Shimura varieties, Galois representations and motives

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I, Gregorio Baldi, confirm that the work presented in this thesis is my own, except the contents of Chapter 5, which is based on a collaboration with Giada Grossi, and Chapter 7, which is based on a collaboration with Emmanuel Ullmo. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

This thesis is about Arithmetic Geometry, a field of Mathematics in which techniques from Algebraic Geometry are applied to study Diophantine equations. More precisely, my research revolves around the theory of *Shimura varieties*, a special class of varieties including modular curves and, more generally, moduli spaces parametrising principally polarised abelian varieties of given dimension (possibly with additional prescribed structures). Originally introduced by Shimura in the '60s in his study of the theory of complex multiplication, Shimura varieties are complex analytic varieties of great arithmetic interest. For example, to an algebraic point of a Shimura variety there are naturally attached a Galois representation and a Hodge structure, two objects that, according to Grothendieck's philosophy of motives, should be intimately related. The work presented here is largely motivated by the (recent progress towards the) *Zilber–Pink conjecture*, a far reaching conjecture generalising the *André–Oort* and *Mordell–Lang* conjectures.

More precisely, we first prove a conjecture of Buium–Poonen which is an instance of the Zilber-Pink conjecture (for a product of a modular curve and an elliptic curve). We then present Galois-theoretical sufficient conditions for the existence of rational points on certain Shimura varieties: the moduli space of K3 surfaces and Hilbert modular varieties (the latter case is joint work with G. Grossi). The main idea underlying such works comes from Langlands programme: to a compatible system of Galois representations one can attach an analytic object (like a classical/Hilbert modular form or a Hodge structure), which in turn determines a motive which eventually gives an algebraic point of a Shimura variety. We then prove a geometrical version of Serre's Galois open image theorem for arbitrary Shimura varieties. We finally discuss representation-theoretical conditions for a variation of Hodge structures to admit an integral structure (joint work with E. Ullmo).

Impact Statement

I expect that the results of this thesis will have impact in various areas of Mathematics. Indeed, I have been using tools from Number Theory, Complex and Algebraic Geometry, and Representation Theory. Thus, I believe my work will influence and generate new research in these areas. To achieve this impact, I have already published in peer-reviewed journals three papers, and have three more posted on the arXiv. As evidence of interest in my work, I have been invited to present my research in various occasions around Europe. For example at the XXI congress of the Italian Mathematical Society (Pavia, IT), at the UMI-PTM-SIMAI conference (Poland), at the Summer school on explicit and computational approaches to Galois representations (Luxemburg), at the university of Amsterdam and at the Workshop on K3 surfaces and Galois representation (Shepperton, UK). More recently I have also been invited to give an online talk at the R.O.W. seminar.

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On the non-mathematical side, thank you to my family for always supporting me. It is hard to describe my gratitude in only a few lines, but I am grateful for your love and support throughout these years.

Contents

Chapter 1

Introduction and main results

> Don't get set into one form, adapt it and build your own, and let it grow, be like water. Empty your mind, be formless, shapeless — like water. Now you put water in a cup, it becomes the cup; you put water into a bottle it becomes the bottle; you put it in a teapot it becomes the teapot. Now water can flow or it can crash. Be water, my friend.
>
> Bruce Lee

We discuss some of the questions that motivated the study of modern Arithmetic Geometry and present the main results of the thesis, explaining where they place in such a broad subject. A more detailed and precise introduction to each of our main results can be found at the beginning of the corresponding chapter.

1.1 What is a rational point?

A *rational point* of an algebraic variety is a point whose coordinates belong to a given field. In this thesis, the given field will either be the field of rational numbers, denoted by $\mathbb{Q}$, or a number field K, that is a finite field extension of $\mathbb{Q}$.

An example may be helpful in clarifying this abstract definition.

Example 1.1.1. Let n be a natural number strictly bigger than three and consider the Fermat curve

$$C_n := \{x^n + y^n = 1\}.$$

The description of the rational points of C_n strongly depends on the chosen field:

- Over $\mathbb{Q}$ the rational points are simply $(x,y) = (1,0), (0,1)$ and, if n is even, $(-1,0)$ and $(0,-1)$, as Wiles [179] proved in 1995;
- Over an arbitrary number field K, a rational point is, by definition, a pair of elements (a,b) in K such that $a^n + b^n = 1$. It is hard in general to say something beyond the definition and, for example, to list them as in the case $K = \mathbb{Q}$.

Faltings [65], in 1983, managed to prove that, over any number field K, C_n has only finitely many rational points. In the proof of both results the idea is that, sometimes, we can attach *linear algebraic data* to a rational point, more precisely *Galois representations*, and then use some representation theory to prove finiteness and, in favourable situations, even count the number of rational points.

Of course one may expect that the description of the $\mathbb{Q}$-points of C_n depends on its nice and simple shape and can ask the following.

Question 1.1.2. What if one considers *more complicated* equations?

A *more complicated* equation, for us, is simply an equation that requires more space to be written down. Two examples of equations that are more complicated than Fermat are given below.

Example 1.1.3. Can the rational points of

$$y^6 + (x+5)y^5 - (4x^2 + 2x - 8)y^4 - (2x^3 + 16x^2 + 14x - 4)y^3 +$$

$$+(6x^4 + 11x^3 - 6x^2 - 12x)y^2 + 2x^2(x+1)(x^2 + 6x + 6)y +$$

$$-x^3(x+1)^2(x+2)^2 = 0$$

be described?

Example 1.1.4. What about the rational points of

$$y^4+5x^4-6x^2y^2+6x^3+26x^2y+10xy^2+$$

$$-10y^3-32x^2-40xy+24y^2+32x-16y=0?$$

The question can be very hard and, in full generality, almost hopeless. For example the negative solution of Hilbert's 10th Problem by Davies, Putnam, Robinson, Matijasevič, Čudnovskiĭ [115] asserts that there can be no algorithm determining whether a given Diophantine equation is soluble in integers $\mathbb{Z}$ or not (the solution of Hilbert's 10th Problem over $\mathbb{Q}$ is not known).

There are however *special* equations that have a *geometrical meaning*. The basic idea of Arithmetic Geometry may be approximated as follows.

To study rational points, it may be useful to think them from a geometrical point of view, so that one can apply powerful results from geometry.

Going back to the above examples, we can explain what it is meant with *thinking rational points of such equation from a geometrical point of view*.

Example 1.1.5 ($X_1(22)$)**.** Over any number field K, a rational point of the equation appearing in Example 1.1.3 *corresponds*[1] to an elliptic curve E defined over K with 22-level structure. The projective curve defined, in some affine chart, by that equation is called *modular curve of level* 22, for its interpretation as moduli space of some algebro-geometric datum. A similar description holds also for the equation of Example 1.1.4. Indeed it describes, in an affine chart, the modular curve $X_s(13)$ (here the subscript s denotes the *split Cartan level structure*).

Let $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$ and ℓ a prime number. A concrete way of applying the basic idea of arithmetic geometry to study $\mathbb{Q}$-rational points of $X_1(22)$ could be as follows:

- Geometrical step. Think about complex elliptic curves as complex tori;

[1] After the equation is homogenised and up to a finite number of points, the cusps, corresponding to *degenerate* elliptic curves.

- Arithmetical step. Study continuous group homomorphisms

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_\ell). \tag{1.1.1}$$

The link between the two steps is that complex tori coming from elliptic curves defined over $\mathbb{Q}$ have a natural action on their ℓ-torsion, of the form described above. In principle, the representation ρ encodes the information needed to translate geometric properties to arithmetic ones. Unfortunately $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a very mysterious group: it is an uncountable profinite group and we can not describe explicitly any elements except for the identity and the complex conjugation.

At this point, two questions appear natural.

Question 1.1.6. Why should we care about complicated equations?

Question 1.1.7. Why should it be easier, for example, to construct elliptic curves, than rational points?

We present a possible answer to Question 1.1.6 that motivated the first main result of our thesis and then discuss how to construct elliptic curves (and generalisations thereof).

1.2 Rational points and elliptic curves

Let C be a smooth projective curve defined over $\mathbb{Q}$. The genus of C is a topological invariant that can be easily computed in most of the cases, for example by the Riemann—Hurwitz formula. To be concrete, the homogenisation of the Fermat curve C_n of Example 1.1.1 has genus $(n-1)(n-2)/2$, which is at least 2 for $n \geq 4$. Even if the genus of C, denoted by g, depends only on the geometry of the curve C, it can distinguish different behaviours of the arithmetic of C:

$g = 0$. Either C has no rational points, or C is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ and so it has infinitely many rational points (Hurwitz [92] 1890);

$g = 1$. Either C has no rational points, or C is an elliptic curve and its rational points form a finitely generated abelian group. That is $C(\mathbb{Q}) \cong \mathbb{Z}^r \oplus C(\mathbb{Q})_{\text{torsion}}$ and

$C(\mathbb{Q})_{\text{torsion}}$ is a finite abelian group (Mordell [126] 1922, answering a question of Poincaré 1901);

$g > 1$. $C(\mathbb{Q})$ is finite (Faltings [65] 1983, settling Mordell's conjecture).

The above thricotomy remains true if $\mathbb{Q}$ is replaced by another number field K. The proofs of the above results are not effective, unless $g = 0$, so that one does not possess an algorithm for finding the rational points. The case of genus zero curves was treated in detail by Hilbert and Hurwitz [92], but the genus one case is more elusive. How to tell whether C has at least one rational point? If this is the case, when does C have infinitely many rational points? Birch and Swinnerton-Dyer proposed an answer to the latter question, which today stands as one of the Millennium Problems [180].

Remark 1.2.1. In the recent years new methods for describing $\mathbb{Q}$-rational points of curves have emerged, notably the Chabauty–Coleman [40] and Chabauty–Kim [99] method. This strategy was recently implemented by Balakrishnan, Dogra, Müller, Tuitman and Vonk [9] to prove that the curve $X_s(13)$, described in Example 1.1.4, has seven $\mathbb{Q}$-points (six corresponding to elliptic curves with complex multiplication and one cusp).

The application we want to discuss concerns the latter question. The work of Wiles, Breuil, Conrad, Diamond and Taylor [179, 169, 23] implies that every elliptic curve $E/\mathbb{Q}$ can be written, for some natural number $N > 3$, as

$$\phi : X_1(N) \twoheadrightarrow E/\mathbb{Q}; \tag{1.2.1}$$

where $X_1(N)$ denotes the *modular curve of level N*. For example for $N = 22$, $X_1(N)$ is the curve discussed in Example 1.1.3.

Remark 1.2.2. Interestingly and certainly surprisingly at first, the *modularity* result of equation (1.2.1) was the missing ingredient to obtain the description of the $\mathbb{Q}$-points of C_n mentioned in Example 1.1.1. Fermat's Last theorem was actually one of the main motivations that led to the study of such problems.

Gross–Zagier and Kolyvagin [87, 104], around the '90s, proved that *special points* of $X_1(N)$ can produce non-torsion points of $E(\mathbb{Q})$, accordingly with the predictions of Birch and Swinnerton-Dyer. Buium and Poonen [25, 26], in 2009, asked whether using other *special subsets* $\Sigma \subset X_1(N)(\overline{\mathbb{Q}})$, one can produce non-torsion points of E. We proved that this is indeed the case. The first main theorem of the thesis, that is discussed in Chapter 3, is the following.

Theorem (=Theorem 3.1.4)**.** *When Σ corresponds to an isogeny class, the points $\phi(\Sigma)$ are "independent" in $E(\overline{\mathbb{Q}})$. For example only finitely many of them can be torsion.*

This is actually an instance of the *Zilber–Pink conjecture* (for mixed Shimura varieties), a far reaching conjecture generalising simultaneously the *André–Oort conjecture* and Faltings' result mentioned above (Mordell conjecture). In its most elementary form the André–Oort conjecture asserts the following (see André [2]). Let V be an irreducible algebraic curve in the complex affine plane, which is neither horizontal nor vertical. Then V is a modular curve $Y_0(N)$, for some $N > 0$, if and only if it contains infinitely many points $(j', j'') \in \mathbb{C}^2$ such that j' and j'' are j-invariants of elliptic curves with complex multiplication.

The proof of the above Theorem is inspired by the strategy that Buium–Poonen employed in [25], using an equidistribution result about Hecke points on modular curves (after Clozel, Eskin, Oh and Ullmo [36, 63]) and Serre's open image theorem [154, 156]. As explained in the previous section the absolute Galois group of a number field is a mysterious object, but its representations are manageable. For example Serre's open image theorem asserts that if

$$\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$$

is the Galois representation attached to an elliptic curve E/K without complex multiplication as in equation (1.1.1) and ℓ is a large enough prime (which depends on K and E), then ρ is surjective. At this point it may also be interesting to point out that studying the dependence of ℓ on E brings again to the study of rational points of modular curves. A famous question of Serre about the uniformity of ℓ was indeed one of the main motivations for the study of the $\mathbb{Q}$-points of $X_s(13)$ discussed in Remark 1.2.1.

1.3 Rational points of Shimura varieties

We now discuss Question 1.1.7 and present some contributions on the study of rational points of *Shimura varieties*. For the time being Shimura varieties could be thought as (smooth quasi-projective) algebraic varieties having a moduli interpretation generalising modular curves (of a certain level) that can be defined over some number field[2]. The following is the main result of Chapter 4. It is inspired by the work of Patrikis, Voloch and Zarhin [139] who have proven a similar theorem for abelian varieties of arbitrary dimension.

Theorem (=Theorem 4.1.3)**.** *Under some standard conjectures in Arithmetic Geometry, necessary and sufficient conditions for the existence of rational points on the moduli space of (principally polarised) K3 surfaces (for any number field K) can be given.*

The following is the main result of Chapter 5 and generalises a result of Helm and Voloch [91] about modular curves.

Theorem (= Theorem 5.1.1 and 5.1.2, joint with Grossi)**.** *Necessary and sufficient conditions for the existence of $\mathbb{Q}$-points on the moduli space of abelian varieties with $\mathcal{O}_F$-multiplication, where F is a totally real number field, can be given. Under the Absolute Hodge conjecture and a weak Serre's modularity conjecture, the same holds for points with coordinates in a totally real number field L.*

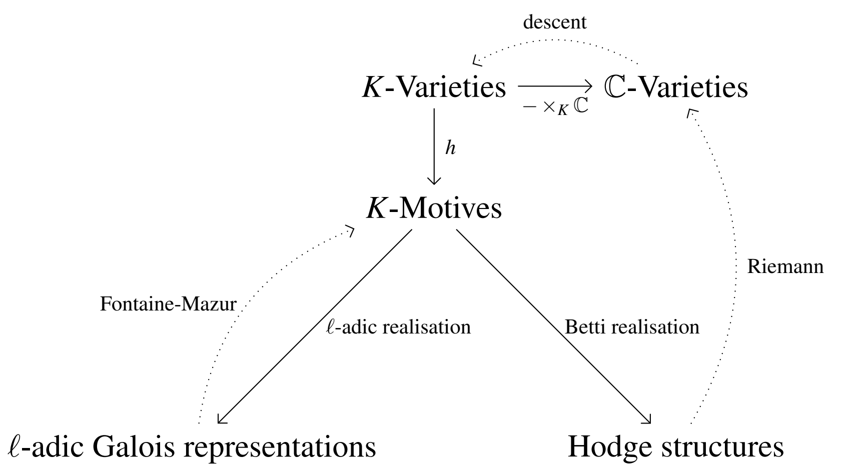
Both results can roughly be thought as receipts to detect the existence of rational points, for a given number field, of certain Shimura varieties. More precisely the former theorem treats *orthogonal Shimura varieties* and the latter *Hilbert modular varieties*.

[2] To be precise the curve $X_1(22)$ appearing in Example 1.1.3 is a compactification of a (one dimensional) Shimura variety, rather than a Shimura variety itself.

The main ideas involved in the proofs come from the work of Langlands and the philosophy of *motives* envisioned by Grothendieck. The idea to find K-rational points is to start with some linear algebra data, more precisely a *compatible system of representations of the absolute Galois group of* K and then argue as follows:

1. Attach to it an analytic object (like a classical/Hilbert modular form or a Hodge structure);
2. This analytic object in turn determines a motive;
3. The motive eventually gives an algebraic point of the Shimura variety.

For example, from Riemann, we know that a polarisable rational Hodge structure of weight one and type $(1,0),(0,1)$ corresponds, up to isogeny, to a complex abelian variety. Starting from a system of Galois representations, satisfying some natural conditions, we can hope to construct a weight one Hodge structure and to descend over K the associated abelian variety. A diagram may be helpful to clarify the picture. Here by varieties we mean smooth projective varieties:



The meaning of the full arrows is explained in Chapter 2. The existence of the dotted arrows requires *ad hoc* arguments and represents the main challenge in proving the results of the section. The problem in the approach sketched above is that sometimes we stumble in some very difficult arithmetic conjectures, like the Fontaine-Mazur conjecture [71] appearing in the above diagram. For example it would be great to have an explicit answer to the following.

Question 1.3.1. When does a Galois representation come from geometry?

1.4 The geometry of Shimura varieties

We have so far described the arithmetic side of the theory of Shimura varieties. Bearing in mind the basic idea of Arithmetic Geometry, such problems should have a geometrical counterpart. For example can we "guess" which Hodge structures come from geometry? For the time being, no one has proposed an answer to this question. However something can be said when we *vary* Hodge structures in families, as explored by Simpson [161]. In the two final chapters we study this situation.

In the geometric setting, the role of Galois representations is played by representations of the *fundamental group* of a variety. Indeed the étale fundamental group of Spec$(\mathbb{Q})$ is the absolute Galois group of $\mathbb{Q}$. Every subvariety of a Shimura variety naturally supports a $\mathbb{Z}$-Variation of Hodge structures (VHS from now on). In the cases described until now, the VHS came indeed from geometry, since the Shimura varieties were moduli spaces of abelian varieties and K3 surfaces (with some extra structure). From a more general and geometric point of view, Shimura varieties are Hermitian locally symmetric domains, that is they can be described and studied using reductive algebraic groups. In a similar fashion as (1.1.1), we can attach to every subvariety Y of a Shimura variety S a representation

$$\rho_Y : \pi_1^{\text{et}}(Y,y) \to G(\mathbb{Z}_\ell),$$

where G is a reductive $\mathbb{Q}$-group that comes with S. The image of ρ_Y is called the *ℓ-adic monodromy* of Y. The following is the main result of Chapter 6.

Theorem (= Theorem 6.1.3)**.** *Over the complex numbers, subvarieties of Shimura varieties without isotrivial components have large ℓ-adic monodromy.*

The above theorem can be thought as a geometric version of Serre's open image theorem described in section 1.2. It may be interesting to notice that such geometric statement is very general, whereas Serre's open image is not known to hold for higher dimensional abelian varieties (once it is suitably stated, see for example

[31]).

The study of subvarieties of Shimura varieties leads to a simple question. Which varieties can arise as subvarieties of Shimura varieties? For example a key step that allowed Faltings to prove the Mordell conjecture is the following observation made by Parshin-Kodaira, see [113] and references therein. Every curve C/K, of genus at least two, admits a finite étale cover $Y \to C$ such that Y admits a morphism with finite fibres to a Shimura variety, which parametrises principally polarised abelian varieties (of a given dimension, depending on the genus of C). Another way of realising curves inside Shimura varieties can be obtained using a famous theorem of Belyi, as established in [39, Theorem 1], relying on special properties of (non-arithmetic) triangle groups $\Delta(a,b,c) \subset \mathrm{SL}_2(\mathbb{R})$. Motivated by such circle of ideas, in Chapter 7, we study complex hyperbolic lattices $\Gamma \subset \mathrm{SU}(1,n)$, for some $n > 1$. Notice that the real algebraic group $\mathrm{SU}(1,n)$ is locally isomorphic to the isometry group of the n (complex) dimensional complex hyperbolic space. Associated to Γ there is a quasi-projective ball quotient S_Γ, thanks to the work of Baily–Borel [7] and Mok [54].

The following is the main result of Chapter 7.

Theorem (joint with E. Ullmo=Theorem 7.3.1)**.** *The standard complex variation of Hodge structures $\mathbb{V}$ on S_Γ admits an integral structure. Moreover S_Γ can naturally be embedded in a domain for polarised $\mathbb{Z}$-VHS.*

A very concrete corollary of the above Theorem, which is of independent interest, is that Γ is always contained in the $\mathcal{O}_K$-points of a K-form of $\mathrm{SU}(1,n)$, for some totally real number field K. If Γ is arithmetic, $\mathbb{V}$ corresponds again to a family of abelian varieties and the existence of an integral structure is obvious. In the non-arithmetic case, Simpson conjectured that $\mathbb{V}$ comes from geometry, but we were not able to prove his conjecture. Nevertheless this was the starting point to interpret totally geodesic subvarieties of S_Γ as *unlikely intersections*. We refer to the preprint [14], where such point of view is carried out to obtain, among other things, new instances of a generalised Zilber–Pink conjecture recently proposed by Klingler [102]. See also the discussion in Chapter 8.

Chapter 2

Preliminaries

We recall some notions of the theory of Shimura varieties, discussing both their geometry and arithmetic properties. We describe in more details the main examples of Shimura varieties that appear in the sequel. Galois representations and motives are then discussed. Finally we present some well known conjectures in Arithmetic Geometry relating the three protagonists of the thesis.

2.1 Notations

We collect here some standard notations we often use. In case we need more notations just for a chapter, we will introduce it when needed.

- Given an algebraic group $G/\mathbb{Q}$, $G^{\text{der}} \subset G$ denotes its derived subgroup, $Z(G)$ its center, $G \to G^{\text{ab}} := G/G^{\text{der}}$ its abelianization and $G \to G^{\text{ad}} := G/Z(G)$ its adjoint quotient. Moreover by reductive group we always mean *connected reductive*;

- We denote by $\mathbb{A}_f$ the (topological) group of finite $\mathbb{Q}$-adeles, i.e. $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$, endowed with the adelic topology. Given a subgroup $K \subset G(\mathbb{A}_f) \subset \prod_\ell G(\mathbb{Q}_\ell)$, we write $K_\ell \subset G(\mathbb{Q}_\ell)$ for the projection of K along $G(\mathbb{A}_f) \to G(\mathbb{Q}_\ell)$. If a compact open subgroup of $G(\mathbb{A}_f)$ may be confused with a number field K, we denote the former by $\widetilde{K}$;

- As in [50, Notation 0.2] we write $(-)^0$ to denote an algebraic connected component and $(-)^+$ for a topological connected component, e.g. $G(\mathbb{R})^+$

is the topological connected component of the identity of the group of the real points of G. We write $G(\mathbb{R})_+$ for the subgroup of $G(\mathbb{R})$ of elements that are mapped into the connected component $G^{\mathrm{ad}}(\mathbb{R})^+ \subset G^{\mathrm{ad}}(\mathbb{R})$, where G^{ad} denotes the adjoint group of G. Finally we set $G(\mathbb{Q})^+ := G(\mathbb{Q}) \cap G(\mathbb{R})^+$ and $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$.

Given a complex algebraic variety S, we denote by S^{an} the complex points $S(\mathbb{C})$ with its natural structure of a complex analytic variety. We denote by $\pi_1(S)$ the topological fundamental group of S^{an} and by $\pi_1^{ét}(S)$ the étale one. Unless it is necessary, we omit the base point in the notation. Given an algebraic variety S defined over a field K, S/K form now on, we write $S_{\overline{K}}$ for the base change of S to $\mathrm{Spec}(\overline{K})$.

2.2 Shimura varieties

Special instances of Shimura varieties were originally introduced by Shimura in the 60's. Deligne outlined the theory of Shimura varieties in [50, 46]. For introductory notes on this vast subject we also mention [121, 62]. This short chapter cannot and it is not intended to be a complete summary of the theory of Shimura varieties. The aim is to introduce and describe the pieces of theory we use later in the text, in order to keep the thesis as self contained as possible.

Let $\mathbb{S}$ denote the real torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$, which is usually called the *Deligne torus*.

Definition 2.2.1. A *Shimura datum* is pair (G,X) where G is a reductive $\mathbb{Q}$-algebraic group and X a $G(\mathbb{R})$-orbit in the set of morphisms of $\mathbb{R}$-algebraic groups $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, such that for some (equivalently all) $h \in X$ the following axioms are satisfied:

SD1. $\mathrm{Lie}(G)_{\mathbb{R}}$ is of type $\{(-1,1),(0,0),(1,-1)\}$;

SD2. The action of the inner automorphism associated to $h(i)$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$. That is, the set

$$\{g \in G^{\mathrm{ad}}(\mathbb{C}) \ : \ h(i)gh(i)^{-1} = g\}$$

is compact;

SD3. For every simple $\mathbb{Q}$-factor H of G^{ad}, the composition of $h : \mathbb{S} \to G_{\mathbb{R}}$ with $G_{\mathbb{R}} \to H_{\mathbb{R}}$ is non trivial.

Let (G,X) be a Shimura datum and $\widetilde{K}$ a compact open subgroup of $G(\mathbb{A}_f)$. Notice that $G(\mathbb{Q})$ acts on the left on $X \times G(\mathbb{A}_f)$ by left multiplication on both factors and $\widetilde{K}$ acts on the right just by right multiplication on the second factor. We set

$$\mathrm{Sh}_{\widetilde{K}}(G,X) := G(\mathbb{Q})\backslash \left(X \times G(\mathbb{A}_f)/\widetilde{K}\right).$$

Let X^+ be a connected component of X and $G(\mathbb{Q})^+$ be the stabiliser of X^+ in $G(\mathbb{Q})$. The above double coset set is a disjoint union of quotients of X^+ by the arithmetic groups $\Gamma_g := G(\mathbb{Q})^+ \cap g\widetilde{K}g^{-1}$ where g runs through a set of representatives for the finite double coset set $G(\mathbb{Q})^+\backslash G(\mathbb{A}_f)/\widetilde{K}$. Baily and Borel [7] proved that $\mathrm{Sh}_{\widetilde{K}}(G,X)$ has a unique structure of a quasi-projective complex algebraic variety. Moreover if $\widetilde{K}$ is neat, then $\mathrm{Sh}_{\widetilde{K}}(G,X)$ is smooth.

Remark 2.2.2. Arithmetic subgroups of G are in particular lattices in $G(\mathbb{R})$ (see section 7.2 for precise definitions). That is discrete subgroups of finite covolume. One may ask to what extent the theory of Shimura varieties can be generalised to treat arbitrary lattices in $G(\mathbb{R})$, where (G,X) is a Shimura datum. This point of view is carried on in [14, Section 3]. More about this is discussed in Chapter 7.

For every inclusion $K_1 \subset K_2$ we have a map $\mathrm{Sh}_{K_1}(G,X) \to \mathrm{Sh}_{K_2}(G,X)$, which is an algebraic map again by a result of Borel. If K_1 is normal in K_2 then it is the quotient for the action by the finite group K_2/K_1 and therefore these morphisms are finite (as morphisms of schemes). That means that we can take the limit (projective limit) of the system of these, in the category of schemes, which we denote by $\mathrm{Sh}(G,X)$. We have

$$\mathrm{Sh}(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

Denote by π the projection

$$\pi : \mathrm{Sh}(G,X) \to \mathrm{Sh}_{\widetilde{K}}(G,X) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/\widetilde{K}).$$

Given a Shimura datum (G,X) there exists an *adjoint Shimura datum* $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ where X^{ad} is the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of the morphism h^{ad}, defined as the composition of any $h : \mathbb{S} \to G$ and $G \to G^{\mathrm{ad}}$. The construction gives a natural morphism of Shimura data $(G,X) \to (G^{\mathrm{ad}},X^{\mathrm{ad}})$ and, choosing a compact open $\widetilde{K}^{\mathrm{ad}}$ in $G^{\mathrm{ad}}(\mathbb{A}_f)$ containing the image of $\widetilde{K}$, we obtain also a finite morphism of Shimura varieties

$$\mathrm{Sh}_{\widetilde{K}}(G,X) \to \mathrm{Sh}_{\widetilde{K}^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}}).$$

2.2.1 Shimura varieties as moduli spaces of Hodge structures

Quoting Edixhoven [62, Section 2], *Hodge structures are really at the start of the theory of Shimura varieties.* We discuss how conditions SD1-3 of Definition 2.2.1 imply that connected components of X are Hermitian symmetric domains and faithful representations of G induce variations of polarisable $\mathbb{Q}$-Hodge structures. For an introduction to Hodge theory we refer to the book [83].

2.2.1.1 Hodge structures

Let $V_{\mathbb{R}}$ be a finite dimensional $\mathbb{R}$-vector space and $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$ it complexification. The complex conjugation acts on $V_{\mathbb{C}}$ by $\lambda \otimes v \mapsto \overline{\lambda} \otimes v$. A *Hodge decomposition* of $V_{\mathbb{R}}$ is a direct sum decomposition of $V_{\mathbb{C}}$ into $\mathbb{C}$-subspaces $V^{p,q}$ indexed by $\mathbb{Z}^2$:

$$V_{\mathbb{C}} = \oplus_{(p,q)\in\mathbb{Z}\times\mathbb{Z}} V^{p,q},$$

such that $\overline{V^{p,q}} = V^{q,p}$. An $\mathbb{R}$-*Hodge structure* is a finite dimensional $\mathbb{R}$-vector space $V_{\mathbb{R}}$ with a Hodge decomposition. The type of this Hodge structure is the set of (p,q) for which $V^{p,q} \neq 0$. Equivalently it is a real representation of the Deligne torus

$$h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}}).$$

With this interpretation we have natural notions of dual, homomorphism, tensor product, direct sum and irreducibility of Hodge structures. Fixing an $n \in \mathbb{Z}$, the subspace

$$V_n := \oplus_{p+q=n} V^{p,q}$$

is stable under complex conjugation and it is referred to as a *Hodge structure of weight n*.

A $\mathbb{Q}$-*Hodge structure* (HS from now on) is a finite dimensional $\mathbb{Q}$-vector space V together with an $\mathbb{R}$-HS on $V \otimes \mathbb{R}$ and a $\mathbb{Z}$-HS is a free[1] $\mathbb{Z}$-module of finite rank $V_{\mathbb{Z}}$ together with a $\mathbb{Q}$-HS on $V_{\mathbb{Z}} \otimes \mathbb{Q}$.

Definition 2.2.3. Let (V,h) be a $\mathbb{Q}$-HS of weight n. A polarisation of (V,h) is a bilinear map of $\mathbb{Q}$-Hodge structures $q : V \otimes V \to \mathbb{Q}(-n)$, that is $q_{\mathbb{R}}(h(z)v, h(z)w) = (z\bar{z})^n q_{\mathbb{R}}(v,w)$, such that

$$q(v, h(i)w)$$

defines a positive definite symmetric form on $V_{\mathbb{R}}$. Finally (V,h) is called *polarisable* if there exists a polarisation q on it.

The category of $\mathbb{Q}$-HS is abelian and the category of polarisable $\mathbb{Q}$-HS is semisimple.

Remark 2.2.4. Since $h(i)^2 = (-1)^n$, q is symmetric if the weight is even, alternating if it is odd.

All the HSs in this thesis will be polarisable (and pure). We therefore simply say *Hodge structure* to mean polarisable Hodge structure. If it is clear from the context, by HS we could also mean $\mathbb{Q}$-HS. Moreover a weight and a type will in general be fixed. We conclude this short section with an important definition.

