Leader-following Consensus Control of a Distributed Linear Multi-agent System using a Sliding Mode Strategy*

Ye Zhang[1], Ning Zhao[1], Dongya Zhao[1], Xinggang Yan[2] and Sarah K. Spurgeon[3]

***Abstract*— A distributed leader-following consensus control framework is proposed for a linear system. The linear system is first transformed into a regular form. Then a linear sliding mode is designed to provide high robustness, and the corresponding consensus protocol is proposed in a fully distributed fashion. When matched disturbances are present, it can be demonstrated that the system states reach the sliding mode in finite time and consensus can be achieved asymptotically using Lyapunov theory and the invariant set theorem. Simulation results validate the effectiveness of the proposed algorithm.**

I. INTRODUCTION

Today is the information era, and fully exploiting available information is a key concern. As a consequence, distributed systems have been studied and developed. In a distributed system, there are several agents forming a network which allows agents to exchange information in order to achieve a given goal. This process is usually called cooperative control. Typical collective behaviors resulting from such cooperative control include synchronization [1]-[3], flocking [4]-[6], swarming [7]-[9] and consensus [10]-[12]. The consensus control of distributed multi-agent systems is a topic that has received much attention in the literature. In general, consensus control focuses on how the agents come to agreement on certain quantities using their own information together with information received from their neighbours. Distributed consensus control can be widely used in practice for diverse applications including control of motion [13], sensor networks [14], robot planning [15] and smart grids [16]. In research on distributed consensus control, the study of leader-following consensus control is of great significance. For instance, in the process of guiding missiles to hit a target, the target is taken as the leader, whereas the missiles, denoted as the followers, track the target until it is hit. In this way leader-following consensus is achieved [17].

From the viewpoint of the current literature, leader-following distributed consensus control mainly focuses on simple systems such as single and double integrator systems. Early work on the single integrator model [18]-[20] considered system states such as voltage or output power and it was concluded that consensus can be reached if the network is connected. The double integrator model [21]-[23] typically is considered to be a dynamic mechanical system, where the states are position and velocity. Corresponding distributed consensus control frameworks based on more general linear system representations are now being developed. To motivate such developments, consider distributed systems within the chemical process control industry. Here different sub-systems can have different dynamics, so it is necessary to design a framework for distributed linear systems which can represent more general classes of systems. In [24], distributed tracking control is considered for multi-agent systems with linear dynamics using two discontinuous controllers with static and adaptive coupling gains. In [25], the leader-follower consensus tracking problem is explored for linear multi-agent systems with unknown external disturbances; in this case a state observer and disturbance observer are deployed in the scheme. In the above literature, distributed consensus for general linear systems is addressed, but design issues remain. Specifically, the consensus protocols typically require the Laplacian matrix to have non-zero eigenvalues. Calculating these eigenvalues results in a heavy computational load, particularly for large scale networks, and the exact weights of the communication graph should be known to every agent [26].

Uncertainty always exists in practical systems due to unmodeled dynamics, parameter variations and external disturbances [27]. In [28], nonlinear multi-agent systems are considered where it is required that each follower node has a lower triangular structure and the developed continuous control which is a function of the system states and consensus error may be difficult to implement in practice. Any control strategy should have high robustness to counteract uncertainty whilst still being straightforward to implement. Sliding mode control is well known to possess these characteristics [29]-[31]. There are a few papers considering a sliding mode approach to the distributed consensus problem. The distributed finite-time consensus problem for second-order multi-agent systems is investigated based on integral sliding mode protocols in [32]. In [33], distributed leader-following consensus for fractional-order multi-agent systems is studied using sliding mode control. In [34], sliding mode control is designed for second order multi-agent systems where uncertainty is not considered and the nonlinear term is required to satisfy a linear growth condition. Further, a distributed tracking problem for first order systems is also considered using second order sliding mode techniques [35]. This paper will consider the distributed consensus problem for linear systems from the viewpoint of sliding mode

*This work is supported by the National Natural Science Foundation (NNSF) of China under Grant 61973315 and 61473312.

[1]Ye Zhang, Ning Zhao and Dongya Zhao are with Department of Chemical Equipment and Control Engineering, China University of Petroleum, Qingdao, China dyzhao@upc.edu.cn

[2]Xinggang Yan is with the School of Engineering & Digital Arts, University of Kent, United Kingdom x.yan@kent.ac.uk

[3]Sarah K. Spurgeon is with the Department of Electronic and Electrical Engineering, University College London, Torrington Place, London WC1E 7JE, UK s.spurgeon@ucl.ac.uk

control. A typical linear sliding surface is selected to achieve consensus asymptotically for linear systems. The sliding motion is usually divided into two phases. In the reaching phase, the system states are driven to the sliding surface, and after that, the system states converge to the equilibrium point asymptotically. The contribution of this paper is as follows: a special regular form is proposed, which makes the consensus sliding surface design feasible. Then a fully distributed leader-following control framework is developed in terms of the sliding mode control principle. Finally, the system consensus is analyzed by using the Lyapunov method and the invariant set theorem.

The rest of this paper is arranged as follows. In Section II, some preliminaries and the problem formulation are stated. In Section III and IV, the sliding mode surface and the sliding mode control are designed respectively. In Section V, simulation results and corresponding analysis are presented. Finally, the conclusions are drawn in Section VI.

