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On Node and Axial Grid Maps:
Distance Measures and Related Topics

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ABSTRACT

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This work defines, in precise terms, what are node and axial maps used in space syntax in order to represent morphological properties of urban form and, also, explores graph theoretic relations between them.
Relativised distance measures are deduced for node and axial grid maps such as to minimise size effects when maps with different number of nodes or axial lines are being compared and allowed to expand in one or two directions. The paper shows that axial maps, expanding in two directions, should be standardised by the root of a diamond shape and those characterised as node maps, expanding in two directions, should be standardised by the corner of a grid with, respectively, the same number of axial lines or nodes.
If we adopt this procedure for maps representing similar urban morphological configurations then:

1. they present similar distances amongst their nodes or lines, regardless of their size.
2. when they are embedded in larger maps they should be placed in such way that expansion could take place in two directions which means that the geometrical centre of larger maps are the preferred location for the smaller ones .

ON NODE AND AXIAL GRID MAPS: DISTANCE MEASURES AND RELATED TOPICS

1. INTRODUCTION

Since the early seventies there has been an extensive use of graph theory to describe morphological properties of architectural and urban form.

March and Steadman(1971) and Tabor(1976 a and b) analyse floor plan designs, the former in terms of an electrical network analogy in order to generate systematically floor plans described as mosaics of rectangles and the later analysing communication and route patterns in terms of circulation cost based on an Euclidean metric or time dimension.

At the urban scale Krüger(1977) analyses the relationship between built form connectivity and urban spatial structure using a graph theoretic approach to describe how built forms are connected on the surface of earth in order to generate, in a town, built forms by type of connectivity.

At the same time several studies were made in order to describe distance measures on polyomino populations (Matela and O'Hare, 1976), as well as in
8H graphs (Entriger et alli, 1976) and , particularly, on ~~the~~ mean distance ~~of~~ ^s in
graphs (Doyle and Graver, 1977, 1978 and 1982) . Later Baglivo and Graver(1983) gave a comprehensive description of these results and Steadman (1983) describes the state of art about the application of graph theory in architectural morphology.

Hillier and Hanson(1983) extend these findings to a new form of description - the axial map - which has been proved to be fruitful in providing condensed descriptions of urban spatial morphology and, also, of the spatial organisation of certain forms of complex buildings for health facilities and research laboratories(Hillier et al~~li~~, 1984 and Hillier and Penn, 1989).

However, when compared with other graph representations of architectural and urban forms, there is no well defined procedure to describe an axial map, neither its distance measures have been systematically analysed against the ones derived from the usual graph-theoretic descriptions, studied by the ~~former~~ authors ^ mentioned above.

The aim of this paper is twofold. The first is to give a mathematical description of the axial map and to analyse the performance of distance measures, when these maps represent grids, against the traditional grid graph representation. The second is to explore how these measures behave if specific procedures are adopted to standardise them in order to
^the minimise the effect of map's size.

2. GENERAL PROPERTIES OF AXIAL AND NODE MAPS

Axial and node maps usually represent different properties of urban form. Axial maps are made by the fewest longest straight lines which cover~~s~~ all urban public spaces, i.e. which pass through all urban public spaces configur~~at~~ed as unified places. These axial lines have two properties: visibility, which is how far one can see, and permeability, which is how far one can walk in straight line.

Node maps are made, generally, by a set of points and lines connecting them. The points can represent, for instance, at an urban scale, road junctions and the lines network arcs joining them and, at an architectural scale, rooms and connections or openings between them.

However, a more precise definition should be thought if we want to achieve an accurate description of these maps in order to explore their properties. is needed

An axial map (AM) consists of a finite non-empty set L = L(A) of K lines together with a prescribed set ~~of~~ X of m unordered pairs of lines of L. Each pair x ={u,v} of lines in X is called a connection (or point) and x is said to join u and v. We write x=uv and say that u and v are adjacent axial lines; point x and line u are incident with each other as are x and v.

An axial map with k lines and m connections is called a (m,k) map, being the (0,1) map a trivial one represented just by an axial line.
case

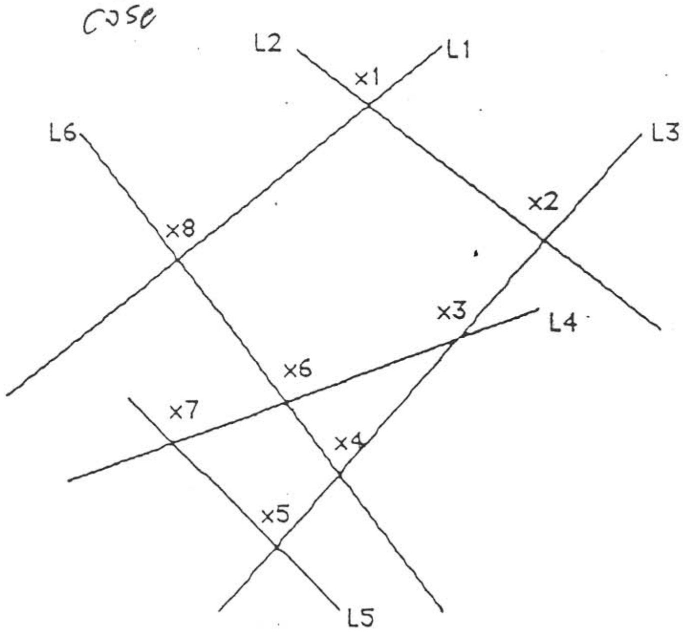


Fig.1 - An example of a (8,6) axial map.

For the (8,6) axial map represented in Fig. 1 the set of lines is defined as being given by L={1,2,3,4,5,6} and the ~~sets~~ of connections as being given by stet
X_1={1,2}, X_2={2,3}, X_3={3,4}, X_4={3,6}, X_5={3,5}, X_6={4,6}, X_7={4,5} and X_8={1,6}.

A graph G of a (m,k) axial map consists of a finite non-empty set V=V(G) of k vertices together with a prescribed set X of m unordered pairs of distinct vertices of V. Each vertex in G represents a line of the (m,k) axial map and each pair y={r,s} of vertices in G represents a connection of the axial map. Each pair y={r,s} of vertices in G is an arc of G and y is said to join u and v. A graph G with k vertices and m arcs is called a (k,m) graph.

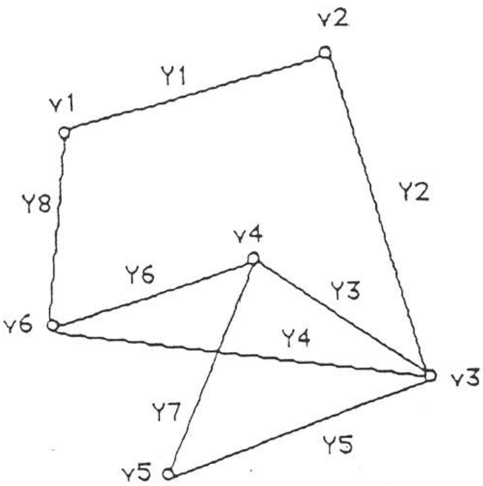


Fig.2 - A (6,8) Graph of the (8,6) Axial Map of Fig.1.

The (6,8) graph represented in Fig.2 is described by the set of vertices V={1,2,3,4,5,6} and by the sets of arcs Y_1={1,2}, Y_2={2,3}, Y_3={3,4}, Y_4={3,6}, Y_5={3,5}, Y_6={4,6}, Y_7={4,5} and Y_8={1,6}.

It should be noticed that the mapping of AM to G is a many-to-one relation . Each axial map has a unique graph representation but not vice-versa. the converse is not true.

In the axial map each line L can be sub-divided in a set of segments defined by S={X_i,X_j}, where X_i and X_j represent connections belonging to the axial line L.
If each segment S is represented by a line or link and each connection or pair X={u,v} by a point or node, then we obtain a node map(NM). By definition, the segments of axial lines which have end-points not incident with to other axial lines are not represented in the node map.