Definition 2.2.5. Let $(V_{\mathbb{Q}}, h)$ be a $\mathbb{Q}$-HS. The *Mumford–Tate group* of (V,h), simply denoted by $\mathrm{MT}(h)$, is the $\mathbb{Q}$-Zariski closure of the image of

$$h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

[1] With this definition the cohomology with $\mathbb{Z}$-coefficients of a smooth projective complex variety does not come with a $\mathbb{Z}$-HS. It does only after the torsion is killed. In our thesis such difference does not play any role.

in $\mathrm{GL}(V_{\mathbb{Q}})$. That is the smallest $\mathbb{Q}$-subgroup H of $\mathrm{GL}(V_{\mathbb{Q}})$ such that $H_{\mathbb{R}}$ contains $h(\mathbb{S})$.

2.2.1.2 Variations of Hodge structures

Let S be a smooth quasi-projective variety and let $\mathcal{V} = (\mathcal{V}_{\mathbb{Z}}, \mathcal{F}, \mathcal{Q}_{\mathbb{Z}})$ a *polarised variation of $\mathbb{Z}$-Hodge structure* on S (VHS from now on). That is the data of

- A local system $\mathcal{V}_{\mathbb{Z}}$ with a flat quadratic form $\mathcal{Q}_{\mathbb{Z}}$;
- A holomorphic locally split filtration $\mathcal{F}$ of $\mathcal{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{S^{\mathrm{an}}}$ such that the flat connection ∇ satisfies Griffiths transversality:

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \text{ for all } p;$$

- $(\mathcal{V}_{\mathbb{Z}}, \mathcal{F}, \mathcal{Q}_{\mathbb{Z}})$ is fiberwise a $\mathbb{Z}$-HS.

In the same way we have definitions of $\mathbb{Z}, \mathbb{Q}$, K and $\mathbb{R}$-VHS, where K is a sub-field of $\mathbb{R}$.

Let $\lambda : \widetilde{S} \to S$ be the universal cover of S and fix a trivialisation $\lambda^*\mathcal{V} \cong \widetilde{S} \times V$. Similarly to Definition 2.2.5, we have the following.

Definition 2.2.6. Let $s \in S$. The *Mumford–Tate group at s of $\mathcal{V}$*, denoted by $\mathrm{MT}_s \subset \mathrm{GL}(V_s)$, is the smallest $\mathbb{Q}$-algebraic group M such that the map

$$h_s : \mathbb{S} \longrightarrow \mathrm{GL}(V_{s,\mathbb{R}})$$

describing the Hodge-structure on V_s, factors through $M_{\mathbb{R}}$. Choosing a point $\tilde{s} \in \lambda^{-1}(s) \subset \widetilde{S}$ we obtain an injective homomorphism $\mathrm{MT}_s \subset \mathrm{GL}(V_s)$. When a point $t \in S$ is such that MT_t is abelian (hence a torus), we say that t is a *special point*[2].

It is well known, see [47, Proposition 7.5], that there exists a countable union $\Sigma \subsetneq S$ of proper analytic subspaces of S such that:

[2] In this thesis a special point could also be called *CM-point* (see also Example 2.2.8 for more details).

- For $s \in S - \Sigma$, $\mathrm{MT}_s \subset \mathrm{GL}(V)$ does not depend on s, nor on the choice of $\tilde{s}$. We call this group *the generic Mumford–Tate group of* $\mathbb{V}$ and we simply write it as G;

- For all s and $\tilde{s}$ as above, with $s \in \Sigma$, MT_s is a proper subgroup of the generic Mumford–Tate group of $\mathbb{V}$.

To be more precise Σ is also known to be a countable union of algebraic subvarieties of S. This follows indeed from the work of Cattani, Deligne and Kaplan [34].

2.2.1.3 Period domains

A useful reference for this section is also [102]. Let $(V_{\mathbb{Z}}, q_{\mathbb{Z}})$ be a polarised $\mathbb{Z}$-Hodge structure. Let G be the $\mathbb{Q}$-algebraic group $\mathrm{Aut}(V_{\mathbb{Q}}, q_{\mathbb{Q}})$. Consider the space D of $q_{\mathbb{Z}}$-polarised Hodge structures on $V_{\mathbb{Z}}$ with specified Hodge numbers (it is homogeneous for G). Fixing a reference Hodge structure, we write $D = G(\mathbb{R})/M$ where M is a subgroup of the compact unitary subgroup $G(\mathbb{R}) \cap U(h)$ with respect to the Hodge form h of the reference Hodge structure.

Let S be a smooth quasi-projective complex variety. By *period map*

$$S^{\mathrm{an}} \to \Gamma \backslash D$$

we mean a holomorphic locally liftable Griffiths transverse map, where Γ is a finite index subgroup of $G(\mathbb{Z}) = \mathrm{Aut}(V_{\mathbb{Z}}, q_{\mathbb{Z}})$. A period map $S^{\mathrm{an}} \to G(\mathbb{Z}) \backslash D$ is equivalent to the datum of a $\mathbb{Z}$-VHS on S with generic Mumford–Tate group $G/\mathbb{Q}$. The period map lifts to $\Gamma \backslash D$ if Γ contains the image of the monodromy representation of the corresponding $\mathbb{Z}$-VHS. See also [102, Section 3].

Remark 2.2.7. Shimura varieties are particular cases of period domains. See indeed [46, Proposition 1.1.14 and Corollary 1.1.17]. Period domains, in the generality defined above, do not have a natural algebraic structure [84]. Shimura varieties could indeed be thought as the *most special* example of period domains.

2.2.2 Canonical models and reflex field

To be of arithmetic interest, Shimura varieties must admit models over a number field. Thanks to the work of Borovoi, Deligne, Milne and Milne-Shih [22, 122, 119], among others, the $\mathbb{C}$-scheme

$$\mathrm{Sh}(G,X) = G(\mathbb{Q})\backslash \left(X \times G(\mathbb{A}_f)\right),$$

together with its $G(\mathbb{A}_f)$-action, can be canonically defined over a number field $E := E(G,X) \subset \mathbb{C}$ called the *reflex field* of (G,X). That is there exists an E-scheme $\mathrm{Sh}(G,X)_E$ with an action of $G(\mathbb{A}_f)$ whose base change to $\mathbb{C}$ gives $\mathrm{Sh}(G,X)$ with its $G(\mathbb{A}_f)$-action. For the precise definition of *canonical model* we refer to [50, section 2.2]. For the easier fact that Shimura varieties can be defined over $\overline{\mathbb{Q}}$, we refer for example to [66].

It follows that for every compact open subgroup $\widetilde{K}$ of $G(\mathbb{A}_f)$, the variety $\mathrm{Sh}_{\widetilde{K}}(G,X)$ admits a canonical model over E in such a way that Hecke correspondences commute with the Galois action. Moreover the map $\pi : \mathrm{Sh}(G,X) \to \mathrm{Sh}_{\widetilde{K}}(G,X)$ is defined over E. In general we write K for a finite extension of E such that $\mathrm{Sh}_{\widetilde{K}}(G,X)(K)$ is not empty.

2.2.3 Some examples of Shimura varieties

We briefly discuss some of the main motivating examples of Shimura varieties. We point out here in which chapter each type of Shimura variety appears:

Chapter 3: Shimura curves (and mixed Shimura varieties);

Chapter 4: Orthogonal Shimura varieties;

Chapter 5: Modular curves and Hilbert modular varieties;

Chapter 6: Arbitrary Shimura varieties and Siegel modular varieties;

Chapter 7: Picard and Hilbert modular varieties.

For brevity we actually describe here five examples of *connected* Shimura varieties.

Example 2.2.8 (Modular curves)**.** Denote by $\mathbb{H}$ the complex upper half plane. It becomes a Hermitian symmetric space when endowed with the metric $y^{-2}dxdy$. The SL_2-action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}, \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ and } z \in \mathbb{H},$$

identifies $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ with the group of holomorphic automorphisms of $\mathbb{H}$, where $I \in \mathrm{SL}_2(\mathbb{R})$ denotes the identity. For any $x+iy \in \mathbb{H}$

$$x+iy = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \cdot i,$$

and so $\mathbb{H}$ is a homogeneous space. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, that is a subgroup containing

$$\Gamma_0(N) := \{M \in \mathrm{SL}_2(\mathbb{Z}) : M \equiv I \bmod N\},$$

for some N. The curve $\Gamma\backslash\mathbb{H}$ is a connected modular curve and can be realized as a moduli variety for elliptic curves with some level structure.

Remark 2.2.9. From Definition 2.2.6, we have a notion of special points in a modular curve $\Gamma\backslash\mathbb{H}$. They are points x that correspond to CM-elliptic curves E_x, that is elliptic curves whose endomorphism ring is different from $\mathbb{Z}$. This is the reason why special points are often referred to as CM points. More about this is discussed in Chapter 3.

Let L be a totally real number field. What if we want to take the quotient of $\mathbb{H}$ by (subgroups of) $\mathrm{SL}_2(\mathcal{O}_L)$? The problem is that $\mathrm{SL}_2(\mathcal{O}_L)$ is in general not discrete in $\mathrm{SL}_2(\mathbb{R})$. This can be circumnavigated by a construction called *Weil restriction*.

Example 2.2.10 (Hilbert modular varieties)**.** Let n_L be the degree of L over $\mathbb{Q}$. The subgroup $\mathrm{SL}_2(\mathcal{O}_L)$ can naturally be identified as $G(\mathbb{Z})$, where G is the Weil restriction from L to $\mathbb{Q}$ of SL_2/L. Indeed, using the n_L-real embeddings of L, we see that $\mathrm{SL}_2(\mathcal{O}_L)$ acts on the product of n_L copies of $\mathbb{H}$. Hilbert modular varieties are obtained as quotients $\Gamma\backslash\mathbb{H}^{n_L}$, where Γ is a congruence subgroup of $G(\mathbb{Z})$ and naturally parametrise principally polarised n_L-dimensional abelian varieties with $\mathcal{O}_L$-multiplication (with some level structure).

Example 2.2.11 (Siegel modular varieties)**.** Let g be an integer ≥ 1. Consider the Siegel upper half space

$$\mathbb{H}_g := \{M \in M_g(\mathbb{C}) : M \text{ is symmetric and } \mathrm{Im}(M) \text{ is positive definite } \}.$$

By Riemann every matrix in $\mathbb{H}_g$ is the period matrix of some principally polarised abelian variety, unique up to isomorphism of polarised abelian varieties. The Siegel modular variety, also denoted as $\mathcal{A}_g$, is the quotient $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$. As in the case of modular curves one can see that there is a bijection between the complex points of the Siegel modular variety and isomorphism classes of principally polarised g-dimensional abelian varieties. For more details we refer to section 5.2.

Example 2.2.12 (Orthogonal Shimura varieties)**.** By *K3 surface* X/K we mean a complete smooth K-variety of dimension two such that $\Omega^2_{X/K} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. The group $H^2(X(\mathbb{C}), \mathbb{Z})$ endowed with the cup product is a even, unimodular lattice with signature $(3, 19)$. Indeed there exists a unique 22-dimensional lattice with these properties which we usually denote by Λ_{K3}. Thanks to the work of Rizov and Madapusi-Pera [110], we have that the moduli space of triples (X, l, u) where X is a K3 surfaces, l a polarisation of degree $2d$ and u an isometry

$$\det(P^2(X, \mathbb{Z}_2)) \to \det(\Lambda_{2d} \otimes \mathbb{Z}_2),$$

is a double cover of the coarse moduli space of polarised K3 surfaces of degree $2d$ (which is a quasiprojective variety defined over $\mathbb{Q}$) and that the former embeds as a Zariski open subset of a Shimura variety associated to the group $SO(2, 19)$. Here we denoted by $P^2(X, \mathbb{Z}_2)$, the primitive part of the $H^2(X, \mathbb{Z}_2)$, with respect to the polarisation l. For details see for example [110, Section 2 and 3] and [31, Section 6].

Finally one of the oldest and simplest examples of Shimura varieties. They were first studied by Picard in 1881 [141].

Example 2.2.13 (Picard modular surfaces)**.** Let E be an imaginary quadratic extension of $\mathbb{Q}$ and V be a 3-dimensional E-vector space (which we consider as a vector space over $\mathbb{Q}$). Fix an integral structure on V, given by an $\mathcal{O}_E$-lattice $L \subset V$ and let

$$J : V \times V \to E$$

be a non-degenerate Hermitian form on V, satisfying $J(\alpha u, \beta v) = \overline{\alpha}\beta\overline{J(u,v)}$ and which is $\mathcal{O}_E$-valued on L and has signature $(1,2)$ over $V \otimes \mathbb{R}$. Let $G' := \mathrm{SU}(J,V)/\mathbb{Q}$ be the special unitary group of J, viewed as a semisimple algebraic group over $\mathbb{Q}$. The group $G'(\mathbb{R})/K'$, for any compact maximal $K' \subset G'(\mathbb{R})$ can be identified with the complex two-dimensional ball. The Picard modular group of E is

$$G'(\mathbb{Z}) := \{\gamma \in G'(\mathbb{Q}) : \gamma L = L\},$$

and given a finite index subgroup Γ of $G'(\mathbb{Z})$ we obtain a *Picard modular surface* $\Gamma \backslash X$. Picard modular surfaces parametrise 3-dimensional homogeneously polarised abelian varieties with $\mathcal{O}_E$-multiplication, signature $(1,2)$ and some level structure. For details and a complete introduction to Picard modular varieties we refer to [81, Sections 1, 2 and 3].

Part of the beauty of the theory of Shimura varieties is that many problems stated in the moduli language translate in the "(G,X)-language". Such translation often allows us to use in a more transparent way powerful group-theoretical tools and, in many cases, it is the first step towards understanding more general period domains. This point of view will be more evident in Chapter 6.

A remark regarding the difference between Shimura varieties and connected Shimura varieties is in order.

Remark 2.2.14. Already in the example of modular curves we described the connected case when we decided to consider the upper half plane $\mathbb{H}$, rather than

$$\mathbb{H}^{\pm} := \mathbb{C} - \mathbb{R},$$

with its natural GL_2-action. The main advantage of Shimura varieties, over their connected counterparts, is that their canonical models do not depend on a realisation of G as the derived group of a reductive group nor on the level structure. For example the curve $\Gamma_0(N)\backslash\mathbb{H}$ admits a (geometrically connected) model over $\mathbb{Q}(\mu_N)$, rather than over $\mathbb{Q}$.

2.3 Galois representation

Let K be a number field, $\overline{K}$ a fixed algebraic closure and denote by $\mathrm{Gal}(\overline{K}/K)$ its absolute Galois group. Being profinite, by construction, $\mathrm{Gal}(\overline{K}/K)$ naturally comes as a topological group.

Definition 2.3.1. Let $\mathbf{G}$ a topological group. A *Galois representation* of K with $\mathbf{G}$-coefficients is a continuous group homomorphism

$$\mathrm{Gal}(\overline{K}/K) \to \mathbf{G}.$$

Here $\mathbf{G}$ is usually given as $G(\mathbb{Q}_\ell)$ or $G(\mathbb{A}_f)$, for some algebraic group $G/\mathbb{Q}$. In this case the topology on $\mathbf{G}$ comes from the fact that $\mathbb{Q}_\ell$ and $\mathbb{A}_f$ have a natural structure of topological rings. The first example of Galois representation to keep in mind is given by the ℓ-adic Tate module of an elliptic curve E/K.

2.3.1 Weakly compatible systems

Given a rational prime ℓ, we denote by Σ_ℓ the set of places of K dividing ℓ. If v is a place of K we write K_v for the local field obtained completing K at v and $\mathrm{Gal}(\overline{K}_v/K_v)$ for its absolute Galois group.

Let $m \in \mathbb{N}$ be non zero. Consider a family of continuous ℓ-adic Galois representation, indexed by every rational prime ℓ

$$\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathbb{Q}_\ell)\}_\ell.$$

The definition of weakly compatible families presented is orginally due to Serre, who called them *strictly compatible* on page I-11 of the book [156]. We say that ρ_ℓ is *unramified at a place* v of K if the image of the inertia at v is trivial. If ρ_ℓ is attached to the ℓ-adic cohomology of a smooth proper variety defined over a number field, the smooth and proper base change theorems, see for example [49, I, Theorem 5.3.2 and Theorem 4.1.1], imply that ρ_ℓ is unramified at every place $v \notin \Sigma_\ell$ such that X has good reduction at v (hence at all but finitely many places).

Definition 2.3.2. A family $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathbb{Q}_\ell)\}_\ell$ is *weakly compatible* if there exists a finite set of places Σ of K such that

(i) For all ℓ, ρ_ℓ is unramified outside $\Sigma \cup \Sigma_\ell$;

(ii) For all $v \notin \Sigma \cup \Sigma_\ell$, denoting by Frob_v a Frobenius element at v, the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$ has rational coefficients and it is independent of ℓ.

Remark 2.3.3. Deligne's work on the Weil conjectures [48, Theorem 1.6], the smooth and proper base change theorems imply that the ℓ-adic representations attached to $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_\ell(j))$ form a weakly compatible system, whenever X is a smooth projective variety defined over a number field K.

2.3.1.1 A quick detour on p-adic Hodge theory

Let p be a rational prime. Galois representations coming from the cohomology of smooth projective varieties satisfy a number of constraints that are best understood when formulated in the language of p-adic Hodge theory. Indeed a deep result of Faltings, [67, Chapter III, Theorem 4.1], shows that if Y is a proper and smooth variety over a p-adic field K_v, then the cohomology groups $H^i_{\mathrm{et}}(Y \times \overline{K_v}, \mathbb{Q}_p)$ give rise to *de Rham representations*. See also [72, Theorem 5.32]. For an accessible introduction to the notions of p-adic Hodge theory (such as de Rham representations and their Hodge–Tate weights), we refer to [24] (in particular sections I.2 and II.6) and to the monograph [72]. We present here just the main definitions we need.

For the definitions of Fontaine's period rings $\mathbb{B}_{dR}, \mathbb{B}_{HT}$ and $\mathbb{B}_{st}$ we refer to [70] and [72, Definition 5.1, 5.13 and 6.10]. They are $\mathbb{Q}_p$-algebras with an action of $\mathrm{Gal}(\overline{K_v}/K_v)$.

Definition 2.3.4. Let $B \in \{\mathbb{B}_{dR}, \mathbb{B}_{HT}, \mathbb{B}_{st}\}$ a period ring and V a p-adic representation of $\mathrm{Gal}(\overline{K_v}/K_v)$. We say that V is *B-admissible* if $B \otimes_{K_v} V$ is a trivial B-representation of $\mathrm{Gal}(\overline{K_v}/K_v)$.

Definition 2.3.5. We say that a p-adic representation V of $\mathrm{Gal}(\overline{K_v}/K_v)$ is *Hodge-Tate* if it is $\mathbb{B}_{HT}$-admissible. We say that V is *de Rham* if it is $\mathbb{B}_{dR}$-admissible.

Since $\mathbb{B}_{st}^{\mathrm{Gal}(\overline{K_v}/K_v)}$ is a field [72, Proposition 6.28], it makes sense to consider the following, which is Definition 6.29. in *op. cit.*. See also Theorem 2.13. in *op. cit.* for the equivalence between various definitions of admissibility.

Definition 2.3.6. A p-adic representation V as above is *semi-stable* if and only if the dimension over $\mathbb{B}_{st}^{\mathrm{Gal}(\overline{K_v}/K_v)}$ of $\left(\mathbb{B}_{st} \otimes_{\mathbb{Q}_p} V\right)^{\mathrm{Gal}(\overline{K_v}/K_v)}$ is $\dim_{\mathbb{Q}_p} V$.

We remark here that semi-stable representations are de Rham and de Rham representations are Hodge-Tate. Each inclusion is strict.

The following is given in [71, Page 193].

Definition 2.3.7. Let K be a number field. An ℓ-adic Galois representation is called *geometric* if it is unramified outside a finite set of places S of K and its restriction to every decomposition group at v (for v ranging trough all non-archimedean places of K) is potentially semistable (i.e. it becomes semistable after a finite field extension K'/K).

2.3.2 Galois representations attached to algebraic points of Shimura varieties

We explain how to attach a Galois representation to an algebraic point of a Shimura variety. For details we refer for example to [174, 65]. Let (G,X) be a Shimura datum as in section 2.2 and suppose that G is the generic Mumford-Tate group on X and let $\widetilde{K} \subset G(\mathbb{A}_f)$ be neat. Let $x = [s, \overline{g}] \in \mathrm{Sh}_{\widetilde{K}}(G,X)_E(K)$, from [174, Section 2] we have:

1. The fibre of x along $\pi : \mathrm{Sh}(G,X) \to \mathrm{Sh}_{\widetilde{K}}(G,X)$ has a transitive and fixed-point free right action of $\widetilde{K}$;

2. $\pi^{-1}(x)$ has a left action of $\mathrm{Gal}(\overline{K}/K)$ which commutes with the action of $\widetilde{K}$ (as explained in section 2.2.2, π is defined over the reflex field).

Fixing a point $\tilde{x} \in \pi^{-1}(x)$, an elementary argument in linear algebra shows that there exists a continuous map

$$\rho_x : \mathrm{Gal}(\overline{K}/K) \to \widetilde{K},$$

such that

$$\sigma(\tilde{x}) = [s, \rho_{x,\tilde{x}}(\sigma)].$$

Projecting $\widetilde{K} \subset G(\mathbb{A}_f)$ to $G(\mathbb{Q}_\ell)$, we obtain also a Galois representation with $\mathbb{Q}_\ell$ coefficients:

$$\rho_{x,\ell} : \mathrm{Gal}(\overline{K}/K) \to G(\mathbb{Q}_\ell).$$

Finally it is interesting to describe [108, Theorem 1.2]. Informally it says that, from a Galois representation point of view, the representations attached to points of Shimura varieties enjoy most of the properties that the Tate module of an abelian variety has.

Theorem 2.3.8 (Liu, Zhu)**.** *Let $x \in \mathrm{Sh}_{\widetilde{K}}(G,X)(K)$ as above. The Galois representation*

$$\rho_{x,\ell} : \mathrm{Gal}(\overline{K}/K) \to G(\mathbb{Q}_\ell)$$

is geometric (in the sense of Definition 2.3.7).

Part of this thesis, in particular Chapter 4 and 5, is motivated by the following.

Question 2.3.9. Let K be a field as above and let $\rho : \mathrm{Gal}(\overline{K}/K) \to \widetilde{K} \subset G(\mathbb{A}_f)$ be a Galois representation. What are necessary and sufficient conditions for the existence of $x \in \mathrm{Sh}_{\widetilde{K}}(G,X)_E(K)$ such that $\rho = \rho_x$?

2.4 Motives

Let $(\mathcal{A},\otimes)$ be a $\otimes$-*category*, that is a symmetric monoidal category. We say that $(\mathcal{A},\otimes)$ is a $\otimes$-*category over* $\mathbb{Q}$ if it is a $\mathbb{Q}$-linear abelian category such that $\otimes$ is $\mathbb{Q}$-bilinear and $\mathrm{End}_{\mathcal{A}}(1) = \mathbb{Q}$. Given a category $\mathcal{A}$ we denote by $\mathcal{A}^{\mathrm{op}}$ the *opposite* category, that is the category where the object are the same as $\mathcal{A}$ and the morphisms are reversed.

Let K be a number field. We explain how the category of K-motives is constructed, for more details we refer to [3] and to [153] for a short introduction. By *nice variety* X/K we mean a smooth projective geometrically irreducible scheme of finite type over K. Let $\mathcal{V}_K$ be the category of finite disjoint unions of nice K-varieties. Given $X \in \mathcal{V}_K$ and a positive integer d we denote by $\mathcal{Z}^d(X)$ the free abelian group generated by irreducible sub-varieties of X of codimension d and by $\mathcal{A}^d(X)$ the quotient of $\mathcal{Z}^d(X) \otimes \mathbb{Q}$ by numerical equivalence[3]. Given $X, Y \in \mathcal{V}_K$, we define $\mathrm{Corr}^r(X,Y)$, the group of correspondences of degree r from X to Y as follows. Assuming for simplicity that X is purely j-dimensional, we set

$$\mathrm{Corr}^r(X,Y) := \mathcal{A}^{j+r}(X \times Y).$$

Thanks to intersection theory, we can define a composition law:

$$\mathrm{Corr}^r(X,Y) \times \mathrm{Corr}^s(Y,Z) \to \mathrm{Corr}^{r+s}(X,Z).$$

We denote by Corr_K the category whose objects are the objects of $\mathcal{V}_K$ and $\mathrm{Corr}^r(X,Y)$ as the Hom-set between X and Y. This is an additive, $\mathbb{Q}$-linear, tensor category, equipped with a tensor functor

$$h : \mathcal{V}_K^{\mathrm{op}} \to \mathrm{Corr}_K.$$

Finally we define $\mathcal{M}_K$, the category of K-motives with rational coefficients, as

[3] Other choices of an *adequate equivalence relation* are possible, see [3, Section 3.1]. We remark here that Jannsen [95] proved that the category of motives for an adequate equivalence relation is semi-simple if and only if that equivalence relation is the numerical equivalence.

the pseudo-abelian envelope of Corr_K, i.e. the category whose objects are triplets (X,i,n), where X is an object of $\mathcal{V}_K$, n is an integer and i an idempotent element of $\mathrm{Corr}^0(X,X)$ and

$$\mathrm{Hom}_{\mathcal{M}_K}((X,i,n),(Y,j,m)) := i\,\mathrm{Corr}^{m-n}(X,Y)\,j.$$

Give $X,Y \in \mathcal{V}_K$, we say that X and Y have *isomorphic motives* if $h(X) := (X,\mathrm{id}_X,0)$ and $h(Y) := (Y,\mathrm{id}_Y,0)$ are isomorphic in the category of K-motives we have just constructed.

2.4.1 Tannakian categories

In this section we follow the monograph [152] and the discussion of [51, Section 5,6 and 7]. Let $(\mathcal{A},\otimes)$ be a $\otimes$-category over $\mathbb{Q}$. We say that $(\mathcal{A},\otimes)$ is a *rigid* $\otimes$-*category over* $\mathbb{Q}$ if in addition it has an autoduality. That is there is an equivalence of categories

$$-^{\vee} : \mathcal{A} \to \mathcal{A}^{\mathrm{op}} \quad \text{such that } -^{\vee} \circ -^{\vee} \cong \mathrm{Id}_{\mathcal{A}},$$

and adjunction isomorphisms:

$$\mathrm{Hom}(-\otimes M^{\vee},+) \cong \mathrm{Hom}(-,+\otimes M), \text{ and } \mathrm{Hom}(M\otimes -,+) \cong \mathrm{Hom}(-,M^{\vee}\otimes +).$$

Definition 2.4.1. A *neutral Tannakian category over* $\mathbb{Q}$ is a rigid $\otimes$-category $(\mathcal{A},\otimes)$ over $\mathbb{Q}$ for which there exists an exact faithful $\mathbb{Q}$-linear tensor functor

$$\omega : (\mathcal{A},\otimes) \to (\mathrm{Vect}_{\mathbb{Q}},\otimes).$$

Any such functor is said to be a *fibre functor* for $(\mathcal{A},\otimes)$.

From every fibre functor ω we obtain a functor in groups:

$$\mathrm{Aut}^{\otimes}(\omega) : \mathbb{Q}-\text{algebras of finite type} \to \text{groups}, \quad R \mapsto \mathrm{Aut}^{\otimes}(\omega_R)$$

where $\omega_R : (\mathcal{A},\otimes) \to (R-\text{modules},\otimes)$ is the functor mapping M to $\omega(M)\otimes R$ and $\mathrm{Aut}^{\otimes}(\omega_R)$ denotes the group of automorphisms of the $\otimes$-functor ω_R.

Example 2.4.2. Let G be an affine $\mathbb{Q}$-group scheme and consider the category $\mathrm{Rep}_{\mathbb{Q}}(G)$ (i.e. the category of $\mathbb{Q}$-linear representation of finite dimension of G). It is a Tannakian category, where a fibre functor is the functor forgetting the G-action.

The following it the main theorem of Tannakian categories [152].

Theorem 2.4.3 (Saavedra Rivano)**.** *Let $(\mathcal{A}, \otimes)$ be a neutral Tannakian category over $\mathbb{Q}$ and $\omega : (\mathcal{A}, \otimes) \to (\mathrm{Vect}_{\mathbb{Q}}, \otimes)$ be a fibre functor. The functor* $\mathrm{Aut}^{\otimes}(\omega)$ *is representable by an affine group $G_{\mathcal{A}}$ over $\mathbb{Q}$ and the fibre functor factorises through an equivalence of category*

$$(\mathcal{A}, \otimes) \cong (\mathrm{Rep}_{\mathbb{Q}}(G_{\mathcal{A}}), \otimes).$$

In particular every neutral Tannakian category is equivalent (possibly in many different ways) to the category of finite-dimensional representations of an affine group scheme.

Grothendieck's idea of the category of Motives was that every "nice" cohomological theory should factorise through the category of motives and that motives are again governed by linear algebra. This means that a good category of motives should encode the information we can read from every cohomological theory, in a universal way. More precisely we have the following far reaching folklore conjecture.

Conjecture 2.4.4. *Motives form a neutral Tannakian category.*

In section 2.5, we will discuss how this conjecture is implied by other classical conjectures.

2.4.2 Realisations

For brevity we do not describe in details all known cohomological theories. We rather present them in an axiomatic and short way, following Deligne. In this thesis we will mainly consider the ℓ-adic/adelic, de Rham and Betti realisations.

The following is given by Deligne in [51, Section 7 (page 147)].

Definition 2.4.5. A *system of realisations* is the data (M1)–(M9), satisfying the axioms (A1)–(A5) explained below.

(M1) *Betti realisation.* A finite dimensional $\mathbb{Q}$-vector space M_B;

(M2) *De Rham realisation.* A finite dimensional $\mathbb{Q}$-vector space M_{dR};

(M3) *Adelic realisation.* An $\mathbb{A}_f$-module of finite type $M_{\mathbb{A}_f}$;

(M4) *Crystalline realisation.* For all but a finite number of primes p, a $\mathbb{Q}_p$-vector space $M_{\mathrm{cris},p}$;

(M5) *Comparisons.* Isomorphisms

$$M_B \otimes \mathbb{C} \cong M_{dR} \otimes \mathbb{C}, \quad M_B \otimes \mathbb{A}_f \cong M_{\mathbb{A}_f}, \quad M_{dR} \otimes \mathbb{Q}_p \cong M_{\mathrm{cris},p};$$

(M6) *Frobenius at infinity.* M_B has an involution F_∞;

(M7) *Hodge filtration.* M_{dR} has a descending filtration;

(M8) *Galois action.* $M_{\mathbb{A}_f}$ has a continuous $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action;

(M9) *crystalline Frobenius.* $M_{\mathrm{cris},p}$ comes with an automorphism

$$\varphi_p : M_{\mathrm{cris},p} \to M_{\mathrm{cris},p}.$$

Axioms:

(A1) M_B is a $\mathbb{Q}$-HS, once equipped with the filtration from $M_B \otimes \mathbb{C} \cong M_{dR} \otimes \mathbb{C}$;

(A2) Over $M_B \otimes \mathbb{C} \cong M_{dR} \otimes \mathbb{C}$ there are two real structures, $M_B \otimes \mathbb{R}$ and $M_{dR} \otimes \mathbb{R}$. They induce two antilinear involutions c_B and c_{dR}. We have

$$F_\infty = c_B c_{dR}.$$

That is c_{dR} respects $M_B \subset M_B \otimes \mathbb{C} \cong M_{dR} \otimes \mathbb{C}$ and c_{dR} restricted to M_B is F_∞;

(A3) There exists a finite set of places S of $\mathbb{Q}$, such that, for every ℓ, the Galois module M_ℓ is unramified outside $S \cup \{\ell\}$, where M_ℓ is the $\mathbb{Q}_\ell$-module obtained from $M_{\mathbb{A}_f}$;

(A4) For S big enough, if $p \notin S$, for all $\ell \neq p$ the eigenvalues of the geometric Frobenius at p acting on M_ℓ and the ones of φ_p acting on $M_{\text{cris},p}$ are algebraic numbers with absolute value $p^{n/2}$ and are ℓ'-units for $\ell' \neq p$;

(A5) Consider $M_B \subset M_{\mathbb{A}_f}$ and let $c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the complex conjugation. The action of c on $M_{\mathbb{A}_f}$ restricts to M_B and induces F_∞.