II. PRELIMINARIES AND PROBLEM FORMULATION

Graph theory will be used to illustrate the communication among agents. Let $\mathscr{G}=(\mathscr{V},\mathscr{E},\mathscr{A})$ denote an undirected graph of order N consisting of a set of vertices $\mathscr{V}=\{v_1,v_2,\ldots,v_N\}$, a set of undirected edges $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$, and a weighted adjacency matrix $\mathscr{A}=(a_{ij})_{N\times N}$. An undirected edge $\mathscr{E}_{ij}$ in the undirected graph $\mathscr{G}$ is denoted by a pair of unordered vertices (v_i,v_j), which indicates that v_i and v_j can communicate with each other. The weights $a_{ij}=a_{ji}$= 1 in the weighted adjacency matrix $\mathscr{A}$ if and only if the edge (v_i,v_j) exists, and $a_{ij}=a_{ji}$=0 otherwise. Define $a_{ii}=0$ when $i=j$. A path is a sequence of connected edges in a graph, and a graph is connected if there is a path between every pair of vertices [36].

Consider a distributed multi-agent system with one leader and $N-1$ followers. The leader is labelled as 1, and the followers are labeled as $2,\cdots,N$. The leader has no incoming information from the followers, but if there exists an interconnection between the leader and the jth follower, $a_{1j}=a_{j1}=1$, where $j=2,\cdots,N$.

Remark 1. It should be noted that the case where $a_{1j}=a_{j1}=1$ will only be used in the consensus proof, and the leader is never influenced by followers in the controller implementation.

1_n and 0_n denote n-dimensional column vectors with all the entries being 1 and 0 respectively. $0_{n\times n}$ denotes an $n\times n$ square matrix with all entries being 0.

The dynamics of the agents have the following form:

$$\dot{x}_i=Ax_i+B(u_i+d_i) \tag{1}$$

where $i=1,\cdots,N$, $x_i\in R^n$ and $u_i\in R^m$ denote the state and the control input of the ith agent respectively, $d_i\in R^m$ denotes the matched disturbances and uncertainties, $A\in R^{n\times n}$ and $B\in R^{n\times m}$ are system matrices.

In this paper, for ease of exposition, it is assumed that all the agents have the same system matrix (A,B) but are subject to different disturbances. It should be pointed out that the results can be applied to agents with different system matrices by slight modification.

Assumption 1. The pair (A,B) is controllable in (1).

The system (1) can be transformed into a suitable regular form by a state transformation T, which can be obtained using the method in the Appendix.

$$z_i=T^{-1}x_i=\begin{bmatrix} z_{i1}^T & z_{i2}^T \end{bmatrix}^T \tag{2}$$

Hence the following dynamics can be obtained:

$$\begin{aligned}\dot{z}_{i1}(t)&=A_{12}z_{i2}(t)\\ \dot{z}_{i2}(t)&=A_{21}z_{i1}(t)+A_{22}z_{i2}(t)+B_2u_i(t)+\xi_i(t)\end{aligned} \tag{3}$$

where $z_{i1}(t)\in R^{n-m}$, $z_{i2}(t)\in R^m$, $u_i(t)\in R^m$, $\xi_i(t)\in R^m$, $A_{12}\in R^{(n-m)\times m}$, $A_{21}\in R^{m\times(n-m)}$, $A_{22}\in R^{m\times m}$, $B_2\in R^{m\times m}$ is nonsingular.

Assumption 2. $\xi_i(t)\in R^m$ denotes the matched disturbances and uncertainties and satisfies the following condition:

$$\left\|\dot{\xi}_i\right\|\leq\beta_i \tag{4}$$

where $\beta_i>0$ is a constant.

Assumption 3. The leader's control input is assumed to be bounded, and there exists $u_1(t)=u_1(z_{11},z_{12},t)$ driving $z_{11}(t)=\delta_1(t)$, $z_{12}(t)=\delta_2(t)$ in system (3), where $\delta_1(t)$ and $\delta_2(t)$ are functions of time. That is to say, the states of the leader, which are directly controlled by the control input $u_1(t)$, will not be influenced by the followers.

Assumption 4. The undirected graph is connected.

Lemma 1 [37]**.** Consider (1), the following statements are equivalent.

(a) The pair (A,B) is controllable.

(b) The controllability matrix $Q_c=\begin{bmatrix} B \mid AB \mid \cdots \mid A^{n-1}B\end{bmatrix}$ has rank n (full row rank).

(c) The $n\times(n+m)$ matrix $\begin{bmatrix}\lambda_lI-A & B\end{bmatrix}$ has full row rank at every eigenvalue λ_l of A.

Theorem 1. Under Assumption 4, the system (1) is controllable if and only if A_{12} in (3) is full row rank.