A node map with m nodes and n links is called a (m,n) node map. It should be noticed that the mapping of AM to NM is a one-to-one relation but not vice-versa. Each axial map has one node map representation and each node map has many axial map representations. the converse mapping is not

A (8,10) node map representation of the (6,8) axial map given in Fig.1 is defined by the set of nodes P={1,2,3,4,5,6,7,8} and by the sets of links Q_1={1,2}, Q_2={2,3}, Q_3={3,4}, Q_4={4,5}, Q_5={5,6}, Q_6={6,7}, Q_7={4,7}, Q_8={3,7}, Q_9={7,8} and Q_{10}={1,8}. stct

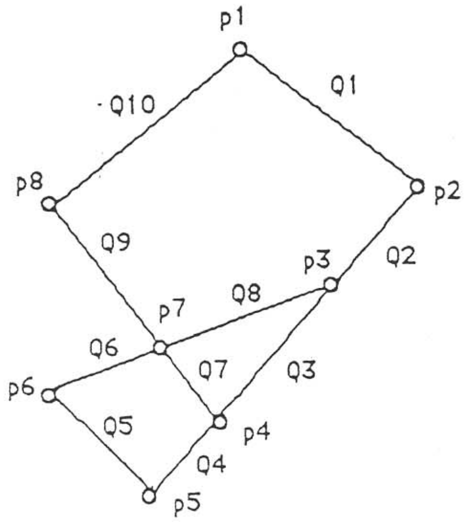


Fig.3 A (8,10) Node Map representation of the (6,8) Axial Map of Fig.1.

Theorem: The number n of links in a node map is given by n = 2m-k, where m represents the number of connections and k the number of lines of the corresponding axial map.

By definition a node map has a number of nodes P identical to the number m of arcs in G or connections in the axial map. The Y_i arcs incident with a vertex v_i in G contribute (Y_i-1) to n, so as there are k vertices in G then

$$n = \sum_{i=1}^{k}(Y_i - 1) = \sum_{i=1}^{k} Y_i - k, \qquad (1),$$

But a well known result in the theory of graphs states that the sum of ~~of~~ arcs incident with v_i vertices in G is twice the number of m lines in G (Harary, 1971, pág.14). Therefore, after simple algebraic substitution,expression (1) becomes

$$n = 2m\text{-}k$$

q.e.d.

For an axial map the maximum number of connections for a given set of k lines is given by

$$m_{max} = K(K-1)/2 \qquad (2),$$

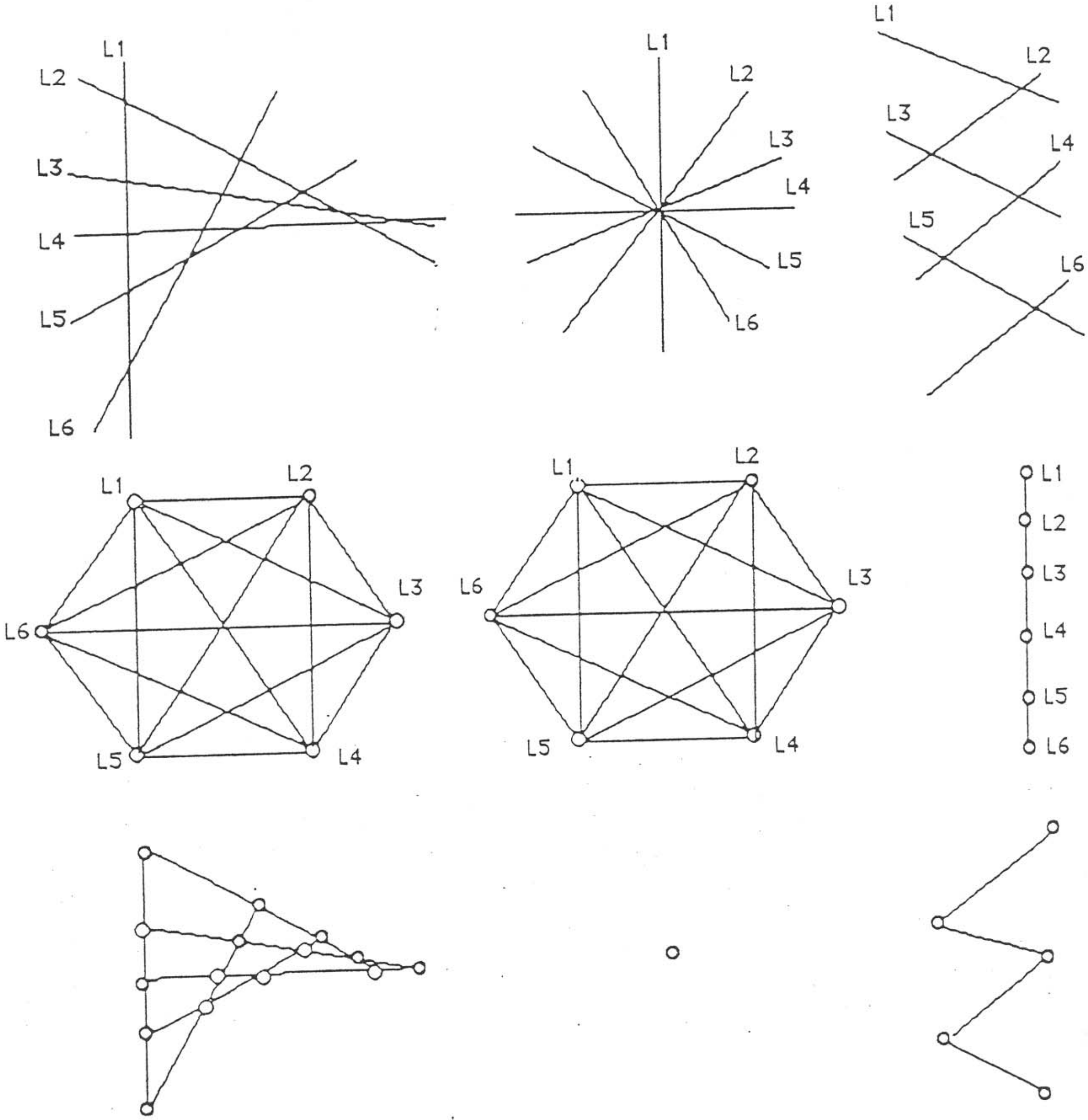
which is identical to the maximum number of lines that a graph G with k points can have (Harary ,1971, pag.16). In graph theory terminology G is called a complete graph since every pair of its k points are adjacent. In ha similar way we can say that a (m_{max},k) axial map is a complete axial map.

Two different forms of complete axial maps can occur in what regards the regard to segmentation of their lines . Their segmentation can be maximal if their axial lines slice the plane in a maximum number of slices (NS) or central and/or polygons, identical to $NS_{max}=\{K(K+1)/2+1\}$ (Steiner, 1826 and Graham et all, 1989), or minimal, if, when extended, they slice the plane in just $NS_{min} = 2k$ slices. In Fig.4a the complete axial map with 6 lines slices the plane in 22 slices and in Fig.4b in 12.

A graph is ~~a~~ said to be connected if every pair of points can be joined by a path, i.e. by an alternating sequence of points and lines, in which all points and lines are distinct and where each line is incident with the two points immediately preceding and following it. A path is said to be closed if the first point in it is identical to the last one. For a minimally connected axial map the corresponding graph G is called a tree, i.e. a connected graph with minimum number of lines, without closed paths or cycles . In a tree with k vertices there must be k-1 lines, thus a lower bound (m_{min}) for the ; number of connections in the axial map is given by k-1 .

Given an axial map with k lines the upper bound for the number of nodes in a node map is identical to the maximum number of connections in the axial map, i.e. m_{max}.

By definition the end points of an axial map not incident with other lines are not represented in the corresponding node map . If the maximal axial map is minimally segmented, then its node map is a trivial one i.e. is a (0,1) node map. It follows, immediately, that a lower bound for the number of nodes in a node map is 1 and for the number of links is 0.



a) b) c)

Fig.4 a) A complete axial map with 6 lines and maximum segmentation, its graph representation and the corresponding node map.

b) A complete axial map with 6 lines and minimum segmentation, its graph representation and the corresponding minimal node map.

c) A minimal axial map with 6 lines, its graph representation and the corresponding node map.

Given a set of k axial lines an upper bound for the number of links in a node map is given by

$$n_{max} = (K^2 - 2K) \qquad (3).$$

Since the number of links in a node map is identical to expression (1) and the maximum number of connections in the axial map is given by expression (2) then, after substitution, we ~~would~~ obtain expression (3). For the example given in Fig.4 a), i.e. for a complete axial map with 6 lines and maximum segmentation we ~~would~~ obtain 24 links for the corresponding node map.

	Axial Map	Graph Representation	Node Map
Number of Lines	k	m {k(k-1)/2} [k-1]	2m-k {k(k-2)} [0]
Number of Points	m {k(k-1)/2} [k-1]	k	m {k(k-1)/2} [1]

Table 1. Summary of the relations between points and lines of the axial, graph and node map representation. Inside the brackets are the expressions for the upper {} and lower [] bounds for these variables.

Table 1 summarises the findings obtained on the relations between points and lines of the axial, graph and node map representation.

Finally it should be noticed that the application AM(m,k)→NM(2m-k,m) is non isomorphic, i.e. not only to does an axial map correspond~~s~~ to just one node map but to the same node map ~~also~~ there correspond~~s~~ many axial maps. Also the lower bound [k-1] for the number of connections of the axial map do not correspond to the lower bound [1] for points in node maps. does

In short, an axial map AM(m,k) corresponds to one graph G(k,m) and to one node map NM(m,2m-k) but not vice-versa. the converse is not true.