The category of systems of realisations from a Tannakian category. Given $X \in \mathcal{V}_{\mathbb{Q}}$ we get a system of realisations often denoted by $h(X)$, which associates to X the corresponding well known cohomology theories. Here we presented system of realisations with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, but the same works with any other number fields. Conjecture 2.4.4 wishes to describe the category of K-motives as the smallest full subcategory of the system of realisations stable under direct sum, tensor product, quotients and containing $h(X)$ for all $X \in \mathcal{V}_K$.

2.5 Some conjectures in Arithmetic Geometry relating the three worlds

Let $S = \text{Sh}_{\widetilde{K}}(G,X)$ be a Shimura variety. So far we have explained that S is canonically defined over a number field $E = E(G,X)$ and that:

- Complex points of S correspond to Hodge structures;
- Algebraic points of S correspond to Galois representation.

Moreover these structures behave in a similar fashion as if they were attached to a motive. We conclude this chapter by describing conjectural properties of motives and how they link Hodge structures to Galois representations.

2.5.1 Tate and semisimplicity

For any field F of characteristic zero, we denote by $\mathcal{M}_{K,F}$ the category of pure motives over K with coefficients in F, i.e. the category described in section 2.4, where the correspondences are tensored with F. Let

$$H_\ell : \mathcal{M}_{K,\mathbb{Q}_\ell} \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\mathrm{Gal}(\overline{K}/K))$$

be the ℓ-adic realisation functor. Tate [166, Conjecture $T^j(X)$ for every j and every X (page 72)] proposed the following.

Conjecture 2.5.1 (Tate conjecture)**.** *The functor H_ℓ is full.*

Another important conjecture is the following.

Conjecture 2.5.2 (Serre's semisimplicity conjecture)**.** *The functor H_ℓ takes values in semisimple Galois representations.*

The following is [125, Theorem 1].

Theorem 2.5.3 (Moonen)**.** *The Tate conjecture implies the semisimplicity conjecture.*

Finally the Tate conjecture implies that $\mathcal{M}_{K,F}$ is a semisimple neutral Tannakian category over $\mathbb{Q}$, in the sense of section 2.4.1. For details see for example [139, Lemma 3.2]. To fix the notation, $\mathcal{M}_{K,F}$ is equivalent to the Tannakian category $\mathrm{Rep}(\mathcal{G}_{K,F})$ for some pro-reductive group $\mathcal{G}_{K,F}$ (choosing an F-linear fibre functor). Moreover the Tate conjecture implies that numerical and homological equivalence agree (in particular the latter does not depend on the choice of a Weil cohomology theory). See also [88, Section (4), page 198]. In particular H_ℓ is a fully faithful functor.

2.5.2 Fontaine–Mazur

In order to describe the essential image of the ℓ-adic realisation functor, one needs to combine the Tate conjecture with the following, which appears as [71, Conjecture 1].

Conjecture 2.5.4 (Fontaine–Mazur)**.** *An irreducible ℓ-adic Galois representation is geometric, in the sense of Definition 2.3.7, if and only if it comes from geometry.*

We fix a family of embeddings $\iota_\ell : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$, for any $F \subset \overline{\mathbb{Q}}$ we write

$$H_{\iota_\ell} : \mathcal{M}_{K,F} \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(\overline{K}/K))$$

for the ℓ-adic realisation functor associated to ι_ℓ.

More precisely, we have the following is [139, Lemma 3.3].

Theorem 2.5.5. *Assume the Tate and the Fontaine–Mazur conjecture. Let* $r_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathbb{Q}_\ell)$ *be an irreducible geometric Galois representation. Then there exists an object* $M \in \mathcal{M}_{K,\overline{\mathbb{Q}}}$ *such that*

$$r_\ell \otimes \overline{\mathbb{Q}}_\ell \cong H_{\iota_\ell}(M) \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(\overline{K}/K)).$$

Theorem 2.3.8 and the Fontaine-Mazur conjecture predict that every point of a Shimura variety is attached to a motive (see also [106, Section 4] for more details).

2.5.3 Hodge

For a complete and precise description of the Hodge conjecture we refer to Deligne's exposition [52]. Hodge [93] conjectured that, on a complex smooth projective algebraic variety, any Hodge class is a rational linear combination of classes $\mathrm{cl}(Z)$ of algebraic cycles. We present here an equivalent conjecture, stated in more motivical terms. Let

$$H_B : \mathcal{M}_{\mathbb{C},\mathbb{Q}} \to \mathbb{Q} - HS$$

be the Betti realisation functor from the category of $\mathbb{C}$-motives with $\mathbb{Q}$-coefficients to the category of $\mathbb{Q}$-Hodge structures.

Conjecture 2.5.6 (Hodge conjecture)**.** *The functor* H_B *is full.*

Together, the Tate and the Hodge conjecture imply another famous conjecture, namely the *Mumford–Tate conjecture*. Its statement for abelian varieties is as follows. Let $K \subset \mathbb{C}$ be a number field, A/K be an abelian variety, $M/\mathbb{Q}$ be the Mumford–Tate group attached to the natural Hodge structure on $H^1(A(\mathbb{C}),\mathbb{Q})$ and G_ℓ be the connected component of the identity of the Zariski closure of the image of

$$\mathrm{Gal}(\overline{K}/K) \to H^1(A_{\overline{K}}, \mathbb{Q}_\ell).$$

Under the comparison isomorphism $H^1(A_{\overline{K}}, \mathbb{Q}_\ell) \cong H^1(A(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell$, the Mumford–Tate conjecture asserts that $G_\ell \cong M \otimes \mathbb{Q}_\ell$.

Remark 2.5.7. Interestingly every representation coming from a rational point of a Shimura variety satisfies half of the Mumford–Tate conjecture. That is given $x \in \mathrm{Sh}_{\widetilde{K}}(G,X)_E(K)$ and $\rho_x : \mathrm{Gal}(\overline{K}/K) \to G(\mathbb{Q}_\ell)$ as in section 2.3.2. Then the image of ρ_x is contained in the $\mathbb{Q}_\ell$-points of the Mumford–Tate group of the $\mathbb{Q}$-HS naturally attached to x. For more details see [174, Proposition 2.9].

2.5.3.1 Absolute Hodge

We only give a brief overview of the category of absolute Hodge cycles. For more details we refer to section 6 of Deligne-Milne's paper in [53], where Deligne's category of *absolute motives* is described. Let $X, Y/\mathbb{C}$ be smooth projective variety. A morphism of Hodge structures between their Betti cohomology groups corresponds to a Hodge class in the cohomology of $X \times Y$:

$$\mathrm{Hom}(H_B^*(X,\mathbb{Q}), H_B^*(Y,\mathbb{Q})) \cong H_B^{2*}(X \times Y, \mathbb{Q}).$$

We say that a Hodge class $\alpha \in H_B^{2i}(X \times Y, \mathbb{Q})$, or a morphism of Hodge structures between their H^i's, is *absolute Hodge* if, for every automorphism σ of $\mathbb{C}$, the class $\alpha^\sigma \in H^{2i}(X^\sigma \times Y^\sigma, \mathbb{C})$ is again a Hodge class. With such definition we can split the Hodge conjecture 2.5.6 in two parts:

Hodge classes = Absolute Hodge classes = Algebraic cycles.

A special case of the absolute Hodge conjecture that we will assume in Chapter 5 is the following.

Conjecture 2.5.8. *If $X, Y/\mathbb{C}$ are smooth projective complex varieties such that, for some i, we have an isomorphism of Hodge structures*

$$H_B^i(X,\mathbb{Q}) \cong H_B^i(Y,\mathbb{Q}),$$

then there exists an absolute Hodge class inducing such isomorphism.

2.5.4 Simpson

Finally we present a conjecture of Simpson [161, 162], predicting that certain VHS come from geometry, in particular implying that they admit an integral structure (see indeed [161, Conjecture 5]). For simplicity assume that S is a smooth projective variety (the quasi-projective case will be described in Chapter 7). Recall that a representation of the topological fundamental group of S is called *rigid* if any nearby representation is conjugate to it. We have the following.

Conjecture 2.5.9 (Simpson)**.** *Suppose ρ is a rigid semisimple representation of $\pi_1(X)$. Then ρ is a direct factor in the monodromy representation of a motive (i.e. family of varieties) over S.*

Example 2.5.10. Let Γ be a discrete subgroup of the classical group $G = \mathrm{PU}(1,n)$. If Γ is an arithmetic subgroup of G, it is a well know fact that the representation

$$\pi_1(\Gamma \backslash \mathbb{B}^n) \to G$$

is induced by a $\mathbb{Z}$-VHS corresponding to a family of principally polarised abelian varieties. See indeed Example 2.2.13. In chapter 7, we will investigate what happens for arbitrary smooth quasi-projective varieties whose universal covering is the complex unit ball $\mathbb{B}^n \subset \mathbb{C}^n$.

Chapter 3

On a conjecture of Buium and Poonen

Given a correspondence between a modular curve S and an elliptic curve A, we prove that the intersection of any finite-rank subgroup of A with the set of points on A corresponding to an isogeny class on S is finite. The question was proposed by A. Buium and B. Poonen in 2009. We follow the strategy proposed by the authors, using a result about the equidistribution of Hecke points on Shimura varieties and Serre's open image theorem. At the end of the chapter we show that the result is an instance of the Zilber–Pink conjecture for mixed Shimura varieties. The work presented here appeared in the paper [11].

3.1 Introduction

A. Buium and B. Poonen [25] studied the problem of independence of points on elliptic curves arising from special points on modular and Shimura curves. As a first approximation the problem can be described as follows. Let $S/\overline{\mathbb{Q}}$ be a modular curve, $A/\overline{\mathbb{Q}}$ an elliptic curve and $\Gamma_0 \leq A(\overline{\mathbb{Q}})$ a finitely generated subgroup. Let

$$\Psi : S \longrightarrow A/\overline{\mathbb{Q}}$$

be a (non-constant) morphism and $\mathrm{CM} \subset S(\overline{\mathbb{Q}})$ be the set of special points of S, i.e. the points corresponding to elliptic curves with complex multiplication (also referred to as CM-elliptic curves). One can ask whether the following are true:

(1) André-Oort-Manin-Mumford: $\Psi(\mathrm{CM}) \cap A_{\mathrm{tors}}$ is finite;

(2) André-Oort-Mordell-Lang: $\Psi(\mathrm{CM}) \cap \Gamma_0$ is finite.

The first statement is an easy consequence of the *André-Oort conjecture* for $S \times A$. We recall the shape of the André-Oort conjecture (AO from now on) for products of a modular curve and an elliptic curve. This is a particular case of a theorem of Pila ([142, Theorem 1.1], see also [144]).

Theorem 3.1.1 (André-Oort-Manin-Mumford)**.** *Let S be a modular curve, A an elliptic curve and consider their product $T := S \times A$. A point $(s,a) \in T$ is said to be special if $s \in \mathrm{CM}$ and $a \in A_{\mathrm{tors}}$. The only irreducible closed subvarieties of T containing a Zariski dense set of special points are: $\{\mathrm{CM}-point\} \times \{torsion\, point\}$, $S \times \{torsion\, point\}$, $\{\mathrm{CM}-point\} \times A$, $S \times A$.*

It is interesting to notice that (1), together with the modularity theorem of Wiles, Breuil, Conrad, Diamond, Taylor dicussed in equation (1.2.1), implies that there are only finitely many torsion Heegner points on any elliptic curve over $\mathbb{Q}$ (first proven in [132]). For a complete discussion about this, we refer to [25, Section 1.2] and the references therein.

Statement (2) is true because there are only finitely many classes of $\overline{\mathbb{Q}}$-isomorphic CM-abelian varieties of a given dimension defined over a given number field. For example, in the case of elliptic curves, it is a classical result in the theory of complex multiplication that the set of CM-points of $X_1(N)$, defined over a given number field, is finite.

A. Buium and B. Poonen [25, Theorem 1.1] were able to deal with finitely generated subgroups and torsion points simultaneously. The main theorem they discuss is as follows.

Theorem 3.1.2 (Buium-Poonen)**.** *Let $A/\overline{\mathbb{Q}}$ be an elliptic curve, $\Psi : X_1(N) \to A$ be a non-constant morphism defined over $\overline{\mathbb{Q}}$. Let $\Gamma \leq A(\overline{\mathbb{Q}})$ be a finite rank subgroup, i.e. the division hull of a finitely generated subgroup $\Gamma_0 \leq A(\overline{\mathbb{Q}})$. Then $\Psi(\mathrm{CM}) \cap \Gamma$ is finite.*

Since the proof is very elegant and our results will follow a similar path, we present here the main points of the strategy. It relies on two deep results from equidistribution theory due to Zhang (namely [184, Corollary 3.3] and [183, Theorem 1.1]) and the Brauer-Siegel theorem. The first result about equidistribution of Galois orbits of CM-points on modular curves was established by Duke [60].

Strategy of the proof. Let μ_S be the hyperbolic measure on $S(\mathbb{C})$ and μ_A be the normalised Haar measure on $A(\mathbb{C})$. Assume that S, A, Ψ are defined over a number field K and that Γ is contained in the division hull of $A(K)$. We have three main facts preventing the existence of infinitely many points in $\Psi(\mathrm{CM}) \cap \Gamma$:

- Let $(x_n)_n$ be an infinite sequence of CM-points in $S(\overline{\mathbb{Q}})$, then the uniform probability measure on the $\mathrm{Gal}(\overline{K}/K)$-orbit of x_n weakly converges, as $n \to \infty$, to the measure μ_S;
- Let $(a_n)_n$ be an infinite sequence of almost division points relative to K, i.e.

$$\lim_{n\to\infty} \sup_{\sigma\in \mathrm{Gal}(\overline{K}/K)} ||a_n^\sigma - a_n|| = 0,$$

  such that $[K(a_n) : K] \to \infty$. Then the uniform probability measure on the $\mathrm{Gal}(\overline{K}/K)$-orbit of a_n weakly converges to the measure μ_A;
- Some measure theoretic lemmas [25, Lemma 3.1, 3.2, 3.3] preventing this.

□

Remark 3.1.3. Such proof allows also to fatten Γ in the following sense. Let $\varepsilon > 0$, we may replace Γ by $\Gamma_\varepsilon := \Gamma + A_\varepsilon$, where A_ε is a set of points of small Néron-Tate height, i.e. $A_\varepsilon := \{a \in A(\overline{\mathbb{Q}}) \text{ such that } h(a) \leq \varepsilon\}$. The theorem then asserts that, for some $\varepsilon > 0$, the set $\Psi(\mathrm{CM}) \cap \Gamma_\varepsilon$ is finite, see [25, Theorem 2.3]. See also [150], where B. Poonen strengthen the Mordell-Lang conjecture by fattening Γ in this way.

3.1.1 Main results

In the subsequent work, Buium and Poonen [26], motivated by some local results involving the theory of arithmetic differential equations, conjectured that the same results hold when the CM-points are replaced by isogeny classes. Recall that a non-cuspidal point $x \in X_1(N)$, defined over $\overline{\mathbb{Q}}$, corresponds to an elliptic curve $E_x/\overline{\mathbb{Q}}$ (with some extra structure) and its isogeny class is defined as the subset of $X_1(N)(\overline{\mathbb{Q}})$ given by the elliptic curves admitting a $\overline{\mathbb{Q}}$-isogeny to E_x (see 3.2.1 for more about the definition). The following is [26, Conjecture 1.7].

Theorem 3.1.4. *Let A be an elliptic curve defined over $\overline{\mathbb{Q}}$ and $\Gamma \leq A(\overline{\mathbb{Q}})$ be a finite rank subgroup. Let $x \in X_1(N)(\overline{\mathbb{Q}})$ be a non-cuspidal point and Σ_x be its isogeny class. Let $X \subset X_1(N) \times A$ be a irreducible closed $\overline{\mathbb{Q}}$-subvariety such that $X(\overline{\mathbb{Q}}) \cap (\Sigma_x \times \Gamma)$ is Zariski dense in X, then X is one of the following: $\{point\} \times \{point\}$, $X_1(N) \times \{point\}$, $\{point\} \times A$, $X_1(N) \times A$.*

The aim of this chapter is to prove Theorem 3.1.4, as a special case of the following more general result.

Theorem 3.1.5. *Let A, Γ, x, Σ_x be as in Theorem 3.1.4. If $X \subset X_1(N) \times A$ is an irreducible closed $\overline{\mathbb{Q}}$-subvariety such that $X(\overline{\mathbb{Q}}) \cap (\Sigma_x \times \Gamma_\varepsilon)$ is Zariski dense in X for every $\varepsilon > 0$, then X is one the following: $\{point\} \times \{point\}$, $X_1(N) \times \{point\}$, $\{point\} \times A$, $X_1(N) \times A$.*

Remark 3.1.6. In particular taking X in Theorem 3.1.5 to be the graph of a non-constant $\overline{\mathbb{Q}}$-morphism $\Psi : X_1(N) \to A$, we get a result analogous to Theorem 3.1.2. Namely we have that, for some $\varepsilon > 0$, the image of Σ_x along Ψ meets Γ_ε in only finitely many points.

Theorem 3.1.4 may be thought as an André-Pink-Mordell-Lang conjecture, as will be discussed in section 3.4. See also [137] for more about the André-Pink conjecture. It is worth noticing that this conjecture appears here in the form of [65, Theorem B] and [146, Theorem 7.6.].

Our approach follows the strategy of Buium-Poonen presented above, using a equidistribution result about Hecke points in place of Duke-Zhang's equidistribution of CM-points on modular curves and Serre's open image theorem for elliptic curves without complex multiplication. The equidistribution result follows from the work of Clozel, Eskin, Oh and Ullmo and it is described in section 3.2.2. Notice that it holds for arbitrary Shimura varieties, in particular it is possible to obtain a result analogue to Theorem 3.1.5 for the isogeny class of Galois generic points in higher dimensional Shimura varieties. We remark that, even if the André-Oort conjecture for $\mathcal{A}_g$ is now a theorem, the equidistribution conjecture for Galois orbits of CM-points is still unsolved. Another advantage of such approach is that it could be applied also replacing A by a torus $\mathbb{G}_m^n/\overline{\mathbb{Q}}$, employing Bilu's result [18] on the equidistribution of small points on algebraic tori, in place of [183].

3.1.2 Related work

Another fruitful approach for problems like the ones discussed in this chapter is to use o-minimality and the Pila-Zannier strategy. This approach relies on the Pila-Wilkie counting theorem and it was used to prove both the Manin-Mumford and the André-Oort conjecture. For example Z. Gao, in [76], obtained important results towards what he calls the André-Pink-Zannier conjecture. After the paper [11] was written, it was pointed out to the author that G. Dill employed such strategy to obtain results about unlikely intersections between isogeny orbits and curves similar to the ones presented here. Dill's progress on a modification of the André-Pink-Zannier conjecture ([59, Conjecture 1.1]) implies indeed Theorem 3.1.1 as well as Theorem 3.1.4 (see [59, Corollary 1.4]). More recently Pila and Tsimerman [143, Theorem 1.5] have also found an improvement of such results. They can indeed bound the intersection $\Psi(\mathrm{CM}) \cap \Gamma$ by a number depending only on the rank of Γ. For more about this see the beginning of section 3.4.1.2.

As explained above, our strategy does not rely on o-minimality at any point. The proofs obtained here are quite short but are confined to isogeny classes of Galois generic points. An advantage of the *equidistributional approach* is that it allows to fatten Γ, by adding points of small Néron-Tate height (as in Theorem 3.1.5). Finally notice that our result does not invoke Masser-Wüstholz Isogeny Theorem, as it often happens in results regarding isogeny classes (see 3.3.4 for a more detailed discussion about this).

Outline of chapter

In the first section we discuss Hecke orbits on modular curves and formulate the equidistribution result of Clozel, Eskin, Oh and Ullmo in the form needed for the main theorem. In the second section we prove the conjecture (Corollary 3.3.3). We end section 3.3 showing how to obtain an analogous statement for quaternionic Shimura curves $X^D(\mathcal{U})$ (Theorem 3.3.8) and discuss in more details the equidistribution of Hecke points on higher dimensional Shimura varieties. We finally present some conjectures about unlikely intersection for a product of a Shimura variety and an abelian variety, inspired by the theorems presented so far. Eventually we prove that they follow from the Zilber–Pink conjecture (Proposition 3.4.9).

3.2 Preliminaries

Let Λ be a neat congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, as in Example 2.2.8 and $X^+ = \mathbb{H}$ be the upper half plane, coming with the action of $\mathrm{SL}_2(\mathbb{Z})$ by fractional linear transformations. For the purpose of the chapter, we may assume Λ to be one of $\Gamma(N), \Gamma_0(N), \Gamma_1(N)$ (and $N > 3$). A (non-compact) modular curve is a Riemann surface of the form

$$S_\Lambda := \Lambda \backslash X^+.$$

Since in this chapter we are interested in maps from modular curves to elliptic curves (which are compact), it is natural to identify the above quotients as the non-cuspidal locus in their Alexandroff compactifications. The compactifications obtained from the choices of Λ mentioned above are denoted by $X(N), X_0(N), X_1(N)$. They can be written as an opportune quotient of

$$\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}),$$

and we denote by ∞_{S_Λ} the projection of $\infty \in \mathbb{P}^1(\mathbb{Q}) \subset \mathbb{H}^*$ onto the compactification of S_Λ.

Points on such complex curves naturally correspond to complex elliptic curves (with some Λ-structure). Using the moduli interpretation one can show that modular curves are naturally defined over a number field. Let $K \subset \mathbb{C}$ be a field such that S_Λ is defined over K and $S_\Lambda(K) \neq \emptyset$. A K-point of a modular curve corresponds to an elliptic curve defined over K. We usually write E_x/K for the elliptic curve associated to $x \in S_\Lambda(K)$.

3.2.1 Hecke operators

For every $a \in \mathrm{SL}_2(\mathbb{Q})$, consider the diagram of (Shimura) coverings

$$\Lambda\backslash X^+ \xleftarrow{\text{pr}} (\Lambda \cap a^{-1}\Lambda a)\backslash X^+ \xrightarrow{a\cdot} \Lambda\backslash X^+.$$

It induces a finite correspondence, called *Hecke operator*

$$T_a : S_\Lambda \longrightarrow S_\Lambda.$$

The Hecke operator T_a maps a point $x \in S_\Lambda$ to the finite set

$$\{a\lambda x \mid \lambda \in (\Lambda \cap a^{-1}\Lambda a)\backslash\Lambda\}.$$

Remark 3.2.1. Since Λ is neat, for all $a, b \in \mathrm{SL}_2(\mathbb{Q})$ the following are equivalent:

- $T_a(x) \cap T_b(x) \neq \emptyset$;
- $T_a(x) = T_b(x)$;
- $\Lambda a\Lambda = \Lambda b\Lambda$.

We set

$$\deg_\Lambda(a) := |\Lambda a\Lambda/\Lambda| = [\Lambda : a^{-1}\Lambda a \cap \Lambda].$$

Given a point $x \in S_\Lambda$ we denote by $T(x)$ its *Hecke orbit*:

$$T(x) := \bigcup_{a \in \mathrm{SL}_2(\mathbb{Q})} T_a(x) \subset S_\Lambda.$$

The Hecke operator T_a also acts on functions f on S_Λ by

$$T_a f(x) := \frac{1}{\deg_\Lambda(a)} \cdot \sum_{s \in T_a(x)} f(s).$$

Example 3.2.2. Let $x \in X_0(N)$, corresponding to a pair (E_x, Ψ_x), where E_x is an elliptic curve and Ψ_x is a $\Gamma_0(N)$-level structure (i.e. subgroup of order N of E_x). Given a prime p, not dividing N, the Hecke operator T_p applied to x gives

$$T_p(x) = T_p(E_x, \Psi_x) = \bigcup_C (E_x/C, (\Psi_x + C)/C)$$

where the union is over all the subgroups $C \subset E_x$ of cardinality p.

3.2.1.1 Hecke orbits and isogeny classes

Let $S = X_1(N)$ and $x \in S(\overline{\mathbb{Q}})$ be a non-cuspidal point. It may be represented as a pair (E_x, P_x), where E_x is an elliptic curve defined over $\overline{\mathbb{Q}}$ and P_x is a point of order N. We set

$$\Sigma_x^{X_1(N)} := \{(E, P) \text{ such that there exists an isogeny between } E \text{ and } E_x\} \subset S(\overline{\mathbb{Q}}). \tag{3.2.1}$$

Otherwise stated, we are looking at elliptic curves isogenous to E and the isogeny is not required to respect the points of order N. This is the notion of isogeny class appearing in Theorem 3.1.4.

Consider $\mathcal{A}_1$ the modular curve parametrizing elliptic curves. It is easy to see that $T(x) = \Sigma_x^{\mathcal{A}_1}$ for any $x \in \mathcal{A}_1(\overline{\mathbb{Q}})$. By forgetting the point of order N, there is a finite Shimura morphism associated to the same Shimura datum

$$\pi : X_1(N) \longrightarrow \mathcal{A}_1.$$

In particular the preimage of an $\mathcal{A}_1$-Hecke orbit can be written as a finite union of

Hecke orbits in $X_1(N)$, in symbols

$$\Sigma_x^{X_1(N)} = \bigcup_{i=1}^{m} T(x_i). \tag{3.2.2}$$

This will be the main step in the deduction of Corollary 3.3.3 from Theorem 3.3.1.

3.2.1.2 Hecke orbits and strictly Galois generic points

Let K be a number field and x a non-cuspidal K-point in S_Λ. As recalled in section 2.3.2, to such x there is a corresponding Galois representation

$$\rho_x : \mathrm{Gal}(\overline{K}/K) \to \overline{\Lambda} \subset \mathrm{GL}_2(\mathbb{A}_f),$$

where $\overline{\Lambda}$ denotes the closure of Λ in $\mathrm{GL}_2(\mathbb{A}_f)$. In terms of the associated elliptic curve E_x, ρ_x is nothing but the representation coming from the inverse limit of the Galois modules $E_x[n]$. As in [146, Definition 6.3], we have:

- x is called *Hodge generic/non-special* if the elliptic curve E_x is not CM;
- x is called *Galois generic* if $\mathrm{Im}(\rho_x)$ is open in $\overline{\Lambda}$;
- x is called *strictly Galois generic* if $\mathrm{Im}(\rho_x)$ is equal to $\overline{\Lambda}$.

Let $x \in X_1(N)$ be a non-special non-cuspidal point defined over a number field. As recalled in the sections 3.1 and 1.2, Serre's open image theorem asserts that $\mathrm{Im}(\rho_x)$ is open in $\overline{\Lambda}$, i.e. that x is Galois generic. See [154, 156] for Serre's proof.

Remark 3.2.3. We remark here that the difference between Galois generic and strictly Galois generic points is not important in this chapter. Indeed let $x \in S_\Lambda$ a Galois generic point, we may shrink Λ in such a way that x lifts along $\pi : S_{\Lambda'} \to S_\Lambda$ and becomes strictly Galois generic in $S_{\Lambda'}$.

It is easy to see that the Hecke orbit $T_a(x)$ of a strictly Galois generic point x is permuted transitively by $\mathrm{Gal}(\overline{K}/K)$. See for example [146, Proposition 6.6]. In particular

$$\forall a \in \mathrm{SL}_2(\mathbb{Q}) \text{ and } s \in T_a(x), \quad \deg_\Lambda(a) = [K(s) : K].$$

3.2.2 Equidistribution of Hecke points

Let $S = S_\Lambda$ be a modular curve and write μ_S for the hyperbolic measure on $S(\mathbb{C})$. If we fix coordinates (x,y) in $\mathbb{H}$, as recalled in example 2.2.8, the measure μ_S is the measure whose pullback to $\mathbb{H}$ equals a multiple of the hyperbolic measure $y^{-2}\mathrm{d}x\mathrm{d}y$. Given a point p in S we denote by δ_p the Dirac distribution at p.

This is the main result of the section.

Theorem 3.2.4. *Let $x \in S$ be a strictly Galois generic point defined over a number field K. Let $(a_n)_n \subset \mathrm{SL}_2(\mathbb{Q})$ be an arbitrary sequence and fix $s_n \in T_{a_n}(x)$ for every n. We have that*

$$T_{a_n}(x) = \mathrm{Gal}(\overline{K}/K)s_n.$$

Moreover, if the cardinality of $\{s_n\}_n$ is not finite then $[K(s_n) : K] \to +\infty$ and the sequence of measures

$$\Delta_{T_{a_n}(x)} := \frac{1}{|\mathrm{Gal}(\overline{K}/K)s_n|} \sum_{p \in \mathrm{Gal}(\overline{K}/K)s_n} \delta_p$$

weakly converges to μ_S as $n \to +\infty$.

Proof. As explained in section 3.2.1.2, since x is a strictly Galois generic point, the Hecke orbit coincides with the Galois orbit. In particular we have

$$\deg_\Lambda(a_n) = [K(s_n) : K],$$

and an equality of measures

$$\frac{1}{\deg_\Lambda(a_n)} \sum_{\lambda \in (\Lambda \cap a_n^{-1}\Lambda a_n)\backslash\Lambda} \delta_\lambda = \frac{1}{|\mathrm{Gal}(\overline{K}/K)s_n|} \sum_{p \in \mathrm{Gal}(\overline{K}/K)s_n} \delta_p.$$

From the former equation we see that $\deg_\Lambda(a_n)$ goes to infinity if and only if $[K(s_n) : K]$ does. The results of [36], together with the existence of infinitely distinct s_n, imply that the $\deg_\Lambda(a_n) \to +\infty$ and the desired weakly convergence of measures (see Theorems 3.3.5 and 3.3.6 for the general statements we are referring to)[1]. □

[1] As the reader may have noticed, Masser-Wüstholz Isogeny Theorem shows that the existence of

3.3 Proofs of the main results

Throughout this section we fix an elliptic curve A defined over $\overline{\mathbb{Q}}$ and a subgroup $\Gamma \leq A(\overline{\mathbb{Q}})$ of finite rank. Let

$$h : A(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0}$$

be a canonical height function attached to some symmetric ample line bundle on A and, for every $\varepsilon \geq 0$, let

$$\Gamma_\varepsilon := \{\gamma + a \text{ such that } \gamma \in \Gamma,\ a \in A(\overline{\mathbb{Q}}),\ h(a) \leq \varepsilon\}.$$

By $S = S_\Lambda$ we denote a modular curve as in the previous section.

