Circle and Torus Actions in Exceptional Holonomy

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I, Udhav Fowdar, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

The work in this thesis is an investigation of the geometric structures arising on S^1 and $\mathbb{T}^2$ quotients of manifolds endowed with G_2 and $Spin(7)$-structures. This was motivated by the work of Apostolov and Salamon who studied the circle reduction of G_2 manifolds and showed that imposing that the quotient is Kähler leads to a rich geometry. We shall consider the following problems:

1. The S^1 quotient of $Spin(7)$-structures

2. The Kähler reduction of $Spin(7)$ manifolds with $\mathbb{T}^2$ actions

3. The S^1-invariant G_2 Laplacian flow

4. The $SU(2)^2 \times U(1)$-invariant G_2 Laplacian flow on $S^3 \times \mathbb{R}^4$

Our key results include expressions relating the intrinsic torsion of S^1-invariant $Spin(7)$-structures to that of the quotient G_2-structures, a new expression for the Ricci curvature of $Spin(7)$-structures only in terms of the intrinsic torsion, infinitely many new examples of (incomplete) $Spin(7)$ metrics arising as $\mathbb{T}^2$ bundles over Kähler manifolds with trivial canonical bundle, the first example of an inhomogeneous shrinking gradient G_2 Laplacian soliton and a local classification of closed $SU(2)^2 \times U(1)$-invariant G_2-structures on $S^3 \times \mathbb{R}^4$.

Impact Statement

This thesis is concerned with the study of geometric objects called G_2 and $Spin(7)$ manifolds arising in the field of exceptional geometry. These are higher dimensional examples of Ricci-flat and hence Einstein manifolds. Despite being a relatively young area of geometry; only a few decades old, it has already witnessed significant development. A few areas of mathematics it has had profound impact on include Gauge theory, Calibrated geometry and Geometric flows. For instance it has led to the definition of higher dimensional analogue of Donaldson-Thomas invariants and to new ideas in Seiberg-Witten theory. Its contribution is however not limited to just mathematics. The link between these objects and theoretical physics was first pointed out by Fields medalist Edward Witten in 1996. G_2 and $Spin(7)$ manifolds are fundamental building blocks for M- and F-theory, which are generalised versions of string theories. This has brought about many interactions between the mathematics and physics community, including the foundation of the Simons Collaboration on Special holonomy in Geometry, Analysis and Physics in 2016. Just as for Calabi-Yau manifolds mirror symmetry type phenomenons are predicted to exist for them as well. In this thesis we find infinitely many new examples of $Spin(7)$ metrics and provide a mathematical framework for some examples which have already appeared in the physics literature. We hope that our results will lead to a better understanding of the underlying features, both from a mathematics and physics perspective. Our construction provides a link between exceptional geometry and Kähler geometry, thereby allowing us to use tools from toric geometry. We also find a new solution to the Laplacian flow; a geometric flow, analogous to Hamilton's Ricci flow, introduced by Robert Bryant as the gradient flow to the

Hitchin's functional in hope of finding new examples of G_2 manifolds. We expect our example to lead to a more in depth study of the flow on new classes of manifolds, aside from homogeneous ones which have been the main focus of most research in the area so far.

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Chapter 1

Introduction

This thesis is concerned with the study of some problems in exceptional holonomy which admit circle or torus symmetry.

What is exceptional holonomy? The holonomy of a Riemannian manifold (M^n, g) is by definition the group generated by parallel transporting a basis of the tangent space at a point around closed loops using the Levi-Civita connection. In his thesis Berger classified the possible holonomy groups in a surprisingly short list [13], excluding symmetric spaces, which were already classified cf. [14], and Riemannian products, as their holonomy is simply the product of the holonomy of each individual components. The list included the groups G_2 and $Spin(7)$ for $n = 7$ and 8 respectively, and whose study are today known as exceptional geometry. It turns out that the geometry of G_2 and $Spin(7)$ manifolds can be fully encoded in a 3-form φ and 4-form Φ respectively, rather than the metric which gives their study an algebro-differential flavour.

Why symmetry? G_2 and $Spin(7)$ holonomy manifolds are in many ways analogous to Calabi-Yau manifolds and many examples have been constructed by exploiting these links [26, 60, 59]. One key difference is however the inexistence of an analogue of Yau's theorem of the Calabi conjecture. The closest analogue currently available is Joyce's theorem, which roughly states that if one can construct forms approximating φ and Φ in suitable Banach spaces then one can deform them to genuine solutions. These analytic hypotheses are in general hard to satisfy, by contrast to the topological one in Yau's theorem. An easier way of finding examples is to impose symmetry, and this is indeed how the first examples were found [18, 21]. As compact Ricci flat manifolds cannot admit continuous symmetry we are led to consider non-compact ones. The downside is that one cannot use standard elliptic theory for compact manifolds but on the upside one can find explicit solutions in several cases [17, 28, 29].

The ‘simplest’ type of symmetry to impose are circles and tori. The prototype for most of our work here is the Gibbons-Hawking ansatz, which gives an elementary way of writing down infinitely many circle invariant Calabi-Yau metrics in dimension 4 starting from only a positive harmonic function on the quotient 3-manifold. This idea was applied in the context of G_2 manifolds by Apostolov and Salamon in [3] and they were able to write down many explicit examples of G_2 metrics starting suitable data on a 6-manifold. Both the Gibbons-Hawking ansatz and the Apostolov-Salamon construction only give incomplete metrics, though in the former it is well-known how to find completions by the addition of fixed points.

In the case of the Gibbons-Hawking ansatz the quotient 3-manifold is an open set in $\mathbb{R}^3$ endowed with the flat Euclidean structure and thus, the structure group reduces from $SU(2)$ to the trivial group. The S^1 quotient of a G_2 manifold instead inherits an $SU(3)$-structure, consisting of a symplectic form ω and a 3-form Ω^+ defining an *almost* complex structure J. Owing to this rich structure Apostolov-Salamon found that if J is a complex structure then the quotient admits a further Kähler reduction to a 4-manifold. Inverting this construction led to the discovery of new type of G_2 metrics. Motivated by this observation, the work in this thesis is a quest to finding ‘interesting geometric structures’ on quotients of manifolds admitting G_2 and $Spin(7)$-structures.

Brief overview

Chapter 2. We cover basic facts about the geometry of G_2, $Spin(7)$ and $SU(3)$-structures and describe the Gibbons-Hawking ansatz. This chapter also serves the purpose of setting up the notation and conventions for the rest of the thesis.

Chapter 3. We consider the S^1 reduction of $Spin(7)$-structures in the case that they are torsion free, locally conformally parallel and balanced. We derive a new formula for the Ricci curvature of $Spin(7)$-structures only in terms of the torsion forms. Finally we study the quotient explicitly in the case when the manifold is the spinor bundle of S^4 endowed with the Bryant-Salamon $Spin(7)$-structure and $\mathbb{R}^8$ with the flat $Spin(7)$-structure. This chapter is based on our work in [44]

Chapter 4. We consider the $\mathbb{T}^2$ reduction of a $Spin(7)$ manifold under the assumption that the quotient 6-manifold is Kähler. Inverting this construction leads us to discover many new examples of $Spin(7)$ metrics. This is a generalisation of the aforementioned Apostolov-Salamon construction. This chapter is based on our work in [45].

Chapter 5. The Laplacian flow is a geometric flow introduced by Bryant as a way of finding torsion free G_2-structures starting from a closed one. If the flow is S^1-invariant then it descends to a flow of $SU(3)$-structure. We derive expressions for these evolution equations. In our search for examples we discover the first example of an inhomogeneous shrinking soliton. This chapter is based of our work in [43]

Chapter 6. We consider the Laplacian flow on $S^3 \times \mathbb{R}^4$ and search for cohomogeneity one solitons with $SU(2)^2 \times U(1)$ symmetry. We rule out conical solitons on $S^3 \times S^3 \times \mathbb{R}^+$ and complete ones on $S^3 \times \mathbb{R}^4$. The search for complete 'neck-type' solitons on $S^3 \times S^3 \times \mathbb{R}$ is an ongoing project.

Chapter 2

Background

The goal of this chapter is to give a quick introduction to the rudimentary of exceptional geometry and some closely related topics that we shall need in the subsequent chapters. This will also serve the purpose of setting up the notation for the rest of this thesis.

In section 2.1 and 2.2 we cover the basics of G_2 and $Spin(7)$ geometry. The material is classical and proofs can be found in the standard references [18, 61, 74]. In section 2.3 we give a quick overview of $SU(3)$-structures which will play a crucial role throughout this thesis. Excellent references for this material include [12, 24, 54]. In the last section we describe the Gibbons-Hawking ansatz, which has been a key motivation for most of the work undertaken here.

2.1 Preliminary on G_2-structures

Definition 2.1.1. A G_2-structure on a 7-manifold L^7 is given by a 3-form φ that can be identified at each point $p \in L^7$ with the standard one on $\mathbb{R}^7$:

$$\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}, \tag{2.1.1}$$

where $x_1, \dots, x_7$ denote the coordinates on $\mathbb{R}^7$ and dx_{ijk} is shorthand for $dx_i \wedge dx_j \wedge dx_k$.

Equivalently, but more abstractly, a G_2-structure can be defined as a reduction of the structure group of the frame bundle of L^7 from $GL(7,\mathbb{R})$ to G_2. By abuse of language we also refer to φ or the pair (L^7,φ) as the G_2-structure. The reason for this nomenclature stems from the fact that the subgroup of $GL(7,\mathbb{R})$ which stabilises φ_0 is isomorphic to the Lie group G_2. Since G_2 is a subgroup of $SO(7)$ [18, 74] it follows that φ defines a Riemannian metric g_φ and volume form vol_φ on L^7. Explicitly these are given by

$$\frac{1}{6}\, \iota_X\varphi\wedge\iota_Y\varphi\wedge\varphi = g_\varphi(X,Y)\,\mathrm{vol}_\varphi,$$

where X,Y are vector fields on L^7. In particular, φ defines a Hodge star operator $*_\varphi$. It is known that a 7-manifold admits a G_2-structure if and only if its first and second Stiefel-Whitney classes vanish [66] so there is a plethora of examples. Moreover the space of G_2-structures is an open set i.e. if $\|\tilde{\varphi}-\varphi\|_\varphi<\varepsilon$ for $\varepsilon>0$ sufficiently small then $\tilde{\varphi}$ is also a G_2-structure. We denote this set by $\Omega^3_+(L^7)$.

One of the main motivations for studying this structure is that if φ is parallel with respect to the Levi-Civita connection ∇^{g_φ} (which is a first order condition) then it has holonomy contained in G_2 and the metric is Ricci-flat. If the holonomy is *equal* to G_2 then (L^7,φ) is called a G_2-manifold and if the holonomy is only contained in G_2 then it is referred to as a torsion free G_2-structure, although some authors use these terminologies interchangeably. Note that in contrast the Ricci-flat system of differential equations are second order. The fact that φ is parallel implies the reduction of the holonomy group of g_φ from $SO(7)$ to (a subgroup of) G_2 and conversely, a holonomy G_2 metric implies the existence of such a 3-form. An alternative way to verify the parallel condition is given by the following theorem.

Theorem 2.1.2 ([36])**.** $\nabla^{g_\varphi}\varphi=0$ *if and only if* $d\varphi=0$ *and* $d*_\varphi\varphi=0$.

The failure of the reduction of the holonomy group to G_2 is measured by the intrinsic torsion. Abstractly, given a general H-structure for a subgroup $H\subset O(n)$ the intrinsic torsion is defined as a section of the associated bundle to $\mathbb{R}^n\otimes\mathfrak{h}^\perp$ where $\mathfrak{so}(n)=\mathfrak{h}\oplus\mathfrak{h}^\perp$ and $\perp$ denotes the orthogonal complement with respect to the Killing form. We shall only give a brief description here but more details can be found in [19, 74]. The space of differential forms on L^7 can be decomposed as

G_2-modules as follows:

$$\Lambda^1 = \Lambda^1_7, \tag{2.1.2}$$

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}, \tag{2.1.3}$$

$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}, \tag{2.1.4}$$

where the subscript denotes the dimension of the irreducible module. Using the Hodge star operator we get the corresponding splitting for Λ^4, Λ^5 and Λ^6. A more concrete description of these modules is given by:

$$\Lambda^2_7 = \{\alpha \in \Lambda^2 \mid *_\varphi(\alpha \wedge \varphi) = 2\alpha\} \tag{2.1.5}$$

$$= \{*_\varphi(\alpha \wedge *_\varphi\varphi) \mid \alpha \in \Lambda^1_7\} \tag{2.1.6}$$

$$\Lambda^2_{14} = \{\alpha \in \Lambda^2 \mid *_\varphi(\alpha \wedge \varphi) = -\alpha\} \tag{2.1.7}$$

$$= \{\alpha \in \Lambda^2 \mid \alpha \wedge *_\varphi\varphi = 0\} \tag{2.1.8}$$

$$\Lambda^3_1 = \{\lambda\varphi \mid \lambda \in \mathbb{R}\} \tag{2.1.9}$$

$$\Lambda^3_7 = \{*_\varphi(\alpha \wedge \varphi) \mid \alpha \in \Lambda^1_7\} \tag{2.1.10}$$

$$\Lambda^3_{27} = \{\alpha \in \Lambda^3 \mid \alpha \wedge \varphi = 0 \text{ and } \alpha \wedge *_\varphi\varphi = 0\} \tag{2.1.11}$$

There is also an isomorphism of G_2 modules $\Lambda^3_1 \oplus \Lambda^3_{27} \cong \langle g_\varphi \rangle \oplus S^2_0$, where S^2_0 denotes the space of traceless symmetric $(0,2)$-tensors. Explicitly this map is defined by

$$j : \Lambda^3 \to S^2$$

$$j(\gamma)(X,Y) = *_\varphi(\iota_X\varphi \wedge \iota_Y\varphi \wedge \gamma). \tag{2.1.12}$$

From characterisation (2.1.10) it is straightforward to verify that Λ^3_7 is the kernel of j.

Before describing the intrinsic torsion of a G_2-structure we first explain the general case following [74]. Given a group $G \subset O(n)$, consider the G-equivariant homomorphism given by

$$\delta : (\mathbb{R}^n)^* \otimes \mathfrak{g} \hookrightarrow (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \Lambda^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n$$

where the first map is simply inclusion and the second is skew-symmetrisation on the first two factors. Suppose now that a manifold M^n is endowed with a G-structure then the difference of any 2 connections on this G-bundle defines a section of the associated bundle to $(\mathbb{R}^n)^* \otimes \mathfrak{g}$. The image under δ (interpreted as a bundle homomorphism via the associated bundle construction) of this difference is, up to a constant factor, the difference of the torsion of the two connections. Thus, the torsion of any connection defines the same element in the associated bundle to the G-module

$$\frac{\Lambda^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n}{\delta((\mathbb{R}^n)^* \otimes \mathfrak{g})} \cong (\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp$$

where we use the Riemannian metric for the identification. This element, which is independent of the choice of connection, is called the intrinsic torsion and vanishes if and only if the G-structure admits a torsion free connection. In practice one can often identify the intrinsic torsion with the exterior derivative of some suitable differential form(s).

In the G_2 case the intrinsic torsion is given by $\dim(\mathbb{R}^7 \otimes \mathfrak{g}_2^\perp) = 49$ equations and can be described using the equations

$$d\varphi = \tau_0 *_\varphi \varphi + 3\,\tau_1 \wedge \varphi + *_\varphi \tau_3 \tag{2.1.13}$$

$$d *_\varphi \varphi = 4\,\tau_1 \wedge *_\varphi \varphi + \tau_2 \wedge \varphi \tag{2.1.14}$$

where $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1_7$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^4_{27}$. Here we are denoting by Ω^i_j the space of smooth sections of Λ^i_j. The fact that τ_1 arises in both equations can be proved using the following lemma.

Lemma 2.1.3 ([19]). *Given $\alpha \in \Lambda^1_7(M)$ and $\beta \in \Lambda^2_7(M)$ we have*

1. $2 *_\varphi (\beta \wedge *_\varphi \varphi) \wedge *_\varphi \varphi = 3\beta \wedge \varphi$
2. $*_\varphi \alpha = -\frac{1}{4} *_\varphi (\alpha \wedge \varphi) \wedge \varphi = \frac{1}{3} *_\varphi (\alpha \wedge *_\varphi \varphi) \wedge *_\varphi \varphi.$

Manifolds with holonomy *equal to* G_2 are very hard to construct and the first examples only appeared in the late 80s cf. [18, 21]. The search for examples is still one of the most active area of research in the field and this has led to the study of weaker notions:

Definition 2.1.4. A G_2-structure (L^7, φ) is called

1. closed or calibrated if $d\varphi = 0$,

2. coclosed or cocalibrated if $d *_\varphi \varphi = 0$,

3. nearly parallel or weakly G_2 if $d\varphi = \lambda *_\varphi \varphi$ for $\lambda \in \mathbb{R} - \{0\}$.

Note that nearly parallel G_2 manifolds are Einstein with positive scalar curvature and their cones have holonomy contained in $Spin(7)$, which is precisely what we describe next.

2.2 Preliminary on $Spin(7)$-structures

Definition 2.2.1. A $Spin(7)$-structure on an 8-manifold N^8 is given by a 4-form Φ that can be identified at each point $q \in N^8$ with the standard one on $\mathbb{R}^8$:

$$\begin{aligned}\Phi_0 &= dx_0 \wedge \varphi_0 + *_{\varphi_0} \varphi_0 && (2.2.1)\\ &= dx_{0123} + dx_{0145} + dx_{0167} + dx_{0246} - dx_{0257} - dx_{0347} - dx_{0356}\\ &\quad + dx_{2345} + dx_{2367} + dx_{4567} - dx_{1247} - dx_{1256} - dx_{1346} + dx_{1357}, && (2.2.2)\end{aligned}$$

where we have augmented the G_2 module $\mathbb{R}^7$ by $\mathbb{R}$ with coordinate x_0.

The subgroup of $GL(8,\mathbb{R})$ which stabilises Φ_0 is isomorphic to $Spin(7)$ cf. [21, 74]. From (2.2.1) it is clear that G_2 is a subgroup of $Spin(7)$. Since $Spin(7)$ is a subgroup of $SO(8)$ it follows that Φ defines a metric g_Φ, volume form vol_Φ and Hodge star $*_\Phi$. Explicitly the volume form is given by

$$vol_\Phi = \frac{1}{14}\,\Phi \wedge \Phi.$$

The expression for g_Φ is more complicated than in the G_2 case cf. [62, section 4.3], but fortunately we will not need it. An 8-manifold admits a $Spin(7)$-structure if and only, if in addition to having zero first and second Stiefel-Whitney classes, either of the following holds

$$p_1(N)^2 - 4p_2(N) \pm 8\chi(N) = 0$$

cf. [51, 66], noting that the '8' factor is accidentally omitted in the former. An important distinction with the G_2 case is that $Spin(7)$ 4-forms do not form an open set in $\Omega^4(N^8)$. If Φ is parallel with respect to the Levi-Civita connection ∇^{g_Φ} then the metric g_Φ has holonomy contained in $Spin(7)$ and the metric is Ricci-flat. A manifold with holonomy *equal to* $Spin(7)$ is called a $Spin(7)$-manifold. Just as in the G_2 situation we have the following alternative formulation of the torsion free condition.

Theorem 2.2.2 ([33])**.** $\nabla^{g_\Phi}\Phi = 0$ *if and only if* $d\Phi = 0$.

The space of differential forms on N^8 can be decomposed as $Spin(7)$-modules as follows:

$$\Lambda^1 = \Lambda^1_8 \tag{2.2.3}$$

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21} \tag{2.2.4}$$

$$\Lambda^3 = \Lambda^3_8 \oplus \Lambda^3_{48} \tag{2.2.5}$$

$$\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}. \tag{2.2.6}$$

As in the G_2 case, the Hodge star defines an isomorphism $\Lambda^k_l \cong \Lambda^{8-k}_l$. A more explicit description of these modules is given by:

$$\Lambda^2_7 = \{\alpha \in \Lambda^2 \mid *_\Phi(\alpha \wedge \Phi) = 3\alpha\} \tag{2.2.7}$$

$$\Lambda^2_{21} = \{\alpha \in \Lambda^2 \mid *_\Phi(\alpha \wedge \Phi) = -\alpha\} \tag{2.2.8}$$

$$\Lambda^3_8 = \{*_\Phi(\alpha \wedge \Phi) \mid \alpha \in \Lambda^1_8\} \tag{2.2.9}$$

$$\Lambda^3_{48} = \{\alpha \in \Lambda^3 \mid \alpha \wedge \Phi = 0\} \tag{2.2.10}$$

$$\Lambda^4_1 = \{\lambda\Phi \mid \lambda \in \mathbb{R}\} \tag{2.2.11}$$

$$\Lambda^4_{35} = \{\alpha \in \Lambda^4 \mid *_\Phi \alpha = -\alpha\} \tag{2.2.12}$$

The space Λ^4_7 is the span of the action of $\Lambda^2_7 \hookrightarrow \Lambda^2 \cong \mathfrak{so}(8)$ on Φ. The first 3 summands of (2.2.6) are the self-dual 4-forms Λ^4_+ (i.e. have eigenvalue $+1$ under $*_\Phi$) and Λ^4_{27} can be defined as the orthogonal complement of $\Lambda^4_1 \oplus \Lambda^4_7 \hookrightarrow \Lambda^4_+$. There is also an injection map $\iota : S^2 \hookrightarrow \Lambda^4$ which restricts to an isomorphism of $Spin(7)$-modules

$$\begin{aligned} \mathsf{i} : \langle g_\Phi \rangle \oplus S^2_0 &\to \Lambda^4_1 \oplus \Lambda^4_{35} \\ a \circ b &\mapsto a \wedge *_\Phi(b \wedge \Phi) + b \wedge *_\Phi(a \wedge \Phi), \end{aligned}$$

Note that $\mathsf{i}(g_\Phi) = 8\Phi$. We denote by j the inverse map extended to Λ^4 as the zero map on $\Lambda^4_7 \oplus \Lambda^4_{27}$. The intrinsic torsion is given by $\dim(\mathbb{R}^8 \otimes \mathfrak{spin}(7)^\perp) = 56$ equations and is completely determined by the exterior derivative of Φ in view of Theorem 2.2.2. This can be written as

$$d\Phi = T^1_8 \wedge \Phi + T^5_{48}, \tag{2.2.13}$$

where $T^1_8 \in \Lambda^1_8$ and $T^5_{48} \in \Lambda^5_{48}$. Of relevance for us will be the following classes of $Spin(7)$-structures.

Definition 2.2.3.

1. If T^1_8 vanishes the $Spin(7)$-structure Φ is called balanced,
2. if T^5_{48} vanishes it is called locally conformally parallel, and
3. if both are vanish then it is called torsion free.

2.3 Preliminary on $SU(3)$-structures

Definition 2.3.1. An $SU(3)$-structure on a 6-manifold P^6 is given by a non-degenerate 2-form ω, a Riemannian metric g_ω, an almost complex structure J and a $(3,0)$-form $\Omega := \Omega^+ + i\Omega^-$ satisfying the two conditions

$$\omega \wedge \Omega^{\pm} = 0, \tag{2.3.1}$$

$$\frac{2}{3}\omega^3 = \Omega^+ \wedge \Omega^-. \tag{2.3.2}$$

Although an $SU(3)$-structure consists of the data $(g_\omega, \omega, J, \Omega^+ + i\Omega^-)$ it is in fact sufficient to specify the pair (ω, Ω^+) satisfying (2.3.1) and (2.3.2). This observation is due to Hitchin in [54] where he shows that Ω^+ (or Ω^-) determines J. Abstractly this follows from the fact that the stabiliser of Ω^+ in $GL^+(6,\mathbb{R})$ is congruent to $SL(3,\mathbb{C}) \subset GL(3,\mathbb{C})$. The metric is then determined by

$$\omega(\cdot,\cdot) := g_\omega(J\cdot,\cdot), \tag{2.3.3}$$

and $\Omega^- := J(\Omega^+) = *_\omega \Omega^+$, where $*_\omega$ is the Hodge star operator determined by g_ω and the volume form

$$vol_\omega := \frac{1}{6}\omega^3 = \frac{1}{4}\Omega^+ \wedge \Omega^-.$$

We say (P^6, ω, Ω^+) is a Calabi-Yau 3-fold if both ω and Ω^+ are covariantly constant with respect to ∇^{g_ω}, and hence so are J and Ω^-. This condition can be equivalently formulated as:

Theorem 2.3.2 ([12, 24])**.** *$\nabla^{g_\omega}\omega = 0$ and $\nabla^{g_\omega}\Omega^+ = 0$ if and only if $d\omega = 0$ and $d\Omega^+ = d\Omega^- = 0$.*

Note that Calabi-Yau 3-folds have holonomy contained in $SU(3)$. Recall that since $J^2 = -1$ we can decompose the complexified space of 1-forms into $+i$ and $-i$ J-eigenspaces denoted by $\Lambda^{1,0}$ and $\Lambda^{0,1}$ respectively. Writing $\Lambda^{p,q}$ for the space of complex $(p+q)$ forms spanned by wedging p elements of $\Lambda^{1,0}$ and q elements of $\Lambda^{0,1}$ we have

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{0\leq i\leq k} \Lambda^{i,k-i}.$$

Hence as $SU(3)$-modules the space of differential forms split as follows:

$$\Lambda^1 = \Lambda^1_6 = [\![\Lambda^{1,0}]\!] \tag{2.3.4}$$

$$\begin{aligned}\Lambda^2 &= \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8 \\ &= \langle \omega \rangle \oplus [\![\Lambda^{2,0}]\!] \oplus [\Lambda^{1,1}_0]\end{aligned} \tag{2.3.5}$$

$$\begin{aligned}\Lambda^3 &= \Lambda^{3+}_1 \oplus \Lambda^{3-}_1 \oplus \Lambda^3_6 \oplus \Lambda^3_{12} \\ &\cong \langle \Omega^+ \rangle \oplus \langle \Omega^- \rangle \oplus [\![\Lambda^{1,0}]\!] \oplus [\Lambda^{2,1}_0].\end{aligned} \tag{2.3.6}$$

Here the $[\![\Lambda^{p,q}]\!]$ and $[\Lambda^{p,p}]$ notation, introduced in [74], refers to taking the corresponding *real* underlying vector space i.e.

$$[\![\Lambda^{p,q}]\!] \otimes \mathbb{C} = \Lambda^{p,q} \oplus \Lambda^{q,p} \ \text{ and } \ [\Lambda^{p,p}] \otimes \mathbb{C} = \Lambda^{p,p}.$$

The above is a refinement of the Gray-Hervalla decomposition in dimension 6 cf. [52]. A more explicit description of these irreducible modules is given by

$$\Lambda^2_6 = \{\alpha \in \Lambda^2 \mid *_\omega (\alpha \wedge \omega) = \alpha\} \tag{2.3.7}$$

$$= \{*_\omega(\alpha \wedge \Omega^+) \mid \alpha \in \Lambda^1_6\} \tag{2.3.8}$$

$$\Lambda^2_8 = \{\alpha \in \Lambda^2 \mid *_\omega (\alpha \wedge \omega) = -\alpha\} \tag{2.3.9}$$

$$\Lambda^3_6 = \{\alpha \wedge \omega \mid \alpha \in \Lambda^1_6\} \tag{2.3.10}$$

$$\Lambda^3_{12} = \{\alpha \in \Lambda^3 \mid \alpha \wedge \omega = 0,\ \alpha \wedge \Omega^\pm = 0\} \tag{2.3.11}$$

Note that as $SU(3)$ modules the spaces $\Lambda^\bullet_6$ are all isomorphic; this is the standard representation $SU(3) \subset SO(6)$. In computations we will often need to interchange between these spaces and to do so we use the following lemma.

Lemma 2.3.3. *Given* 1*-form* $\alpha \in \Lambda^1_6$*, let* $\beta := *_\omega(\alpha \wedge \Omega^-)$ *then the following hold*

1. $J(\alpha) \wedge \Omega^+ = \alpha \wedge \Omega^- = \beta \wedge \omega$

2. $\beta \wedge \Omega^- = 2 *_\omega (\alpha) = -(J\alpha) \wedge \omega^2$

3. $\beta \wedge \Omega^+ = 2 *_\omega (J\alpha) = \alpha \wedge \omega^2$

Proof. Since this is an algebraic statement it is sufficient to prove the above hold on $\mathbb{R}^6$. With coordinates $x_2, \dots, x_7$ on $\mathbb{R}^6$ we can express

$$\begin{aligned}\omega &= dx_{23} + dx_{45} + dx_{67},\\ \Omega^+ &= dx_{246} - dx_{257} - dx_{347} - dx_{356},\\ \Omega^- &= dx_{256} + dx_{247} + dx_{346} - dx_{357}.\end{aligned}$$

Furthermore, since $SU(3)$ acts transitively on S^5 it suffices to verify the above hold for $\alpha = dx_2$. □

In view of theorem 2.3.2 and the above characterisation of Λ^k_l, the intrinsic torsion of an $SU(3)$-structure is determined by

$$d\omega = -\frac{3}{2}\sigma_0\, \Omega^+ + \frac{3}{2}\pi_0\, \Omega^- + \nu_1 \wedge \omega + \nu_3, \tag{2.3.12}$$

$$d\Omega^+ = \pi_0\, \omega^2 + \pi_1 \wedge \Omega^+ - \pi_2 \wedge \omega, \tag{2.3.13}$$

$$d\Omega^- = \sigma_0\, \omega^2 + (J\pi_1) \wedge \Omega^+ - \sigma_2 \wedge \omega, \tag{2.3.14}$$

where $\sigma_0, \pi_0 \in \Omega^0$, $\nu_1, \pi_1 \in \Omega^1_6$, $\pi_2, \sigma_2 \in \Omega^2_8$ and $\nu_3 \in \Omega^3_{12}$, cf. [12]. Many well known geometric structures can be redefined using this formulation.

Definition 2.3.4. The $SU(3)$-structure (P^6, ω, Ω^+) is said to be

1. nearly Kähler if $\sigma_0 = -2$ and all other torsion forms vanish,
2. Kähler if all torsion forms aside from π_1 vanish,
3. complex if $\pi_0 = \sigma_0 = 0$ and $\pi_2 = \sigma_2 = 0$,
4. half-flat if $\pi_0 = 0$, $\pi_1 = \nu_1 = 0$ and $\pi_2 = 0$.

On notations and conventions

Throughout this thesis we shall often use the suggestive notation β^l_m for an l-form to mean that $\beta^l_m \in \Omega^l_m$ or write $(\beta)^l_m$ for the Ω^l_m-component of an l-form β. We will specify the underlying space N^8, L^7 or P^6 if there is any risk of ambiguity.

$d^* := (-1)^{n(k-1)+1} * d *$ will denote the codifferential on k-forms on an n-manifold. The inner product on decomposable k-forms is defined by

$$g(\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) = \det(g(\alpha_i, \beta_j)),$$

which might differ from other conventions in the literature by a '$k!$' factor, and the Hodge star is defined by

$$\alpha \wedge *\beta = g(\alpha, \beta)\,\mathrm{vol},$$

for k-forms α and β.

2.4 The Gibbons-Hawking ansatz

This thesis is mainly concerned with the study of (free) circle and torus actions on manifolds with G_2- and $Spin(7)$-structures. The prototype for this construction is the so-called Gibbons-Hawking ansatz [48]. This is an elementary, yet powerful, way of constructing infinitely many (local) examples of hyperKähler 4-manifolds with triholomorphic Killing circle actions. Before describing this construction we first introduce some notions from Kähler geometry.

Definition 2.4.1. A Riemannian manifold (M^{4n}, g_{HK}) is said to be hyperKähler if there exists a triple of complex structures I, J, K compatible with the metric which are parallel with respect to the Levi-Civita connection $\nabla^{g_{HK}}$ and satisfy $K = IJ = -JI$.

HyperKähler manifolds have holonomy contained in $Sp(n)$. To each of the complex structures we associate real $(1,1)$-forms ω_I, ω_J, ω_K using (2.3.3). By analogy to Theorems 2.1.2, 2.2.2 and 2.3.2 we give the following alternative characterisation of the parallel condition on I, J, K.

Theorem 2.4.2 ([6])**.** *The complex structures I, J, K are parallel if and only if*

$$d\omega_I = d\omega_J = d\omega_K = 0.$$

We can now describe the Gibbons-Hawking ansatz. Given an open set $B \subset \mathbb{R}^3$

with the Euclidean metric and coordinates (x,y,z), together with a positive harmonic function $V : B \to \mathbb{R}^+$ satisfying the 'integrality' condition $[-*dV] \in H^2(B,\mathbb{Z})$ (ignoring a factor of 2π), then we can define a hyperKähler triple on the total space $\pi : M^4 \to B$ of the circle bundle by

$$\omega_I = \theta \wedge dx + V\ dy \wedge dz,$$
$$\omega_J = \theta \wedge dy + V\ dz \wedge dx,$$
$$\omega_K = \theta \wedge dz + V\ dx \wedge dy,$$

where θ is a connection 1-form satisfying $d\theta = -\pi^*(*dV)$. The closedness of ω_i is immediate by construction and the hyperKähler metric is given by

$$g_{HK} = V^{-1}\theta^2 + V(dx^2 + dy^2 + dz^2).$$

Thus, the Gibbons-Hawking ansatz reduces the problem of constructing S^1-invariant hyperKähler metrics to simply choosing a positive harmonic function V. Of course given V, we still have the freedom of varying the connection form by the addition of a flat connection.

In this thesis we study some generalised G_2 and $Spin(7)$ versions of this construction. A fundamental work in this direction was carried out by Apostolov and Salamon in [3] where the authors studied free circle actions on G_2 manifolds whose quotients are Kähler. Related works have also appeared previously in the physics literature cf. [27, 64, 63].

Chapter 3

S^1-quotient of $Spin(7)$-structures

3.1 Overview of chapter

In this chapter we investigate the quotient of $Spin(7)$-structures under free circle actions. The case when N^8 is a $Spin(7)$ manifold has also been studied by Foscolo in [39]. One motivation for studying the non-torsion free cases lies in the fact that they also have interesting geometric properties, for instance, balanced $Spin(7)$-structures admit harmonic spinors [57] and compact locally conformally parallel ones are fibred by nearly parallel G_2 manifolds [58]. A further motivation is that $Spin(7)$-structures have only two torsion classes, see (2.2.13), and thus have only four types whereas G_2-structures have four classes, see (2.1.13) and (2.1.14), thus allowing for a more refined decomposition of the $Spin(7)$ torsion classes. The outline for the rest of this chapter is as follows.

In section 3.2 we describe the quotient of $Spin(7)$-structures which are invariant under a free circle action. The foundational result is proposition 3.2.2, which gives explicit expressions relating the torsion of the $Spin(7)$-structure on the 8-manifold N^8 to the torsion of the quotient G_2-structure on L^7 together with a positive function s and the curvature of the S^1 bundle. The key observation is that this construction is reversible. In the subsequent subsections we specialise to the three cases when the $Spin(7)$-structure is torsion free, locally conformally parallel and balanced. In the torsion free situation we show that quotient manifold cannot have holonomy *equal to* G_2 unless N is a Calabi-Yau 4-fold and L^7 is the Riemannian product of a Calabi-Yau 3-fold and a circle. We also give explicit expressions for the $SU(4)$-structure in terms of the data on the quotient manifold, see Theorem 3.2.7. In the locally conformally parallel situation, we show that L has vanishing Λ^3_{27} torsion component and furthermore, if the Λ^3_1 torsion component is non-zero then $N^8 = L^7 \times S^1$, see Theorem 3.2.8. In the balanced situation, we show that the existence of an invariant $Spin(7)$-structure is equivalent to the existence of a suitable section of Λ^2_{14} of the quotient space, see Theorem 3.2.10. We provide several examples to illustrate each case.

In section 3.3 we derive formulae for the Ricci and scalar curvatures of $Spin(7)$-structures in terms of the torsion forms à la Bryant cf. [19], see Proposition 3.3.1. As a corollary, under our free S^1 action hypothesis, we show that the Λ^2_7 component of the curvature form corresponds to the mean curvature vector of the circle fibres.

In the last two sections we demonstrate how our construction can be applied to the Bryant-Salamon $Spin(7)$-structure on the (negative) spinor bundle of S^4 and on the flat $Spin(7)$-structure on $\mathbb{R}^8$. In the former case the quotient space is the anti-self-dual bundle of S^4 and in the latter it is the cone on $\mathbb{CP}^3$. We interpret the quotient of the spinor bundle as a fibrewise (reverse) Gibbons-Hawking ansatz and give an explicit expression for the quotient (non-torsion free) G_2-structure. In both case we compare the $SU(3)$-structure on the link $\mathbb{CP}^3$.

3.2 The quotient construction

Given an 8-manifold N^8 endowed with a $Spin(7)$-structure Φ which is invariant under a free circle action generated by a vector field X the quotient manifold L^7 inherits a natural G_2-structure $\varphi := \iota_X \Phi$. We can write the $Spin(7)$ form as

$$\Phi = \eta \wedge \varphi + s^{4/3} *_\varphi \varphi, \tag{3.2.1}$$

where $s := \|X\|^{-1}_\Phi$ and $\eta(\cdot) := s^2 g_\Phi(X, \cdot)$. For the sake of visual clarity we will identify basic (i.e. invariant and horizontal) tensors on N^8 with the corresponding tensors on L^7 omitting any pullback signs. The proof that Φ can be expressed as

(3.2.1) with the scaling factor $s^{4/3}$ in front of $*_\varphi\varphi$ is analogous to that of lemma 3.2.1 below. The assumption that the action is free i.e. X is nowhere vanishing implies that s is a well-defined strictly positive function. Moreover s is invariant under X, since $\mathscr{L}_X g_\Phi = 0$, and hence descends to L^7. The metrics and volume forms of (L^7, φ) and (N^8, Φ) are related by

$$g_\Phi = s^{-2}\eta^2 + s^{2/3}g_\varphi, \tag{3.2.2}$$

$$vol_\Phi = s^{4/3}\eta \wedge vol_\varphi. \tag{3.2.3}$$

In this setup η can be viewed as a connection 1-form on the S^1-bundle N^8 over L^7 and $d\eta$ is its curvature, which by Chern-Weil theory defines an element in $H^2(L,\mathbb{Z})$. We denote by $(d\eta)^2_7$ and $(d\eta)^2_{14}$ the curvature components in view of decomposition (2.1.3). Under the inclusion $G_2 \hookrightarrow Spin(7)$ we may decompose the torsion forms of (2.2.13) further as

$$T^1_8 = f \cdot \eta + T^1_7, \tag{3.2.4}$$

$$T^5_{48} = T^5_7 + T^5_{14} + \eta \wedge (T^4_7 + T^4_{27}), \tag{3.2.5}$$

where f is (the pullback of) a function on L^7 and all the differential forms on the right hand side, aside from η, are basic. Note that $56 = 8+48 = (1+7)+(7+14+27) = 49+7$ where 56 and 49 are respectively the dimensions of the space of intrinsic torsions of $Spin(7)$ and G_2 structures. This simple dimension count confirms the absence of any $T^4_1 \in \Lambda^4_1(L^7)$ term in T^5_{48}. More explicitly, since $\Phi \wedge *_\Phi T^5_{48} = 0$ we have that

$$T^4_1 \wedge \varphi = 0$$

i.e. $T^4_1 = 0$. So this says that the intrinsic torsion of Φ is determined by that of φ together with a section of a rank 7 vector bundle. In order to relate the intrinsic torsion of the $Spin(7)$-structure to that of the G_2-structure we first need to relate their Hodge star operators.