Hillier conjectures that

However, according to a conjecture of Prof. Bill Hillier, an axial map is topologicaly specified, simultaneously, by its graph representation and by the corresponding node graph. This conjecture opens, if proved, the possibility of extending space syntax methods from the realm of graph theory, i.e. topology, to geometry.

3. DEFINITION OF DISTANCE MEASURES ON NODE AND AXIAL GRID MAPS

Several distance measures have been proposed in the literature to analyse the performance of the graph representation of the axial map as well as of the node map.

In general, we can speak of the distance d_{ij} between two points i and j in graph G as being the length of the shortest path joining them, if any; otherwise $d_{ij}=\infty$. In a connected graph, distance presents always metric properties i.e. for all points i,j and k (Harary, 1971, pag.14), the following set of axioms holds :

1. $d_{ij} \geq 0$, with $d_{ij}=0$ if and only if i=j,
2. $d_{ij} = d_{ji}$,
3. $d_{ij} + d_{jk} \geq d_{ik}$.

In axial maps the distance between line i and j is, generally, measured by the number ~~of depths of~~ axial lines located on the shortest path joining them.

depths, that is to say the number of

In node maps, specially if they represent road networks, as those used in transportation studies, the distance can be measured in terms of depth but, generally, it is in terms of a generalised cost function which can be defined as being dependent of the shortest distance, time and also a monetary cost of locomotion, by a specific mode of transport, between node i and j, which in turn, generally, represent the centroid of a study area.

like

measured

on

For the purpose of systematic comparison between node and axial maps we consider that distance is measured in terms of depths for both types of maps.

propose

be

Mean depth of line i in an axial map or node i in a node map is defined by

$$MD_i = \sum_{j=1}^{k} d_{ij}/(k-1) \quad ,$$

where k represents the number of lines in the axial map or the number of nodes in the node map.

Mean depth measures how segregated a given line or node i is from the remaining lines or nodes of each map (called from now on simply ~~as~~ graphs). In that sense it can be called a global property of a specific node. line or

In order to standardise the variation of mean depth between zero and one, Hillier and Hanson (1983) proposed the following measure, known as the relative asymmetry of a line or node i

$$RA_i = 2(MD_i-1)/(k-2) \qquad (4),$$

have where the variables present the usual meaning.

To obtain expression (4) we need to know the maximum and minimum value s that a node can have in terms of mean depth.

The minimum value is given when node i in a graph G is at minimum depth from all other ones, i.e. when it is at depth 1 from all other nodes. In that case the minimum mean depth is 1, i.e. MD_{min}=1.In graph theory terminology this corresponds to the centre of a star or to the root of a bush.

The maximum value for MD_i is given when node i is the end point of a chain i.e. of a tree with 2 points incident to 1 line and the remaining (k-2) points incident with 2 lines .

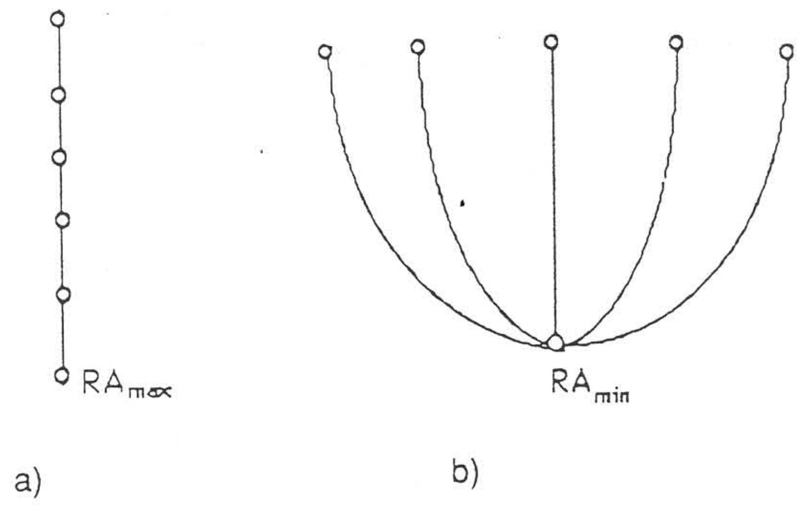


Fig.5 a) Chain presenting end point with maximum relative asymmetry. having
b) Star presenting centre with minimum relative asymmetry. having

The total depth of an end point i in a chain is given by the following expression

$$\sum_{j=1}^{k} d_{ij} = \sum_{m=1}^{k-1} m \quad ,$$

i.e. is identical to the summation of natural numbers, from m=1, which corresponds to the node j at depth 1 from i, up to k-1, which corresponds to the deepest node j from i.

The summation of the series of natural numbers, from 1 up to k-1,is given by k(k-1)/2. Therefore, the mean depth of an end node i in a chain is given by $MD_{max} = k/2$.

The expression of RA_i, defined to vary between 0 and 1, is given in its standardised form as

$$RA_i = (MD_i - MD_{min})/(MD_{max} - MD_{min}) \qquad (5).$$

Substituting the values of MD_{min} and MD_{max} in expression (5) we ~~would~~ obtain expression (4) which gives the value of the relative asymmetry of point i. Values close to 1 represent segregated points in relationship to the whole graph and values close to 0 represent points integrated in the system.

However, as it stands, expression (4) does not allow us to compare directly the values of relative asymmetry for points located in maps with of different sizes. In fact, as k increases the mean depth decreases, ceteribus paribus, in proportionate terms. This means that RA measures also decrease in proportionate terms when the number of axial lines increases; being therefore impossible to compare systems with different sizes. it

There are two ways in which we can minimise this size effect. Either we compare RA values of for each point with a root of a diamond shape or with the corner of a grid with the same number of points. The reason for adopting this procedure resides in rests on the fact that, in both cases, the depths from the root or from the corner are approximately normally distributed.

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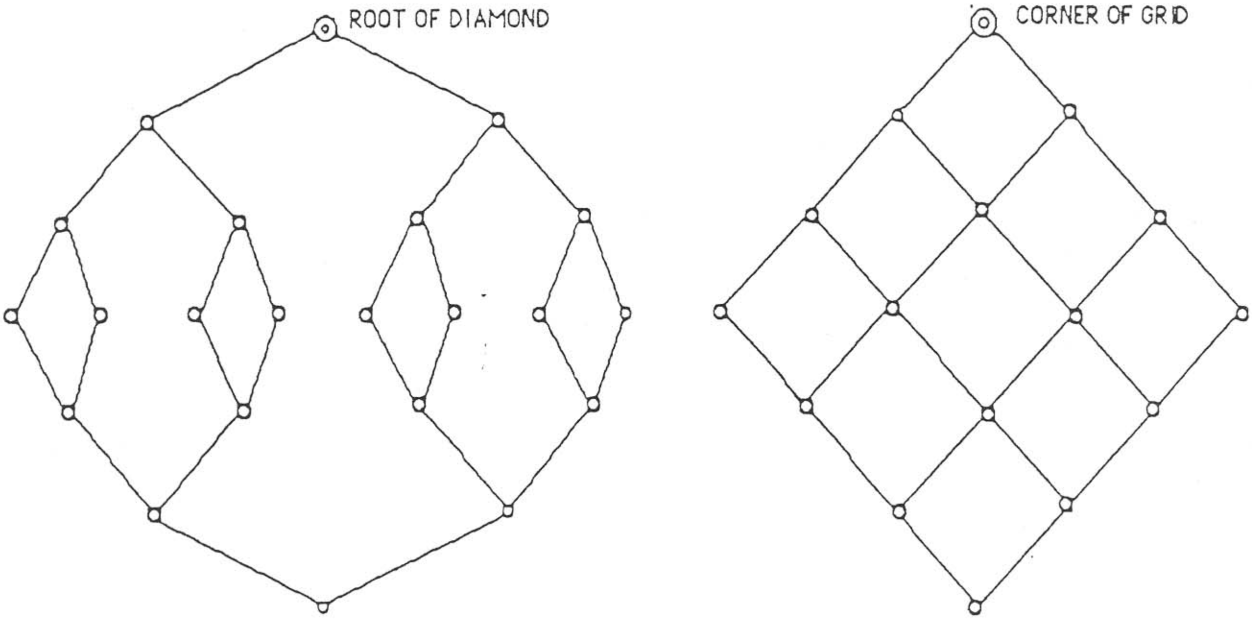


a) b)

Fig. 6 a) Diamond shape with 22 points .
b) Grid shape with 16 points.

