now W, the tensor product of Λ with $\mathbb{Q}$. It is easily seen that

$$W \equiv \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda \simeq \mathbb{Q}^{s+r} \oplus \mathbb{Q}(\zeta_p)^{n+r} .$$

If p is odd then the simple summand $\mathbb{Q}(\zeta_p)$ is of type IV in Albert's classification, since it is totally imaginary and quadratic over the totally real field $\mathbb{Q}(\zeta_p+\bar{\zeta}_p)$. Combining Theorems (2.3.6) and (5.1.3) and preserving the same notation, we get

Corollary (5.1.4): Let (s, r, n; [α]) be a quadruple as in Theorem (5.1.3). We may choose the flat C_p-manifold corresponding to it to be complex projective if and only if s+r is even and also n+r if p = 2 .

C. Estimating the Size of the Set of Positive Line Bundles of a C_p-manifold.

A holomorphic line bundle $\mathcal{L}$ over a C_p-manifold M is said to be *positive* if there exists kählerian metric ω on M such that

$$\delta(\mathcal{L}) \otimes 1 = \frac{1}{2\pi i}\,[\omega]\ ,$$

where $\delta(\omega) \in H^2(M;\mathbb{Z})$ is the Chern class of $\mathcal{L}$ and $\delta(\mathcal{L}) \otimes 1 \in H^2(M;\mathbb{C})$. For us, $\mathcal{L}$ will be positive if and only if $\mathrm{Im}(\delta(\mathcal{L}))$ is a Riemann form. We know that from §4.2 that Pic(M) admits a decomposition as in (4.2.15). Given such a decomposition, the set of all positive line bundles being closed under multiplication is easily seen to be contained in the $\mathbb{Z}^N$ summand.

We know by Corollary (4.2.14) that

$$N = \mathrm{rank}_{\mathbb{Z}}[NS(\hat{M})]^{\Phi} \leq \mathrm{rank}_{\mathbb{Z}}[H^2(\hat{M};\mathbb{Z})]^{\Phi} = \dim_{\mathbb{Q}}[H^2(\Lambda;\mathbb{Q})]^{\Phi}$$

where the notation is being kept as in chapter IV and $\Lambda \equiv \mathbb{Z}^{2m}$. Applying Corollaries (3.2.7), (3.2.8) we have that

$$\dim_{\mathbb{Q}} H^1(\Lambda;\mathbb{Q}) = 2m \quad \text{and} \quad \dim_{\mathbb{Q}} H^2(\Lambda;\mathbb{Q}) = \overset{2}{\Lambda}\ H^1(\Lambda;\mathbb{Q}) = m(2m-1).$$

If Λ admits the decomposition in Theorem (5.1.1) one easily sees that

$$H^1(\Lambda;\mathbb{Q}) = \mathbb{Q}^{r+s} \oplus \mathbb{Q}(\zeta_p)^{r+n}$$

where $2m = r+s+(p-1)(r+n)$. We need further

Lemma (5.1.5): Let N be a $\mathbb{Q}[C_p]$-module. Assume that $N \simeq \mathbb{Q}^{\mu} \oplus V^{\nu}$, as $\mathbb{Q}[C_p]$-modules; where $V \equiv \mathbb{Q}(\zeta_p)$ is the (p−1)-dimensional $\mathbb{Q}[C_p]$-irreducible. Then

$$\wedge^2(N) \simeq \mathbb{Q}^{\binom{\mu}{2}} \oplus V^{\nu\mu} \oplus (\wedge^2(V))^{\nu} \oplus (V \otimes V)^{\frac{\nu(\nu-1)}{2}}.$$

proof:

$$\wedge^2(\mathbb{Q}^{\mu} \oplus V^{\nu}) = \wedge^2(\mathbb{Q}^{\mu}) \otimes \wedge^0(V^{\nu})$$

$$\oplus \wedge^1(\mathbb{Q}^{\mu}) \otimes \wedge^1(V^{\nu}) \oplus \wedge^0(\mathbb{Q}^{\mu}) \otimes \wedge^2(V^{\nu})$$

$$= \mathbb{Q}^{\binom{\mu}{2}} \otimes \mathbb{Q} \oplus V^{\nu\mu} \oplus \wedge^2(V^{\nu})$$

We are now left to find how $\wedge^2(V^{\nu})$ breaks down into irreducibles. We are going to use induction on s to prove that

$$\wedge^2(V^{\nu}) = (\wedge^2(V))^{\nu} \oplus (V \otimes V)^{\frac{\nu(\nu-1)}{2}}.$$

For $s = 2$

$$\wedge^2(V \oplus V) = \wedge^2(V) \otimes \wedge^0(V) \oplus \wedge(V) \otimes \wedge(V) \oplus \wedge^0(V) \otimes \wedge^2(V)$$

$$= \wedge^2(V) \oplus V \otimes V \oplus \wedge^2(V) = (\wedge^2(V))^2 \oplus V \otimes V.$$

We assume it is true for s and prove it for s+1.

$$\wedge^2(V^{\nu} \oplus V) = \wedge^2(V^{\nu}) \otimes \wedge^0(V) \oplus \wedge(V^{\nu}) \otimes \wedge(V) \oplus \wedge^0(V^{\nu}) \otimes \wedge^2(V)$$

$$= (\wedge^2(V))^{\nu} \oplus (V \otimes V)^{\frac{\nu(\nu-1)}{2}} \oplus (V \otimes V)^{\nu} \oplus \wedge^2(V)$$

$$= (\wedge^2(V))^{\nu+1} \oplus (V \otimes V)^{\frac{\nu(\nu+1)}{2}}.$$

To describe the action of C_p on $\wedge^2(N)$ we now need to find the $\mathbb{Q}[C_p]$-module structure of $V \otimes V$ and $\wedge^2(V)$ respectively, where V is as above.

Lemma (5.1.6): Let V denote the (p−1)-dimensional irreducible $\mathbb{Q}[C_p]$-

module. Then (as $\mathbb{Q}[C_p]$-modules):

i) $V \otimes V \simeq \mathbb{Q}^{p-1} \oplus V^{p-2}$, and ii) $\wedge^2(V) \simeq \mathbb{Q}^{\frac{p-1}{2}} \oplus V^{\frac{p-3}{2}}$.

proof: For the first part of the statement one can proceed by finding the dimension of the fixed point set by finding the nullity of the matrix $g-1$, where g is expresses the diagonal action of $C_p = \langle g \rangle$ on the $\mathbb{Q}$-vector space $V \otimes V$. The matrix g has a particularly nice when considered with respect to basis $\{\zeta_p \otimes \zeta_p, \zeta_p \otimes \zeta_p{}^2, \ldots, \zeta_p{}^{p-1} \otimes \zeta_p{}^{p-2}, \zeta_p{}^{p-1} \otimes \zeta_p{}^{p-1}\}$.

$$g := \begin{bmatrix} O_p & O_p & O_p & \cdots & O_p & -J_p \\ J_p & O_p & O_p & & & \\ O_p & J_p & O_p & & & \\ \vdots & & & \ddots & & \\ O_p & & & & O_p & -J_p \\ O_p & & & & J_p & -J_p \end{bmatrix}$$

where O_p is the $(p-1) \times (p-1)$ matrix that is identically everywhere zero and J_p

$$J_p := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 0 & -1 \\ 0 & & & & 1 & -1 \end{bmatrix}.$$

Alternatively, it is not difficult to see that the set $\{v_k\}_k$, $k = 1, \ldots, p-1$, with

$$v_k = \zeta_p \otimes \zeta_p{}^k + \zeta_p{}^2 \otimes \zeta_p{}^k + \zeta_p{}^2 \otimes \zeta_p{}^{k+1}$$

$$+ \zeta_p{}^3 \otimes \zeta_p{}^k + \zeta_p{}^3 \otimes \zeta_p{}^{k+1} + \zeta_p{}^3 \otimes \zeta_p{}^{k+2} + \cdots$$

$$+ \zeta_p{}^l \otimes \zeta_p{}^k + \cdots + \zeta_p{}^l \otimes \zeta_p{}^{k+l-1}$$

$$- \zeta_p^{\,l+1} \otimes \zeta_p - \cdots - \zeta_p^{\,l+1} \otimes \zeta_p^{\,k-1} - \cdots$$

$$- \zeta_p^{\,l+2} \otimes \zeta_p^{\,2} - \cdots - \zeta_p^{\,l+2} \otimes \zeta_p^{\,k-1} - \cdots$$

$$- \zeta_p^{\,l+j} \otimes \zeta_p^{\,j} - \cdots - \zeta_p^{\,l+j} \otimes \zeta_p^{\,k-1} \;,$$

where $l = p-k$ and $r = p-l-1$, forms a basis for the fixed point set of $V \otimes V$ under the action of C_p .