Lemma 3.2.1. *Given $\alpha \in \Lambda^2_7(L)$, $\beta \in \Lambda^2_{14}(L)$, $\gamma \in \Lambda^1_7(L)$ and using the same nota-*

tion for their pullbacks to N^8 we have

1. $*_\Phi(\alpha \wedge \varphi) = -2s^{-2}\eta \wedge \alpha$
2. $*_\Phi(\beta \wedge \varphi) = s^{-2}\eta \wedge \beta$
3. $*_\Phi\gamma = -s^{2/3}\eta \wedge *_\varphi\gamma$
4. $*_\Phi\eta = s^{10/3}\,\mathrm{vol}_\varphi$
5. $*_\Phi(\eta \wedge \alpha) = \frac{1}{2}s^2\alpha \wedge \varphi$
6. $*_\Phi(\eta \wedge \beta) = -s^2\beta \wedge \varphi$
7. $*_\Phi(\eta \wedge \gamma) = s^{8/3} *_\varphi \gamma$

Proof. This is a straightforward computation using (3.2.2), (3.2.3) and the characterisation of Λ^2_7 and Λ^2_{14} as having eigenvalues $+2$ and -1 under wedging with φ and taking the Hodge star cf. (2.1.5) and (2.1.7). We prove (1) as an example. Since we only need to show the above formula holds at each point we may pick coordinates at a point $q \in N$ such that Φ is identified with Φ_0, $\eta = s \cdot dx_0$ and $\varphi = s^{-1} \cdot \varphi_0$. Then for any given $\vartheta \in \Omega^2(L)$ we see easily that

$$*_\Phi(\vartheta \wedge \varphi) = -s^{-2}\eta \wedge *_\varphi(\vartheta \wedge \varphi).$$

If $\vartheta = \alpha$ we have $*_\varphi(\alpha \wedge \varphi) = 2\alpha$, which completes the proof of (1). $\square$

Proposition 3.2.2. *The intrinsic torsion of the $Spin(7)$-structure and G_2-structure are related by*

1. $f = -s^{-4/3}\tau_0$
2. $7T^1_7 = 24\tau_1 + 3s^{-4/3}d(s^{4/3}) + 2s^{-4/3} *_\varphi ((d\eta)^2_7 \wedge *_\varphi\varphi)$
3. $7T^5_7 = 4(d\eta)^2_7 \wedge \varphi + 4d(s^{4/3}) \wedge *_\varphi\varphi + 4s^{4/3}\tau_1 \wedge *_\varphi\varphi$
4. $T^5_{14} = (d\eta)^2_{14} \wedge \varphi + s^{4/3}\tau_2 \wedge \varphi$
5. $T^4_{27} = - *_\varphi \tau_3$

6. *T_7^4 and T_7^5 are G_2-equivalent up to a factor of $s^{-4/3}$; explicitly, the composition*

$$\mathscr{F} : \Lambda^5_7 \xrightarrow{*} \Lambda^2_7 \xrightarrow{\wedge *\varphi} \Lambda^6_7 \xrightarrow{*} \Lambda^1_7 \xrightarrow{\wedge\varphi} \Lambda^4_7$$

is a bundle isomorphism and $\mathscr{F}(7T_7^5) = 4s^{-4/3}T_7^4$.

Moreover the occurrence of τ_1 in both (2) and (3) shows that

$$T_7^5 - \frac{1}{6}s^{4/3}T_7^1 \wedge *_\varphi\varphi = \frac{1}{2}(d(s^{4/3}) \wedge *_\varphi\varphi + (d\eta)^2_7 \wedge \varphi) \tag{3.2.6}$$

and

$$3\tau_1 \wedge *_\varphi\varphi = T_7^1 \wedge *_\varphi\varphi - \frac{3}{4}s^{-4/3}T_7^5, \tag{3.2.7}$$

in other words given the data $(s,(d\eta)^2_7)$ on L^7 any of the 7-dimensional torsion components $\tau_1, T_7^1, T_7^4, T_7^5$ determine the other three.

Proof. Using lemmas 2.1.3 and 3.2.1 we compute

$$\begin{aligned}*_\Phi d\Phi = {} & s^{-2}\eta \wedge (\mathrm{d}\eta)^2_{14} - 2s^{-2}\eta \wedge (d\eta)^2_7 - 3s^{2/3} *_\varphi (\tau^1 \wedge \varphi) - \tau_0\, s^{2/3}\varphi - s^{2/3}\tau_3 \\ & - s^{-2} *_\varphi (d(s^{4/3}) \wedge *_\varphi\varphi) \wedge \eta + s^{-2/3}\tau_2 \wedge \eta - 4s^{-2/3}\eta \wedge *_\varphi(\tau^1 \wedge *_\varphi\varphi).\end{aligned}$$

It now suffices to use the identity $7 *_\Phi T_8^1 = *_\Phi(d\Phi) \wedge \Phi$ and compare terms in the different G_2 modules. We demonstrate this for (1). From (2.2.13) we have

$$*_\Phi d\Phi = *_\Phi T^5_{48} + *_\Phi(T_7^1 \wedge \Phi) + *_\Phi(f\eta \wedge \Phi).$$

The last term can be expressed as

$$*_\Phi(f\eta \wedge \Phi) = fs^2\varphi$$

and thus comparing with the above expression we see that $f = -s^{-4/3}\tau_0$. This proves (1). The proofs of the rest are analogous. □

Remark 3.2.3. Note that the above construction can also be extended to non-free S^1 actions by working on the complement of the fixed point locus. The fixed point locus then corresponds to the region where s blows up. We shall in fact see an example of this below when we look at the Bryant-Salamon $Spin(7)$ metric.

Equipped with above proposition we can now proceed to studying the quotient of different types of $Spin(7)$-structures.

3.2.1 The torsion free quotient

Theorem 3.2.4 (Gibbons-Hawking ansatz for $Spin(7)$ manifolds)**.** *Assuming (N^8,Φ) is a $Spin(7)$-manifold, the quotient G_2-structure φ is calibrated and the curvature of the connection form η defined above is determined by*

$$(d\eta)^2_7 \wedge *_\varphi \varphi = -\frac{3}{2} *_\varphi d(s^{4/3}) \tag{3.2.8}$$

and

$$(d\eta)^2_{14} = s^{4/3}\tau_2, \tag{3.2.9}$$

or equivalently by

$$2d\eta = *_\varphi(d^{*_\varphi}(s^{4/3}\varphi)\wedge\varphi) - d^{*_\varphi}(s^{4/3}\varphi). \tag{3.2.10}$$

Proof. This follows directly from proposition 3.2.2. From (1), (3.2.7) and (5) we see that τ_0, τ_1 and τ_3 must vanish. The curvature equations follow from (3.2.6) and (4). □

The above equations have also been studied by Foscolo in [39], where the author studies 'adiabatic limits' of the equations to produce new complete non-compact $Spin(7)$ manifolds. The pair (3.2.8) and (3.2.9) generally constitute a complicated system of PDEs. A strategy to solving this system and hence constructing $Spin(7)$ metrics on the total space involves taking a formal limit of the equations as the size of the circle fibres tend to zero and thus, allowing for the system to degenerate to the torsion free G_2 equations. One then employs analytical techniques to perturb the latter equations to construct solutions to the original system. This limiting procedure of shrinking the fibres is referred to as the 'adiabatic limit'. In the G_2 setting a similar, but substantially harder, strategy was outlined in [30] to construct $K3$-fibred G_2 manifolds.

Remark 3.2.5.

1. First we note that if (N, Φ) has holonomy *equal to* $Spin(7)$ then it is necessarily non-compact. If N was compact then from the Bochner formula and the fact that g_Φ is Ricci-flat we know that every Killing vector field is parallel cf. [14, Theorem 1.84]. Thus, the holonomy group must be a strict subgroup of $Spin(7)$.

2. If the size of the circle orbits are constant i.e. s is constant then τ_2 is proportional to $d\eta$ so in particular τ_2 is closed. But from equation (4.35) of [19]

$$d\tau_2 = \frac{1}{7}\|\tau_2\|^2_\varphi + (d\tau_2)^3_{27}$$

   and hence $\tau_2 = 0$ i.e. $d\eta = 0$ and $N^8 = S^1 \times L^7$ (up to a finite quotient).

If we now further demand that (L^7, φ) is also torsion free then from Theorem 3.2.1 this forces the connection to be a G_2-anti-instanton i.e. $d\eta \in \Lambda^2_7$. More generally:

Definition 3.2.6. A G_2 instanton on a principal G-bundle Q over a G_2 manifold (L^7, φ) is a connection A whose curvature form $F_A \in \Gamma(L^7, ad(Q) \otimes \Lambda^2)$ satisfies $(F_A)^2_7 = 0$. A is called a G_2 anti-instanton if instead $(F_A)^2_{14} = 0$.

In our situation $G = U(1)$. Since ds is closed, $\nabla ds \in S^2(T^*L) \cong \Omega^3_1 \oplus \Omega^3_{27}$ (cf. 2.1.12) but we also have

$$d\eta \wedge *_\varphi \varphi = -\frac{3}{2} *_\varphi d(s^{4/3}).$$

As $d\eta$ and $d(s^{4/3})$ are related by a G_2-equivariant map it follows that the two components of $\nabla ds \in \langle g_\varphi \rangle \oplus S^2_0(L^7)$ are completely determined by the Λ^3_1 and Λ^3_{27} components of $dd\eta = 0 \in \Lambda^3$. Hence ds is a covariantly constant 1-form and as such *hol* $\subsetneq G_2$ [21, Theorem 4]. If s is constant then $d\eta = 0$ and (N^8, Φ) is the Riemannian product of manifold (L^7, φ) with holonomy contained in G_2 and a circle. If s is not constant then from Berger's classification of holonomy groups the universal cover of L^7 endowed with the pullback metric must have holonomy contained in $SU(3)$. Thus, we have proven the following.

Theorem 3.2.7. *If (N^8, Φ) is a torsion free $Spin(7)$-structure which is invariant under a free S^1 action generated by a non-constant vector field such that the quotient structure has holonomy contained in G_2 then $L^7 = P^6 \times \mathbb{R}^+$, where $(P^6, g_\omega, \omega, \Omega := \Omega^+ + i\Omega^-)$ is a Calabi-Yau 3-fold. Furthermore (N^8, Φ) is a Calabi-Yau 4-fold and is given by $\Phi = \frac{1}{2}\hat{\omega}^2 + Re(\hat{\Omega})$ where*

$$\hat{\omega} = s^{2/3}\omega + \eta \wedge d(s^{2/3}), \tag{3.2.11}$$

$$\hat{\Omega} = \Omega \wedge (-\eta - i\frac{2}{3}s^{5/3}ds), \tag{3.2.12}$$

$$g_{\hat{\omega}} = s^{2/3}g_\omega + s^{-2}\eta^2 + (\frac{2}{3}s^{2/3}\,\mathrm{ds})^2, \tag{3.2.13}$$

defines the $SU(4)$-structure and s is the coordinate on the $\mathbb{R}^+$ factor. The curvature form is $d\eta = -\omega$, corresponding to a G_2-anti-instanton, and the product G_2-structure is given by

$$\varphi = \frac{2}{3}s^{1/3}ds \wedge \omega + \Omega^+,$$

$$*_\varphi\varphi = \frac{1}{2}\omega^2 - \frac{2}{3}s^{1/3}ds \wedge \Omega^-.$$

Moreover this construction is reversible i.e. starting from a CY 3-fold $(P^6, g_\omega, \omega, \Omega)$ with $[-\omega] \in H^2(P^6, \mathbb{Z})$, we can choose a connection form η satisfying $d\eta = -\omega$ on the S^1 bundle defined by $[-\omega]$ together with a positive function s and thus define an irreducible CY 4-fold $(N^8, g_{\hat{\omega}}, \hat{\omega}, \hat{\Omega})$ by (3.2.11) ,(3.2.12) and (3.2.13).

Proof. Observe that it suffices to verify that $\hat{\omega}$ and $\hat{\Omega}$ defined by (3.2.11) and (3.2.12) are indeed closed. Firstly

$$d\tilde{\omega} = d(s^{2/3}) \wedge \omega + d\eta \wedge d(s^{2/3}) = 0,$$

where the last equality follows from the curvature assumption, and secondly

$$d\tilde{\Omega} = -\Omega \wedge (-d\eta) = 0$$

from (2.3.1). $\square$

The above theorem in fact recovers the so-called *Calabi model space*. More precisely, given a CY manifold P together with an ample line bundle L_P, the Calabi ansatz [23] gives a way of defining a new CY metric on an open set of L_P (this holds in all dimensions). Although incomplete, the Calabi model space provides a good approximation for the asymptotic behaviour of the complete Tian-Yau metrics [78] and has recently been employed in [53] to study new degenerations of hyperKähler metrics on $K3$ surfaces. Note that the Tian-Yau metrics are constructed on the complement of an anti-canonical divisor P in a (compact) Fano manifold and the normal bundle of divisor is then precisely L_P. The Tian-Yau metrics on the complement approximate the Calabi model space given above near infinity. We refer the reader to [53, Section 3] for a more precise statement.

Observe that taking (P^6, g_ω) to be the Riemannian product of a hyperKähler metric obtained by the Gibbons-Hawking ansatz and a flat torus $\mathbb{T}^2$ we get infinitely many holomomy $SU(4)$ metrics. We give a simple example to illustrate this construction. The metric below has also been described in [46] as a solution to the Hitchin flow starting from a 7-nilmanifold endowed with a cocalibrated G_2-structure.

Example. Consider $\mathbb{T}^6$, with coordinates $\theta_i \in [0, 2\pi)$, endowed with the flat CY-structure

$$\omega = e^{12} + e^{34} + e^{56},$$
$$\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6),$$

where $e^i = d\theta_i$. $[-\omega] \in H^2(\mathbb{T}^6, \mathbb{Z})$ defines a non-trivial S^1- bundle diffeomorphic to the nilmanifold Q^7 with nilpotent Lie algebra $(0,0,0,0,0,0,12+34+56)$ where we are using Salamon's notation cf. [75]. The connection form is locally given by

$$\eta = d\theta_7 + \theta_2 e^1 + \theta_4 e^3 + \theta_6 e^5,$$

where θ_7 denotes the coordinate of the S^1 fibre. Writing $s = r^3$ the CY metric on $Q^7 \times \mathbb{R}^+$ can be written as

$$g_{\hat{\omega}} = r^2 g_{\mathbb{T}^6} + r^{-6}(d\theta_7 + \theta_2 e^1 + \theta_4 e^3 + \theta_6 e^5)^2 + 4r^8 dr^2.$$

Using MAPLE we have been able to verify that indeed the matrix of curvature 2-form has rank 15 everywhere, confirming that the holonomy is *equal* to $SU(4)$. If we set $\rho = \frac{2}{5} r^5$ then the metric can be written as

$$g_{\hat{\omega}} = \big(\frac{5}{2}\,\rho\big)^{2/5} g_{\mathbb{T}^6} + \big(\frac{5}{2}\,\rho\big)^{-6/5}(d\theta_7 + \theta_2 e^1 + \theta_4 e^3 + \theta_6 e^5)^2 + d\rho^2$$

and in this form we can easily show that the volume growth $\sim \rho^{8/5}$ and the curvature tensor $|\mathrm{Rm}| \sim \rho^{-2}$ as $\rho \to \infty$. This metric is in fact incomplete at the end $\rho \to 0$ and complete at the end $\rho \to \infty$. By way of comparing with the approach in [46], the $SU(4)$ holonomy metric can also be obtained by evolving the cocalibrated G_2-structure on P determined by

$$\varphi = \eta \wedge \omega + Re(\Omega),$$

in the notation of Theorem 3.2.7. Our approach however avoids the problem of having to solve the Hitchin flow evolution equations and moreover, it explains why one only obtains $SU(4)$ holonomy metrics rather than $Spin(7)$ ones. As we have just seen one cannot obtain a holonomy G_2 metric from a $Spin(7)$ manifold via this construction. This suggests to study instead the geometric structure of the quotient calibrated G_2-structure. We shall do so in detail for the Bryant-Salamon $Spin(7)$-metric in section 3.4.2.

3.2.2 The locally conformally parallel quotient

Theorem 3.2.8. *If (N^8, Φ) is a locally conformally parallel $Spin(7)$-structure which is S^1-invariant then at least one of the following holds:*

1. *$N^8 \simeq L^7 \times S^1$ and the G_2-structure on L has $\tau_3 = 0$ in the notation of (2.1.13), or*

2. *(L^7, φ) is locally conformally calibrated i.e. τ_0 and τ_3 are both zero, and hence τ_1 is closed.*

Proof. Since $T^5_{48} = 0$ we know that T^5_7, T^5_{14},T^4_7 and T^4_{27} all vanish. From Proposition 3.2.2 it follows that $\tau_0 = -s^{4/3} f$, $\tau_1 = -s^{-4/3}(d(s^{4/3}) + \frac{2}{3} *_\varphi ((d\eta)^2_7 \wedge *_\varphi \varphi))$, $\tau_2 = -s^{-4/3}(d\eta)^2_{14}$ and $\tau_3 = 0$. From Proposition 3.2.2 we also get

$$T^1_7 = -3s^{-4/3} d(s^{4/3}) - 2s^{-4/3} *_\varphi ((d\eta)^2_7 \wedge *_\varphi \varphi)$$

Furthermore, differentiating $d\Phi = T^1_8 \wedge \Phi$ we have

$$dT^1_8 \wedge \Phi = 0.$$

As wedging with Φ defines an isomorphism of Λ^2 and Λ^6 it follows that T^1_8 is closed. Since $\mathscr{L}_X \Phi = 0$ we have

$$d(\iota_X d\Phi) = 0$$

and this shows that

$$\mathscr{L}_X T^1_8 \wedge \Phi = d(\iota_X d\Phi) = 0.$$

Thus $f = T^1_8(X)$ is constant and if non-zero then from (3.2.4) we have

$$d\eta = -\frac{1}{f} dT^1_7.$$

Since the latter is exact, the Chern class is zero and the bundle is topologically trivial i.e. $N^8 \simeq L^7 \times S^1$. Otherwise if $f = 0$ then $\tau_0 = 0$. $\square$

In [58, Theorem B] Ivanov et al. prove that any compact locally conformally parallel $Spin(7)$-structure fibres over an S^1 and each fibre is endowed with a nearly parallel G_2-structure i.e. the only non-zero torsion form is τ_0. Thus, it follows from Proposition 3.2.2 that one can construct many such examples by taking $N^8 = L^7 \times S^1$ where L^7 is a nearly parallel G_2-manifold and endow N^8 with the product $Spin(7)$-structure. In particular these examples cover case (1) above where the S^1 is only acting on the second factor. We also point out that aside from the fact that the cone metric on a nearly parallel G_2 manifold has holonomy contained in $Spin(7)$, there exists another Einstein metric, with instead positive scalar curvature, on $(0,\pi)\times L^7$ given by the sine-cone construction:

$$g_{sc} := dt^2 + \sin(t)^2 g_{L^7}.$$

The latter metric however does not seem to have been studied in detail in the literature. The fact that g_{sc} is Einstein is easily deduced since its Riemannian cone is Ricci-flat. Let us now show how situation (2) can arise. The reader might find it helpful to compare the following example with section 3.5.

Example. As above let $N^8 = S^7 \times S^1$, where S^7 is given the nearly parallel G_2-structure induced by restricting Φ_0 to $S^7 \hookrightarrow \mathbb{R}^8$. The induced G_2-structure φ_{S^7} satisfies

$$d\varphi_{S^7} = 4 \ *_{S^7} \varphi_{S^7}$$

and defines the standard round metric on S^7. Consider any free S^1 action, generated by a unit vector field X (say given by multiplication by an imaginary octonion), on S^7 preserving φ_{S^7}. We can then write

$$\varphi_{S^7} = \eta \wedge \omega + \Omega^+ \ \text{ and } \ *_{S^7} \varphi_{S^7} = \frac{1}{2}\omega\wedge\omega - \eta\wedge\Omega^-$$

cf. [3]. The intrinsic torsion of the quotient G_2-structure on $L^7 = \mathbb{C}P^3 \times S^1$, with coordinate θ on the circle, is then given by

$$d\varphi = 3(-\frac{4}{3}\, d\theta)\wedge\varphi,$$

$$d*_\varphi \varphi = 4(-\frac{4}{3}\, d\theta)\wedge *_\varphi\varphi - (\frac{2}{3}\omega + d\eta)\wedge\varphi,$$

confirming that indeed τ_0 and τ_3 vanish but τ_1 and τ_2 do not, cf. (2.1.13) and (2.1.14).

3.2.3 The balanced quotient

Since $T_8^1 = 0$, from (1) of proposition 3.2.2 we have $\tau_0 = 0$ and (2) gives

$$\tau_1 = -\frac{1}{24}(3s^{-4/3}d(s^{4/3}) + 2s^{-4/3} *_\varphi ((d\eta)^2_7 \wedge *_\varphi\varphi)). \tag{3.2.14}$$

Remark 3.2.9. Differentiating the balanced condition $*_\Phi(d\Phi)\wedge\Phi = 0$ we get

$$\|d\Phi\|^2_\Phi vol_\Phi = -(d *_\Phi d\Phi)\wedge\Phi = (\Delta_\Phi\Phi)\wedge\Phi.$$

In particular this shows that $d\Phi = 0$ i.e. Φ is torsion free iff

$$\Delta_\Phi\Phi\wedge\Phi = 0.$$

It is well-known that a $Spin(7)$-structure can be equivalently characterised by the existence of a non-vanishing spinor ψ, instead of the 4-form Φ. Following Theorem 4.3.4, the induced metric has holonomy contained in $Spin(7)$ if and only if the spinor is parallel. From this perspective the action of the Dirac operator $\not{D}$ on the spinor was shown to be completely determined by the torsion form T_8^1 cf. [57, (7.21)]. As a consequence, it follows that balanced $Spin(7)$-structures are characterised by the fact that they admit harmonic spinors i.e. $\not{D}\psi = 0$.

In [10] the authors construct many such examples on nilmanifolds by adopting a spinorial point of view. We instead here describe, via a few simple examples, a construction of balanced $Spin(7)$-structures starting from suitable G_2-structures. Henceforth we shall restrict to the case when $s = 1$ so that (3.2.14) can be equivalently written as

$$(d\eta)^2_7 = -4 *_\varphi (\tau_1 \wedge *_\varphi\varphi). \tag{3.2.15}$$

Theorem 3.2.10. (N^8, Φ) *is a free S^1-invariant balanced $Spin(7)$-structure if and* *only if the G_2-structure (L^7, φ) has $\tau_0 = 0$ and admits a section $\beta \in \Omega^2_{14}$ such that*

$$[\beta - 4 *_\varphi (\tau_1 \wedge *_\varphi \varphi)] \in H^2(M, \mathbb{Z})$$

or equivalently,

$$\{\kappa + 4 *_\varphi (\tau_1 \wedge *_\varphi \varphi) \mid [\kappa] \in H^2(M, \mathbb{Z})\} \cap \Omega^2_{14} \neq \emptyset. \tag{3.2.16}$$

Moreover, the $Spin(7)$-structure on the total space can be written as

$$\Phi = \eta \wedge \varphi + *_\varphi \varphi \tag{3.2.17}$$

where the connection form η satisfies $d\eta = \beta - 4 *_\varphi (\tau_1 \wedge *_\varphi \varphi)$ i.e.

$$(d\eta)^2_7 = -4 *_\varphi (\tau_1 \wedge *_\varphi \varphi) \ \textit{and} \ (d\eta)^2_{14} = \beta.$$

Proof. The *if* statement is clear since given β we can always choose a connection η with $d\eta = \beta - 4 *_\varphi (\tau_1 \wedge *_\varphi \varphi)$. Then define Φ by (3.2.17). The *only if* statement follows by setting $\beta = (d\eta)^2_{14}$. □

The reader might find such a theorem of little practical use in general. However, as we shall illustrate below via concrete examples, when L^7 is a nilmanifold Theorem 3.2.10 provides a systematic way of constructing balanced $Spin(7)$-structures.

Example. Let $L^7 = Q^5 \times \mathbb{T}^2$, where Q is a nilmanifold with an orthonormal coframing given by e^i for $i = 0, ..., 4$ and satisfying

$$de^i = 0, \ \text{ for } i \neq 4$$
$$de^4 = e^{02} + e^{31},$$

and for the flat $\mathbb{T}^2$ by e^6 and e^7. The G_2-structure defined by

$$\varphi = e^{137} + e^{104} + e^{162} + e^{306} + e^{324} - e^{702} - e^{746}.$$

has $\tau_0 = 0$. Hence from (3.2.15), to construct a balanced $Spin(7)$-structure we need to find a connection η whose Λ^2_7-curvature component satisfies

$$\begin{aligned}(d\eta)^2_7 &= -4 *_\varphi (\tau_1 \wedge *_\varphi \varphi) \\ &= \frac{2}{3}(e^{03} + e^{12} - e^{47}).\end{aligned}$$

Choosing $(d\eta)^2_{14}$ to be either of following 2-forms in Ω^2_{14}:

$$\begin{aligned}&\frac{1}{3}(e^{03} + e^{12} + 2e^{47}), \\ &\frac{2}{3}(2e^{12} - e^{03} + e^{47})\end{aligned}$$

gives connections with curvature forms $e^{03} + e^{12}$ and $2e^{12}$ respectively, and thus we obtain two distinct balanced $Spin(7)$-structures. Denoting η by e^5 the $Spin(7)$ form can once again be written in the standard form (2.2.1). This construction shows that given a balanced $Spin(7)$-structure on an S^1-bundle we can modify the Λ^2_{14}-component of the curvature form while keeping its Λ^2_7-component, already determined by τ_1, unchanged to construct a new balanced structure.

Suppose that we have fixed $d\eta = de^5 = 2e^{12}$. We can now take the S^1-quotient with respect to the Killing vector field e_4. In other words, the total space can be viewed as a different circle bundle with the new connection form $\tilde{\eta} := e^4$. We can repeat the above procedure with the new G_2-structure $\tilde{\varphi} := e_4 \lrcorner \Phi$, explicitly given by

$$\tilde{\varphi} = e^{501} + e^{523} + e^{567} + e^{026} + e^{073} - e^{127} - e^{136},$$

which of course has $\tilde{\tau}_0 = 0$. Once again to construct a balanced $Spin(7)$-structure we need a connection 1-form ξ satisfying

$$\begin{aligned}(d\xi)^2_7 &= (d\tilde{\eta})^2_7 \\ &= \frac{2}{3}(e^{02} + e^{31} - e^{57}).\end{aligned}$$

If we choose

$$(d\xi)^2_{14} = (d\tilde{\eta})^2_{14} + e^{51} + 2e^{26} + e^{37}$$

then $d\xi = e^{02} + e^{31} + e^{51} + 2e^{26} + e^{37}$ indeed defines an element in $H^2(\tilde{L}, \mathbb{Z})$. Thus this gives yet another balanced $Spin(7)$-structure. These three examples were found in [10] denoted by $\mathcal{N}_{6,22}$, $\mathcal{N}_{6,23}$ and $\mathcal{N}_{6,24}$, by instead using the spinorial approach described above and computing the Dirac operator.

The above examples in fact illustrate a new procedure for constructing balanced $Spin(7)$-structures on nilmanifolds: Starting from an S^1-invariant balanced $Spin(7)$-structure on a nilmanifold we know that the quotient G_2-structure φ has $\tau_0 = 0$. Given that the de Rham complex of the quotient nilmanifold $\check{L}^7$ is completely determined by the Chevalley-Eilenberg complex of the associated nilpotent Lie algebra, it is relatively straightforward to compute the set (3.2.16), via say MAPLE. Thus, by choosing distinct βs we can classify all invariant balanced $Spin(7)$-structures on *different* nilmanifolds which arise as circle bundles over $(\check{L}^7, \varphi)$. A general classification however appears to be quite hard. Closed G_2-structures on nilpotent Lie algebras, hence with $\tau_0 = 0$, were classified in [25]. Although a classification of 7-dimensional nilpotent Lie algebras is known cf. [50], those admitting G_2-structures with only vanishing τ_0 is not known.

Having encountered several examples of $Spin(7)$-structures it seems worth making a brief digression from our main example and derive some curvature formulae of $Spin(7)$-structures in terms of the torsion forms, rather than the metric, that the reader might find quite practical in specific examples.

3.3 Ricci and Scalar curvatures

In this section we derive formulae for the Ricci and scalar curvatures of $Spin(7)$-structures in terms of the torsion forms.

Formulae for the Ricci and scalar curvatures of G_2-structures in terms of the torsion forms seem to have first appeared in [19, (4.28), (4.30)] and for the $Spin(7)$ case in [57, (1.5), (7.20)]. The approach taken in each paper to derive the curvature formulae differ greatly. While Ivanov uses the equivalent description of $Spin(7)$-

structures as corresponding to the existence of certain parallel spinors, Bryant uses a representational theoretic argument. In [57], however, it is not obvious from the Ricci formula that it is a symmetric tensor and moreover the presence of a term involving the covariant derivative of the torsion form makes explicit computations quite hard. We instead adapt the technique outline in [19, Remark 10] to the $Spin(7)$ setting and derive an alternative formula.

Proposition 3.3.1. *The Ricci and scalar curvatures of a $Spin(7)$-structure (N,Φ) are given by*

$$\begin{aligned} Ric(g_\Phi) = & \left(\frac{5}{8}\,\delta T_8^1 + \frac{3}{8}\,\|T_8^1\|_\Phi^2 - \frac{2}{7}\,\|T_{48}^5\|_\Phi^2\right) g_\Phi \\ & + \mathrm{j}\left(-3\,\delta(T_8^1\wedge\Phi) + 4\,\delta T_{48}^5 - 2\,(T_8^1\wedge *_\Phi T_{48}^5) - \frac{9}{4} *_\Phi (T_8^1\wedge\Phi)\wedge T_8^1)\right) \\ & + \frac{1}{2}\, g_\Phi(\cdot \lrcorner *_\Phi T_{48}^5, \cdot \lrcorner *_\Phi T_{48}^5), \end{aligned}$$

$$Scal(g_\Phi) = \frac{7}{2}\,\delta T_8^1 + \frac{21}{8}\,\|T_8^1\|_\Phi^2 - \frac{1}{2}\,\|T_{48}^5\|_\Phi^2,$$

where $\delta = -*_\Phi d *_\Phi$ is the codifferential and $\mathrm{j} : \Lambda^4 \to S^2$ is defined in section 2.2.

Proof. Following Bryant's argument in [19] for the G_2 case, we first define the two $Spin(7)$-modules V_1 and V_2 by

$$(\mathfrak{gl}(8,\mathbb{R})/\mathfrak{so}(7)) \otimes S^k(\mathbb{R}^8) = V_k \oplus (\mathbb{R}^8 \otimes S^{k+1}(\mathbb{R}^8)),$$

where $S^k(\mathbb{R}^8)$ denotes the k^{th} symmetric power. Note that for any vector space V and $k \geq 2$ we have $S^2(V)\otimes S^k(V) = V\otimes S^{k+1}(V)\oplus \mathbb{S}_{(k,2)}(V)$, where the last summand is the Weyl module cf. [47, Chapter 6]. In particular, this shows that $V_k = \mathbb{S}_{(k,2)}(\mathbb{R}^8)\oplus\mathfrak{so}(7)^\perp\otimes S^k(\mathbb{R}^8)$ for $k \geq 2$. We shall refer to these modules to also mean the corresponding associated vector bundles on N. Representing irreducible $Spin(7)$-modules by the highest weight vector we have the following decomposition:

$$V_1 = V_{0,0,1} \oplus V_{1,0,1},$$

$$V_2 = V_{0,0,0} \oplus V_{1,0,0} \oplus V_{0,1,0} \oplus V_{1,1,0} \oplus V_{2,0,0} \oplus V_{0,2,0} \oplus 2V_{0,0,2} \oplus V_{1,0,2},$$

$$S^2(V_1) = 2V_{0,0,0} \oplus V_{1,0,0} \oplus V_{0,1,0} \oplus 2V_{1,1,0} \oplus 2V_{2,0,0} \oplus V_{0,2,0} \oplus 4V_{0,0,2} \oplus 2V_{1,0,2} \oplus V_{2,0,2}.$$

It is known that the second order term of the scalar curvature (i.e. terms involving two derivatives of Φ or equivalently one derivative of the intrinsic torsion) values in the trivial component of V_2 of which there is only one cf. [19, Remark 10]. This is spanned by δT_8^1. The first order terms are at most quadratic in sections of V_1 of which there are only two components. These are just the norm squared of the torsion forms: $\|T_8^1\|_\Phi^2$ and $\|T_{48}^5\|_\Phi^2$. So the scalar curvature can be expressed in terms of these three terms and to determine the coefficients it suffices to test it on a few examples. A similar argument applies for the traceless part of the Ricci tensor. The second order terms correspond to sections of the module $V_{0,0,2} \cong S_0^2(\mathbb{R}^8)$ in V_2 and there are exactly two of those. These are spanned by the projections of $\delta(T_8^1 \wedge \Phi)$ and δT_{48}^5. For the first order terms, they are given by sections of the module $V_{0,0,2}$ in $S^2(V_1)$. There are in fact four of those; one quadratic in T_8^1, two quadratic in T_{48}^5 and one mixed term. All but one quadratic term in T_{48}^5 appear in the Ricci formula. Again to determine the coefficients it suffices to test the formula on a few examples. This can be done quite easily using MAPLE. □

From the results of section 3.2 we have the following lemma.

Lemma 3.3.2. *In the S^1-invariant setting, δT_8^1, $\|T_8^1\|_\Phi^2$ and $\|T_{48}^5\|_\Phi^2$ are given in terms of the data (L^7, φ, η, s) by*

$$\delta T_8^1 = \frac{1}{7} s^{-4/3} \delta_\varphi (24 s^{2/3} \tau_1 + 4 s^{-1/3} ds + 2 s^{-2/3} *_\varphi ((d\eta)_7^2 \wedge *_\varphi \varphi))) \tag{3.3.1}$$

$$\|T_8^1\|_\Phi^2 = s^{-2/3} \tau_0^2 + \frac{1}{49} s^{-2/3} \|24\tau_1 + 4 s^{-1} ds + 2 s^{-4/3} *_\varphi ((d\eta)_7^2 \wedge *_\varphi \varphi)\|_\varphi^2 \tag{3.3.2}$$

$$\begin{aligned}\|T_{48}^5\|_\Phi^2 =& s^{-2/3} \|\tau_3\|_\varphi^2 + s^{-4/3} \|s^{-1} (d\eta)_{14}^2 + s^{1/3} \tau_2\|_\varphi^2 \\ &+ s^{-10/3} \|\frac{8}{7} (d\eta)_7^2 + \frac{4}{7} *_\varphi (d(s^{4/3}) \wedge *_\varphi \varphi) + \frac{4}{7} s^{4/3} *_\varphi (\tau_1 \wedge *_\varphi \varphi)\|_\varphi^2 \\ &+ 4 \|\frac{3}{7} s^{2/3} \tau_1 + \frac{2}{7} s^{-2/3} *_\varphi ((d\eta)_7^2 \wedge *_\varphi \varphi) + \frac{3}{7} s^{-2/3} d(s^{4/3})\|_\varphi^2\end{aligned} \tag{3.3.3}$$

where δ_φ is the codifferential of φ acting on k-forms by $\delta_\varphi = (-1)^k *_\varphi d *_\varphi$.

Proof. This is a straightforward albeit long computation using the expressions for the torsion forms of the $Spin(7)$-structure from Proposition 3.2.2. □

Of course these formulae are far from practical to compute the scalar curvature but nonetheless in the case of Riemannian submersions they do simplify considerably.

Corollary 3.3.3. *In the case of a Riemannian submersion i.e. $s = 1$,*

$$Scal(g_\Phi) = Scal(g_\varphi) - \frac{1}{2}\|d\eta\|^2_\varphi.$$

Proof. Combining the above lemma with our formula for scalar curvature and the one in the G_2 case from [19, (4.28)] we find that

$$\begin{aligned} Scal(g_\Phi) = Scal(g_\varphi) - \frac{1}{2}\|d\eta\|^2_\varphi - g_\varphi((d\eta)^2_{14}, \tau_2) + \delta_\varphi(*_\varphi((d\eta)^2_7 \wedge *_\varphi \varphi)) \\ + 4g_\varphi(*_\varphi \tau_1, (d\eta)^2_7 \wedge *_\varphi \varphi). \end{aligned}$$

On the other hand we also have that

$$\begin{aligned} \delta_\varphi(*_\varphi((d\eta)^2_7 \wedge *_\varphi \varphi)) &= - *_\varphi (d\eta \wedge d(*_\varphi \varphi)) \\ &= *_\varphi((d\eta)^2_{14} \wedge *_\varphi \tau_2) - *_\varphi((d\eta)^2_7 \wedge *_\varphi \varphi \wedge 4\tau_1) \\ &= g_\varphi(\tau_2, (d\eta)^2_{14}) - 4g_\varphi(*_\varphi \tau_1, (d\eta)^2_7 \wedge *_\varphi \varphi), \end{aligned}$$

where we used the fact that $(d\eta)^2_{14} \wedge *_\varphi \varphi = 0$ for the first equality. Combining the two expressions gives the result. □

Remark 3.3.4. Corollary 3.3.3 gives another way of showing that (L^7, φ) and (N^8, Φ) cannot both be Ricci-flat unless the S^1 bundle is (locally) trivial. The curvature form $d\eta$ measures the obstruction to integrability of the horizontal distribution and is a special case of the O'Neil formula [14, (9.37)].

We now turn to our main example namely the S^1 quotient of the Bryant-Salamon $Spin(7)$ metric.

3.4 S^1-quotient of Bryant-Salamon cone metric on squashed S^7

Let us first outline our general strategy to performing the quotient construction. In what follows we shall interpret the cone on S^7 as the spinor bundle of S^4 without the zero section. Recall that the fibres of the spinor bundle of S^4 are diffeomorphic to $\mathbb{R}^4 \simeq \mathbb{C}^2$. We shall consider the action of the diagonal $U(1)$ in $SU(2)$ on the fibres. This fibrewise quotient can be interpreted as a reverse Gibbons-Hawking (GH) ansatz as described in section 2.4. Since we will be working with the *anti-self dual* bundle of S^4 we use a slight variant of the usual GH ansatz so that the hyperKähler triples are anti-self dual rather than self-dual i.e. we consider

$$\omega_1 = \eta \wedge dx_1 - hdx_2 \wedge dx_3,$$
$$\omega_2 = \eta \wedge dx_2 - hdx_3 \wedge dx_1,$$
$$\omega_3 = \eta \wedge dx_3 - hdx_1 \wedge dx_2,$$

and the positive harmonic function h on $B \subset \mathbb{R}^3$ satisfies $*dh = d\eta$. We shall illustrate this construction in detail for the Hopf map by viewing our quotient construction as a fibrewise Hopf fibration in subsection 3.4.1.