A diamond shape, as a graph, is a special form of ~~a~~ justified graph .A justified graph is one in which a point, called the root, "is put at the base and then all points of depth 1 are aligned horizontally above it, all points at depth 2 from that point above those at depth 1, and so on until all levels of depth from that point are accounted for"(Hillier and Hanson, 1983). In a diamond shape there are k points at mean depth level, k/2 at one level above and below, k/4 at two levels above and below, and so on until there is one point at the shallowest(the root) and deepest levels.

both

In a regular grid all the points are incident with four, three or two lines. Those points on its border are incident with three or two lines. Of the points incident with two lines we choose one as being its corner. In a justified grid there are k points at mean depth level, k-1 at one level above and below, k-2 at two levels above and below, and so on until there is also one point at the shallowest(the corner) and deepest levels.

both

For

If we have an axial map with k lines the general procedure has been (see Hillier and Hanson, 1983) to estimate the D_k i.e. the relative asymmetry of the root of a diamond shape with k points and to divide the RA value found for a specific line of the axial map by the value obtained for D_k. This new value has been called in the literature (see Hillier and Hanson, 1983) Real Relative Asymmetry(RRA) and varies above and bel~~l~~ow 1. Values well bel~~l~~ow 1, such as those lower than 0.6, indicate strongly integrated lines in the axial map and values above 1 more segregated lines.

Alternatively, we could estimate G_k i.e. the relative asymmetry of the corner of a grid with k points and, similarly, ~~to~~ divide the RA value found by G_k. The new RRA value, padronised by the RA value of a grid corner, will also vary above and bellow 1. ?

In order to compare the performance of both procedures to standardise RA values, such as to compare axial maps with different sizes, we need to estimate, formally, the mean depth of a grid corner as well as of a diamond root. in standardising as for example in comparing

Let us concentrate first on generalised distance measures for a grid and , latter, for the diamond shape.

4. RELATIVE ASYMMETRY OF A GRID CORNER

Let us assume, without loss of generality, a regular node grid with $K=N^2$ points located on a Cartesian coordinate system which ranges from coordinate (1,1) up to (N,N) (see Fig.7). l.c.

The first task is to estimate the distance D_{ij-lm} between two generic points with coordinates (i,j) and (l,m) given by the following expression:

$$D_{ij-lm} = |i-l| + |j-m| \quad (6) ,$$

which can be called a rectangular or taxi-cab distance (Krause, 1975) between points with coordinates ij and lm.

Although this distance function differs from the usual Euclidean distance, defined, for the same type of coordinate system, as $D_{ij-lm} = \sqrt{(i-l)^2+(j-m)^2}$, it obeys all the axioms set out in section 3 presenting, therefore, a metric which can be used to make comparisons ~~between~~ points set in this Cartesian coordinate system. and provides l.c.

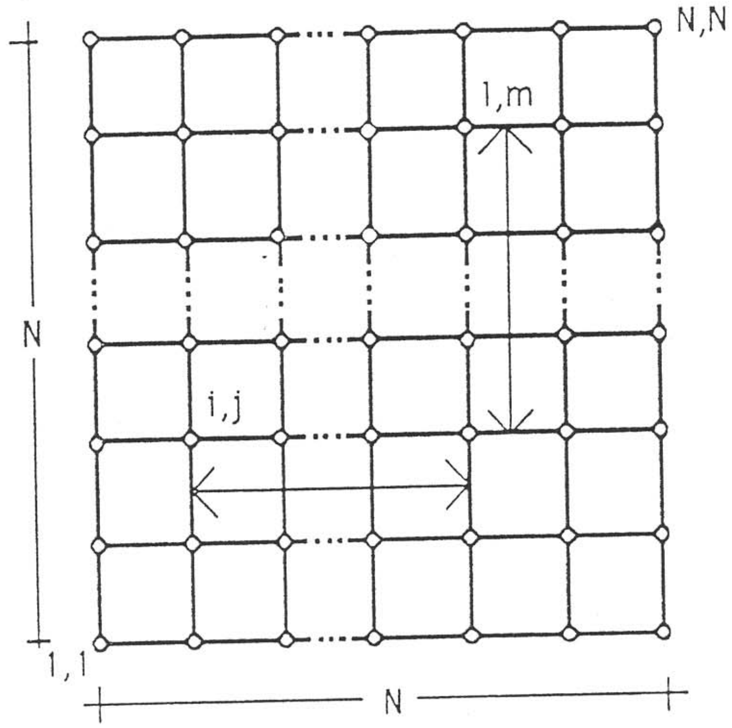


Fig.7 A generic grid with $k=N^2$ points.

From expression (6) we can estimate the mean depth MD_{ij} of a generic point ij in the grid, which is given by

$$MD_{ij} = \sum_{l=1}^{N} \sum_{m=1}^{N} D_{ij\text{-}lm}/(K-1) \qquad (7).$$

Substituting expression (6) in (7) we ~~would~~ obtain

$$MD_{ij} = \left(\sum_{l=1}^{N} \sum_{m=1}^{N} \{|i-l| + |j-m|\}\right)/(K-1) \qquad (8),$$

and expanding the summation signs in (8) we ~~would~~ have

$$MD_{ij} = \left(\sum_{l=1}^{N} |i-l|N + \sum_{m=1}^{N} |j-m|N\right)/(K-1) \qquad (9).$$

In order to transform the absolute expressions which occur in (9) into a form more amenable to algebraic treatment, we can expand it into

$$MD_{ij} = \left(N\left\{\sum_{l=1}^{i}(i-l) + \sum_{l=i+1}^{N}(l-i)\right\} + N\left\{\sum_{m=1}^{j}(j-m) + \sum_{m=j+1}^{N}(m-j)\right\}\right)/(K-1) \qquad (10).$$

As $\sum_{l=1}^{i} l=i(i+1)/2$ and $\sum_{m=1}^{j} m=j(j+1)/2$, then, after algebraic manipulation, expression (10) becomes

$$MD_{ij}=\left\{N\left[i^2-i(i+1)/2+(N-i)(N-i+1)/2+j^2-j(j+1)/2+(N-j)(N-j+1)/2\right]\right\}/(K-1)\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

For a generalised point, with Cartesian coordinates ij , expression (11) gives its mean depth as a function of the total number of points $K(=N^2)$ of the grid as well as a function of ~~the~~ these coordinates.

Expression (4) gives the relative asymmetry of a point i in a graph with k points. If we adapt this expression for a system with Cartesian coordinates and substitute the mean depth MD_{ij} of a generic point in a grid, given by (11), in expression (4), we ~~would~~ obtain

$$RA_{ij} = 2\left\{N(i^2 -i-Ni+N^2+N+j^2-j-Nj)-(K-1)\right\}/\left\{(K-1)(K-2)\right\} \qquad (12).$$

The corner of a grid can have the following sets of Cartesian coordinates: (1,1), (1,N), (N,1) and (N,N). If we substitute any of these coordinates in expression (12) we ~~would~~ end up with G_k, i.e. the relative asymmetry for the corner of a grid with k points. Therefore,

$$G_k = 2\{N(N^2-N)-(K-1)\}/\{(K-1)(K-2)\} \qquad (13).$$

As $N=\sqrt{k}$, if we substitute this in (13) then we ~~would~~ have G_k as a function of k ~~and~~, and (13) transforms into

$$G_k = 2\{K\sqrt{K}-2K+1\}/\{(K-1)(K-2)\} \qquad (14).$$

It should be noticed that $\lim_{k\to\infty} G_k = 0$, which means that as the number of nodes expands the relative asymmetry of a grid corner tend to zero. s

5. RELATIVE ASYMMETRY OF A DIAMOND ROOT

In order to compare the performance of different procedures to standardise the relative asymmetry of an axial or node map we need to obtain an expression for the relative asymmetry of the diamond root ~~just~~ as a function of the numbers of its points, as we did for the relative asymmetry of a grid corner, given by expression (14). just

In a diamond shape, exemplified in Fig.8 for 46 points, the total depth from its root(TD_r), in relationship with all other points, is given by the following expression

$$TD_r = \sum_{q=0}^{d/2} q\,(2^q) + \sum_{q=0}^{d/2-1} (d-q)(2^q) \qquad (15),$$

where d represents the maximum depth from the root and q the depth, also from the root, of the points located in each level. on

The first term ($S_1 = \sum_{q=0}^{d/2} q\left(2^{q}\right)$) on the right hand side of equation (15) represents the total depth of the root in relationship to those points located from depth 0 to depth d/2 and the second term ($S_2 = \sum_{q=0}^{d/2-1}(d-q)\left(2^{q}\right)$) represents the same in relationship to those points situated at depth (d/2+1) up to maximum depth d from the root.