For the second part of the statement, define $e : V \longrightarrow V$ to be the map defined by $e(\zeta_p^{\,i} \otimes \zeta_p^{\,j}) = (\zeta_p^{\,j} \otimes \zeta_p^{\,i})$. From the bilinearity of the tensor product and the symmetry of the basis above, one notices immediately that an element $x \in V$ is fixed under the C_p-action if and only if $e(x)$ is fixed. Let now $\{x_i\}$, $i = 1,\ldots,\frac{p-1}{2}$ be linearly independent elements of the C_p-fixed subspace of $V \otimes V$. Then $\{x_i, e(x_i)\}$ form a basis for the C_p-fixed subspace of $V \otimes V$, and $\{x_i - e(x_i)\}$ a basis for the C_p-fixed subspace of $\wedge^2 V$, and the proof of the lemma is now complete.

Combining the above, we obtain $H^2({}_\rho\mathbb{Z}^{2m}; \mathbb{Q}) = \wedge^2 H^1({}_\rho\mathbb{Z}^{2m}; \mathbb{Q}) \simeq \mathbb{Q}^{\alpha} \oplus V^{\beta}$, where the notation is being kept as in §2.3 and

$$\alpha = \binom{r+s}{2} + \frac{(r+n)(p-1)}{2} + \frac{(r+n)(r+n-1)(p-1)}{2}\;, \text{ and}$$

$$\beta = (r+n)(r+s) + \frac{(r+n)(p-3)}{2} + \frac{(r+n)(s-1)(p-2)}{2}\;.$$

We have thus proved

Proposition (5.1.7): The set of positive line bundles of a flat, complex projective C_p-manifold classified by a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow G \longrightarrow C_p \longrightarrow 1,$$

where Λ as in Theorem (5.1.1) is contained in a free abelian group of rank

$$\binom{r+s}{2} + \frac{(r+n)(p-1)}{2} + \frac{(r+n)(r+n-1)(p-1)}{2}\;.$$

D. Linear Systems and Projective Embeddings of C_p-manifolds.

To provide a linear system and subsequently a projective embedding, we produce an ample line bundle and use afterwards Lefschetz's Theorem (3.6.8) or even better Corollary (4.2.15). The generalized version for flat manifolds of the Appell-Humbert Theorem (4.2.13) will be of great use since to produce such a line bundle it suffices to exhibit an element of $[NS(\hat{M})]^{\Phi}$, where $\hat{M}$ is always the universal covering of the flat manifold M .

First recall that if p is odd then the simple summand $\mathbb{Q}(\zeta_p)$ is of type IV in Albert's classification, since it is totally imaginary and quadratic over the totally real field $\mathbb{Q}(\zeta_p+\bar{\zeta}_p)$. We now have the following

Proposition (5.1.8): $\left(\mathbb{Q}(\zeta_p),\ \mathbb{Q},\ \tau,\ \zeta_p-\bar{\zeta}_p\right)$, where τ is the complex conjugation, is a CM-algebra. Moreover its canonical Riemann β form is given by

$$\beta(\underline{x},\underline{y}) = -2p\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+\cdots+(x_{p-2}y_{p-1}-x_{p-1}y_{p-2})\}.$$

Furthermore, β is C_p-invariant.

proof: The first statement is quite clear by simply going through the definition and checking its validity. To find the canonical Riemann form we have to calculate

$$\beta(\underline{x},\underline{y}) = \mathrm{Tr}_{\mathbb{Q}}\left(\alpha\cdot(\underline{y}\ \tau(\underline{x})-\underline{x}\ \tau(\underline{y}))\right)$$

where $\alpha = \zeta_p-\bar{\zeta}_p$. We regard $\mathbb{Q}(\zeta_p)$ as a vector space over $\mathbb{Q}$ with a basis $\{\zeta_p{}^1,\ldots,\zeta_p{}^{p-1}\}$. With respect to this basis

$$\underline{x} = x_1\zeta_p{}^1+\cdots+x_{p-1}\zeta_p{}^{p-1} \quad\text{and}\quad \underline{y} = y_1\zeta_p{}^1+\cdots+y_{p-1}\zeta_p{}^{p-1}, \quad\text{also}$$

$$\tau(\underline{x}) = x_{p-1}\zeta_p{}^1+\cdots+x_1\zeta_p{}^{p-1} \quad\text{and}\quad \tau(\underline{y}) = y_{p-1}\zeta_p{}^1+\cdots+y_1\zeta_p{}^{p-1}.$$

The following lemmata and remarks are going to be of great help.

Lemma 1: $\mathrm{Tr}_{\mathbb{Q}}(f_1\zeta_p{}^1+\cdots+f_{p-1}\zeta_p{}^{p-1}) = -\sum_{i=1}^{p-1} f_i$

Notice also that

$$(\zeta_p-\overline{\zeta_p})\cdot(k_1\zeta_p{}^1+\cdots+k_{p-1}\zeta_p{}^{p-1}) = (-k_2+k_1-k_{p-1})\zeta_p$$

$$+ (k_1-k_3+k_1-k_{p-1})\zeta_p{}^2 +\cdots$$

$$+ (k_{p-3}-k_{p-1}+k_1-k_{p-1})\zeta_p{}^{p-2}$$

$$+ (k_{p-2}+k_1-k_{p-1})\zeta_p{}^{p-1}.$$

Lemma 2: Let l_1 and l_{p-1} denote the coefficients of $\underline{y}\ \tau(\underline{x})-\underline{x}\ \tau(\underline{y})$ when expanded in terms of the above $\mathbb{Q}$ basis for $\mathbb{Q}(\zeta_p)$, then $l_1=-l_{p-1}$.