Extending this to the total space we obtain the quotient G_2-structure on the anti-self dual bundle of S^4, see subsection 3.4.2. From the results of section 3.2.1 we know that the quotient G_2-structure cannot be torsion free but on the other hand, it is well-known that the anti-self dual bundle of S^4 also admits a holonomy G_2 metric cf. [21]. Motivated by the fact that both of these G_2-structures are asymptotic to a cone metric on $\mathbb{CP}^3$ we study the induced $SU(3)$-structures. In subsection 3.4.3 we give explicit formulae for the $SU(3)$-structures on the link and show that in both cases the induced almost complex structure corresponds to the nearly-Kähler one.

3.4.1 S^1-quotient of a fixed fibre of the spinor bundle

We begin by reminding the reader of the construction of the Bryant-Salamon $Spin(7)$ manifold. Given S^4 with the standard round metric and orientation, we denote by $\mathscr{P} \simeq SO(5)$ the total space of the $SO(4)$-structure. Since $H^2(S^4, \mathbb{Z}) = 0$, in particular the second Stiefel-Whitney class vanishes hence it is a spin manifold so we can lift $\mathscr{P}$ to its double cover $\tilde{\mathscr{P}} \simeq Sp(2)$. The spin group can be described explicitly via the well-known isomorphism

$$Spin(4) \cong Sp(1)_+ \times Sp(1)_- \cong SU(2)_+ \times SU(2)_-$$

where the $\pm$ subscripts distinguish the two copies of $SU(2)$. Taking the standard representation of $SU(2)_-$ on $\mathbb{C}^2_-$, we construct the (negative) spinor bundle $V_- := \tilde{\mathscr{P}} \times_{SU(2)_-} \mathbb{C}^2_-$ as an associated bundle.

There is also an action of $SU(2)$ on the fibres of V_- which can be described as follows. If we ignore the complex structure the fibres of V_- are simply $\mathbb{R}^4$ and its complexification is isomorphic to $\mathbb{C}^2_- \otimes \mathbb{C}^2$. The desired $SU(2)$ action is the standard action on $\mathbb{C}^2$ and is well-defined on the realification of $V_- \otimes \mathbb{C}$. In the description of the Bryant-Salamon construction in [21], this action on the fibre can also be viewed as a global $Sp(1)$ action (acting on the right) on $\mathbb{H}$ in

$$\tilde{\mathscr{P}} \times \mathbb{H} \xrightarrow{/Sp(1)_-} V_-,$$

thus commuting with the left action of $Sp(1)_-$ and hence passes to the quotient. Having now justified the existence of this $SU(2)$ action, we fix a point, $p \in S^4$ and describe the action of an $S^1 \hookrightarrow SU(2)$ on the fibre of V_-. This will enable us to describe a fibrewise HK quotient and then reconstruct the $\mathbb{R}^4$ fibre using the Gibbons-Hawking ansatz with harmonic function $h = 1/2R$ where R denotes the radius in $\mathbb{R}^3 - \{0\}$ as described in the previous section. Note that topologically the base manifold is just the anti-self dual bundle of S^4 which we denote by $\Lambda^2_- S^4$. This is due to the fact that the quotient construction reduces the $Sp(1)_-$ action on the $\mathbb{R}^4_-$ fibre to an action of $SO(3)_-$ on $\mathbb{R}^3 = \mathbb{R}^4/S^1$, as we shall see below, and the associated bundle construction for this representation is $\Lambda^2_- S^4$ cf. [74].

Let (x_1, x_2, x_3, x_4) denote the coordinates on the fibre, so that we may write the fibre metric as

$$g = \sum_{i=1}^{4} dx_i \otimes dx_i$$

i.e. g denotes the restriction of the Bryant-Salamon metric g_Φ to the vertical space. Denoting by r the radius function in the fibre, i.e. $r^2 = \sum_{i=1}^4 x_i^2$, we have $rdr = \sum_{i=1}^4 x_i dx_i$. We make the identifications $\mathbb{R}^4 \cong \mathbb{C}^2 \cong \mathbb{H}$ by

$$(x_1, x_2, x_3, x_4) = (x_1 + ix_2, x_3 + ix_4) = x_1 + ix_2 + jx_3 - kx_4$$

Consider now the $U(1)$ action on $\mathbb{R}^4 \cong \mathbb{C}^2$ given by

$$e^{i\theta}(z_1, z_2) = (e^{-i\theta} z_1, e^{+i\theta} z_2)$$

or equivalently by left multiplication by $-i$ on $\mathbb{H}$. Note that this S^1 is just the diagonal torus of $SU(2)$. The Killing vector field X generating this action is given by

$$X = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}.$$

and thus $\|X\|_g = r$. We also endow the fibre with a HK structure given by the triple

$$\begin{aligned}
\gamma_1 &= dx_1 \wedge dx_2 - dx_3 \wedge (-dx_4) \\
\gamma_2 &= dx_1 \wedge dx_3 - (-dx_4) \wedge dx_2 \\
\gamma_3 &= dx_1 \wedge (-dx_4) - dx_2 \wedge dx_3
\end{aligned}$$

They can be extended to a local orthonormal basis of the bundle $\Lambda^2_- S^4$ but the resulting forms will *not* be closed. The spin bundle *does* have a global HK structure, but arising from $SU(2)_+$ and since we have already fixed one of its complex structures, this HK structure is not relevant. In view of our quotient construction, we define

$$\eta := r^{-2} g_\Phi(X, \cdot) = r^{-2}(x_2 dx_1 - x_1 dx_2 - x_4 dx_3 + x_3 dx_4)$$

i.e. η is a connection 1-form on V_-. The map

$$\mu: \mathbb{R}^4 \to \mathbb{R}^3$$
$$(x_1, x_2, x_3, x_4) \mapsto (\mu_1, \mu_2, \mu_3),$$

where

$$\begin{aligned}\mu_1 &= \tfrac{1}{2}(x_1^2 + x_2^2 - x_3^2 - x_4^2),\\ \mu_2 &= x_1x_4 + x_2x_3,\\ \mu_3 &= x_1x_3 - x_2x_4\end{aligned}$$

is the HK moment map for the $U(1)$ action. By identifying $\mathbb{R}^3$ with $\mathrm{Im}(\mathbb{H})$, μ can also be expressed using quaternions as:

$$\mu(q) = \frac{1}{2}\bar{q}iq, \qquad q = x_1 + x_2 i + x_3 j - x_4 k,$$

making the S^1-invariance clear. Thus μ induces a diffeomorphism

$$\mathbb{R}^4/U(1) \simeq \mathbb{R}^3.$$

Note that strictly speaking this action is not free but nonetheless the construction can be carried out on $\mathbb{R}^4 - \{0\}$ and can be extended smoothly to the origin. A direct computation gives

$$\begin{aligned}\gamma_3 &= dx_{32} + dx_{41}\\ &= r^{-2}\Big((x_2dx_1 - x_1dx_2 - x_4dx_3 + x_3dx_4) \wedge (x_1dx_3 + x_3dx_1 - x_2dx_4 - x_4dx_2)\\ &\quad - (x_1dx_1 + x_2dx_2 + x_3dx_3 + x_4dx_4) \wedge (x_1dx_4 + x_4dx_1 + x_2dx_3 + x_3dx_2)\Big)\\ &= \eta \wedge d\mu_3 - h\, d\mu_1 \wedge d\mu_2,\end{aligned}$$

where $h = \frac{1}{2R}$ and $R = \sqrt{\mu_1^2 + \mu_3^2 + \mu_3^2}$ is the radius on $\mathbb{R}^3$. Similarly we obtain

$$\begin{aligned}\gamma_1 &= \eta \wedge d\mu_1 - h\, d\mu_2 \wedge d\mu_3\\ \gamma_2 &= \eta \wedge d\mu_2 - h\, d\mu_3 \wedge d\mu_1\end{aligned}$$

This confirms that η is the connection form that features in the Gibbons-Hawking ansatz with

$$g_{\mathbb{R}^4} = h^{-1}\eta \otimes \eta + h\pi^* g_U,$$

where $g_U = d\mu_1^2 + d\mu_2^2 + d\mu_3^2$ is the Euclidean metric on $\mathbb{R}^3$ with volume form $vol_{\mathbb{R}^3} = d\mu_{123}$. Using $R^2 = \sum_{i=1}^3 \mu_i^2 = \frac{1}{4}r^4$ we can directly verify that

$$*_{\mathbb{R}^3} dh = d\eta.$$

Having described the GH ansatz for the Euclidean space we proceed to our main example.

3.4.2 S^1-quotient of the Bryant-Salamon cone metric

We shall now take the quotient of Bryant-Salamon cone metric by applying the above construction to each $\mathbb{R}^4 - \{0\}$ fibre. The conical Bryant-Salamon $Spin(7)$ 4-form is given (pointwise) in our notation by

$$\Phi = 16r^{-8/5} dx_{1234} + 20r^{2/5} \sum \gamma_i \wedge \varepsilon_i + 25r^{12/5} dvol_{S^4},$$

where $\{\varepsilon_i\}$ is a local basis of ASD forms on S^4 and $dvol_{S^4}$ is the (pullback of) the volume form. The $Spin(7)$ metric is then given by

$$g_\Phi = 4r^{-4/5} \sum_{i=1}^{4} dx_i{}^2 + 5r^{6/5} g_{s^4},$$

and so the 1-forms $d\mu_i$ (or rather, $\pi^* d\mu_i = d(\mu_i \circ \pi)$) have norm

$$\|d\mu_i\|_\Phi^2 = \frac{1}{4} r^{14/5}.$$

On the other hand, from (3.2.1) we compute

$$s^{-2} = g_\Phi(X,X) = 4r^{6/5},$$

so $s = r^{-3/5}$. We know that the G_2 metric g_φ satisfies

$$g_\varphi = s^{-2/3}(g_\Phi - s^{-2}\eta^2) = r^{2/5}(g_\Phi - 4r^{6/5}\eta^2).$$

Considering the volume form of the fibre of the quotient we have

$$\begin{aligned} -r^{-2}d\mu_{123} &= -x_3dx_{123} - x_4dx_{124} + x_1dx_{134} + x_2dx_{234} \\ &= X \lrcorner\, dx_{1234}. \end{aligned}$$

Defining $d\nu_i = \iota_X\gamma_i$ we have that $d\nu_{123} = -d\mu_{123}$. Putting all together we have

$$\begin{aligned} X \lrcorner\, \Phi &= X \lrcorner\, 16r^{-8/5}dx_{1234} + 20r^{2/5}\sum(X \lrcorner\, \gamma_i)\wedge\varepsilon_i \\ &= 2^{11/5}(R^{-9/5}d\nu_{123} + 5R^{1/5}\textstyle\sum_{i=1}^3 d\nu_i\wedge\varepsilon_i). \end{aligned}$$

We can now extend this pointwise construction to the whole of V^-. From our construction, the induced G_2-structure on the quotient is given (after rescaling) by

$$\varphi_{GH} = \frac{1}{6}R^{-9/5}\beta + 5R^{1/5}d\tau.$$

We are here using the globally well-defined forms defined in [74, pg 94] (see also the appendix below) where τ is tautological 2-form on the ASD bundle and $\frac{1}{6}\beta$ is the volume form of the fibre which was pointwise denoted by dx_{1234}. By contrast the holonomy G_2 form is given by

$$\varphi_{BS} = \frac{1}{6}R^{-3/2}\beta + 2R^{1/2}d\tau.$$

Since the Bryant-Salamon metric on $\mathbb{R}^+\times\mathbb{CP}^3$ is just the cone metric on $\mathbb{CP}^3$ endowed with its nearly-Kähler (NK) metric we may also write it as

$$g_{BS} = dt^2 + t^2\,(\frac{1}{2}g_{S^4} + \frac{1}{4}\hat{g}_{S^2}),$$

where t denotes the coordinate of $\mathbb{R}^+$ and $g_{NK} := \frac{1}{2}g_{S^4} + \frac{1}{4}\hat{g}_{S^2}$is the NK metric (up to homothety). Here we are interpreting g_{NK} as a metric on the twistor space of

S^4 where g_{S^4} denotes the pullback of the round metric and $\hat{g}_{S^2}$ the metric on the S^2 fibres (see the appendix for more details). Comparing φ_{BS} with φ_{GH} and using our expression for g_{BS} we can perform a pointwise computation as above and show that

$$g_{GH} = dt^2 + \frac{8}{5}t^2\,(\frac{1}{2}g_{S^4} + \frac{1}{10}\hat{g}_{S^2}).$$

The quotient metric is thus the cone metric on the twistor space of S^4 but with "smaller" S^2 fibres. In order to gain better understanding of the geometric structure on the $\mathbb{CP}^3$ we look at the induced $SU(3)$-structure.

3.4.3 Remarks on the induced $SU(3)$-structure on $\mathbb{CP}^3$

Note that an oriented hypersurface in a 7-manifold with a G_2-structure naturally inherit an $SU(3)$-structure. If $\mathbf{n}$ denotes the unit normal to a hypersurface P^6 then the defining forms are given by:

$$\omega = \mathbf{n}\lrcorner\,\varphi\big|_{P^6},$$

$$\Omega^+ = \varphi\big|_{P^6} \quad \text{and} \quad \Omega^- = -\mathbf{n}\lrcorner *_\varphi \varphi\big|_{P^6}.$$

From definition 2.3.4 (1), we know that the NK structure on $\mathbb{C}P^3$ satisfies

$$d\omega_{NK} = 3\,\Omega^+_{NK} \ \text{ and } \ d\Omega^-_{NK} = -2\,\omega^2_{NK}. \tag{3.4.1}$$

In contrast the $SU(3)$-structure $(\omega_{GH}, \Omega^+_{GH}, \Omega^-_{GH})$ on the link (for $t = 1$) of the quotient G_2-structure satisfies

$$d\omega_{GH} = 3\,\Omega^+_{GH}$$
$$d\Omega^-_{GH} = -2\,\omega^2_{GH} - \frac{1}{5}\,(\frac{1}{5}\,\sigma - \tau)\wedge\omega_{GH}$$

The proof is a straightforward computation using the formulae in the appendix. Two things worth noting are that $\Omega^+_{GH} = \frac{32}{25}\,\Omega^+_{NK} = \frac{8}{25}\,d\tau$ so in particular both define the same almost complex structure and the extra-torsion component $\frac{1}{5}\sigma - \tau$ lives in Λ^2_8. Using the formulae from [12, Thm 3.4-3.6] we can confirm directly that this metric is not Einstein which is consistent with the canonical variation approach [14, pg. 258] which asserts that there are only two Einstein metrics in this family, the Fubini-Study metric and the NK one. Moreover it was conjectured in [40], based on numerical evidence, that in fact there are no other cohomogeneity one NK structure on $\mathbb{C}P^3$. Nonetheless the scalar curvature of g_{GH} is still constant and positive:

$$\begin{aligned} Scal(g_{GH}) &= 30 - \frac{1}{2} \cdot \| \frac{1}{5}(\frac{1}{5}\sigma - \tau) \|^2_{g_{GH}} \\ &= \frac{477}{16} > 0. \end{aligned}$$

It is also worth pointing out that this $SU(3)$-structure is half-flat, see definition 2.3.4 (4), and as such can be evolved by the Hitchin flow (for $SU(3)$-structures):

$$\frac{\partial}{\partial t}\Omega^+ = d\omega, \tag{3.4.2}$$

$$\frac{\partial}{\partial t}(\omega^2) = -2d\Omega^-, \tag{3.4.3}$$

to construct a torsion free G_2-structure. The resulting metric belongs to the general class of metrics of the form

$$g = dt^2 + a(t)^2 \hat{g}_{S^2} + b(t)^2 g_{S^4},$$

which were considered in [28, Sect. 5B]. It was also shown, after suitable normalisations, that the Bryant-Salamon metrics are the only solutions to this system.

Remark 3.4.1. Observe that, as in the GH ansatz for the Hopf map, this construction extends to the smooth Bryant-Salamon $Spin(7)$ metric with the same circle action but which now has as fixed point locus an S^4 corresponding to the zero section of the spinor bundle. Extending the above construction to the smooth metric simply amounts to replacing R by $R+1$ in the expressions φ_{BS} and φ_{GH}. Thus, we obtain a closed G_2-structure on all of $\Lambda^2_- S^4$.

3.5 S^1-quotient of flat $Spin(7)$ metric

We now consider a simpler situation: that of the S^1-reduction of the flat $Spin(7)$ structure $\Phi_0 = \frac{1}{8}(-d\alpha_1^2 + d\alpha_2^2 + d\alpha_3^2)$ on $\mathbb{R}^8$ where

$$\alpha_1 = -x_1dx_0 + x_0dx_1 + x_3dx_2 - x_2dx_3 - x_5dx_4 + x_5dx_4 + x_7dx_6 - x_6dx_7,$$

$$\alpha_2 = -x_2dx_0 + x_0dx_2 + x_1dx_3 - x_3dx_1 - x_6dx_4 + x_4dx_6 + x_5dx_7 - x_7dx_5,$$

$$\alpha_3 = -x_3dx_0 + x_0dx_3 + x_2dx_1 - x_1dx_2 - x_7dx_4 + x_4dx_7 + x_6dx_5 - x_5dx_6.$$

This explicit construction was motivated by the work of Acharya, Bryant and Salamon [2] where they investigate the S^1-reduction of the conical G_2 metric on $\mathbb{R}^+ \times \mathbb{C}P^3$. We can identify $\mathbb{R}^8$ with coordinates $(x_0, x_1, ..., x_7)$ with $\mathbb{H}^2$ by $(x_0 + ix_1 + jx_2 + kx_3, x_4 + ix_5 + jx_6 + kx_7)$. There are natural actions given by $Sp(2)$ acting by left multiplication and $Sp(1)$ acting by multiplication on the right. The 1-forms α_i are simply the dual of the S^1 actions given by right multiplication by the imaginary quaternions. We consider the S^1 action generated by the vector field

$$X = -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3 - x_5\partial_4 + x_4\partial_5 - x_7\partial_6 + x_6\partial_7$$

given by a diagonal $U(1) \subset Sp(2)$. A simple computation shows that

$$d(X \lrcorner d\alpha_i \wedge d\alpha_i) = 0 \quad \text{for} \quad i = 1, 2, 3$$

from which it follows that $\mathscr{L}_X\Phi_0 = 0$. Thus we get a closed G_2-structure on the quotient space $\mathbb{R}^+ \times \mathbb{C}P^3$ given by $\varphi = \iota_X\Phi$ from (3.2.1). Noting that Φ_0 is also invariant by the right S^1 action generated by the vector field

$$Y = -x_1\partial_0 + x_0\partial_1 + x_3\partial_2 - x_2\partial_3 - x_5\partial_4 + x_4\partial_5 + x_7\partial_6 - x_6\partial_7$$

i.e $\mathscr{L}_Y\Phi_0 = 0$ and that both S^1 actions commute, we can take the (topological) $\mathbb{T}^2$ reduction to the 6-manifold $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$. More concretely, we can split $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$ with coordinates x_0, x_1, x_4, x_5 on the first factor and x_2, x_3, x_6, x_7 on the second and we consider the equivalent $\mathbb{T}^2$ action given by the vector fields $\frac{1}{2}(X+Y)$ and $\frac{1}{2}(X-Y)$, each acting non-trivially on only one $\mathbb{R}^4$ factor. Using the HK moment maps as in the previous section we get coordinates u_i and v_i on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$ given by

$$\begin{aligned}
u_1 &= x_0^2 + x_1^2 - x_4^2 - x_5^2, & v_1 &= x_2^2 + x_3^2 - x_6^2 - x_7^2,\\
u_2 &= 2\,(x_0x_4 + x_1x_5), & v_2 &= 2\,(x_2x_6 + x_3x_7),\\
u_3 &= 2\,(x_0x_5 - x_1x_4), & v_3 &= 2\,(x_2x_7 - x_3x_6).
\end{aligned}$$

These coordinates can now be pulled back to $\mathbb{R}^+ \times \mathbb{C}P^3$ and will allow us to give an explicit expression for φ. From this point of view we have the S^1-bundle:

$$\mathbb{R}^+ \times \mathbb{C}P^3 \xrightarrow{/S^1} \mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}.$$

Following the Apostolov-Salamon construction [3] we can write

$$\varphi = \xi \wedge \omega + H^{3/2}\,\Omega^+ \tag{3.5.1}$$

$$*_\varphi \varphi = \frac{1}{2}\,H^2\omega^2 - \xi \wedge H^{1/2}\,\Omega^-, \tag{3.5.2}$$

where $H := \|Y\|_\varphi^{-1}$, ξ is the connection 1-form defined by

$$\xi(\cdot) := H^2\, g_\varphi(Y,\cdot)$$

and $(\omega, \Omega^+, \Omega^-)$ is the $SU(3)$-structure induced on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$. We now give coordinate expressions for the aforementioned differential forms.

Proposition 3.5.1. *In the above notation the closed G_2-structure on $\mathbb{R}^+ \times \mathbb{C}P^3$ given by $\varphi = \iota_X \Phi_0$ can be expressed as*

$$\varphi = \xi \wedge \frac{1}{2}\sum_{i=1}^{3} dv_i \wedge du_i + \frac{1}{8}\left(\frac{1}{u}\big(du_{123} - \{dv,du,du\}\big) + \frac{1}{v}\big(dv_{123} - \{dv,dv,du\}\big)\right),$$

where $\{dv, dv, du\}$ *denotes*

$$dv_1 \wedge dv_2 \wedge du_3 + dv_2 \wedge dv_3 \wedge du_1 + dv_3 \wedge dv_1 \wedge du_2,$$

similarly for $\{dv, du, du\}$. *Moreover we have*

$$H^{\frac{1}{2}}\Omega^- = \frac{1}{4R^{\frac{2}{3}}}\left(\{dv, dv, du\} - \{du, du, dv\} + \frac{u}{v}dv_{123} - \frac{v}{u}du_{123}\right),$$

$$H = \frac{R^{2/3}}{2\,u^{1/2}\,v^{1/2}},$$

where $R^2 := x_0^2 + \cdots + x_7^2$, $u^2 := u_1^2 + u_2^2 + u_3^2 = (x_0^2 + x_1^2 + x_4^2 + x_5^2)^2$ *and likewise for* v. *The curvature of the* S^1*-bundle over* $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$ *is given by*

$$d\xi = -\frac{\{v, dv, dv\}}{4(v_1^2 + v_2^2 + v_3^2)^{3/2}} + \frac{\{u, du, du\}}{4(u_1^2 + u_2^2 + u_3^2)^{3/2}},$$

where $\{v, dv, dv\}$ *denotes*

$$v_1 dv_2 \wedge dv_3 + v_2 dv_3 \wedge dv_1 + v_3 dv_1 \wedge dv_2,$$

and likewise for $\{u, du, du\}$.

The proof is a long computation which was carried out with the help of MAPLE. One can directly verify the above formulae hold using the definitions of u_i, v_i and expressing them in terms of x_i. The reader might find it interesting to compare our expressions to those in [2] for the torsion free G_2 quotient. In [54] Hitchin shows that an $SU(3)$-structure is completely determined by the pair (ω, Ω^+). Note that here Ω^+ can easily be read off from the expressions for φ and H in Proposition 3.5.1 and formula (3.5.1). Thus, we can explicitly compute the induced complex structure and metric on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$.

Proposition 3.5.2. *The metric induced by* (ω, Ω^+) *on* $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$ *is given by*

$$g_\omega = \frac{1}{2}\left(\frac{v^{1/2}}{u^{1/2}}\,(du_1^2 + du_2^2 + du_3^2) + \frac{u^{1/2}}{v^{1/2}}\,(dv_1^2 + dv_2^2 + dv_3^2)\right)$$

and the almost complex structure J by

$$J(u^{1/2}\partial_{u_i}) = v^{1/2}\partial_{v_i}, \quad \textit{for} \;\; i = 1,2,3.$$

Note that since φ is closed from (2.1.14) we have that

$$d *_\varphi \varphi = \tau_2 \wedge \varphi.$$

We shall now derive an explicit expression for the torsion of the G_2-structure φ. Under the inclusion $SU(3) \hookrightarrow G_2$ we can write the torsion form as

$$\tau_2 = \xi \wedge \tau_v + \tau_h$$

where τ_v and τ_h are basic 1-form and 2-form respectively i.e. they are (pullback of) forms on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$. It is not hard to show that $\tau_h \in \Lambda^2_6 \oplus \Lambda^2_8$ and that the Λ^2_6-component of τ_h is $SU(3)$-equivalent to τ_v (the proof is differed to Chapter 5). We compute τ_h and τ_v as

$$\begin{aligned}
\tau_h \cdot (3uv \cdot R^{8/3}) = & -u \cdot (\frac{1}{2}(\{u,dv,dv\} + \{v,dv,dv\}) + \frac{3u}{2v}\{v,dv,dv\}) \\
& - v \cdot (\frac{1}{2}(\{v,du,du\} + \{u,du,du\}) + \frac{3v}{2u}\{u,du,du\}) \\
& - \frac{1}{2}(u\{v,dv,du\} + v\{u,dv,du\}) \\
& - \frac{1}{2}(v\{u,du,dv\} + u\{v,du,dv\})
\end{aligned}$$

and

$$\begin{aligned}
\tau_v \cdot (\frac{3}{2} \cdot R^{8/3}) &= \sum_{i=1}^{3} \left(\frac{1}{v}(vu_i - 3uv_i)\, dv_i - \frac{1}{u}(uv_i - 3vu_i)\, du_i \right) \\
&= \Big(u \cdot dv - v \cdot du - 3(udv - vdu) \Big)
\end{aligned}$$

where $u \cdot dv$ denotes $\sum_{i=1}^{3} u_i dv_i$ and likewise for $v \cdot du$. From these expressions one can show that the Λ^2_8-component of τ_h is non zero and hence J is non-integrable.

Remark 3.5.3. Another way of viewing the above quotient construction is as follows. If we restrict the $Spin(7)$ 4-form Φ_0 on $\mathbb{R}^8$ to S^7 we get a G_2 4-form $*_{S^7}\varphi_{S^7}$ and the flat metric restricts to give the standard round metric. Since the cone metric is just the flat metric again, this means that this cocalibrated G_2 structure is inducing the round metric. We can now take the S^1-quotient with respect to any free S^1 action (say generated by multiplication by an imaginary octonion) preserving the round nearly parallel G_2-structure. Since this quotient is also a Riemannian submersion (as the size of the circle orbits are constant) the quotient metric is just the Fubini-Study metric. However by contracting the 4-form with the vector field generated by the S^1 we get the (negative) imaginary part of a $(3,0)$ form on the $\mathbb{C}P^3$. The latter induces an almost complex structure compatible with the Fubini-Study metric but which definitely cannot be the integrable one, otherwise this contradicts the fact that the canonical bundle of $\mathbb{C}P^3$ with the standard complex structure is non-trivial. From this perspective we see that the metric by the above closed G_2-structure is in fact the cone metric on $\mathbb{C}P^3$ endowed with the Fubini-Study metric. More explicitly, we can write the flat metric on $\mathbb{R}^8$ as

$$g_{\mathbb{R}^8} = dR^2 + R^2(\eta^2 + g_{FS}) = R^2\eta^2 + R^{-2/3}g_\varphi$$

where η is just the connection form of the S^1 action for the Fubini-Study quotient $S^1 \hookrightarrow S^7 \to \mathbb{C}P^3$ as above and $s := \|X\|_\Phi^{-1} = R^{-1}$. Thus, the metrics of proposition 3.5.1 can be equivalently expressed as

$$\begin{aligned} g_\varphi &= R^{2/3} \cdot dR^2 + R^{8/3} g_{FS} = dr^2 + \frac{16}{9} r^2 g_{FS}, \\ g_\omega &= 2(u\cdot v)^{1/2}\,(dR^2 + R^2\, g_{FS} - \frac{4\, u\cdot v}{R^2}\xi^2), \end{aligned}$$

in terms of more standard metrics. Note that by construction, the latter metric is invariant under the vector field Y and thus, passes to the quotient $(\mathbb{R}^+ \times \mathbb{C}P^3)/S^1$.

The above construction shows that the $\mathbb{T}^2$ reduction of $\mathbb{R}^8$ with its torsion free $Spin(7)$-structure has a rich underlying geometry. This motivates us to study the quotient of more general $Spin(7)$ manifolds, which is precisely the purpose of the next chapter.

Chapter 4

Kähler reduction of $Spin(7)$ metrics

4.1 Overview of chapter

In this chapter we study the Kähler reduction of torsion free $Spin(7)$-structures. More specifically we consider an eight-manifold N^8 endowed with a torsion free $Spin(7)$-structure which is invariant under the free action of a certain two-torus. In general the quotient six-manifold P^6 is only almost Kähler. Under the further assumption that the almost complex structure is integrable i.e. P^6 is Kähler, we discover that it inherits naturally either a $\mathbb{C}^\times$ or $(\mathbb{C}^\times)^2$-action. This allows us to perform a Kähler reduction, in the sense that this is both a symplectic and holomorphic quotient, to a complex surface M^4 or a complex curve Σ^2. Our main result is that one can reverse this construction i.e. starting from a Kähler manifold M^4 or Σ^2 with some additional data we can construct a $Spin(7)$ holonomy metric. By solving these equations in special cases we give many new explicit, though incomplete, examples of $Spin(7)$ metrics. The precise statements of our main results are given in Corollary 4.4.4 and Theorem 4.10.1.

Our construction, in the $\mathbb{C}^\times$ action case recovers the results of Apostolov and Salamon [3] in the special situation when N^8 is the product of a G_2 manifold L^7 and a circle. The key point of our construction relies on the fact that the Kähler assumption on P^6 implies that M^4 is endowed with a holomorphic symplectic form $\omega_2 + i\omega_3$. The $\mathbb{T}^2$ bundle (up to finite covers) is then determined by the cohomology classes $[\omega_2], [\omega_3] \in H^2(M, \mathbb{Z})$, ignoring factors of 2π. In the simplest instance, our construction can be viewed an as extension of the Gibbons-Hawking ansatz to $Spin(7)$ metrics, see Corollary 4.6.2 and 4.7.1. Thus, this gives an elementary way of constructing local $Spin(7)$ metrics starting from just a harmonic function on an open set of $\mathbb{R}^3$. Another interesting aspect of our construction is that it can also be viewed as a special case of the $\mathbb{T}^3$ reduction of $Spin(7)$ metrics via multi-moments as described by Madsen in [70]. This $\mathbb{T}^3$ action is obtained in our setting by considering, in addition to the original $\mathbb{T}^2$ action, the horizontal lift of the $S^1 \subset \mathbb{C}^\times$. The multi-moment map turns out to be the actual symplectic moment map for the Kähler reduction from P^6 to M^4. This shows that the name multi-moment map is indeed befitting.

In the case of $(\mathbb{C}^\times)^2$ action, we show that a similar theory holds. We show that the general problem of constructing a $Spin(7)$ metric can be reduced to choosing a positive harmonic function and solving a single PDE on a (real) 2-dimensional surface Σ^2 with trivial first Chern class. From this we are able to construct explicit examples of $Spin(7)$ metrics starting from just an elliptic curve and the punctured complex plane. In contrast to the previous situation, the horizontal lift of the $\mathbb{T}^2 \subset (\mathbb{C}^\times)^2$ action on P^6 does not preserve the $Spin(7)$-structure. Thus, our examples correspond to torus bundles over torus bundles; which we aptly call 'nilbundles'. In particular, our examples differ from those discovered by Madsen and Swann in the context of toric $Spin(7)$ manifolds [72] which instead have $\mathbb{T}^4$ symmetry.

Outline. In section 4.2 we carry out the $\mathbb{T}^2$ reduction of a torsion free $Spin(7)$-structure and describe the intrinsic torsion of the induced $SU(3)$-structure on the quotient 6-manifold. In section 4.3 we impose that the $SU(3)$-structure is Kähler i.e. that the almost complex structure is in fact integrable. We show that the quotient manifold is naturally equipped with Hamiltonian vector fields U and W which also preserve the complex structure. These vector fields can either span a line or a 2-plane in TN. We consider the two cases separately. In the former case, we carry out a Kähler reduction to a four-manifold M^4 endowed with a holomorphic symplectic form. We then explain how this procedure may be inverted in exactly two cases;

one corresponding to the situation when one of the circle bundle is trivial (which corresponds to the Apostolov-Salamon construction) and the second one when both circle bundles are non-trivial, see Theorem 4.3.4 and Corollary 4.4.4. We also explain how this reduces the local problem of finding $Spin(7)$ metrics to solving a *single* second order PDE (for a 1-parameter family of Kähler potentials) on an open set of $M^4 \times \mathbb{R}$. After stating a general existence result in the case when we have real analytic initial data on M^4, we then proceed to describe the simplest examples that can arise from our construction starting from hyperKähler four-manifolds. In section 4.6 we describe the examples of Gibbons et al. in our setup and in section 4.7 we give a new example of a $Spin(7)$ metric. In section 4.8 we explain how the simplest examples may also be obtained from the Hitchin flow of cocalibrated G_2-structures on certain nilmanifolds. In section 4.9 we show that one can perturb the (Kähler potential of the) examples of section 4.5 to construct more complicated ones, which are no longer of cohomogeneity one type. We illustrate this construction by giving an explicit example of a $Spin(7)$ metric by perturbing the GLPS example. In section 4.10 we address the situation when the commuting vector fields U and W are orthogonal. We carry out once again a Kähler reduction but now to a complex curve Σ^2. In this case we reduce the local problem of constructing a $Spin(7)$-metric to choosing a positive harmonic function on Σ^2 and solving a *single* PDE on an open set of $\Sigma^2 \times \mathbb{R}$. By inverting this construction we construct more examples of $Spin(7)$ metrics in sections 4.11 and 4.12.

4.2 $\mathbb{T}^2$-reduction of torsion free $Spin(7)$-structures

4.2.1 The general setup

We consider the problem of taking the quotient of a torsion free $Spin(7)$-structure (N^8, Φ) under the free action of a 2-torus generated by 2 *orthogonal* vector fields (this hypothesis will be assumed throughout this chapter). Since (N^8, Φ) is Ricci-flat, the hypothesis that the action is free and preserves Φ implies that if N^8 is compact then it is locally the Riemannian product of the flat $\mathbb{T}^2$ and a six-manifold with holonomy contained in $SU(3)$ cf. remark 3.2.5 (1). Thus, we shall assume that N^8 is non-compact, although our calculations are always valid in a small neighbourhood where such an action is free.

Denoting a pair of *perpendicular*[1] commuting vector fields generating this torus action by X and Y, our hypothesis is that

$$\mathscr{L}_X \Phi = \mathscr{L}_Y \Phi = 0.$$

The quotient six-manifold P^6 then inherits an $SU(3)$-structure. From a linear algebra point of view this follows from the fact that

$$\frac{Spin(7)}{G_2} = S^7 \quad \text{and} \quad \frac{G_2}{SU(3)} = S^6,$$

whereby the 2-plane in the tangent space of N^8 invariant under the $SU(3)$ action is generated by the span $\langle X, Y \rangle$. Denoting by $(\omega, \Omega = \Omega^+ + i\Omega^-)$ the induced $SU(3)$-structure on P^6, these relate to the $Spin(7)$ form Φ by

$$\Phi = \eta \wedge (\xi \wedge \omega + H^{3/2}\Omega^+) + s^{4/3}(\frac{1}{2}H^2 \omega \wedge \omega - H^{1/2}\xi \wedge \Omega^-),$$

where η and ξ denote the connection 1-forms defined by

$$\eta(\cdot) := s^2 g_\Phi(X, \cdot),$$
$$\xi(\cdot) := H^2 g_\varphi(Y, \cdot),$$

with $s := \|X\|_\Phi^{-1}$ and $H := \|Y\|_\varphi^{-1}$, and where $\varphi := \iota_X \Phi$ is denoting (the pullback of) the G_2-structure on the seven-manifold L^7 obtained from quotienting by the circle action generated by X;

$$(N^8, \Phi, g_\Phi) \xrightarrow{/S^1_X} (L^7, \varphi, g_\varphi) \xrightarrow{/S^1_Y} (P^6, \omega, g_\omega, \Omega).$$

[1] In appendix B we give the general expressions for the $Spin(7)$ and $SU(3)$-structures without this hypothesis on the vector fields

Note that ω and Ω^+ can equivalently be expressed as

$$\omega = Y \lrcorner X \lrcorner \Phi \quad \text{and} \quad \Omega^+ = H^{-3/2}(X \lrcorner \Phi - \xi \wedge \omega).$$

Remark 4.2.1. A priori the reader might find it unnatural that we are distinguishing the vector fields X and Y, since rather than performing a direct $\mathbb{T}^2$ reduction we are instead performing two circle quotients in succession. The advantage of this procedure of going through the intermediate G_2 quotient is that it makes it easier to reconcile our construction with the Kähler reduction of G_2 metrics [3].

The positive functions s and H are $\mathbb{T}^2$-invariant and as such are pullbacks of functions on P^6, which by abuse of notation we also denote by s and H. The associated metrics are then related by:

$$g_\Phi = s^{-2}\eta^2 + s^{2/3}H^{-2}\xi^2 + s^{2/3}Hg_\omega \tag{4.2.1}$$

A direct computation shows that the condition $d\Phi = 0$ is equivalent to $d\omega = 0$ together with the system

$$d\Omega^+ = -\frac{3}{2}d(\ln H)\wedge\Omega^+ - H^{-3/2}d\xi\wedge\omega, \tag{4.2.2}$$

$$d\Omega^- = -(\frac{4}{3}d^c(\ln s) + \frac{1}{2}d^c(\ln H))\wedge\Omega^+ - s^{-4/3}H^{-1/2}d\eta\wedge\omega, \tag{4.2.3}$$

$$H^{3/2}d\eta\wedge\Omega^+ + \frac{1}{2}d(H^2s^{4/3})\wedge\omega^2 - s^{4/3}H^{1/3}d\xi\wedge\Omega^- = 0, \tag{4.2.4}$$

where $d^c := J\circ d$. Here we follow the convention that J acts on a 1-form β by $J\beta(\cdot) = \beta(J\cdot)$, which differs from the convention in [3] by a minus sign.

From the results of the previous chapter we know that φ is closed. Moreover, from Theorem 3.2.7 we also know that φ is also coclosed, hence torsion free, if and only if g_Φ has holonomy contained in $SU(4)$.