Proof. It can be seen that $rank\begin{bmatrix}\lambda_lI-A & B\end{bmatrix}=n$ for system (1) by Assumption 1 and Lemma 1. Because of the special structure of system (3) and using the fact that B_2 is nonsingular, it follows that

$$\begin{aligned}rank\begin{bmatrix}\lambda_lI-A & B\end{bmatrix}&=rank\begin{bmatrix}\lambda_lI & -A_{12} & 0\\ -A_{21} & \lambda_lI-A_{22} & B_2\end{bmatrix}\\ &=rank\begin{bmatrix}\lambda_lI & A_{12}\end{bmatrix}+m\end{aligned} \tag{5}$$

which indicates that

$$rank\begin{bmatrix}\lambda_lI-A & B\end{bmatrix}=n\Leftrightarrow rank\begin{bmatrix}\lambda_lI & A_{12}\end{bmatrix}=n-m \tag{6}$$

Due to the controllability properties of system (1), it can be obtained that $rank\begin{bmatrix}\lambda_lI-A & B\end{bmatrix}=n\Leftrightarrow rankQ_c=rank\begin{bmatrix}B \mid AB \mid \cdots \mid A^{n-1}B\end{bmatrix}=n$, then for system (2):

$$\begin{aligned}&rank\begin{bmatrix}\lambda_lI & A_{12}\end{bmatrix}=n-m\Leftrightarrow\\ &rankQ_c\\ &=rank\begin{bmatrix}A_{12} \mid 0_{(n-m)\times(n-m)}A_{12} \mid \cdots \mid 0_{(n-m)\times(n-m)}^{n-m-1}A_{12}\end{bmatrix}\\ &=n-m\end{aligned} \tag{7}$$

According to (7), A_{12} is row full rank. That is to say, the system (1) is controllable if and only if A_{12} is row full rank in system (3). Then $n-m \le m$ because of $A_{12} \in R^{(n-m)\times m}$.

Definition 1 [25]**.** The leader-following consensus in the distributed multi-agent system (3) is said to be achieved if for any initial conditions, $\lim_{t\to\infty} \|x_i(t)-x_1(t)\| = 0,\ i=1,2,\cdots,N$.

Remark 2. In Definition 1, $i=1$ always qualifies as a special case. This is consistently assumed in this paper unless otherwise stated.

Definition 2. $\operatorname{sgn}(.) : R^k \to R^k$ is a sign function that defined as $\operatorname{sgn}(y) = [\operatorname{sgn}(y_1), \operatorname{sgn}(y_2), \ldots, \operatorname{sgn}(y_k)]^T$, where $y = [y_1, y_2, \ldots, y_k]^T$.

Lemma 2 [38]**.** Consider the autonomous system $\dot{x} = f(x)$ with f continuous, and let $V(x)$ be a scalar function with continuous first partial derivatives. Assume that $V(x) \to \infty$ as $\|x\| \to \infty$, and $\dot{V}(x) \le 0$ over the whole state space. Let $\mathscr{R}$ be the set of all points where $\dot{V}(x) = 0$, and $\mathscr{M}$ be the largest invariant set in $\mathscr{R}$. Then all solutions globally asymptotically converge to $\mathscr{M}$ as $t \to \infty$.

Lemma 3 [39]**.** If $a_1, a_2, \cdots, a_n \ge 0$ and $0 < p < q$, then $\left(\sum\limits_{i=1}^{n} a_i^q\right)^{1/q} \le \left(\sum\limits_{i=1}^{n} a_i^p\right)^{1/p}$.

III. SLIDING MODE SURFACE DESIGN

The switching function is defined in the following form:

$$s_i(t) = \dot{z}_{i2}(t) - c\sum_{j=1}^{N} a_{ij}\left(z_{j2}(t) - z_{i2}(t) + A_{12}^T\left(z_{j1}(t) - z_{i1}(t)\right)\right) \tag{8}$$

where $c > 0$ influences the convergence rate and the amplitudes of the states and the control input. The corresponding sliding surface is

$$\left\{\left(z_{21}^T, \cdots, z_{N1}^T, z_{22}^T, \cdots, z_{N2}^T\right)^T | s_i = 0, \forall i = 2, \cdots, N\right\} \tag{9}$$

where s_i is defined in (8).

Theorem 2. Under Assumption 4, if the states in (3) can reach the sliding surface (9), then the leader-following consensus in the distributed multi-agent system (3) can be asymptotically achieved.

Proof. When the states reach the sliding mode, it can be obtained that

$$\dot{z}_{i2}(t) = c\sum_{j=1}^{N} a_{ij}\left(z_{j2}(t) - z_{i2}(t) + A_{12}^T\left(z_{j1}(t) - z_{i1}(t)\right)\right) \tag{10}$$

Combining the first equation in (3) and (10), it follows that

$$\begin{array}{l} \dot{z}_{i1}(t) = A_{12} z_{i2}(t) \\ \dot{z}_{i2}(t) = \zeta_i(t) \end{array} \tag{11}$$

where $\zeta_i(t) = c\sum\limits_{j=1}^{N} a_{ij}\left(z_{j2}(t) - z_{i2}(t) + A_{12}^T\left(z_{j1}(t) - z_{i1}(t)\right)\right)$.