Combining the three remarks above, one can easily see that we only need to calculate the coefficient of ζ_p in $\underline{y}\ \tau(\underline{x})-\underline{x}\ \tau(\underline{y})$ for if

$$\underline{y}\ \tau(\underline{x})-\underline{x}\ \tau(\underline{y}) = l_1\zeta_p{}^1+\cdots+l_{p-1}\zeta_p{}^{p-1}$$

$$\mathrm{Tr}_{\mathbb{Q}}\Big(\alpha\cdot(\underline{y}\ \tau(\underline{x})-\underline{x}\ \tau(\underline{y}))\Big) = -\Big((-l_2+l_1-l_{p-1}) + (l_1-l_3+l_1-l_{p-1}) +\cdots$$

$$+ (l_{p-3}-l_{p-1}+l_1-l_{p-1}) + (l_{p-2}+l_1-l_{p-1})\Big)$$

$$= -p(l_1-l_{p-1}) = -2pl_1.$$

We, now, do so:

$$\underline{y}\cdot\tau(\underline{x}) = \sum_{i,j} (y_i\zeta_p{}^i)\,(x_j\zeta_p{}^{p-j}) = \sum_{i,j} (y_ix_j)\zeta_p{}^{p-j+i} \quad \text{where } 1 \le i,j \le p-1.$$

It is clear

$$(*): \qquad \underline{y}\ \tau(\underline{x})-\underline{x}\ \tau(\underline{y}) = \sum_{i,j} (y_ix_j - x_iy_j)\zeta_p{}^{p-j+i}\,.$$

Thus

$$l_1 = \sum_{j=1}^{p-2} (y_{j+1}x_j - x_{j+1}y_j)\,.$$

proof of lemma 1: We need to look at the action of $f \equiv f_1\zeta_p{}^1+\cdots+f_{p-1}\zeta_p{}^{p-1}$ on the random element $\zeta_p{}^i$ of the above specified $\mathbb{Q}$ basis for $\mathbb{Q}[\zeta_p]$.

$$f \cdot \zeta_p{}^i = f_{p-i} + \sum_{\substack{j=1 \\ j \neq i}}^{p-1} f_j \zeta_p{}^{p-j} = -f_{p-i}\zeta_p{}^i + \sum_{\substack{j=1 \\ j \neq i}}^{p-1} (f_j - f_{p-i})\zeta_p{}^{p-j}.$$

It is clear now that $\mathrm{Tr}_{\mathbb{Q}}(f_1\zeta_p{}^1+\cdots+f_{p-1}\zeta_p{}^{p-1}) = -\sum_{i=1}^{p-1} f_i$.

proof of lemma 2: $l_1 = -l_{p-1}$ is an easy consequence of $(*)$.

To complete the proof of the proposition, we only need to show now the C_p-invariance of β . But,

$$\beta(\zeta_p\, \underline{x}, \zeta_p\, \underline{y}) = \beta(\underline{x},\underline{y}) = \mathrm{Tr}_{\mathbb{Q}}\Big(\alpha \cdot (\underline{y}\ \tau(\underline{x}) - \underline{x}\ \tau(\underline{y}))\Big)$$

$$= \mathrm{Tr}_{\mathbb{Q}}\Big(\big(\alpha \cdot (\underline{y}\ \tau(\underline{x}) - \underline{x}\ \tau(\underline{y}))\big)\big(\zeta_p\ \tau(\zeta_p)\big)\Big)$$

$$= \mathrm{Tr}_{\mathbb{Q}}\Big(\alpha \cdot (\underline{y}\ \tau(\underline{x}) - \underline{x}\ \tau(\underline{y}))\Big) = \beta(\underline{x},\underline{y}) ,$$

and this completes the proof of Proposition (5.1.8).

We recall that the irreducible $\mathbb{Z}[C_p]$-modules, according to the Reiner-Diederichsen theorem, are either $\mathbb{Z}$ with the trivial C_p-action or a fractional ideal in the cyclotomic field of fractions $\mathbb{Q}(\zeta_p)$, say A, with the obvious action, or even the more complicated ones (A,α) where A a fractional ideal as above and $\alpha \in A$ but $\alpha \notin (\zeta_p - 1)\cdot A$ and the action is given by $g\cdot(x, n) = (\zeta_p x + n\alpha, n)$, where $\langle g \rangle = C_p$. We proceed by considering these irreducible blocks separately.

Proposition (5.1.9): If A is a fractional ideal in $\mathbb{Q}(\zeta_p)$, and $\mathbb{Z}(\zeta_p)$ is a principal ideal domain, there always exists a basis $\{u_1, \ldots ,u_{p-1}\}$ of A over $\mathbb{Z}$ such that the integral skew-symmetric form defined, with respect to this basis, by

$$\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & & & \\ 0 & 1 & 0 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 0 & -1 \\ 0 & & & & 1 & 0 \end{bmatrix}$$

is a C_p-invariant Riemann form on (A,τ) , where τ is the the complex conjugation.

<u>*proof:*</u> Because the class group is trivial in a principal ideal domain $A \simeq q\mathbb{Z}(\zeta_p)$, where $q \in \mathbb{Q}(\zeta_p)$. We can therefore take as basis for A over $\mathbb{Z}$ the set $\{ q\zeta_p, \dots ,q\zeta_p^{p-1} \}$ and let β be the skew-form defined on A, with respect to this basis, by the above matrix. Proposition (5.1.8) says that the matrix representing β above commutes with the matrix

$$g := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 0 & -1 \\ 0 & & & & 1 & -1 \end{bmatrix}$$

which gives the C_p-action on A with respect to the above choice of basis. This settles the C_p-invariance of β . Notice futher, that the complex structure t commutes with g and so does any rational polynomial expression of g . The invariance of β under the complex structure follows by

$$\beta(t(qx),t(qy)) = \beta(qtx),qty)) = \frac{1}{2p}\beta(tx,ty) = \frac{1}{2p}\beta(x,y) = \beta(qx,qy) \ .$$

The third condition of the definition of the Riemann form follows in a similar way.

<u>*Remark (5.1.10):*</u> In fact one can easily see that the form β of

Proposition (5.1.9) is equivalent over $\mathbb{Q}$ to the form β of Proposition (5.1.8).

The next proposition tackles the irreducible C_p-modules of a projective type (those of second kind in Reiner's theorem) suitably modified to satisfy the dimension requirements for Λ.

Proposition (5.1.11): If A is a fractional ideal in $\mathbb{Q}(\zeta_p)$ and $\mathbb{Z}(\zeta_p)$ is a principal ideal domain, then there is always a basis $\{u_1, \dots ,u_{p-1},u_p,u_{p+1}\}$ of $(A,\alpha)\oplus\mathbb{Z}$ as a free abelian group such that the integral skew-symmetric form defined, with respect to this basis, by

$$\begin{bmatrix} 0 & -p & 0 & \cdots & 0 & -pM_1 & 0 \\ p & 0 & -p & & & pM_2 & \\ 0 & p & 0 & & & & \vdots \\ \vdots & & & \ddots & & & \\ 0 & & & & 0 & pM_{p-1} & 0 \\ pM_1 & -pM_2 & & & -pM_{p-1} & 0 & -1 \\ 0 & & \cdots & & 0 & 1 & 0 \end{bmatrix}$$

is C_p-invariant, Riemann form for the Riemann algebra $((A,\alpha)\oplus\mathbb{Z},\tau)$, where τ is complex conjugation. Here $\mathbb{Z}$ is regarded as a C_p-module with the trivial action and $\alpha = a_1u_1+\cdots+a_pu_{p-1} \in A$ but $\notin (\zeta_p-1)\cdot A$. Furthermore

$M_1 = X_2$, $M_i = X_{i-1}-X_{i+1}$ for $i = 2, \dots ,p-2$, and $M_{p-1} = X_{p-1}$, where

$$X_i = \frac{p(a_1+\cdots+a_i)-i(a_1+\cdots+a_{p-1})}{p}.$$

proof: Since the ideal class group is trivial, any fractional ideal A in

$\mathbb{Q}(\zeta_p)$ is of the form $A = q \cdot \mathbb{Z}(\zeta_p)$, where $q = q_1\zeta_p + \cdots + q_p\zeta_p^{p-1} \in \mathbb{Z}(\zeta_p)$.
We shall denote by 11 a specific generator for a copy of $\mathbb{Z}$ in $(A,\alpha) \simeq \mathbb{Z}(\zeta_p) \oplus \mathbb{Z}$ (abelian group isomorphism), and by 111 a generator of $\mathbb{Z}$ in $(A,\alpha) \oplus \mathbb{Z}$. Let our $\mathbb{Z}$-basis for $(A,\alpha) \oplus \mathbb{Z}$ be the following

$u_i = q\zeta_p^i$, for $i = 1,\ldots,p-1$; and $u_p = 11$, while $u_{p+1} = 111$.