From equations (4.2.2) and (4.2.3) it follows that $d\eta$ and $d\xi$ have no ω-component. Thus, $d\eta, d\xi \in \Lambda^2_6 \oplus \Lambda^2_8$ and we may write

$$d\eta\wedge\omega = \alpha_\eta\wedge\Omega^+ + (d\eta)^2_8\wedge\omega,$$

$$d\xi \wedge \omega = \alpha_\xi \wedge \Omega^+ + (d\xi)^2_8 \wedge \omega,$$

for 1-forms α_η and α_ξ on P^6 from (2.3.8). Condition (4.2.4) can then equivalently be expressed as

$$J(\alpha_\eta) - s^{4/3}H^{-1}\alpha_\xi = \frac{1}{2}H^{-3/2}d(H^2 s^{4/3}).$$

In view of (2.3.12), (2.3.13), (2.3.14), we can decompose the system (4.2.2), (4.2.3) into irreducible $SU(3)$-modules. The fact that the torsion form π_1 arises in both (2.3.13) and (2.3.14) allows us to express the 1-forms α_ξ and α_η only in terms of s and H. The result of this calculation is summed up in the following lemma:

Lemma 4.2.2. *The condition $d\Phi = 0$ is equivalent to $d\omega = 0$ and the system*

$$d\Omega^+ = d(\ln(H^{-1/2}s^{-1/3})) \wedge \Omega^+ - H^{-3/2}(d\xi)^2_8 \wedge \omega,$$
$$d\Omega^- = d^c(\ln(H^{-1/2}s^{-1/3})) \wedge \Omega^+ - s^{-4/3}H^{-1/2}(d\eta)^2_8 \wedge \omega,$$

with

$$J(\alpha_\eta) = H^{1/2}s^{1/3}ds \quad \text{and} \quad \alpha_\xi = -H^{1/2}dH + \frac{1}{3}H^{3/2}s^{-1}ds.$$

We shall be primarily interested in the case when the $SU(3)$-structure is Kähler, but before proceeding ahead we make the following important observation.

Proposition 4.2.3. *If s is constant then (L^7,φ) has holonomy contained in G_2 and (N^8,Φ) is the Riemannian product of L^7 and S^1. If furthermore, H is also constant then (P^6,ω,Ω) has holonomy contained in $SU(3)$ and (N^8,Φ) is the Riemannian product of P^6 and a flat 2-torus. Hence ξ and η cannot both be Hermitian Yang-Mills connections if (N^8,Φ) has holonomy $Spin(7)$.*

Proof. The argument is analogous to remark 3.2.5 (2). If s is constant then $d\eta \in \Lambda^2_8$. By differentiating the relation

$$d\eta \wedge \Omega^- = 0$$

we get that $\|d\eta\|_\omega = 0$. It follows that $[d\eta]$ defines a trivial class in $H^2(L,\mathbb{Z})$ and this proves the first claim. If H is also constant we can apply the same argument to $d\xi$. The last assertion now follows directly from Lemma 4.2.2. $\square$

Remark 4.2.4. Our construction also includes the $\mathbb{T}^2$ quotient of hyperKähler eight-manifolds and CY 4-folds under the group inclusions: $Sp(2) \subset SU(4) \subset Spin(7)$. As differential forms these can be expressed as

$$\begin{aligned}\Phi &= \frac{1}{2}(\omega_I \wedge \omega_I + \omega_J \wedge \omega_J - \omega_K \wedge \omega_K) \\ &= \frac{1}{2}(\hat{\omega} \wedge \hat{\omega}) + Re(\hat{\Omega}),\end{aligned}$$

where $(\omega_I, \omega_J, \omega_K)$ defines the hyperKähler triple and $(\hat{\omega}, \hat{\Omega})$ denotes the $SU(4)$-structure of the CY 4-fold. Note that from the results of section 3.5 we know that even if N^8 is a hyperKähler manifold it is not generally the case that the quotient $SU(3)$-structure is torsion free.

4.3 The Kähler reduction

4.3.1 The first reduction

We shall now impose that J is an integrable almost complex structure so that (P^6, ω, J) is a Kähler manifold. This implies that $d\eta, d\xi \in \Lambda^2_6$ (see 2.3.4) and thus we have

$$d\Omega^+ = d(\ln(H^{-1/2}s^{-1/3})) \wedge \Omega^+ \tag{4.3.1}$$

$$d\Omega^- = d^c(\ln(H^{-1/2}s^{-1/3})) \wedge \Omega^+ \tag{4.3.2}$$

with

$$J(\alpha_\eta) = H^{1/2}s^{1/3}ds \quad \text{and} \quad \alpha_\xi = -H^{1/2}dH + \frac{1}{3}H^{3/2}s^{-1}ds$$

satisfying

$$[*_\omega(\alpha_\eta \wedge \Omega^+)], [*_\omega(\alpha_\xi \wedge \Omega^+)] \in H^2(P^6, \mathbb{Z}).$$

Note that the latter requirement is the natural higher dimensional analogue of the 'integrality' condition that figures in the Gibbons-Hawking ansatz, see section 2.4

for comparison.

Remark 4.3.1. Since
$$d(H^{1/2}s^{1/3}\Omega) = 0$$
i.e. it is a *holomorphic* $(3,0)$-form, it follows that the Ricci form of (P^6,ω,Ω) is given by

$$\begin{aligned}\rho &= i\partial\bar{\partial}(\ln(Hs^{2/3}))\\ &= i\partial\bar{\partial}(\ln H) + \frac{2}{3}i\partial\bar{\partial}(\ln s).\end{aligned}$$

and the scalar curvature is

$$S = -d^{*_\omega}d(\ln(Hs^{2/3})),$$

where d^{*_ω} denotes the codifferential on P^6, cf. [65, Pg. 158].

Proposition 4.3.2. *The intrinsic torsion τ_2 of the closed G_2-structure φ is given by*

$$\begin{aligned}\tau_2 &= *_\omega(\frac{1}{3}H^{1/2}s^{-1}d^cs\wedge\Omega^+) - \frac{2}{3}H^{-1}s^{-1}\xi\wedge d^cs\\ &= -\frac{1}{3}s^{-4/3}d\eta - \frac{2}{3}H^{-1}s^{-1}\xi\wedge d^cs.\end{aligned}$$

Thus, it follows that Apostolov-Salamon construction [3], which considers the Kähler S^1 reduction of torsion free G_2-structures, corresponds to the case when the first circle bundle is just a trivial bundle i.e. $\alpha_\eta = 0$, or equivalently $d\eta = 0$ or s is constant (which by rescaling we can assume is 1). In our notation their result can be stated as follows:

Proposition 4.3.3. *Given a Kähler 6-manifold (P^6,ω,J) with an $SU(3)$-structure determined by the $(3,0)$-form $\Omega = \Omega^+ + i\Omega^-$ and a positive function H such that*

$$d(H^{1/2}\Omega^+) = 0$$

and

$$[-*_\omega(\frac{2}{3}d(H^{3/2})\wedge\Omega^+)]\in H^2(P,\mathbb{Z}), \qquad (4.3.3)$$

then

$$\varphi := \xi\wedge\omega + H^{3/2}\Omega^+$$

defines a torsion free G_2-structure on the S^1-bundle determined by (4.3.3), where ξ is a connection 1*-form on the circle bundle satisfying*

$$d\xi = -*_\omega(\frac{2}{3}d(H^{3/2})\wedge\Omega^+).$$

Moreover, the Hamiltonian vector field corresponding to $-H$ also preserves Ω, and hence J, and thus one can perform a Kähler reduction to a four-manifold endowed with a holomorphic symplectic structure.

Since we shall give explicit examples corresponding to the case when $s = H^{3/4}$ in sections 4.6 and 4.7, it is worth stating the corresponding proposition in this situation.

Proposition 4.3.4. *Given a Kähler 6-manifold (P^6,ω,J) with an $SU(3)$-structure determined by the $(3,0)$-form $\Omega=\Omega^+ + i\Omega^-$ and a positive function H such that*

$$d(H^{3/4}\Omega^+) = 0$$

and

$$[-*_\omega(\frac{1}{2}d(H^{3/2})\wedge\Omega^+)],\ [-*_\omega(\frac{1}{2}d^c(H^{3/2})\wedge\Omega^+)]\in H^2(P,\mathbb{Z}), \qquad (4.3.4)$$

then

$$\Phi := \eta\wedge\xi\wedge\omega + H^{3/2}\eta\wedge\Omega^+ + \frac{1}{2}H^3\omega^2 - H^{3/2}\xi\wedge\Omega^-$$

defines a torsion free Spin(7) structure on the $\mathbb{T}^2$-bundle determined by (4.3.4), where η and ξ are connection 1*-forms on the torus bundle satisfying*

$$d\xi = -*_\omega(\frac{1}{2}d(H^{3/2})\wedge\Omega^+),$$

$$d\eta = -*_\omega\left(\frac{1}{2}d^c(H^{3/2})\wedge\Omega^+\right).$$

Proof. The proof is simply a matter of setting $s = H^{3/4}$ in (4.3.1) and (4.3.2), and using the expressions for α_η and α_ξ. Unwinding the definitions of $d\xi$, $d\eta$ and Φ from the previous section gives the result. □

4.3.2 A second reduction

In order to perform a further reduction, we define, in hindsight, two Hamiltonian vector fields U and W by

$$\omega(U,\cdot) = -d(Hs^{-1/3})$$

and

$$\omega(W,\cdot) = ds.$$

Using these vector fields together with expressions (4.3.1), (4.3.2) and lemma 2.3.3, the curvature 2-forms of η and ξ can be equivalently expressed as

$$d\xi = -U\lrcorner\,(H^{1/2}s^{1/3}\Omega^+), \tag{4.3.5}$$

$$d\eta = -JW\lrcorner\,(H^{1/2}s^{1/3}\Omega^+). \tag{4.3.6}$$

Thus, by differentiating these equations and using (4.3.1) and (4.3.2) it follows that U and W, in addition to being Hamiltonian, also preserve the complex structure J. In other words, they define an infinitesimal symmetry of the (torsion free) $U(3)$-structure determined by (P,ω,J,g_ω).

Remark 4.3.5. Recall from section 2.3 that $stab(\Omega^+)\cong SL(3,\mathbb{C})$ and since

$$SL(3,\mathbb{C})\hookrightarrow GL(3,\mathbb{C})$$

it follows that changing Ω^+ by a positive factor leaves the induced complex structure J unchanged.

Indeed it is not generally true that U and W preserve the whole $SU(3)$-

structure. In fact, we have that

$$\mathscr{L}_U \Omega^+ = \mathscr{L}_W \Omega^+ = 0 \quad \text{if and only if} \quad \mathscr{L}_U s = 0.$$

We shall henceforth assume that this is indeed the case. The idea is now to perform a Kähler reduction using the action generated by these vector fields. In particular we shall investigate the following two situations:

1. $s = s(H)$ i.e. s is a function of H
2. s and $y := Hs^{-1/3}$ are independent functions, and the vector fields U and W are orthogonal i.e.
$$g_\omega(U,W) = \omega(W,JU) = 0.$$

Let us explain the geometry of these hypotheses. We shall always be working on an open set where U and W don't vanish. The assumption that s is invariant by U implies that W and JU are orthogonal. The two possibilities are either that W lies in the complex span of U and hence, W and U are equal up to some non-vanishing function, or that W has a non-trivial component orthogonal to the span $\langle U, JU \rangle$. So geometrically condition (1) is saying that the vector fields U and W define the same line field on P^6, whereas condition (2) means that we are assuming that the complex planes defined by $\langle U, JU \rangle$ and $\langle W, JW \rangle$ are in fact orthogonal to each other.

We shall consider these two cases separately, though our general strategy will the same in both cases. We first focus on situation (1) and defer the study of case (2) to section 4.10.

4.4 Further reduction I

4.4.1 S^1 Kähler reduction

Working under the assumption that $s = s(H)$ we can perform a Kähler reduction, with respect to the vector field U, to a four-manifold M^4. The reader will find the general theory of Kähler reduction elaborated in [56, Sect. 3C]. We shall describe this construction in our context in more detail.

First we introduce a connection 1-form α on P^6 by

$$\alpha(\cdot) := u\, g_\omega(U,\cdot),$$

where $u := \|U\|_\omega^{-2}$, so that $\alpha(U) = 1$. From the definition of U, we can express α and ω as

$$\alpha = ug(H)s^{-1/3}d^cH, \tag{4.4.1}$$

$$\omega = \tilde{\omega}_1(H) + s^{-1/3}g(H)\alpha \wedge dH, \tag{4.4.2}$$

where $g(H) := -1 + \frac{1}{3}Hs^{-1}s'$ and $'$ denotes the derivative with respect to H. We define a holomorphic $(2,0)$-form $\omega_2 + i\omega_3$, invariant under the complexified $U(1)$ action generated by the vector field U on M^4, by

$$H^{1/2}s^{1/3}\Omega = (\omega_2 + i\omega_3) \wedge (\alpha - iJ\alpha). \tag{4.4.3}$$

The symplectic form on the Marsden-Weinstein quotient M^4 of (P,ω), with moment map $-Hs^{-1/3}$, can then be identified with $\tilde{\omega}_1$. On the other hand, viewed as a GIT or holomorphic quotient a compatible complex structure J_1 on the quotient is defined by $\omega_2(\cdot,\cdot) = \omega_3(J_1\cdot,\cdot)$, cf. [74, Sect. 8]. We are assuming here that the quotient is carried out for regular values of the moment map or equivalently that this is the stable GIT quotient.

The final step of our construction is to impose the Kähler constraint on (ω,Ω) and to express it solely in terms of u, α, $\tilde{\omega}_1$, ω_2 and ω_3. In other words, we formulate the Kähler condition on P^6 purely in terms of the data on M^4.

Denoting by d_M and d_P the exterior differential on M^4 and P^6 respectively, and defining $d_M^c := J_1 \circ d_M$, we find that the condition $d\omega = 0$ implies that

$$\tilde{\omega}_1' = -s^{-1/3}g(H)d_M\alpha$$

and $d(H^{1/2}s^{1/3}\Omega) = 0$ implies that

$$\alpha' = -g(H)s^{-1/3}d^c_M u$$

and $d_M\alpha \wedge \omega_2 = d_M\alpha \wedge \omega_3 = 0$. The result of combining these two equations is (4.4.5) whereas the normalisation condition (2.3.2) leads to the algebraic constraint (4.4.4).

The fact that this construction is reversible follows by noting that given initial data on M^4 satisfying the above constraint we can define N^8 as the product of $\mathbb{R}^+_H$ and the bundle determined by the cohomology classes $[d\alpha], [d\xi]$ and $[d\eta]$. The result of this construction can be summed up in the following theorem.

Theorem 4.4.1. *Let* (M^4, J_1) *be a complex four-manifold endowed with a* 1*-parameter family of Kähler forms* $\tilde{\omega}_1(H)$*, a* 1*-parameter family of positive functions* $u(H)$ *and a holomorphic* $(2,0)$*-form given by* $\omega_2 + i\omega_3$ *satisfying the two conditions:*

$$\frac{1}{2}u(\omega_2 + i\omega_3) \wedge (\omega_2 - i\omega_3) = Hs^{2/3}\ \tilde{\omega}_1 \wedge \tilde{\omega}_1, \tag{4.4.4}$$

$$d_M d^c_M u = s^{2/3}g^{-2}\tilde{\omega}_1'' + \frac{1}{2}(s^{2/3}g^{-2})'\tilde{\omega}_1'. \tag{4.4.5}$$

Then

$$\varphi = \xi \wedge (\tilde{\omega}_1 + s^{-1/3}g\ \alpha \wedge dH) + Hs^{-1/3}\omega_2 \wedge \alpha - uHs^{-2/3}g\ \omega_3 \wedge dH$$

defines a closed G_2*-structure on* L^7*; the* $\mathbb{T}^2_{\alpha,\xi}$ *bundle determined by the integral cohomology classes* $[d\xi]$ *and* $[d\alpha]$ *on* $M^4 \times \mathbb{R}^+_H$*, where*

$$d\xi = -\omega_2,$$

$$d\alpha = s^{-1/3}g\ d^c_M u \wedge dH - s^{1/3}g^{-1}\tilde{\omega}_1'.$$

If we further assume that

$$[*_\omega(H^{1/2}s^{1/3}d^c s \wedge \Omega^+)] \in H^2(P^6, \mathbb{Z}) \tag{4.4.6}$$

so that there is another connection 1*-form* η *satisfying*

$$d\eta = - *_{\omega} (H^{1/2} s^{1/3} d_P^c s \wedge \Omega^+),$$

then the 4*-form*

$$\Phi = \eta \wedge \varphi + s^{4/3} *_{\varphi} \varphi$$

defines a torsion free $Spin(7)$*-structure on* N^8*; the total space of the* S^1 *bundle on* (L^7, φ) *defined by* $[d\eta] \in H^2(P^6, \mathbb{Z})$*, and the induced metric is given by*

$$g_{\Phi} = s^{-2}\eta^2 + (s^{2/3}H^{-2})\xi^2 + (s^{2/3}Hu^{-1})\alpha^2 + (g^2Hu)dH^2 + (s^{2/3}H)g_{\tilde{\omega}_1}. \quad (4.4.7)$$

Remark 4.4.2. For *generic* data on M^4, satisfying the hypothesis of the theorem, the holonomy of Φ is *not* a subgroup of G_2. If the holonomy is contained in G_2 then there exists a non-trivial parallel vector field, which also commutes with X, Y and U as they preserve Φ cf. [21, Theorem 4]. Since the curvature forms of η and α are non-trivial unless s is constant or $d_M u = 0$ and $\tilde{\omega}_1' = 0$, this vector field does not lie in the span of $\langle X, Y, U \rangle$ in general. Assuming this is the case, it must therefore descend to an infinitesimal symmetry of the Kähler structure on M^4 and $u(H)$. Thus, if we further assume that the data $(M^4, \omega_1(H), \omega_2 + i\omega_3, u(H))$ has no infinitesimal symmetry then the holonomy must be *either* $Spin(7)$, $SU(4)$ or $Sp(2)$ (at least locally). Note however that this is only a sufficient but not necessary condition as the horizontal lift of an infinitesimal symmetry of the data on M^4 will not preserve Φ in general. Although generically our construction should produce either $Spin(7)$, $SU(4)$ or $Sp(2)$ holonomy metrics, so far we have only been able find $Spin(7)$ metrics. Eliminating $SU(4)$ and $Sp(2)$ holonomy amounts to showing the inexistence of any parallel 2-form ω, which we have not been able to prove or disprove yet.

Proposition 4.4.3. *The Ricci form of* $(M^4, \tilde{\omega}_1, J_1, g_{\tilde{\omega}_1})$ *is given by*

$$\rho_M = \frac{1}{2} d_M d_M^c (\ln u).$$

Proof. This follows immediately from the fact that

$$\|\omega_2 + i\omega_3\|_{\tilde{\omega}_1} = c_0 \cdot u^{-1/2} H^{1/2} s^{1/3},$$

where c_0 is some positive constant, and that H and s are constants on M^4. $\square$

Thus, the induced metric on M^4 is Ricci-flat if and only if $\ln u$ is a harmonic function on M^4, for each value of H. In particular, if M^4 is compact then this means that u is only a function of H i.e. it is constant on M^4.

To sum up, what we have shown so far is that if a $Spin(7)$ manifold admits a $\mathbb{T}^2$-invariant 4-form Φ with $s = s(H)$ and that the resulting quotient six-manifold is Kähler then in fact there exists a third S^1 action preserving the $Spin(7)$-structure. To be more precise, the horizontal lift of the vector field U to N^8, still denoted by U by abuse of notation, also preserves Φ since

$$\mathscr{L}_U \eta = \mathscr{L}_U \xi = 0,$$

and commutes with X and Y. In fact, our construction fits into the more general framework investigated by Madsen in the context of multi-moment maps on $Spin(7)$-manifolds with $\mathbb{T}^3$ symmetry [70]. In our present situation the multi-moment map $\nu : N^8 \to \mathbb{R}$, defined by

$$d\nu = \Phi(X, Y, U, \cdot\ ),$$

corresponds to the Hamiltonian function $-Hs^{-1/3}$ and the four-manifold M^4 can be identified with the "multi-moment $Spin(7)$ reduction". Our perspective has however the advantage of inheriting a richer structure owing to the Kähler condition which we shall exploit in the next sections.

Note that one can generally solve equations (4.4.4) and (4.4.5) for many different choices of the function s and thus construct many closed G_2-structures. However, it is condition (4.4.6) that determines when we can lift such a G_2-structure to a torsion free $Spin(7)$-structure. This is precisely what we investigate next i.e. we shall solve equation (4.4.6) and thus determine for which function $s(H)$ we get a torsion free $Spin(7)$ structure.

4.4.2 The $Spin(7)$ condition

From equations (4.3.5) and (4.3.6) the curvature forms can be expressed as:

$$d\xi = -s^{1/3}H^{1/2}(U \lrcorner \Omega^+),$$
$$d\eta = \frac{H^{1/2}s^{1/3}s'}{s^{-1/3} - \frac{1}{3}Hs^{-4/3}s'}(JU \lrcorner \Omega^+).$$

We also recall that the holomorphic $(2,0)$-form defined by

$$\begin{aligned}\omega_2 + i\omega_3 &= \frac{1}{2}(U - iJU)\lrcorner\,(H^{1/2}s^{1/3})(\Omega^+ + i\Omega^-)\\ &= H^{1/2}s^{1/3}((U \lrcorner \Omega^+) + i(-JU \lrcorner \Omega^+))\end{aligned}$$

is closed, since $d(H^{1/2}s^{1/3}\Omega) = 0$, and by definition is invariant on the leaves of the foliation generated by holomorphic vector field $U - iJU$ and thus passes to the Kähler quotient M^4. It is now easy to see that the curvature forms are equivalently given by

$$d\xi = -\omega_2 \quad \text{and} \quad d\eta = -\Big(\frac{s'}{s^{-1/3} - \frac{1}{3}Hs^{-4/3}s'}\Big)\omega_3.$$

Remark on integrality and anti-instantons. Although ω_2 and ω_3 do not generally define *integral* classes in $H^2(M,\mathbb{R})$ this is nonetheless always true locally (say on a small ball since it is contractible) . In what follows we shall assume that the classes are indeed integral and the reader is welcome to interpret the results as always valid in a suitable open set. In the situation when M^4 is compact then, from Kodaira's classification of complex surfaces, M^4 is either a torus $\mathbb{T}^4$ or a $K3$ surface with

$$[H^{0,2} \oplus H^{2,0}] \cap H^2(M,\mathbb{Z}) = \langle [\omega_2], [\omega_3] \rangle_{\mathbb{Z}}.$$

The latter condition implies that the $K3$ surface has maximal Picard rank cf. [5, (2.15)] and [8, Pg. 325]. These have been classified by Shioda and Inose in [77]. In particular, our assumption implies that the connection forms ξ and η are abelian *anti-instantons* i.e. their curvature lies in $\Lambda^2_1 \oplus \Lambda^2_6 \cong \mathfrak{su}(3)^{\perp}$.

Thus, condition (4.4.6) now becomes equivalent to solving the non-linear ODE:

$$s' = A \cdot (s^{-1/3} - \frac{1}{3} H s^{-4/3} s'),$$

for $A \in \mathbb{Z}$. The solution is implicitly given by

$$AH = s^{1/3}(s+c), \tag{4.4.8}$$

where c is a constant of integration. If $A = 0$ then the positivity assumption on s forces c to be negative, and by rescaling s we can assume $c = -1$. Thus, $s = 1$ and Proposition 4.3.2 implies that Theorem 4.4.1 reduces to [3, Theorem 1]. In other words, setting $A = 0$ truncates the $Spin(7)$ equations to the G_2 equations considered in [3]. In what follows it will be more convenient to use s as the independent variable, rather than H.

Corollary 4.4.4. *Suppose that constants $A \neq 0$ and c are chosen such that s is positive in (4.4.8). Given a four-manifold M^4 with the data $(\tilde{\omega}_1, \omega_2, \omega_3, J_1, u)$ as in Theorem 4.4.1 and satisfying the two conditions:*

$$\frac{1}{2} u(\omega_2 + i\omega_3) \wedge (\omega_2 - i\omega_3) = A^{-1} s(s+c)\ \tilde{\omega}_1 \wedge \tilde{\omega}_1, \tag{4.4.9}$$

$$d_M d^c_M u = A^2 \frac{\partial^2}{\partial s^2}(\tilde{\omega}_1) \tag{4.4.10}$$

Then the 4-form

$$\Phi = \eta \wedge \varphi + s^{4/3} *_{\varphi} \varphi$$

defines a torsion free $Spin(7)$-structure on N^8; the total space of the $\mathbb{T}^3_{\alpha,\xi,\eta}$ bundle on $M^4 \times \mathbb{R}^+_H$ defined by

$$d\xi = -\omega_2,$$

$$d\eta = -A\cdot\omega_3,$$
$$d\alpha = -A^{-1}d^c_M u\wedge ds + A\frac{\partial}{\partial s}(\tilde{\omega}_1)$$

and the induced metric is given by

$$g_\Phi = s^{-2}\eta^2 + \frac{A^2}{(s+c)^2}\xi^2 + \frac{s(s+c)}{A\cdot u}\alpha^2 + \frac{s(s+c)u}{A^3}ds^2 + \frac{s(s+c)}{A}g_{\tilde{\omega}_1}.$$

Here the calibrated G_2-structure φ on L^7 is given by

$$\varphi = \ \xi\wedge(\tilde{\omega}_1 + \frac{1}{A}ds\wedge\alpha) + \Big(\frac{s+c}{A}\Big)\omega_2\wedge\alpha\ + \frac{u(s+c)}{A^2}\ \omega_3\wedge ds.$$

Before proceeding to the construction of explicit examples, we first give a general existence result and for simplicity we shall set $A=1$.

4.4.3 A general local existence result

Since $\tilde{\omega}_1$ is a 1-parameter family of Kähler forms there exists, on each suitably small open set $B\subset M^4$, a Kähler potential $f: B\to\mathbb{R}$, depending on s in a small interval, such that

$$\tilde{\omega}_1 = d_M d^c_M f.$$

Thus, we may always solve equation (4.4.10) by setting

$$u = \ddot{f} + \ddot{r}(s),$$

where $\dot{}$ refers to derivative with respect to s and $\ddot{r}$ is a non-negative function of *s only*, chosen to ensure that u is positive. Picking complex Darboux coordinates (z_1,z_2) on B, we can express the (2,0)-form as

$$\omega_2 + i\omega_3 = \ dz_1\wedge dz_2.$$

Defining $F := f + r$, equation (4.4.9) can now be expressed in these coordinates as

$$\Big(\frac{1}{s(s+c)}\Big)\ddot{F} = 4\det\Big(\frac{\partial^2}{\partial z_i \partial \bar{z}_j}F\Big)_{1\leq i,j\leq 2}, \tag{4.4.11}$$

where $\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$ for $j = 1,2$. Under the assumption that we are given real analytic initial data to (4.4.11), we may then appeal to the Cauchy-Kovalevskaya theorem for the general existence and uniqueness of a real analytic solution.

Corollary 4.4.5. *Given a real analytic Kähler potential F_0 on (an open set of) a complex surface $(M^4, J_1, \omega_2 + i\omega_3)$ and an additional real analytic function F_1, then there exists a unique real analytic solution $F(s)$, for s is a small interval, to (4.4.11) with $F(0) = F_0$ and $\dot{F}(0) = F_1$, and hence by Corollary 4.4.4 a torsion free $Spin(7)$-structure.*

Remark 4.4.6. Thus, we have abstractly proven that there exists a large class of $Spin(7)$ metrics admitting Kähler reduction. Our general solution is determined by 2 functions, namely the two initial conditions to (4.4.11), of 4 variables. By contrast, Bryant shows using Cartan-Kähler theory that an arbitrary $Spin(7)$ metric is determined by 12 functions of 7 variables cf. [18]. This difference is essentially due to the fact that the Kähler condition has allowed us to reduce the general problem to a *single* second order PDE involving a family of Kähler potentials.

For the sake of concreteness, we shall now investigate special cases when the pair (4.4.9) and (4.4.10), or equivalently (4.4.11), can be solved explicitly.

4.5 Constant solutions I

We first consider the simplest case in Corollary 4.4.4 when u is only a function of H i.e. $d_M u = 0$. Solving equation (4.4.10) we get

$$\tilde{\omega}_1 = s\check{\omega}_0 + \hat{\omega}_0,$$

where $\check{\omega}_0$ and $\hat{\omega}_0$ are closed 2-forms on M^4, independent of s. It is well-known that the wedge product on $\Lambda^2 := \Lambda^2(TM)$ defines a non-degenerate symmetric bilinear form B of signature $(3,3)$ given by

$$S^2(\Lambda^2) \to \Lambda^4 \cong \mathbb{R}$$
$$(\beta_1, \beta_2) \mapsto B(\beta_1, \beta_2)\theta,$$

where $\theta := \frac{1}{2}\omega_2 \wedge \omega_2 = \frac{1}{2}\omega_3 \wedge \omega_3$. The orientation form θ on M^4 allows for a splitting

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-,$$

with B positive definite on Λ^2_+ and negative definite on Λ^2_-. Restricting B to the 2-plane in Λ^2 spanned by $\langle \check{\omega}_0, \hat{\omega}_0 \rangle$ it follows from the theory of four-manifolds, cf. [74, Chap. 7], together with the fact that $\tilde{\omega}_1$ is of type $(1,1)$ and $\omega_2 + i\omega_3$ of type $(2,0)$ that there exist $(1,1)$-forms ω_0 and ω_1 spanning $\langle \check{\omega}_0, \hat{\omega}_0 \rangle$ and functions a, b, p, q on M^4 such that

$$\tilde{\omega}_1 = (a+bs)\omega_0 + (p+qs)\omega_1, \tag{4.5.1}$$

where,

$$\frac{1}{2}\omega_0 \wedge \omega_0 = -\theta, \quad \frac{1}{2}\omega_1 \wedge \omega_1 = \theta, \quad \omega_0 \wedge \omega_1 = 0.$$

The fact that $d_M \tilde{\omega}_1 = 0$ becomes equivalent to

$$d_M(a \cdot \omega_0 + p \cdot \omega_1) = d_M(b \cdot \omega_0 + q \cdot \omega_1) = 0.$$

We consider for simplicity the case when a, b, p, q are only constants. The fact that $\tilde{\omega}_1$ defines a positive definite metric implies that we need

$$p + qs > |a + bs|. \tag{4.5.2}$$

Hence it follows that the triple $(\omega_1, \omega_2, \omega_3)$ define a hyperKähler structure on M^4, while ω_0 is a closed anti-self-dual 2-form. Finally from equation (4.4.9) we have that

$$u = \frac{s(s+c)}{A} \cdot ((p+qs)^2 - (a+bs)^2). \tag{4.5.3}$$

The $\mathbb{T}^3_{\alpha,\xi,\eta}$ bundle on M^4 is then determined by

$$(d\alpha, d\xi, d\eta) = (A \cdot (b\omega_0 + q\omega_1), -\omega_2, -A \cdot \omega_3). \tag{4.5.4}$$

A trichotomy of the total space of the $\mathbb{T}^3$ bundle arises from whether $b > q$, $b = q$ or $b < q$. In summary we have

Theorem 4.5.1. *Given a hyperKähler manifold $(M^4, g_{HK}, \omega_1, \omega_2, \omega_3)$ with an almost Kähler form ω_0 compatible with g_{HK} but with the opposite orientation and suppose that (a,b,p,q) and u satisfy (4.5.2) and (4.5.3) respectively, then*

$$\begin{aligned}\Phi = \eta \wedge \Big(\xi \wedge (\tilde{\omega}_1 + \frac{1}{A} ds \wedge \alpha) + \frac{(s+c)}{A}(\omega_2 \wedge \alpha) + \frac{u \cdot (s+c)}{A^2}(\omega_3 \wedge ds)\Big) \\ + \frac{s^2(s+c)^2}{A^2}\Big(\frac{1}{2}\tilde{\omega}_1^2 + \frac{1}{A}(\tilde{\omega}_1 \wedge ds \wedge \alpha)\Big) - s \cdot \xi \wedge (\omega_3 \wedge \alpha - \omega_2 \wedge \frac{u}{A} ds)\end{aligned}$$

defines a torsion free $Spin(7)$-structure on the product of the $\mathbb{T}^3$ bundle determined by (4.5.4) and $\mathbb{R}^+_s$. If $a = b = 0$ then one can set $\omega_0 = 0$.

When $M^4 = \mathbb{T}^4$ these $\mathbb{T}^3$ bundles correspond to certain 2-step nilmanifolds. In the next two sections we give explicit examples which arise from our construction when we take $M^4 = \mathbb{T}^4$ with its flat hyperKähler structure.

4.6 Examples with holonomy $Spin(7)$, G_2, $SU(3)$ and $SU(4)$.

Our aim in this section is to describe certain special holonomy metrics admitting Kähler reductions. We shall explain how all these metrics can be obtained via a generalised version of the Calabi ansatz.

4.6.1 The GLPS examples

The $Spin(7)$ example we describe here was first discovered by Gibbons, Lü, Pope and Stelle (GLPS) in [49]. This is a special case of the constant solution when

$M = \mathbb{T}^4$ with $c = 0$, $A = 1$ and $(a,b,p,q) = (0,0,0,1)$. Choosing different integers A corresponds to pulling back their $Spin(7)$ 4-form Φ to covers of the circle bundle determined by $d\eta$. The induced $Spin(7)$ metric is then rescaled by a factor $A^{1/2}$ on the covering space. The G_2 example has also appeared in [3, 24]. An interesting feature in the following examples is that the symplectic form on P^6 is always the same but the complex structure (on the fibre) changes. Put differently, this means that each example below corresponds to a different integrable section of the associated bundle on (P^6,ω) with fibre $\frac{Sp(6,\mathbb{R})}{SU(3)}$.

Spin(7): Let $P^6 = Q^5 \times \mathbb{R}_t$, where Q^5 is a nilmanifold whose Lie algebra is given by

$$(0,0,0,0,13+42).$$

Recall that this means that we can choose a coframing e^i on Q^5 satisfying

$$de^5 = e^{13} + e^{42},$$
$$de^i = 0, \ \text{ for } i = 1,2,3,4.$$

We define a Kähler $SU(3)$-structure on P^6 by

$$\omega = d(t \cdot e^5),$$
$$\Omega = t(\sigma_3 + i\sigma_1) \wedge (-t^{-2}e^5 + it^2 dt),$$

where $\sigma_1 := e^{12} + e^{34}$, $\sigma_2 := e^{13} + e^{42}$ and $\sigma_3 := e^{14} + e^{23}$ denote the standard self-dual 2-forms on $\mathbb{T}^4$. The torsion forms, cf. Lemma 4.2.2, are then given by

$$d\Omega^+ = -t^{-1} dt \wedge \Omega^+,$$
$$d\Omega^- = t^{-5} e^5 \wedge \Omega^+.$$

Taking $H = t^{4/3}$ and $s = t$, we have $d\xi = \sigma_3$ and $d\eta = \sigma_1$. Hence from Proposition 4.3.4 it follows that Φ is torsion free. In fact one can verify that the holonomy group is *equal* to $Spin(7)$, using MAPLE for instance. This can be done by verifying that the dimension of the holonomy algebra, or equivalently by the Ambrose-Singer Theorem the rank of the curvature operator, is equal to 21. A curious observation is that the G_2 torsion form on L^7, given by

$$\tau_2 = -\frac{1}{3}t^{-4/3}\sigma_1 - \frac{2}{3}t^{-19/3}e^5 \wedge e^6$$

has as stabiliser $U(2)^- \hookrightarrow G_2$ acting by the adjoint representation on $\Lambda^2_{14} \cong \mathfrak{g}_2$. By contrast a generic element of $\mathfrak{g}_2$ only has $\mathbb{T}^2$ (the maximal abelian subgroup of G_2) as the identity component of the stabiliser group [19]. There are in fact two distinguished copies of $U(2)$ in G_2; denoted by $U(2)^+$ and $U(2)^-$, in the notation introduced by Ball in [7].

G_2: If we keep ω unchanged but modify the complex structure so that

$$\Omega = t(\sigma_3 + i\sigma_1) \wedge (-t^{-3/2}e^5 + it^{3/2}dt)$$

and take $H = t$ and $s = 1$, then from Proposition 4.3.3 we see that φ is torsion free. Here $d\xi$ is again defined as in the $Spin(7)$ example. One can once again verify that the holonomy group is *equal* to G_2 cf. [3, 49].

Remark 4.6.1. A natural question one might ask is whether there exist any extremally Ricci-pinched (ERP) closed G_2-structures cf. [19] on the S^1 bundle determined by $[\sigma_3] \in H^2(P, \mathbb{Z})$ given by

$$\varphi := \xi \wedge \omega + H^{3/2}\Omega^+$$

in the family of Kähler $SU(3)$-structures defined by

$$\omega = d(t \cdot e^5),$$
$$\Omega = t(\sigma_3 + i\sigma_1) \wedge (-fe^5 + if^{-1}dt),$$

for a suitable function $f(t)$. By definition an ERP G_2-structure satisfies

$$d\tau_2 = \frac{1}{6}(\|\tau_2\|^2_\varphi\ \varphi + *_\varphi(\tau_2 \wedge \tau_2)).$$

However, the answer turns out to be negative; the only ERP solution in this family is the torsion free one described above.

CY: Following the same strategy, it is easy to see that we obtain a torsion free $SU(3)$-structure by keeping ω unchanged and taking

$$\Omega = t(\sigma_3 + i\sigma_1) \wedge (-t^{-1}e^5 + it^1 dt).$$

Note that from Theorem 3.2.1, we can also construct a metric with holonomy $SU(4)$ from this Calabi-Yau 3-fold. The $SU(4)$-structure on $\hat{N}^8$ is given by

$$\hat{\omega} = s^{2/3}\omega + \hat{\eta} \wedge d(s^{2/3}),$$
$$\hat{\Omega} = \Omega \wedge (-\hat{\eta} - i\frac{2}{3}s^{5/3}ds),$$

with $d\hat{\eta} = -\omega$. Topologically $\hat{N}^8 = \hat{L}^7 \times \mathbb{R}^+_s$, where $\hat{L}^7$ is the S^1 bundle determined by $[-\omega] \in H^2(P^6, \mathbb{Z})$. This gives an example of a cohomogeneity two Einstein metric. Explicitly it is given by

$$\hat{g} = s^{2/3}(t^2dt^2 + t^{-2}(e^5)^2 + tg_{\mathbb{T}^4}) + s^{-2}\hat{\eta}^2 + (\frac{2}{3}s^{2/3}ds)^2.$$

By analogy to our construction, this can also be viewed as an 'inversion' of the Kähler reduction, from a CY 3-fold to a CY 4-fold, with the moment map is $s^{2/3}$.

4.6.2 $Spin(7)$ metrics from Gibbons-Hawking ansatz

It is clear that one can replace $\mathbb{T}^4$ by any hyperKähler manifold M^4 in the above example. Although it is not generally true that the triple of hyperKähler forms define integral cohomology classes this is nonetheless always true locally. Thus, combined with the Gibbons-Hawking ansatz this gives infinitely many local examples of $Spin(7)$ metrics starting from just a positive harmonic function on an open set in $\mathbb{R}^3$ cf. section 2.4.