The consensus problem is transformed into the following stabilisation problem.

$$\left\{\begin{array}{l} \dot{e}_i^a(t) = A_{12} e_i^b(t) \\ \dot{e}_i^b(t) = \zeta_i(t) \end{array}\right. \tag{12}$$

where $e_i^a \triangleq \left(e_{i1}^a, \ldots, e_{i,n-m}^a\right)^T = z_{i1}(t) - z_{11}(t)$, $e_i^b(t) \triangleq \left(e_{i1}^b, \ldots, e_{i,m}^b\right)^T = z_{i2}(t) - z_{12}(t)$.

Based on the definition of the error in (12), $\zeta_i(t)$ can be re-expressed as

$$\zeta_i(t) = c\sum_{j=1}^{N} a_{ij}\left(e_j^b(t) - e_i^b(t) + A_{12}^T\left(e_j^a(t) - e_i^a(t)\right)\right) \tag{13}$$

A Lyapunov candidate function is chosen as

$$V = \frac{1}{2}\sum_{i=2}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m}\int_0^{e_{ik}^a - e_{jk}^a} c a_{ij} y dy + \frac{1}{2}\sum_{i=2}^{N}\left(e_i^b\right)^T e_i^b \tag{14}$$

Note that $a_{ij} = a_{ji}$, then the derivative of $V(x)$ is

$$\begin{aligned} \dot{V} &= \frac{1}{2}\sum_{i=2}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m} c a_{ij}\left(e_{ik}^a - e_{jk}^a\right)\left(\dot{e}_{ik}^a - \dot{e}_{jk}^a\right) + \sum_{i=2}^{N}\left(e_i^b\right)^T \dot{e}_i^b \\ &= \frac{1}{2}\sum_{i=2}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m} c a_{ij}\left(\left(e_{ik}^a - e_{jk}^a\right)\dot{e}_{ik}^a + \left(e_{jk}^a - e_{ik}^a\right)\dot{e}_{jk}^a\right) \\ &\quad + \sum_{i=2}^{N}\left(e_i^b\right)^T \dot{e}_i^b \\ &= \sum_{i=2}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m} c a_{ij}\left(e_{ik}^a - e_{jk}^a\right)\dot{e}_{ik}^a + \sum_{i=2}^{N}\left(e_i^b\right)^T \dot{e}_i^b \\ &= \sum_{i=2}^{N}\left(\dot{e}_i^a\right)^T \sum_{j=1}^{N} c a_{ij}\left(e_i^a - e_j^a\right) + \sum_{i=2}^{N}\left(e_i^b\right)^T \dot{e}_i^b \\ &= \sum_{i=2}^{N}\left(A_{12} e_i^b\right)^T \sum_{j=1}^{N} c a_{ij}\left(e_i^a - e_j^a\right) \\ &\quad + \sum_{i=2}^{N}\left(e_i^b\right)^T \sum_{j=1}^{N} c a_{ij}\left(e_j^b - e_i^b + A_{12}^T\left(e_j^a - e_i^a\right)\right) \\ &= \sum_{i=2}^{N}\left(e_i^b\right)^T \sum_{j=1}^{N} c a_{ij}\left(e_j^b - e_i^b\right) \\ &= \frac{1}{2}\sum_{i=2}^{N}\sum_{j=1}^{N}\sum_{p=1}^{m} c a_{ij}\left(e_{ip}^{\mathrm{b}} - e_{jp}^{\mathrm{b}}\right)\left(e_{jp}^b - e_{ip}^b\right) \\ &= -\frac{1}{2}\sum_{i=2}^{N}\sum_{j=1}^{N}\sum_{p=1}^{m} c a_{ij}\left(e_{ip}^{\mathrm{b}} - e_{jp}^{\mathrm{b}}\right)^2 \\ &\le 0 \end{aligned} \tag{15}$$

The analysis of (15) is presented as follows:

(a) $V(t)$ is radially unbounded over e_i^a and e_i^b.

(b) Since the undirected graph is connected, if $\dot{V} = 0$, then $e_{ip}^b = e_{jp}^b$, $p = 1, \ldots, m$, $\forall i \ne j$, that is, $e_i^b = e_j^b$. As $e_1^b = 0$, it can be obtained that $e_i^b = e_j^b = 0$.

(c) In the second equation of (12), $\dot{e}_i^b(t) = \zeta_i(t)$, then $\zeta_i(t) =$

$\zeta_j(t)$, $\forall i \neq j$. Since $a_{ij} = a_{ji}$, it follows that

$$\begin{aligned}\sum_{i=2}^{N}\zeta_i(t) &= \sum_{i=2}^{N}\sum_{j=1}^{N} ca_{ij}\left(e_j^b(t) - e_i^b(t) + A_{12}^T\left(e_j^a(t) - e_i^a(t)\right)\right) \\ &= \frac{1}{2}\sum_{i=2}^{N}\sum_{j=1}^{N} ca_{ij}\left(e_j^b(t) - e_i^b(t) + e_i^b(t) - e_j^b(t)\right) \\ &+ \frac{1}{2}A_{12}^T\sum_{i=2}^{N}\sum_{j=1}^{N} ca_{ij}\left(e_j^b(t) - e_i^b(t) + e_i^b(t) - e_j^b(t)\right) \\ &= 0\end{aligned} \tag{16}$$