With respect to this $\mathbb{Z}$-basis, it is immediate to observe that the C_p-action on $(A,\alpha) \oplus \mathbb{Z}$ is given by a matrix of the form

$$\begin{bmatrix} \mathfrak{C} & {}^t\underset{\sim}{0} \\ \underset{\sim}{0} & 1 \end{bmatrix},$$

where $\underset{\sim}{0} = (0,\ldots,0) \in GL_{\mathbb{Z}}[1 \times (p-1)]$ and $\mathfrak{C}$ is the matrix describing the C_p-action on (A,α). Thus, keeping the notation as above, with respect to the set of generators $\{u_1, \ldots ,u_{p-1},u_p\}$ for (A,α) over $\mathbb{Z}$

$$\mathfrak{C} := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 & a_1 \\ 1 & 0 & 0 & & & & a_2 \\ 0 & 1 & 0 & & & & \\ & & & \ddots & & & \vdots \\ \vdots & & & & 0 & -1 & a_{p-2} \\ & & & & 1 & -1 & a_{p-1} \\ 0 & & \cdots & & & 0 & 1 \end{bmatrix},$$

where $\alpha = a_1\zeta_p + \cdots + a_{p-1}\zeta_p^{p-1}$.
This same collection of elements $\{u_1, \ldots ,u_{p-1},u_p,u_{p+1}\}$ of $(A,\alpha) \oplus \mathbb{Z}$ as defined above can be thought as a $\mathbb{Q}$-basis for the extended module $\big((A,\alpha) \oplus \mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{Q}$ with coefficients in $\mathbb{Q}$. Although easy, it is instructive to find an explicit $\mathbb{Q}$-basis of the extended module so that the isomorphism $\big((A,\alpha) \oplus \mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}(\zeta_p) \oplus \mathbb{Q} \oplus \mathbb{Q}$ is completely materialized. To this end, it suffices to find a fixed element of $(A,\alpha) \otimes_{\mathbb{Z}} \mathbb{Q}$ under the action given by $\mathfrak{C}$, above. Direct calculation shows that the space fixed under $\mathfrak{C}$ is 1-dimensional and given with respect to $\{u_1, \ldots ,u_{p-1},u_p\}$ by $\mathfrak{F} = \big\{(x_1, \ldots ,x_{p-1},x_p)\big\}$

where $x_i = \frac{p(a_1+\cdots+a_i)-i(a_1+\cdots+a_{p-1})}{p} \cdot x_p$, i=1,...,p−1, $x_p \in \mathbb{Q}$.

So, we may as well take $x_p = 1$, to get the following explicit $\mathbb{Q}$-basis $\mathcal{A}$ of $\left((A,\alpha) \oplus \mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}(\zeta_p) \oplus \mathbb{Q} \oplus \mathbb{Q}$

$$\mathcal{A} := \left\{ q\zeta_p, \dots, q\zeta_p^{p-1}, \sum_{i=1}^{p-1} \frac{p(a_1+\cdots+a_i)-i(a_1+\cdots+a_{p-1})}{p} q\zeta_p^i + 11, 111 \right\}.$$

Let β_q (dependence on q) denote the form defined by the matrix in the statement of this proposition with respect to the $\mathbb{Z}$-basis $\{u_1, \dots ,u_{p-1},u_p,u_{p+1}\}$ defined previously in the proof. This same integral matrix also represents the rationalization, say ${}_{\mathbb{Q}}\beta_q$, of β_q on $\mathbb{Q}(\zeta_p) \oplus \mathbb{Q} \oplus \mathbb{Q}$ with respect to the same basis $\{u_1, \dots ,u_{p-1},u_p,u_{p+1}\}$, now regarded as a $\mathbb{Q}$-basis of $\mathbb{Q}(\zeta_p) \oplus \mathbb{Q} \oplus \mathbb{Q}$. It is then straightforward to see that if we change the basis of $\mathbb{Q}(\zeta_p) \oplus \mathbb{Q} \oplus \mathbb{Q}$ from $\{u_1, \dots ,u_{p-1},u_p,u_{p+1}\}$ to $\mathcal{A}$ the matrix representing ${}_{\mathbb{Q}}\beta_q$ is nothing else but

$$\begin{bmatrix} 0 & -p & 0 & \cdots & 0 & 0 & 0 & 0 \\ p & 0 & -p & & & & & \\ 0 & p & 0 & & & & & \vdots \\ \vdots & & & \ddots & & & & \\ 0 & & & & 0 & -p & & \\ 0 & & & & p & 0 & & 0 \\ 0 & & & & & & 0 & -1 \\ 0 & & \cdots & & & 0 & 1 & 0 \end{bmatrix}.$$

But then

$${}_{\mathbb{Q}}\beta_q(u_i,u_j) = \begin{cases} \frac{1}{2p}\beta(\zeta_p{}^i,\zeta_p{}^j) \ , & \text{for } i,j = 1,\dots,p-1 \ , \\ -1 \text{ and } 1 \ , & \text{for } (u_p,u_{p+1}) \text{ and } (u_{p+1},u_p) \text{ respectively,} \\ 0 \ , & \text{everywhere else.} \end{cases}$$

which is C_p-invariant, Riemann form on $\mathbb{Q}(\zeta_p) \oplus \mathbb{Q} \oplus \mathbb{Q}$ by Proposition (5.1.8), Theorem (2.3.6) and arguments identical as in the proof of Proposition (5.1.9).

It is not very difficult to observe that we can "normalize" the Riemann form and therefore find its elementary divisors by choosing a suitable $\mathbb{Z}$-basis for our lattice .

Thus, if our lattice Λ is an irreducible C_p-module consisting of a fractional ideal A as above then we can choose as a new $\mathbb{Z}$-basis the set

$$\{u_{p-2}, u_{p-4}, \cdots, u_1, u_{p-1}, u_{p-1}+u_{p-3}, u_{p-1}+u_{p-3}+\cdots+u_2\},$$

where u_i for i=1,...,p-1 are as in the Proposition (5.1.9). It is straightforward to see that the Riemann form β is represented , with respect to this basis, by a matrix of the form

$$\begin{bmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{bmatrix}, \text{ with } \Delta_\delta = \begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{bmatrix}$$

where Δ_δ is a $\frac{p-1}{2} \times \frac{p-1}{2}$ integral diagonal matrix.

In the case where our lattice Λ consists of the direct sum of one of the more complicated irreducible C_p-modules along with a trivial copy of $\mathbb{Z}$, namely $(A, \alpha) \oplus \mathbb{Z}$, we choose as a new $\mathbb{Z}$-basis the set

$$\{u_p, u_{p-2}-2M_{p-2}u_{p+1}, u_{p-4}-2M_{p-4}u_{p+1}, \cdots, u_3-2M_3u_{p+1},$$

$$u_1+2M_1u_{p+1}, u_{p+1}, u_{p-1}-2M_{p-1}u_{p+1},$$

$$u_{p-1}+u_{p-3}-2(M_{p-1}+M_{p-3})u_{p+1},$$

$$u_{p-1}+u_{p-3}+\cdots+u_2-2(M_{p-1}+M_{p-3}+\cdots+M_2)u_{p+1}\} .$$

The notation here again is consistent with the Proposition (5.1.11). Again it only requires a simple calculation to see that the Riemann form β is represented , with respect to this basis, by a matrix of the form

$$\begin{bmatrix} 0 & \Delta'_\delta \\ -\Delta'_\delta & 0 \end{bmatrix}, \text{ with } \Delta'_\delta = \begin{bmatrix} -1 & & & 0 \\ & -p & & \\ & & \ddots & \\ 0 & & & -p \end{bmatrix},$$

where Δ'_δ is a $\frac{p+1}{2} \times \frac{p+1}{2}$ integral diagonal matrix.