Theorem 3.3.1. *Let x be Galois generic point of S and $(a_n)_n$ be an arbitrary sequence in $\mathrm{SL}_2(\mathbb{Q})$. Let $X \subset S \times A$ be an irreducible closed $\overline{\mathbb{Q}}$-subvariety which is not of the form $S \times \{point\}$, $\{point\} \times A$, $S \times A$. For some $\varepsilon > 0$, $X(\overline{\mathbb{Q}})$ contains only finitely many points lying in $(\bigcup_n T_{a_n}(x)) \times \Gamma_\varepsilon$.*

Remark 3.3.2. In particular, by listing all the elements of $\mathrm{SL}_2(\mathbb{Q})$, we have also that

$$X(\overline{\mathbb{Q}}) \cap (T(x) \times \Gamma_\varepsilon)$$

is finite, where $T(x)$ is defined as $\bigcup_{g \in \mathrm{SL}_2(\mathbb{Q})} T_g(x)$.

Regarding isogeny classes, as in section 3.2.1.1, and in the direction of Theorem 3.1.5 we obtain the following.

Corollary 3.3.3. *Suppose S is the modular curve $X_1(N)$ over $\overline{\mathbb{Q}}$. Let A, Γ, X be as in Theorem 3.3.1, and x be a non-cuspidal $\overline{\mathbb{Q}}$-point of $X_1(N)$. For some $\varepsilon > 0$, $X(\overline{\mathbb{Q}})$ contains only finitely many points lying in $\Sigma_x^{X_1(N)} \times \Gamma_\varepsilon$.*

Proof of Corollary 3.3.3. Let E_x be the elliptic curve (with some extra structure) corresponding to x. A non-cuspidal point in a modular curve is either special or

infinitely many distinct (s_n) forces the degree to grow (as explained in section 3.3.4). However we are showing this by invoking an equidistribution results which does not rely on the Isogeny Theorem and holds for arbitrary Shimura varieties.

Hodge generic. In terms of the endomorphisms ring of E_x this means that $\mathrm{End}(E_x)\otimes\mathbb{Q}$ is either a quadratic imaginary field, or the field of rational numbers. In the former case the corollary follows from Theorem 3.1.2. Indeed elliptic curves isogenous to a CM-elliptic curve are again CM and therefore the set $\Sigma_x^{X_1(N)}$ is contained in the set of special points of S, so

$$X(\overline{\mathbb{Q}})\cap\left(\Sigma_x^{X_1(N)}\times\Gamma_\varepsilon\right)\subset X(\overline{\mathbb{Q}})\cap(\mathrm{CM}\times\Gamma_\varepsilon)\,.$$

Theorem 3.1.2, in the more general form of [25, Theorem 2.3], shows precisely that the right hand side is finite (for some $\varepsilon>0$).

Suppose now that x is Hodge generic. Serre's open image theorem implies that x is Galois generic. The result then follows from Theorem 3.3.1 since $\Sigma_x^{X_1(N)}$ is a finite union of Hecke orbits (as explained in section 3.2.1, in particular 3.2.1.1). □

3.3.1 Proof of Theorem 3.3.1

In the statement of Theorem 3.3.1 x is assumed to be Galois generic. As explained in the remark of section 3.2.1.2, we may and do assume that x is *strictly Galois generic*. Indeed there exist Λ' and $x_{\Lambda'}\in S_{\Lambda'}$ such that

$$\pi: S_{\Lambda'}\longrightarrow S$$

maps $x_{\Lambda'}$ to x and $x_{\Lambda'}$ is strictly Galois generic. Since π is a finite map, we may replace X by an $X'\subset S_{\Lambda'}\times A$ which projects onto $X\subset S\times A$ and the validity of the result does not change.

Denote by μ_S the hyperbolic measure on $S(\mathbb{C})$ and by μ_A the normalised Haar probability measure on $A(\mathbb{C})$. Define B_r to be the open disk in $S(\mathbb{C})$ with center ∞_S and radius r with respect to the metric. Lemma 3.3 in [25] shows that μ_S blows up relative to the Riemannian metric near the cusp ∞_S. Using also [25, Lemma 3.1] we can choose a compact annulus $C\subset B_r-\{\infty_S\}$ such that

$$\mu_S(C)>\mu_A(\Psi(B_r-\{\infty_S\})),\tag{3.3.1}$$

for more details see also [25, pp.6, last but second paragraph]. From now on we fix such a C.

Finally we say that a sequence $(y_n)_n$ in a scheme X is *generic* if it converges to the generic point of X with respect to the Zariski topology, i.e. each proper subvariety of X contains at most finitely many y_n.

Proof of Theorem 3.3.1. Of course if the set $\bigcup_n T_{a_n}(x)$ is finite the theorem trivially holds true. Therefore we may and do assume that the set

$$\Sigma_x^{(a_n)} := \bigcup_n T_{a_n}(x)$$

is infinite.

Heading for a contradiction let us suppose that $X(\overline{\mathbb{Q}}) \cap (\Sigma_x^{(a_n)} \times \Gamma_\varepsilon)$ is Zariski dense in X for every $\varepsilon > 0$. Since X has only countably many subvarieties, we may choose a generic sequence of points $y_n = (s_n, \gamma_n) \in X(\overline{\mathbb{Q}})$ with $s_n \in \Sigma_x^{(a_n)}$ and $\gamma_n \in \Gamma_{\varepsilon_n}$ where $\varepsilon_n \to 0$. In particular, each s_n appears only finitely often.

Up to enlarging the base field, we may assume that A, S, X, x are all defined over a number field K and that Γ is contained in the division hull of $A(K)$. Theorem 3.2.4 implies that $[K(s_n) : K] \to +\infty$. Since, by assumption, X surjects onto S and A and $X \neq S \times A$ we have that the projection $X \to A$ is generically finite, say of degree d. Since $[K(s_n) : K] \leq d[K(\gamma_n) : K]$, we have also that $[K(\gamma_n) : K] \to +\infty$ (as n goes to infinity). The γ_ns form a sequence of almost division points relative to K in the sense of [183] and, by passing to a subsequence, we may assume that they admit a coherent limit. Moreover, as $\dim A = 1$, the only possibility for the coherent limit of the γ_ns is $(A, \{0\})$.

The combination of the next two facts implies the contradiction we were aiming for:

- As explained in Theorem 3.2.4, the uniform probability measure associated to the points $\mathrm{Gal}(\overline{K}/K)s_n$ weakly converges to μ_S on $S(\mathbb{C})$, i.e.

$$\left(\frac{1}{|\mathrm{Gal}(\overline{K}/K)s_n|}\sum_{p\in\mathrm{Gal}(\overline{K}/K)s_n}\delta_p\right)\longrightarrow \mu_S, \text{ as } n\to+\infty;$$

- Zhang's result, [183, Theorem 1.1], implies that the uniform probability measure on $\mathrm{Gal}(\overline{K}/K)\gamma_n$, as $n\to+\infty$, weakly converges to the Haar measure μ_A on $A(\mathbb{C})$.

In particular, arguing as in [25, pp.7, first paragraph], they imply that

$$\mu_S(C)\le\mu_A(\Psi(B_r-\{\infty_S\})),$$

contradicting the choice of C in 3.3.1. The theorem is eventually proven. □

3.3.2 Shimura varieties

We now discuss the more general case of Shimura varieties, using the notation of section 2.2. Let G be an almost $\mathbb{Q}$-simple group, (G,X^+) be a connected Shimura datum and Λ an arithmetic subgroup of $G(\mathbb{Q})_+$. In this section we present the general setting for arbitrary connected Shimura varieties

$$S_\Lambda := \Lambda\backslash X^+.$$

Remark 3.3.4. The definitions presented in section 3.2 naturally generalise to arbitrary connected Shimura data (G,X^+). Notice that there is a more general notion of *generalized Hecke orbit* which takes into account non-inner automorphisms of (G,X), see [146, Definition 3.1]. This generalisation does not substantially change the content of the chapter. Indeed, when the group G is of adjoint type, the quotient

$$\mathrm{Aut}(G,X^+)/G(\mathbb{Q})^+$$

is finite.

The main theorem about equidistribution of Hecke points (after Clozel, Eskin, Oh and Ullmo) is the following.

Theorem 3.3.5. *Let $(a_n)_n \subset G(\mathbb{Q})_+$ be an arbitrary sequence of points and $x \in \Lambda\backslash X^+$. Exactly one of the following happens:*

1. *The set $\bigcup_n T_{a_n}(x)$ is finite and $\deg_\Lambda(a_n)$ is bounded;*

2. *The set $\bigcup_n T_{a_n}(x)$ is Zariski dense in S and the sequence of measures $\Delta_{T_{a_n}(x)}$ weakly converges to the canonical Haar measure on S. Where we set*

$$\Delta_{T_{a_n}(x)} := \frac{1}{\deg_\Lambda(a_n)} \sum_{\lambda \in (\Lambda \cap a^{-1}\Lambda a)\backslash\Lambda} \delta_\lambda$$

and δ_λ denotes the Dirac distribution at λ.

Proof. See [65, Corollary 7.2.3], which follows from [65, Theorem 7.2.2]. In [65, Section 9.1] it is also explained how the result can be deduced from [63] (using [63, Proposition 2.1]). When $\Lambda = \mathrm{GSp}_{2g}(\mathbb{Z})$, see also [146, Theorem 7.5], which builds on [36]. □

Theorem 3.3.5 implies the next result.

Theorem 3.3.6. *Let S_Λ be a connected Shimura variety and $x \in S$ be a strictly Galois generic point defined over a number field K. Let $(a_n)_n \subset G(\mathbb{Q})_+$ be an arbitrary sequence and fix $s_n \in T_{a_n}(x)$ for every n. We have that*

$$T_{a_n}(x) = \mathrm{Gal}(\overline{K}/K)s_n.$$

Moreover, if the cardinality of $\{s_n\}_n$ is not finite then $[K(s_n):K] \to +\infty$ and the sequence of measures

$$\Delta_{T_{a_n}(x)} = \frac{1}{|\mathrm{Gal}(\overline{K}/K)s_n|} \sum_{p \in \mathrm{Gal}(\overline{K}/K)s_n} \delta_p$$

weakly converges to the hyperbolic measure μ_S on $S(\mathbb{C})$ as $n \to +\infty$.

3.3.3 Quaternionic Shimura curves

Let D be a non-split indefinite quaternion algebra over $\mathbb{Q}$ and fix a maximal order $\mathcal{O}_D$. In this section we prove a statement analogous to Corollary 3.3.3 for quaternionic Shimura curves $X^D(\mathcal{U})/\overline{\mathbb{Q}}$, i.e. the Shimura curves attached to $(D,\mathcal{U})$, where $\mathcal{U}$ is a sufficiently small compact subgroup of $(\mathcal{O}_D\otimes\widehat{\mathbb{Z}})^*$ such that $X^D(\mathcal{U})$ is connected (see [27] for more details). This curves parametrise *fake elliptic curve* (with a $\mathcal{U}$-level structure), i.e. abelian surfaces E with an embedding $\mathcal{O}_D\subset \mathrm{End}(E)$. Using such interpretation we have a notion of isogeny class $\Sigma_x\in X^D(\mathcal{U})$ as in section 3.2.1.1.

We have a version of Serre's open image theorem which holds for arbitrary Shimura curves.

Theorem 3.3.7. *Let S_Λ be a Shimura curve. A $\overline{\mathbb{Q}}$-point $x\in S$ is either special or Galois generic.*

Proof. On a Shimura curve a point is either special or Hodge generic. The main theorem of [135] shows precisely that Hodge generic points are Galois generic (the proof is similar to the methods used by Serre). □

The above theorem, combined with Theorem 3.3.6 implies the equidistribution of the Hecke orbit associated to a Hodge generic point of a quaternioinc SHimura curve. The equidistribution of the Galois orbit of CM-points, as used in [25, Theorem 2.5 and Theorem 2.6], follows again from Brauer-Siegel and Zhang's paper [184].

To obtain a contradiction in this case, it is enough to use [25, Lemma 3.6]. Indeed let $\Psi: S\to A$ be a map from a Shimura curve[2] to an elliptic curve, [25, Lemma 3.6] shows that $\Psi_*\mu_S\neq\mu_A$. Therefore we cannot have a sequence of measures Δ_{s_n} weakly converging to μ_S, whose pushforward, $\Psi^*\Delta_{s_n}$, weakly converges to μ_A.

We have eventually proved the following.

[2] In the previous theorem we had to use a different strategy since the compact Riemann surface $X_1(N)$ is *only* the compactification of a Shimura curve.

Theorem 3.3.8. *Let $S/\overline{\mathbb{Q}}$ a quaternionic Shimura curve. Let A, Γ, Ψ be as in Theorem 3.3.1. Let x be a $\overline{\mathbb{Q}}$-point of S. There exists an $\varepsilon > 0$, such that image of a isogeny class $\Sigma_x \subset S(\overline{\mathbb{Q}})$ along Ψ intersects Γ_ε in only finitely many points.*

3.3.4 A remark on Masser-Wüstholz Isogeny Theorem

This is the main theorem of [114] (see also [78] for a bound that does not depend on the polarisations).

Theorem 3.3.9 (Masser-Wüstholz)**.** *Let A,B be principally polarised abelian varieties of dimension g over a number field K and suppose that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are isogenous. Then if we let N be the minimal degree of an isogeny between them over $\mathbb{C}$, we have*

$$N \leq b_g \max(h_{\mathrm{Fal}}(A), [K:\mathbb{Q}])^{c_g},$$

where b_g, c_g are positive constants depending only on g and $h_{\mathrm{Fal}}(A)$ denotes the semistable Faltings height of A.

It has the following amusing consequence regarding the field of definition of the Hecke points. For the proof see [65, Lemma 9.2.1].

Corollary 3.3.10. *Let (G, X^+) be a connected Shimura datum of abelian type. Let $x \in S_\Lambda$ be a point with residue field K. For every integer d there are only finitely many $t \in T(x)$ such that the degree of $K(t)$ over K is bounded by d.*

It is interesting to notice that Theorem 3.3.9 is used in the proof of partial results towards conjectures about unlikely intersections. For example in the AO and André-Pink conjectures. Corollary 3.3.10 may be used for the result of this chapter. Indeed, for Shimura varieties of abelian type, it implies the existence of finitely many Hecke operators of bounded degree. In our approach we deduced this from Theorem 3.3.5, which builds on different techniques.

Remark 3.3.11. For example Corollary 3.3.10 may be applied to arbitrary Shimura curves. This is possible since all Shimura curves are of abelian type, as proven by Deligne in [46, Section 6].

3.4 Mixed Shimura varieties and the Zilber–Pink conjecture

In section 3.1, we discussed characterisations of subvarieties of a product of a modular curve and an elliptic curve intersecting a dense set of *special* points (Theorems 3.1.1, 3.1.2, 3.1.4). We formulate analogous conjectures for products of higher dimensional Shimura varieties and abelian varieties. In section 3.4.1.2, we prove that they follow from the Zilber–Pink conjecture about unlikely intersections in *mixed* Shimura varieties (Conjecture 3.4.7).

Throughout this section let $T := S \times A$ be the product of S a Shimura variety and A an abelian variety (of dimension $g > 0$). When we do not specify the field of definition of an object, we assume that it is defined over the field of complex numbers.

Recall that we have notions of being special and weakly special for both subvarieties of Shimura and abelian varieties, in particular we denote by $\mathrm{CM} \subset S(\overline{\mathbb{Q}})$ the subset of special points of S. In this section we combine the two as follows. For an overview about special subvarieties and the André-Oort conjecture, we refer the reader to [103].

Definition 3.4.1. A *special* (resp. *weakly special*) subvariety of T is a subvariety of the form $S' \times A'$ where S' is a special (resp. weakly special) subvariety of S and A' is a special (resp. weakly special) subvariety of A. We say that a subvariety of T is *weakly special generic* if it not contained in any smaller weakly special subvariety of T.

We state three conjectures about a weakly special generic closed irreducible subvariety $X \subsetneq T$.

Conjecture 3.4.2 (André-Oort-Manin-Mumford)**.** *The subset of special points of T is not Zariski dense in X.*

We denote by $A^{[>d]}$ the union of all algebraic subgroups of A of codimension $> d$ and we fix a subgroup of finite rank $\Gamma \leq A(\mathbb{C})$.

Conjecture 3.4.3 (André-Oort-Mordell-Lang). *The set* $(\mathrm{CM}\times(A^{[>\dim X]}+\Gamma))\cap X$ *is not Zariski dense in* X.

In the next conjecture, by isogeny class of a point $s\in S\subset\mathcal{A}_{g'}$, corresponding to an abelian variety A_s, we mean the set of points $s'\in S$ corresponding to abelian varieties $A_{s'}$ isogenous to A_s.

Conjecture 3.4.4 (André-Pink-Mordell-Lang). *Assume* S *is a sub-Shimura variety of* $\mathcal{A}_{g'}$ *for some* $g'>0$ *and let* Σ_s *be the isogeny class of a point* $s\in S(\mathbb{C})$. *The set* $(\Sigma_s\times\Gamma)\cap X$ *is not Zariski dense in* X.

When the abelian variety A is defined over $\overline{\mathbb{Q}}$, we can also fatten Γ, by replacing Γ by Γ_ε, to formulate an André-Oort-Mordell-Lang-Bogomolov Conjecture and an André-Pink-Mordell-Lang-Bogomolov Conjecture. Theorem 3.1.5 is a special case of such formulation, requiring all the objects to be defined over $\overline{\mathbb{Q}}$. Since the aim of the section is a comparison with the conjectures appearing in the work of Pink [146, 147], we discuss only the case of subgroups of finite rank.

Remark 3.4.5. Combining the recent proof of the AO conjecture for Shimura varieties of abelian type (culminated in [171]) and the proof of Manin-Mumford [144], it is possible to prove Conjecture 3.4.2 whenever S is a Shimura variety of abelian type.

For recent developments, using o-minimality, towards the André-Pink-Mordell-Lang we point out to the main theorems of G. Dill [59]. See also the main theorems of [76]. Indeed, as mentioned in section 3.1, Conjecture 3.4.4 formally follows from Gao's André-Pink-Zannier[3].

3.4.1 Zilber–Pink conjecture

To state the Zilber–Pink conjecture [147, Conjecture 1.1] we need to introduce some vocabulary from the theory of mixed Shimura varieties. For a complete treatment we refer the reader to [146, Section 2], [120, Chapter VI] and [145].

[3]Only a small modification is needed, in order to take into account non-polarised isogenies and subgroups of arbitrary finite rank.

3.4.1.1 Mixed Shimura varieties

In section 2.2 we discussed only the case of pure Shimura varieties. Since (connected) *mixed Shimura varieties* appear only here, we briefly recall their definition.

Definition 3.4.6. A connected mixed Shimura datum is a pair (P, X^+) where

- P is a connected linear algebraic group defined over $\mathbb{Q}$, with unipotent radical W and an algebraic subgroup $U \subset W$ which is normal in P;
- $X^+ \subset \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ is a connected component of an orbit under the subgroup $P(\mathbb{R}) \cdot U(\mathbb{C}) \subset P(\mathbb{C})$;

satisfying axioms (i)-(vi) in [146, Definition 2.1]. A connected mixed Shimura variety associated to (P, X^+) is a complex manifold of the form $\Lambda \backslash X^+$ where Λ is a congruence subgroup of $P(\mathbb{Q})_+$ acting freely on X^+.

A mixed Shimura datum allows to take into consideration groups of the form $\mathrm{GSp}_{2g} \ltimes \mathbb{G}_a^{2g}$. For suitable congruence subgroups the associated connected mixed Shimura variety is the universal family of abelian varieties over the moduli space of principally polarised abelian varieties (with some n-level structure). The point is that every (principally polarised) abelian variety can be realised as a fibre of such a family.

As for the pure case, there is a notion of special and weakly special subvarieties of mixed Shimura varieties (see [146, Section 4]). Moreover every irreducible component of the intersection of special subvarieties (resp. weakly special) is again special (resp. weakly special) and a weakly special subvariety containing a special point is itself special. For example special points in the universal family of abelian varieties correspond to torsion points in the fibers A_s over all special points $s \in \mathcal{A}_g$.

Finally pure Shimura varieties, as described in section 2.2, are also mixed Shimura varieties (they occur precisely when P is reductive) and a product of (finitely many) mixed Shimura varieties is again a mixed Shimura variety.

3.4.1.2 The conjecture

Conjecture 3.4.7 (Zilber–Pink)**.** *Consider a mixed Shimura variety M over $\mathbb{C}$ and a Hodge generic irreducible closed subvariety $X \subset M$. Then the intersection of X*

with the union of all special subvarieties of M of codimension $> \dim X$ is not Zariski dense in X.

It is interesting to notice that Conjecture 3.4.7 implies Mordell-Lang [146, Section 5]. Moreover Stoll has recently proven that (a special case of) the Zilber-Pink conjecture actually implies the following, see [165, Section 2.1].

Conjecture 3.4.8 (Uniform Mordell–Lang for curves)**.** *Given $g \geq 2$ and $r \geq 0$, there is a constant $N(g,r)$ such that for any genus g curve $C/\mathbb{C}$ with an embedding $i : C \to J$ into its Jacobian and for any subgroup $\Gamma \subset J(\mathbb{C})$ of rank r, one has that the cardinality of $i^{-1}(\Gamma)$ is at most $N(g,r)$.*

The proof of the next proposition is similar to the arguments appearing in Theorem 3.3, 5.3 and 5.7 of the preprint [147]. See also [76, Section 8] and [146], where it is explained that André-Pink(-Zannier) for mixed Shimura varieties implies Mordell-Lang ([146, Theorem 5.4]).

Proposition 3.4.9. *Conjecture 3.4.7 implies both Conjecture 3.4.3 and Conjecture 3.4.4.*

We first fix some notations. Let $a \in \mathcal{A}_{g,n}$ be the point corresponding to the abelian variety A (for some $n \geq 3$), S^a the smallest Shimura subvariety of $\mathcal{A}_{g,n}$ containing a and M the universal abelian scheme over S^a. The variety $S \times M$ is a mixed Shimura variety and it contains $S \times A = S \times M_a$, where M_a denotes the fibre of

$$\pi : M \to S^a$$

over the point a.

Finally fix a maximal sequence of linearly independent elements $a_1, \dots, a_n \in \Gamma$, and let C the Zariski closure of the subgroup of A^n generated by the point $\underline{a} := (a_1, \dots, a_n)$. We may assume C is an abelian variety. Moreover, since a is Hodge generic in S^a, we may view C as the fibre over a of an S^a flat subgroup scheme Z of the n-th fibred power of M (cf. the discussion at the beginning of the proof of [147, Theorem 5.7]). We will apply the Zilber–Pink conjecture to the subvarieties of the mixed Shimura variety

$$B := S \times (M \times_{S^a} Z)\,.$$

Zilber–Pink implies André–Oort–Mordell–Lang. Equivalently we may suppose that X is not contained in any special subvariety of T and deduce that $(\mathrm{CM} \times \Gamma) \cap X$ is not Zariski dense in X. Consider the irreducible closed subvariety of B defined by

$$Y := X \times \{\underline{a}\}.$$

Since a is Hodge generic in S^a, X is weakly special generic in T, $\underline{a}$ is Zariski dense in C, then Y is a Hodge generic subvariety of B of dimension $\dim X$.

As for abelian varieties, we denote by $M^{[>d]}$ the union of $M_x^{[>d]}$, varying x in S^a. To conclude, applying the Zilber–Pink conjecture to $Y \subset B$, we only need to show that the set

$$(X \cap (\mathrm{CM} \times (M^{[>\dim X]} + \Gamma))) \times \{\underline{a}\}$$

is contained in the intersection between Y and the union of all special subvarieties of B of codimension $> \dim X = \dim Y$. Let G be a S^a-flat algebraic subgroup of M of codimension $> \dim X$, and let $x = (c, g + \gamma) \in X$, where c is a special point in S and γ an element of Γ. For some integer $m > 0$ we may write

$$m\gamma = m_1 a_1 + \cdots + m_n a_n,$$

then we have $m\gamma = \varphi(\underline{a})$ for the homomorphism of S^a-group schemes $\varphi := (m_1, \ldots, m_n) : M^n \to M$. We may therefore write $(x, \underline{a})$ as an element in the set

$$H := \mathrm{CM} \times (m^{-1}(G \times \{0\} + (\varphi, m)(C))).$$

Since the codimension of H in B is bigger than $\dim X$, we have proved the desired inclusion.

The result on the fiber over a then follows in virtue of the following remark (see proof of [147, Theorem 5.7]). Let $X \subset A$ be an irreducible closed subvariety of A, X is contained in a proper algebraic subgroup of A if and only it is contained in a special subvariety of M of codimension > 0. $\square$

Zilber–Pink implies André–Pink–Mordell–Lang. By applying Hecke operators, we may assume that $X \times \{\underline{a}\}$ and $\{s\} \times \{\underline{a}\}$ lie in a given connected component of B. Let S^s be the smallest Shimura subvariety containing s, and S' the smallest Shimura subvariety of $B \times \mathcal{A}_{g'}$ containing $Y \times \{s\}$.

Suppose $(X \cap (\Sigma_s \times (M^{[>\dim X]} + \Gamma)))$ is not Zariski dense in X, we want to prove that X is weakly special, more precisely we show that $X \times \{s\}$ is an irreducible component of a fibre of $S' \to S^s$. To do so we apply Zilber–Pink (actually in the equivalent form appearing in [147, Conjecture 1.1]) to

$$\Sigma_s \times \{s\} \times (M^{[>\dim X]} + \Gamma) \times \{\underline{a}\}.$$

The result follows by combining the argument presented in the previous proof, which allows to see the points in Γ as special points in an opportune Shimura variety (see also the last paragraph in the proof of [147, Theorem 5.3]), and the argument of [147, Theorem 3.3] (noticing that given two points $s,t \in \mathcal{A}_{g'}$ such that the underlying abelian varieties are isogenous, then the defect of $s \in \mathcal{A}_{g'}$ is equal to the defect of $(s,t) \in \mathcal{A}^2_{g'}$, as in [137, Lemma 2.2]). $\square$

Chapter 4

Local to Global principle for the moduli space of K3 surfaces

In 2016, S. Patrikis, J.F. Voloch and Y. Zarhin proved, assuming several well known conjectures, that the finite descent obstruction holds on the moduli space of principally polarised abelian varieties. We show an analogous result for K3 surfaces, under some technical restrictions on the Picard rank. This is possible since abelian varieties and K3s are quite well described by "Hodge-theoretical" results. In particular the theorem we present can be interpreted as follows: a family of ℓ-adic representations that *looks like* the one induced by the transcendental part of the ℓ-adic cohomology of a K3 surface (defined over a number field) determines a Hodge structure which in turn determines a K3 surface (which may be defined over a number field). The work presented here appeared in the paper [10].

4.1 Introduction

Let X be an algebraic K3 surface defined over a number field K and ℓ a rational prime. We consider $T_\ell(X_{\overline{K}})$ the transcendental part of the second ℓ-adic cohomological group of $X_{\overline{K}}$, i.e. $T_\ell(X_{\overline{K}})$ is the orthogonal complement of the image of the Néron-Severi group of $X_{\overline{K}} = X \times_K \overline{K}$ in $H^2_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. It is a free $\mathbb{Q}_\ell$-module of rank $22-\rho$, where $\rho \in \{1,2,\ldots,20\}$ denotes the rank of the Néron-Severi group of $X_{\overline{K}}$, usually called the (geometric) Picard rank of X. For every rational prime ℓ, there is a continuous ℓ-adic Galois representation of the absolute Galois group of K

$$\rho_{X,\ell} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(T_\ell(X_{\overline{K}})).$$

The family $\{\rho_{X,\ell}\}_\ell$ encodes many algebro-geometric properties of X that can expressed in the language of representation theory.

The problem discussed in this chapter is motivated by the following question, which can be thought as a refinement of the Fontaine–Mazur conjecture, as recalled in section 2.5.2, since it aims to describe the essential image of the ℓ-adic realisations of K3 surfaces. See also Example 2.2.12 and Question 2.3.9.

Question 4.1.1. Given a family of ℓ-adic representations of the absolute Galois group of a number field K, can we *understand* if it is of the form $\{\rho_{X,\ell}\}_\ell$ for some K3 surface X/K (possibly after a finite field extension L/K)?

As explained in section 2.3.1.1, Question 4.1.1 requires some p-adic Hodge theory. The analogous of Question 4.1.1 for abelian varieties has been addressed and solved, assuming some of the conjectures recalled in section 2.5, in [139, Theorem 3.1] (at least for abelian varieties with endomorphism ring equal to $\mathbb{Z}$). More precisely the authors proved the following.

Theorem 4.1.2 (Patrikis, Voloch, Zarhin)**.** *Let $N > 0$ be a natural number. Assume the Hodge, Tate, Fontaine–Mazur and the semisimplicity conjectures, as recalled in Section 2.5. Let*

$$\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2N}(\mathbb{Q}_\ell)\}_\ell$$

be a weakly compatible family (in the sense of Definition 2.3.2) of ℓ-adic representations such that:

i) For some prime ℓ_0, ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;

ii) For some prime ℓ_1, ρ_{ℓ_1} is absolutely irreducible;

iii) For some prime ℓ_2 and at least one place v above ℓ_2, $\rho_{\ell_2|\mathrm{Gal}(\overline{K}_v/K_v)}$ is de Rham with Hodge–Tate weights $-1, 0$ each with multiplicity N, where K_v denotes the completion of K at the place v.

Then there exists a N-dimensional abelian variety A defined over K such that $\rho_\ell \cong V_\ell(A)$ for all ℓ, where $V_\ell(A)$ denotes the rational ℓ-adic Tate module of A with its natural Galois action.

Notice that conditions i) and iii) are satisfied by the cohomology of every abelian variety and condition ii) holds for the generic abelian variety. For the link between Theorem 4.1.2, the section conjecture of anabelian geometry and the sufficiency of the finite descent obstruction to the Hasse principle for $\mathcal{A}_g$ we refer to [139, Section 2, Theorem 3.7]. Interestingly, thanks to the Kodaira– Parshin construction [113], such result could also have applications to the section conjecture for curves. See indeed Theorem 5.2 in *op. cit.*. For expository article describing some related ideas, we refer also to [100].

It is reasonable to expect a result of the same fashion for varieties whose geometry is well captured from cohomological invariants. For example the above theorem can not tell the difference between a curve of genus $N > 1$ and its Jacobian (see [139, Section 5] for a more detailed discussion about this). From this point of view K3 surfaces (and hyperkähler varieties) are very similar to abelian varieties. Indeed, over the complex numbers, they enjoy a Torelli type theorem [148] and the surjectivity of the period map [170], see also the discussion in Example 2.2.12. We refer to Proposition 4.3.2 for precise statements.

4.1.1 Main results

In Section 4.3 we prove the following, which is the main theorem of the chapter.

Theorem 4.1.3. *Assume the Tate, Fontaine–Mazur and the Hodge conjectures. Let ρ be a natural number such that such that $2 < 22 - \rho \leq 19$ and let*

$$\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{22-\rho}(\mathbb{Q}_\ell)\}_\ell,$$

be a weakly compatible family of ℓ-adic representations satisfying the following conditions:

1. *For some prime ℓ_0, ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;*

2. *For some prime ℓ_1, ρ_{ℓ_1} is absolutely irreducible;*

3. *For some prime ℓ_2 and at least one place v above ℓ_2, $\rho_{\ell_2|\mathrm{Gal}(\overline{K}_v/K_v)}$ is de Rham with Hodge–Tate weights $0,1,2$, with multiplicities, respectively, $1,20-\rho,1$.*

Then there exists a K3 surface X defined over a finite extension L/K with geometric Picard rank ρ, such that the restriction of ρ_ℓ to $\mathrm{Gal}(\overline{L}/L)$ is isomorphic to $T_\ell(X_{\overline{L}})$ for all ℓ.