Thus $\zeta_i = 0$, and $\sum_{i=1}^{N}\left(A_{12}^T e_i^a\right)^T \zeta_i = 0$ accordingly. Under the condition that $e_i^b = e_j^b$, $\zeta_i(t)$ becomes $\zeta_i(t) = cA_{12}^T\sum_{j=1}^{N} a_{ij}\left(e_j^a - e_i^a\right)$, so $c\sum_{i=1}^{N}\left(A_{12}^T e_i^a\right)^T A_{12}^T\sum_{j=1}^{N} a_{ij}\left(e_j^a - e_i^a\right) = 0$, and it follows that $-\frac{c}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(A_{12}^T\left(e_i^a - e_j^a\right)\right)^T A_{12}^T\left(e_i^a - e_j^a\right) = 0$, which requires $A_{12}^T\left(e_i^a - e_j^a\right) = 0$. Since A_{12} is row full rank by Theorem 1, it can be established that A_{12}^T is column full rank. Because $A_{12}^T \in R^{m\times(n-m)}$, $\left(e_j^a - e_i^a\right) \in R^{n-m}$ and $n-m \leq m$, consider taking $n-m$ linear independent rows from A_{12}^T to form a new matrix $\overset{\frown}{A}{}_{12}^T \in R^{(n-m)\times(n-m)}$, then $\overset{\frown}{A}{}_{12}^T\left(e_j^a - e_i^a\right) = 0$. Thus it can be seen that $e_j^a - e_i^a = 0$. Therefore $A_{12}^T\left(e_j^a - e_i^a\right) = 0$ results in $e_j^a - e_i^a = 0$, which becomes $e_i^a = e_j^a$. As $e_1^a = 0$, it can be obtained that $e_i^a = e_j^a = 0$.

On the basis of the above analysis and Lemma 2, it can be seen that $\lim_{t\to\infty}\|e_i^a(t)\| = 0$, $\lim_{t\to\infty}\left\|e_i^b(t)\right\| = 0$. Finally $\lim_{t\to\infty}\|z_{i1}(t) - z_{11}(t)\| = 0$, $\lim_{t\to\infty}\|z_{i2}(t) - z_{12}(t)\| = 0$, $i = 1, 2, \cdots, N$. That is, $\lim_{t\to\infty}\|x_i(t) - x_1(t)\| = 0$, $i = 2, \cdots, N$. Under Assumption 4, when the states in (3) reach the sliding surface (9), the leader-following consensus is achieved asymptotically.

IV. SLIDING MODE CONTROL DESIGN

Let the sliding mode control law be

$$\begin{cases} u_j(t) = B_2^{-1}\left(u_{eqj}(t) + u_{nj}(t)\right) \\ u_{eqj}(t) = c\sum_{k=1}^{N} a_{jk}\left(z_{k2} - z_{j2} + A_{12}^T\left(z_{k1} - z_{j1}\right)\right) \\ \qquad -A_{21}z_{j1} - A_{22}z_{j2} \\ \dot{u}_{nj}(t) = -\eta_j \operatorname{sgn}(s_j) \end{cases} \tag{17}$$

where $j = 2, \cdots, N$, $k = 1, \cdots, N$, $\eta_j > \beta_j$.

Theorem 3. Under Assumptions 2, 3 and 4, the sliding mode control law (17) can drive the system states in (3) onto the sliding surface (9) in finite time.

Proof. Substitute the second equation of (3) into (8), then the sliding function can be represented as follows:

$$\begin{aligned} s_j &= \dot{z}_{j2} - c\sum_{k=1}^{N} a_{jk}\left(z_{k2} - z_{j2} + A_{12}^T\left(z_{k1} - z_{j1}\right)\right) \\ &= A_{21}z_{j1} + A_{22}z_{j2} + c\sum_{k=1}^{N} a_{jk}\left(z_{k2} - z_{j2} + A_{12}^T\left(z_{k1} - z_{j1}\right)\right) \\ &-A_{21}z_{j1} - A_{22}z_{j2} + u_{nj} + \xi_j - c\sum_{k=1}^{N} a_{jk}\left(z_{k2} - z_{j2} + A_{12}^T\right. \\ &\left.\left(z_{k1} - z_{j1}\right)\right) \\ &= u_{nj} + \xi_j, j = 2, \cdots, N. \end{aligned} \tag{18}$$

A Lyapunov candidate function is constructed as

$$V(t) = \frac{1}{2}\sum_{j=2}^{N}(s_j)^T s_j \tag{19}$$

Differentiating (19), combining (18) and Lemma 3 yeilds

$$\begin{aligned} \dot{V}(t) &= \sum_{j=2}^{N}(s_j)^T \dot{s}_j \\ &= \sum_{j=2}^{N}(s_j)^T\left(\dot{u}_{nj} + \dot{\xi}_j\right) \\ &= -\eta_j\sum_{j=2}^{N}(s_j)^T \operatorname{sgn}(s_j) + \sum_{j=2}^{N}(s_j)^T\dot{\xi}_j \\ &\leq \sum_{j=2}^{N}\sum_{k=1}^{m}|s_{jk}|\left(-\eta_j + \beta_j\right) \\ &\leq \theta\sqrt{2V} \end{aligned} \tag{20}$$

where $-\eta_j + \beta_j < \theta < 0$.

Therefore, the system states reach the sliding surface in finite time using the control law (17) [38].