It is now obvious that if $\Lambda = \mathbb{Z}^s \oplus A^n \oplus (A,\alpha_1) \oplus \cdots \oplus (A,\alpha_r)$ where A is a fractional ideal in $\mathbb{Q}(\zeta_p)$ and r,s,n as in Corollary (5.1.4) we can find a unitary transformation to modify the $\mathbb{Z}$-base of the lattice Λ so that the direct sum of the Riemann forms of Propositions (5.1.9) (5.1.11) as in Theorem (2.3.6) is represented by a matrix of the form

$$\begin{bmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{bmatrix},$$

with

$$\Delta_\delta = \begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -p \end{bmatrix},$$

a $c \times c$ matrix, $c = \frac{s-r}{2} + n\frac{p-1}{2} + r\frac{p+1}{2}$ and the number of -1's is equal to $\frac{s-r}{2} + n\frac{p-1}{2} + r$ while the number of $-p$'s is equal to $r\frac{p-1}{2}$.

Corollary (4.2.15) tells us that if H is the C_p-invariant hermitian form corresponding to the Riemann form defined above on Λ , then $L^{\otimes 3}$ is very ample, where $L(H,\psi)$ is as in Corollary (4.2.15). Furthermore, the comments preceding Lemma (3.6.3) and Remark (3.6.7) provides a basis for the space of theta functions $\mathbf{Th}(3H,\psi^3)$ whose dimension is given by the Frobenius Theorem (3.6.4) to be three times the pfaffian of the above Riemann form on Λ .

We have thus proved

Proposition (5.1.12): Let p be a prime such that $\mathbb{Z}(\zeta_p)$ is a principal ideal domain. Let further, (s, r, n; $[\alpha]$) be a quadruple, with the integers s, r, n and the fractional ideal α are as in Corollary (5.1.4), classifying a flat complex projective C_p-manifold M . Then,

(i) M is embeddable in $\mathbb{CP}(3^{\frac{s-r}{2}+n\frac{p-1}{2}+r\frac{p+1}{2}}\,p^{r\frac{p-1}{2}})$ and

(ii) the set of theta functions $\{\phi(z,T,j)\}_j$,

$$\phi(z,T,j) = \exp\{\tfrac{1}{2}\,{}^t z(T-\overline{T})^{-1}z\} \cdot \sum_{m\,\in\,\mathbb{Z}^n} \exp\{\tfrac{1}{2}\,{}^t(j+m)T(j+m) + {}^t(j+m)z\}$$

j ranges over a complete system of representatives for $\Delta_\delta{}^{-1}\mathbb{Z}^n \,/\, \mathbb{Z}^n$, and T is the period matrix $(V \equiv \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \,,\, \Lambda)$ with respect to the Frobenius decomposition of Λ above provides a basis for the corresponding space of theta functions.

§ 5.2 Projective Flat Manifolds of Dihedral Holonomy.

We first recall the integral representation theory of the dihedral group D_{2p} , where p a prime number. This was done by Lee in [L3].

A. Integral Representations of the Dihedral Group of Order 2p.

The dihedral group of order 2p has the following presentation:

$$D_{2p} = [\, f , g : \; f^2 = g^p = 1 \, , \, gfg = f \,] \, .$$

Although the first example of a dihedral group is the Klein 4-group, we do not discuss it here because by Theorem (2.1.7) one has an infinite number of indecomposable integral representations, described however in [Na]. For p odd prime, according to [L3], one has essentially ten different types of indecomposable $\mathbb{Z}[D_{2p}]$-modules.

Let ζ_p denote a pth primitive root of unity over $\mathbb{Q}$, and let $K \equiv \mathbb{Q}(\zeta_p)$. We also let R denote the ring of all algebraic integers in K , $R_0 \equiv \mathbb{Z}[\zeta_p+\bar{\zeta_p}]$, $C_0 \equiv C(\mathbb{Q}(\zeta_p+\bar{\zeta_p}))$ the ideal class group of $\mathbb{Q}(\zeta_p+\bar{\zeta_p})$ and $A_1,\dots,A_n$ a full set of representatives for these ideal classes, n being the ideal class number. As in §5.1.A, we shall consider elements $a_i \in A_i$, but $a_i \notin (1-\zeta_p)A_iR$. The following list describes all ten types, $\{M_1,\dots,M_{10}\}$, of indecomposable $\mathbb{Z}[D_{2p}]$-modules:

$M_1 \simeq_{\mathbb{Z}} \mathbb{Z}$	f and g act trivially;
$M_2 \simeq_{\mathbb{Z}} \mathbb{Z}$	$f\cdot k = -k$, $g\cdot k = k$;
$M_3 \simeq_{\mathbb{Z}} \mathbb{Z}\oplus\mathbb{Z}$	$f\cdot(k_1,k_2) = (k_2,k_1)$, $g\cdot(k_1,k_2) = (k_1,k_2)$;
$M_4^i \simeq_{\mathbb{Z}} A_iR$	$f\cdot x = \bar{x}$, $g\cdot x = \zeta_p x$;
$M_5^i \simeq_{\mathbb{Z}} A_iR$	$f\cdot x = -\bar{x}$, $g\cdot x = \zeta_p x$;

$M_6^i \simeq_{\mathbb{Z}} A_iR \oplus \mathbb{Z}$ $\quad f\cdot(x,k) = (-\bar{x}+ka_i,k)$, $g\cdot(x,k) = (\zeta_p x+ka_i,k)$;

$M_7^i \simeq_{\mathbb{Z}} A_iR \oplus M_2$ $\quad f\cdot(x,k) = (\bar{x}-ka_i,-k)$, $g\cdot(x,k) = (\zeta_p x+ka_i,k)$;

$M_8^i \simeq_{\mathbb{Z}} A_iR \oplus M_3$ $\quad f\cdot(x,k_1,k_2) = (\bar{x}+(k_1-k_2)a_i,k_2,k_1)$,

$g\cdot(x,k_1,k_2) = (\zeta_p x+[k_1+k_2(1-2\zeta_p)]a_i,k_1,k_2)$;

$M_9^i \simeq_{\mathbb{Z}} A_iR \oplus M_3$ $\quad f\cdot(x,k_1,k_2) = (-\bar{x}+(k_1+k_2)a_i,k_2,k_1)$,

$g\cdot(x,k_1,k_2) = (\zeta_p x+(k_1+k_2)a_i,k_1,k_2)$;

$M_{10}^i \simeq_{\mathbb{Z}} A_iR \oplus A_iR \oplus M_3$ where

$f\cdot(x,y,k_1,k_2) = (\bar{x}+(k_1-k_2)a_i,\bar{x}+(k_1+k_2)a_i,k_2,k_1)$,

$g\cdot(x,y,k_1,k_2) = (\zeta_p x+[k_1+k_2(1-2\zeta_p)]a_i,\zeta_p y+(k_1+k_2)a_i k_1,k_2)$;

i runs from 1 to n , for all types M_4 to M_{10} .

<u>*Theorem (Charlap)(5.2.1):*</u> Every $\mathbb{Z}[D_{2p}]$-module, say Λ , that is finitely generated and torsion-free as abelian group, is isomorphic to a direct sum of modules of the above ten types. The invariants determining uniquely such a decomposition are

$$\text{rank}_{\mathbb{Z}}\Lambda \ , \quad H^j(D_{2p},\Lambda) \text{ for } j=1,2,3,4 \ , \quad E(\Lambda) \text{ and } I(\Lambda) \ ,$$

where $E(\Lambda)$:= the total number of modules of type M_6 , M_8 and M_{10} that occur in the decomposition of Λ into indecomposables

and $I(\Lambda)$:= the ideal class of the product of all ideals occuring in the decompositon of Λ into indecomposables.