Corollary 4.6.2. *Given a hyperKähler four-manifold* $(M^4, g_{HK}, \omega_1, \omega_2, \omega_3)$ *such that* $[\omega_1], [-\omega_2], [-\omega_3] \in H^2(M, \mathbb{Z})$*, let* K^7 *denote the total space of this* $\mathbb{T}^3$ *bundle. Then we can define a metric with holonomy contained in* $Spin(7)$ *on* $K^7 \times \mathbb{R}^+_s$ *by*

$$g_\Phi = s^{-2}\eta^2 + (s+c)^{-2}\xi^2 + (s+p)^{-2}\alpha^2 \\ + s^2(s+c)^2(s+p)^2\, ds^2 + s(s+c)(s+p)g_{HK}, \tag{4.6.1}$$

where $c,\ p \in [0, +\infty)$ *and the connection* 1*-forms* α*,* η*,* ξ *satisfy*

$$(d\alpha, d\xi, d\eta) = (\omega_1, -\omega_2, -\omega_3).$$

Moreover, if M^4 *admits a triholomorphic* S^1 *action then we can locally write*

$$g_{HK} = V^{-1}\theta^2 + V(dx^2 + dy^2 + dz^2)$$

via the Gibbons-Hawking ansatz and hence g_Φ *is completely determined by* V*.*

The metric (4.6.1) corresponds to the constant solution with $A = 1$, $(a,b,q) = (0,0,1)$ and is defined for $s \in (0,+\infty)$. This metric is incomplete at $s = 0$ since the circle fibre corresponding to the connection form η always blows up while the length of the other two circles fibres converge to c^{-1} and p^{-1} as $s \to 0$. It is not hard to see that g_Φ is complete as $s \to \infty$. The family of metrics given by (4.6.1), especially in the case when M^4 is $\mathbb{T}^4$ or a suitable $K3$ surface, might be useful in future gluing problems as in the hyperKähler case recently investigated in [53].

A remark on the 'generalised Calabi ansatz.' The $SU(3)$ and $SU(4)$ holonomy metrics appearing in this section in fact arise from a special case of the Calabi ansatz [23]. In our setting this can be neatly described as follows: given a Calabi-Yau n-fold $\hat{N}^{2n}$ with symplectic form σ and holomorphic volume form Ψ we define a connection 1-form γ on the line bundle $L_{\hat{N}}$ with Chern class determined by

$$d\gamma = -\sigma.$$

We then obtain a torsion free $SU(n+1)$-structure on an open set of the total space $L_{\hat{N}}$ given by

$$\hat{\sigma} = -d(r^2\gamma),$$
$$\hat{\Psi} = \Psi \wedge (\gamma + i\frac{2}{n+2}d(r^{n+2})),$$

where r denotes a radial coordinate from the zero section. The examples above can thus be interpreted as a '*generalised Calabi ansatz*' for exceptional holonomy metrics, whereby one uses the hyperKähler forms ω_1, ω_2 and ω_3 in succession to construct $SU(3)$, G_2 and $Spin(7)$ holonomy metrics. The relations between the Einstein manifolds described above can be expressed in the diagram below.

$$\begin{tikzcd}
& (N^8,\Phi) \arrow[dl, "\mathbb{T}^2"'] \arrow[dr, "S^1"] \arrow[dd, "\mathbb{T}^3\times\mathbb{R}^+"] & \\
(P^6,\omega,\Omega) \arrow[dr, "\mathbb{C}^*"'] & & (L^7,\varphi) \arrow[dl, "\mathbb{T}^2\times\mathbb{R}^+"] \\
& (M^4,\omega_1,\omega_2,\omega_3) &
\end{tikzcd}$$

The submersions of P^6 and L^7 with their Ricci flat metric to M^4 correspond to the Calabi ansatz and Apostolov-Salamon construction, respectively.

By contrast to the above examples, where all the circle bundles were determined by the hyperKähler forms (since we had $d\alpha = \sigma_2$), in the next section we give examples corresponding to the case when $b \neq 0$.

4.7 More examples

The G_2 example we describe in this section has also appeared in [3, 24] but the $Spin(7)$ metric does not seem to have been mentioned in the literature.

Let $P^6 = Q^5 \times \mathbb{R}_t$, where Q^5 is a nilmanifold whose Lie algebra is given by

$$(0,0,0,0,24).$$

So Q^5 is again topologically a circle bundle over $\mathbb{T}^4$. We define a Kähler $SU(3)$-structure on P^6 by

$$\begin{aligned}\omega &= e^{13} - d(t^2e^5),\\ \Omega &= t(-\sigma_1 + i\sigma_3) \wedge (-2t^3dt + it^{-2}e^5),\end{aligned}$$

where σ_i denote the standard self-dual 2-forms as before. The torsion forms are given by

$$\begin{aligned}d\Omega^+ &= -t^{-1}dt \wedge \Omega^+,\\ d\Omega^- &= -\frac{1}{2}t^{-6}e^5 \wedge \Omega^+.\end{aligned}$$

Taking $H = t^2$ so that $d\xi = -\sigma_3$, one can verify directly that the hypothesis of Proposition 4.3.3 are satisfied. We thus get a holonomy G_2 metric as described in [3]. This was also shown to arise from the Hitchin flow of half-flat $SU(3)$-structures in [24]. As before we keep ω unchanged and consider

$$\Omega = t(-\sigma_1 + i\sigma_3) \wedge (-2t^4dt + it^{-3}e^5).$$

Taking $H = t^{8/3}$ we see that the hypothesis of Proposition 4.3.4 are satisfied, with $d\eta = -\sigma_1$ and $d\xi = -\sigma_3$. Thus we get a metric with holonomy *equal* to $Spin(7)$ explicitly given by

$$g_\Phi = (t^2e^1)^2 + (t^3e^2)^2 + (t^2e^3)^2 + (t^3e^4)^2 + (t^{-1}e^5)^2 + (t^{-2}\eta)^2 + (t^{-2}\xi)^2 + 4t^{12}dt^2.$$

Of course, we can also construct holonomy $SU(3)$ and $SU(4)$ metrics by carrying out an analogous argument as in the previous section.

4.7.1 $Spin(7)$ metrics from Tod's ansatz

As in section 4.6 there is a natural way of obtaining many local examples using the so-called Tod's ansatz cf. [4, Prop. 3.1]. The idea is again to apply the Gibbons-Hawking ansatz but choosing the harmonic function to depend on only two variables.

In the notation of section 2.4, if we choose $V : B \to \mathbb{R}^+$, independent of say coordinate x, then in addition to the Gibbons-Hawking hyperKähler triple, we can also define an almost Kähler form by

$$\omega_0 = \theta \wedge dx - V\, dy \wedge dz.$$

Thus, we can again appeal to the result of section 4.5 to construct $Spin(7)$ metrics. In particular, for $A = 1$ and $(a,b,q) = (0,1,1)$ we have:

Corollary 4.7.1. *Given a hyperKähler four-manifold $(M^4, g_{HK}, \omega_1, \omega_2, \omega_3)$ together with an almost Kähler form ω_0 compatible with the opposite orientation such that $[\omega_0 + \omega_1], [-\omega_2], [-\omega_3] \in H^2(M,\mathbb{Z})$, let K^7 denote the total space of this $\mathbb{T}^3$ bundle. Then we can define a metric with holonomy contained in $Spin(7)$ on $K^7 \times \mathbb{R}^+_s$ by*

$$\begin{aligned} g_\Phi = s^{-2}\eta^2 + (s+c)^{-2}\xi^2 + p^{-1}(2s+p)^{-1}\alpha^2 \\ + ps^2(s+c)^2(2s+p)\, ds^2 + s(s+c)g_{\tilde{\omega}_1}, \end{aligned}$$

where c, $p \in (0,+\infty)$, $g_{\tilde{\omega}_1}$ is defined by (4.5.1) and the connection 1-forms α, η, ξ satisfy

$$(d\alpha, d\xi, d\eta) = (\omega_0 + \omega_1, -\omega_2, -\omega_3).$$

Moreover, if M^4 admits a triholomorphic S^1 action then g_Φ is completely determined by the harmonic function $V(y,z)$, as in Tod's ansatz.

4.8 Hypersurfaces and Hitchin's flow

In this section we explain how the aforementioned metrics may also be obtained by evolving suitable cocalibrated G_2-structures. It is well-known that an oriented hypersurface $\tilde{L}$ in a $Spin(7)$ manifold (N^8, Φ) inherits a cocalibrated G_2-structure defined by

$$\phi = \mathbf{n} \lrcorner\ \Phi\big|_{\tilde{L}},$$

where $\mathbf{n}$ denotes the unit normal vector field. As a converse Hitchin shows that given a cocalibrated G_2-structure ϕ_0 on a compact seven-manifold $\tilde{L}$ one can define a torsion free $Spin(7)$-structure on $N = \tilde{L} \times (0,T)$ by solving the system

$$d_{\tilde{L}}(*_{\phi_t}\phi_t) = 0, \tag{4.8.1}$$

$$\frac{\partial}{\partial t}(*_{\phi_t}\phi_t) = d_{\tilde{L}}\phi_t, \tag{4.8.2}$$

where $t \in (0,T)$, [55, Theorem 7]. Furthermore, Bryant shows that if ϕ_0 is real analytic then there always exists a local solution to (4.8.1) and (4.8.2), cf. [20, Theorem 7]. The resulting $Spin(7)$ form on N^8 is then given by

$$\Phi = dt \wedge \phi_t + *_{\phi_t}\phi_t.$$

Recall that we introduced the analogous Hitchin's flow of $SU(3)$-structures in section 3.4.3. In fact the G_2 holonomy metrics appearing in the last two sections have been described via this technique in [24]. We shall now explain how the $Spin(7)$ examples corresponding to constant solutions may also be obtained via the Hitchin flow (of G_2-structures). From the definition of α and the expression relating the metrics g_Φ and g_ω, it is straightforward to compute

$$\|dH\|_{g_\Phi} = \frac{A}{u^{1/2}H^{1/2}s^{1/3}s'}.$$

Thus we can define a geodesic coordinate t on N^8 by

$$t = \frac{1}{A^{3/2}}\int s(s+c)((p+qs)^2-(a+bs)^2)^{1/2}ds.$$

The hypersurfaces $\tilde{L}_t$ in N^8 corresponding to level sets of t are the $\mathbb{T}^3_{\alpha,\xi,\eta}$ bundles over M^4 defined by (4.5.4) and are endowed with cocalibrated G_2-structures ϕ_t. From our expression for Φ we have that

$$*_{\phi_t}\phi_t = \eta\wedge\xi\wedge\tilde{\omega}_1 + \Big(\frac{s+c}{A}\Big)\eta\wedge\alpha\wedge\omega_2 + \frac{s^2(s+c)^2}{2A^2}\tilde{\omega}_1\wedge\tilde{\omega}_1 + s\alpha\wedge\xi\wedge\omega_3.$$

It is easy to see, from the expressions of the curvature forms (4.5.4), that (4.8.1) holds. It is straightforward to verify that (4.8.2) is also satisfied.

For instance, in the GLPS $Spin(7)$ example we find that $4t = H^3 = s^4$ and an orthonormal coframing for ϕ_t is given by

$$s^{3/2}e^1,\ s^{3/2}e^2,\ s^{3/2}e^3,\ s^{3/2}e^4,\ s^{-1}\eta,\ s^{-1}\xi,\ s^{-1}\alpha.$$

Remark 4.8.1. Although there are many cocalibrated G_2-structures on nilmanifolds, the scarcity of finding explicit metrics with holonomy *equal* to $Spin(7)$ stems from the fact that the Hitchin flow is generally hard to solve and moreover, it often leads to $SU(4)$ holonomy metrics rather than $Spin(7)$ cf. [46] and theorem 3.2.1.

4.9 Perturbation of constant solutions

In this section we describe explicit solutions to Corollary 4.4.4 which vary on M^4 i.e. with $d_M u \neq 0$. Our solutions are obtained by perturbing the Kähler potential of the constant solution examples. We shall again assume that $(M^4,\omega_1,\omega_2,\omega_3)$ is a hyperKähler manifold together with an anti-self-dual 2-form ω_0 as defined in section 4.5. We look for solutions to (4.4.9) and (4.4.10) with $\tilde{\omega}_1$ of the form

$$\tilde{\omega}_1 = (a+bs)\omega_0 + (p+qs)\omega_1 + d_M d_M^c G.$$

When $G=0$ we recover the constant solution metrics. We also know from the global dd^c lemma that any Kähler form in the same cohomology class can be expressed in this form for some function G. Equation (4.4.10) can now be written as

$$d_M d_M^c(u - A^2\ddot{G}) = 0, \tag{4.9.1}$$

and condition (4.4.9) becomes

$$u\cdot\omega_1^2 = \frac{s(s+c)}{A}\Big(\Big((p+qs)^2-(a+bs)^2+(p+qs)\Delta_M G\Big)\cdot\omega_1^2 \qquad (4.9.2)$$
$$+2(a+bs)(d_M d_M^c G)\wedge\omega_0+(d_M d_M^c G)^2\Big),$$

where Δ_M denotes the Hodge Laplacian on (M^4, g_{ω_1}, J_1). Note that we can also express the last term as

$$(d_M d_M^c G)^2 = \Big(\frac{1}{4}(\Delta_M G)^2 - \frac{1}{2}\|(d_M d_M^c G)_0\|^2_{g_{\omega_1}}\Big)\cdot\omega_1^2,$$

where $(d_M d_M^c G)_0$ denotes the traceless component of $d_M d_M^c G$ or equivalently its projection in Λ^2_-. The system (4.9.2) and (4.9.1) is still quite hard to solve in full generality, so we shall make some further simplifying assumptions.

From [11, Theorem 2.4, 3.2] we know that a smooth real function F on M^4 satisfies

$$(d_M d_M^c F)^2 = 0$$

if and only if M^4 admits a foliation by complex submanifolds, with the leaves corresponding to the integral (complex) curves of the ideal generated by $d_M d_M^c F$. In this case we may assume there exists locally a fibration $\pi : M^4 \to \Sigma^2$, where Σ^2 is a complex curve and that F descends to a function on Σ^2. Under this hypothesis on G, for each s, we can eliminate the quadratic term in (4.9.2) .

We shall now illustrate how one can construct metrics with holonomy *equal* to $Spin(7)$ under these assumptions by perturbing the GLPS example.

Example. As before, consider $M = \mathbb{T}^4$ with local coordinates (x_1, x_2, x_3, x_4) and endowed with the standard flat hyperKähler structure. We set $(a,b,p,q) = (0,0,0,1)$, $A = 1$ and consider G of the form:

$$G(s,x_1,x_2) = v(s)\cdot F(x_1,x_2) + \frac{1}{12}s^4.$$

Σ^2 here is the elliptic curve $\mathbb{T}^2$ with coordinates (x_1, x_2). Defining u by

$$u = \ddot{G}$$

automatically solves (4.9.1), and (4.9.2) becomes equivalent to the pair;

$$\Delta_M F = \mu F,$$
$$\ddot{v} = \mu s^2 v,$$

where μ is a constant. The reader might recognise that the second equation is the well-known Weber equation. With $\mu = 1$, a simple solution is given by

$$F = \sin(x_1),$$
$$v = U(0, \sqrt{2}s),$$

with $U(a,t)$ denoting the parabolic cylinder function, see [1, Chapter 19] for a precise definition in terms of hypergeometric functions. From Corollary 4.4.4 we find that the connection form α is can be expressed as

$$\alpha = dx_5 - \dot{v}\cos(x_1)dx_2,$$

where x_5 denotes the angular coordinate on the S^1 fibre. One can verify that g_Φ, well-defined for $\{s \mid U(0, \sqrt{2}s) < 1\}$, has holonomy *equal* to $Spin(7)$. Thus this gives a $Spin(7)$ perturbation of the GLPS metric.

Setting $f(x_1, s) = 1 + \sin(x_1)v(s)$ and denoting by (x_6, x_7) the coordinates on the $\mathbb{T}^2$ fibres, we can express the connection forms as

$$\xi = dx_6 - x_3 dx_1 - x_2 dx_4,$$
$$\eta = dx_7 - x_4 dx_1 - x_3 dx_2.$$

and hence, the perturbed metric can be expressed in local coordinates as

$$g_\Phi = s^2(f(dx_1^2+dx_2^2)+dx_3^2+dx_4^2)+f^{-1}\alpha^2+s^{-2}(\xi^2+\eta^2)+s^4 f ds^2.$$

One can get other similar examples by choosing $\mu=-1$ and allowing F to depend on both x_1 and x_2 for instance.

Remark 4.9.1. Another source of compact examples fitting in the above construction are elliptic $K3$ surfaces. These examples however require more sophisticated tools to study as the metrics are no longer explicit.

We conclude our study of the S^1 Kähler reduction and now proceed to the $\mathbb{T}^2$ case.

4.10 Further reduction II

4.10.1 $\mathbb{T}^2$ Kähler reduction

Recall from subsection 4.3.2 that there are two natural constraints to impose on the function s. The first is that s is a function of H, and the second that s and $y := Hs^{-1/3}$ are independent functions on P^6 with U and W orthogonal. Having investigated the former situation, we shall now study the latter case and give yet more examples of $Spin(7)$ metrics.

We follow the same strategy as in the proof of Theorem 4.4.1. We first define connection 1-forms α and κ on P by

$$\alpha(\cdot) = g_\omega(U,\cdot\,)u, \tag{4.10.1}$$

$$\kappa(\cdot) = g_\omega(W,\cdot\,)w, \tag{4.10.2}$$

where $u := \|U\|_\omega^{-2}$ and $w := \|W\|_\omega^{-2}$. From our assumptions on U and W, it is easy to see that they commute and that they are infinitesimal symmetries of the $SU(3)$-structure. Hence they define a $(\mathbb{C}^\times)^2$ action on P^6 and we can once again carry out a Kähler reduction:

$$(P^6,\omega,\Omega,J) \xrightarrow{//\mathbb{T}^2} (\Sigma^2,\tilde{\omega},\Upsilon,\tilde{J}).$$

The holomorphic $(1,0)$-form Υ on Σ^2 is defined by

$$\Upsilon_1 - i\Upsilon_2 := \frac{1}{4}(W - iJW)\lrcorner(U - iJU)\lrcorner(H^{1/2}s^{1/3}\Omega)$$

and the quotient symplectic form $\tilde{\omega}(s,y)$ is given by

$$\omega = -\alpha \wedge dy + \kappa \wedge ds + \tilde{\omega}(s,y). \tag{4.10.3}$$

Note that if Σ^2 is compact then it must be an elliptic curve, since it has trivial first Chern class. Unlike in the previous case however the horizontal lifts of U and W do not preserve the $Spin(7)$-structure as

$$\mathscr{L}_U \eta = \Upsilon_2 \quad \text{and} \quad \mathscr{L}_W \xi = -\Upsilon_1.$$

One easily shows the above using expressions (4.3.5), (4.3.6) for the curvature forms and expression (4.10.6) below for Ω. Hence, for each fixed s and H, the six dimensional hypersurface in N^8 corresponds to a $\mathbb{T}^2$ bundle over a $\mathbb{T}^2$ bundle over the surface Σ^2. In the case when $\Sigma = \mathbb{T}^2$, this hypersurface is just a nilmanifold. Thus, we shall generally refer to these hypersurfaces as 'nilbundles'.

From (4.3.5) and (4.3.6) we can equivalently write α and κ as

$$\alpha = -ud^c y, \tag{4.10.4}$$

$$\kappa = wd^c s. \tag{4.10.5}$$

As in subsection 4.4.1, we can once again express the data $(P^6, \omega, \Omega, \alpha, \kappa)$ purely in terms of $(\Sigma^2, \tilde{\omega}, \Upsilon, u(s,y), w(s,y))$, and thus provide a way to invert the Kähler reduction. More precisely imposing that the $(3,0)$ form given by

$$y^{1/2}s^{1/2}\Omega = (\alpha - iudy) \wedge (\kappa + iwds) \wedge (\Upsilon_1 - i\Upsilon_2) \tag{4.10.6}$$

and ω, given by (4.10.3), are both closed leads to the system (4.10.8)-(4.10.10) while the normalisation condition (2.3.2) leads to (4.10.7). For instance from $d\omega = 0$, we have

$$\frac{\partial \tilde{\omega}}{\partial s} = -d_\Sigma \kappa,$$

while using the closedness of (4.10.6) and the fact that $\beta \wedge \Upsilon_1 = J(\beta) \wedge \Upsilon_2$ for a 1-form β on Σ we have

$$\frac{\partial \kappa}{\partial s} = -d^c_\Sigma w,$$

combining both equations give (4.10.9). The proof for the rest is completely analogous and the result is summed up as follows:

Theorem 4.10.1. *Given a complex curve $(\Sigma^2, \tilde{J})$ with a holomorphic $(1,0)$-form $\Upsilon_1 - i\Upsilon_2$, a 1-parameter family of positive functions $u = u(y)$ and $w = w(s)$, and a family of Kähler forms $\tilde{\omega}(s,y)$ satisfying*

$$-(s \cdot y)\ \tilde{\omega} = (u \cdot w)\Upsilon_1 \wedge \Upsilon_2, \tag{4.10.7}$$

$$\frac{\partial^2 \tilde{\omega}}{\partial y^2} = d_\Sigma d^c_\Sigma u, \tag{4.10.8}$$

$$\frac{\partial^2 \tilde{\omega}}{\partial s^2} = d_\Sigma d^c_\Sigma w, \tag{4.10.9}$$

$$\frac{\partial^2 \tilde{\omega}}{\partial y \partial s} = 0, \tag{4.10.10}$$

where d_Σ denotes the exterior differential on Σ^2 and $d^c_\Sigma := \tilde{J} \circ d_\Sigma$. Then there exists, on the 'nilbundle' over $\Sigma^2 \times \mathbb{R}^+_s \times \mathbb{R}^+_y$, defined by the curvature 2-forms:

$$\begin{aligned} d\alpha &= -d^c_\Sigma u \wedge dy + \frac{\partial \tilde{\omega}}{\partial y}, \\ d\kappa &= d^c_\Sigma w \wedge ds - \frac{\partial \tilde{\omega}}{\partial s}, \\ d\xi &= \Upsilon_1 \wedge \kappa + \Upsilon_2 \wedge w\ ds, \\ d\eta &= \alpha \wedge \Upsilon_2 + u\ dy \wedge \Upsilon_1, \end{aligned}$$

a torsion free $Spin(7)$-structure Φ inducing the metric:

$$g_\Phi = s^{-2}\eta^2 + y^{-2}\xi^2 + y \cdot s\ (u^{-1}\alpha^2 + u\ dy^2 + w^{-1}\kappa^2 + w\ ds^2 + g_{\tilde{\omega}}), \tag{4.10.11}$$

where $g_{\tilde{\omega}}$ denotes the Kähler metric on $(\Sigma^2, \tilde{J})$ determined by $\tilde{\omega}$.

4.10.2 A general existence result

Before constructing explicit examples we first describe how to find a general solution to Theorem 4.10.1.

We pick complex coordinate $z = x_1 + ix_2$ on Σ^2 so that we can write $\Upsilon = dx_1 + idx_2$ and the Kähler form is given by

$$\tilde{\omega} = F(y,s)\ dx_1 \wedge dx_2,$$

where F is a positive function on Σ^2 depending on y and s. From equation (4.10.10) we have that

$$F(s,y) = F_1(y) + F_2(s).$$

Thus, equations (4.10.8) and (4.10.9) are equivalent to the pair:

$$\frac{\partial^2 F_1}{\partial y^2} = -(u_{x_1,x_1} + u_{x_2,x_2}), \tag{4.10.12}$$

$$\frac{\partial^2 F_2}{\partial s^2} = -(w_{x_1,x_1} + w_{x_2,x_2}), \tag{4.10.13}$$

while equation (4.10.7) reduces to

$$s \cdot y\ (F_1(y) + F_2(s)) = u(y) \cdot w(s). \tag{4.10.14}$$

It follows, without loss of generality, that either F_1 or F_2 must be zero and hence, that either $u(y)$ or $w(s)$ is a 1-parameter family of harmonic functions on Σ^2. In particular if $\Sigma = \mathbb{T}^2$ then either u or w is constant.

Assuming that $F_2 = 0$, (4.10.13) and (4.10.14) implies that

$$F_1(y) = \left(\frac{u(y)}{y}\right) \cdot G(x_1,x_2) \quad \text{and} \quad w(s) = s \cdot G(x_1,x_2),$$

for a positive harmonic function $G : \Sigma \to \mathbb{R}^+$, *independent* of s and y. Therefore, solving the general system of Theorem 4.10.1 amounts to solving the single PDE

$$G \cdot \frac{\partial^2}{\partial y^2}(\tilde{u}(y)) = y \cdot \Delta_\Sigma \tilde{u}(y), \tag{4.10.15}$$

where $\tilde{u}(y) := \frac{u(y)}{y}$ and Δ_Σ denotes the Hodge Laplacian on Σ^2. Given real analytic initial data we can once again appeal to the Cauchy-Kovalevskaya theorem for the existence and uniqueness of a real analytic solution.

Corollary 4.10.2. *Given real analytic functions u_0 and u_1 on (an open set of) a complex curve $(\Sigma^2, J_1, \Upsilon_1 - i\Upsilon_2)$ with $u_0 > 0$, then there exists a unique real analytic solution $\tilde{u}(y)$, for y in a small interval, to (4.10.15) with $\tilde{u}(0) = u_0$ and $\frac{\partial \tilde{u}}{\partial y}(0) = u_1$, and hence by Theorem 4.10.1 a torsion free $Spin(7)$-structure.*

Remark 4.10.3. If we look for separable solutions $\tilde{u} = A(y) \cdot B(x_1, x_2)$, then (4.10.15) becomes equivalent to the pair

$$\frac{\partial^2}{\partial y^2} A(y) = \mu \cdot y \cdot A(y), \tag{4.10.16}$$

$$\Delta_\Sigma B = \mu \cdot G \cdot B, \tag{4.10.17}$$

where μ is a constant and equation (4.10.16) is the well-known Airy equation for $\mu \neq 0$.

In summary, we have reduced the problem of finding $Spin(7)$ metrics admitting Kähler reduction with $\mathbb{T}^2$ symmetry to choosing a positive harmonic function G and solving (4.10.15). We now proceed to describe some explicit examples.

4.11 Constant solutions II

In this section we describe the simplest solutions which arise when u and w are both constants on Σ^2. Without loss of generality, this corresponds to setting $\mu = 0$, $B = 1$ and $G = c$ is a positive constant, in (4.10.16) and (4.10.17). The general solution is then given by

$$w(s) = cs,$$

$$u(y) = y(p+qy),$$
$$\tilde{\omega} = c(p+qy)\ dx_{12},$$

where $p, q \in \mathbb{R}$ and the positivity condition on u implies that the solution is valid for $p + qy > 0$.

Denoting the coordinates on the fibres by (x_3, x_4, x_5, x_6), we can express the connection 1-forms as

$$\alpha = cqx_1 dx_2 + dx_3,$$
$$\kappa = dx_4,$$
$$\xi = dx_5 + x_1 dx_4 - csx_2 ds,$$
$$\eta = dx_6 + x_2 dx_3 - yx_1(p+qy)dy.$$

If we fix y and s, then we have that

$$(d\alpha, d\kappa, d\xi, d\eta) = (cqdx_{12}, 0, dx_{14}, dx_{23}).$$

Thus, it follows that if $\Sigma = \mathbb{T}^2$ then these codimension 2 submanifolds are diffeomorphic to nilmanifolds with nilpotent Lie algebra isomorphic to either

$$(0,0,0,0,12,34) \quad \text{or} \quad (0,0,0,12,13,24),$$

depending on whether q is zero or not. The former corresponds to the 2-step nilpotent Lie algebra of the product of two real Heisenberg groups while the latter corresponds to an indecomposable 3-step nilpotent Lie algebra. By computing the rank of the associated curvature operator we find:

Theorem 4.11.1. *The metrics determined by expression (4.10.11) when $\Sigma = \mathbb{T}^2$ endowed the flat metric, explicitly given by*

$$g_\Phi = s^{-2}\eta^2 + y^{-2}\xi^2 + s(p+qs)^{-1}\alpha^2 + c^{-1}y\kappa^2$$
$$+ y^2 s(p+qs)dy^2 + cys^2 ds^2 + csy(p+qy)(dx_1^2 + dx_2^2),$$

have holonomy equal to $Spin(7)$.

Thus, this classifies the constant solution examples. We shall now consider some non-constant solutions.

4.12 Examples of non-constant solutions

In this section we give explicit examples of $Spin(7)$ metrics which vary on Σ^2. To illustrate the different cases that can arise from our construction, in the first example we consider a non-compact surface so that we may choose non-constant harmonic functions on Σ^2 and in the second example we consider a separable solution with $\mu = 1$ on $\mathbb{T}^2$. As in the previous section we shall denote the fibre coordinates by (x_3, x_4, x_5, x_6).

Example 1. We take $\Sigma = \mathbb{C} - B_1(0)$, where $B_1(0)$ denotes the unit ball centred at the origin, with the holomorphic form $\Upsilon = dx_1 + idx_2$ as before. Following the strategy outline in subsection 4.10.2 we find that a solution is given by choosing $F_1(y) = y\ln(r)$, $F_2(s) = 0$, $w(s) = s\ln(r)$ and $u(y) = y^2$, where $r := x_1^2 + x_2^2$. The connection 1-forms are given in local coordinates by:

$$\begin{gathered}
\alpha = dx_3 + (x_1 \ln(r) - 2x_1 + 2x_2 \arctan\left(\frac{x_1}{x_2}\right))dx_2, \\
\kappa = dx_4 - \frac{1}{2}s^2 d^c_\Sigma \ln(r), \\
\xi = dx_5 + x_1 dx_4 + \frac{1}{2}s^2 \ln(r) dx_2 \\
\eta = dx_6 + x_2 dx_3 - x_1 y^2 dy.
\end{gathered}$$

One can again check that the induced metric has holonomy *equal* to $Spin(7)$.

Example 2. We now take $\Sigma = \mathbb{T}^2$ endowed with the standard flat Kähler structure. With $\mu = 1$ the general solution to (4.10.16) is the Airy function $\mathrm{Ai}(y)$. Thus, picking $F_1(y) = \mathrm{Ai}(y)\sin(x_1)$, $F_2(s) = 0$, $w(s) = s$ and $u(y) = y\mathrm{Ai}(y)\sin(x_1)$, we obtain another solution. The connection 1-forms are now given by:

$$\alpha = dx_3 - \mathrm{Ai}'(y)\cos(x_1)dx_2,$$
$$\kappa = dx_4,$$
$$\xi = dx_5 + x_1 dx_4 - s x_2 ds,$$
$$\eta = dx_6 + x_2 dx_3 + y\mathrm{Ai}(y)\cos(x_1)dy.$$

The resulting $Spin(7)$ metric is well-defined on the set where $u > 0$. By taking $\Sigma = \mathbb{C}$ and $\mu = -1$ instead we find yet more examples of $Spin(7)$ holonomy metrics.

Concluding Remarks. In this chapter we investigated the $\mathbb{T}^2$-reduction of torsion free $Spin(7)$ structures under the assumption that the quotient is Kähler. However as shown in section 4.2 the quotient $SU(3)$-structure is generally only almost Kähler. Thus, it would interesting to investigate if other distinct types of $SU(3)$-structures, aside from the Kähler case considered here, can arise as well. From our results, it follows that, even locally, such a quotient cannot be a Calabi-Yau 3-fold unless N^8 is the Riemannian product $P^6 \times \mathbb{T}^2$. Furthermore, we have been able to prove that the quotient cannot be a special generalised CY 3-fold (defined by $d\omega$=0 and $d\Omega^+$=0) as well. This still leaves plenty other cases to study. By contrast, in the G_2 case it is not hard to see that only two types of $SU(3)$-structures can arise, namely the Kähler one or the generic one i.e. with neither π_1 nor π_2 zero. Another interesting problem would be to investigate if one can find smooth completions to our $Spin(7)$ metrics. This will likely necessitate the study of non-free $\mathbb{T}^2$ actions.

Chapter 5

S^1-invariant closed G_2-structure and the Laplacian flow

5.1 Overview of chapter

In this chapter we study S^1-invariant closed G_2-structures (L^7, φ) evolving under the Laplacian flow. Let us first give some motivation for our work.

Motivation. The interest in closed G_2-structures is mainly due to the fundamental works of Joyce and Bryant-Hitchin. Joyce's theorem roughly states that a closed G_2-structure on a compact 7-manifold with sufficiently small torsion can be deformed to a torsion free one cf. [61, Theorem 11.3.4]. This theorem underpins all the currently known constructions of compact G_2 manifolds. Although compact G_2 manifolds cannot admit continuous symmetry closed ones can, which makes them much easier to construct, see for instance [25] for the classification of closed G_2-structures on nilmanifolds. Of more relevance for us here is the G_2 Laplacian flow, which was introduced by Bryant in [19] as a way of deforming a closed G_2-structure within its cohomology class to a torsion free one. In [54] Hitchin introduces a functional on the cohomology class of a closed G_2-structure and shows that critical points are exactly the torsion free ones (in that cohomology class) [54, Theorem 19]. Bryant's Laplacian flow then turns out to be the gradient flow of this functional for a suitable metric. The flow can be interpreted as a Kähler Ricci type flow for G_2-structures. Due to the complex nature of the Laplacian flow most of the works in this area have been carried out on homogeneous spaces. The first inhomogeneous Laplacian soliton examples only appear at the time of writing of this thesis cf. [7]. In the final section of this chapter we shall give another new example.

In an attempt to simplify the Laplacian flow we initiate the investigation when the flow is invariant under a free circle action. Our goal in this chapter is two-fold; we first give a 'Gibbons-Hawking ansatz' type theorem for the construction of closed G_2-structures starting from suitable data on a symplectic manifold (P^6, ω) and secondly we derive the evolution equations for S^1-invariant closed G_2-structure under the Laplacian flow, or more precisely the evolution equation of the quotient $SU(3)$-structure. The latter problem was motivated by the works of Fino and Raffero in [38], where they studied the flow equations for warped G_2-structures on $P^6 \times S^1$, and of Fine and Yao in [37], where they interpreted the Laplacian flow on $M^4 \times \mathbb{T}^3$ as a hypersymplectic flow on M^4 in a bid to find hyperKähler triples. In the final section we consider the flow on two specific examples. In particular we will construct the first example of an inhomogeneous shrinking gradient soliton.

5.2 S^1 reduction of closed G_2-structure

Our aim in this section is to characterised S^1-invariant closed G_2-structures purely in terms of the data on the quotient space. In other words we aim to derive a Gibbons-Hawking type construction for closed G_2-structures, see theorem 5.2.2.

Let (L^7, φ) denote a closed G_2-structure which is invariant under a free S^1 action generated by a vector field Y. Following the conventions of section 4.2.1 we write

$$\varphi = \xi \wedge \omega + H^{3/2}\Omega^+, \tag{5.2.1}$$

$$*_\varphi \varphi = -\xi \wedge H^{1/2}\Omega^- + \frac{1}{2}H^2\omega^2, \tag{5.2.2}$$

$$g_\varphi = H^{-2}\xi^2 + Hg_\omega, \tag{5.2.3}$$

where $\xi(\cdot) := H^2 g_\varphi(Y, \cdot)$ and $H := \|Y\|_\varphi^{-1}$. Since $\mathscr{L}_X \varphi = 0$ and φ is closed, we have

$$d\omega = 0, \tag{5.2.4}$$

$$d\Omega^+ = -\frac{3}{2}H^{-1}dH \wedge \Omega^+ - H^{-\frac{3}{2}}d\xi \wedge \omega, \tag{5.2.5}$$

$$d\xi \wedge \omega^2 = 0. \tag{5.2.6}$$

Note that the last equation follows by differentiating $\varphi \wedge \omega$ and using (5.2.4). Under the inclusion $SU(3) \subset G_2$, we can express the torsion form as $\tau_2 = \tau_h + \xi \wedge \tau_v$ for a 2-form τ_h and a 1-form τ_v which are basic (since $\mathscr{L}_X \tau = 0$). As $\tau_2 \in \Lambda^2_{14}$ we find from (2.1.8) that

$$\tau_h \wedge \omega^2 = 0, \tag{5.2.7}$$

$$\tau_v \wedge \frac{1}{2}H^{3/2}\omega^2 = \tau_h \wedge \Omega^-. \tag{5.2.8}$$

Note that (5.2.6) and (5.2.7) imply that $d\xi$ and τ_h have no ω-component i.e. $d\xi = (d\xi)^2_6 + (d\xi)^2_8$ and $\tau_h = \tau_6 + \tau_8 \in \Lambda^2_6 \oplus \Lambda^2_8$. In terms of the $SU(3)$-structure we can express the condition $d\varphi = \tau_2 \wedge \varphi$ as

$$d\Omega^- = H^{\frac{1}{2}}\tau_6 \wedge \omega + (H\tau_v - \frac{1}{2}H^{-1}d^cH) \wedge \Omega^+ + H^{\frac{1}{2}}\tau_8 \wedge \omega, \tag{5.2.9}$$

$$HdH \wedge \omega^2 - (d\xi)^2_6 \wedge H^{1/2}\Omega^- = \tau_6 \wedge H^{\frac{3}{2}}\Omega^+. \tag{5.2.10}$$

Observe that the forms $(d\xi)^2_6$, τ_v and τ_6 are related by (5.2.8), (5.2.10) and the fact that π_1 appears in both (2.3.13) and (2.3.14). Thus, these forms are all essentially equivalent, more precisely we have:

Lemma 5.2.1.

$$\tau_v = -2H^{-2}(d^cH + J(\gamma^1_6)) \quad \textit{and} \quad -2\tau_6 = H^{\frac{3}{2}} *_\omega (\tau_v \wedge \Omega^+),$$

where the 1*-form* γ^1_6 *is defined by* $H^{-\frac{1}{2}}(d\xi)^2_6 \wedge \omega = \gamma^1_6 \wedge \Omega^+$.

Proof. Let $\tau_6 \wedge \omega = H^{\frac{3}{2}}\beta_6 \wedge \Omega^+$ for a 1-form β_6, then using lemma 2.3.3 we can express the $SU(3)$ torsion forms into irreducible summands as

$$d\Omega^+ = (-\frac{3}{2}H^{-1}dH - H^{-1}\gamma_6^1)\wedge\Omega^+ - H^{-\frac{3}{2}}(d\xi)_8^2\wedge\omega, \tag{5.2.11}$$

$$d\Omega^- = (H(\tau_v+\beta_6) - \frac{1}{2}H^{-1}d^cH)\wedge\Omega^+ + H^{-\frac{1}{2}}\tau_8\wedge\omega, \tag{5.2.12}$$

and hence we have

$$d^cH + J\gamma_6^1 = -H^2(\tau_v+\beta_6).$$

From (5.2.8) we find that $\tau_v = -2\beta_6$ and together with (5.2.10) this completes the proof. □

We can express the $SU(3)$ torsion forms in terms of suitable derivatives of the $SU(3)$-structure as:

$$\tau_6 = H^{-\frac{1}{2}} *_\omega ((d^cH + J\gamma_6^1)\wedge\Omega^+)$$
$$\tau_8 = -H^{\frac{1}{2}} *_\omega d\Omega^- - *_\omega((\frac{3}{2}H^{-\frac{1}{2}}d^cH + J\gamma_6^1)\wedge\Omega^+)$$

and the G_2 torsion form as:

$$\tau_2 = -H^{\frac{1}{2}} *_\omega d\Omega^- - \frac{1}{2}H^{-\frac{1}{2}} *_\omega -(d^cH\wedge\Omega^+) - 2\xi\wedge(H^{-2}d^cH + H^{-2}J\gamma_6^1). \tag{5.2.13}$$

In appendix C we give the corresponding expressions relating the torsion of an arbitrary (i.e. not necessarily closed) G_2-structure to that of the quotient $SU(3)$-structure. We can now prove the main result of this section.