The proof shows something stronger: there exists a motive M defined over K, in the sense of section 2.4, inducing the representations ρ_ℓ and a finite extension L/K, such that the base change of M to L is isomorphic to the transcendental part (in the sense of Section 4.2) of the motive of a K3 surface defined over L. It is not clear whether or not the extension L/K is needed, more about this is discussed in Section 4.1.3.

In the proof, from the motive M/K, we will first produce a complex (algebraic) K3 surface and descend it to a number field. This is shown in the last section and may be of independent interest. For a complex K3 surface X and an element $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ we set

$$^{\sigma}X := X \times_{\mathbb{C},\sigma} \mathbb{C}$$

for the conjugate of X with respect to σ. Here we write $\mathbb{C}$ to denote its spectrum. Let $T(X)_{\mathbb{Q}}$ be the rational polarised Hodge structure given by the transcendental part of the $H^2(X(\mathbb{C}),\mathbb{Q})$, i.e. the orthogonal complement of the image of $\mathrm{NS}(X)\otimes\mathbb{Q}$ in $H^2(X,\mathbb{Q})$.

Theorem 4.1.4. *Let $X/\mathbb{C}$ be a K3 surface such that*

$$T(X)_{\mathbb{Q}} \cong T(^{\sigma}X)_{\mathbb{Q}} \quad \textit{for all } \sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}}),$$

where the isomorphism is an isomorphism of rational polarised Hodge structures. Then X admits a model defined over a number field, i.e. there exists a number field $L \subset \mathbb{C}$ and a K3 surface Y/L, such that $Y \times_L \mathbb{C}$ is isomorphic to X.

The above condition can be thought as an *isogeny* relation between X and its $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$-conjugates. With this interpretation the theorem is analogous of the descent for abelian varieties established in [139, Lemma 3.6]. Let $A/\mathbb{C}$ be an abelian variety such that all its $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$-conjugates are isogenous to A, then A descends to a number field. Theorem 4.1.4 will follow from a general criterion proven by González-Diez (see Lemma 4.4.3) and the fact that there are only finitely many complex K3 surfaces with given transcendental lattice (see Lemma 4.4.2). It is quite different from the proof of [139, Lemma 3.6] and can be used also to reprove such result, as explained in Remark 4.4.4.

In the next remark we show that Theorem 4.1.4 implies that complex K3 surfaces with complex multiplication (or CM, for brevity) admit models over number fields. As from Definition 2.2.6, by complex multiplication we intend that the Mumford-Tate group associated to the Hodge structure $T(X)_{\mathbb{Q}}$ is commutative; for example every K3 surface of geometric Picard rank 20 has such property (cf. [94, Remark 3.10 (page 54)]). If X/K is a CM K3 surface, it is a consequence of the Kuga–Satake construction and Deligne's work on absolute Hodge cycles (see indeed Remark 2.5.7), that the ℓ-adic monodromy associated to $T(X_{\overline{K}})_{\mathbb{Q}_\ell}$ is commutative as well.

Remark 4.1.5. One can check that complex K3 surfaces with complex multiplication satisfy the condition of the above theorem. Therefore they admit a model over a number field, reproving a result originally due to Pjateckiĭ-Šapiro and Šafarevič [149, Theorem 4]. For example, when $X/\mathbb{C}$ has maximal Picard rank, it is enough to notice that, thanks to the comparison between Betti and étale cohomology recalled in section 2.4.2, the quadratic forms associated to $T(X)$ and $T({}^{\sigma}X)$ are in the same genus.

Finally we point out that recently C. Klevdal [101] considered an analogue of Theorem 4.1.3 for K3 surfaces of Picard rank 1. Therefore only the case of geometric Picard rank two and twenty are left out of the picture. We hope to come back to this in the future. We refer the reader to [101, Section 1.1] for a comparison between the two results and a precise statement of Klevdal's result.

4.1.2 Comments on the assumptions of Theorem 4.1.3

As remarked above and in section 2.3.1.1, the cohomology of K3 surfaces (defined over local fields) gives rise to de Rham representations. In particular condition (3) is satisfied if there exists a K3 surface X_v/K_v of geometric Picard rank ρ and $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is isomorphic to the representation induced by $T_{\ell_2}(X_{\overline{K}_v})$[1]. So (1) and (3) are necessary conditions for the existence of the K3 surface the theorem aims to prove.

Condition (2), which appears also in Theorem 4.1.2 as ii), is crucial to make the argument work (see Theorem 2.5.5) but it is not satisfied by every K3 surface (even if it holds for the generic K3 surface of Picard rank ρ). Klevdal considers the representations arising from the full H^2 of K3 surface of geometric Picard rank 1 and works with a different irreducibility condition (see condition (3) in [101, Theorem1.1]). His conditions, which are satisfied by the cohomology of the generic K3 surface, imply that the representation attached to the motive M splits as a sum of the trivial representation and an absolutely irreducible one and then works with the latter.

The absolutely irreducibility of condition (2) can not be weakened to require only the irreducibility of the ℓ-adic Galois representations, as the beginning of section 4.3 shows. Notice also that the ℓ-adic and Betti realisations of simple motives may be reducible and the irreducibility of the Hodge structure associated to a motive wont imply the irreducibility of the Galois representations attached to it. For example, if X is a K3 surface defined over a number field K, $T(X)_{\mathbb{Q}}$, as Hodge structure, is always irreducible (see [94, Chapter 3, Lemma 2.7 and Lemma 3.1]), but the Galois representations $T_\ell(X_{\overline{K}})$ may be reducible. This happens, for some ℓ, whenever X has complex multiplication.

The restriction on the Picard rank is due to the way we obtain a K3 surface from a polarised Hodge structure of K3 type (see Proposition 4.3.2). As we observed, the conditions in the theorem require the representations to have non-commutative im-

[1] In condition iii) of Theorem 4.1.2, the Hodge–Tate weights are $-1, 0$ since they want to relate $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ to the Tate module of an abelian variety, which is the dual representation attached to the H^1. This explain the change of sings between the two theorems.

age (unless they have values in $\mathrm{GL}_1(\mathbb{Q}_\ell)$). Therefore our approach can not deal with case of Picard rank 20 (as Theorem 4.1.2 excludes abelian varieties with complex multiplication).

4.1.3 How to get rid of the extension L/K

A positive answer to the following would allow to take $L = K$ in Theorem 4.1.3.

Question 4.1.6. Assume the conjectures as in Theorem 4.1.3. Let M be a simple motive defined over some number field K. Assume there exists a finite (Galois) extension L/K and a K3 surface Y/L such that M_L is isomorphic to the transcendental part of the motive of Y_L (in the sense of Section 4.2). Is there a K3 surface X defined over K such that the transcendental part of the motive associated to X is isomorphic to M.

In the case of abelian varieties (see the discussion before [139, Proof of Lemma 3.6]), the authors give an affirmative answer. The proof works by considering the Weil restriction to K of the abelian variety Y_L and, using Frobenius reciprocity, producing an endomorphism of it whose image corresponds to a K-abelian variety with the desired property. Unfortunately the argument does not apply to K3 surfaces and it is not clear if this field extension is necessary or not.

4.1.4 Examples and applications

We explain how to obtain examples of Galois representations to which Theorem 4.1.3 applies, without writing down a K3 surface and how it produces a *phantom isogeny class* of K3 surfaces, in analogy with the phantom isogeny class of abelian varieties defined by Mazur in [117, page 38].

Let Y/K be a smooth projective variety defined over a number field K such that, for some i, the Hodge decomposition induced on the primitive cohomology $H^{2i}_{\mathrm{prim}}(Y(\mathbb{C}),\mathbb{Q})$ *looks like* the one of a K3 surface. More precisely its Hodge numbers are all zero but $h^{i,i}$ and $h^{p,q} = 1$ for a unique, up to reordering, pair (p,q) with $p+q = 2i$ and the transcendental part of the $H^{i,i}$ has positive dimension less or equal than 18. Examples are provided by the H^4 of cubic fourfolds (where the Hodge numbers are $h^{0,4} = 0, h^{1,3} = 1$ and $h^{2,2} = 21$) and many varieties with $h^{2,0} = 1$.

Consider the family of representations attached to the transcendental part of the $H^{2i}_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_\ell(i-1))$, simply denoted by $T_\ell(Y_{\overline{K}})$, where we considered a Tate twist by $(i-1)$ to obtain the weight of an H^2:

$$\{\rho_{Y,\ell} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(T_\ell(Y_{\overline{K}}))\}_\ell.$$

The geometric origin of such representations implies that $\{\rho_{Y,\ell}\}_\ell$ is a weakly compatible system (cf. Remark 2.3.3) and that condition (1) of Theorem 4.1.3 is satisfied. The assumptions on the Hodge decomposition of the $H^{2i}(Y(\mathbb{C}), \mathbb{Q})$ imply that condition (3) is satisfied.

Assume now that Y is such that also condition (2) is satisfied, i.e. for some prime ℓ_1, ρ_{ℓ_1} is absolutely irreducible (such condition is satisfied by the generic cubic fourfold). Theorem 4.1.3, after a finite extension L/K, associates a K3 surface X/L to Y, with an isomorphism of $\mathrm{Gal}(\overline{L}/L)$-representations between $\rho_{Y,\ell}$ and $\rho_{X,\ell}$. Such K3 surface need not to be unique and we think about K3s satisfying such condition as a *phantom isogeny class*. The existence of such K3s could greatly simplify the study of Galois representations attached to such Ys. It would be interesting to construct them (over a number field!) without assuming any conjectures. For a survey explaining how K3 surfaces can help the study of the geometry of cubic fourfolds we refer the reader to [90]. In particular in [90, Section 3] it is discussed how to associate K3 surfaces to *special* cubic fourfolds via Hodge theoretical methods.

Theorem 4.1.3 has also the following amusing consequence, purely expressed in the ℓ-adic language.

Corollary 4.1.7. *Assume the Tate, Fontaine–Mazur and the Hodge conjectures. Let*

$$\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{22-\rho}(\mathbb{Q}_\ell)\}_\ell,$$

be a family of ℓ-adic representations as considered in Theorem 4.1.3. Then there exists a reductive group $G/\mathbb{Q}$ such that, after a finite extension K'/K, for every ℓ, the image of $\mathrm{Gal}(\overline{K}/K')$ *via ρ_ℓ has finite index in $G(\mathbb{Z}_\ell)$ and the index is bounded when ℓ varies.*

Proof. Theorem 4.1.3 shows that the weakly compatible system $\{\rho_\ell\}_\ell$, up to replacing K with a larger number field is associated to a K3 surface X. By a result of Serre there exists a finite extension K'/K such that the ℓ-adic monodromy of the $\rho_{\ell|\mathrm{Gal}(\overline{K}/K')}$ is connected for every ℓ (see [31, Section 1.1] for precise references). Let G be the Mumford-Tate group of the K3 surface $X_{\mathbb{C}}$. The combination of the Tate and Hodge conjectures implies that, for every ℓ, the image of $\rho_{\ell|\mathrm{Gal}(\overline{K}/K')}$ is contained in $G(\mathbb{Q}_\ell)$ as a subgroup of finite index. Thanks to [31, Theorem 6.6], which is peculiar to K3 surfaces and abelian varieties, we have furthermore that the index is bounded independently from ℓ. □

Remark 4.1.8. In the proof of Theorem 4.1.3 it is first produced a motive M whose ℓ-adic realisations induce the family $\{\rho_\ell\}_\ell$. This weaker conclusion is not enough to obtain the corollary. Indeed the proof uses the *Integral Mumford-Tate conjecture* which is known to follow from the classical Mumford-Tate conjecture, thanks to the work of Cadoret and Moonen, only for Galois representations attached to K3 surface and abelian varieties. More details about this can be found in [31, Sections 1 and 2].

To conclude this section we point out that Theorem 4.1.3 can be interpreted in the setting of anabelian geometry and the section conjecture for the moduli space classifying primitively polarised K3 surfaces of degree $2d$. For more about this we refer the reader to the introduction of [139] and the second part of Section 1.1 in [101].

Outline of chapter

In Section 4.2 we review the motive of a surface and how it splits in the algebraic and the transcendental part. In Section 4.3 we prove the first main result (assuming Theorem 4.1.4). The beginning of the proof closely follows the proof of [139, Theorem 3.1] and we present here only the main steps. The last section of the chapter, which is independent from the previous ones, proves Theorem 4.1.4.

Notations

By K3 surface X/K we mean a complete smooth K-variety of dimension two such that $\Omega^2_{X/K} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. In this chapter K will always denote a subfield of $\mathbb{C}$. For a complete overview of the theory of K3 surfaces we refer to the book [94]. We will make free use of the following standard notations:

- We denote by $(\Lambda_{\mathrm{K3}}, q)$ the K3-lattice, where q is the quadratic form: it is the unique even unimodular lattice of signature $(3,19)$ (i.e. $E_8(-1)^2 \oplus U^3$);

- We write *PHS* as an acronym for *rational* polarised Hodge structure and $\mathbb{Z}$-PHS for *integral* polarised Hodge structure (in particular we require that the underlying $\mathbb{Z}$-module is torsion free). See also section 2.2.1.1. Morphism in the category of PHS are maps of Hodge structure preserving the induced pairing;

- By *Hodge structure of K3 type* we mean an irreducible PHS of weight two such that $h^{2,0} = h^{0,2} = 1$. In other references the irreducibility is not part of the definition and they refer to *irreducible PHS of K3 type*;

- Let X be a complex K3 surface, we denote by $T(X)_{\mathbb{Q}}$ the transcendental part of the $H^2(X(\mathbb{C}), \mathbb{Q})$, i.e. the orthogonal complement of $\mathrm{NS}(X) \otimes \mathbb{Q} \subset H^2(X(\mathbb{C}), \mathbb{Q})$. It is a Hodge structure of K3 type (the irreducibility was first established in [182]);

- Analogously, if X is a K3 surface defined over a number field K, we define $T_\ell(X_{\overline{K}})$ to be the orthogonal complement of the image of $\mathrm{NS}(X_{\overline{K}}) \otimes \mathbb{Q}_\ell$ in $H^2_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ with respect to the cup product in ℓ-adic cohomology.

4.2 The motive of a surface

Here we use the same notation of section 2.4. Let $M = h(X)$ be the motive of a (smooth projective connected) surface. The class of the diagonal $\Lambda \in \mathrm{Corr}^0(X \times X)$ can be written as

$$[\Lambda] = \sum_i \pi^i \in H^4(X \times X).$$

Since the decomposition is algebraic, in the sense that each π^i can be seen as the class of some orthogonal projector π^i in $\mathrm{Corr}^0(X\times X)$, we can decompose the motive M as follows (Chow-Künneth decomposition):

$$M = \mathbb{1}\oplus h^1(X)\oplus h^2(X)\oplus h^3(X)\oplus \mathbb{L}^2,$$

where $\mathbb{1}$ is the motive of a point, $\mathbb{L}$ is the Lefschetz motive (defined by the equation $h(\mathbb{P}^1_k) = \mathbb{1}\oplus\mathbb{L}$) and $h^i(X) = (X,\pi^i,0)$.

Moreover, in [97, Proposition 2.3], it is explained that there exists a unique splitting

$$\pi^2 = \pi^2_{alg} + \pi^2_{tr}$$

inducing a refined Chow-Künneth decomposition for the motive M:

$$h^2(X) = \left(h^2_{alg}(X)\oplus t^2(X)\right)$$

where $h^2_{alg}(X) = (X,\pi^2_{alg},0)$ and $t^2(X) = (X,\pi^2_{tr},0)$.

In this chapter we will be interested in the case of a K3 surface, so, from now on we will consider just the weight-two part of the motive of X. In particular the Betti realisation satisfies the following relation:

$$H_B(h^2_{alg}(X)\oplus t^2(X)) = \mathrm{NS}(X)_{\mathbb{Q}}\oplus T(X)_{\mathbb{Q}}.$$

4.3 Proof of Theorem 4.1.3

The proof of Theorem 4.1.3 begins like the argument in [139], so we only recall the main steps. As explained in section 2.5.2, thanks to the Tate, Fontaine–Mazur (and semisimplicity) conjectures the essential image of the ℓ-adic realisation functor from the category of motives over K with coefficients in $\overline{\mathbb{Q}}$ can be described explicitly, as recalled in Theorem 2.5.5. In particular, choosing a place ℓ_0 as in (1), there exists a representation of the group $\mathcal{G}_{K,E}$ (as explained after Conjecture 2.4.4)

$$\rho : \mathcal{G}_{K,E} \to \mathrm{GL}_{22-\rho,E}$$

for some number field E, such that $H_{\ell_0}(\rho) \cong \rho_{\ell_0} \otimes \overline{\mathbb{Q}}_{\ell_0}$. Since the family $\{\rho_\ell\}_\ell$ is weakly compatible and we assumed ρ_{ℓ_1} to be absolutely irreducible, it follows that ρ induces every ρ_ℓ. This allows to read the assumptions imposed at some prime l_i in every ρ_ℓ. Finally, since every ρ_ℓ is a representation with $\mathbb{Q}_\ell$ coefficients (rather than with coefficients in the completions of E), hence [139, Lemma 3.4] guarantees that the representation ρ can be defined over $\mathbb{Q}$.

To summarise, we know that the compatible family $\{\rho_\ell\}_\ell$ arises from a representation

$$\rho : \mathcal{G}_K \to \mathrm{GL}_{22-\rho,\mathbb{Q}},$$

or, in equivalent terms, from a motive $M \in \mathcal{M}_K$ of rank $22-\rho$. By construction M is also absolutely simple and $\mathrm{End}(\rho) = \mathbb{Q}$.

By hypothesis there exists a prime ℓ_2 and a place v dividing ℓ_2 such that $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is de Rham with Hodge–Tate numbers equal to those of the transcendental lattice of a K3 surface of rank ρ. Denote by $H_{dR} : \mathcal{M}_K \to \mathrm{Fil}_K$ the de Rham realisation functor into the category of filtered K-vector spaces. From the comparison theorem between de Rham and étale cohomology[2] we have

$$H_{dR}(M) \otimes_K \mathbb{B}_{dR,K_v} \cong H_{\ell_2}(M) \otimes_{\mathbb{Q}_{\ell_2}} \mathbb{B}_{dR,K_v}$$

where $\mathbb{B}_{dR,K_v}$ is the de Rham period ring over K_v, as recalled in section 2.3.1.1. The fact that the above isomorphism is compatible with the filtration and the Galois action, the definition of D_{dR,K_v} and the fact that $\mathbb{B}_{dR}^{\mathrm{Gal}(\overline{K}_v/K_v)} = K_v$, imply that

$$H_{dR}(M) \otimes_K K_v \cong D_{dR,K_v}(H_{\ell_2}(M)). \tag{4.3.1}$$

[2]Such comparison was conjectured by Fontaine in [69, Conjecture A.6] and proved by Faltings in [68].

We write $M_{|\mathbb{C}}$ for the base change of $M \in \mathcal{M}_K$ in the category $\mathcal{M}_{K,\mathbb{C}}$ (we fixed from the beginning an embedding of K into the complex numbers). Recall, as in section 2.4.2, the Betti-de Rham comparison isomorphism:

$$H_{dR}(M) \otimes_K \mathbb{C} \cong H_B(M_{|\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

By 4.3.1, $H_B(M_{|\mathbb{C}})$ is a polarisable rational Hodge structure of weight two and with Hodge numbers $1-(20-\rho)-1$.

Remark 4.3.1. While the previous part of the proof works in general (and closely follows the beginning of the proof of [139, Theorem 3.1]), from now on we will use in a substantial way the condition on the Hodge–Tate weights to produce a K3 surface. Our aim is to use $H_B(M_{|\mathbb{C}})$ to produce a period and then a K3 surface, invoking the surjectivity of the period map of Todorov. To apply this strategy we need the conjunction of the Tate and the Hodge conjecture, so that we can deduce properties of the Hodge structure from the properties imposed on the family $\{\rho_\ell\}_\ell$ (especially condition (3)). To do so, the Hodge conjecture will be used from now on.

Since M is absolutely simple, $H_B(M_{|\mathbb{C}})$ is an irreducible Hodge structure. Fixing a polarisation ψ on $H_B(M_{|\mathbb{C}})$, the pair $(H_B(M_{|\mathbb{C}}), \psi)$ becomes $\mathbb{Q}$-PHS of K3 type. Moreover, since $\mathrm{End}(M) = \mathbb{Q}$, we have have that the endomorphism field of $(H_B(M_{|\mathbb{C}}), \psi)$ is $\mathbb{Q}$ as well (here again the fact that the representation is absolutely irreducible and the Hodge conjecture are fundamental).

Invoking the surjectivity of the period map, we want to produce a complex K3 surface from the rational polarised Hodge structure associated to $M_{|\mathbb{C}}$. We argue as follows.

Proposition 4.3.2. *Let (V, h, ψ) be a $\mathbb{Q}$-PHS of K3 type of dimension $22-\rho$. If $2 \leq 22-\rho \leq 19$, then there exists a complex K3 surface X with $T(X)_{\mathbb{Q}}$ isomorphic to (V, h, ψ) as rational Hodge structures.*

Remark 4.3.3. Proposition 4.3.2 requires ρ to be different from 1 and 2, where some restriction on the square class of the determinant of (V, ψ) and its Hasse invariant appears (see [136, Section IV]). Even if the above proposition applies, the case $\rho = 20$ has to be excluded from the theorem, as remarked in section 4.1.2. Indeed if X has Picard rank 20 then the endomorphism field of $T(X)_{\mathbb{Q}}$ has to be larger than $\mathbb{Q}$ (see [94, Remark 3.10 (page 54)] for an elementary proof of this fact).

Proof of Proposition 4.3.2. Notice that the quadratic form (V, ψ) is rationally represented by the K3-lattice $(\Lambda_{\mathrm{K3}}, q)$. Indeed, as explained in [136, Section IV] (see also [96, Theorems 17 and 31]), this is true whenever

$$\text{defect} := \dim \Lambda_{\mathrm{K3}} - \dim V = \rho \geq 3.$$

We can therefore interpret V as a subspace of $\Lambda_{\mathrm{K3}} \otimes \mathbb{Q}$ and let T be the intersection of V with Λ_{K3} (seen in $\Lambda_{\mathrm{K3}} \otimes \mathbb{Q}$). By definition T is a primitive sub-lattice of Λ_{K3}. Since the Hodge structure h on V is of K3 type, the quadratic form on V (and thus on T) has signature $(2, 19 - \rho)$. Transporting the Hodge structure h from V to T we obtain an irreducible integral Hodge structure with the right signature. Finally we can apply the surjectivity of the period map (see for example [94, Theorem 6.3.1 and Remark 6.3.3 (page 114)]), to obtain a complex (algebraic) K3 surface X such that $T(X) \cong T$. □

Let X be the complex K3 surface obtained as in the above proposition from $(H_B(M_{|\mathbb{C}}), \psi)$. Thanks to the Hodge conjecture 2.5.6, we can lift the isomorphism of Hodge structures

$$T(X)_{\mathbb{Q}} \cong H_B(M_{|\mathbb{C}}),$$

to get an isomorphism at the level of motives. We indeed have

$$t^2(X) \cong M_{|\mathbb{C}} \in \mathcal{M}_{\mathbb{C}},$$

where $t^2(X)$ is the transcendental part of the motive of X, introduced in Section 4.2.

To complete the proof we need a model Y_L of X defined over a finite extension L of K, such that

$$t^2(Y_L) \cong M_{|L} \in \mathcal{M}_L.$$

Since M is defined over a number field, for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, we have the following chain of isomorphisms:

$$^{\sigma}t^2(X) \cong {}^{\sigma}M_{|\mathbb{C}} = M_{|\mathbb{C}} \cong t^2(X) \in \mathcal{M}_{\mathbb{C}}. \tag{4.3.2}$$

Notice that $^{\sigma}t^2(X) = (^{\sigma}X, {}^{\sigma}\pi^2_{alg}, 0) = t^2(^{\sigma}X)$ from the uniqueness of the splitting $\pi^2 = \pi^2_{alg} + \pi^2_{tr}$ in $X \times X$.

In particular we have

$$H_B(t^2(X)) = T(X)_{\mathbb{Q}} \text{ and } H_B(^{\sigma}t^2(X)) = T(^{\sigma}X)_{\mathbb{Q}}.$$

Taking the Betti realisation (with $\mathbb{Q}$-coefficients, as usual) of the equation 4.3.2, we observe that $T(X)_{\mathbb{Q}} \cong T(^{\sigma}X)_{\mathbb{Q}}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$. Applying Theorem 4.1.4, that will be proved in the next section, this condition is enough to obtain a model Y_L of $X/\mathbb{C}$ defined over some number field L/K where $t^2(Y_L) \cong M_{|L} \in \mathcal{M}_L$.

Theorem 4.1.3 is finally proven: Y_L is the K3 surface, defined over a finite extension L of K, we were looking for. As remarked in the section 4.1 we actually proved something more: there exists a simple motive M defined over K inducing the representations ρ_ℓ and a finite extension L/K such that the base change of M to L, denoted by M_L, is isomorphic to the transcendental part of the motive of a K3 surface defined over L.

4.4 Descent to a number field

In this last section, we prove Theorem 4.1.4. The result will follow from the combination of the following:

- The number of complex K3 surfaces, up to isomorphism, Y such that $T(Y)_{\mathbb{Q}}$ is isomorphic to $T(X)_{\mathbb{Q}}$ is at most countable, cf. Lemma 4.4.2;
- If all the conjugates of X, with respect to $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, fall into countably many isomorphism classes, then X descends to a number field, cf. Lemma 4.4.3.

The first point resembles the fact the the isogeny class of a given complex abelian variety consists of a countable set of (complex) abelian varieties (up to an isomorphism).

Remark 4.4.1. In the integral case we have the following. Let K3 be the full subcategory of complex varieties whose objects are K3 surfaces and X be a K3 surface. The set

$$FM(X) := \{Y \in K3 \text{ such that there exists a Hodge isometry } T(Y) \cong T(X)\}$$

contains only finitely many isomorphism classes. The proof of this result is due to Mukai [129] and builds on the derived Torelli theorem and the finiteness of the Fourier-Mukai partners. See also [94, Proposition 16.3.10, Corollary 16.3.7 and Corollary 16.3.8] and [138, Proposition 4.4] for a direct argument which we emulate in the next lemma.

Lemma 4.4.2. *Let $X/\mathbb{C}$ be a K3 surface. The set*

$$S := \{Y \in K3 \textit{ such that } T(Y)_{\mathbb{Q}} \cong T(X)_{\mathbb{Q}} \textit{ as } \mathbb{Q}-PHS\}/\textit{isomorphism},$$

is either finite or countable.

Proof. Let $Y \in S$, by reasoning as in [138, Prop 4.4], it is enough to show that there are at most countably many choices for the rank and the discriminant of $T(Y)$. The rank is clearly fixed, so we have only to explain how the discriminant may vary. We notice that the discriminant of $T(Y)$ has to be equal to the discriminant of $T(X)$ modulo $(\mathbb{Q}^*)^2$, since the quadratic forms are non-degenerate and so there are countably many choices. □

Lemma 4.4.3. *Let $X/\mathbb{C}$ be a K3 surface such that the set*

$$\{{}^{\sigma}X\}/\textit{isomorphism}$$

varying $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ is at most countable. Then there exists a K3 surface $Y/\overline{\mathbb{Q}}$ such that $Y \times_{\overline{\mathbb{Q}}} \mathbb{C}$ is isomorphic to X.

Proof. This is by no means specific to K3 surfaces. Indeed it follows from a more general result due to González-Diez (and the fact that $\overline{\mathbb{Q}}$ is countable). Let X be an irreducible complex projective variety, in [80, Criterion 1 (page 3)] is proven that the following are equivalent:

a) X can be defined over $\overline{\mathbb{Q}}$;

b) The set $\{{}^{\sigma}X : \sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})\}$ contains only finitely many isomorphism classes of complex projective varieties;

c) The set $\{{}^{\sigma}X : \sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})\}$ contains only countably many isomorphism classes of complex projective varieties.

□

Remark 4.4.4. Using González-Diez's result as above, we can also offer another proof of [139, Lemma 3.6], which, for example, does not invoke the existence of the moduli space of abelian varieties (with some extra structure) established by Mumford in [130, Part II, Section 6]. Let $A/\mathbb{C}$ be an abelian variety such that all its $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$-conjugates are isogenous to A. In particular the set $\{{}^{\sigma}A : \sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})\}$ is contained in the set of complex abelian varieties isogenous to A which, up to isomorphism, is a countable set. As above, the implication c) $\Rightarrow$ a) shows that A can be defined over $\overline{\mathbb{Q}}$.

Proof of Theorem 4.1.4. Thanks to Lemma 4.4.2, we may apply Lemma 4.4.3 which produces the desired model of X. □

Chapter 5

Finite descent obstruction for Hilbert modular varieties (joint with G. Grossi)

Let S be a finite set of primes. We prove that a form of finite Galois descent obstruction is the only obstruction to the existence of $\mathbb{Z}_S$-points on integral models of twists of Hilbert modular varieties, extending a result of D.Helm and F.Voloch about modular curves. Let L be a totally real field. Under (a special case of) the absolute Hodge conjecture and a weak Serre's modularity conjecture for mod ℓ representations of the absolute Galois group of L, we prove that the same holds also for the $\mathcal{O}_{L,S}$-points. The work presented here appeared in the paper [13].

5.1 Introduction

As discussed in Chapter 1, a leading problem in arithmetic geometry is to determine whether or not an equation with coefficients in a number field F has any solutions. Since there can be no algorithm determining whether a given Diophantine equation is soluble in the integers $\mathbb{Z}$, one usually tries to understand the problem under strong constraints of the geometry of the variety defined by such equation or assuming the existence of many *local* solutions. In the case of curves, for example, Skorobogatov [163] asked whether the Brauer–Manin obstruction is the only obstruction to the existence of rational points. The question, or variations thereof, attracted the attention of Bruin, Harari, Helm, Poonen, Stoll and Voloch among others. In particular Helm and Voloch [91] studied a form of the finite Galois descent obstruction for the integral points of modular curves. To goal of our chapter is to present a case of arbitrary large dimensional Shimura varieties that can be treated similarly to modular curves. More precisely, we give sufficient conditions for the existence of $\mathcal{O}_{L,S}$-integral points on (twists of) Hilbert modular varieties associated to K, where both L and K are totally real fields.

The philosophy underlying our strategy is the relation, predicted by the Langlands' programme, between the worlds:

$$\text{Automorphic forms} \leftrightarrow \text{Motives} \leftrightarrow \text{Galois representations.}$$

We refer to [35] for an introduction to such circle of ideas (the latter arrow was already discussed in section 2.5 and in the previous chapter). More precisely, from a system of Galois representations that "looks like" the one coming from an abelian variety with $\mathcal{O}_K$-multiplication, we want to produce, via Serre's conjecture, a Hilbert modular form over L with Fourier coefficients in K. To this modular form we attach, via Eichler–Shimura theory, an abelian variety over L with $\mathcal{O}_K$-multiplication, which will correspond to an L-point on the Hilbert modular variety over K. If $L = \mathbb{Q}$, Serre's conjecture is known to hold true by the work of Khare and Wintenberger [98] and the Eichler–Shimura theory has been worked out by Shimura [160]. If $L \neq \mathbb{Q}$, to make such a strategy work, we will need to assume Serre's conjecture and also (a special case of) the absolute Hodge conjecture, where the latter is required by Blasius in [19] in order to attach abelian varieties to Hilbert modular forms. In the next section we present in more details the main results of the chapter.