For a proof of the above theorem, see [C3] Theorem 4.12.

According to the scheme described in §1.3.C, to classify the flat manifolds we need faithful D_{2p}-representations. It is clear that this is the case if and only if there are indecomposables of types other than M_1 , M_2 , M_3 . We also need those cohomology 2-classes that give rise to torsion-free extensions of D_{2p} by a module Λ . To this end, we need the following well known

Proposition (5.2.2): Let K be a normal subgroup of a finite group Φ such that $(|K|,|\Phi/K|) = 1$. Let further, Λ be a $\mathbb{Z}[\Phi]$-module and denote by Λ_K its restriction to K . Then

$$H^i(\Phi,\Lambda) = [H^i(K,\Lambda_K)]^{\Phi/K} \oplus H^i(\Phi/K,[\Lambda]^K) .$$

The proof of this proposition follows from the Lyndon-Hochschild-Serre spectral sequence applied on the short exact sequence

$$0 \longrightarrow K \xrightarrow{i} \Phi \xrightarrow{\sigma} \Phi/K \longrightarrow 1 .$$

We have the following corollary of Proposition (5.2.2):

Corollary (5.2.3): If i^* and σ^* are the induced maps in cohomology, then σ^* is surjective and $\mathrm{Ker}(i^*) = H^*(\Phi/K,[\Lambda]^K)$.

The dihedral group of order 2p has a unique cyclic subgroup of order p , namely C_p , and precisely p subgroups of order 2 , all conjugate by the Sylow theorems. If Λ is a $\mathbb{Z}[D_{2p}]$-module, denote by Λ_p its restriction to C_p and by Λ_i its restriction the 2-Sylow subgroup $S_i = < fg^i >$. One can check through a simple case-by-case analysis the following

Lemma (5.2.4): $\Lambda_i \simeq \Lambda_j$ as $\mathbb{Z}[C_2]$-modules.

We recall that there is a split short exact sequence

$$0 \longrightarrow C_p \xrightarrow{i} D_{2p} \xrightarrow{\sigma} C_2 \longrightarrow 1 ,$$

Lemma (5.2.4) says that $H^2(S_i,\Lambda_i) \simeq H^2(S_j,\Lambda_j)$

and Corollary (5.2.3) that $i^* : H^2(D_{2p},\Lambda_i) \longrightarrow H^2(S_i,\Lambda_i)$.

Therefore to find special points we only need to consider the cyclic subgroups of D_{2p} , $C_2 = < f >$ and $C_p = < g >$.

B. The Classification Theorem.

Let B be a fractional ideal of K . Recall the classification of $\mathbb{Z}[C_p]$-modules as given in §5.1.A. We shall describe the idecomposables of the first type by presenting a generic representative of the ideal class group, say B , and of the second type by (B,b) where b obeies the obvious restrictions. The following table is immediate:

Table I

	M^i_j	$[M^i_j]_2$	$[M^i_j]_p$
i =	1	$\mathbb{Z}$	$\mathbb{Z}$
	2	B	$\mathbb{Z}$
	3	(B,b)	$2\mathbb{Z}$
	4	$\frac{1}{2}(p-1)(B,b)$	B
	5	$\frac{1}{2}(p-1)(B,b)$	B
	6	$\frac{1}{2}(p-1)(B,b) \oplus \mathbb{Z}$	(B,b)
	7	$\frac{1}{2}(p-1)(B,b) \oplus B$	(B,b)
	8	$\frac{1}{2}(p+1)(B,b)$	$(B,b) \oplus \mathbb{Z}$
	9	$\frac{1}{2}(p-1)(B,b)$	$(B,b) \oplus \mathbb{Z}$
	10	p(B,b)	2(B,b)

These fully determine the second cohomology

Table II

M^i_j	$H^2(D_{2p};M^i_j)$	$H^2(\mathbb{Z}_2;M^i_j)$	$H^2(\mathbb{Z}_p;M^i_j)$
i = 1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_p$
2	$\mathbb{Z}_p$	0	$\mathbb{Z}_p$
3	$\mathbb{Z}_p$	0	$\mathbb{Z}_p \oplus \mathbb{Z}_p$
4	0	0	0
5	0	0	0
6	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
7	0	0	0
8	0	0	$\mathbb{Z}_p$
9	$\mathbb{Z}_p$	0	$\mathbb{Z}_p$
10	0	0	0

From Table II , the remarks preceeding Table I and Theorem (1.3.10), we deduce

Proposition (5.2.5): $H^2(D_{2p};\Lambda)$ has special points if and only if Λ contains indecomposables of type M_1 .

To complete the classification of D_{2p}-manifoldes one needs to decide on conditions preventing us from receiving isomorphic extensions from different 2-cocycles and and different D_{2p}-modules. We do not find convenient doing this. Nevertheless, from the remarks following Theorem (1.3.10) one can state the following

Theorem (5.2.6): The class of groups which occur as the fundamental group of a D_{2p}-manifold consists precisely of those groups G that occur in extensions

$$0 \longrightarrow \Lambda \longrightarrow G \longrightarrow D_{2p} \longrightarrow 1 ,$$

where Λ contains indecomposables of type M_1 and at least one of types $M_4 , \dots , M_{10}$.

We shall now consider the tensor product of Λ with $\mathbb{Q}$. We do this in Table III, but before let us mention here that by Corollary 1 on page 103 of [S], there are only three types of $\mathbb{Q}[D_{2p}]$-modules, two 1-dimensional, the trivial and non-trivial one, and a (p−1)-dimensional one, $\mathbb{Q}(\zeta_p)$. It is not difficult to verify the following table

Table III

	M^i_j	$M^i_j \otimes_{\mathbb{Z}} \mathbb{Q}$
i =	1	$\mathbb{Q}$
	2	$\mathbb{Q}'$
	3	$\mathbb{Q} \oplus \mathbb{Q}'$
	4	$\mathbb{Q}(\zeta_p)$
	5	$\mathbb{Q}(\zeta_p)$
	6	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)$
	7	$\mathbb{Q}' \oplus \mathbb{Q}(\zeta_p)$
	8	$\mathbb{Q} \oplus \mathbb{Q}' \oplus \mathbb{Q}(\zeta_p)$
	9	$\mathbb{Q} \oplus \mathbb{Q}' \oplus \mathbb{Q}(\zeta_p)$
	10	$\mathbb{Q}[D_{2p}]$

where $\mathbb{Q}$ ($\mathbb{Q}'$) is the trivial (non-trivial) 1-dimensional $\mathbb{Q}[D_{2p}]$-module. It is clear from Corollary (2.1.3) that

$$\mathbb{Q}[D_{2p}] \simeq \mathbb{Q} \oplus \mathbb{Q}' \oplus \mathbb{Q}(\zeta_p) \oplus \mathbb{Q}(\zeta_p) .$$

Also

$$\mathrm{End}_{\mathbb{Q}[D_{2p}]}(\mathbb{Q}') \simeq \mathbb{Q} \text{ and } \mathrm{End}_{\mathbb{Q}[D_{2p}]}(\mathbb{Q}(\zeta_p)) \simeq \mathbb{Q}(\zeta_p+\bar{\zeta}_p) .$$

These are of type I in Albert's classification. Let s_i denote the number of indecomposables of type i , i = 1,...,10 in the decomposition of Λ according to Theorem (5.2.1). Combining Theorems (2.3.6), (5.2.6) and Table III, we have

Corollary (5.2.7): The class of groups which occur as the fundamental group of a projective D_{2p}-manifold consists precisely of those groups G that occur in extensions

$$0 \longrightarrow \Lambda \longrightarrow G \longrightarrow D_{2p} \longrightarrow 1 ,$$

where $s_1>0$, $s_4s_5s_6s_7s_8s_9s_{10} \neq 0$ and $s_1+s_3+s_6+s_8+s_9+s_{10}$ and $s_2+s_3+s_7+s_8+s_9+s_{10}$ are even.