Theorem 5.2.2 (Gibbons-Hawking ansatz for closed G_2-structures)**.** *Given a symplectic manifold (P^6,ω) with trivial first Chern class admitting an $SU(3)$-structure (ω,Ω) and a positive function $H : P^6 \to \mathbb{R}^+$ satisfying*

$$d^{*_\omega}(d^{*_\omega}(H^{3/2}\Omega^-)\wedge\omega) = 0 \tag{5.2.14}$$

with

$$[-*_\omega (d^{*_\omega}(H^{3/2}\Omega^-)\wedge\omega)] \in H^2(P^6,\mathbb{Z}) \tag{5.2.15}$$

then $\varphi = \xi \wedge \omega + H^{3/2}\Omega^+$ defines a closed G_2-structure on the total space of this S^1 bundle and the curvature of the connection form ξ is given by

$$d\xi = - *_\omega (d^{*_\omega}(H^{3/2}\Omega^-) \wedge \omega). \tag{5.2.16}$$

Proof. In view of the above quotient construction we only need to prove that $\xi(\cdot) := H^2 g_\varphi(Y,\cdot)$ satisfies (5.2.16). Applying $*_\omega$ to (5.2.5) we find

$$(d\xi)^2_8 = (d\xi)^2_6 + *_\omega(d(H^{3/2}\Omega^+))$$

and thus, $d\xi = 2(d\xi)^2_6 + *_\omega(d(H^{3/2}\Omega^+))$. Equivalently one may consider the automorphism $\mathbf{L} : \Lambda^2_6 \oplus \Lambda^2_8 \to \Lambda^2_6 \oplus \Lambda^2_8$ given by

$$\mathbf{L}(\alpha) = *_\omega(\alpha \wedge \omega)$$

which acts as the identity on Λ^2_6 and minus identity on Λ^2_8. Then

$$d\xi = -\mathbf{L}(*_\omega(d(H^{3/2}\Omega^+))) = - *_\omega (d^{*_\omega}(H^{3/2}\Omega^-) \wedge \omega),$$

using $\Omega^+ = - *_\omega \Omega^-$. $\square$

Remark 5.2.3. Note that in the above theorem condition (5.2.15) is the higher dimensional analogue of the 'integrality' condition that figures in the Gibbons-Hawking ansatz and the (linear) harmonic condition on V replaced by the (non-linear) condition (5.2.14) on the pair (H,Ω). We should emphasise that $*_\omega$ depends on both ω and Ω.

From lemma 5.2.1 we can characterise torsion free G_2-structures in terms of the data on the base by the following proposition.

Proposition 5.2.4 ([3]).

$$d *_\varphi \varphi = 0 \quad \textit{if and only if} \quad \gamma^1_6 = -dH \;\; \textit{and} \;\; d(H^{\frac{1}{2}}\Omega^-) = 0.$$

Moreover (P^6, ω, Ω) *is a Calabi-Yau* 3*-fold if and only if* H *is constant.*

Proof. The first part follows immediately by imposing $\tau_6 = \tau_8 = 0$. If furthermore H is constant then from lemma 5.2.1 we have $(d\xi)^2_6 = 0$. Differentiating the relation

$$d\xi \wedge \Omega^+ = 0$$

and using (5.2.5) shows that $\|d\xi\|_\omega = 0$, which completes the proof. □

5.3 The S^1-invariant Laplacian flow

5.3.1 Basics

Given a closed G_2-structure φ_0 the Laplacian flow (LF) is defined as the initial value problem

$$\frac{\partial}{\partial t}(\varphi) = \Delta_\varphi \varphi, \tag{5.3.1}$$

$$\varphi(0) = \varphi_0, \tag{5.3.2}$$

where $\Delta_\varphi := dd^{*_\varphi} + d^{*_\varphi} d$. The LF preserves the closed condition i.e. $d\varphi_t = 0$ for all t (when the flow exists) and thus (5.3.1) is equivalent to $\frac{\partial}{\partial t}\varphi = d\tau_2$. In the compact setting, Hitchin gives the following interpretation of the flow. Consider the functional $\Psi : [\varphi_0]^+ \to \mathbb{R}^+$ defined by

$$\Psi(\rho) := \frac{1}{7}\int_L \rho \wedge *_\rho \rho = \int_L \mathrm{vol}_\rho$$

where $[\varphi_0]^+ := \{\varphi_0 + d\beta \in \Omega^3_+(L) \mid \beta \in \Omega^2(L)\}$ denotes an open set of closed G_2-structures in the cohomology class $[\varphi_0] \in H^3(L, \mathbb{R})$. Then computing the Euler-Langrange equation Hitchin finds that the critical points of Ψ satisfy $d *_\varphi \rho = 0$. The LF is the gradient flow of Ψ with respect to the L^2 norm induced by g_ρ on $[\varphi_0]^+$. The Hessian of Ψ at a critical point is non-degenerate transverse to the action of the diffeomorphism group and in fact is negative definite. Ψ can thus be interpreted as a Morse-Bott functional and the torsion free G_2-structures correspond to the local maxima. In the non-compact setting this interpretation is not valid but the LF is still well-defined.

In [19], Bryant computes the evolution equations for following geometric quantities under the LF

$$\frac{\partial}{\partial t}(*_\varphi \varphi) = \frac{1}{3}\|\tau_2\|_\varphi^2 *_\varphi \varphi - *_\varphi d\tau_2, \tag{5.3.3}$$

$$\frac{\partial}{\partial t}(g_\varphi) = -2Ric(\varphi) + \frac{1}{6}\|\tau_2\|_\varphi^2 g_\varphi + \frac{1}{4}j(*_\varphi(\tau_2 \wedge \tau_2)), \tag{5.3.4}$$

$$\frac{\partial}{\partial t}(vol_\varphi) = \frac{1}{3}\|\tau_2\|_\varphi^2 vol_\varphi. \tag{5.3.5}$$

Note that the coefficient in front of g_φ in (5.3.4) differs from that in [19, (6.15)] but a calculation in local coordinates show that (5.3.4) agrees with [38, (4.2)]. In [19, (4.30)] Bryant also derives an expression for the Ricci tensor in terms of the torsion form only and thus, one can express (5.3.4) only in terms of the torsion form as

$$\frac{\partial}{\partial t}(g_\varphi) = -\frac{1}{3}\|\tau_2\|^2 g_\varphi + \frac{1}{2}j(d\tau_2). \tag{5.3.6}$$

The simplest solutions to (5.3.1) are those that evolve by the symmetry of the flow. If φ_0 satisfies

$$\Delta_{\varphi_0}\varphi_0 = \lambda \cdot \varphi_0 + \mathscr{L}_V \varphi_0 \tag{5.3.7}$$

for a vector field V and constant λ then

$$\varphi_t := (1 + \frac{2}{3}\lambda t)^{\frac{3}{2}} F_t^* \varphi_0,$$

where F_t is the diffeomorphism group generated by $U(t) = (1 + \frac{2}{3}\lambda t)^{-\frac{2}{3}} V$, is a solution to LF and φ_0 is called a Laplacian solition [68]. Depending whether λ is positive, zero or negative the soliton is called expanding, steady or shrinking respectively. If V is a gradient vector field then we called them gradient solitons.

5.3.2 S^1-invariant flow equations

Consider now the LF starting from an S^1-invariant G_2-structure. Then by existence and uniqueness of the flow (at least in the compact case) cf. [22, 68] it follows that this symmetry persists. Thus, in the S^1-invariant setting, using (5.2.13) we see that the LF equation is equivalent to the following evolution equations on the $SU(3)$ structure together with the connection 1-form ξ and Higgs field H:

$$\frac{\partial}{\partial t}(\omega) = -2dd^c(H^{-1}) + 2d(H^{-2}J\gamma_6^1), \tag{5.3.8}$$

$$\frac{\partial}{\partial t}(\xi)\wedge\omega + \frac{\partial}{\partial t}(H^{3/2}\Omega^+) = -d*_\omega d(H^{\frac{1}{2}}\Omega^-) + 2d\xi\wedge d^c(H^{-1}) - 2H^{-2}d\xi\wedge J\gamma_6^1. \tag{5.3.9}$$

Observe that (5.3.8) agrees with the fact that since φ remains in its cohomology class so does ω. In this section we derive the evolution equations for the data $(\omega,\Omega,\xi,H,vol_\omega)$ on P^6, but before we derive expressions for quantities that will appear in the evolution equations.

Lemma 5.3.1.

1. $\|\gamma_6^1\|^2_\omega = \frac{1}{2}H^{-1}\|(d\xi)^2_6\|^2_\omega$.
2. $\|\tau_2\|^2_\varphi = H^{-2}(\|\tau_8\|^2_\omega + 3\|\tau_6\|^2_\omega)$
3. $\|d\Omega^-\|^2_\omega = H^{-1}\|\tau_8\|^2_\omega + H^{-2}(\frac{9}{2}\|dH\|^2_\omega + 2\|\gamma_6^1\|^2_\omega + 6g_\omega(dH,\gamma_6^1))$
4. $\|\tau_6\|^2_\omega = 2H^{-1}\|dH+\gamma_6^1\|^2_\omega$

Proof. The proof is analogous to lemma 3.2.1 using the expressions given in the previous section together with lemma 2.3.3. We prove *1.* as an example,

$$\gamma_6^1\wedge*_\omega\gamma_6^1 = \frac{1}{2}H^{-1/2}\gamma_6^1\wedge(d\xi)^2_6\wedge\Omega^+ = \frac{1}{2}H^{-1}(d\xi)^2_6\wedge*_\omega(d\xi)^2_6,$$

where the first equality follows from *2.* of lemma 2.3.3 and the definition of γ_6^1. The second equality is again just by the definition of γ_6^1. The proofs for the rest are similar. □

Proposition 5.3.2.

$$\frac{\partial}{\partial t}(\xi)\wedge\omega^2 = -(d*_\omega d)(H^{\frac{1}{2}}\Omega^-)\wedge\omega + 2d\xi\wedge d^c(H^{-1})\wedge\omega \tag{5.3.10}$$

$$-2H^{-2}d\xi\wedge J(\gamma_6^1)\wedge\omega-2H^{\frac{3}{2}}\Omega^+\wedge dd^c(H^{-1}) \tag{5.3.11}$$

$$+2H^{\frac{3}{2}}\Omega^+\wedge d(H^{-2}J(\gamma_6^1)) \tag{5.3.12}$$

Proof. In order to extract the evolution equation of the connection form ξ in (5.3.9), we differentiate the relation $H^{\frac{3}{2}}\Omega^+\wedge\omega=0$ and use (5.3.8). □

Proposition 5.3.3.

$$\frac{\partial}{\partial t}(H)=-H^{-1}d^{*_\omega}d(H)-2H^{-2}g_\omega(dH,\gamma_6^1)-H^{-2}\|dH\|^2_\omega$$
$$+\frac{1}{6}H^{-1}\|\tau_8\|^2_\omega+\frac{1}{2}H^{-3}\|(d\xi)^2_8\|^2_\omega \tag{5.3.13}$$

Proof. Since $H^{-2}=g_\varphi(Y,Y)$ we can use (5.3.6) to write down its evolution equation. Thus, we only need to simplify the term

$$j(d\tau_2)(Y,Y)=*_\varphi(\omega\wedge\omega\wedge\xi\wedge d(H^{-2}d^cH+H^{-2}J\gamma_6^1))$$

which is straightforward, except for the term involving $d(J\gamma_6^1)$. From lemma 2.3.3 it suffices to compute

$$d(H^{1/2}J\gamma_6^1\wedge\omega\wedge\omega)=-d((d\xi)^2_6\wedge\Omega^+).$$

To compute $d((d\xi)^2_6)\wedge\Omega^+$, we first note that $d(d\xi)^2_6=-d(d\xi)^2_8$. Now differentiating the relation $(d\xi)^2_8\wedge\Omega^+=0$ we have

$$d((d\xi)^2_6\wedge\Omega^+)=(d\xi)^2_6\wedge d\Omega^++(d\xi)^2_8\wedge d\Omega^+$$

and using (5.2.11) this finishes the proof. □

Observe that even if H is initially constant (i.e. the S^1 orbits have constant size) this is not preserved in time. Using the above two propositions we can also extract the evolution equation for Ω^+ in (5.3.9) but the resulting expression is quite involved and we have been unable to simplify it so we don't write it out.

Proposition 5.3.4.

$$\frac{\partial}{\partial t}(g_\omega) = -\left(\frac{\partial}{\partial t}(\ln H) + \frac{1}{3}\|\tau_2\|^2_\varphi - \frac{1}{3}g_\omega(d\tau_v,\omega) - 2H^{-3/2}(d\xi\wedge\tau_v + d\tau_h)^{3+}_1\right)g_\omega$$
$$+ j((d\xi\wedge\tau_v + d\tau_h)^3_{12} + \xi\wedge(d\tau_v)^2_8). \tag{5.3.14}$$

Proof. The idea is to again use equation (5.3.6) for g_φ. Since $\frac{\partial}{\partial t}(g_\omega)$ only evolves on the base P^6 we can ignore terms involving ξ. Thus, we have that

$$\frac{\partial}{\partial t}(g_\omega) = -\frac{\partial}{\partial t}(\ln H)g_\omega - \frac{1}{3}\|\tau_2\|^2_\varphi g_\omega + \frac{1}{2}H^{-1}j(d\tau_2)\bigg|_{P^6}$$

As $SU(3)$ modules we have the following decomposition

$$\begin{aligned}
\Lambda^3_1(L)\oplus\Lambda^3_{27}(L) \cong S^2(\mathbb{R}^7) &= S^2(\mathbb{R}\oplus\mathbb{R}^6)\\
&= \langle\xi^2\rangle\oplus(\xi\odot\mathbb{R}^6)\oplus S^2(\mathbb{R}^6)\\
&= \langle\xi^2\rangle\oplus(\xi\odot\mathbb{R}^6)\oplus\langle g_\omega\rangle\oplus S^2_0(\mathbb{R}^6)\\
&\cong \langle\xi^2\rangle\oplus(\xi\odot\mathbb{R}^6)\oplus\langle g_\omega\rangle\oplus\Lambda^2_8(P)\oplus\Lambda^3_{12}(P)
\end{aligned}$$

It follows that the only terms in $j(d\tau)$ that contribute to the evolution of g_ω belong to the last 3 summands. Since we have that $d\tau_2 = d\xi\wedge\tau_v + d\tau_h - \xi\wedge d\tau_v$, the only terms that can arise in the evolution of g_ω are the $\langle\Omega^+\rangle\oplus\langle\Omega^-\rangle\oplus\Lambda^3_{12}$ components of $d\xi\wedge\tau_v + d\tau_h$ which we write as

$$(d\xi\wedge\tau_v + d\tau_h)^{3+}_1\Omega^+ + (d\xi\wedge\tau_v + d\tau_h)^{3-}_1\Omega^- + (d\xi\wedge\tau_v + d\tau_h)^3_{12}$$

and the $\Lambda^2_1\oplus\Lambda^2_8$ components of $*_\omega(*_\varphi(\xi\wedge d\tau_v)) = Hd\tau_v$. A direct computation using pointwise coordinates to identify φ with φ_0 shows that

$$j(H^{3/2}\Omega^+) = 4Hg_\omega$$

and since $j(\varphi) = 6 \cdot g_\varphi$ we also have

$$j(\xi \wedge \omega) = 6H^{-2}\xi^2 + 2Hg_\omega.$$

Lastly since $\Omega^- = -Y \lrcorner (H^{-1/2} *_\varphi \varphi) \in \Lambda^3_7$ we see that

$$j(H^{3/2}\Omega^-) = 0.$$

It follows now from the above that as $SU(3)$ modules we have

$$\Lambda^3_{27} \cong \langle g_\omega - 6\xi^2 \rangle \oplus \langle \xi \odot v \rangle_{v \in T^*P} \oplus \Lambda^2_8 \oplus \Lambda^3_{12}$$

and this concludes the proof. □

The reader might find the presence of the map j in (5.3.14) rather unusual as the latter is strictly speaking a G_2-equivariant map but one can replace it by the corresponding $SU(3)$-equivariant map

$$\iota \oplus \gamma : S^2_0(P) \cong \Lambda^2_8(P) \oplus \Lambda^3_{12}(P)$$

defined in [12, Sect. 2.3].

Proposition 5.3.5.

$$\frac{\partial}{\partial t}(\Omega^-) = \left(\frac{1}{3}\|\tau_2\|^2_\varphi - \frac{1}{2}\frac{\partial}{\partial t}(\ln H)\right)\Omega^- - H^{-3/2} *_\omega (d\tau_h + d\xi \wedge \tau_v) \qquad (5.3.15)$$

Proof. It suffices to use the evolution equation (5.3.3) for $*_\varphi\varphi$ and look the terms involving only ξ. □

Proposition 5.3.6.

$$\frac{\partial}{\partial t}(vol_\omega) = (\frac{1}{3}\|\tau_2\|^2_\varphi - H^{-2}\frac{\partial}{\partial t}(H^2))vol_\omega \qquad (5.3.16)$$

Proof. This follows directly from (5.3.5) and the relation $vol_\varphi = H^2\xi \wedge vol_\omega$. □

Remark 5.3.7. The evolution equations derived in this section generalise those derived in [38] in the special case that $L^7 = S^1 \times P^6$ is a warped product. Note however their choice of $SU(3)$-structure $(P^6, \check{\omega}, \check{\Omega})$ differs from ours by a conformal factor so that $(\check{\omega}, \check{\Omega}) = (H\omega, H^{3/2}\Omega)$. In particular, $\check{\omega}$ is not symplectic. Since the induced flow on the data (H, ξ, Ω) is still generally quite complicated we shall only study it on a couple of simple examples in the next section, which exclude their case.

5.4 Examples of Laplacian solitons

5.4.1 The Bryant-Fernández example

The compact nilmanifold L^7 associated to the 2-step nilpotent Lie algebra $(0,0,0,0,0,12,13)$ admits a closed G_2-structure given by

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

This example was discovered by Fernández in [34] and Bryant worked out the LF on this example in [19]. The solution to the Laplacian flow is given by

$$\varphi_t = f^3 e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $f := (\frac{10}{3}t + 1)^{\frac{1}{5}}$. Note that the solution in [19] contains a mistake, which was subsequently corrected in [35] and more examples were found. This solution is immortal and the volume grows as $\sim t^{1/5}$ in time. Bryant also shows that L^7 cannot admit a torsion free G_2-structure for topological reasons and hence one cannot expect the LF to converge. Nonetheless the torsion $\|\tau_t\|^2_{g_t} = 2f^{-5}$ converges to zero.

We choose the vector field Y generating an S^1 action preserving φ_0 to be e_6 so that the connection form $\xi = e^6$. The solution to the induced flow on the quotient nilmanifold P^6 is then given by

$$H = f^{1/2}, \tag{5.4.1}$$

$$\omega_t = \omega_0 = -e^{17} + e^{24} - e^{35}, \tag{5.4.2}$$

$$\Omega_t^+ = f^{\frac{9}{4}} e^{123} + f^{-\frac{3}{4}}(e^{145} - e^{257} - e^{347}), \tag{5.4.3}$$

$$\Omega_t^- = -f^{-\frac{9}{4}}e^{457} - f^{\frac{3}{4}}(e^{237} + e^{125} + e^{134}), \tag{5.4.4}$$

$$g_{\omega_t} = f^{\frac{3}{2}}((e^1)^2 + (e^2)^2 + (e^3)^2) + f^{-\frac{3}{2}}((e^4)^2 + (e^5)^2 + (e^7)^2), \tag{5.4.5}$$

$$\gamma_6^1 = \frac{1}{2}f^{-5/2}e^5, \;\; d\xi = \frac{1}{2}(e^{12} - e^{47}) + \frac{1}{2}(e^{12} + e^{47}) \in \Lambda_6^2 \oplus \Lambda_8^2 \tag{5.4.6}$$

We see that the symplectic form, and hence the volume form, stay constant while the metric (equivalently the complex structure) degenerates at infinity. Note that neither τ_6 nor τ_8 is zero in this example.

5.4.2 The Apostolov-Salamon examples

Consider the manifold $L^7 = N^6 \times \mathbb{R}_u$ where N^6 is a compact nilmanifold with Lie algebra $(0,0,0,0,13-24,14+23)$. The G_2-structure

$$\varphi = -f^2h(\omega_1 \wedge du) + g^2h(e^{56} \wedge du) - gf^2(\omega_3 \wedge e^5 - \omega_2 \wedge e^6), \tag{5.4.7}$$

defines an orthonormal G_2 coframing on L^7 given by $E^1 = fe^3, E^2 = fe^2, E^3 = ge^5$, $E^4 = -ge^6, E^5 = -fe^1, E^6 = -fe^4$ and $E^7 = hdu$, where f,g,h functions of u only and ω_i denote the standard self-dual 2-forms in $\langle e^1, e^2, e^3, e^4 \rangle$. A direct calculation shows:

Lemma 5.4.1.

1. *$d\varphi = 0$ if and only if $\frac{\partial}{\partial u}(gf^2) = g^2h$.*
2. *$d *_\varphi \varphi = 0$ if and only if $\frac{\partial}{\partial u}(fg) = 0$ and $\frac{\partial}{\partial u}(f) = \frac{gh}{f}$.*

The explicit torsion free G_2-structure given by setting $f = (3u)^{1/3}$, $g = (3u)^{-1/3}$ and $h = 1$ corresponds to the GLPS G_2 metric of section 4.6.

Let us now impose that φ is closed, so that h is determined by condition 1 of lemma 5.4.1, and consider the S^1 action generated by the vector field $Y = e^6$. Then applying the construction of section 5.2 we find

$$\omega = (g^2h)du \wedge e^5 + (gf^2)\omega_2,$$
$$\Omega^+ = -(f^2hg^{3/2})\omega_1 \wedge du - (g^{5/2}f^2)\omega_3 \wedge e^5,$$

$$\Omega^- = -(f^2 g^{5/2})\omega_1 \wedge e^5 + (g^{3/2} h f^2)\omega_3 \wedge du$$
$$g_\omega = g f^2((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + g^3 (e^5)^2 + h^2 g (du)^2,$$
$$\xi = e^6, \quad \gamma_6^1 = -h f^{-2} du, \ d\xi = \omega_3 \in \Lambda^2_6$$
$$H = g^{-1}, \quad \pi_1 = \frac{\partial}{\partial u}(\ln(g^{5/2} f^2))du, \quad \tau_8 = 0.$$

Note that the above form for expressions of the $SU(3)$-structure was to be expected since P^6 is a complex line bundle on $\mathbb{T}^4$ so the $SU(3)$-structure had to involve forms pullbacked from the base and the fibre. Since $\tau_8 = (d\xi)^2_8 = 0$ it follows that these closed G_2-structures all admit Kähler reductions.

Lemma 5.4.2. *The torsion form is computed as*

$$\tau_2 = \frac{\partial}{\partial u}(f^2 g^2)\frac{1}{hg^2}\omega_1 + 4(\frac{g^3}{f^2} - \frac{g^2}{fh}\frac{\partial}{\partial u}(f))e^{56}.$$

Computing the LF for φ of the form (5.4.7) gives a pair

$$\frac{\partial}{\partial t}(f^2 h) = -\frac{\partial}{\partial u}(\frac{1}{hg^2}\frac{\partial}{\partial u}(f^2 g^2)), \tag{5.4.8}$$
$$\frac{\partial}{\partial t}(g f^2) = 4g^2(\frac{g}{f^2} - \frac{1}{hf}\frac{\partial}{\partial u}f). \tag{5.4.9}$$

This shows that the flow preserves ansatz (5.4.7) and hence the Kähler condition. We have been unable to find the general solution to (5.4.8), (5.4.9), though numerics show that there exists many local solutions. An explicit particular solution is given as follows.

A shrinking gradient soliton.

With $f(u) = 2^{-1/4} e^{u/2}$, $g(u) = 2^{1/2} e^u$ and $h(u) = 1$ we have

$$\varphi_0 = -2^{-1/2} e^u (\omega_1 \wedge du) + 2e^{2u}(e^{56} \wedge du) - e^{2u}(\omega_3 \wedge e^5 - \omega_2 \wedge e^6).$$

Taking $\lambda = -18$ and $V = 15 \cdot \partial_u$, we verify directly that the soliton equation (5.3.7) is satisfied. Thus, it defines a gradient shrinking soliton and moreover the induced metric is complete. To the best of our knowledge this example appears to be new.

To derive the soliton equation we first observe that the general invariant vector field V is of the form $V = a \cdot \partial_u + b \cdot e_5 + c \cdot e_6$, for functions $a(u)$, $b(u)$ and $c(u)$. Comparing with the expressions for τ_2 and φ it is easy to see that we get a consistent system only if $b = c = 0$. By rescaling the u-coordinate we can set $h = 1$ and defining $F = f^2 g$ and $G = g^2$, the closed condition becomes equivalent to $G = F'$. We compute the soliton equation for the unknowns $(F(u), a(u))$ as

$$(\ln(F^2F'))' = \frac{\lambda}{(\ln(F))'} + a, \tag{5.4.10}$$

$$\left(\frac{(F(F')^{1/2})'}{F'}\right)' = -\lambda F(F')^{-1/2} - (aF(F')^{-1/2})'. \tag{5.4.11}$$

With the ansatz $F = e^{ku}$, we find the solution $\lambda = -\frac{9}{2}k^2$ and $a = \frac{15}{2}k > 0$. The scalar curvature is

$$Scal = -\frac{1}{2}\|\tau_2\|^2_{\varphi_0} = -\frac{27}{4}k^2.$$

Observe that this example is not only S^1-invariant but is in fact of cohomogeneity one type under the action of the nilpotent Lie group. As the general S^1-invariant LF is quite complicated motivated by the above example we initiate instead a search for complete cohomogeneity one examples in the next chapter.

Chapter 6

Cohomogeneity one closed G_2-structures

6.1 Overview of chapter

Our aim in this chapter is to search for solutions to the Laplacian flow on $S^3 \times \mathbb{R}^4$ which are of cohomogeneity one type.

Recall that a cohomogeneity one manifold is one that admits an action of a compact Lie group G such that M/G is diffeomorphic to either $[0,1]$, $(0,1)$, $[0,1)$ or S^1 (thought of as $[0,1]$ with $0 \sim 1$). The generic G orbit whose quotient corresponds to a point $s \in (0,1)$ is a codimension 1 hypersurface diffeomorphic to a homogeneous manifold G/K (known as the principal orbit) while for the endpoints (if any) corresponding to $s = 0,1$ the orbit is a homogeneous space G/H_i for $i = 0,1$ (known as the singular orbits) and has codimension greater than 1. Thus, we have a group inclusion $K \subset H_i \subset G$ known as the group diagram of the cohomogeneity one manifold M. We shall only be concerned with the case when $M/G = [0,1)$ so that M can be viewed as $(0,1) \times G/K$ compactified at the end $s = 0$ by G/H (writing $H = H_0$ since there is only one singular orbit).

The space $S^3 \times \mathbb{R}^4$ is known to admit three 1-parameter families of cohomogeneity one G_2 metrics, each family corresponding to a different group diagram, see section 6.2. We shall only consider one of these three families in this chapter. The first step to finding solutions to the Laplacian flow is to construct closed cohomogeneity one G_2-structures as initial data. In section 6.3 we show that any smooth closed $SU(2)^3$-invariant G_2-structure is necessarily torsion free and in fact corresponds to the Bryant-Salamon one. Considering the less symmetric case of $SU(2)^2 \times U(1)$-invariant ones we find in section 6.4 that there exists a function's worth of smooth closed G_2-structures and a 1-parameter family of closed cones. The simplest solutions to the Laplacian flow being the soliton ones it makes sense to understand these first. We rule out the existence of invariant conical and smooth solitons on $S^3 \times S^3 \times \mathbb{R}^+$ and $S^3 \times \mathbb{R}^4$ respectively. Since solitons arise in the blow-up analysis of singularities of geometric flows the latter result seems to hint that no finite time singularity occurs, thus suggesting long time existence of the flow.

6.2 Examples of explicit G_2 metrics

The purpose of this section is to give a brief overview of the explicitly known torsion free G_2-structures on $S^3 \times \mathbb{R}^4$ with $SU(2)^2 \times U(1)$ symmetry and also to set up notation for the subsequent sections. We shall adhere to the conventions of [67].

6.2.1 The setup

Consider the Lie algebra $\mathfrak{su}_2$ with its standard basis

$$T_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

such that $[T_i, T_j] = 2\varepsilon_{ijk}T_k$. Denoting by σ_i the dual basis and following the conventions

$$v \wedge w = \frac{1}{2}(v \otimes w - w \otimes v)$$

and

$$2d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$

we find $d\sigma_i = -\varepsilon_{ijk}\sigma_j \wedge \sigma_k$. By identifying the tangent spaces of $S^3 \times S^3$ with $\mathfrak{su}_2 \oplus \mathfrak{su}_2$ we can define a basis of the tangent bundle by $T_i^+ = (T_i, T_i)$ and $T_i^- = (T_i, -T_i)$. The vectors $\{T_i^+\}_{i=1}^3$ define a Lie subalgebra and as such corresponds to the tangent space of a diagonal S^3 embedded in $S^3 \times S^3$. If we now denote by $\eta_i^\pm$

the corresponding dual 1-forms, then the Maurer-Cartan relations give

$$d\eta_i^+ = -\varepsilon_{ijk}(\eta_{jk}^{++} + \eta_{jk}^{--}),$$
$$d\eta_i^- = -2\varepsilon_{ijk}\eta_{jk}^{-+},$$

where we follow the convention that adjacent 1-forms are wedged together. Equipped with these forms we define $SU(2)^2 \times U(1)$ invariant $SU(3)$-structures on $S^3 \times S^3$ by

$$h = (2A_1)^2(\eta_1^+ \otimes \eta_1^+) + (2A_2)^2(\eta_2^+ \otimes \eta_2^+ + \eta_3^+ \otimes \eta_3^+) + (2B_1)^2(\eta_1^- \otimes \eta_1^-) + (2B_2)^2(\eta_2^- \otimes \eta_2^- + \eta_3^- \otimes \eta_3^-), \tag{6.2.1}$$

$$\omega = 4A_1B_1(\eta_{11}^{-+}) + 4A_2B_2(\eta_{22}^{-+} + \eta_{33}^{-+}), \tag{6.2.2}$$

$$\Omega^+ = 8B_1B_2^2\eta_{123}^{---} - 8A_1A_2B_2(\eta_{123}^{++-} + \eta_{123}^{+-+}) - 8A_2^2B_1\eta_{123}^{-++}, \tag{6.2.3}$$

$$\Omega^- = -8A_2^2A_1\eta_{123}^{+++} + 8B_1B_2A_2(\eta_{123}^{--+} + \eta_{123}^{-+-}) + 8A_1B_2^2\eta_{123}^{+--}, \tag{6.2.4}$$

where $A_i, B_i : \mathbb{R}_s^+ \to \mathbb{R}$ for $i = 1, 2$. Here the $U(1)$ is acting diagonally and is generated by the vector field T_1^+. It is easy to check that $\mathscr{L}_{T_1^+}h = 0$ if and only if the coefficient functions of η_i^+ and η_i^- for $i = 2, 3$ are equal. We refer the reader to [9] and section 2 of [42] for more details on these group actions. Having now introduced the basic geometric objects we can now describe the $SU(2)^2 \times U(1)$ invariant G_2 structures on $\mathbb{R}_s^+ \times S^3 \times S^3$ given by the 3-form

$$\varphi = ds \wedge \omega(s) + \Omega^+(s)$$

inducing the metric

$$g_\varphi = ds \otimes ds + h(s).$$

Note that there is an extra $\mathbb{Z}_2$ isometric action which swaps the two S^3s such that $\eta_i^- \leftrightarrow -\eta_i^-$ while η_i^+ remain unchanged. It is now easy to see this action corresponds to multiplying both ω and Ω^+ (and hence φ) by -1. We shall encounter this symmetry again in subsection 6.3.2.

6.2.2 Bryant-Salamon metric

In the Bryant-Salamon (BS) case we have that $A_1 = A_2$ and $B_1 = B_2$, as such this enhances the aforementioned $U(1)$-symmetry to $SU(2)$. More concretely, $S^3 \times S^3$, viewed as $SU(2)^2$, admits in addition to the usual left action by $SU(2)^2$, a diagonal right action by $SU(2)$ and if all the A_i and B_i are equal then the metric is invariant by all three $SU(2)$s. In this case Hitchin's equations (3.4.2) and (3.4.3) become

$$A_1' = \frac{1}{2}(1 - \frac{A_1^2}{B_1^2}) \text{ and } B_1' = \frac{A_1}{B_1}.$$

Defining $s = \int_1^r \frac{dx}{\sqrt{1-x^{-3}}}$, one can solve the above system to get

$$A_1 = \frac{r}{3}\sqrt{1-r^{-3}} \text{ and } B_1 = \frac{r}{\sqrt{3}}.$$

Observe that when $r = 1$, $A_1 = 0$ and $B_1 = \frac{1}{\sqrt{3}}$ so that the principal $SU(2)^2$ orbits collapse to an S^3. This S^3 is an associative submanifold i.e. φ restricts to a volume form on it. This metric is asymptotically conical (AC) to the cone metric on $S^3 \times S^3$ with its nearly Kähler structure.

6.2.3 Brandhuber-Gomis-Gubser-Gukov metric

We now consider the BGGG metric. In this case, by defining

$$s = \int_{9/4}^r \frac{\sqrt{(x-3/4)(x+3/4)}}{(x-9/4)(x+9/4)} dx,$$

we can again solve Hitchin's equation to get

$$A_1 = \frac{\sqrt{(r-9/4)(r+9/4)}}{\sqrt{(r-3/4)(r+3/4)}}, \; A_2 = \sqrt{\frac{(r-9/4)(r+3/4)}{3}}$$
$$B_1 = \frac{2r}{3}, \; B_2 = \sqrt{\frac{(r-9/4)(r+3/4)}{3}}$$

As r tends to infinity, we see that A_1 tends to 1 while the other coefficients grow at the rate r. This metric is said to be of type ALC, i.e. it is asymptotic to a metric on an S^1 bundle over a 6 dimensional CY cone over a Sasaki-Einstein 5-manifold.

In this example the latter is just $S^2 \times S^3$. Bogoyavlenskaya proved that one can deform this G_2 structure to get a 1-parameter family of $SU(2)^2 \times U(1)$ invariant torsion free G_2 structure by varying the relative size of the singular S^3 orbit and that of the S^1 at infinity [15]. As such the Bryant-Salamon metric arises in the limiting case when the size of the S^1 tends infinity while when it shrinks to zero we get the Stenzel metric on the deformation of the conifold which is topologically just $T^*S^3 = S^3 \times \mathbb{R}^3$. The Stenzel metric is a cohomogenity one AC CY metric with principal orbits $S^2 \times S^3$ or more precisely $SU(2)^2/\Delta U(1)$, where the $U(1)$ is generated by T_1^+ as mentioned above.

Recently Foscolo-Haskins-Nordström showed the existence of another family of G_2 metric on $S^3 \times \mathbb{R}^4$, referred to as the $\mathbb{D}^7$ family in physics literature [41]. This 'dual' family arises when instead of considering the smoothing (i.e deformation) of the conifold, one looks at its resolution (i.e blow-up[1]) whereby the singular orbit is replaced by an S^2 rather than an S^3. They then construct a family of G_2 metrics on a non-trivial S^1 bundle on the resolution which is topologically again $S^3 \times \mathbb{R}^4$. The group diagram of this family is $\{1\} \subset \{1\} \times SU(2) \subset SU(2)^2$ which is different from those of the BGGG family; $\{1\} \subset \Delta SU(2) \subset SU(2)^2$. Nonetheless both families are $SU(2)^2 \times U(1)$ invariant. These 2 families are related by a "G_2 flop"; if one fixes the size of the S^1 at infinity and vary the parameter within each family so that the singular S^3 collapses to a point, then we get the same conically singular asymptotically locally conical (CS ALC) metric in the limit. The difference between these two families is that the S^1 quotient of the BGGG family is (topologically) the smoothing of the conifold while the S^1 quotient of the $\mathbb{D}^7$ family is the resolved conifold; the S^1 here being the one corresponding to the asymptotically constant sized S^1 at infinity. This is duality is the G_2 analogue of the concept of geometric transition for CY 3-folds.

[1] strictly speaking the blow-up introduces 2 $\mathbb{C}P^1$s and one can contract either to get the same topological space although distinct as complex manifolds.

6.3 $SU(2)^3$-invariant closed G_2-structures on $\mathbb{R}^4 \times S^3$

6.3.1 $SU(2)^3$-invariant $SU(3)$-structures on $S^3 \times S^3$

As seen above the BS metric on the spinor bundle of S^3 is $SU(2)^3$-invariant. Our aim in this section is to classify all $SU(2)^3$-invariant $SU(3)$-structures on $S^3 \times S^3$, cf. [71, 76] for $SU(2)^2$-invariant ones.

To do so we first remind the reader of the group action. Given a point $(a,b) \in S^3 \times S^3$, the group $SU(2)^3$ acts as

$$(g,h,k)(a,b) = (gak^{-1}, hbk^{-1}).$$

An alternative way of describing this action is to identify $S^3 \times S^3$ with $SU(2)^3/\Delta SU(2)$ where the $\Delta SU(2)$ acts diagonally on the right and the natural action of $SU(2)^3$ on the left passes to the quotient.
As an $SU(2)^3$ module, we have the decomposition

$$\Lambda^2(\mathbb{R}^6) \cong \Lambda^2(\mathbb{R}^3 \oplus \mathbb{R}^3) \cong 3\mathbb{R}^3 \oplus \mathbb{R} \oplus S^2_0(\mathbb{R}^3)$$

into irreducible summands. Without loss of generality, the generator of the 1-dimensional component can be taken to be

$$\omega = 12 + 34 + 56,$$

where for visual clarity we denote by $1,3,5$ and $2,4,6$ the left invariant 1-forms on each S^3. We now look at the space of invariant 3-forms. As $SU(2)^3$ modules, we have

$$\begin{aligned}\Lambda^3(\mathbb{R}^3 \oplus \mathbb{R}^3) &\cong \Lambda^3(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3 \oplus \mathbb{R}^3 \otimes \Lambda^2(\mathbb{R}^3) \oplus \Lambda^3(\mathbb{R}^3)\\ &\cong 2\mathbb{R}^3 \oplus 4\mathbb{R} \oplus 2S^2_0(\mathbb{R}^3).\end{aligned}$$

The four invariant subspaces are generated by 135, 246 (since they correspond to the volume forms of each S^3), 146+245+236 and 235 +145+136. Thus, an arbitrary invariant 3-form will be a linear combination of these forms.

Our strategy can now be summed up as follows. By imposing the normalisation condition, it will follow that there exist a $4 = (5-1)$ parameter family of $SU(3)$ structures on $SU(2)^2$ which are $SU(2)^3$ invariant. We can then use 'half' of the Hitchin's equations to construct closed G_2 structures on $\mathbb{R}^+ \times S^3 \times S^3$. Finally we will need to ensure that these solutions extend smoothly across a singular S^3 orbit and are globally well-defined.