5.1.1 Main results

Let L, K be totally real extensions of $\mathbb{Q}$ and set

$$n_L := [L : \mathbb{Q}], \quad n_K := [K : \mathbb{Q}].$$

We denote by w a place of L and by v a place of K. In what follows, one should think of L as the *field of definition* and K as the *Hecke field*. We denote by $\mathcal{O}_L$ and

$\mathcal{O}_K$ the rings of integers of L and K, by L_w (respectively K_v) the completion of L at w (respectively of K at v) and by $\mathcal{O}_{L_w}$ (respectively $\mathcal{O}_{K_v}$) the ring of integers of L_w (respectively K_v). Finally $\mathrm{Gal}(\overline{L}/L)$ will denote the absolute Galois group of L.

Let S be a finite set of places of L (including all archimedean places). Our initial datum is a system of Galois representations

$$\rho_v : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(K_v) \tag{$\mathcal{S}$}$$

for every v finite place of K such that:

($\mathcal{S}$.1) $\{\rho_v\}_v$ is a weakly compatible system of Galois representations;

($\mathcal{S}$.2) $\det(\rho_v) = \chi_\ell$, where χ_ℓ is the ℓ-adic cyclotomic character and $v \mid \ell$;

($\mathcal{S}$.3) The residual representation $\bar{\rho}_v$ is finite flat at $w \mid \ell$, for all $v \mid \ell$ such that $\bar{\rho}_v$ is irreducible and ℓ is not divisible by any prime in S;

($\mathcal{S}$.4) $\bar{\rho}_v$ is absolutely irreducible for all but finitely many v;

($\mathcal{S}$.5) The field generated by the trace of $\rho_v(\mathrm{Frob}_w)$ for every w is not smaller than K.

We discuss separately the case when $n_L = 1$ and $n_L > 1$, to make clear which case is conjectural and which one is not.

Theorem 5.1.1. *If $L = \mathbb{Q}$, there exists an n_K-dimensional abelian variety $A/\mathbb{Q}$ with $\mathcal{O}_K$-multiplication, such that, for every v, the v-adic Tate module of A, denoted by T_vA, is isomorphic to ρ_v as representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Theorem 5.1.2. *Assume $n_L > 1$. Under the validity of the absolute Hodge–conjecture (more precisely Conjecture 2.5.8) and a suitable generalisation of Serre's conjecture (Conjecture 5.1.4), there exists an n_K-dimensional abelian variety A/L with $\mathcal{O}_K$-multiplication, such that, for every v, T_vA is isomorphic to ρ_v as representations of* $\mathrm{Gal}(\overline{L}/L)$.

We apply the above theorems to study the finite descent obstruction of Hilbert modular varieties. For an introduction to descent obstruction, we refer to [163, 164, 89]; a recap is given in section 5.4.1. More precisely denote by Y_K the Hilbert modular variety associated to K. Let $\mathfrak{N}$ be an ideal in $\mathcal{O}_K$ and denote by $Y_K(\mathfrak{N})$ the moduli space of n_K-dimensional abelian variety, principally $\mathcal{O}_K$-polarized and with $\mathfrak{N}$-level structure (see section 5.2.1.1 for a precise definition). As a corollary of the above theorems, we prove that the finite Galois descent obstruction (as defined in section 5.4) is the only obstruction to the existence of S-integral points on integral models of twists of Hilbert modular varieties, denoted by $\mathcal{Y}_K(\mathfrak{N})$, over the ring of S-integers of a totally real field L, generalising [91, Theorem 3]. Assume that S contains the places of bad reduction of $Y_K(\mathfrak{N})$. The set $\mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$ will be defined in section 5.4.1. We prove the following.

Theorem 5.1.3. *If $n_L > 1$, assume the conjectures of Theorem 5.1.2. Let $\mathcal{Y}_\rho$ be the S-integral model of a twist of $\mathcal{Y}_K(\mathfrak{N})$, corresponding to a representation $\rho : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(\mathcal{O}_K/\mathfrak{N})$. If $\mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$ is non-empty then $\mathcal{Y}_\rho(\mathcal{O}_{L,S})$ is non-empty.*

In the work of Helm–Voloch, $\mathcal{Y}$ is the integral model of an affine curve. In the case of curves there are also other tools to establish (variants) of such results, without invoking Serre's conjecture. Indeed, as noticed after the proof of [91, Theorem 3], Stoll [164, Corollary 8.8] proved a similar result, under some extra assumptions, using the fact that a factor of the Jacobian of such modular curves has finite Mordell-Weil and Tate-Shafarevich groups. The goal of this chapter is to push Helm–Voloch's strategy to a particular class of varieties of arbitrary large dimension and whose associated Albanese variety is trivial (see Theorem 5.2.2) showing that the method could be applied to study also L-points.

5.1.2 Serre's weak conjecture over totally real fields

Having already discussed the Absolute Hodge conjecture in Section 2.5.3.1, we now briefly state the conjecture of Serre appearing in Theorem 5.1.2 needed to obtain the main theorem when $L \neq \mathbb{Q}$. See for example [29, Conjecture 1.1], where it is referred to as a folklore generalisation of Serre's conjecture. For more details we refer to the introduction of [29] and references therein.

Conjecture 5.1.4. *Let ℓ be any odd prime and $\overline{\rho} : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be an irreducible and totally odd*[1] *Galois representation. Then there exists some Hilbert modular eigenform f for L such that $\overline{\rho}$ is isomorphic to the reduction mod λ of $\rho_{f,\lambda}$, where $\rho_{f,\lambda}$ is the λ-adic Galois representation attached to f and λ is a prime of the Hecke field of f dividing ℓ.*

Remark 5.1.5. This is usually referred to as *weak* Serre's conjecture, because there is no explicit recipe to compute the weight $k(\overline{\rho})$ and the level $N(\overline{\rho})$. It has been proven (see [79] and [75]) that such *refined* version follows from the weak version under some assumptions. We state the needed results in Theorem 5.3.5.

When $L = \mathbb{Q}$, this was proven by Khare and Wintenberger [98]. When $L \neq \mathbb{Q}$, Conjecture 5.1.4 is known when the coefficient field is $\mathbb{F}_3$ (Langlands-Tunnell [107, 172]), but we need to assume that the conjecture holds for all (but finitely many) ℓ's. Our strategy follows indeed the lines of the proof of modularity theorems assuming Serre's conjecture: starting from a system of Galois representations, we produce a Hilbert modular form whose Fourier coefficients are equal to the traces of Frobenii modulo infinitely many primes and hence are equal as elements of $\mathcal{O}_K$.

Finally, a potential version of the above conjecture has been proven in [168, Theorem 1.6]. Taylor proves a potential modularity result, i.e. if $\bar{\rho} : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ is a totally odd irreducible representation with determinant equal to the cyclotomic character, then there exists L'/L a finite totally real Galois extension such that all the primes of L above ℓ split in L' and there exists f a Hilbert modular form for L' such that $\bar{\rho}_{f,\lambda'}$ is isomorphic to $\bar{\rho}$ restricted to $\mathrm{Gal}(\overline{L'}/L')$.

5.1.3 Related work

We compare our results with [139, Theorem 3.1 and Theorem 3.7], the main theorem of Chapter 4 and the work of Klevdal [101], where, however, a finite extension of the base field is required. In the approach of Patrikis, Voloch and Zarhin there

[1] Here totally odd means that $\det(\overline{\rho}(c)) = -1$ for all n_L complex conjugations c.

are no restrictions on the base field, wheras here it is crucial that L is a totally real field. We believe that it is easier to make the results of this chapter unconditional. We notice here that the absolute Hodge conjecture is not enough for such results: in [139] and Chapter 4 the Hodge conjecture is not only needed to descend complex abelian varieties over number fields. Finally the version of Serre's conjecture we are assuming here is always about GL_2-coefficients and so is certainly easier than the full Fontaine-Mazur conjecture [71].

Outline of chapter

In Section 5.2 we collect all the results we need about Hilbert modular forms (especially how Eichler-Shimura works in this setting). In Section 5.3, which is the heart of the chapter, we prove Theorem 5.1.1 and 5.1.2. We then explain how these results are related to the finite descent obstruction for Hilbert modular varieties in Section 5.4, eventually proving Theorem 5.1.3.

5.2 Recap on Hilbert modular varieties and modular forms

5.2.1 Hilbert modular varieties

In Example 2.2.10, we briefly discussed Hilbert modular varieties. We present here a more complete and precise description of such Shimura varieties. Let $F/\mathbb{Q}$ be a totally real extension of degree n_F and fix $\{\sigma_i\}_{i=1}^{n_F}$ the set of real embeddings of F into $\mathbb{C}$. We let G be the $\mathbb{Q}$-algebraic group obtained as the Weil restriction of GL_2 from F to $\mathbb{Q}$ and X be n_F copies of $\mathbb{H}^{\pm} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) \neq 0\}$, on which $G(\mathbb{Q}) = \mathrm{GL}_2(F)$ acts on the ith component via σ_i, i.e.

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau_1, \dots, \tau_{n_F})\right)_i = \frac{\sigma_i(a)\tau_i + \sigma_i(b)}{\sigma_i(c)\tau_i + \sigma_i(d)}.$$

In this case, the reflex field of (G,X) is $\mathbb{Q}$ and the set of geometrically connected components of the Shimura variety $S := \mathrm{Sh}_{G(\widehat{\mathbb{Z}})}(G,X)$ is $\mathrm{Pic}(\mathcal{O}_F)^+$ (different choices of level structure will appear later).

Remark 5.2.1. To obtain a Shimura variety from the above construction, it is fundamental that F is totally real. Indeed if F is a number field, G an algebraic F-group, then the real points of $\mathrm{Res}_{F/\mathbb{Q}} G$ have a structure of Hermitian symmetric space if and only if F is a totally real field and the symmetric space associated to each real embedding of F is Hermitian.

It is interesting to notice here a first difference between modular curves (i.e. when $F = \mathbb{Q}$) and higher dimensional Hilbert modular varieties. We recall the following folklore result, see [19, Section 2.3.2.], to see how it follows from Matsushima's formula [21, Theorem VII.5.2].

Theorem 5.2.2. *Let (G,X) be a Shimura datum as above and let $S_{\widetilde{K}}$ be the Hilbert modular variety associated to (G,X) and a neat $\widetilde{K}$. Unless $n_F = 1$, the first group of Betti cohomology of $S_{\widetilde{K}}$, with rational coefficients, is trivial. In particular there are no non-constant maps from $S_{\widetilde{K}}$ to an abelian variety.*

To have a better interpretation as moduli space we can actually consider the subgroup G^* of G given by its elements whose determinant is in $\mathbb{Q}$. More precisely we let G^* be the pull-back of

$$\det : G \to \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m$$

to $\mathbb{G}_m/\mathbb{Q}$. The Shimura variety $Y_F := \mathrm{Sh}_{G^*(\widehat{\mathbb{Z}})}(G^*, X^*)(\mathbb{C})$ is connected and comes with a finite map to $S/\mathbb{C}$. It is a quasi projective n_F-dimensional $\mathbb{Q}$-scheme.

In the next section we present the moduli problem solved by Y_F. It will be clear also from such moduli interpretation that the reflex field of Y_F is the field of rational numbers.

5.2.1.1 Hilbert modular varieties as moduli spaces

As explained for example in [61, Section 3], the Shimura variety Y_F represents the (coarse) moduli space for triplets (A, α, λ) where:

- A is a complex abelian variety of dimension n_F;
- $\alpha : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$ is a morphism of rings;

- $\lambda : A \to A^*$ is a principal $\mathcal{O}_F$-polarisation.

By A^* we denoted the $\mathcal{O}_F$-dual abelian variety of A, i.e. it is defined as $\mathrm{Ext}^1(A, \mathcal{O}_F \otimes \mathbb{G}_m)$. Otherwise one can obtain such abelian variety considering $A^\vee$ (the dual of A, in the standard sense) and tensoring it over $\mathcal{O}_F$ with the different ideal of the extension $F/\mathbb{Q}$. By principal $\mathcal{O}_F$-polarisation we mean an isomorphism $\lambda : A \to A^*$, such that the induced map $A \to A^\vee$ is a polarisation.

Furthermore, the Shimura variety of level

$$U_0(\mathfrak{N}) := \left\{ \gamma \in G(\widehat{\mathbb{Z}}) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod \mathfrak{N} \right\},$$

where $\mathfrak{N}$ is an integral ideal of $\mathcal{O}_F$, parametrises triplets as above, equipped with a $\mathfrak{N}$-level structure as follows. We fix an isomorphism of $\mathcal{O}_F$-modules

$$(\mathcal{O}_F/\mathfrak{N}\mathcal{O}_F)^2 \to A[\mathfrak{N}]$$

making the following diagram commutative:

$$\begin{array}{ccc} \left((\mathcal{O}_F/\mathfrak{N}\mathcal{O}_F)^2\right)^2 & \longrightarrow & A[\mathfrak{N}]^2 \\ \downarrow{\scriptstyle \psi_{\mathfrak{N}}} & & \downarrow{\scriptstyle e_{\lambda,\mathfrak{N}}} \\ \mathcal{O}_F \otimes \mathbb{Z}/N\mathbb{Z} & \longrightarrow & \mathcal{O}_F \otimes \mu_N, \end{array}$$

where $(N) = \mathbb{Z} \cap \mathfrak{N}$, $\psi_{\mathfrak{N}}$ is the pairing given by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $e_{\lambda,\mathfrak{N}}$ is the perfect Weil pairing on $A[\mathfrak{N}]$ induced by the $\mathcal{O}_F$-polarisation λ and $\mathcal{O}_F \otimes \mathbb{Z}/N\mathbb{Z} \to \mathcal{O}_F \otimes \mu_N$ is an arbitrarily chosen isomorphism. When a level structure is needed, we always assume that $N > 3$ to have a *fine* moduli space. A rational point of $Y_F(\mathfrak{N}) := \mathrm{Sh}_{U_0(\mathfrak{N}) \cap G^*}(G^*, X)$ represents then a triple as above together with such level structure. In section 5.4.2, we will see a similar description for $\mathcal{O}_{L,S}$-points of an *integral* model of $Y_F(\mathfrak{N})$.

5.2.2 Eichler–Shimura theory

We discuss Eichler–Shimura theory for classical and Hilbert modular forms, reviewing results that attach opportune abelian varieties to modular forms.

5.2.2.1 Classical modular eigenform

The following is [160, Theorem 7.14, page 183 and Theorem 7.24, page 194].

Theorem 5.2.3 (Shimura). *Let f be a holomorphic newform of weight 2 with rational Fourier coefficients $(a_n(f))_n$. There exists an elliptic curve $E/\mathbb{Q}$ such that, for all primes p at which E has good reduction one has*

$$a_p(f) = 1 - N_p(E) + p,$$

where $N_p(E)$ denotes the number of points of the reduction mod p of E, over the field with p-elements. In other words, up to a finite number of Euler factors, $L(s,E/\mathbb{Q})$ and $L(s,f)$ coincide.

More generally, let $K(f)$ be the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by $(a_n(f))_n$ for all n. Then there exists an abelian variety $A/\mathbb{Q}$ and an isomorphism $K(f) \cong \mathrm{End}^0(A)$ with the following properties:

- $\dim(A) = [K(f):\mathbb{Q}]$;
- *Up to a finite number of Euler factors at primes at which A has good reduction $L(s,A/\mathbb{Q},K(f))$ coincides with $L(s,f)$;*

where the L-function $L(s,A/\mathbb{Q},K(f))$ is defined by the product of the following local factors, where v is a prime of K not dividing ℓ

$$\det(1-\ell^{-s}\mathrm{Frob}_\ell|T_v(A)).$$

Shimura considers the Jacobian of the modular curve of level N and takes the quotient by the kernel of the homomorphism giving the Hecke action on f. What happens if we want to produce an abelian variety with such properties, defined over our totally real field F, when $\mathbb{Q} \subsetneq F$? Here is where Hilbert modular forms come into play. In the next section we discuss what we need from such theory and explain Blasius' generalisation of Theorem 5.2.3 and why the absolute Hodge conjecture is needed.

5.2.2.2 Hilbert modular forms for F

With the same notation as in the previous sections, we now recall the definition of Hilbert modular forms. Consider $\mathbb{H}_F$, which is defined to be n_F copies of the upper half plane $\mathbb{H}^+$. We fix $\{\sigma_1,\ldots,\sigma_{n_F}\}$ the set of all real embeddings of $F \hookrightarrow \mathbb{C}$. We consider subgroups $\Gamma \subset \mathrm{GL}_2(\mathcal{O}_F)$ of the form $U_0(\mathfrak{N}) \cap G(\mathbb{Q})^+$. Moreover for $\lambda \in F$ and $\underline{r} = (r_1,\ldots,r_{n_F}) \in \mathbb{Z}^{n_F}$, we write $\lambda^{\underline{r}} = \lambda_1^{r_1}\cdots\lambda_{n_F}^{r_{n_F}}$, where $\lambda_i = \sigma_i(\lambda)$. Similarly if $\underline{\tau} = (\tau_1,\ldots,\tau_{n_F}) \in \mathbb{H}_F$, we write $\underline{\tau}^{\lambda} = \tau_1^{r_1}\cdots\tau_{n_F}^{r_{n_F}}$.

Definition 5.2.4. A Hilbert modular form of level $\mathfrak{N}$ and weight $(\underline{r},w) \in \mathbb{Z}^{n_F}\times\mathbb{Z}$, with $r_i \equiv w \mod 2$ (and trivial nebentype character) is a holomorphic function $f: \mathbb{H}_F \to \mathbb{C}$ such that

$$f(\gamma\cdot\underline{\tau}) = (\det\gamma)^{-\underline{r}/2}(c\underline{\tau}+d)^{\underline{r}}f(\underline{\tau}),$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in U_0(\mathfrak{N}) \cap G(\mathbb{Q})^+$ and for every $\underline{\tau} = (\tau_1,\ldots,\tau_{n_L}) \in \mathbb{H}_F$.

Since Hilbert modular forms are holomorphic functions on $\mathbb{H}_F$ invariant under the lattice $\mathcal{O}_F$, they admit a q-expansion over $\mathcal{O}_F^{\vee} = \mathfrak{d}^{-1}$, where $q = e^{2\pi i \sum \tau_i}$, see [82, Definition 3.1] for more details.

5.2.2.3 Hecke operators and Hilbert eigenforms

On the space of Hilbert modular forms of level

$$U_0(\mathfrak{N}) \cap G(\mathbb{Q})^+$$

one has Hecke operators $T(\mathfrak{n})$ for every integral ideal of $\mathcal{O}_F$ coprime with $\mathbb{N}$. The definition is analogous to the one for classical modular forms. For example, if $\mathfrak{p}$ does not divide $\mathbb{N}$ and x is a totally positive generator of $\mathfrak{p}$, one defines

$$(T(\mathfrak{p})f)(\underline{\tau}) := \mathrm{Nm}(\mathfrak{p})f(x\cdot\underline{\tau}) + \frac{1}{\mathrm{Nm}(\mathfrak{p})}\sum_{a\in\mathcal{O}_F/\mathfrak{p}} f(\gamma_a\cdot\underline{\tau}),$$

where $\gamma_a := \left(\begin{smallmatrix} 1 & a \\ 0 & x \end{smallmatrix}\right)$. We have the following.

Definition 5.2.5. A cuspidal Hilbert modular form (i.e. such that the 0-th Fourier coefficient $a_0(f)$ vanishes) is an eigenform if it is an eigenvector for every Hecke operator $T(\mathfrak{n})$.

As in the case of classical modular forms, if f is an eigenform, normalised so that $a_1(f) = 1$, then the eigenvalues of the Hecke operators are the Fourier coefficients, i.e. $T(\mathfrak{n})f = a_{\mathfrak{n}}(f) \cdot f$; moreover they are algebraic integers lying in the number field $K(f) := \mathbb{Q}((a_{\mathfrak{n}}(f))_{\mathfrak{n}})$, as shown in [159, §2].

5.2.2.4 Eichler–Shimura for Hilbert modular form

Blasius and Rogawski [20], Carayol [33] and Taylor [167] proved that to any Hilbert eigenform, one may attach representations of $\mathrm{Gal}(\overline{F}/F)$ in a similar way to the classical case. They proved the following.

Theorem 5.2.6. *If f is a Hilbert eigenform for F of weight $(\underline{r}, t)$ and level $\mathfrak{N}$, trivial nebentype character and $K(f)$ is the number field generated by its eigenvalues, then for every finite place λ of $K(f)$ there is an irreducible 2-dimensional Galois representation*

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(K(f)_\lambda)$$

such that for every prime $w \nmid \mathfrak{N}\mathrm{Nm}_{K(f)/\mathbb{Q}}(\lambda)$ in F, $\rho_{f,\lambda}$ is unramified at w and

$$\det(1 - X\rho_{f,\lambda}(\mathrm{Frob}_w)) = X^2 - a_w(f)X + \mathrm{Nm}_{F/\mathbb{Q}}^{t-1}(w).$$

Assume that f is of weight $(2, \dots, 2)$. As in the classical case, we would like to have such Galois representations to be attached to opportune abelian varieties. The existence of abelian varieties associated to f was first considered by Oda in [134]; Blasius gave a conjectural solution to the problem.

Theorem 5.2.7 (Blasius)**.** *Let f be a Hilbert eigenform for F of parallel weight 2. Denote by $K(f)$ the number field generated by the $a_w(f)$ for all w. Assume Conjecture 2.5.8. There exists an $[K(f) : \mathbb{Q}]$-dimensional abelian variety A_f/F*

with $\mathcal{O}_{K(f)}$-multiplication such that for all but finitely many of the finite places w of F at which A_f has good reduction, we have

$$L(s, A_f, K(f)) = L(f, s),$$

where the L-function $L(s, A_f, K(f))$ is defined by the product of local factors

$$\det(1 - \mathrm{Nm}(w)^{-s} \mathrm{Frob}_w \,|\, T_v(A)^{I_w}),$$

where v is a prime of K and w is a prime of F lying above distinct rational primes.

Proof. If $K(f) = \mathbb{Q}$, this is precisely [19, Theorem 1 on page 3]. As noticed by Blasius [19, 1.10], the proof easily adapts to the general case (where the necessary changes are hinted in the remarks in section [19, 5.4., 5.7., 7.6.]). □

Remark 5.2.8. The proof is completely different from the one of Shimura, since, as noticed in Theorem 5.2.2, we can not obtain a non-trivial abelian variety as quotient of the Albanese variety of a Hilbert modular variety. Blasius instead considers the symmetric square of the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F^f)$ associated to the Hilbert eigenform; it is an automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F^f)$ and its base change to a quadratic imaginary field appears in the middle degree cohomology of a Picard modular variety (see Example 2.2.13). He then considers the associated motive and shows that its Betti realisation is the symmetric square of a polarised Hodge structure of type $(1,0),(0,1)$. This gives a complex abelian variety A and Conjecture 2.5.8 (applied to the product of the Picard modular variety and A) lets him conclude that A is defined over a number field containing F. He then finds the desired abelian variety inside the restriction of scalars of A over F.

Remark 5.2.9. Theorem 5.2.7 is known to hold unconditionally in many interesting cases (for example when n_F is odd, by the work of Hida). For more details we refer to [19, Theorem 3] and references therein. The proof of such unconditional cases actually follows Shimura's proof of Theorem 5.2.3, rather than the strategy described in the previous remark.

5.2.3 A remark on polarisations

To use Theorem 5.2.3 and 5.2.7 to produce F-points of a Hilbert modular variety, we need of course the abelian varieties produced to be principally polarised (up to isogeny would actually be enough for our applications, if the isogeny is defined over the base field F). An abelian variety over an algebraically closed base field always admits an isogeny to a principally polarised abelian variety. But since the same does not hold over number fields, some considerations are needed. The first observation is that every weight one Hodge structure of dimension 2 with an action by $\mathcal{O}_{K(f)}$, is automatically $\mathcal{O}_{K(f)}$-polarised, as explained for example in [61, Appendix B].

As noticed in [19, Remark 5.7.], Blasius first finds a principally polarised abelian variety A over a finite extension L'/F. Actually we can assume that A has a principal $\mathcal{O}_{K(f)}$-polarisation λ. As explained above, the proof considers then the Weil restriction of A to F, which is again principally $\mathcal{O}_{K(f)}$-polarised and finds here the desired abelian variety. It is not hard to see that the construction of [19, section 7] behaves well with respect to the $\mathcal{O}_{K(f)}$-polarisation and so the proof actually produces an $\mathcal{O}_{K(f)}$-polarised abelian variety over F.

5.3 Producing abelian varieties via Serre's conjecture

In this section we prove Theorems 5.1.1 and 5.1.2. As in Section 5.1.1, here L and K are totally real fields of degree n_L and n_K respectively. We work with a compatible system of Galois representations of $\mathrm{Gal}(\overline{L}/L)$ with values in $\mathrm{Res}_{K/\mathbb{Q}}(\mathrm{GL}_2(\mathbb{A}_f)) = \mathrm{GL}_2(\mathbb{A}_{f,K})$ that “looks like” an algebraic point of the Hilbert modular variety for K. We then produce a Hilbert modular form for L of weight $(2,\dots,2)$ and opportune explicit conductor and from this we obtain an abelian variety over L that will allow us to produce a L-rational point on the Hilbert modular variety for K.

5.3.1 Weakly compatible systems with coefficients

Definition 5.3.1 (Weakly compatible system)**.** A family $\{\rho_v : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(K_v)\}_v$ is *weakly compatible* if there exists a finite set of places S of L such that:

(i) For all places w of L, ρ_v is unramified outside the set S_v. Here we denoted by S_v the union of S and all the primes of L dividing ℓ where ℓ is the residue characteristic of K_v.

(ii) For all $w \notin S_v$, denoting by Frob_w a Frobenius element at w, the characteristic polynomial of $\rho_v(\mathrm{Frob}_w)$ has K-rational coefficients and it is independent of v.

From the Weil conjectures, it can be easily shown that the ℓ-adic Tate modules of abelian varieties form a weakly compatible family of Galois representations. The case of $\mathbb{Q}_\ell$-coefficients was already discussed in Chapter 2, see indeed section 2.3.1.

5.3.2 Key proposition

Let S be a finite set of places of L, including all archimedean places. Let $K/\mathbb{Q}$ be a totally real field extension of degree n_K. We now work with a system of Galois representations

$$\rho_v : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(K_v)$$

for every v finite place of K satisfying the conditions $(\mathcal{S}.1)$-$(\mathcal{S}.4)$ as in $(\mathcal{S})$. In order to prove Theorem 5.1.1 and Theorem 5.1.2 we need the following.

Proposition 5.3.2. *Assume Conjecture 5.1.4 and let* $(\rho_v)_v$ *a system of representations satisfying conditions* $(\mathcal{S}.1)$-$(\mathcal{S}.4)$. *For every* $w \notin S$, *let* $a_w \in \mathcal{O}_K$ *be the trace of* $\rho_v(\mathrm{Frob}_w)$. *Then there exists* f *a normalised Hilbert eigenform for* L *with Fourier coefficients in* $\mathcal{O}_K$, *such that for every* $w \notin S$, $a_w(f) = a_w$. *Moreover* f *is of weight* $(2,\dots,2)$ *and conductor divisible only by primes in* S.

Remark 5.3.3. Given an abelian variety A/L with $\mathcal{O}_K$-multiplication, we can produce a system

$$\rho_v : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}(T_v(A))$$

which satisfies the four conditions of $(\mathcal{S})$ for S the union of infinite places and the set of places of bad reduction of A, see [151, §3]. Proposition 5.3.2 hence implies that A is modular, i.e. there exists a Hilbert modular form for the totally real field L, such that

$$L(A/L,s) = L(f,s)$$

up to a finite number of Euler factors. Unconditionally, it has been proven that elliptic curves over real quadratic fields are modular (see [74]) and, more in general, the work of Taylor and Kisin implies that elliptic curves over L become modular (in this sense) after a totally real extension L'/L. See [28, Theorem 1.16] and reference therein.

In the proof of Proposition 5.3.2 we use Conjecture 5.1.4 and the following result due to Serre (for the proof see [155, 4.9.4]).

Proposition 5.3.4 (Serre). *Let q be a power of ℓ. Let $r : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_2(\mathbb{F}_q)$ a continuous homomorphism, where $q = \ell^t$ and E is a local field of residue characteristic $p \neq \ell$ and discrete valuation v_E. Let $e_E := v_E(p)$ and $c \geq 0$ be an integer such that the image via r of the wild inertia of E has cardinality p^c. We denote by $n(r,E)$ the exponent of the conductor of r. We have*

$$n(r,E) \leq 2\left(1 + e_E \cdot c + \tfrac{e_E}{p-1}\right).$$

We also need to compute the weight and the conductor of the modular forms produced by Conjecture 5.1.4. As anticipated in Remark 5.1.5, this is a known result under some assumptions. The weight part stated in the following theorem is a special case of the work [79]; the conductor part follows from automorphy lifting methods or can be seen as a consequence of the main theorem of [75].

Theorem 5.3.5 ([79, 75]). *Let $\ell > 5$ and $\bar{\rho} : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be an irreducible totally odd representation such that its determinant is the cyclotomic character and it is finite flat at all places $w \mid \ell$. Assume furthermore that $\bar{\rho}$ satisfies the Taylor–Wiles assumption, namely*

$$\bar{\rho}_{|\mathrm{Gal}(\overline{L(\zeta_\ell)}/L(\zeta_\ell))} \text{ is irreducible,} \tag{TW}$$

where ζ_ℓ is a primitive ℓth root of unity. Then if $\bar{\rho}$ is modular, there exists a Hilbert modular form of parallel weight 2 and conductor equal to the Artin conductor of $\bar{\rho}$ giving rise to $\bar{\rho}$.

We are ready to prove Proposition 5.3.2.

Proof of Proposition 5.3.2. Our goal is to apply Serre's conjecture to $\bar{\rho}_v$, the reduction modulo v of the representation ρ_v, for infinitely many $v \notin S_K$, where S_K is the following finite set of primes of K:

$$S_K = \{v :\ v \mid \ell \text{ and } w \mid \ell \text{ for some } w \in S \text{ or } \ell \text{ is ramified in } L\}.$$

Let v be such a prime, let ℓ be its residue characteristic.