C. Estimating the Size of the Set of Positive Line Bundles of a D_{2p}-manifold.

We follow a similar approach as in §5.1.C. Once again by Corollary (4.2.14) we know that

$$N = \mathrm{rank}_{\mathbb{Z}}[\mathrm{NS}(\hat{M})]^{\Phi} \leq \mathrm{rank}_{\mathbb{Z}}[H^2(\hat{M};\mathbb{Z})]^{\Phi} = \dim_{\mathbb{Q}}[H^2(\Lambda;\mathbb{Q})]^{\Phi}$$

where the notation is being kept as in chapter IV and $\Lambda \equiv \mathbb{Z}^{2m}$. Applying Corollaries (3.2.7), (3.2.8) we have that

$$\dim_{\mathbb{Q}} H^1(\Lambda;\mathbb{Q}) = 2m \quad \text{and} \quad \dim_{\mathbb{Q}} H^2(\Lambda;\mathbb{Q}) = \overset{2}{\wedge}\ H^1(\Lambda;\mathbb{Q}) = m(2m-1).$$

If Λ admits a decomposition as in Theorem (5.2.1), by use of theAdjoint Isomorphism Theorem, see [R3] page 37 Theorem 2.11, and Table III

(5.2.8) $$H^1(\Lambda;\mathbb{Q}) = \Lambda \otimes \mathbb{Q} = \mathbb{Q}^r \oplus (\mathbb{Q}')^q \oplus \mathbb{Q}(\zeta_p)^n$$

where $2m = r+q+(p-1)n$, $r = s_1+s_3+s_6+s_8+s_9+s_{10}$, $q = s_2+s_3+s_7+s_8+s_9+s_{10}$, $n = s_4+s_5+s_6+s_7+s_8+s_9+2s_{10}$ and the s_i's are as in Corollary (5.2.7).

We now need

Lemma (5.2.9): Let N be a $\mathbb{Q}[D_{2p}]$-module. Assume that $N \simeq \mathbb{Q}^\mu \oplus (\mathbb{Q}')^k \oplus V^\nu$, as $\mathbb{Q}[D_{2p}]$-modules; where $V \equiv \mathbb{Q}(\zeta_p)$ is the (p−1)-dimensional $\mathbb{Q}[D_{2p}]$-irreducible. Then

$$\wedge^2(N) \simeq \mathbb{Q}^{\binom{\mu}{2} + \frac{k(k-1)}{2}} \oplus (\mathbb{Q}')^{\nu\mu} \oplus V^{\nu(\mu+k)} \oplus (\wedge^2(V))^\nu \oplus (V \otimes V)^{\frac{\nu(\nu-1)}{2}}.$$

proof:

$$\begin{aligned} \wedge^2((\mathbb{Q}^\mu \oplus (\mathbb{Q}')^k) \oplus V^\nu) &= \wedge^2((\mathbb{Q}^\mu \oplus (\mathbb{Q}')^k)) \otimes \wedge^0(V^\nu) \\ &\oplus \wedge^1((\mathbb{Q}^\mu \oplus (\mathbb{Q}')^k)) \otimes \wedge^1(V^\nu) \\ &\oplus \wedge^0((\mathbb{Q}^\mu \oplus (\mathbb{Q}')^k)) \otimes \wedge^2(V^\nu) \\ &= \wedge^2((\mathbb{Q}^\mu \oplus (\mathbb{Q}')^k)) \oplus V^{\nu(\mu+k)} \oplus \wedge^2(V^\nu). \end{aligned}$$

From the proof of Lemma (5.1.5), we know

$$\wedge^2(V^\nu) = (\wedge^2(V))^\nu \oplus (V \otimes V)^{\frac{\nu(\nu-1)}{2}}.$$

We further need the decomposition of $\wedge^2((\mathbb{Q}^\mu \oplus (\mathbb{Q}')^k))$. Notice that because D_{2p} acts diagonally on the tensor product

$$\mathbb{Q} \otimes \mathbb{Q}' \simeq \mathbb{Q}' \quad \text{and} \quad \mathbb{Q}' \otimes \mathrm{V} \simeq \mathrm{V}\ , \ \text{so}$$

$$\begin{aligned} \wedge^2((\mathbb{Q}^\mu \oplus (\mathbb{Q}')^k)) &= \wedge^2(\mathbb{Q}^\mu) \otimes \wedge^0((\mathbb{Q}')^k)) \\ &\oplus \wedge^1(\mathbb{Q}^\mu) \otimes \wedge^1((\mathbb{Q}')^k)) \oplus \wedge^0(\mathbb{Q}^\mu) \otimes \wedge^2((\mathbb{Q}')^k)) \\ &= \mathbb{Q}^{\binom{\mu}{2}} \oplus (\mathbb{Q}')^{\nu\mu} \oplus \wedge^2((\mathbb{Q}')^k)) \ . \end{aligned}$$

As above

$$\wedge^2((\mathbb{Q}')^k)) = (\wedge^2(\mathbb{Q}'))^k \oplus (\mathbb{Q}' \otimes \mathbb{Q}')^{\frac{k(k-1)}{2}} = \mathbb{Q}^{\frac{k(k-1)}{2}}$$

for $\mathbb{Q}'$ is 1-dimensional and $\mathbb{Q}' \otimes \mathbb{Q}' \simeq \mathbb{Q}$.

To describe the action of D_{2p} on $\wedge^2(N)$ we now need to find the $\mathbb{Q}[D_{2p}]$-module structure of $V \otimes V$ and $\wedge^2(V)$, V is as above.

<u>*Lemma (5.2.10):*</u> Let V denote the (p–1)-dimensional irreducible $\mathbb{Q}[D_{2p}]$-module. Then (as $\mathbb{Q}[D_{2p}]$-modules):

i) $V \otimes V \simeq \mathbb{Q}^{\frac{p-1}{2}} \oplus (\mathbb{Q}')^{\frac{p-1}{2}} \oplus V^{p-2}$, and ii) $\wedge^2(V) \simeq (\mathbb{Q}')^{\frac{p-1}{2}} \oplus V^{\frac{p-3}{2}}$.

<u>*proof:*</u> First of all notice that if x is an element of the fixed C_p-subspace of $V \otimes V$ then so is $f \cdot x$, where f , g the generators of D_{2p} as in §5.2.A. This is true because $f \cdot x = (gfg) \cdot x = g \cdot (f \cdot x)$. Therefore the action of f respects the C_p-decomposition of Lemma (5.1.6). We now simly check that the nullity of f–1 when acting on $\mathbb{Q}^{p-1}$ is indeed $\frac{p-1}{2}$, and the proof follows.

For part ii), consider $\{x_i - e(x_i)\}$ as in the proof of Lemma (5.1.6). Then f acts on these in the non-trivial way, essentially because

$$\begin{aligned} f \cdot [(\zeta_p{}^i \otimes \zeta_p{}^j) - e(\zeta_p{}^i \otimes \zeta_p{}^j)] &= f \cdot [(\zeta_p{}^i \otimes \zeta_p{}^j) - \zeta_p{}^{p-i} \otimes \zeta_p{}^{p-j})] \\ &= -[(\zeta_p{}^i \otimes \zeta_p{}^j) - (\zeta_p{}^{p-i} \otimes \zeta_p{}^{p-j})] \ . \end{aligned}$$

This completes the proof of the lemma.