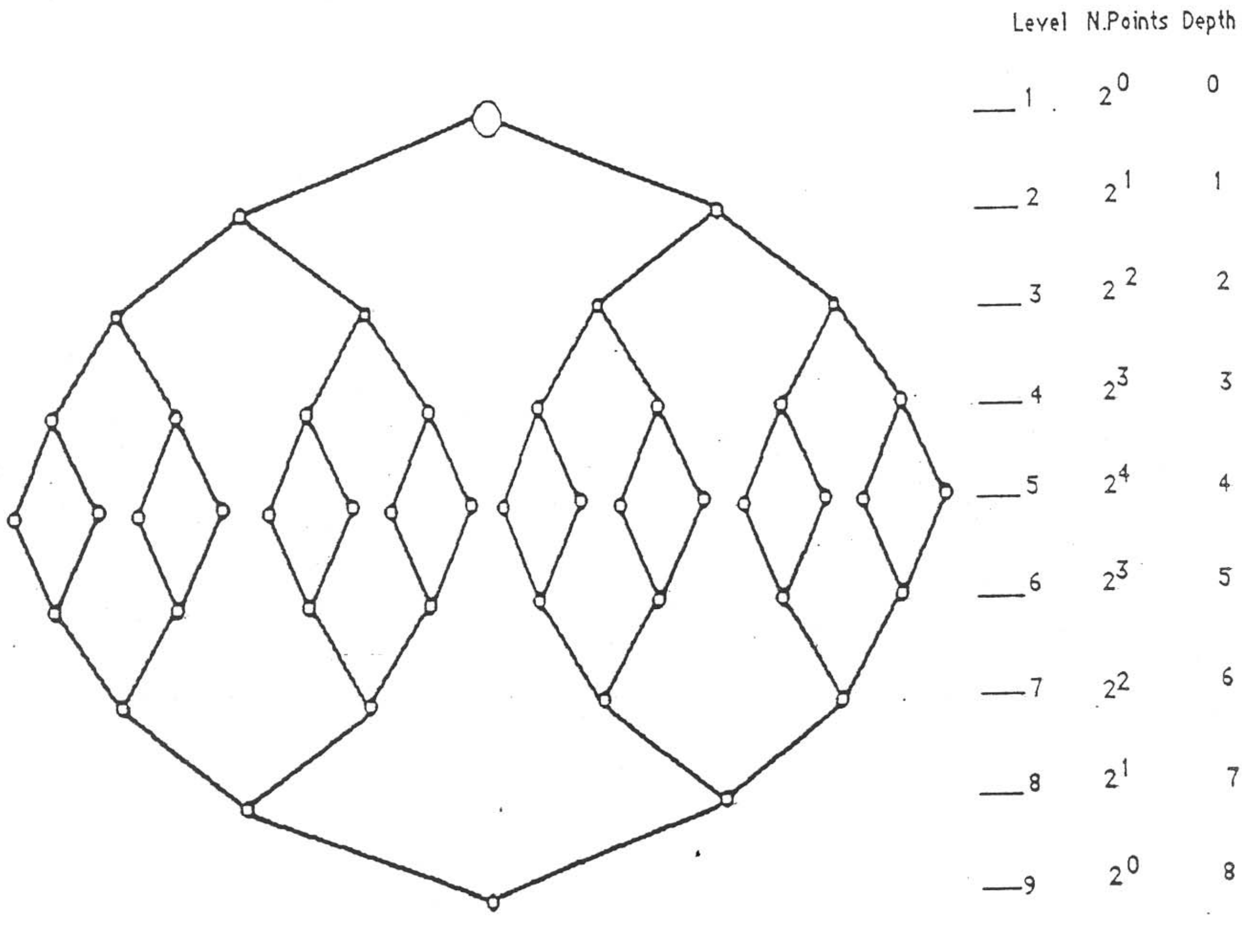


Fig.8 Diamond Shape with 46 points and 9 levels of depth.

As, in general, $\sum_{q=0}^{n} q\left(2^{q}\right) = (n-1)\left(2^{(n+1)}\right) + 2$ (see Graham et al,1989)

then the first term in the right hand side of expression (15) becomes $S_1 = (d/2-1)\left(2^{d/2+1}\right)+2$. For the second term, after expansion, it becomes $S_2 = \sum_{q=0}^{d/2-1} d\left(2^{q}\right) - \sum_{q=0}^{d/2-1} q\left(2^{q}\right)$.

Substitution of these expanded terms S_1 and S_2 in (15) gives the following result

$$TD_r = S_1 + S_2 = \sum_{q=0}^{d/2} q(2^q) + \sum_{q=0}^{d/2-1} q(2^q) + \sum_{q=0}^{d/2-1} d(2^q) \qquad (16).$$

The first two terms in the right hand side of expression (16) partially cancel out giving the following result

$$TD_r = (d/2)\, 2^{d/2} + \sum_{q=0}^{d/2-1} d\, 2^q \qquad (17).$$

But, in general, as $\sum_{k=0}^{n} a\, x^k = (a - a\, x^{n+1})/(1-x)$ then, developing the second term in the right hand side of expression (17), we ~~would~~ obtain

$$TD_r = (3/2)\, d\, 2^{d/2} - d \qquad (18).$$

Expression (18) gives the total depth of a root of a diamond shape as a function of d i.e. as a function of the maximum depth from that root.

As in a diamond shape (see Fig.8) d/2=n, where n in expression 2^n represents the depth of the diameter of a diamond i.e. of the diamond's level with the greatest number of points and 2^n represents the number of points at that level. Then, if we substitute this result in (18) we ~~would~~ obtain, after algebraic manipulation, for the total depth of a diamond

$$TD_r = 2n\,(3 * 2^{(n-1)} - 1) \qquad (19).$$

Expression (19) gives the total depth of a diamond root as a function of its diameter depth.

The total number k of points in a diamond shape can be given as a function of its diameter depth i. e. as a function of n, by the following expression

$$K = 2^n + 2\sum_{i=0}^{n-1} 2^i \quad ,$$

where the first expression in the right hand side represents the number of points at diameter level and the second one the number of points at all other levels.

As, in general, $\sum_{i=0}^{n} 2^i = (1-2^{(n+1)})/(1-2)$ then, after algebraic manipulation, the last equation for K could be transformed, by substitution, into

$$K = 3*2^n - 2 \qquad (20).$$

If we substitute (20) in (19) we ~~would~~ obtain an expression for the root total depth as a function of the number of diamond points (k) as well as a function of its diameter level (n) , i.e. simply as

$$TD_r = K*n \qquad (21).$$

Then the mean depth of a diamond root (MD_r) can now be given by

$$MD_r = (K*n)/(K-1) \qquad (22).$$

If we substitute expression (22) in (4) we ~~would~~ obtain the relative asymmetry of a root of a diamond (D_k) as a function of the number of points k and the depth of its diameter n, i.e. by

$$D_k = 2\{k(n-1)+1\}/\{(k-1)(k-2)\} \qquad (23).$$

However, from expression (20) we can estimate n as a function of K, which is given by

$$n = lg_2((K+2)/3) \qquad (24).$$

If we substitute the value of n, given by expression (24), in (23) we ~~would~~ finally obtain the relative asymmetry of a diamond root simply as a function of the number of its k points i.e. as