The most general ansatz for the $SU(3)$-structure is given by

$$\omega = f(12+34+56), \tag{6.3.1}$$

$$\Omega^+ = a(135) + b(146+245+236) + c(235+145+136) + d(246), \tag{6.3.2}$$

where the parameters a, b, c, d and f need to be determined. The compatibility condition $\omega \wedge \Omega^+ = 0$ is already satisfied. Following [54], we view $(\mathfrak{su}_2 \oplus \mathfrak{su}_2, 123456)$ as an oriented vector space (V, ε). We define an operator

$$\begin{aligned} K_{\Omega^+} : V &\to V \otimes \Lambda^6(V^*) \\ v &\mapsto A((v \lrcorner \Omega^+) \wedge \Omega^+) \end{aligned}$$

where $A : \Lambda^5 V^* \cong V \otimes \Lambda^6 V^*$ is the inverse of the interior product. The orientation form ε allows us to view K_{Ω^+} as an endomorphism of V. With respect to the standard basis we can write

$$K_{\Omega^+} = \begin{pmatrix} -ad+bc & 2b^2-2cd & 0 & 0 & 0 & 0 \\ 2ad-2c^2 & ad-bc & 0 & 0 & 0 & 0 \\ 0 & 0 & -ad+bc & 2b^2-2cd & 0 & 0 \\ 0 & 0 & 2ad-2c^2 & ad-bc & 0 & 0 \\ 0 & 0 & 0 & 0 & -ad+bc & 2b^2-2cd \\ 0 & 0 & 0 & 0 & 2ad-2c^2 & ad-bc \end{pmatrix}.$$

We see that indeed $\mathrm{tr}(K_{\Omega^+}) = 0$, in agreement with (6) in [54]. The linear map

$$\mathrm{J} := \frac{1}{\sqrt{-\lambda}} K_{\Omega^+},$$

where $\lambda := \frac{1}{6}\mathrm{tr}(K_{\Omega^+}^2) = (ad-bc)^2 + 4(ab-c^2)(b^2-cd)$, is an almost complex structure iff $\lambda < 0$. By construction J is independent of the choice of basis. Thus, the 4-parameter space of $SU(2)^3$-invariant $SU(3)$-structures on $S^3 \times S^3$ is parametrised by the open set $\{(a,b,c,d) \mid \lambda < 0\}$.

Lemma 6.3.1. *Writing $A = -ad + cb$, $B = 2ab - 2c^2$ and $C = 2b^2 - 2cd$, the complex structure* J *is explicitly given by,*

$$\begin{aligned}
\sqrt{-\lambda}\,\mathrm{J}(1) &= A1 + C2\\
\sqrt{-\lambda}\,\mathrm{J}(2) &= B1 - A2\\
\sqrt{-\lambda}\,\mathrm{J}(3) &= A3 + C4\\
\sqrt{-\lambda}\,\mathrm{J}(4) &= B3 - A4\\
\sqrt{-\lambda}\,\mathrm{J}(5) &= A5 + C6\\
\sqrt{-\lambda}\,\mathrm{J}(6) &= B5 - A6.
\end{aligned}$$

This allows us to compute $\Omega^- := \mathrm{J}(\Omega^+)$ as

$$\begin{aligned}
\Omega^- = \left(\frac{1}{\sqrt{-\lambda}}\right)^3 &((135)a_{135} + (146+245+236)a_{146}\\
&+(246)a_{246} + (145+235+136)a_{145}),
\end{aligned}$$

where

$$\begin{aligned}
a_{146} &= aAC^2 + b(A^3 - 2ABC) + c(BC^2 - 2A^2C) + d(A^2B),\\
a_{145} &= aA^2C + b(B^2C - 2A^2B) + c(-A^3 + 2ABC) + d(-AB^2),\\
a_{135} &= aA^3 + b(3AB^2) + c(3A^2B) + d(B^3),\\
a_{246} &= aC^3 + b(3A^2C) + c(-3AC^2) - d(A^3).
\end{aligned}$$

As already mentioned above the normalisation condition (2.3.2) imposes one constraint on quintuple (a,b,c,d,f). Explicitly this corresponds to

$$f^3 = \frac{1}{4(\sqrt{-\lambda})^3}(3ca_{146} - 3ba_{145} - aa_{246} + da_{135}).$$

6.3.2 $SU(2)^3$-invariant closed G_2-structures

Since we have now derived explicit expressions for a general $SU(2)^3$-invariant $SU(3)$-structure we can compute the closed G_2 equations as follows.

Lemma 6.3.2. *A 1-parameter family $(a(s),b(s),c(c),d(s))$ defines a closed G_2-structure on $\mathbb{R}^+_s \times S^3 \times S^3$ if and only if $b=-c$, $a'=d'=0$ and $b'=-f$, where f is determined by (a,b,c,d) via the formula given in the previous subsection.*

Proof. Differentiating a 1-parameter family of $SU(2)^3$ invariant $SU(3)$ structure on $SU(2)^2$ we get

$$d\omega = f'\,\mathrm{ds}\wedge(12+34+56) + f(235-146+145-236+136-245)$$

and

$$\begin{aligned} d\Omega^+ = {} & b(3546+2413-2516)+c(4635-1625+1342) \\ & + b'\,\mathrm{ds}\wedge(146+245+236)+c'\,\mathrm{ds}\wedge(235+145+136) \\ & + a'\,\mathrm{ds}\wedge(135)+d'\,\mathrm{ds}\wedge(246). \end{aligned}$$

The proof is completed by imposing $d\varphi = 0$, or equivalently that $\frac{\partial}{\partial s}\Omega^+ = d_6\omega$ and $d_6\Omega^+ = 0$, where d_6 refers to the exterior differential on $S^3 \times S^3$. $\square$

Hence we see that a and d are constants, in other words the cohomology class of the stable 3-form Ω^+ in $H^3(S^3\times S^3,\mathbb{R}) \cong \mathbb{R}^2$ is fixed. The latter isomorphism simply maps $[\Omega^+]$ to (a,d).

In this notation the BS solution of section 6.2.2 can be expressed as

$$g_\varphi = (A_1^2((1+2)^2+(3+4)^2+(5+6)^2)+B_1^2((1-2)^2+(3-4)^2+(5-6)^2))/4$$

$$\begin{aligned}\omega(s) &= A_1B_1((1-2)(1+2)+(3+4)(3-4)+(5+6)(5-6))/4\\ \Omega^+(s) &= -((B_1^3-3A_1^2B_1)(135)+(-B_1^3+3A_1^2B_1)(246)+\\ &\quad (B_1^3+A_1^2B_1)(-145+146-235+245+236-136))/2^3\end{aligned}$$

and one checks directly that indeed $d=-a=1/(24\sqrt{3})$ are constants. Thus, after rescaling we see that Ω^+ belongs to the cohomology class $(1,-1)$. Since our ultimate goal is to find solutions to the Laplacian flow and we know that the flow preserves the cohomology class of φ (equivalently of Ω^+) it makes sense to restrict to those classes which are known to admit torsion free solutions. In fact, a torsion free $SU(2)^3$-invariant G_2-structure extends smoothly across a singular S^3 orbit if and only if $[\Omega^+]$ belongs to $(1,-1)$, $(0,1)$ or $(-1,0)$, after suitable rescaling, cf. [16]. The first case corresponds to the situation when the diagonal S^3 in $S^3\times S^3$ collapses at $s=0$ and the latter two correspond to the case when either the first or second S^3 factor collapses. This relates to our discussion in section 6.2.3 whereby $S^3\times S^3$ can be viewed in 3 different ways

$$\begin{gathered}\{1\}\subset \Delta SU(2)\subset SU(2)^2,\\ \{1\}\subset\{1\}\times SU(2)\subset SU(2)^2,\\ \{1\}\subset SU(2)\times\{1\}\subset SU(2)^2.\end{gathered}$$

The $\mathbb{Z}_2$ outer automorphism which swaps the two S^3s, mapping $1\leftrightarrow 2$, $3\leftrightarrow 4$, $5\leftrightarrow 6$, preserves the metric in the first case only. Note however that it swaps $(\omega,\Omega^+)\leftrightarrow(-\omega,-\Omega^+)$. In the last two cases this map is essentially a lift of the Atiyah flop and swaps the two $SU(2)^3$-invariant metrics.

From lemma 6.3.2 we see that the closed equations correspond to a determined system; a single ODE in b. Although the equation $b'=-f$ is hard to solve explicitly, when $(a,d)=(1,-1)$ and $b=-c$ the ansatz (6.3.1), (6.3.2) can be equivalently expressed as (6.2.2), (6.2.3) with $A_1=A_2$ and $B_1=B_2$. The closed equation is then easily seen to be equivalent to the torsion free one, cf. lemma 6.4.1 and 6.4.2 below. An inspection of the formula for J in lemma 6.3.1 in the case when $(a,d)=(0,1)$

shows that the metric h induced by ω and J can be diagonalised by choosing the coframing

$$\mathbf{2},\ \mathbf{4},\ \mathbf{6},\ 2(\mathbf{1})-\mathbf{2},\ 2(\mathbf{3})-\mathbf{4},\ 2(\mathbf{5})-\mathbf{6}$$

on $S^3 \times S^3$, where we denote the 1-forms in bold to distinguish from the constant '2' factor. With respect to these coordinates one can express ω and Ω^+ by similar expression as in (6.2.2) and (6.2.3) with $\eta_i^{\pm}$ replaced by the above coframing. We can see once again that the closed equation is equivalent to the torsion free one as in the previous case. A similar argument holds in the $(-1,0)$ case. Up to a diffeomorphism these 3 metrics are the same cf. [16] and [55]. To sum up;

Theorem 6.3.3. *A closed $SU(2)^3$-invariant G_2-structure on $\mathbb{R}^4 \times S^3$ is in fact torsion free and the induced metric is isometric to the Bryant-Salamon one.*

Remark 6.3.4. Note that when (a,d) is not equivalent to one of the 3 cases above the local existence of a solution to the ODE $b' = -f$ imply that there does exists a closed G_2-structure on $(\varepsilon_1, \varepsilon_2) \times S^3 \times S^3$ but these will not give complete metrics.

6.4 $SU(2)^2 \times U(1)$-invariant closed G_2-structures on $\mathbb{R}^4 \times S^3$

Since the $SU(2)^3$ symmetry is a too strong condition to find strictly closed G_2 structures, we move to the less symmetric situation of $SU(2)^2 \times U(1)$. Throughout this section we shall use the ansatz introduced in section 6.2, hence we shall restrict only to the case when the metric has the extra $\mathbb{Z}_2$ symmetry.

6.4.1 The basic quantities

The purpose of this section to compute expressions for various quantities that will be essential in our study in subsequent sections.

Lemma 6.4.1. *The closed condition is equivalent to the underdetermined system;*

$$\frac{\partial}{\partial s}(B_1 B_2^2) = A_1 B_1 + 2A_2 B_2 \tag{6.4.1}$$

$$\frac{\partial}{\partial s}(A_1 A_2 B_2) = A_1 B_1 \tag{6.4.2}$$

$$\frac{\partial}{\partial s}(A_2^2 B_1) = -A_1 B_1 + 2A_2 B_2 \tag{6.4.3}$$

In particular, we have the conserved quantity $B_1 B_2^2 - A_2^2 B_1 - 2A_1 A_2 B_2$ (see Proposition 6.4.11 below for a geometric interpretation).

Proof. This is a direct computation from $\frac{\partial}{\partial s}\Omega^+ = d_6\omega$, using

$$\begin{aligned} d_6\omega = &-8A_1B_1\eta_{123}^{+-+} - 8A_1B_1\eta_{123}^{++-} + (8A_1B_1 - 16A_2B_2)\eta_{123}^{-++} \\ &+ (8A_1B_1 + 16A_2B_2)\eta_{123}^{---}. \end{aligned}$$

The condition $d_6\Omega^+ = 0$ is automatic from the ansatz. □

Lemma 6.4.2. *The coclosed equations are given by;*

$$\frac{\partial}{\partial s}(A_1B_1A_2B_2) - A_2^2 A_1 - A_1 B_2^2 = 0$$
$$\frac{\partial}{\partial s}((A_2B_2)^2) - A_1A_2^2 - 2B_1B_2A_2 + A_1B_2^2 = 0$$

Proof. The condition $d_6(\omega \wedge \omega) = 0$ is automatic from the ansatz, so it suffices to impose $\frac{\partial}{\partial s}(\omega) \wedge \omega = -d_6(\Omega^-)$. □

Lemma 6.4.3. *The intrinsic torsion τ_2 of the closed G_2-structure is given by*

$$\begin{aligned} \tau_2 = &\Big((2A_2B_2)\frac{\partial}{\partial s}(A_2B_2) - A_1A_2^2 - 2A_2B_1B_2 + A_1B_2^2\Big)\Big(\frac{2}{A_2B_2}(\eta_{22}^{-+} + \eta_{33}^{-+}) \\ &- 4\frac{A_1B_1}{A_2^2B_2^2}(\eta_{11}^{-+})\Big). \end{aligned}$$

Proof. This is again a direct computation using

$$\begin{aligned} d_6\Omega^- = 16(&\eta_{2323}^{--++}(A_1A_2^2 + 2B_1B_2A_2 - A_1B_2^2) + \eta_{1313}^{--++}(A_2^2A_1 + A_1B_2^2) \\ &+ \eta_{1212}^{--++}(A_2^2A_1 + A_1B_2^2)), \end{aligned}$$

$$\frac{1}{2}\frac{\partial}{\partial s}(\omega^2) = 16\frac{\partial}{\partial s}(A_1A_2B_1B_2(-\eta_{1212}^{--++} - \eta_{1313}^{--++}) - (A_2B_2)^2\eta_{2323}^{--++}).$$

and imposing (6.4.1), (6.4.2) and (6.4.3). □

Remark 6.4.4.

1. Notice that the intrinsic torsion is determined by a single function. The reason for this is that the closed condition is given by three equations whereas the coclosed condition is given by two, however one of these equations coincide, namely the one in Λ^2_7.

2. Observe that if one imposes the coclosed condition and $SU(2)^3$ invariance then we get only one equation;

$$\frac{\partial}{\partial s}A + \frac{A}{B}\frac{\partial}{\partial s}B = \frac{1}{2}(1+\frac{A^2}{B^2}).$$

   This suggests that there exist coclosed G_2-structures which are not closed, which is in sharp contrast to the converse as demonstrated in the previous section. As such these provide possible initial conditions for studying the $SU(2)^3$-invariant Laplacian coflow.

In what follows we shall always assume that φ is closed.

Lemma 6.4.5. *The mean curvature H_s of the principal $S^3 \times S^3$ orbit is given by*

$$\begin{aligned} H_s &= \frac{\partial}{\partial s}(\ln(A_1 A_2^2 B_1 B_2^2)) \\ &= \frac{B_1}{A_2 B_2} - \frac{1}{2}\frac{A_1}{A_2^2} + \frac{B_2}{A_2 B_1} + \frac{1}{2}\frac{A_1}{B_2^2} + \frac{A_2}{B_1 B_2} \end{aligned} \tag{6.4.4}$$

The above expression is rather surprising since imposing that the G_2-structure is closed determines the mean curvature; a quantity involving first order terms in g_φ, in fact given by $h(s)^{-1}h'(s)$ cf. [31, (2.1)] up to a factor of 2, by only an algebraic expression. In particular we record that in the BS situation we have:

$$H_s = \frac{3}{2A}(1+\frac{A^2}{B^2}) = \frac{3}{2}\frac{(4-r^{-3})}{r(1-r^{-3})^{1/2}},$$

where $s = \int_1^r \frac{dx}{\sqrt{1-x^{-3}}}$. Note the mean curvature is never zero for $r \in (1,\infty)$ but converges to zero.

Remark 6.4.6. Proposition 6.1 of [42] states that a local torsion free G_2 structure defined for $s \in (\varepsilon_1, \varepsilon_2)$ extends to a complete solution at infinity if and only if there does not exists any $s^* \in [\varepsilon_2, \infty)$ such that $H_{s^*} = 0$. In other words the forward completeness of the G_2 metric is equivalent to asking that there exists no minimal principal orbit i.e. a principal orbit with is also a minimal hypersurface. The proof relies on the fact that G_2 manifolds are Ricci flat (or more generally Einstein is a sufficient condition). In our case however we only have closed G_2 structures so we cannot use their argument. Hence we need to use the closed condition together with (6.4.4) to find a lower bound for H. One might still require further constraints to guarantee forward completeness of local closed G_2-structure though but we will not address this issue in this thesis.

Lemma 6.4.7. *The Laplacian flow equations are given as follows;*

$$\begin{aligned}\frac{\partial}{\partial t}(4A_2B_2) = &\frac{\partial^2}{\partial s^2}(4A_2B_2) + \frac{\partial}{\partial s}(A_2B_2)(\frac{2A_1}{B_2^2} + \frac{4B_1}{A_2B_2} - \frac{2A_1}{A_2^2}) \\ &- \frac{4A_1B_1}{B_2^2} + \frac{4A_1B_1}{A_2^2} + \frac{2A_1^2}{A_2B_2} \\ &+ \frac{A_1^2A_2}{B_2^3} + \frac{A_1^2B_2}{A_2^3} - \frac{4A_2}{B_2} \\ &- \frac{4B_2}{A_2} + \frac{2A_1A_2^2}{B_1B_2^2} - \frac{2A_1B_2^2}{A_2^2B_1} \end{aligned} \tag{6.4.5}$$

$$\begin{aligned}\frac{\partial}{\partial t}(4A_1B_1) = &\Big(-\frac{8A_1B_1}{A_2B_2}\Big)\frac{\partial}{\partial t}(A_2B_2) \\ &-4\frac{\partial}{\partial s}\Big(\frac{A_1B_1}{A_2B_2}\Big)(2\frac{\partial}{\partial s}(A_2B_2) - \frac{A_1A_2}{B_2} - 2B_1 + \frac{A_1B_2}{A_2}) \end{aligned} \tag{6.4.6}$$

$$\frac{\partial}{\partial t}(B_1B_2^2) = (1 - \frac{A_1B_1}{A_2B_2})(2\frac{\partial}{\partial s}(A_2B_2) - \frac{A_1A_2}{B_2} - 2B_1 + \frac{A_1B_2}{A_2}) \tag{6.4.7}$$

$$\frac{\partial}{\partial t}(A_1A_2B_2) = -(\frac{A_1B_1}{A_2B_2})(2\frac{\partial}{\partial s}(A_2B_2) - \frac{A_1A_2}{B_2} - 2B_1 + \frac{A_1B_2}{A_2}) \tag{6.4.8}$$

$$\frac{\partial}{\partial t}(A_2^2 B_1) = (1 + \frac{A_1 B_1}{A_2 B_2})(2\frac{\partial}{\partial s}(A_2 B_2) - \frac{A_1 A_2}{B_2} - 2B_1 + \frac{A_1 B_2}{A_2}) \tag{6.4.9}$$

Since we know that the flow preserves the closed condition and that the torsion form is determined by only 1 function, it follows that the above system is determined by only one of these equations, essentially (6.4.5). Moreover we observe that equation (6.4.5) is parabolic which is consistent with the fact that the LF is parabolic (modulo diffeomorphism). An immediate consequence of the above equations is:

Lemma 6.4.8. *The conserved quantity $B_1 B_2^2 - A_2^2 B_1 - 2A_1 A_2 B_2$ of lemma 6.4.1 is also preserved in time under the flow.*

6.4.2 Smooth extension to singular orbit

As seen in the previous section the closed G_2 equations form an underdetermined system (3 equations for 4 variables) and as such one expects that there exists many local closed G_2-structures. However these may not extend smoothly to the singular S^3 orbit. The purpose of this section is to derive sufficient conditions to ensure this smooth extension.

We shall now use the Eschenburg-Wang technique [31] to determine when an invariant 3-form defined on the principal orbits extends smoothly on the singular orbit. Their method can be summarised as follows; Consider a smooth manifold M with the action of compact Lie group G such that the principal orbits have codimension 1 i.e. the quotient space M/G is 1-dimensional. We shall be interested in the situation when this quotient is isomorphic to $[0,1)$. The singular orbit Q, corresponding to the orbit at 0 with isotropy group H, has codimension strictly greater than 1. A neighbourhood of Q can be equivariantly identified with its normal bundle, say with fibre V and the principal orbits, say with isotropy group K, are then identified with its sphere bundle. A G-invariant tensor $T \in C^\infty(\bigotimes^l TM \otimes \bigotimes^m T^*M)$ can be identified with an H equivariant map

$$T : V \to \bigotimes^l (V \oplus \mathfrak{p}) \otimes \bigotimes^m (V \oplus \mathfrak{p})^*$$

where we have chosen an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ so that the tangent at $eH \in Q$ is identified with $\mathfrak{p}$. Since H acts transitively on the sphere in V and T is G-invariant, it suffices to know its value along a ray in V starting from the origin to completely determine T. The Eschenburg-Wang technique answers the question as to when can one smoothly extend a 1-parameter family of H equivariant map

$$T_s : S \to \bigotimes^{l}(V \oplus \mathfrak{p}) \otimes \bigotimes^{m}(V \oplus \mathfrak{p})^*$$

where S is the unit sphere in V and $H \subset O(V)$ (for some metric on V), onto Q. The answer can be found by the following steps.

S1: Find K-invariant tensors in $\bigotimes^l(V \oplus \mathfrak{p}) \otimes \bigotimes^m(V \oplus \mathfrak{p})^*$ and compute their degree by which we mean the degree of the H-equivariant homogeneous polynomial map $S \subset V \to \bigotimes^l(V \oplus \mathfrak{p}) \otimes \bigotimes^m(V \oplus \mathfrak{p})^*$ (which by H-equivariance necessarily take values in the subspace of K-invariant tensors). Then evaluate them at any point on the unit sphere S in V.

S2: Write T_s as a finite sum of these tensors. Then T_s extends smoothly if and only if the coefficient functions, which are just functions of s, written as Taylor series are even (respectively odd) functions if the degree of its invariant tensor is even (respectively odd) and r is greater than or equal to the degree of the tensor, where $a_r s^r$ is the first non-zero term in the expansion.

Let us illustrate concretely how to apply this technique. In our situation $G = SU(2)^2$, $K = \{1\}$, $H = \Delta SU(2)$, $T_s = \varphi_s$ and $V = \mathbb{R}^4$. So the first step is to find the degrees of the 3-forms we will need to write φ_s i.e. we need to find the degree of $Sp(1)$ equivariant homogeneous polynomial maps $f : \mathbb{H} \to \Lambda^3(\mathbb{H} \oplus \mathrm{im}\,\mathbb{H})$. For $x \in Sp(1)$, we have by equivariance that $f(x)$ is determined by $f(1)$ i.e. explicitly

$$f(x)((a,b),(c,d),(e,f)) = f(1)((\bar{x}ax, \bar{x}b),(\bar{x}cx, \bar{x}d),(\bar{x}ex, \bar{x}f)),$$

where $(a,b),(c,d),(e,f) \in \mathbb{H} \oplus \mathrm{im}\,\mathbb{H}$.

Lemma 6.4.9. *The list of relevant* 3*-forms evaluated at* $x = 1$ *are:*

- *degree 0 :* $dq_1dq_2dq_3$, *and* $dp_3dp_2dq_1 + dp_2dp_0dq_2 + dq_1dp_1dp_0 +$

$dp_1dp_3dq_2 + dp_2dp_1dq_3 + dp_3dp_0dq_3$

- *degree 1 :* $dp_3dq_2dq_1 - dp_2dq_3dq_1 + dp_1dq_3dq_2$
- *degree 2 :* $dp_0dp_2dq_2 + dp_1dp_3dq_2 + dp_0dp_3dq_3 + dp_2dp_1dq_3$, $dp_0dp_2dq_2 + dp_0dp_3dq_3 - dq_1dp_3dp_2$, *and* $dp_0dp_1dq_1 + dp_3dp_2dq_1$
- *degree 3 :* $dp_1dq_3dq_2$
- *degree 4 :* $dp_1dp_3dq_2 - dp_1dp_2dq_3 + dp_2dp_3dq_1$,

where $p = (p_0, p_1, p_2, p_3)$ *and* $q = (q_1, q_2, q_3)$ *denote the coordinates on* $\mathbb{H} \oplus \mathrm{im}\,\mathbb{H}$.

Proof. Consider the degree zero, i.e. constant, polynomial $f(x) = dq_1dq_2dq_3$. This is clearly $SU(2)$ invariant since it corresponds to the volume form of S^3 so is trivially $SU(2)$ equivariant of degree 0. Likewise $f(x) = dp_3dp_2dq_1 + dp_2dp_0dq_2 + dq_1dp_1dp_0 + dp_1dp_3dq_2 + dp_2dp_1dq_3 + dp_3dp_0dq_3$ is also invariant since it corresponds to the cyclic permutation of the self-dual forms on $\mathbb{H}$ with the 1-forms on $\mathrm{im}\,\mathbb{H}$.

Consider now the degree 1 polynomial given by

$$f(x)((a,b),(c,d),(e,f)) = \bar{x}(acf + ceb + ead - aed - caf - ecb).$$

It determines four 3-forms corresponding to the real and imaginary parts. Evaluating at 1 these correspond to

$$\begin{aligned}
&dp_3dq_2dq_1 - dp_2dq_3dq_1 + dp_1dq_3dq_2, \\
&dp_2dq_2dq_1 + dp_3dq_3dq_1 - dp_0dq_3dq_2, \\
&dp_0dq_3dq_1 - dp_1dq_2dq_1 + dp_3dq_3dq_2, \\
&dp_0dq_2dq_1 + dp_1dq_3dq_1 + dp_2dq_3dq_2.
\end{aligned}$$

Consider now the degree 2 polynomial given by

$$f(x)((a,b),(c,d),(e,f)) = (\bar{x}a)(d\bar{x}f - f\bar{x}d) + (\bar{x}c)(f\bar{x}b - b\bar{x}f) + (\bar{x}e)(b\bar{x}d - d\bar{x}b).$$

Again as above we get four 3-forms but we only consider the one relevant for us. Evaluating it at 1, we get the 3-form

$$dp_1dp_3dq_2 + dq_3dp_2dp_1 + dq_1dp_3dp_2.$$

Similarly the degree 3 polynomial is given by

$$f(x)((a,b),(c,d),(e,f)) = \bar{x}axi\bar{x}cf + \bar{x}cxi\bar{x}eb + \bar{x}exi\bar{x}ad - \bar{x}axi\bar{x}ed - \bar{x}cxi\bar{x}af - \bar{x}exi\bar{x}cb$$

and the degree 4 by

$$f(x)((a,b),(c,d),(e,f)) = \bar{x}axi(\bar{x}d\bar{x}f - \bar{x}f\bar{x}d) + \bar{x}cxi(\bar{x}f\bar{x}b - \bar{x}b\bar{x}f) + \bar{x}exi(\bar{x}b\bar{x}d - \bar{x}d\bar{x}b).$$

Evaluated at 1, they give the forms stated in the lemma. □

For the second step we have to express φ_s in terms of these invariant forms. We first need to embed $\mathbb{R}^4 \times S^3$ into $\mathbb{H} \oplus \mathbb{H}$ so that we can rewrite φ in terms of the coordinates p and q. Recall that the singular S^3 is given as the homogeneous space $SU(2)^2/\Delta SU(2)$. The normal bundle N is constructed as the associated bundle $(SU(2)^2 \times \mathbb{C}^2)/\Delta SU(2)$, where $\Delta SU(2)$ acts on $\mathbb{C}^2$ in the usual way by left multiplication. The resulting left action of $SU(2)^2$ on N can be identified with $(g,h)\cdot(p,q) = (gp, gqh^{-1})$. Recall that the Lie algebra elements $T_i^{\pm}$ induced global the vector fields on $\mathbb{R}^4 \times S^3$ which we denoted by the same letter. Along the ray $(s,1)$, the vector field generated by the action of T_1^+ is simply $s\frac{\partial}{\partial p_1}$. More explicitly,

$$\left.\frac{d}{dt}\right|_{t=0} (\exp(T_1t)s, \exp(T_1t)\exp(-T_1t)) = (sT_1, 0) = (si, 0) = s\frac{\partial}{\partial p_1}.$$

Similarly we compute

$$T_i^+ = s\frac{\partial}{\partial p_i} \ , \ T_i^- = s\frac{\partial}{\partial p_i} + 2\frac{\partial}{\partial q_i} \text{ and } \frac{\partial}{\partial s} = \frac{\partial}{\partial p_0}.$$

Indeed when $s = 0$, the vector field $T_i^+ = 0$ in accordance with the fact that the diagonal S^3 has collapsed. The corresponding dual 1-forms are then given by

$$\eta_i^+ = \frac{1}{s}dp_i - \frac{1}{2}dq_i, \;\; \eta_i^- = \frac{1}{2}dq_i \;\text{ and }\; dt = dp_0$$

To conclude step 2, we can now rewrite φ as;

$$\begin{aligned}
\varphi = {} & dq_1dq_2dq_3(B_1B_2^2 - 2A_1A_2B_2 - A_2^2B_1) \\
& - C(dp_3dp_2dq_1 + dp_2dp_0dq_2 + dq_1dp_1dp_0 + dp_1dp_3dq_2 + dp_2dp_1dq_3 \\
& \quad + dp_3dp_0dq_3) \\
& + (-\frac{2A_1A_2B_2}{s} - \frac{2A_2^2B_1}{s})(dp_3dq_2dq_1 - dp_2dq_3dq_1 + dp_1dq_3dq_2) \\
& + (-\frac{A_2B_2}{s} + \frac{A_1B_1}{s} + \frac{2A_2^2B_1}{s^2} + \frac{2A_1A_2B_2}{s^2} + C) \\
& (dp_0dp_2dq_2 + dp_1dp_3dq_2 + dp_0dp_3dq_3 + dp_2dp_1dq_3) \\
& - (\frac{2A_1B_1}{s} + C)(dp_0dp_1dq_1 + dp_3dp_2dq_1) \\
& - (\frac{A_2B_2}{s} + \frac{A_1B_1}{s} + \frac{2A_2^2B_1}{s^2} + \frac{2A_1A_2B_2}{s^2} + 2C) \\
& (dp_0dp_2dq_2 + dp_0dp_3dq_3 - dq_1dp_3dp_2) \\
& + (\frac{2A_2^2B_1}{s} - \frac{2A_1A_2B_2}{s})(dp_1dq_3dq_2) \\
& + (-\frac{2A_2^2B_1}{s^2} - \frac{A_1B_1}{s} + \frac{A_2B_2}{s} + \frac{2A_1A_2B_2}{s^2})(dp_1dp_3dq_2 - dp_1dp_2dq_3 \\
& \quad + dp_2dp_3dq_1),
\end{aligned}$$

where the coefficients functions in the terms above need to have degrees 0, 0, 1, 2, 2, 2, 3, 4 respectively, and be even or odd functions depending on the parity of the degree in order for φ to extend smoothly to the singular S^3 orbit. Simplifying the above expression we get the following key lemma.

Lemma 6.4.10. *The* 3*-form φ extends smoothly across the singular S^3 orbit if and only if the following holds in a neighbourhood of $s = 0$;*

$$2A_1B_1 = -c_0s + (k_3 - a_4 + d_4)s^3 + \sum_{i=2}^{\infty} b_{2i+1}s^{2i+1}$$

$$2A_2B_2 = -c_0 s + k_3 s^3 + \sum_{i=2}^{\infty} k_{2i+1} s^{2i+1}$$
$$4A_2^2 B_1 = -c_0 s^2 + a_4 s^4 + \sum_{i=3}^{\infty} a_{2i} s^{2i}$$
$$4A_1A_2B_2 = -c_0 s^2 + d_4 s^4 + \sum_{i=3}^{\infty} d_{2i} s^{2i}$$
$$B_1 B_2^2 = -c_0^3 - 3c_0^2(a_4 - 2k_3)s^2 + \sum_{i=2}^{\infty} r_{2i} s^{2i}$$

where c_0, a_i, d_i, k_i, r_i are arbitrary constants.

Note that although we have given five equations in the above lemma, we only require any four of them to hold as the fifth one will be determined. Nonetheless it will be useful to have all five expressions handy. For the sake of comparison we also include the torsion free case computed in [67]:

$$2A_1B_1 = bs + (2cb + \frac{1}{4b})s^3 + \cdots$$
$$2A_2B_2 = bs + (\frac{1}{8b} - cb)s^3 + \cdots$$
$$4A_2^2B_1 = bs^2 + (-2cb)s^4 + \cdots$$
$$4A_1A_2B_2 = bs^2 + (\frac{1}{8b} + bc)s^4 + \cdots$$
$$B_1B_2^2 = b^3 + (\frac{3}{4}b)s^2 + (\frac{1}{16}b)s^4 + \cdots$$

In the latter case all the terms are determined by the constants b and c only. Geometrically $b^3 = -c_0^3$ corresponds to the volume of the singular S^3 orbit. So fixing $b = 1$ allows us to vary the parameter c and this is precisely the 1-parameter family of ALC G_2 metrics we alluded to in section 6.2. Henceforth we shall always assume that $c_0 < 0$ so that the S^3 has positive volume.

6.4.3 The closed and smooth condition

Recall that our aim is to construct *closed* G_2-structures which extend smoothly across the singular S^3 orbit. But before addressing the above problem we make two important observations which follow immediately from lemmas 6.4.1, 6.4.8 and 6.4.10.

Proposition 6.4.11. *The conserved quantity*

$$B_1B_2^2 - A_2^2B_1 - 2A_1A_2B_2 = -c_0^3$$

corresponds to the volume of the singular S^3 orbit and is preserved under the LF.

Remark 6.4.12. The existence of the above preserved quantity can be explained by the fact that the Laplacian flow preserves the cohomology class of φ, here determined by a pair $(a,d) \in H^3(S^3 \times S^3, \mathbb{R}) \cong \mathbb{R}^2$. As shown in the previous section however the additional $\mathbb{Z}_2$ symmetry imposes that $a = -d$ and thus, this explains why there is only one conserved quantity.

Proposition 6.4.13. *If the closed 3-form φ extends smoothly to S^3 then it calibrates the S^3 and induces the round metric. Moreover this persists under the Laplacian flow.*

Proof. Evaluating φ and g_φ at $s = 0$ gives

$$\begin{aligned} \varphi\big|_{s=0} &= -8c_0^3\eta_{123}^{---}, \\ g_\varphi\big|_{s=0} &= 4c_0^2(\eta_1^- \otimes \eta_1^- + \eta_2^- \otimes \eta_2^- + \eta_3^- \otimes \eta_3^-). \end{aligned}$$

□

In order to analyse the closed equations in a neighbourhood of the S^3 orbit it will be more convenient to introduce the following new variables rather than the A_i and B_i. Let

$$x = B_1B_2^2, \; y = A_1A_2B_2, \; z = A_2^2B_1 \; \text{ and } \; w = A_2B_2.$$

The closed condition in lemma 6.4.1 can then equivalently be expressed as

$$\frac{dx}{dy} = 1 + 2\frac{w^3}{yx^{1/2}(c_0^3 + x - 2y)^{1/2}}$$

together with $z = x - 2y + c_0^3$. Note that the functions x, y, z and w are all required to be positive for $s > 0$ to ensure that g_φ is non-degenerate. Hence from equation (6.4.2) we can interchange between the variables s and y since $y'(s) > 0$ i.e. y is strictly increasing. So we can eliminate the auxiliary variable s by instead reparametrising the arc length by y. It is now easy to see that the function w completely determines the closed G_2-structure. Thus, in order to ensure that the closed G_2-structure extends smoothly to S^3 we only need to choose appropriate w. Before making this rigorous we shall need the following Theorem of Malgrange cf. [40, 73].

Theorem 6.4.14. *Consider the singular initial value problem*

$$\Upsilon' = \frac{1}{s}M_{-1}(\Upsilon) + M(s,\Upsilon), \quad \Upsilon(0) = \Upsilon_0 \tag{6.4.10}$$

where Υ takes value in $\mathbb{R}^k$, $M_{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is a smooth function of Υ in a neighbourhood of Υ_0 and $M : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is smooth in (s,Υ) in a neighbourhood of $(0,\Upsilon_0)$. If the two conditions:

1. $M_{-1}(\Upsilon_0) = 0$
2. $hId - dM_{-1}\big|_{\Upsilon_0}$ *is invertible for all $h \in \mathbb{N}$, $h \geq 1$*

hold then there exists a unique solution $\Upsilon(s)$ to (6.4.10) and the solution depends continuously on Υ_0 satisfying (1) and (2).

Equipped with the above theorem we can now state the smooth extension result for closed G_2-structures.

Proposition 6.4.15. *The $SU(2)^2 \times U(1)$-invariant closed G_2-structure determined by w extends smoothly to the singular S^3 orbit if and only if there exists a function $f(s)$ (unrelated to the function f of section 6.3) given by*

$$f(s) = -\frac{1}{2}c_0^3 - \frac{1}{4}c_0 s^2 + \frac{1}{8}k_3 s^4 + F(s),$$

where F is an even power series of order at least 6 such that $w = f'(s)$. In which case we have that $x = 2f + y$, $z = 2f - y + c_0^3$ and y is defined by

$$\frac{dy}{ds} = \frac{y(c_0^3 - y + 2f(s))^{1/2}(2f(s)+y)^{1/2}}{f'(s)^2}. \tag{6.4.11}$$

and satisfies $y(0) = 0$ *and* $y''(0) \neq 0$.

Proof. If $w(s)$ extends smoothly and φ is closed then from Lemma 6.4.10 we see that f can be defined by $\int w(s)ds$ with $f(0) = -c_0^3/2$ and takes the form stated in the Proposition. This proves the 'only if' part.

From the expressions for x, y, z and w given in terms of $f(s)$ in the Proposition it is straightforward to verify that the closed equations in lemma 6.4.1 are satisfied. To see that the smoothness condition is also satisfied we can directly compute the local power series solution to (6.4.11) and we get

$$\begin{aligned}
x &= -c_0^3 - \frac{3}{4}c_0 s^2 - \frac{1}{16c_0}s^4 + \cdots \\
4y &= -c_0 s^2 - (\frac{1}{4c_0} + k_3)s^4 + \cdots \\
4z &= -c_0 s^2 + (\frac{1}{4c_0} + 2k_3)s^4 + \cdots \\
2w &= -c_0 s + k_3 s^3 + \cdots
\end{aligned}$$

It easy to see these indeed satisfy the conditions of lemma 6.4.10. The only point that remains to be proven is the existence of a solution to (6.4.11) i.e. we need to prove that the power series solution for y converges. Note that the latter is a singular initial value problem since $f'(0) = 0$, so we cannot appeal to the standard existence theorem. However Malgrange's theorem is exactly designed to address this type of problem. To do so we first write

$$y = -\frac{c_0}{4}s^2 + s^4\Upsilon(s)$$

and from the above power series for y we see that $\Upsilon(0) = -(\frac{1}{16c_0} + \frac{k_3}{4})$. Using (6.4.11) we find that Υ satisfies the ODE

$$\Upsilon' = \frac{1}{s}(-3\Upsilon - \frac{3}{4}k_3 - \frac{3}{16c_0}) + O(s),$$

where $O(s)$ denotes a power series of order at least 1 in s with coefficients involving Υ, which is precisely in the form of (6.4.10). The smoothness condition on y implies that $M_{-1}(\Upsilon(0)) = 0$. We also have that $dM_{-1}\big|_{\Upsilon_0} = -3$ and hence condition (2) of Theorem 6.4.14 is also satisfied. Thus, there exists a unique solution depending continuously on $\Upsilon(0)$. □

Thus, Proposition 6.4.15 says that there is a function's worth of local closed G_2-structures, determined by $F(s)$, which extend smoothly to the S^3. We digress for a moment to comment on the torsion free solutions, which are of course included in the closed ones we have just constructed. Using y as the independent variable and from lemma 6.4.3, we can express the torsion free condition as follows.