Remark 3. It should be noted that the control law (17) is only applicable to the followers and the leaders states are not influenced by the followers.

V. SIMULATION AND ANALYSIS

Consider a distributed multi-agent system with 4 agents, whose topology connection is shown as Fig.1. Here ① represents the leader, ②③④ represent the followers. The weighted adjacency matrix can be obtained as $\mathscr{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

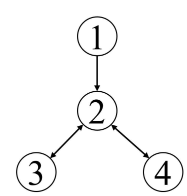

Fig. 1. The topology connection with 4 agents

The dynamics [40] of each agent is given by

$$\dot{x}_i = \begin{bmatrix} 2 & 10 & 10 \\ 1 & 4 & 5 \\ -2 & -8 & -9 \end{bmatrix} x_i + \begin{bmatrix} 2 & 16 \\ 0.5 & 2.5 \\ -1 & -7 \end{bmatrix} u_i \tag{21}$$

The corresponding dynamics in the regular form can be obtained by the state transformation $T = \begin{bmatrix} 3 & 5 & 2 \\ -1 & 0.5 & 0.5 \\ 0 & -2 & -1 \end{bmatrix}$.

$$\begin{bmatrix} \dot{z}_{i1}(t) \\ \dot{z}_{i2}(t) \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ -2 & -4 & -3 \end{array}\right] \begin{bmatrix} z_{i1}(t) \\ z_{i2}(t) \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & 2 \\ 1 & 3 \end{array}\right] u_i(t) \tag{22}$$

where the states of the leader are shown as follows: when $0 \le t \le 50$, $z_{11}(t) = t$, $z_{12}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and when $t > 50$, $z_{11}(t) = 50$, $z_{12}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The initial states of the followers are $z_{21}(0) = 2$, $z_{22}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $z_{31}(0) = 1$, $z_{32}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $z_{41}(0) = -1$, $z_{42}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $u_{n2}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $u_{n3}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $u_{n4}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $c = 1.3$, $\eta_j = 1.0$ $(j = 2,3,4)$. Disturbances $d_j = 0.5\sin(t)$ are applied to the system when $t \ge 75$.

The simulation results are shown in Figs. 2-5.

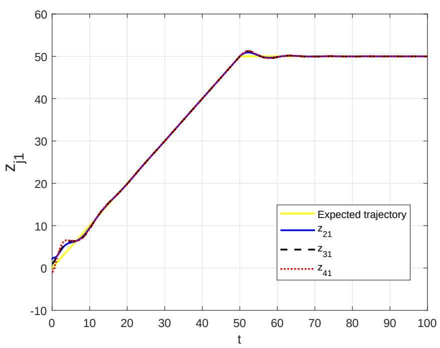

Fig. 2. The tracking trajectories z_{j1}

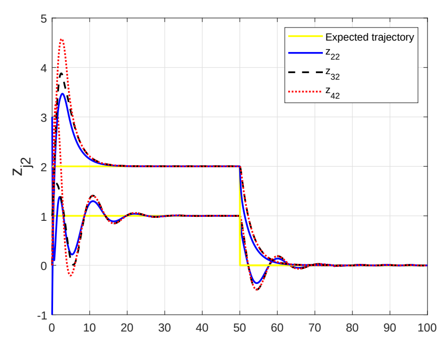

Fig. 3. The tracking trajectories z_{j2}

Figs. 2-3 show the tracking performance. It can be seen that the followers track the expected trajectory, so that leader-following consensus is achieved. The system exhibits good robustness when the disturbances are present. Fig.4 shows

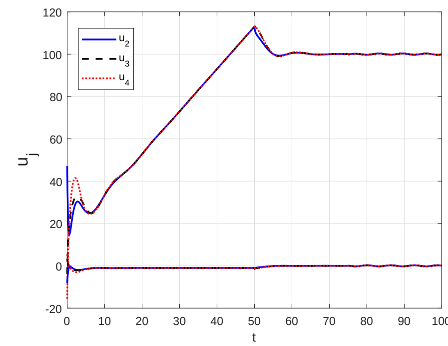

Fig. 4. The control inputs u_j

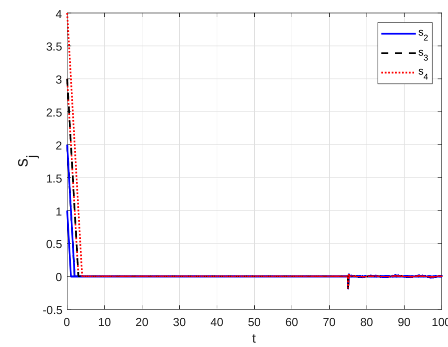

Fig. 5. The sliding modes s_j

the control inputs, which are bounded. Fig. 5 illustrates that the state errors first approach the sliding surface and then asymptotically converge to zero along it.

VI. CONCLUSIONS

In this paper leader-following consensus is achieved for a linear multi-agent system. A consensus protocol is proposed based on a sliding mode strategy. The system states first reach the sliding surface and consensus is achieved asymptotically. In future work, the directed topology graph will be introduced, and mismatched disturbances and uncertainties will also be considered.