$$D_k = 2\{K(lg_2((K+2)/3)-1)+1\}/\{(k-1)(k-2)\} \qquad (25).$$

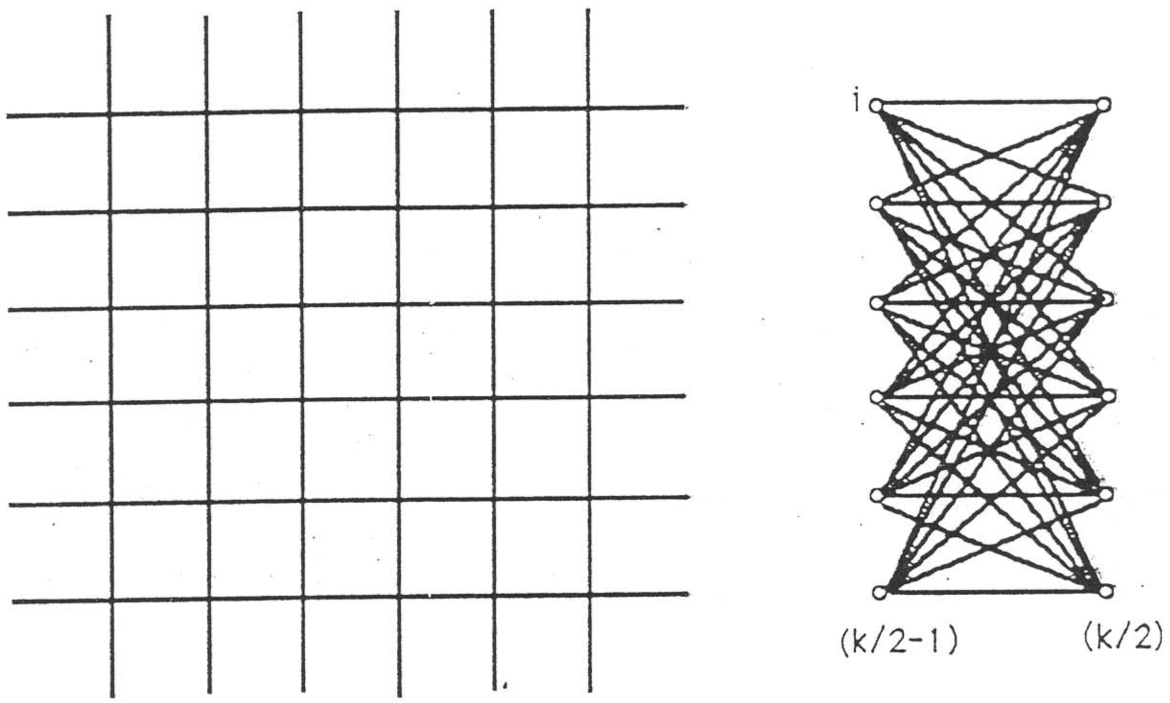
We are now able to compare what means to standardise the relative asymmetry either by the corner of a grid (expression 14) or by the root of a diamond shape (expression 25), since both are given just as a function of the total number k of its points. *the consequences of standardising*

We have already seen that the relative asymmetry of a grid corner, i.e. G_k tends to zero when k tends to infinity. The same happens with expression (25) where $\lim_{k\to\infty} D_k = 0$. However, the limit, as k approaches higher values, is higher for G_k than for D_k, being proportional to $(\sqrt{k}/k)$ in the former case and to $\{\log_2(k)/k\}$ in the latter.

It remains to be seen with axial and node grids, with k lines or nodes, *what is* the performance of the real relative asymmetry measure when standardised either by G_k or D_k.

6. REAL RELATIVE ASYMMETRY OF AN AXIAL GRID

A regular (m,k) axial grid is, by definition, composed by k lines and by of $m = (k/2)^2$ connections, where each of its k/2 lines intersects all the remaining k/2 lines. Its graph representation is called a bipartite graph, i.e. a graph whose point set can be partitioned in two subsets such that every line joins one subset with the other one. Furthermore, a graph representation of a regular axial grid is a complete bipartite graph since it contains every line joining the two subsets.



a) b)

Fig.9 a) A (36,12) Axial Grid . b) A (12,36) complete bipartite graph.

In a graph we say that a closed path or cycle is a triangle if it is made by three distinct points. In bipartite graphs there are no triangles (König, 1936, p.170) being the closed paths made, at least, by four points. This means that in the corresponding axial map a closed path has, at least, four connections.

Estimating the mean value of the Real Relative Asymmetry of an axial grid means that we need to know first the mean depth of a generic line i.

Each line in the axial map (see Fig.9) is at depth 1 in relationship to k/2 lines and at depth 2 in relation to the remaining (k/2-1) lines . Then the mean depth MD_i for line i is given by

$$MD_i = \{(k/2-1)2+k/2\}/\{k-1\} \qquad (26),$$

which, after simple algebraic manipulation, gives

$$MD_i = (3k-4)/(2k-2) \qquad (27).$$

If we substitute (27) in (4) we ~~would~~ obtain an expression for the relative asymmetry of line i in an axial grid map, with k lines, given by

$$RA_i = 1/(k-1) \qquad (28).$$

If we standardise expression (28) either by G_k or by D_k (given, respectively by expressions 14 and 25) we ~~would~~ obtain the real relative asymmetry of a line in an axial grid map.

It is important to study the behaviour of these measures as the number of k lines expands, since this ~~would~~ allow us to compare maps with different sizes. hs of

There are now two possibilities for the expansion of an axial map. It can either expand in one or two directions, i.e. the two subsets of k/2 lines can either expand their number simultaneously, or one at each time remaining the other sub-set constant.

Consider Let us admit that we have an axial grid map with two subsets of lines where there are s lines in the first subset and k-s=r in the second and where $s \geq k-s$. Furthermore, without loss of generality, let us admit that line i belongs to the set of s lines. This means that (k-s)=r lines are at depth 1 from i and (s-1) lines are at depth 2 from i. agree

Then, the mean depth from i is given by

$$MD_i = \{2(s-1)+(k-s)\}/\{k-1\} \qquad (29),$$

and the relative asymmetry by

$$RA_i = \{2(s-1)\}/\{(k-1)(k-2)\} \qquad (30).$$

Let us substitute K by r+s in expression (30). Then we ~~would~~ have

$$RA_i = \{2(s-1)\}/\{(r+s-1)(r+s-2)\} \qquad (31).$$

It is clear, from expression (31), that $\lim_{r\to\infty} RA_i = 0$, $\lim_{s\to\infty} RA_i = 0$ and $\lim_{\substack{r\to\infty \\ s\to\infty}} RA_i = 0$, i.e. either expanding the axial map in just one direction ($r\to\infty$ or $s\to\infty$) or in both ($r\to\infty$ and $s\to\infty$) we ~~would~~ end up with a relative asymmetry tending to 0.

The mean values for the relative asymmetry of an axial grid map are given for regular grids by

$$RA_{mean} = \left(\sum_{i=1}^{k} RA_i\right)/k = 1/(k-1) \qquad (32),$$

and for axial grids with two subsets of axial lines with different cardinalships i.e. with one subset with s lines and the other one with (k-s) lines, we ~~would~~ obtain

$$RA_{mean} = \left(\sum_{i=1}^{k} RA_i\right)/k = \{2(s-1)\}/\{(k-1)(k-2)\} \qquad (33).$$

It is now worth~~while to~~ looking at similar expressions for node grid maps in order to compare their behaviour as k expands.

7. REAL RELATIVE ASYMMETRY OF A NODE GRID MAP

In order to estimate the mean relative asymmetry of a node grid map we should recall expression (12) which gives the mean relative asymmetry of a generic point with Cartesian coordinates ij.

The mean relative asymmetry RA_{mean} of a grid is given by the following expression

$$RA_{mean} = \Big(\sum_{i=1}^{N} \sum_{j=1}^{N} RA_{ij}\Big)/k \qquad (34).$$

If we substitute the value of RA_{ij} given by expression (12) in (34), expand it and cancel out identical terms, we ~~would~~ end up with an expression for RA_{mean} which is a function of the number of nodes in the grid, given by

$$RA_{mean} = \big(4\sqrt{k}-6\big)/\big(3(k-2)\big) \qquad (35).$$

From these results it can be easily proved that for a grid node map ,as k expands, we ~~would~~ obtain the following result : $\lim_{k \to \infty} RA_{mean} = 0$.

We have seen that expression (11) allows us to estimate the mean depth of a generic point and (35) the mean depth of all the points in a grid node map. However, we admitted that its Cartesian coordinates could run from (1,1) up to (N,N), i.e. we supposed that the grid's shape was quadrangular. square ?

In order to extrapolate these results to grids of a more general kind, i.e. to grids with rectangular shape, we need to estimate an expression which enables us to calculate the mean depth of a point with Cartesian coordinates running from (1,1) to (r,s). We assume that $r \cdot s = k$ and, without loss of generality, that $r \geq s$.

The specification of the distance between two points with Cartesian coordinates was already given in expression (6). From this it is possible to define the mean depth of a point with coordinates ij as being given by

$$MD_{ij} = \Big(\sum_{l=1}^{s} \sum_{m=1}^{r} \big\{ |i-l| + |j-m| \big\}\Big)/(k-1) \qquad (36),$$

where grid coordinates run from 1 to s and from 1 to r (see Fig. 10).