Proposition 6.4.16. *The G_2-structure determined by φ is torsion free and extends smoothly to the singular S^3 orbit if and only if $x = 2f + y$, $z = 2f - y + c_0^3$ and $f = f(y)$ is a solution to*

$$\ddot{f}(y) - \frac{2\dot{f}(y)}{y} + \frac{\dot{f}(y)}{c_0^3 - y + 2f(y)}(1 + \dot{f}(y)) + \frac{\dot{f}(y)}{y + 2f(y)}(-1 + \dot{f}(y)) = 0, \quad (6.4.12)$$

where $\dot{}$ denotes the derivative with respect to y and satisfies the initial conditions $f(0) = -c_0^3/2$ and $\dot{f}(0) = 1$. Moreover the solution is then completely determined by the constant k_3, where $\ddot{f}(0) = 2/c_0^3 + 12k_3/c_0^2$.

The proof of the above Proposition is simply a matter of expressing the torsion free condition in terms of $f(y)$ and y only, and the initial conditions are chosen to guarantee the smooth extension at the singular S^3 orbit. The argument is analogous to the one in Proposition 6.4.15. We shall not address the issue of local existence of a solution f, which can be proven by an application of Theorem 6.4.14 as above, nor the issue of extending the solution for $s \in [0, \infty)$ which was carried out in [15] and also in [42] albeit in the former case the author formulated the problem as a system of *first* order equations instead. We only content ourselves by stating two explicit solutions;

$$f(y) = y + \frac{1}{6\sqrt{3}}$$

which corresponds to the BS solution with $c_0^3 = -1/(3\sqrt{3})$, and

$$f(y) = \frac{1}{2}(\frac{1}{6}y + \frac{3}{16})\sqrt{48y + 81} + \frac{27}{32}$$

which corresponds to the BGGG solution with $c_0^3 = -27/8$, both described in section 6.2.

Having now established the existence of infinitely many *strictly* closed $SU(2)^2 \times U(1)$-invariant G_2-structures in a neighbourhood of the singular S^3 orbit we now proceed to studying the Laplacian flow.

6.5 $SU(2)^2 \times U(1)$-invariant Laplacian flow

6.5.1 Closed conical G_2-structures and solitons

It is known that the only torsion free G_2 cone metric with this symmetry is the BS one corresponding to the cone on $S^3 \times S^3$ with its nearly Kähler structure. In particular, this cone has the enhanced $SU(2)^3$ symmetry. We shall now show that there exists a 1-parameter family of closed $SU(2)^2 \times U(1)$-invariant G_2 cones which are not torsion free, however none of which are solitons. A motivation for considering conical solitons stems from the fact that in the non-compact setting the analysis of geometric flows is significantly harder than in the compact case. Thus, in many cases one instead considers non-compact manifolds with one end whose geometry is asymptotically conical. For instance there is an extensive literature on asymptotically conical Ricci solitons, cf. [32, 69] and references therein. In the case of the Laplacian flow however, to the best of our knowledge, there are no known (non-trivial) such examples.

Since we are searching for conical G_2-structures, we write $A_i = a_i s$ and $B_i = b_i s$ for constants a_i and b_i to be determined, and applying lemma 6.4.1 we get:

Lemma 6.5.1. *The closed equations for G_2 cones is given by*

$$a_2^2 + b_2^2 = \left(\frac{2}{3}\right)^2, \tag{6.5.1}$$

$b_1 = 3a_2 b_2$ *and* $2a_1 = 3(b_2^2 - a_2^2)$.

Proof. This is immediate from lemma 6.4.1. □

Since we require the metric to be non-degenerate we have to exclude the eight points corresponding to $a_2 = 0$, $b_2 = 0$ and $a_2 = \pm b_2$ on the circle determined by equation (6.5.1). We now observe that the $\mathbb{Z}_2$-symmetry

$$(a_2, b_2) \leftrightarrow (-a_2, -b_2)$$

leaves φ unchanged while the $\mathbb{Z}_2$-symmetry

$$(a_2, b_2) \leftrightarrow (a_2, -b_2)$$

swaps φ and $-\varphi$. There is also a third $\mathbb{Z}_2$-symmetry

$$(a_1, a_2, b_1, b_2) \leftrightarrow (-a_1, b_2, b_1, a_2)$$

which amounts to an outer automorphism of $S^3 \times S^3$. Up to these symmetries we have:

Proposition 6.5.2. *There is a* 1*-parameter of closed* G_2 *cones determined by equation (6.5.1) with* $b_2 > a_2 > 0$. *Only the point* $(1/3, 1/\sqrt{3})$ *gives a torsion free solution. The limiting point* $(\sqrt{2}/3, \sqrt{2}/3)$ *corresponds to the collapsing a circle fibre so that the metric collapses to a conical metric on* $S^2 \times S^3$ *whereas the limiting point* $(0, 2/3)$ *corresponds to a degeneration of the metric.*

We should point out that the cone on $S^2 \times S^3$ in the above Proposition is different from the AC Calabi-Yau cone i.e the $SU(3)$-structure on $S^2 \times S^3$ is not the Sasaki-Einstein one.

Since we have proven the existence of a 1-parameter family of closed cones, a natural question to ask is whether there exists any conical solitons. We shall answer this question negatively.

Proposition 6.5.3. *The soliton equation for a cone is given by*

$$\tau_2 - V \lrcorner \varphi = \frac{\lambda}{3} s^3 \omega + \gamma \tag{6.5.2}$$

where V *is a vector field and* γ *is a closed* 2*-form.*

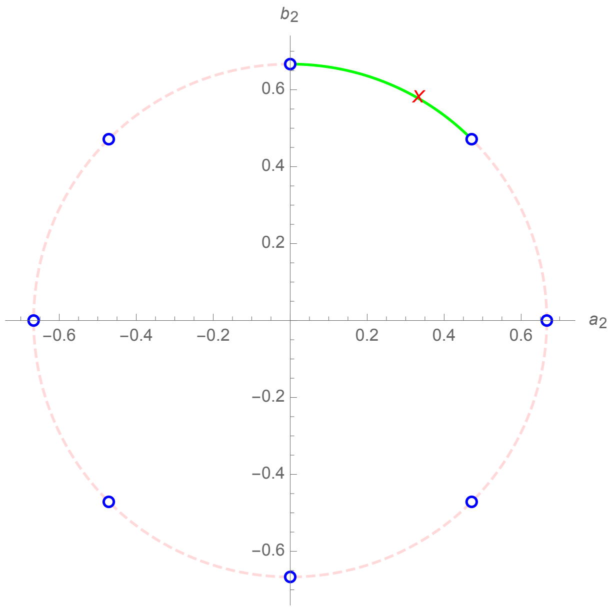

Figure 6.1: 1-parameter family of closed G_2 cones — with x denoting the BS cone

Proof. Recall that Laplacian soliton equation is given by

$$d(\tau_2 - V \lrcorner \varphi) = \lambda \cdot \varphi. \tag{6.5.3}$$

For conical closed G_2-structures we have that $d_6\Omega^+ = 0$ and $d_6\omega = 3\Omega^+$, hence $\varphi = d(s^3\omega/3)$. (6.5.2) now follows from this. □

Since $H^2(S^3 \times S^3, \mathbb{R}) = \{0\}$, in our case $\gamma = d\beta$ is in fact exact. From the $SU(2)^2 \times U(1)$-invariant of the problem we can restrict to V and β of the form:

$$V = v_0\partial_s + v_1^- T_1^- + v_2^-(T_2^- + T_3^-) + v_1^+ T_1^+ + v_2^+(T_2^+ + T_3^+),$$

and

$$\beta = \beta_0 ds + \beta_1^- \eta_1^- + \beta_2^-(\eta_2^- + \eta_3^-) + \beta_1^+ \eta_1^+ + \beta_2^+(\eta_2^+ + \eta_3^+),$$

where $v_0, v_i^\pm$ and $\beta_0, \beta_i^\pm$ are functions of s.

Proposition 6.5.4. *There does not exist any $SU(2)^2 \times U(1)$-invariant cone solitons on $S^3 \times S^3 \times \mathbb{R}^+$, aside from the torsion free one.*

Proof. Using the definitions of V and β as above we can express the soliton equation (6.5.3) as a system of ODEs. Restricting only to terms involving η_{11}^{+-}, η_{22}^{+-} and η_{33}^{+-}, we find that we require

$$\tau_2 = (\frac{\lambda}{3}s^3 + s^2 v_0)\omega \ + \ \text{terms not involving } \eta_{ii}^{+-}. \tag{6.5.4}$$

On the other hand from lemma 6.4.3 we see that τ_2 is equal to

$$-4(a_1 b_1)(\eta_{11}^{+-}) + 2(a_2 b_2)(\eta_{22}^{+-} + \eta_{33}^{+-}) \in \Lambda^2_{14}$$

up to some factor. Thus, (6.5.4) can only hold if $\tau_2 = 0$ since $\omega \in \Lambda^2_7(L^7)$. □

6.5.2 Invariant smooth solitons

Having ruled out the existence of any cone solitons, we shall now rule out the existence of any smooth $SU(2)^2 \times U(1)$-invariant solitons as well.

From the soliton equation (6.5.3) we see immediately that if $\lambda \neq 0$ then φ is necessarily exact. On the other hand from Proposition 6.4.13 we know that φ calibrates the singular S^3 orbit and hence corresponds to the generator of $H^3(S^3 \times \mathbb{R}^4, \mathbb{R}) \cong H^3(S^3, \mathbb{R}) \cong \mathbb{R}$. Thus, there are no shrinking nor expanding solitons which are $SU(2)^2 \times U(1)$-invariant on $S^3 \times \mathbb{R}^4$. This argument in fact applies to any 7-manifold with a closed G_2-structure which calibrates a closed associative. Note that we could not use this argument in the cone case since φ was exact.

To give a more explicit argument we can compute the soliton equation following the same notation introduced in the previous sections. As a consequence we find that

$$\begin{aligned}
\lambda x &= R(w-R)\left(2\frac{d}{dy}(\ln w) - \frac{1}{x} - \frac{2}{y} + \frac{1}{z}\right) - 2v_0 w - v_0 R,\\
\lambda y &= -R^2\left(2\frac{d}{dy}(\ln w) - \frac{1}{x} - \frac{2}{y} + \frac{1}{z}\right) - v_0 R,\\
\lambda z &= R(w+R)\left(2\frac{d}{dy}(\ln w) - \frac{1}{x} - \frac{2}{y} + \frac{1}{z}\right) - 2v_0 w - v_0 R,
\end{aligned}$$

where $R := \frac{yz^{1/2}x^{1/2}}{w^2}$. It is easy to see that $\lambda(x - z - 2y) = 0$. On the other hand from

Proposition 6.4.11 we know that $x - z - 2y = -c_0^3$ is the volume of the associative S^3. Hence if $\lambda \neq 0$ then φ does not extend to the S^3 orbit but is only defined on $\mathbb{R} \times S^3 \times S^3$. If $\lambda = 0$ then it follows that $x_0 = 0$ and

$$2\frac{d}{dy}(\ln w) - \frac{1}{x} - \frac{2}{y} + \frac{1}{z} = 0$$

i.e. $\tau_2 = 0$. We can sum up the result of this section into:

Proposition 6.5.5. *There does not exist any $SU(2)^2 \times U(1)$-invariant soliton on $S^3 \times \mathbb{R}^4$, aside from the torsion free ones.*

Remark 6.5.6. A consequence of the results in this chapter is that if a finite time singularity occurs in the LF on $S^3 \times R^4$ with $SU(2)^2 \times U(1) \times \mathbb{Z}_2$ symmetry then it cannot occur at the associative S^3. The other possibility is that a principal $S^3 \times S^3$ orbit develops a singularity. A blow-up analysis of such a singularity will likely give rise to a soliton on the cylinder $\mathbb{R} \times S^3 \times S^3$ or a torsion free solution. A natural question to ask is whether there exists any $SU(2)^2 \times U(1)$-invariant solitons on $\mathbb{R} \times S^3 \times S^3$ since in this case φ is indeed exact. This is currently work in progress. We have been able to eliminate the existence of steady solitons.

Appendix A

Appendix for S^1 quotient of $Spin(7)$-structures

We give a brief overview of the construction of the Bryant-Salamon metrics on the anti-self dual bundle of S^4. We follow the approach described in [74]. The reader will find proofs of the assertions made therein.

Consider S^4 (of unit radius) embedded in $\mathbb{R}^5$ with coordinates $x_1,...,x_5$ we may choose the following local orthonormal frame

$$v_1 = \frac{1}{R}\begin{pmatrix} x_2 \\ -x_1 \\ x_4 \\ -x_3 \\ 0 \end{pmatrix},\ v_2 = \frac{1}{R}\begin{pmatrix} -x_3 \\ x_4 \\ x_1 \\ -x_2 \\ 0 \end{pmatrix},\ v_3 = \frac{1}{R}\begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ -x_1 \\ 0 \end{pmatrix},\ v_4 = \frac{1}{\sqrt{-1+\frac{1}{x_5^2}}}\begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \\ -x_4 \\ -x_5+\frac{1}{x_5} \end{pmatrix},$$

where $R^2 := x_1^2+x_2^2+x_3^2+x_4^2$. Denoting by e^i the corresponding coframe we compute

$$\begin{aligned}
de^1 &= \frac{2}{R}e^{23} + \frac{\sqrt{1-R^2}}{R}e^{14}, \\
de^2 &= \frac{2}{R}e^{31} + \frac{\sqrt{1-R^2}}{R}e^{24}, \\
de^3 &= \frac{2}{R}e^{12} + \frac{\sqrt{1-R^2}}{R}e^{34}, \\
de^4 &= 0
\end{aligned}$$

In the language of Cartan moving frames the structure equations are given by $d\mathbf{e} = -\omega \wedge \mathbf{e}$ and $F = d\omega + \omega \wedge \omega \in \Lambda^2 \otimes \mathfrak{so}(4)$ where ω is the Levi-Civita connection form and F the curvature. We compute them as

$$\omega = -\begin{pmatrix} 0 & -\frac{1}{R}e^3 & \frac{1}{R}e^2 & \frac{\sqrt{1-R^2}}{R}e^1 \\ \cdot & 0 & -\frac{1}{R}e^1 & \frac{\sqrt{1-R^2}}{R}e^2 \\ \cdot & \cdot & 0 & \frac{\sqrt{1-R^2}}{R}e^3 \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & e^{12} & e^{13} & e^{14} \\ \cdot & 0 & e^{23} & e^{24} \\ \cdot & \cdot & 0 & e^{34} \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

Here we are only writing the upper triangular entries since the matrices are skew-symmetric. The second equation confirms that the round metric has constant curvature and that the scalar curvature is 12. We can define a local orthonormal basis of the anti-self dual bundle by $c^1 := e^{12} - e^{34}$, $c^2 := e^{13} - e^{42}$ and $c^3 := e^{14} - e^{23}$. ω induces a connection on this bundle given by

$$\nabla c^i = \psi^i_j \otimes c^i$$

Since the connection is torsion free we can compute ψ^i_j by

$$\begin{aligned} dc^1 &= \psi^1_2 \wedge c^2 + \psi^1_3 \wedge c^3 \\ dc^2 &= \psi^2_1 \wedge c^1 + \psi^2_3 \wedge c^3 \\ dc^3 &= \psi^3_1 \wedge c^1 + \psi^3_2 \wedge c^2 \end{aligned}$$

where $\psi^2_1 = \frac{\sqrt{1-R^2}+1}{R}e^1$, $\psi^1_3 = \frac{\sqrt{1-R^2}+1}{R}e^2$, $\psi^2_3 = \frac{\sqrt{1-R^2}+1}{R}e^3$ and $\psi^i_j = -\psi^j_i$. These forms can all be pulled back to the total space of the ASD bundle which we denote by the same letter. We introduce fibre coordinates (a_1, a_2, a_3) with respect to the coordinate system defined by c^i. We can define vertical 1-forms by

$$b^i = da_i + a_j \psi^j_i$$

i.e. they vanish on horizontal vectors. Together with the pull back of the e^i they give an absolute parallelism of the ASD bundle. The following forms are all $SO(4)$-

invariant and are hence globally well-defined on the total space:

$$
\begin{aligned}
\rho &= a_1 a_1 + a_2 a_2 + a_3 a_3 \\
\sigma &= 2\,(a_1 b^2 b^3 + a_2 b^3 b^1 + a_3 b^1 b^2) \\
\alpha &= a_1 b^2 c^3 + a_2 b^3 c^1 + a_3 b^1 c^2 - a_1 b^3 c^2 - a_2 b^1 c^3 - a_3 b^2 c^1 \\
\tau &= a_1 c^1 + a_2 c^2 + a_3 c^3 \\
\beta &= 6\, b^{123}.
\end{aligned}
$$

The unit ($\rho = 1$) sphere bundle is diffeomorphic to $\mathbb{C}P^3$ and restricting the above forms we have

$$
\begin{aligned}
g_{FS} &= \frac{1}{2}((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + \frac{1}{2}((b^1)^2 + (b^2)^2 + (b^3)^2)\Big|_{S^2} \\
\omega_{FS} &= \frac{1}{2}\tau - \frac{1}{4}\sigma \\
g_{NK} &= \frac{1}{2}((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + \frac{1}{4}((b^1)^2 + (b^2)^2 + (b^3)^2)\Big|_{S^2} \\
\omega_{NK} &= \frac{1}{2}\tau + \frac{1}{8}\sigma \\
\Omega_{NK} &= \frac{1}{4}(d\tau - i\alpha)
\end{aligned}
$$

The subscript FS refers to the Fubini-Study metric and NK to the nearly-Kähler one. Our choice of scaling was made to fit the conventions of section 3.4.2. The Bryant-Salamon form is then given by

$$
\varphi_{BS} = u^2 v d\tau + \frac{1}{6} v^3 \beta,
$$

where $u = (2\rho + 1)^{1/4}$ and $v = (2\rho + 1)^{-1/4}$.

Appendix B

Appendix for Kähler reduction of $Spin(7)$

As in section 4.2 consider a pair of commuting vector fields X and Y on a 8-manifold N^8 generating a free $\mathbb{T}^2$ action preserving a torsion free $Spin(7)$-structure Φ. We can define 3 $\mathbb{T}^2$-invariant functions by $s_1 := \|X\|_\Phi^{-1}$, $s_2 := \|Y\|_\Phi^{-1}$ and $R := g_\Phi(X,Y)^{1/2}$. There also exists natural 1-forms given by

$$\eta_1(\cdot) := \frac{g_\Phi(X,\cdot)}{g_\Phi(X,X)}, \quad \eta_2(\cdot) := \frac{g_\Phi(Y,\cdot)}{g_\Phi(Y,Y)}.$$

These 1-forms are *not* connection forms, since for instance $\eta_1(Y) = s_1^2R^2$ is generally not constant, instead the canonical Riemannian connection 1-form $A = (A_1,A_2) \in \Omega^1(N^8)\otimes\mathbb{R}^2$ is given by

$$A_1 := (1-R^4s_1^2s_2^2)^{-1}(\eta_1 - R^2s_1^2\eta_2), \quad A_2 := (1-R^4s_1^2s_2^2)^{-1}(\eta_2 - R^2s_2^2\eta_1).$$

Note that these satisfy $A_1(X) = A_2(Y) = 1$, $A_1(Y) = A_2(X) = 0$ and vanish on the orthogonal complement of $\langle X,Y\rangle$.

Proposition B.0.1. *Let $H_1 := s_1^{1/2}s_2^{3/2}(1-R^4s_1^2s_2^2)^{-3/4}$ and $H_2 := s_1^{3/2}s_2^{1/2}(1-R^4s_1^2s_2^2)^{-1/4}$, then we can write*

$$\Phi = A_1\wedge A_2\wedge\sigma + H_1\cdot(A_1 + R^2s_1^2A_2)\wedge\psi^+ - H_2\cdot(A_2)\wedge\psi^- + \frac{1}{2}H_1H_2\cdot(\sigma\wedge\sigma),$$

where $(\sigma,\psi^{\pm})$ defines a symplectic $SU(3)$-structure on the quotient $P^6 := N^8/\mathbb{T}^2$. Furthermore, we have

$$g_\Phi = s_1^{-2}\eta_1^2 + s_2^{-2}(1-R^4 s_1^2 s_2^2)^{-1}(\eta_2 - s_2^2 R^2\eta_1)^2 + s_1 s_2 (1-R^4 s_1^2 s_2^2)^{-1/2} g_\sigma.$$

Proof. By definition the covectors $s_1^{-1}\eta_1$ and $s_2^{-1}(1-R^4s_1^2s_2^2)^{-1/2}(\eta_2 - s_2^2R^2\eta_1)$ are orthonormal. Since we can identify Φ with Φ_0 at each point and $Spin(7)$ acts transitively on $Gr(2,8)$ it follows that we can express it as

$$\Phi = (s_1^{-1}\eta_1)\wedge(s_2^{-1}(1-R^4s_1^2s_2^2)^{-1/2}(\eta_2 - s_2^2R^2\eta_1))\wedge\tilde{\omega} + (s_1^{-1}\eta_1)\wedge\tilde{\Omega}^+ \\ -(s_2^{-1}(1-R^4s_1^2s_2^2)^{-1/2}(\eta_2 - s_2^2R^2\eta_1))\wedge\tilde{\Omega}^- + \frac{1}{2}\cdot(\tilde{\omega}\wedge\tilde{\omega}),$$

where $(\tilde{\omega},\tilde{\Omega}^{\pm})$ defines an $SU(3)$-structure on P^6. Setting $(\sigma,\psi^{\pm}) = (u\tilde{\omega}, u^{3/2}\tilde{\Omega}^{\pm})$ for $u = s_1^{-1}s_2^{-1}(1-R^4s_1^2s_2^2)^{1/2}$ and using the above definitions for A_i proves the expressions for Φ and g_Φ. Taking the exterior derivative of $\sigma := Y \lrcorner X \lrcorner \Phi$ we get

$$d\sigma = d(Y\lrcorner X\lrcorner\Phi) = \mathscr{L}_Y(X)\lrcorner\Phi + X\lrcorner\mathscr{L}_Y\Phi - Y\lrcorner\mathscr{L}_X\Phi = 0$$

which completes the proof. □

Remark B.0.2.

1. The reader will notice that the expression for Φ in the above Proposition is not symmetric in (A_1,s_1) and (A_2,s_2) (with suitable). The reason is because we chose to distinguish the vector field X in the proof which in turn distinguished Ω. However one can equivalently express g_Φ as

$$g_\Phi = s_1^{-2}A_1^2 + s_2^{-2}A_2 + R^2(A_1\otimes A_2 + A_2\otimes A_1) + (H_1H_2)^{1/2}g_\sigma.$$

   showing it is indeed symmetric.

2. When $g_\Phi(X,Y)=0$, we recover the situation investigated in section 3.2 with $s_1 = s$, $s_2 = Hs^{-1/3}$, $\eta_1 = A_1 = \eta$ and $\eta_2 = A_2 = \xi$.

A study of the situation when $(P^6, \sigma, \psi^{\pm})$ is Kähler in this more general setting will be investigated in future work.

Appendix C

Appendix for S^1-invariant closed G_2-structure and the Laplacian flow

We give an analogous result to Proposition 3.2.2 for the quotient of an arbitrary G_2-structure (L^7, φ) under a free S^1 action. Writing $\tau_4 = *_\varphi \tau_3$, and from the inclusion of $SU(3) \subset G_2$ we decompose the G_2 torsion forms as $SU(3)$-modules:

$$\tau_1 = f\eta + T_6^1, \tag{C.0.1}$$

$$\tau_2 = \tau_6 + \tau_8 + \eta \wedge \tau_v, \tag{C.0.2}$$

$$\tau_4 = f_4\omega^2 + (\tau_4)_6^4 + (\tau_4)_8^4 - \eta \wedge (f_+\Omega^+ + f_-\Omega^- + (\tau_4)_6^3 + (\tau_4)_{12}^3), \tag{C.0.3}$$

where we use the notation $(\alpha)_l^k$ to denote a k-form on P^6 which belongs to the $SU(3)$-module of dimension l, associated to the differential form α on L^7.

Proposition C.0.1. *The intrinsic torsion of (L^7, φ) relates to that of (P^6, ω, Ω) by*

1. $7H^2\tau_0 = 6(d\eta)_1^0 + 12\pi_0 H^{3/2}$
2. $f_1 = \frac{1}{2}H^{-3/2}\sigma_0$
3. $6T_6^1 = H^{-1}\gamma_6^1 + \frac{3}{2}H^{-1}dH + \pi_1 + \nu_1$
4. $\tau_8 = -H^{1/2}\sigma_2$
5. $\tau_6 \wedge \omega = (H^{-1/2}d^cH + H^{1/2}J\nu_1 + H^{-1/2}J\gamma_6^1) \wedge \Omega^+$

6. $f_+ = 0$

7. $f_- = -\frac{6}{7}H^{-3/2}(d\eta)^0_1 - \frac{3}{14}\pi_0$

8. $f_4 = \frac{1}{2}(d\eta)^0_1 + \frac{1}{12}\tau_0 H^2$

9. $(\tau_4)^4_6 = \frac{1}{2}(d\eta)^2_6 \wedge \omega + \frac{1}{2}(\frac{3}{2}H^{1/2}dH + H^{3/2}\pi_1 - H^{3/2}\nu_1) \wedge \Omega^+$

10. $(\tau_4)^4_8 = ((d\eta)^2_8 - H^{3/2}\pi_2) \wedge \omega$

11. $(\tau_4)^3_{12} = \nu_3$

12. $(\tau_4)^3_6 = \nu_1 \wedge \omega - 3T^1_6 \wedge \omega$

13. $H^{3/2}\tau_v \wedge \Omega^+ = \frac{1}{2}H^{-1/2}dH \wedge \Omega^- + H^{1/2}J\pi_1 \wedge \Omega^+ - \tau_6 \wedge \omega - 4T^1_6 \wedge H^{1/2}\Omega^-$

A similar result was also given in [24, Thm 5.1] under the simplifying assumption that $H = 1$. The proof of proposition C.0.1 is analogous to the calculations in the closed case in section 5.2. Indeed setting all the G_2 torsion forms aside from τ_2 to zero recovers the results of section 5.2.

Bibliography

[1] Milton Abramowitz. *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables*. Dover Publications, Inc., 1974.

[2] Bobby Samir Acharya, Robert L. Bryant, and Simon Salamon. A circle quotient of a G_2 cone. *arXiv:1910.09518*, 2019.

[3] Vestislav Apostolov and Simon Salamon. Kähler reduction of metrics with holonomy G_2. *Communications in Mathematical Physics*, 246(1):43–61, 2004.

[4] John Armstrong. An ansatz for almost-Kähler, Einstein 4-manifolds. *Journal fur die Reine und Angewandte Mathematik*, 542:53–84, 2002.

[5] Paul S Aspinwall and Renata Kallosh. Fixing all moduli for M-theory on $K3 \times K3$. *Journal of High Energy Physics*, 2005(10):001, 2005.

[6] Michael Francis Atiyah and Nigel Hitchin. *The geometry and dynamics of magnetic monopoles*. Princeton University Press, 1988.

[7] Gavin Ball. Quadratic closed G_2-structures. *arXiv:2006.14155*, 2020.

[8] Wolf Barth, Klaus Hulek, Chris Peters, and Antonius Van de Ven. *Compact Complex Surfaces*, volume 4. Springer-Verlag Berlin Heidelberg, 2004.

[9] Ya. V. Bazaikin and O. A. Bogoyavlenskaya. Complete Riemannian metrics with holonomy group G_2 on deformations of cones over $S^3 \times S^3$. *Mathematical Notes*, 93(5):643–653, 2013.

[10] Giovanni Bazzoni, Lucia Martin-Merchan, and Vicente Munoz. Spin-harmonic structures and nilmanifolds. *arXiv:1904.01462*, 2019.

[11] Eric Bedford and Morris Kalka. Foliations and complex Monge-Ampère equations. *Communications on Pure and Applied Mathematics*, 30(5):543–571, 1977.

[12] Lucio Bedulli and Luigi Vezzoni. The Ricci tensor of SU(3)-manifolds. *Journal of Geometry and Physics*, 57(4):1125–1146, 2007.

[13] Marcel Berger. Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés Riemanniennes. *Bulletin de la Société Mathématique de France*, 83:279–330, 1955.

[14] A. L. Besse. *Einstein Manifolds*. Springer, 1987.

[15] O. A. Bogoyavlenskaya. On a new family of complete G_2-holonomy riemannian metrics on $S^3 \times \mathbb{R}^4$. *Siberian Mathematical Journal*, 54:431–440, 2013.

[16] Andreas Brandhuber. G_2 holonomy spaces from invariant three-forms. *Nuclear Physics B*, 629(1):393–416, 2002.

[17] Andreas Brandhuber, Jaume Gomis, Steven S. Gubser, and Sergei Gukov. Gauge theory at large N and new G_2 holonomy metrics. *Nuclear Physics B*, 611(1):179–204, 2001.

[18] Robert L. Bryant. Metrics with exceptional holonomy. *Annals of Mathematics*, 126(3):525–576, 1987.

[19] Robert L. Bryant. Some remarks on G_2-structures. In *Proceedings of Gökova Geometry-Topology Conference 2005*. International Press, 2006.

[20] Robert L. Bryant. Non-embedding and non-extension results in special holonomy. *The Many Facets of Geometry: A Tribute to Nigel Hitchin*, pages 346–367, 2010.

[21] Robert L Bryant and Simon M Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.*, 58(3):829–850, 1989.

[22] Robert L. Bryant and Feng Xu. Laplacian flow for closed G_2-structures: Short time behavior. *arXiv:1101.2004*, 2011.

[23] Eugenio Calabi. On Kähler manifolds with vanishing canonical class. In *Algebraic Geometry and Topology*, pages 78–89. Princeton University Press, 1957.

[24] Simon Chiossi and Simon Salamon. The intrinsic torsion of $SU(3)$ and G_2 structures. In *Differential Geometry, Valencia 2001*, pages 115–133. World Scientific, 2002.

[25] Diego Conti and Marisa Fernández. Nilmanifolds with a calibrated G_2-structure. *Differential Geometry and its Application*, 29(4):493–506, 2011.

[26] Alessio Corti, Mark Haskins, Johannes Nordström, and Tommaso Pacini. G_2 -manifolds and associative submanifolds via semi-fano 3 -folds. *Duke Mathematical Journal*, 164(10):1971–2092, 2015.

[27] M. Cvetič, G.W. Gibbons, H. Lü, and C.N. Pope. Almost special holonomy in type IIA and M-theory. *Nuclear Physics B*, 638(1):186–206, 2002.

[28] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope. Cohomogeneity one manifolds of $Spin(7)$ and G_2 holonomy. *Phys. Rev. D*, 65:106004, 2002.

[29] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope. New complete noncompact Spin(7) manifolds. *Nuclear Physics B*, 620(1):29–54, Jan 2002.

[30] Simon Donaldson. *Adiabatic Limits of co-associative Kovalev-Lefschetz Fibrations*, volume 324, pages 1–29. Birkhäuser, Cham, 2017.

[31] J.-H Eschenburg and Mckenzie Y. Wang. The Initial Value Problem for Cohomogeneity One Einstein Metrics. *The Journal of Geometric Analysis*, 10(1), 2000.

[32] Mikhail Feldman, Tom Ilmanen, and Dan Knopf. Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons. *J. Differential Geom.*, 65(2):169–209, 2003.

[33] Marisa Fernández. A classification of Riemannian manifolds with structure group $Spin(7)$. *Annali di Matematica Pura ed Applicata*, 143(1):101–122, 1986.

[34] Marisa Fernández. An example of a compact calibrated manifold associated with the exceptional Lie group G_2. *J. Differential Geom.*, 26(2):367–370, 1987.

[35] Marisa Fernández, Anna Fino, and Víctor Manero. Laplacian flow of closed G_2-structures inducing nilsolitons. *The Journal of Geometric Analysis*, 26(3):1808–1837, 2016.

[36] Marisa Fernández and Alfred Gray. Riemannian manifolds with structure group G_2. *Annali di Matematica Pura ed Applicata*, 132(1):19–45, 1982.

[37] Joel Fine and Chengjian Yao. Hypersymplectic 4-manifolds, the G_2-Laplacian flow, and extension assuming bounded scalar curvature. *Duke Math. J.*, 167(18):3533–3589, 2018.

[38] Anna Fino and Alberto Raffero. Closed warped G_2-structures evolving under the Laplacian flow. *Annali Scuola Normale Superiore - Classe Di Scienze*, 2017.

[39] Lorenzo Foscolo. Complete non-compact Spin(7) manifolds from self-dual Einstein 4-orbifolds. *arXiv:1901.04074*, 2019.

[40] Lorenzo Foscolo and Mark Haskins. New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$. *Annals of Mathematics*, 185(1):59–130, 2017.

[41] Lorenzo Foscolo, Mark Haskins, and Johannes Nordström. Complete non-compact G_2-manifolds from asymptotically conical Calabi-Yau 3-folds. *arXiv:1709.04904*, 2017.

[42] Lorenzo Foscolo, Mark Haskins, and Johannes Nordström. Infinitely many new families of complete cohomogeneity one G_2-manifolds: G_2 analogues of the Taub-NUT and Eguchi-Hanson spaces. *arXiv:1805.02612*, 2018.

[43] Udhav Fowdar. S^1-invariant Laplacian flow. *arXiv:2007.05130*, 2020.

[44] Udhav Fowdar. S^1-quotient of $Spin(7)$-structures. *Annals of Global Analysis and Geometry*, 57(4):489–517, 2020.

[45] Udhav Fowdar. Spin(7) metrics from Kähler Geometry. *arXiv:2002.03449*, 2020.

[46] Marco Freibert. $SU(4)$-holonomy via the left-invariant hypo and Hitchin flow. *Annali di Matematica Pura ed Applicata*, pages 1051–1084, 12 2016.

[47] William Fulton and Joe Harris. *Representation Theory*. Springer-Verlag New York, 2004.

[48] G. W. Gibbons and S. W. Hawking. Gravitational multi-instantons. *Physics Letters B*, 78(4):430–432, 1978.

[49] G.W. Gibbons, H. Lü, C.N. Pope, and K.S. Stelle. Supersymmetric domain walls from metrics of special holonomy. *Nuclear Physics B*, 623(1):3–46, 2002.

[50] Gong, Ming-Peng. Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed field and $\mathbb{R}$). *PhD thesis at University of Waterloo*, 1998.

[51] Alfred Gray and Paul S. Green. Sphere transitive structures and the triality automorphism. *Pacific J. Math.*, 34(1):83–96, 1970.

[52] Alfred Gray and Luis M. Hervella. The sixteen classes of almost hermitian manifolds and their linear invariants. *Annali di Matematica Pura ed Applicata*, 123:35–58, 1980.

[53] Hans-Joachim Hein, Song Sun, Jeff Viaclovsky, and Ruobing Zhang. Nilpotent structures and collapsing Ricci-flat metrics on K3 surfaces. *arXiv:1807.09367*, 2018.

[54] Nigel Hitchin. The geometry of three-forms in six and seven dimensions. *Journal Differential Geometry*, 55(3):547–576, 2000.

[55] Nigel Hitchin. Stable forms and special metrics. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), number 288 in Contemporary Mathematics*, pages 70–89, 2001.

[56] Nigel Hitchin, Anders Karlhede, Ulf Lindström, and Martin Roček. HyperKähler metrics and Supersymmetry. *Communications in Mathematical Physics*, 108(4):535–589, 1987.

[57] Stefan Ivanov. Connection with torsion, parallel spinors and geometry of Spin(7) manifolds. *Mathematical Research Letters*, 11(2):171 – 186, 2004.

[58] Stefan Ivanov, Maurizio Parton, and Paolo Piccinni. Locally conformal parallel G_2 and $Spin(7)$ manifolds. *Mathematical Research Letters*, 13(2):167–177, 2006.

[59] Dominic Joyce. Compact Riemannian 7-manifolds with holonomy. *Journal of Differential Geometry*, 43(2):329–375, 1996.

[60] Dominic Joyce. A new construction of compact 8-manifolds with holonomy $\mathrm{Spin}(7)$. *J. Differential Geom.*, 53(1):89–130, 1999.

[61] Dominic Joyce. *Riemannian holonomy groups and calibrated geometry*, volume 12. Oxford University Press, 2007.

[62] Spiro Karigiannis. Deformations of G_2 and Spin(7) Structures on Manifolds. *math/0301218*, 2003.

[63] P. Kaste, R. Minasian, M. Petrini, and A. Tomasiello. Kaluza-Klein bundles and manifolds of exceptional holonomy. *Journal of High Energy Physics*, 2002(9):33, 2002.

[64] P. Kaste, R. Minasian, M. Petrini, and A. Tomasiello. Nontrivial RR two-form field strength and $SU(3)$-structure. *Fortschritte der Physik*, 51(7-8):764–768, 2003.

[65] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry II*. Volume 61 of Wiley Classics Library. Wiley, 1963.

[66] H. Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry (PMS-38)*. Princeton University Press, 1989.

[67] Jason D. Lotay and Goncalo Oliveira. $SU(2)^2$-invariant G_2-instantons. *Mathematische Annalen*, 371(1):961–1011, 2018.

[68] Jason D. Lotay and Yong Wei. Laplacian flow for closed G_2 structures: Shi-type estimates, uniqueness and compactness. *Geometric and Functional Analysis*, 27(1):165–233, 2017.

[69] John Lott and Patrick Wilson. Note on aymptotically conical expanding Ricci solitons. *Proceedings of American Mathematical Society*, 145(8):3525–3529, 2017.

[70] Thomas Bruun Madsen. Spin(7)-manifolds with three-torus symmetry. *Journal of Geometry and Physics*, 61(11):2285 – 2292, 2011.

[71] Thomas Bruun Madsen and Simon Salamon. Half-flat structures on $S^3 \times S^3$. *Annals of Global Analysis and Geometry*, 44(4):369–390, 2013.

[72] Thomas Bruun Madsen and Andrew Swann. Toric geometry of spin(7)-manifolds. *International Mathematics Research Notices*, 2019. rnz279.

[73] Bernard Malgrange. Sur les points singuliers des équations ifférentielles. *Séminaire Équations aux dérivées partielles (Polytechnique)*, 1971-1972. talk:20.

[74] Simon Salamon. *Riemannian geometry and holonomy groups*. Longman Scientific & Technical, 1989.

[75] Simon Salamon. Complex structures on nilpotent Lie algebras. *Journal of Pure and Applied Algebra*, 157(2):311 – 333, 2001.

[76] Fabian Schulte-Hengesbach. Half-flat structures on lie groups. *PhD thesis at University of Hamburg*, 2010.

[77] T. Shioda and H. Inose. *On Singular $K3$ Surfaces*, pages 119–136. Cambridge University Press, 1977.

[78] Gang Tian and Shing Tung Yau. Complete Kähler manifolds with zero Ricci curvature. I. *Journal of the American Mathematical Society*, 3(3):579–609, 1990.