Putting together (5.2.8) and Lemmas (5.2.9), (5.2.10), we get

$$H^2(\Lambda;\mathbb{Q}) = \wedge^2 H^1(\Lambda;\mathbb{Q}) \simeq \mathbb{Q}^\alpha \oplus (\mathbb{Q}')^\beta \oplus V^\gamma , \text{ where}$$

$$\alpha = \binom{r}{2} + \frac{q(q-1)}{2} + \frac{n(n-1)(p-1)}{4} ,$$

$$\beta = nr + \frac{n(p-1)}{2} + \frac{n(n-1)(p-1)}{4} ,$$

$$\gamma = n(r+q) + \frac{n(p-3)}{2} + \frac{n(n-1)(p-2)}{2} .$$

We have thus proved

Proposition (5.2.11): The set of positive line bundles of a flat, complex projective D_{2p}-manifold classified by a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow G \longrightarrow D_{2p} \longrightarrow 1,$$

where Λ as in Theorem (5.2.1) is contained in a free abelian group of rank

$$\binom{r}{2} + \frac{q(q-1)}{2} + \frac{n(n-1)(p-1)}{4} ,$$

where $r = s_1+s_3+s_6+s_8+s_9+s_{10}$, $q = s_2+s_3+s_7+s_8+s_9+s_{10}$, $n = s_4+s_5+s_6+s_7+s_8+s_9+2s_{10}$.

D. Linear Systems and Projective Embeddings of D_{2p}-manifolds.

We shall use once again Corollary (4.2.15) to provide a very ample line bundle once an ample line bundle is produced. The generalized version for flat manifolds of the Appell-Humbert Theorem (4.2.13) will be of great use since to produce such a line bundle it suffices to exhibit an element of $[NS(\hat{M})]^\Phi$, where $\hat{M}$ is always the universal covering of the flat manifold M .

We recall that

$$\mathrm{End}_{\mathbb{Q}[D_{2p}]}(\mathbb{Q}') \simeq \mathbb{Q} \text{ and } \mathrm{End}_{\mathbb{Q}[D_{2p}]}(\mathbb{Q}(\zeta_p)) \simeq \mathbb{Q}(\zeta_p+\bar{\zeta}_p) .$$

These are of type I in Albert's classification. We have the following

Proposition (5.2.12): $D(\mathbb{Q}(\zeta_p)) = \mathbb{Q}(\zeta_p) \oplus \mathbb{Q}(\zeta_p)$ admits a Riemann form β over $\mathbb{Q}$, which is expressed with respect to the basis $\{(\zeta_p{}^i,0),(0,\zeta_p{}^j)\}$, i,j = 1,...,p−1, by a matrix of the form

$$\begin{bmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{bmatrix}, \text{ with } \Delta_\delta = \begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{bmatrix}.$$

Further, β is D_{2p}-invariant.

proof: This is straightforward application of the doubling construction of §2.2, as in Lemma (2.3.4). Notice that if

$$f_i(\zeta_p{}^j) = \begin{cases} 1 & \text{, if } i=j \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

is a basis for the dual space of $\mathbb{Q}(\zeta_p)$, it is easy to see that

$$\beta((\zeta_p{}^i,0),(\zeta_p{}^j,0)) = \sum_k f_k(0)f_k(\zeta_p{}^j)-f_k(\zeta_p{}^i)f_k(0) = 0$$

$$\beta((\zeta_p{}^i,0),(0,\zeta_p{}^j)) = \sum_k f_k(0)f_k(0)-f_k(\zeta_p{}^i)f_k(\zeta_p{}^j) = \begin{cases} -1 & \text{, if } i=j \\ 0 & \text{, otherwise} \end{cases}$$

$$\beta((0,\zeta_p{}^i),(0,\zeta_p{}^j)) = \sum_k f_k(\zeta_p{}^i)f_k(0)-f_k(0)f_k(\zeta_p{}^j) = 0$$

$$\beta((0,\zeta_p{}^i),(\zeta_p{}^j,0)) = \sum_k f_k(\zeta_p{}^i)f_k(\zeta_p{}^j)-f_k(0)f_k(0) = \begin{cases} 1 & \text{, if } i=j \\ 0 & \text{, otherwise} \end{cases}$$

and the proof follows now immediately. The D_{2p}-invariance is straightforward as well.

Ideally, one would now want integral reductions of this form β on each one of the ten kinds of $\mathbb{Z}[D_{2p}]$-indecomposables, as these are described in §5.2.A. We do not find this convenient so, the propositions that follow are ruther indicative of what can be done.

Proposition (5.2.13): If p is a prime number such that $\mathbb{Z}[\zeta_p+\bar{\zeta}_p]$ is a principal ideal domain, there always exists a basis $\{u_1, \dots ,u_{p-1}\}$ of M_4^i over $\mathbb{Z}$ such that the integral skew-symmetric form defined, with respect to this basis, by

$$\begin{bmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{bmatrix}, \text{ with } \Delta_\delta = \begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{bmatrix}$$

is a D_{2p}-invariant Riemann form on M_4^i .

proof: Because the class group is trivial in a principal ideal domain $A_i \simeq q_i\mathbb{Z}[\zeta_p+\bar{\zeta}_p]$, where $q_i \in \mathbb{Q}(\zeta_p+\bar{\zeta}_p)$. We can therefore take as basis for M_4^i over $\mathbb{Z}$ the set $\{ q_i\zeta_p, \dots ,q_i\zeta_p^{p-1} \}$ and let β be the skew-form defined on M_4^i, with respect to this basis, by the above matrix. The result follows now directly form the fact that f acts on the above basis in precisely the same way as on the $\mathbb{Q}$-basis $\{ \zeta_p, \dots ,\zeta_p^{p-1} \}$ of $\mathbb{Q}(\zeta_p)$ and the same exact reasoning as in the proof (5.1.9).

Corollary (4.2.15) tells us that if H is the D_{2p}-invariant hermitian form corresponding to the Riemann form defined above on Λ , then $L^{\otimes 3}$ is very ample, where $L(H,\psi)$ is as in Corollary (4.2.15). Furthermore, the comments preceding Lemma (3.6.3) and Remark (3.6.7) provide a basis for the space of theta functions $Th(3H,\psi^3)$ whose dimension is given by the Frobenius Theorem (3.6.4) to be three times the pfaffian of the above Riemann form on Λ .

We have thus proved

Proposition (5.2.14): Let p be a prime such that $\mathbb{Z}[\zeta_p+\bar{\zeta}_p]$ is a principal ideal domain. Let further, M be a complex projective flat D_{2p}-manifold classified by a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow G \longrightarrow D_{2p} \longrightarrow 1 ,$$

where $s_5^2+s_6^2+s_7^2+s_8^2+s_9^2+s_{10}^2 = 0$. Then,

(i) M is embeddable in $\mathbb{CP}(3^{\frac{s_1}{2}+\frac{s_2}{2}+s_3+s_4\frac{p-1}{2}})$ and

(ii) the set of theta functions $\{\phi(z,T,j)\}_j$,

$$\phi(z,T,j) = \exp\{\tfrac{1}{2}\,{}^t z(T-\overline{T})^{-1}z\} \cdot \sum_{m \in \mathbb{Z}^n} \exp\{\tfrac{1}{2}\,{}^t(j+m)T(j+m) + {}^t(j+m)z\}$$

j ranges over a complete system of representatives for $\Delta_\delta^{-1}\mathbb{Z}^n \,/\, \mathbb{Z}^n$, and T is the period matrix $(V \equiv \Lambda \otimes_{\mathbb{Z}} \mathbb{R}\ ,\ \Lambda)$ with respect to the Frobenius decomposition of Λ above, provides a basis for the corresponding space of theta functions.

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