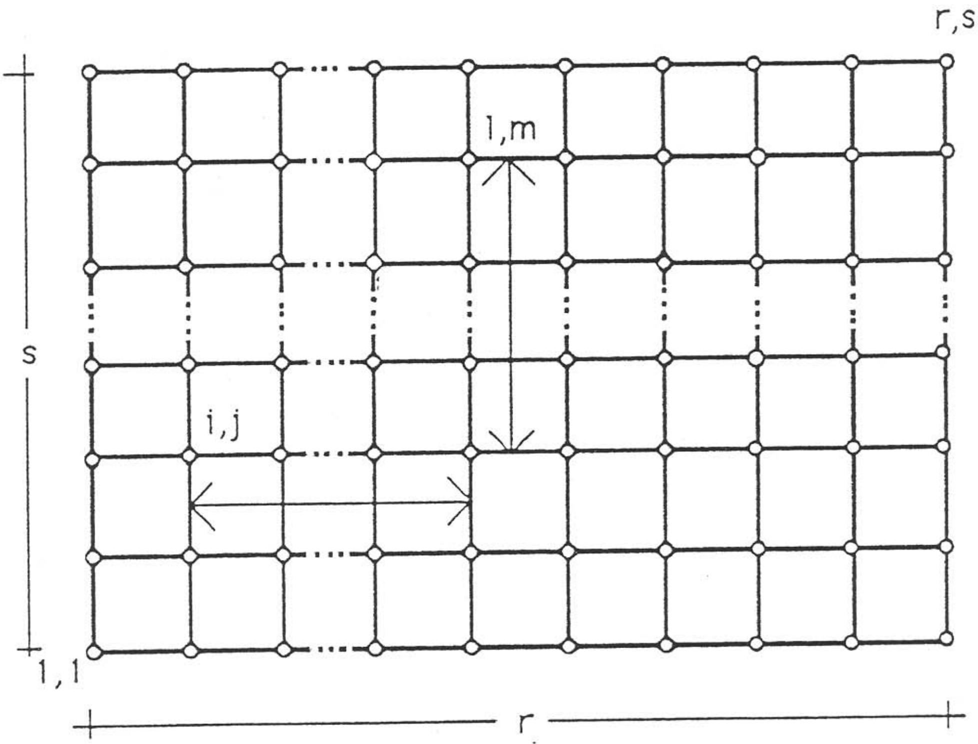


Fig. 10 A node grid map with k(= rs) points.

Expression (36) can be expanded into the following ~~one~~

$$MD_{ij} = \left(r\left\{\sum_{l=1}^{i}(i-l)+\sum_{l=i+1}^{s}(l-i)\right\}+s\left\{\sum_{m=1}^{j}(j-m)+\sum_{m=j+1}^{r}(m-j)\right\}\right)/\left(k-1\right) \qquad (37),$$

which, in turn, can be expanded and compressed, by cancelling out identical terms, into

$$MD_{ij} = \left(2ri^2+rs^2-2sri+2sj^2+sr^2-2srj+2sr\right)/\left(2(k-1)\right) \qquad (38).$$

If we substitute the expression (38) in (4) we ~~would~~ obtain the relative asymmetry RA_{ij} of a node with Cartesian coordinates ij which can now be used in expression (34) to obtain the mean relative asymmetry RA_{mean} of a grid node map with k(=r*s) nodes . The final result, assuming that K=rs, after these substitutions, is given by

$$RA_{mean} = \left(2Ks+r+s+2Kr+6\right)/\left(3(k-1)(k-2)\right) \qquad (39).$$

Since

As r=k/s, if we substitute this value in (39) then, as K expands and s remains constant, we ~~would~~ obtain $\lim_{k\to\infty} RA_{mean} = 2/3s$, which means that when

is expanded ~~expanding~~ continuously, the node grid map in one direction its mean relative asymmetry will tend to a value inversely proportionally related to the other direction's highest coordinate value. In other words, the higher the latter coordinate for one direction the lower the mean relative asymmetry value for a grid node map expanding in the other direction.

8. SUMMARY OF FINDINGS AND RESULTS

Table 2 summarises the findings obtained for the mean relative asymmetry (RA_{mean}) of axial and node grid maps assuming expansion either in both directions or just in one direction. For the latter case variable s represents the highest Cartesian coordinate in the fixed direction.

		AXIAL MAP	NODE MAP
Expansion in Two Directions	RA_{mean}	$\frac{1}{(k-1)}$	$\frac{4\sqrt{k}-6}{3(k-2)}$
	$\lim_{k\to\infty} RA_{mean}$	0	0
Expansion in One Direction	RA_{mean}	$\frac{2(s-1)}{(k-1)(k-2)}$	$\frac{2Ks+r+s+2Kr+6}{3(k-1)(k-2)}$
	$\lim_{k\to\infty} RA_{mean}$	0	$\frac{2}{3s}$

Table 2. Summary of results obtained for the mean relative asymmetry (RA_{mean}) of axial and node grid maps.

If we divide each of the RA_{mean} values given in table 2 either by G_k or D_k we ~~would~~ obtain the mean real relative asymmetry (RRA_{mean}) for axial and node grid maps standardised, respectively, by the real asymmetry of a grid corner or by the diamond root.

If, in these new found expressions for RRA_{mean}, we allow k to approach infinity then we would obtain, in the limit, the results given in table 3.

		AXIAL MAP	NODE MAP
Expansion in Two Directions	$\lim_{k\to\infty} \frac{RA_{mean}}{G_k}$	0	2/3
	$\lim_{k\to\infty} \frac{RA_{mean}}{D_k}$	0	∞
Expansion in One Direction	$\lim_{k\to\infty} \frac{RA_{mean}}{G_k}$	0	∞
	$\lim_{k\to\infty} \frac{RA_{mean}}{D_k}$	0	∞

Table 3. Summary of results obtained for the mean real relative asymmetry (RRA_{mean}) of axial and node grid maps i.e. of their mean relative asymmetry (RA_{mean}) standardised either by the relative asymmetry of a grid corner (G_k) or a diamond root (D_k) when K, i.e. the number of their points or lines, is allowed to approach infinity.

Several comments can be made on these results but what is remarkable is the constancy of the values obtained for axial maps, when they are either standardised by G_k or by D_k, which always approach zero when k approaches infinity. That means that some sort of "edge effect" is still present in axial maps, even when we standardise their RA_{mean} with either G_k or D_k.

However, at a more realistic scale, where k runs from one hundred units of observed lines up to thirteen hundred, ~~it has been verified~~ a constancy for RRA_{mean} values in the town of London. In other words, for several studies made in central London ~~it~~ has been empirically observed a constant value for RA_{mean} when standardised by D_k (Hanson, 1989).

has been verified

city

The question now is to choose a standardisation procedure for axial maps in order to get acceptable results, such as to obtain values for the RRA_{mean} so which do not vary considerably, regardless the size of the axial map but of assuming a constant spatial morphology, ~~h~~as happens to be case of axial the for grid maps as well as the urban axial maps of central London.

Suppose Let us admit that we have one axial (or node) grid map with K lines (or nodes) which is expanded up to K+ΔK lines (or nodes), maintaining, in both cases, the same grid morphology. Let us define the auxiliary parameter given by $\beta = (1+\Delta K/K)$. Then, if we estimate the limit of the relationship between $RRA_{mean(k)}$ and $RRA_{mean(\beta k)}$ as $\Delta K \rightarrow K$ and $K \rightarrow \infty$ i.e. if we estimate the following limit

$$\lim_{\substack{\Delta K \to K \\ K \to \infty}} \frac{(RRA_{mean(k)})}{(RRA_{mean(\beta k)})} \qquad (40),$$

where RRA, in both cases, can be either standardised by G_k or D_k, we ~~would~~ obtain the relative increase of the mean RRA when we allow the number of lines (or nodes) in a map, firstly, to double its size and, secondly, to approach, in the limit, an infinite value..

Table 4. summarises the results obtained when expression (40) is either applied to axial or node maps, being standardised either by G_k or D_k and the allowed expansion of the grid being developed in one or two directions.

Several conclusions can be drawn based on the results so far obtained. If we look at Table 2 we can see that in all cases, except one, i.e. either for axial or grid maps expanding in one or two directions the RA values decrease as the number of lines(for axial maps) or nodes (for node maps) increases, approaching zero if we allow these lines or points approach to infinity. The only exception is for node maps expanding continuously in one direction. In this case the limiting value for RA is inversely proportional to the fixed size of the node map grid.

These results show us the importance of developing some systematic procedure in order to compare maps with different sizes.

STANDARDISATION of RA_{mean}		AXIAL MAP	NODE MAP
Expansion in Two Directions	G_k	$\sqrt{\beta}$	1
	D_k	1	$1/\sqrt{\beta}$
Expansion in One Direction	G_k	$\beta\sqrt{\beta}$	$(s\sqrt{\beta})/\beta$
	D_k	β	$2s/\beta$

Table 4. Summary of results obtained for $\lim\limits_{\substack{\Delta K \to K \\ K \to \infty}} \dfrac{(RRA_{mean(k)})}{(RRA_{mean(\beta k)})}$

where s represents the fixed size of a grid that expands in just one direction, ΔK represents the number of expanded lines (or points) in the axial (or node) map, β is an auxiliary coefficient given by $(1+\Delta K/K)$ and G_k or D_k means that mean RA were standardised, respectively, by mean RA of a grid corner or a diamond root.