APPENDIX

According to [38] and [40], let $B = [b_1, b_2, \ldots, b_m]$, and assume that n linear independent column vectors of the controllability matrix Q_c are $b_1, Ab_1, \ldots, A^{\upsilon_1-1}b_1; b_2, Ab_2, \ldots, A^{\upsilon_2-1}b_2; \ldots; b_l, Ab_l, \ldots, A^{\upsilon_l-1}$ b_l, where $\upsilon_1 + \upsilon_2 + \ldots + \upsilon_l = n$.

$A^{\upsilon_k-1}b_k$ can be represented as the linear combination of $\{b_1, Ab_1, \ldots, A^{\upsilon_1-1}b_1; b_2, Ab_2, \ldots, A^{\upsilon_2-1}b_2; \ldots; b_k, Ab_k, \ldots,$ $A^{\upsilon_k-1}b_k\}$, where $k = 1, 2, \ldots, l$ with $l \le m$.

Based on the linear combination given above, the corresponding bases are derived as follows.

$$A^{\upsilon_k-1}b_k = -\sum_{j=0}^{v_k-1} \alpha_{kj}A^j b_k + \sum_{i=1}^{k-1}\sum_{j=1}^{v_i} \gamma_{kji}e_{ij} \tag{23}$$

where α_{kj} and γ_{kji} are the characteristic polynomial coefficients.

Define the corresponding basis as

$$\begin{cases} e_{k1} \triangleq A^{\upsilon_k-1}b_k + \alpha_{k,\upsilon_k-1}A^{\upsilon_k-2}b_k + \ldots + \alpha_{k1}b_k \\ e_{k2} \triangleq A^{\upsilon_k-2}b_k + \alpha_{k,\upsilon_k-1}A^{\upsilon_k-3}b_k + \ldots + \alpha_{k2}b_k \\ \cdots\cdots \\ e_{k\upsilon_k} \triangleq b_k \end{cases} \tag{24}$$

Then the state transformation matrix T can be obtained as:

$$T = \underbrace{[e_{11}, e_{21}, \ldots, e_{l1}, \ldots]}_{n} \tag{25}$$

REFERENCES

[1] X. Li and G. Yang, FLS-based adaptive synchronization control of complex dynamical networks with nonlinear couplings and state-dependent uncertainties, IEEE Transactions on Cybernetics, vol. 46, January 2016, pp. 171-180.

[2] A. Ouannas, Z. Odibat, N. Shawagfeh, A. Alsaedi and B. Ahmad, Universal chaos synchronization control laws for general quadratic discrete systems, Applied Mathematical Modelling, vol. 45, 2017, pp. 636-641.

[3] L. Harnefors, M. Hinkkanen, U. Riaz, F. M. Mahafugur Rahman and L. Zhang, Robust analytic design of power-synchronization control, IEEE Transactions on Industrial Electronics, vol. 66, March 2018, pp. 5810-5819.

[4] H. Zhang, Z. Cheng, G. Chen and C. Li, Model predictive flocking control for second-order multi-agent systems with input constraints, IEEE Transactions on Circuits and Systems, vol. 62, June 2015, pp. 1599-1606.

[5] Q. Luo and H. Duan, Distributed UAV flocking control based on homing pigeon hierarchical strategies, Aerospace Science and Technology, vol. 70, 2017, pp. 257-264.

[6] W. Yuan , N. Ganganath , C. Cheng , S. Valaee, Q. Guo and F. C. M. Lau, Energy-efficient semi-flocking control of mobile sensor networks on rough terrains, IEEE Transactions on Circuits and Systems, vol. 66, April 2019, pp. 622-626.

[7] S. Chiew, W. Zhao and T. Go, Swarming coordination with robust control Lyapunov function approach, Journal of Intelligent & Robotic Systems, vol. 78, 2015, pp. 499-515.

[8] H. Liang, H. Su, X. Wang and M. Chen, Swarming of heterogeneous multi-agent systems with periodically intermittent control, Neurocomputing, vol. 207, 2016, pp. 213-219.

[9] F. Wang and H. Yang, Swarming movement of dynamical multi-agent systems with modeling errors, in 2017 IEEE Int. Conf. on Computational Science and Engineering and IEEE Int. Conf. on Embedded and Ubiquitous Computing, vol. 1, pp. 255-259.

[10] C. L. P. Chen, G. Wen, Y. Liu and Z. Liu, Observer-based adaptive backstepping consensus tracking control for high-order nonlinear semi-strict-feedback multiagent systems, IEEE Transactions on Cybernetics, vol. 46, July 2016, pp. 1591-1601.

[11] H. Zhang and J. Chen, Bipartite consensus of multiagent systems over signed graphs: State feedback and output feedback control approaches, Int. J. Robust Nonlinear Control, vol. 27, 2017, pp. 3-14.

[12] W. He , B. Xu, Q. Han and F. Qian, Adaptive consensus control of linear multiagent systems with dynamic event-triggered strategies, IEEE Transactions on Cybernetics, June 2019, pp. 1-13.

[13] X. Wang, Z. Zeng and Y. Cong, Multi-agent distributed coordination control: Developments and directions via graph viewpoint, Neurocomputing, vol. 199, 2016, pp. 204-218.