We now want to compute the conductor $N(\bar{\rho}_v)$. Since $\bar{\rho}_v$ is unramified outside S_v, the conductor is divisible only by primes in S. We then apply Proposition 5.3.4 to $E = L_w$ and $r = \bar{\rho}_v$. The image of $\bar{\rho}_v$ s contained in $\mathrm{GL}_2(\mathbb{F}_{\ell^t})$, where $t \leq [K:\mathbb{Q}] = n_K$. The cardinality of this group is $\ell^t(\ell^{2t}-1)(\ell^t-1)$. Let W_w denote the wild inertia subgroup of $\mathrm{Gal}(\overline{L_w}/L_w)$. If ℓ satisfies the following congruences

$$\ell^{n_K} \not\equiv \begin{cases} \pm 1 \mod p & \text{if } p \neq 2,3 \\ \pm 1 \mod 8 & \text{if } p = 2 \\ \pm 1, 4, 7 \mod 9 & \text{if } p = 3, \end{cases} \tag{$\star$}$$

then the same congruences hold for ℓ^t and hence $\bar{\rho}_v(W_w)$ is trivial if $p \neq 2,3$ and is at most p^5 if $p = 2$ and at most p if $p = 3$. Hence for $v \notin S$ laying above ℓ satisfying the above conditions, using that $e_E \leq [L:\mathbb{Q}] = n_L$, the inequality of Proposition 5.3.4 implies

$$n_K(\bar{\rho}_v, L_w) \leq \begin{cases} 2(1+n_L) & \text{if } p \neq 2,3 \\ 2(1+6n_L) & \text{if } p = 2 \\ 2(1+2n_L) & \text{if } p = 3. \end{cases}$$

Writing, $\mathfrak{p}_w$ for the prime ideal of L corresponding to w, we hence find that the conductor of $\bar{\rho}_v$ divides

$$\mathfrak{C} := \prod_{\substack{w\in S,\\ w\nmid 2,3}} \mathfrak{p}_w^{2+2n_L} \cdot \prod_{\substack{w\in S,\\ w\mid 2}} \mathfrak{p}_w^{2+12n_L} \cdot \prod_{\substack{w\in S,\\ w\mid 3}} \mathfrak{p}_w^{2+4n_L}.$$

Finally, notice that ρ_v is odd thanks to the condition on the determinant and, moreover, [15, Proposition 5.3.2] implies that there exists a density one set of primes such that $(\bar{\rho}_v)_{|G_{L(\zeta_\ell)}}$ is irreducible, i.e. (TW) is satisfied.

We can hence apply Conjecture 5.1.4 to $\bar{\rho}_v$ for $v \in \Sigma$, where Σ is the infinite set of primes $v \mid \ell$ such that $v \notin S_K$, ℓ satisfies $(\star)$, $\bar{\rho}_v$ is absolutely irreducible and satisfies (TW). We have produced infinitely many f_v Hilbert modular eigenform, which by Theorem 5.3.5 are of parallel weight 2 and level dividing $\mathfrak{C}$. Their Fourier coefficients are defined over a ring $\mathcal{O}(v) \subset \mathcal{O}_K$ and at a prime $\lambda \mid v$ the associated Galois representation $\rho_{f_v,\lambda}$ is isomorphic to $\bar{\rho}_v$ modulo λ. Since the space of Hilbert modular form of fixed weight and with conductor dividing $\mathfrak{C}$ is finite dimensional (see [73, Theorem 6.1]), we can find at least one Hilbert modular eigenform f of parallel weight 2 and level dividing $\mathfrak{C}$ defined over some $\mathcal{O} \subset \mathcal{O}_K$ such that for infinitely many of the v above the same property holds for $\rho_{f,\lambda}$ for $\lambda \mid v$. This implies that for all $w \notin S$ the congruence

$$a_w(f) \equiv a_w \mod \lambda \mid v$$

holds for infinitely many primes λ and hence $a_w(f) = a_w$ as required. □

5.3.3 Proof of Theorems 5.1.1 and 5.1.2

Recall that, as in Section 5.1.1, the system $(\rho_v)_v$ is required to satisfy the following additional property:

$$\text{the field generated by } a_w \text{ for every } w \text{ is exactly } K. \qquad (\mathcal{S}.5)$$

In other words, with have $K(f) = K$, where f is the Hilbert modular form for L produced in Proposition 5.3.2.

Proof of the Theorems. Starting with our initial datum of Galois representations, we have produced a Hilbert modular form f for L. We can then apply Theorem 5.2.7, which gives an abelian variety A_f over L of dimension $[K:\mathbb{Q}]$ and an embedding of $\mathcal{O}_K$ into $\mathrm{End}(A)$. For all but finitely many $w \mid p$ at which A_f has good reduction

$$\det(1 - X\rho_{A_f,v}(\mathrm{Frob}_w)) = 1 - a_w(f)X + N_w X^2,$$

where v is a finite prime of K not dividing p and $\rho_{A_f,v}$ is the $\mathrm{Gal}(\overline{L}/L)$-representation on $T_v(A_f)$, the v-adic Tate module of A_f. We therefore have produced an abelian variety A_f as stated in Theorem 5.1.1 and 5.1.2.

We just need to stress that we do not require any conjectural statement in the case $n_L = 1$. Indeed we can use Theorem 5.2.3 in place of Blasius' conjectural version and Serre's conjecture is fully known thanks to the work of Khare–Wintenberger [98, Theorem 1.2]. □

5.3.4 A corollary

We rephrase the main results of the section as needed to prove Theorem 5.1.3.

Corollary 5.3.6. *Assume that, in the setting of Theorems 5.1.1 and 5.1.2, we also have a representation*

$$\rho : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(\mathcal{O}_K/\mathfrak{N})$$

for some integral ideal $\mathfrak{N} \subset \mathcal{O}_K$, such that for all pairs (v,b), where v is a place of K and b a natural number, such that v^b divides $\mathfrak{N}$, the reductions of ρ and ρ_v modulo v^b agree. Then there exists a n_K-dimensional abelian variety A/L with good reduction at all w outside S and the action of $\mathrm{Gal}(\overline{L}/L)$ *on $A[\mathfrak{N}]$ is given by ρ.*

Proof. Theorems 5.1.1 and 5.1.2 give an n_K-dimensional abelian variety A/L and, using the Néron-Ogg-Shafarevich criterion, we can see that it has good reduction at all w outside S. Finally $\mathrm{Gal}(\overline{L}/L)$ acts on $A[\mathfrak{N}]$ via ρ, since the reduction modulo v^b of ρ and ρ_v agree. □

5.4 Finite descent obstruction and proof of Theorem 5.1.3

In this final section we discuss the finite descent obstruction for integral points, explaining how it relates to the system of Galois representations considered in the previous section. Using the main theorems of the chapter, we indeed produce an $\mathcal{O}_{L,S}$-point of integral models of twists of Hilbert modular varieties, therefore proving Theorem 5.1.3.

5.4.1 Recap on the integral finite descent obstruction

Let Y/F be a smooth, geometrically connected variety (not necessarily proper) over a number field F. Let S be a finite set of places of F and, as before, assume that S contains the archimedean places and all places of bad reduction of Y. Choose and fix a smooth model $\mathcal{Y}$ of Y over $\mathcal{O}_{F,S}$. In this section we present the definition of the set $\mathcal{Y}^{f-cov}$, which corresponds to the adelic points of $\mathcal{Y}$ that are unobstructed by all Galois covers. To make the chapter self contained we recall the discussion from [91, Section 2] (where they work with affine curves). We then explain that, for Hilbert modular varieties, a point unobstructed by finite covers admits an infinite tower of twists of covers with a compatible system of lifts of adelic points along the tower (following [91, Proposition 1]).

Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a map of $\mathcal{O}_{F,S}$-schemes, such that it becomes a Galois covering over $\overline{F}$. Such map is called geometrical Galois cover of $\mathcal{Y}$. Denote by $\mathrm{Tw}(\pi)$ the set of isomorphism classes of twists of π, i.e. of maps $\pi' : \mathcal{X}' \to \mathcal{Y}$ that become isomorphic to π over $\overline{F}$. We have

$$\mathcal{Y}(\mathcal{O}_{F,S}) = \bigcup_{\pi' \in \mathrm{Tw}_0(\pi)} \pi'\left(\mathcal{X}'(\mathcal{O}_{F,S})\right),$$

where $\mathrm{Tw}_0(\pi)$ is a suitable finite subset of $\mathrm{Tw}(\pi)$ (for a more detailed discussion we refer to [163, Page 105 and 106]) and $\pi' : \mathcal{X}' \to \mathcal{Y}$ is a twist of π. In what follows, w denotes a place of F.

Definition 5.4.1. We define $\mathcal{Y}^{f-cov}(\mathcal{O}_{F,S}) = \mathcal{Y}^{f-cov}$ as the set of $(P_w)_w \in$

$\prod_{w\notin S}\mathcal{Y}(\mathcal{O}_{F_w})$ such that, for each geometrical Galois cover π, we can write

$$P_w = \pi'(Q_w), \quad \forall w \notin S$$

for some $\pi' \in \mathrm{Tw}_0(\pi)$ and $(Q_w)_w \in \prod_{w\notin S}\mathcal{X}'(\mathcal{O}_{F_w})$.

Proposition 5.4.2. *A point $(P_w)_w$ lies in $\mathcal{Y}^{f-cov}$ if and only if, for each geometrical Galois cover $\pi : \mathcal{X} \to \mathcal{Y}$, we can choose a twist $\pi' : \mathcal{X}' \to \mathcal{Y}$ and a point $(P_w)_\pi \in \prod_{w\notin S}\mathcal{X}'(\mathcal{O}_{F_w})$ lifting $(P_w)_w$ in a compatible way (i.e. if π_1, π_2 are Galois covers and π_2 dominates π_1, then π_2' dominates π_1' and $(P_w)_{\pi_2'}$ maps to $(P_w)_{\pi_1'}$).*

A few words to justify the equivalence between the two definitions are needed. This is explained in [91, Proposition 1] for curves and it relies on results from [164] (notably [164, Lemma 5.7][2]). In [164], Stoll works with projective varieties and their rational points, but what he says still holds true for the integral points of non-projective varieties. Once such differences are taken into account, the proof works in the same way in our setting.

Clearly we have $\mathcal{Y}(\mathcal{O}_{F,S}) \subset \mathcal{Y}^{f-cov}$ and so if $\mathcal{Y}^{f-cov}$ is empty, then $\mathcal{Y}(\mathcal{O}_{F,S})$ has to be empty as well. What can be said when $\mathcal{Y}^{f-cov}$ contains a point?

Definition 5.4.3. We say that the S-integral finite descent obstruction is the only obstruction for the existence of S-integral points, whenever $\mathcal{Y}^{f-cov}(\mathcal{O}_{F,S}) \neq \emptyset$ implies that $\mathcal{Y}(\mathcal{O}_{F,S})$ is non-empty.

From now on we specialise to the case of Hilbert modular varieties (and their twists).

5.4.2 Integral points on Hilbert modular varieties

Recall the notation from section 5.2.1.1. Let $Y_K(\mathfrak{N})$ be the n_K-dimensional $\mathbb{Q}$-scheme described in Section 5.2.1.1 and let N be the integer such that $\mathfrak{N} \cap \mathbb{Z} = (N)$. The set of twists of $\pi : Y_K(\mathfrak{N}) \to Y_K(1)$ over a number field F corresponds to the set of Galois representations $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathcal{O}_K/\mathfrak{N})$ whose determinant is the

[2] It is actually better to refer to the corrected version of [164] available on the author's website (www.mathe2.uni-bayreuth.de/stoll/papers/Errata-FiniteDescent-ANT.pdf).

cyclotomic character $\chi : \mathrm{Gal}(\overline{F}/F) \to (\mathbb{Z}/N\mathbb{Z})^\times$. Moreover a point $x \in Y_K(1)(F)$ lifts to a F-rational point of the twist of $Y_K(\mathfrak{N})$ corresponding to a representation ρ, if and only if ρ describes the action of $\mathrm{Gal}(\overline{F}/F)$ on the $\mathfrak{N}$-torsion of the underlying abelian variety A_x (as an $\mathcal{O}_K$-module).

Using its moduli interpretation, as described in Section 5.2.1.1, we can construct a model of $Y_K(\mathfrak{N})$ over $\mathbb{Z}$, which is smooth over $\mathbb{Z}[1/b]$, for some natural number b, divisible by N. To be more precise b depends on the level structure and the discriminant of K. Fixing such model, that we denote by $\mathcal{Y}_K(\mathfrak{N})$, we can talk about $\mathcal{O}_{F,S}$-points of $Y_K(\mathfrak{N})$, for any number field F and set of places S containing the archimedean places and the ones dividing b. Such $\mathcal{O}_{F,S}$-points then correspond to abelian varieties (with some extra structure), having good reduction outside S. Notice that N is assumed to be bigger than 3, since it is important to have a fine moduli space. For example the affine line is the moduli space of elliptic curves and has plenty of $\mathbb{Z}$-points, even though there are no elliptic curves defined over $\mathbb{Z}$.

We are ready to study the finite descent obstruction for the $\mathcal{O}_{L,S}$-points of $\mathcal{Y}_K(\mathfrak{N})$, where L is a totally real field (the fact that L is totally real is used only in the next section) and its twists. Let

$$\rho : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(\mathcal{O}_K/\mathfrak{N}).$$

be a representation whose determinant is the cyclotomic character. Assume that S contains the places w of ramification of ρ. Under such assumption, arguing as above, we can consider $\mathcal{Y}_\rho$ the S-integral model of the twist of $\mathcal{Y}_K(\mathfrak{N})$ corresponding to ρ. From now on we assume that $\mathcal{Y}_\rho(\mathcal{O}_{L,S})$ is non-empty. The next lemma relates a point $(P_w)_w \in \mathcal{Y}_\rho^{f-cov}$ to a system of Galois representations as considered in the previous section.

Lemma 5.4.4. *A point $(P_w)_w \in \mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$ corresponds to the following data:*

- *For each v finite place of K a representation $\rho_v : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(K_v)$;*
- *For each w finite place of L such that $w \notin S$ an abelian variety A_w/L_w of dimension n_K, with good reduction and $\mathcal{O}_K$-multiplication;*

such that

- *For every place v in K the action of* $\mathrm{Gal}(\overline{L_w}/L_w)$ *on* $T_v(A_w)$ *is given by the restriction of ρ_v to the decomposition group at w;*
- *For all pairs (v,a) such that v^a divides $\mathfrak{N}$, the reductions of ρ and ρ_v modulo v^a agree.*

Moreover every such system satisfies the first four conditions of ($\mathcal{S}$).

Proof. We first check, using Lemma 5.4.2, that an unobstructed point corresponds to a family of Galois representations as described above and then we show that every such family enjoys the desired properties.

Thanks to Proposition 5.4.2, we can fix a compatible system of lifts of $(P_w)_w$ on $\mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$. In particular, for each $\mathfrak{M}$ divisible by $\mathfrak{N}$, we obtain a twist $\mathcal{Y}_K(\mathfrak{M})'$ of $\mathcal{Y}_K(\mathfrak{M})$ and a compatible family of points $(P_w)_{\mathfrak{M}}$ of $\mathcal{Y}_K(\mathfrak{M})'$ lifting $(P_w)_w$. We remark here that the latter compatible family of points depends on $(P_w)_w$. Indeed, a priori, we can not simply lift ρ to mod $\mathfrak{M}$ coefficients.

By the interpretation of $\mathcal{Y}_K(\mathfrak{M})'$ as moduli space of abelian varieties, as discussed above, the point $(P_w)_{\mathfrak{M}}$ corresponds to an abelian variety A_w/L_w of dimension n_K, with good reduction and $\mathcal{O}_K$-multiplication and prescribed $\mathfrak{M}$-torsion. The other conditions are easily checked as at the end of proof of [91, Theorem 2].

The fact that the action of $\mathrm{Gal}(\overline{L_w}/L_w)$ on $T_v(A_w)$ is given by the restriction of ρ_v to the decomposition group at w ensures that ($\mathcal{S}$.1) and ($\mathcal{S}$.2) are satisfied. Moreover, since A_w has good reduction, then $A_w[v] \simeq \bar{\rho}_v$ is a finite flat group scheme over O_{K_w} for all $w \mid \ell$ if $v \mid \ell$, giving that ($\mathcal{S}$.3) also holds.

Finally, we need to show that ($\mathcal{S}$.4) is satisfied. With the three conditions above one can show, as in the proof of Proposition 5.3.2, that the conductor of $\bar{\rho}_v$ divides a fixed ideal $\mathfrak{C}$ of L. If w is such that A_w/L_w is supersingular at v, then $A_w[v] \simeq \bar{\rho}_v$ is absolutely irreducible. If there existed infinitely many v such that $\bar{\rho}_v$ is absolutely reducible, we could then write the semisemplification of $\bar{\rho}_v$ as direct sum of ϕ and $\chi_\ell\phi^{-1}$, for some character ϕ. Since $\bar{\rho}_v \simeq A_w[v]$ for $w \mid p$, we then have that A_w is ordinary and hence ϕ is unramified at w. We also know that the conductor of ϕ

divides $\mathfrak{C}$, hence if $w \equiv 1 \mod \mathfrak{C}$ we have

$$a_w(A_w) := \mathrm{Tr}(\mathrm{Frob}_w, T_v(A_w)) \equiv \chi_\ell(\mathrm{Frob}_w) + 1 \mod v.$$

Since $\chi_\ell(\mathrm{Frob}_w) = p^{[L_w:\mathbb{Q}_p]}$, we showed that if we had infinitely many v such that $\bar{\rho}_v$ is absolutely reducible, we would find $a_w(A_w) = p^{[L_w:\mathbb{Q}_p]} + 1$. Since the Weil bound says that $|a_w| \leq 2\sqrt{p^{[L_w:\mathbb{Q}_p]}}$, we reached a contradiction. □

Remark 5.4.5. As discussed above, for any number field F, we have a map from $Y_K(F)$ to systems of Galois representations satisfying $(\mathcal{S}.1) - (\mathcal{S}.4)$. Thanks to Faltings [65, Satz 6], this map has finite fibres. Indeed if two points give rise to the same system, the two corresponding abelian varieties have the same locus of bad reduction, that we denote by S, it follows from the Shafarevich conjecture that Shimura varieties of abelian type have only finitely $\mathcal{O}_{F,S}$-points. For more details we refer to [173, Theorem 3.2(A)].

We are now ready to prove the main theorem about descent obstruction for Hilbert modular varieties.

5.4.3 Proof of Theorem 5.1.3

We do not treat the cases $n_L = 1$ and $n_L > 1$ separately, but, as in the proof of Theorem 5.1.1 and 5.1.2, we emphasize that we do not need any conjectural statement in the case $n_L = 1$, since we have Shimura's unconditional result, Theorem 5.2.3.

Proof of Theorem 5.1.3. Thanks to Lemma 5.4.4, a point in $\mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$, which is assumed to be non-empty, gives rise to a compatible system of representations of $\mathrm{Gal}(\overline{L}/L)$, denoted by $\{\rho_v\}_v$. Let E be the the subfield of K generated by $\mathrm{tr}(\rho_v(\mathrm{Frob}_w))$ for all w. If $E = K$, Corollary 5.3.6 produces an $\mathcal{O}_{L,S}$-abelian variety A with $\mathcal{O}_K$-multiplication, such that $\mathrm{Gal}(\overline{L}/L)$ acts on $A[\mathfrak{N}]$ via ρ. To conclude the proof we just need to see how A corresponds to a point $P \in \mathcal{Y}_\rho(\mathcal{O}_{L,S})$. The only issue that is not clear from the quoted corollary is whether A is principally $\mathcal{O}_K$-polarised, but this follows from the discussion in section 5.2.3.

If E is strictly contained in K, i.e. condition $(\mathcal{S}.5)$ is not satisfied, we consider S_E, the Hilbert modular variety associated to E and of level $\mathfrak{N} \cap \mathcal{O}_E$. For the same reason as above, the system $\{\rho_v\}$ corresponds to a $\mathcal{O}_{L,S}$-point P of the twist by ρ of S_E. The embedding $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL}_2 \hookrightarrow \mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_2$ induces a map of Shimura varieties

$$r : S_E \to Y_K(\mathfrak{N}),$$

and therefore on their twists by ρ. Via r, we can regard P as an $\mathcal{O}_{L,S}$-point of $\mathcal{Y}_\rho$, therefore concluding the proof of the theorem. The only difference is that the abelian variety constructed in this case is not primitive. The proof of Theorem 5.1.3 is concluded. $\square$

Chapter 6

On the geometric Mumford–Tate conjecture for subvarieties of Shimura varieties

We study the image of ℓ-adic representations attached to subvarieties of Shimura varieties $\mathrm{Sh}_K(G,X)$ that are not contained in a smaller Shimura subvariety and have *no isotrivial components*. We show that, for ℓ large enough (depending on the Shimura datum (G,X) and the subvariety), such image contains the $\mathbb{Z}_\ell$-points coming from the simply connected cover of the derived subgroup of G. This can be regarded as a geometric version of the integral ℓ-adic Mumford–Tate conjecture. The work presented here appeared in the paper [12].

6.1 Introduction

6.1.1 Geometric and ℓ-adic monodromy

Let (G,X) be a Shimura datum, as in section 2.2 and K a compact open subgroup of $G(\mathbb{A}_f)$. For the length of this chapter we fix a faithful rational representation $G \subset \mathrm{GL}(V)$ and an integral structure $V_{\mathbb{Z}} \subset V$ such that the image of $\Gamma := G(\mathbb{Q})^+ \cap K$ in $\mathrm{GL}(V)$ is contained in $\mathrm{GL}(V_{\mathbb{Z}})$.

Let S be the the image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(G,X)$. Write $\pi_1^{\acute{e}t}(S)$ for the étale fundamental group of S, with respect to some base point. One can attach to S (and its subvarieties) an adelic representation

$$\pi_1^{ét}(S) \to K \subset G(\mathbb{A}_f),$$

corresponding to the tower of étale covering of $\mathrm{Sh}_K(G,X)$ indexed by open subgroups of K (see for example [65, Section 4] and [174, Section 2]).

Let $C \subset S$ be a smooth irreducible complex subvariety. We define the *ℓ-adic monodromy of C*, denoted by Π_C^ℓ, as the image of

$$\pi_1^{ét}(C_{\mathbb{C}}) \to \pi_1^{ét}(S_{\mathbb{C}}) \to K \xrightarrow{\pi_\ell} G(\mathbb{Q}_\ell),$$

where $\pi_\ell : G(\mathbb{A}_f) \to G(\mathbb{Q}_\ell)$ denotes the projection to the ℓ-th component. See Section 6.3.1 for a more detailed description of Π_C^ℓ.

Here we prove that, for ℓ-large enough, the ℓ-adic monodromy of subvarieties $C \subset S$, satisfying two conditions we introduce in the next subsection, is *large*.

6.1.2 Main theorem

To study the geometric and ℓ-adic monodromy of subvarieties of S we may assume they are not contained in any smaller Shimura subvariety of S (Theorem 6.2.5 will explain what happens if this is not the case). This is made precise in the next definition.

Definition 6.1.1. Let S be a connected component of a Shimura variety $\mathrm{Sh}_K(G,X)$ and C be an irreducible smooth complex subvariety of S. We say that C is *Hodge generic* if there exists a point $c \in C$ whose corresponding morphism of real algebraic groups $\mathbb{S} \to G_{\mathbb{R}}$ does not factor through $H_{\mathbb{R}} \subset G_{\mathbb{R}}$ for any rational algebraic subgroup $H \subsetneq G$.

Let G^{ad} be the adjoint group of G and X^{ad} be the $G^{\mathrm{ad}}(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{S}, G^{\mathrm{ad}})$ that contains the image of X in $\mathrm{Hom}(\mathbb{S}, G^{\mathrm{ad}})$. Choose a compact open subgroup $K^{\mathrm{ad}} \subset G^{\mathrm{ad}}(\mathbb{A}_f)$ containing the image of K. Then $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a Shimura datum and we call $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ the *adjoint Shimura variety* associated to $\mathrm{Sh}_K(G,X)$. The last ingredient needed to state our main theorem is the following.

Definition 6.1.2. Let S be a connected component of a Shimura variety $\mathrm{Sh}_K(G,X)$ and C be an irreducible smooth complex subvariety of S. We say that C has *no isotrivial components* if, for every decomposition $(G^{\mathrm{ad}},X^{\mathrm{ad}},K^{\mathrm{ad}}) \sim (G_1,X_1,K_1) \times \cdots \times (G_n,X_n,K_n)$, the image of $C \to S \to \mathrm{Sh}_{K_i}(G_i,X_i)$ has dimension > 0 for all $i = 1,\dots,n$.

Set $G(\mathbb{Z}_\ell) = G(\mathbb{Q}_\ell) \cap \mathrm{GL}(V_\mathbb{Z} \otimes \mathbb{Z}_\ell)$. Let G^{der} be the derived subgroup of G and $\lambda : G^{\mathrm{sc}} \to G^{\mathrm{der}}$ be the simply connected cover of G^{der}. We write $G(\mathbb{Z}_\ell)^+$ for the subgroup of $G(\mathbb{Z}_\ell)$ given by the image of $\lambda_{\mathbb{Z}_\ell} : G^{\mathrm{sc}}(\mathbb{Z}_\ell) \to G^{\mathrm{der}}(\mathbb{Z}_\ell)$. It is an open subgroup of $G^{\mathrm{der}}(\mathbb{Z}_\ell)$ of index bounded independently of ℓ. We now state the main theorem of the chapter.

Theorem 6.1.3. *Let S be a connected component of a Shimura variety* $\mathrm{Sh}_K(G,X)$ *and C be an irreducible smooth complex subvariety of S which is Hodge generic and has no isotrivial components. For all ℓ big enough (depending only on (G,X,K) and C), we have that $G(\mathbb{Z}_\ell)^+ \subset \Pi_C^\ell$.*

Remark 6.1.4. In general one cannot expect the equality $\Pi_C^\ell = G(\mathbb{Z}_\ell)$, even for large ℓ. Indeed this may only happen when the Hodge structure associated to C, $h_C : \mathbb{S} \to G_\mathbb{R}$, is *Hodge maximal*, i.e. there is no non-trivial isogeny of connected $\mathbb{Q}$-groups $H \to G$ such that h_C lifts to a homomorphism $h_C : \mathbb{S} \to H_\mathbb{R} \to G_\mathbb{R}$ (see [157, Definition 11.1] and [31, Definition 2.1]). Of course if G is simply connected, there are no such isogenies. Theorem 6.1.3, when G is semisimple simply connected, shows indeed that $G(\mathbb{Z}_\ell)^+ = G(\mathbb{Z}_\ell) = \Pi_C^\ell$, for all but finitely many primes ℓ.

6.1.3 Main theorem for Shimura varieties of Hodge type

Let g be a positive natural number. The prototype of all Shimura varieties is the Siegel moduli space of principally polarised (complex) abelian varieties of dimension g with a level structure, in which case $(G,X) = (\mathrm{GSp}_{2g}, \mathcal{H}_g^\pm)$, as recalled in Example 2.2.11. For more details see also [46, Section 4]. We end the introduction of the chapter reformulating Theorem 6.1.3 for subvarieties of Shimura varieties parametrising abelian varieties. We remark that the notion of having no isotrivial components for $C \subset \mathrm{Sh}_K(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ we proposed in the (G,X)-language (Definition 6.1.2) does not translate as parametrising a family of abelian varieties $A \to C$ with no non-zero constant factors (not even after passing to a finite cover of C). This is discussed in [1, Scholie (page 13)] and [124, Section 4.5]. Here we say that a g-dimensional abelian scheme $A \to C$ is *with no isotrivial components* if it corresponds to a subvariety with no isotrivial components (as in Definition 6.1.2) of the Shimura variety associated to $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$. We have the following.

Corollary 6.1.5. *Let C be a complex smooth irreducible variety and η its geometric generic point. Let $A \to C$ be a g-dimensional abelian scheme with no isotrivial components, write M for the Mumford–Tate group of A_η and $T_\ell(A_\eta)$ for the ℓ-adic Tate module of A_η. For all ℓ large enough, the image of the map*

$$\pi_1^{\acute{e}t}(C_{\mathbb{C}}) \to \mathrm{GL}(T_\ell(A_\eta))$$

contains $M(\mathbb{Z}_\ell)^+$.

In proving the corollary, we may replace C with a finite cover and the family $A \to C$ with an isogenous abelian variety. Hence we may assume that the family $A \to C$ gives rise to a subvariety of $\mathrm{Sh}_{K(3)}(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$, where $K(3) \subset \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ is the subgroup of elements that reduce to the identity modulo 3. By assumption such subvariety has no isotrivial components (in the sense of Definition 6.1.2). Since such family is contained in a Shimura subvariety whose defining group is the Mumford–Tate group of the abelian variety A_η, the corollary follows from Theorem 6.1.3 applied to the smooth locus of the image of C in $\mathrm{Sh}_{K(3)}(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$.

Remark 6.1.6. We explain why Corollary 6.1.5 can be thought as a geometric analogue of the *integral Mumford–Tate* conjecture for abelian varieties (see [157, Conjectures 10.3, 10.4 ,10.5] and [31]). As recalled in section 2.2.2, Shimura varieties admit canonical models over number fields. Definition 6.1.2 implies, among other things, that C has positive dimension. Let us make the analogy with the case of an abelian variety B over a number field L. Let C be the spectrum of L. We still have a continuous morphism

$$\rho_\ell : \pi_1^{ét}(C) \to \mathrm{GL}(T_\ell(B))$$

describing the action of the absolute Galois group of L on the torsion points of B. The Mumford–Tate conjecture, as discussed at the end of section 2.5.3, predicts that the image of ρ_ℓ is open in the $\mathbb{Z}_\ell$-points of the Mumford–Tate group M of B and its integral refinement that $\mathrm{Im}(\rho_\ell)$ is as large as possible and its index in $M(\mathbb{Z}_\ell)$ can be bounded independently of ℓ.

Outline of chapter

In Section 6.2 we discuss how to compute the geometric monodromy of subvarieties of (connected components of) Shimura varieties and explain the importance of Definition 6.1.2. In Section 6.3 we prove Theorem 6.1.3, combining the results of Section 6.2 with a theorem of Nori.

6.2 Monodromy of subvarieties, after Deligne, André and Moonen

We explain how to produce, starting form a subvariety $C \subset \mathrm{Sh}_K(G,X)$, a Shimura subvariety of $\mathrm{Sh}_K(G,X)$ containing C and such that C becomes Hodge generic and has no isotrivial components in such Shimura subvariety. This section builds on the interpretation of Shimura varieties in terms of Hodge theory, as recalled in section 2.2.1.

6.2.1 Deligne–André Monodromy Theorem

Let S be a connected, smooth complex algebraic variety and $\mathcal{V} = (\mathcal{V}, \mathcal{F}, \mathcal{Q})$ a polarised variation of $\mathbb{Z}$-Hodge structure on S, as described in section 2.2.1.2.

Let $s \in S$ be a Hodge generic point. From the local system underlying $\mathcal{V}$ we obtain a representation $\rho : \pi_1^{\mathrm{top}}(S(\mathbb{C})) \to \mathrm{GL}(V)$, where π_1^{top} denotes the *topological* fundamental group of S^{an} with base point s.

Definition 6.2.1. We denote by M_s the connected component of the identity of the Zariski closure of the image of ρ and we call M_s the *(connected) monodromy group*.

Since we fixed a trivialisation for $\lambda^*\mathcal{V}$, we have that $M_s \subset \mathrm{GL}(V)$ and M_s does not depend on the choice of s and $\tilde{s}$.

The following will be crucial for proving the results of the chapter. The proof can be found in [1, Theorem 1] and [45] (see also [103, Theorem 4.10.]). As in section 2.2.1.2, here Σ denotes a suitable countable union of proper analytic subspaces of S.