The results obtained show that, as K is allowed to expand to infinity, ~~shows~~ (see Table 3) ~~that~~ it does not matter if we standardise RA values either by the root of a diamond shape or by a corner of a grid, since the results obtained w~~h~~ere similar for axial maps, on the one hand, and for node maps, on the other hand.

The only exception found is when the node grid map is allowed to expand in both directions and the mean RA values are standardised by the mean RA of a grid corner with the same number of K points.

Apart from that exception, we can see that mean RA values for axial grid maps tend to zero when K expands to infinity and for the case of node grid maps their mean RA values tend, in similar conditions, to infinity. In short, axial and node grid maps present opposite behaviour when they expand, regardless of the number of directions (one or two) for expansion. hs

However if we look for the ratio of RRA mean values at the limit (see Table 4.) we are able to detect the following conclusions: draw

Firstly, a general pattern emerges from the previous findings. For axial maps, as the number of axial lines increases the limit of the relation between mean RRA values for maps with k and βk lines also increases and vice-versa for node maps.

However, a remarkable exception occurs. For axial grid maps, standardised by the mean RA of a diamond root, and for the node grid maps, standardised by the mean RA of a grid corner, when the expansion runs, in both cases, in two directions, we are able to identify a limiting constant
equal ? identical to 1, which means that, in the limit, mean RRA values are
of independent in relationship to the initial number of lines or points in the
of map and also in relationship to the number of lines or points in which expansion takes place. In all other cases the limit is proportional to β coefficient and/or to the size s of the fixed direction in the expansion.

For the remaining cases in axial grid maps as β = (1+ΔK/K) the bigger the greater
expanded values ΔK the bigger the limiting value and, vice-versa, for the greater
remaining node grid maps (see Table 4.).

What is important to retain is that we only obtain a limit identical to 1 when, for axial and node grids, the expansion occurs in two directions.

From that follows the following postulates:

1. When axial grid or node maps are embedded in larger maps they should be placed in such way that expansion could take place in two directions. The geometrical centre of the larger maps is the preferred location for the
s smaller ones, since that correspond to similar rates of expansion in both directions.

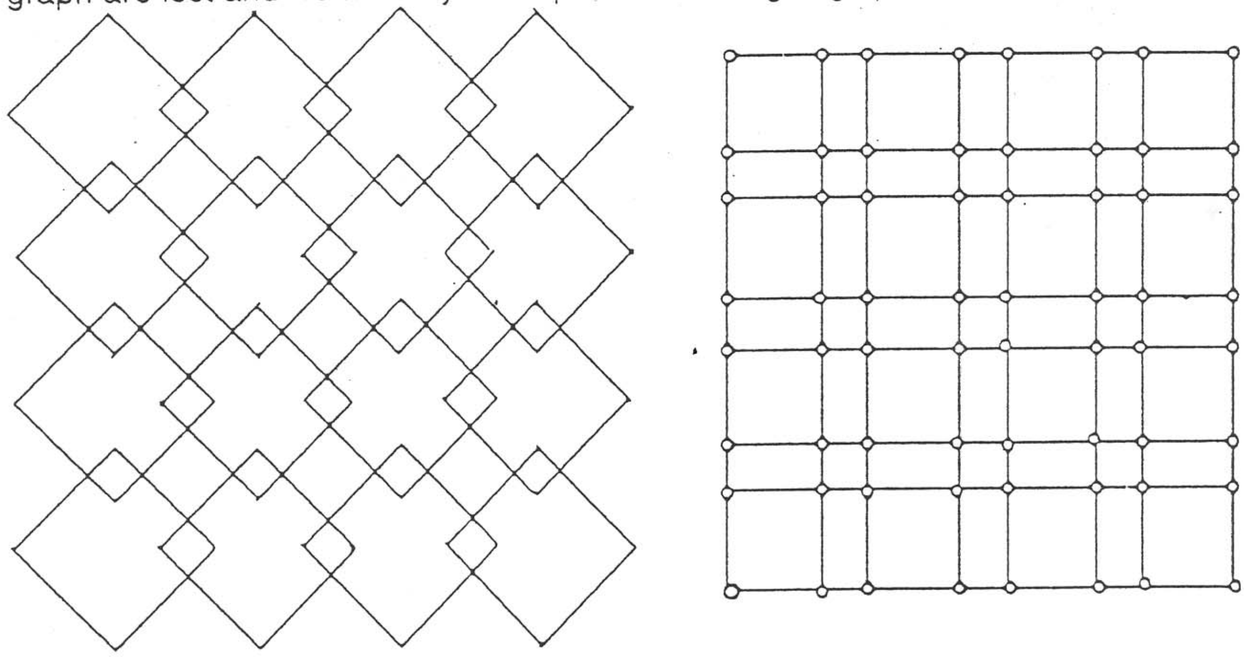
2. Axial grid maps should, in general, be standardised by the mean RA for a root of a diamond (D_k) and node grid maps should be standardised by the mean RA of a grid corner (G_k) in order to the ratio of these mean RA for values
to approach a constant limit, identical to 1, remaining, as far as possible, independent of the initial and expanded number of axial lines or nodes.

In relationship to the axial and node maps of urban areas, axial and node grid maps are highly improbable of occurrence in its pure form. In general, in urban grids the graph representation of the axial map is not a bipartite graph neither its node map is a node grid map. to occur in their

The classical cases of ~~an~~ urban grids, like Cerda's Barcelona or Taylor's New York, is made by orthogonal lines intersected by diagonals, in order to introduce shortest paths which would be otherwise longer on a taxi-cab geometry. are

In fact, we could say that in the majority of real urban situations the graph of an axial map representing an urban grid is neither itself a grid neither a bipartite graph but a deformed grid i.e. one where only some prop~~rie~~erties of axial grid maps are present. In that case we need to develop a strategy in order to standardise RA values in real urban maps and, in particular, in real urban grids. such cases

When a G(k,m) graph representation of an AM(m,k) axial map is a (k,m) Node Grid Graph we say that the Axial Map and the Node Map are duals (see Fig.11). In that case the RA values of the lines of the axial map should be standardised by the mean RA values of a grid corner and not by the diamond root since all the properties, in terms of depth, of a complete bipartite graph are lost and we are only in the presence of a grid graph. this



a) b)

Fig.11 a) A (112,64) Axial Map and b) The corresponding (64,112) Graph representation. In that particular case the graph representation is a (68,112) Node Grid Map and the Axial Map and this Node Grid Map are said to be dual.

such a

However, ~~that~~ clear relationship between the graph representation of the axial map and a node grid map is unlikely to occur in reality. What is likely probable to occur is, for certain axial maps/ to have a graph representation closer more close to a bipartite graph, and ~~for~~ others more close to a node grid map. closer

? context

In that sense, in order to generalise postulate 2 not just or axial grid to maps but to all kind of axial maps we need to introduce the notion of quasi-axial map.

A quasi-axial map has a graph representation close or approximate to a node grid graph. The degree of closeness can be given by several statistics but one that is possible to define is to estimate the mean RA of the (m,k) involves estimating axial map and ing compare it not only with the mean RA of a (k,m) node grid map given by expression (35) but also with the mean depth of a complete bipartite graph, with k nodes, given by expression (28).

Knowing the standard deviation of these maps it is possible to compute the t-Test statistic to estimate, with 2K-2 degrees of freedom, at what significance level the RA means are equal and choose the one, either to related with the bipartite graph or with node grid map, which presents the to the smallest critical region.

Depending on which is of s

The more significant ~~is~~ the t-Test to compare the mean RA of the axial either map of a deformed grid to each one of these forms - node grid or bipartite graph - ~~then~~ so the RA values of the lines of the axial map should be accordingly standardised. If the test is more significant in relation to the mean RA of a node grid map then we are in presence of what was called a quasi-axial map, which means that it should be standardised by the mean RA of a grid corner; and if the test is in relation more significant to the mean RA of a bipartite graph, then it should be standardised by the mean RA of a diamond root.

this way

In that sense we can preserve, partially, the property related with the limit for the ratio of mean RRA values, given in Table 4, as the axial map expands. As the ratio between these RRA values approaches 1 as k increases, it means that, in the initial axial map as well as in its expanded form, we are able to find some underlying pattern which allows us to compare RRA values for specific lines in axial maps with different sizes, and to compare similar morphological configurations and approximate mean RRA values.

In short, in order to compare RA values of axial maps with different sizes they should be standardised either by the root of a diamond or by the corner of a grid, according to the significance level of the difference of their mean RA, respectively, to the mean RA of a bipartite graph or a node grid/ with the same number k of axial lines or nodes.

For the same purpose RA values of node grid graphs should always be standardised by the grid corner.

A subsequent paper will describe empirical results obtained from condensed descriptions of the real urban world, in order to evaluate the preliminary findings already developed.

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