[14] W. Li, G. Wei, F. Han and Y. Liu, Weighted average consensus-based unscented Kalman filtering, IEEE Transactions on Cybernetics, vol. 46, February 2016, pp. 558-567.

[15] Y. Cheng, R. Jia, G. Wen and W. Zhu, Robust finite-time consensus formation control for multiple nonholonomic wheeled mobile robots via output feedback, Int. J. Robust Nonlinear Control, October 2017, pp. 1-15.

[16] A. A. Saad, S. Faddel and O. Mohammed, A secured distributed control system for future interconnected smart grids, Applied Energy, vol. 243, March 2019, pp. 57-70.

[17] Q. Zhao, X. Dong, Z. Liang and Z. Ren, Distributed group cooperative guidance for multiple missiles with fixed and switching directed communication topologies, Nonlinear Dynammics, vol. 90, September 2017, pp. 2507-2523.

[18] S. Djaidja and Q. Wu, Leader-following consensus for single-integrator multi-agent systems with multiplicative noises in directed topologies, International Journal of Systems Science, vol. 46, 2015, pp. 2788-2798.

[19] Z. Ma, Y. Wang and X. Li, Cluster-delay consensus in first-order multi-agent systems, Nonlinear Dynamics, vol. 83, 2016, pp. 1303-1310.

[20] S. Djaidja and Q. Wu, Leader-following consensus of single-integrator multi-agent systems under noisy and delayed communication, International Journal of Control, Automation and Systems, vol. 14, 2016, pp. 357-366.

[21] Y. Zhao, Z. Duan, G. Wen and G. Chen, Distributed finite-time tracking for a multi-agent system under a leader with bounded unknown acceleration, Systems and Control Letters, vol. 81, May 2015, pp. 8-13.

[22] C. Wei, J. Luo, Z. Yin and J. Yuan, Leader-following consensus of second-order multi-agent systems with arbitrarily appointed-time prescribed performance, IET Control Theory and Applications, vol. 12, 2018, pp. 2276-2286.

[23] B. Tian, H. Lu, Z. Zuo and W. Yang, Fixed-time leader-follower output feedback consensus for second-order multiagent systems, IEEE Transactions on Cybernetics, vol. 49, April 2019, pp. 1545-1550.

[24] Z. Li, X. Liu, W. Ren and L. Xie, Distributed tracking control for linear multiagent systems with a leader of bounded unknown input, IEEE Transactions on Automatic Control, vol. 58, February 2013, pp. 518-523.

[25] W. Cao, J. Zhang, W. Ren, Leader-follower consensus of linear multi-agent systems with unknown external disturbances, Systems and Control Letters, vol. 82, June 2015, pp. 64-70.

[26] Z. Li, Z. Duan and F. L. Lewis, Distributed robust consensus control of multi-agent systems with heterogeneous matching uncertainties, Automatica, vol. 50, 2014, pp. 883-889.

[27] V. Utkin and J. Shi, Integral Sliding mode in systems operating under uncertainty conditions, in Proc. 35th Conf. Decision and Control, Japan, 1996, pp.4591-4596.

[28] C. Hua, X. You and X. Guan, Leader-following consensus for a class of high-order nonlinear multi-agent systems, Automatica, vol. 73, 2016, pp. 138-144.

[29] V. I. Utkin, Sliding Modes in Control and Optimization. New York, NY, USA: Springer, 1992.

[30] C. Edwards and S.K. Spurgeon. Sliding Mode Control: Theory and Applications. Taylor and Francis Ltd, 1998.

[31] B. Bandyopadhyay, S. Janardhanan and S. K. Spurgeon, Advances in Sliding Mode Control: Concept, Theory and Implementation, Springer, 2013.

[32] S. Yu and X. Long, Finite-time consensus for second-order multi-agent systems with disturbances by integral sliding mode, Automatica, vol. 54, 2015, pp. 158-165.

[33] J. Bai, G. Wen, A. Rahmani and Y. Yu, Distributed consensus tracking for the fractional-order multi-agent systems based on the sliding mode control method, Neurocomputing, vol. 235, January 2017, pp. 210-216.

[34] C. Ren and C.L.P. Chen, Sliding mode leader-following consensus controllers for second-order non-linear multi-agent systems, IET Control Theory & Applications, vol. 9, 2015, pp. 1544-1552.

[35] J. Dávila, Distributed tracking of first order systems using second-order sliding modes, in IFAC Proc. Volumes, vol. 47, 2014, pp. 1392-1397.

[36] W. Ni and D. Cheng, Leader-following consensus of multi-agent systems under fixed and switching topologies, Systems and Control Letters, vol. 59, 2010, pp. 209-217.

[37] C. Chen, Linear System Theory and Design, Oxford University Press, 1999, pp. 145.

[38] J-J. E. Slotine and W. Li, Applied Nonlinear Control. Prentice Hall, 1991, pp. 73.

[39] G.H. Hardy, J.E. Littlewood and G. Plya, Inequalities. Cambridge, U.K.: Cambridge University Press, 1988.

[40] N. Zhao, D. Zhao, Y. Song and X. Yan, A new form of a class of MIMO linear systems for nonsingular terminal sliding mode control, submittied to International Journal of Systems Science.