Theorem 6.2.2 (Deligne, André)**.** *Let $s \in S - \Sigma$. We have:*

Normality. *M_s is a normal subgroup of the derived group G^{der};*

Maximality. *Suppose S contains a special point. Then $M_s = G^{der}$.*

An immediate application of Theorem 6.2.2 is the following. Let S be a connected component of a Shimura variety $\mathrm{Sh}_K(G,X)$. For simplicity assume that G is a semi-simple algebraic group of adjoint type and that G is the generic Mumford–Tate group on X. Fixing a rational representation of G in $\mathrm{GL}(V)$, as we did from the section 6.1, we obtain a polarised variation of $\mathbb{Q}$-Hodge structure, denoted by $\mathcal{V}$, on the constant sheaf V_{X^+}. Moreover, since $\Gamma = G(\mathbb{Q})^+ \cap K$ acts freely on X^+, $\mathcal{V}$ descends to a variation of Hodge structures on S. To obtain a $\mathbb{Z}$-structure and to apply the previous theorem, we may choose $V_{\widehat{\mathbb{Z}}}$, a K-invariant lattice in $V \otimes \mathbb{A}_f$ and define $V_{\mathbb{Z}}$ as $V \cap V_{\widehat{\mathbb{Z}}}$. Let $c \in C$ be a Hodge-generic point on C, since $\pi_1(C,c)$ acts on $\mathcal{V}_{\mathbb{Z},c}$, it acts on $V_{\mathbb{Z}}$. Let Π be its image in $\mathrm{GL}(V_{\mathbb{Z}})$, it is a finitely generated group. We have

Corollary 6.2.3. *Let C be a smooth irreducible Hodge generic subvariety of S containing a special point t. We have $\Pi \subset \Gamma$ and both of them are Zariski dense in G.*

The next section explains what happens when C does not contain a special point and it is not Hodge generic. More details can be found in [124, Sections 2.9, 3.6 and 3.7].

6.2.2 Monodromy of subvarieties with no isotrivial components

Let C be an irreducible smooth complex subvariety of a Shimura variety $\mathrm{Sh}_K(G,X)$. Since the irreducible components of the intersection of two Shimura subvarieties of $\mathrm{Sh}_K(G,X)$ are again Shimura subvarieties, there exists a unique smallest sub-Shimura variety $S_C \subset \mathrm{Sh}_K(G,X)$ containing C. By definition there exists a $\mathbb{Q}$-group M such that S_C is an irreducible component of the image of $X_M^+ \times \eta K$ in $\mathrm{Sh}_K(G,X)$, for some $\eta \in G(\mathbb{A}_f)$, where X_M^+ is the restriction of X^+ to M. Moreover we may take M to be the generic Mumford–Tate group on C (recall that we fixed from the beginning a faithful representation of G). By construction, C is Hodge generic in S_C.

Let H the connected monodromy group associated to the polarised variation of $\mathbb{Z}$-Hodge structures $\mathcal{V}$ restricted to C. Theorem 6.2.2 implies that H is a normal subgroup of the derived subgroup of M and, since M is reductive, we can find a normal algebraic subgroup H_2 in M such that M is the almost direct product of H and H_2. This induces a decomposition of the adjoint Shimura datum:

$$(M^{\mathrm{ad}}, X_{M^{\mathrm{ad}}}) = (H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}}) \times (H_2^{\mathrm{ad}}, X_{H_2^{\mathrm{ad}}}).$$

As in Corollary 6.2.3, when C contains a special point, H is the derived subgroup of M, X_M is isomorphic to X_H and X_{H_2} is nothing but a point. But if C does not contain a special point, Moonen, in [124, Proposition 3.7], proves the following.

Proposition 6.2.4 (Moonen)**.** *Let $\mathcal{C}$ be an irreducible component in the preimage of C in X_M. The image of $\mathcal{C}$ under the projection $X_M \to X_{H_2}$ is a single point, say $y_2 \in X_{H_2}$. We have that C is contained in the image of $(Y_1 \times \{y_2\}) \times \eta' K$ in* $\mathrm{Sh}_K(G,X)$ *for some connected component $Y_1 \subset X_H$ and a class $\eta' K \in G(\mathbb{A}_f)/K$.*

To summarise our discussion, from Corollary 6.2.3 and Proposition 6.2.4 applied to C, we have

Theorem 6.2.5. *Let C be an irreducible smooth complex Hodge generic subvariety of a Shimura variety* $\mathrm{Sh}_{K_M}(M,X_M)$. *There exists a unique sub-Shimura datum*

$(H, X_H) \hookrightarrow (M, X_M)$ *such that*

$$(M^{\mathrm{ad}}, X_{M^{\mathrm{ad}}}) = (H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}}) \times (H_2^{\mathrm{ad}}, X_{H_2^{\mathrm{ad}}}),$$

and

- *the projection of C to $\mathrm{Sh}_{K_{H^{\mathrm{ad}}}}(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}})$, denoted by $\widetilde{C}$, is Hodge generic and has no isotrivial components;*
- *the projection of C to $\mathrm{Sh}_{K_{H_2^{\mathrm{ad}}}}(H_2^{\mathrm{ad}}, X_{H_2})$ is a single point.*

Moreover

- *the fundamental groups of $\widetilde{C}$ and of $\mathrm{Sh}_{K_{H^{\mathrm{ad}}}}(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}})$ are both Zariski dense in H^{ad}.*

6.3 ℓ-adic monodromy

After describing the interplay between the geometric and the ℓ-adic monodromy of subvarieties of S, we eventually prove Theorem 6.1.3.

6.3.1 A commutative diagram

Let $\mathcal{V}$ be a variation of polarised $\mathbb{Z}$-Hodge structures on S (as explained in Section 6.2) and let

$$\pi_1^{\mathrm{top}}(C(\mathbb{C})) \to \Gamma \subset \mathrm{GL}(V_{\mathbb{Z}})$$

be the monodromy representation of the induced variation on C. Let $\overline{\Gamma}$ be the closure of Γ in $\mathrm{GL}(V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}})$. The map $\pi_1^{\mathrm{top}}(C(\mathbb{C})) \to \Gamma \to \overline{\Gamma}$ canonically factorises trough the profinite completion of $\pi_1^{\mathrm{top}}(C(\mathbb{C}))$, which is canonically isomorphic to $\pi_1^{\acute{e}t}(C_{\mathbb{C}})$, the étale fundamental group of $C_{\mathbb{C}}$. Therefore we have a commutative diagram

$$\begin{array}{ccc} \pi_1^{\mathrm{top}}(C(\mathbb{C})) & \longrightarrow & \Gamma \subset \mathrm{GL}(V_{\mathbb{Z}}) \\ \downarrow & & \downarrow \\ \pi_1^{\acute{e}t}(C_{\mathbb{C}}) & \longrightarrow & \overline{\Gamma} \subset \mathrm{GL}(V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}) \end{array} .$$

Let $\pi_\ell : \mathrm{GL}(V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}) \to \mathrm{GL}(V_{\mathbb{Z}} \otimes \mathbb{Z}_\ell)$ be the projection to the ℓ-th component. The image of the map

$$\pi_1^{\acute{e}t}(C_{\mathbb{C}}) \to \overline{\Gamma} \xrightarrow{\pi_\ell} \mathrm{GL}(V_{\mathbb{Z}} \otimes \mathbb{Z}_\ell)$$

is precisely the ℓ-adic monodromy of C, that we denoted in section 6.1 by Π_C^ℓ.

6.3.2 Proof of Theorem 6.1.3

First of all notice that, to prove Theorem 6.1.3, the difference between $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and (G,X) is irrelevant. Indeed the arithmetic data play no role in the problem. Therefore we may and do assume that G is of adjoint type. Since Theorem 6.1.3 is about large enough primes ℓ, we may ignore finitely many primes and assume that G is the generic fiber of a semisimple adjoint group scheme over $\mathbb{Z}$.

Before starting the proof we fix some notations. By $(-)_\ell$ we denote the reduction modulo-ℓ of subgroups of $G(\mathbb{Z})$ and we write $G(\mathbb{F}_\ell)$ (resp. $G(\mathbb{F}_\ell)^+$) for the reduction modulo-ℓ of $G(\mathbb{Z})$ (resp. $G(\mathbb{Z}_\ell)^+$).

Let $C \subset S = \Gamma \backslash X^+$ be a subvariety as in the statement of Theorem 6.1.3 and let $\Pi \subset \Gamma$ be the image of the geometric monodromy representation

$$\pi_1^{\mathrm{top}}(C(\mathbb{C})) \to \pi_1^{\mathrm{top}}(S(\mathbb{C})) \to \Gamma \subset \mathrm{GL}(V_{\mathbb{Z}}).$$

Since C is Hodge generic and has no isotrivial components, Corollary 6.2.3 shows that Π is Zariski dense in G.

The following is a theorem of Nori [133, Theorem 5.1] about Zariski dense subgroups of semisimple groups.

Theorem 6.3.1 (Nori)**.** *Let $H \subset \mathrm{GL}_n/\mathbb{Q}$ be a semisimple group and $\Pi \leq H(\mathbb{Q})$ be a discrete finitely generated Zariski-dense subgroup. Then for all sufficiently large prime numbers ℓ (depending only on H and Π), the reduction modulo-ℓ of Π contains $H(\mathbb{F}_\ell)^+$.*

Reducing modulo-ℓ is well defined for ℓ large enough: indeed, since Π is finitely generated, there are only finitely many primes $\ell_1, \dots, \ell_k$ such that Π belongs to

$$H(\mathbb{Z}[\ell_i^{-1}]_i) := H \cap \mathrm{GL}_n(\mathbb{Z}[\ell_1^{-1}, \ldots, \ell_k^{-1}]),$$

and the reduction mod ℓ is well-defined for all other primes.

Since the group G is assumed to be semisimple and adjoint we may apply Theorem 6.3.1 to get, for ℓ large enough, a chain of inclusions

$$G(\mathbb{F}_\ell)^+ \subset \Pi_\ell \subset \Gamma_\ell \subset G(\mathbb{F}_\ell).$$

We are left to lift such chain of inclusions to the $\mathbb{Z}_\ell$-points of G. Denote by α_ℓ the reduction modulo-ℓ map:

$$\alpha_\ell : G(\mathbb{Z}_\ell) \to G(\mathbb{F}_\ell).$$

It is a well known fact that, for ℓ large enough, if G is a connected semisimple group, α_ℓ is a *Frattini cover*, i.e. α_ℓ is surjective and $G(\mathbb{Z}_\ell)$ contains no strict subgroups mapping surjectively onto $G(\mathbb{F}_\ell)$ (equivalently, the kernel of α_ℓ is contained in the Frattini subgroup of $G(\mathbb{Z}_\ell)$). A proof of this fact can be found in [109, Lemma 16.4.5 (page 403)], see also [30, Section 2.3] and [116, Proposition 7.3].

Nori, in [133, Section 3] (in particular [133, Remark 3.6]), shows also that, for ℓ large enough, $G(\mathbb{F}_\ell)^+$ can be identified with the subgroup of $G(\mathbb{F}_\ell)$ generated by its ℓ-Sylow subgroups. Hence we have that the index $[G(\mathbb{F}_\ell) : G(\mathbb{F}_\ell)^+]$, for ℓ-large enough, is prime to ℓ. From this and the fact that α_ℓ is Frattini, we deduce that the maps

$$\alpha_\ell^{-1}(G(\mathbb{F}_\ell)^+) \to G(\mathbb{F}_\ell)^+, \text{ and } \alpha_\ell^{-1}(\Pi_\ell) \to \Pi_\ell$$

are also Frattini covers (see also [30, Section 2.3]). Since $\Pi_C^\ell \subset \alpha_\ell^{-1}(\Pi_\ell)$ surjects onto Π_ℓ and $G(\mathbb{Z}_\ell)^+$ onto $G(\mathbb{F}_\ell)^+$, the inclusions

$$G(\mathbb{Z}_\ell)^+ \subset \alpha_\ell^{-1}(G(\mathbb{F}_\ell)^+), \text{ and } \Pi_C^\ell \subset \alpha_\ell^{-1}(\Pi_\ell)$$

are actually equalities. Eventually we have

$$G(\mathbb{Z}_\ell)^+ \subset \Pi_C^\ell,$$

as desired. This ends the proof of Theorem 6.1.3.

Chapter 7

Non-arithmetic ball quotients and Hodge theory (joint with E.Ullmo)

We study complex hyperbolic lattices $\Gamma \subset \mathrm{PU}(1,n)$, for $n > 1$, using Hodge theory. Let S_Γ be the ball quotient associated to Γ. By the work of Baily, Borel and Mok, S_Γ is a quasi projective variety. We prove that S_Γ *naturally* admits a holomorphic map to a period domain for polarised integral variation of Hodge structures (not necessarily of Shimura type). The work presented here appeared in the fourth section of the preprint [14].

7.1 Introduction

The study of lattices of semisimple Lie groups G (without compact factors) is a field rich in open questions and conjectures. Complex hyperbolic lattices and their finite dimensional representations are certainly far from being understood. A lattice $\Gamma \subset G$ is archimedean superrigid if for any simple noncompact Lie group G' with trivial centre, every homomorphism $\Gamma \to G'$ with Zariski dense image extends to a homomorphism $G \to G'$. Thanks to the work of Margulis [112], Corlette [42] and Gromov–Schoen [86], all irreducible lattices in simple Lie groups are superrigid unless G is $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ for some $n \geq 1$. Real hyperbolic lattices are known to be softer and more flexible than their complex counterpart and non-arithmetic lattices in $\mathrm{SO}(1,n)$ can be constructed for every n [85]. Non-arithmetic complex hyperbolic lattices have been constructed only in $\mathrm{SU}(1,n)$, for $n = (1,)2,3$ [128, 54].

One can consider the quotient by Γ of the symmetric space X associated to G. In the complex hyperbolic case we obtain a ball quotient, denoted by S_Γ, which has a natural structure of a quasi-projective variety, as proven by Baily-Borel [7] in the arithmetic case and by Mok [54] in general. Indeed, if Γ is arithmetic, S_Γ is a Shimura variety as described in section 2.2 (see also Example 2.2.13). In this chapter we prove, using a recent work of Esnault and Groechenig [64] and Simpson's theory [162], that S_Γ always embeds in a period domain for polarised integral variations of Hodge structures. In particular we show that the traces of elements of Γ, under the adjoint representation, lie in the ring of integers of a totally real number field. Applications of such result to a generalisation of the Zilber-Pink conjecture, which generalises the one described in section 3.4.1, are discussed in Chapter 8.1.

Notations

In this and the next chapter we denote by G real algebraic groups and by $\mathbf{G}$ algebraic groups defined over some number field, which is either $\mathbb{Q}$, or a (totally) real number field. Let K be a real field, the K-forms of $G = \mathrm{SU}(1,n)$ are known, by the work of Weil [176], to be obtained as $\mathrm{SU}(h)$ for some Hermitian form h on F^r, where F is a division algebra with involution over a quadratic imaginary extension L of K and $n+1 = r\deg(F)$.

7.2 Preliminaries on lattices

A discrete subgroup Γ of a locally compact group G is a *lattice* if G/Γ has a finite invariant Haar measure. All lattices considered in this chapter are also assumed to be torsion free. Selberg's Lemma asserts if G is a semisimple Lie group, then Γ has a torsion-free subgroup of finite index, see for example [127, Theorem 4.8.2]. Finally a subgroup $\Gamma \subset G$ is *arithmetic* if there exists a semisimple linear algebraic group $\mathbf{G}/\mathbb{Q}$ and a surjective morphism with compact kernel $p : \mathbf{G}(\mathbb{R}) \to G$ such that Γ lies in the commensurability class of $p(\mathbf{G}(\mathbb{Z}))$. Here we denote by $\mathbf{G}(\mathbb{Z})$ the group $\mathbf{G}(\mathbb{Q}) \cap v^{-1}(\mathrm{GL}(V_\mathbb{Z}))$ for some faithful representation $v : G \to \mathrm{GL}(V_\mathbb{Q})$, where $V_\mathbb{Q}$ is a finite dimensional $\mathbb{Q}$-vector space and $V_\mathbb{Z}$ a lattice in $V_\mathbb{Q}$. Torsion free arithmetic subgroups are lattices. We remark here that there is an isogeny from $\lambda : \mathrm{SU}(1,n) \to \mathrm{PU}(1,n)$ and we may assume that, up to replacing the lattice by a finite index subgroup, that any lattice in $\mathrm{PU}(1,n)$ comes from a lattice in $\mathrm{SU}(1,n)$.

7.2.1 Local and cohomological rigidity

Let G be a semisimple algebraic group without compact factors and Γ be a subgroup of G. Denote by

$$\mathrm{Ad} : \Gamma \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g}) \tag{7.2.1}$$

the adjoint representation in the automorphism of the Lie algebra $\mathfrak{g}$ of G.

Definition 7.2.1. We define the *trace field* of Γ as the field generated over the rational by

$$\{\mathrm{tr}\,\mathrm{Ad}\,\gamma : \gamma \in \Gamma\}.$$

If Γ is a lattice, its trace field is a finitely generated field extension of $\mathbb{Q}$, which depends only on the commensurability class of Γ. Indeed it is well known that lattices are finitely generated (and even finitely presented), see for example [127, Theorem 4.7.10] and references therein.

Definition 7.2.2. An irreducible lattice $\Gamma \subset G$ is *locally rigid* if there exists a neighbourhood U of the inclusion $i : \Gamma \hookrightarrow G$ in $\mathrm{Hom}(\Gamma, G)$, such that any elements of U is conjugated to i.

The trace field K of a locally rigid lattice is a number field. By Borel density theorem, lattices in G are Zariski dense (see for example [127, Corollary 4.5.6]). In particular Γ determines a K-form of G, which we denote by $\mathbf{G}$, such that, up to conjugation by an element in G, Γ lies in $\mathbf{G}(K)$ (and K is minimal with this property). For references see [175], [112, Chapter VIII, Proposition 3.22] and [54, Proposition 12.2.1].

Lattices in $G = \mathrm{SU}(1,n)$, necessarily for $n > 1$, are known to be locally rigid (and the trace field is a totally real number field, as we prove in section 7.4). For completeness we describe a more general[1] result [77, Theorem 0.11], which builds

[1] More general in the sense that it allows also to consider $G = \mathrm{SL}_2(\mathbb{C})$. Even if local rigidity for non-cocompact lattices in $\mathrm{SL}_2(\mathbb{C})$ can fail.

on the study of lattices initiated by Selberg, Calabi and Weil [177].

Theorem 7.2.3. *If G is not locally isomorphic to* $\mathrm{SL}_2(\mathbb{R})$*, then, for every lattice Γ in G there exists $g \in G$ and a subfield $K \subset \mathbb{R}$ of finite degree over $\mathbb{Q}$, such that $g\Gamma g^{-1}$ is contained in the set of K-rational points of $\mathbf{G}$.*

To construct a $\mathbb{Z}$-VHS we need a stronger rigidity, namely *cohomological rigidity* (without boundary conditions). Bulding on a study initated by Weil [177] in the cocompact case, Garland-Raghunathan [77, Theorem 1.10], proved also the following.

Theorem 7.2.4 (Garland-Raghunathan)**.** *Let G be a semisimple Lie group, not locally isomorphic to* SL_2*, nor to* $\mathrm{SL}_2(\mathbb{C})$*. For any lattice Γ in G, the first Eilenberg–MacLane cohomology group of Γ with respect to the adjoint representation is zero. In symbols:*

$$H^1(\Gamma, \mathrm{Ad}) = 0.$$

Too see why such vanishing is related to a rigidity result, observe that the the space of first-order deformations of $\rho : \Gamma \hookrightarrow G$ is naturally identified with $H^1(\Gamma, \mathrm{Ad})$, where Ad is the adjoint representation, as in 7.2.1.

Since lattices are finitely generated, we know that there exist a finite set of finite places Σ of K such that Γ lies in $\mathbf{G}(\mathcal{O}_{K,\Sigma})$ (once a representation is fixed and up to conjugation by G). In the next subsection we discuss what happens when the lattices Γ are contained in $\mathbf{G}(\mathcal{O}_K)$.

7.2.2 Lattices and integral points

Let $K \subset \mathbb{R}$ be a totally real number field and $\mathcal{O}_K$ its ring of integers. The following is [54, Corollary 12.2.8], see also [128, Lemma 4.1].

Theorem 7.2.5 (Mostow–Vinberg arithmeticity criterion)**.** *Let $\Gamma \subset \mathbf{G}(\mathcal{O}_K)$ be a lattice in G. Γ is arithmetic if and only if for every embedding $\sigma : K \to \mathbb{R}$ different form the fixed embedding $K \subset \mathbb{R}$, the group G_σ is compact.*

Remark 7.2.6. If $\Gamma \subset \mathbf{G}(\mathcal{O}_K)$ is a non-arithmetic lattice, then $\mathbf{G}(\mathcal{O}_K)$ is not discrete in G and Γ has infinite index in $\mathbf{G}(\mathcal{O}_K)$.

Let $\mathbf{G}$ be a semisimple algebraic group defined over a totally real number field $K \subset \mathbb{R}$ and write G for its real points. For each place σ of K, let G_σ be the real group $\mathbf{G} \otimes_{K,\sigma} \mathbb{R}$ by σ. Denote by Ω_∞ the set of all archimedean places of K and write $\widehat{\mathbf{G}}$ for the Weil restriction from K to $\mathbb{Q}$ of $\mathbf{G}$, see [178, Section 1.3]. It has a natural structure of $\mathbb{Q}$-algebraic group and

$$\widehat{\mathbf{G}}(\mathbb{R}) = \prod_{\sigma \in \Omega_\infty} G_\sigma.$$

For details and proofs we refer to [127, Section 5.5]. See also Example 2.2.10, where we described Hilbert modular varieties.

Proposition 7.2.7. *The subgroup* $\mathbf{G}(\mathcal{O}_K)$ *of* G *embeds as an arithmetic subgroup of* $\widehat{\mathbf{G}}$ *via the natural embedding*

$$\mathbf{G}(\mathcal{O}_K) \hookrightarrow \widehat{\mathbf{G}}, \quad g \mapsto (\sigma(g))_{\sigma \in \Omega_\infty}.$$

Indeed $\mathbf{G}(\mathcal{O}_K)$ is identified with $\widehat{\mathbf{G}}(\mathbb{Z})$. We also have:

- If $\mathbf{G}$ is simple, then $\mathbf{G}(\mathcal{O}_K)$ gives rise to an irreducible lattice in $\widehat{\mathbf{G}}(\mathbb{R})$;
- If, for some $\sigma \in \Omega_\infty$, G_σ is compact, then $\mathbf{G}(\mathcal{O}_K)$ gives rise to a cocompact lattice.

7.2.3 Examples of complex hyperbolic lattices

Regarding commensurability classes of non-arithmetic lattices in $\mathrm{PU}(1,n)$, we have:

$n = 2$. By the work of Deligne, Mostow and Deraux, Parker, Paupert, there are 22 known commensurability classes in $\mathrm{PU}(1,2)$. See [57, 58] and references therein;

$n = 3$. By the work of Deligne, Mostow and Deraux, there are 2 commensurability classes of non-arithmetic lattices in $\mathrm{PU}(1,3)$. In both cases the trace field is $\mathbb{Q}(\sqrt{3})$ and the lattices are not cocompact. See [56] and references therein.

For $n > 3$ non-arithmetic lattices are currently not known to exist. One of the biggest challenge in the study of complex hyperbolic lattices is to understand for each n how many commensurability classes non-arithmetic lattices exist in $\mathrm{PU}(1,n)$. We hope the main results of this chapter could shed some light towards such question.

7.3 Main result

The main result of the Chapter is the following (the fact that K is totally real is proven in the next section).

Theorem 7.3.1. *Let Γ be a lattice in $G = \mathrm{SU}(1,n)$, for some $n > 1$. Then there exists a finite index subgroup $\Gamma' \subset \Gamma$ with integral traces. Equivalently, up to conjugation by G, Γ' lies in $\mathbf{G}(\mathcal{O}_K)$, for some number field $K \subset \mathbb{R}$.*

The following is [64, Theorem 1.1], whose argument relies on Drinfeld's theorem on the existence of ℓ-adic companions over a finite field.

Theorem 7.3.2 (Esnault–Groechenig)**.** *Let S be a smooth connected quasi-projective complex variety. Then a complex local system $\mathbb{V}$ on S is integral, i.e. it comes as extension of scalars from a local system of projective $\mathcal{O}_L$-modules of finite type (for some number field $L \subset \mathbb{C}$), whenever it is:*

1. *Irreducible;*
2. *Quasi-unipotent local monodromies around the components at infinity of a compactification with normal crossings divisor $i : S \hookrightarrow \overline{S}$;*
3. *Cohomologically rigid, that is $\mathbb{H}^1(\overline{S}, i_{!*}\,\mathrm{End}^0(\mathbb{V}))$ vanishes;*
4. *Of finite determinant.*

Here $i_{!*}\,\mathrm{End}^0(\mathbb{V})$ denotes the intermediate extension seen as a perverse sheaf as in [17]. See [64, Remark 2.4] for more details. Moreover $\mathbb{H}^1(\overline{S}, i_{!*}\,\mathrm{End}^0(\mathbb{V}))$ is the Zariski tangent space at the moduli point of $\mathbb{V}$ of the Betti moduli stack of complex local systems of given rank with prescribed determinant and prescribed local monodromies along the components of the normal crossing divisor $\overline{S} - i(S)$. Theorem 7.3.2 is predicted by the conjectures of Simpson described in section 2.5.4, since cohomologically rigid local systems are rigid. We remark here that is not know of a single example of a rigid local system which is not cohomologically rigid (in the sense of (3)).

Proof of Theorem 7.3.1. The fact that $\mathbb{V}$ is integral in the sense of Theorem 7.3.2 if and only if Γ has integral traces is content of Bass-Serre theory [16], see also [43, Lemma 7.1]. In the proof we are free to replace Γ with a finite index subgroup and so may and do assume that Γ is torsion free and with only unipotent parabolic elements. The latter condition is needed to have a *nice* toroidal compactification of the associated ball quotient, see for example [14, Section 3.3], for a more detailed discussion[2]. Let S_Γ be the ball quotient $\Gamma\backslash X$. It is a smooth quasi-projective variety and admits a smooth toroidal compactification

$$i : S_\Gamma \hookrightarrow \overline{S_\Gamma},$$

with smooth boundary. The boundary is actually a disjoint union of N abelian varieties. Moreover, since G has rank one, the toroidal compactification of S_Γ does not depend on any choices. See indeed the main theorem of [123] and, even if it is written for arithmetic lattices, the monograph [4].

Consider the standard complex $n+1$-dimensional representation of $\mathrm{SU}(1,n)$ in $\mathrm{GL}(V_\mathbb{C})$ and let

$$\rho : \pi_1(S_\Gamma) \cong \Gamma \to \mathrm{GL}(V_\mathbb{C}),$$

be the associated complex representation of the fundamental group of S_Γ. Let $\mathbb{V}$ be the corresponding local system on $S_\Gamma/\mathbb{C}$. By construction the local system $\mathrm{End}^0(\mathbb{V})$ corresponds to the adjoint representation described in equation (7.2.1).

Notice that, since Γ is irreducible and Zariski dense in G, also $\mathbb{V}$ is irreducible (this of course depends on our choice of the faithful representation of G in $\mathrm{GL}(V_\mathbb{C})$). To prove the integrality of $\mathbb{V}$ we are left to check conditions (2), (3) and (4) of Theorem 7.3.2. By construction, since we can assume Γ lies in $\mathrm{SU}(1,n)$, ρ is of finite determinant, checking therefore (4).

Proof of (2). Denote by $\Delta \subset \mathbb{C}$ the complex disk and by Δ^* the punctured disk.

[2]This condition is implicitly used also in [123].

There exists an open cover $\{U_\alpha\}_\alpha$ of $\overline{S_\Gamma}$ such that $U_\alpha = \Delta^n$ and $U_\alpha \cap S_\Gamma = \Delta^{n-1} \times \Delta^*$. We notice here that the singular locus of B is empty and therefore $\overline{S_\Gamma}$ is equal to what is denoted by U in [64, Section 2]. As proven in [123, Section 1.3], for any Γ-rational parabolic P^3, the set $U_P \cap \Gamma$ where U_P is the centre of the unipotent radical of P, is isomorphic to $\mathbb{Z}$. We want to prove that the image of fundamental group of $U_\alpha \cap S_\Gamma \cong \Delta^{n-1} \times \Delta^*$ in the fundamental group of $\overline{S_\Gamma}$ lies in $\Gamma \cap U_P \cong \mathbb{Z}$. We work with the local coordinates as in [131, pages 255-256]. Let $X^\vee$ be the compact dual of X and $\mathcal{X} \subset X^\vee$ be the Borel embedding. Assume that we are working with a Γ-rational boundary component $F = \{b\}$ corresponding to the Γ-rational parabolic P and let V be the quotient of the unipotent radical of P by it centre. It is a real vector space of rank $n-1$ (where $n = \dim S_\Gamma$). For any boundary component $b \in X$, set

$$\mathcal{X}_b := \bigcup_{g \in U_P \otimes \mathbb{C}} g \cdot \mathcal{X} \subset X^\vee.$$

There exists a canonical holomorphic isomorphism

$$j : \mathcal{X} \cong \mathbb{C}^{n-1} \times \mathbb{C} \times F,$$

where $\mathbb{C}^{n-1} = V_\mathbb{C}$ and the latter copy of $\mathbb{C}$ is $U_P \otimes \mathbb{C}$. We can naturally identify the universal cover of $U_\alpha \cap S_\Gamma = \Delta^{n-1} \times \Delta^*$ with

$$\mathcal{D} \cong \{(z_1, \dots, z_{n-1}, z_n, b) \in \mathbb{C}^{n-1} \times \mathbb{C} \times F : \mathrm{Im}(z_n) \geq 0\} \tag{7.3.1}$$

The group $U_P \otimes \mathbb{C}$ acts on $\mathcal{D}$, in these coordinates, by $(z_1, \dots, z_{n-1}, z_n, b) \mapsto (z_1, \dots, z_{n-1}, z_n + a, b)$. Observe that we can factorise the map $\pi : \mathcal{X} \to S_\Gamma$ as

$$\mathcal{X} \xrightarrow{\exp_F} \exp_F(\mathcal{X}) \to S,$$

where $\exp_F : \mathbb{C}^{n-1} \times \mathbb{C} \times F \to \mathbb{C}^{n-1} \times \mathbb{C}^* \times F$ is simply $\exp(2\pi i -)$ on the $\mathbb{C}$-component and the identity on $\mathbb{C}^{n-1}$. Moreover $\exp_F(\mathcal{X})$ is $\Gamma_{U_P} \backslash \mathcal{X}$. To conclude,

[3] Here a parabolic subgroup P of $\mathrm{SU}(1,n)$ is said to be Γ-*rational* if its its unipotent radical intersects Γ as a lattice.

notice that we have a commutative diagram:

$$\begin{array}{ccc} \exp_F(\mathcal{X}) & \hookrightarrow & \exp_F(\mathcal{X})^\vee \\ \downarrow & & \downarrow \\ S_\Gamma & \hookrightarrow & \overline{S_\Gamma} \end{array},$$

since, as in the arithmetic case (see for example [4, Theorem 7.2]), the boundary of $\exp_F(\mathcal{X})^\vee$ is mapped onto the boundary of $\overline{S_\Gamma}$. □

Proof of (3). Here we show that $\mathbb{V}$ is cohomologically rigid (without boundary conditions). We have that

$$H^1(S_\Gamma, \mathrm{End}^0(\mathbb{V})) = H^1(\Gamma, \mathrm{Ad}) = 0,$$

where the last equality follows from Theorem 7.2.4. To show that $\mathbb{H}^1(\overline{S_\Gamma}, i_{!*}\mathrm{End}^0(\mathbb{V}))$ vanishes, it is enough that it injects in $H^1(S_\Gamma, \mathrm{End}^0(\mathbb{V})) = 0$. This follows from the description of $\mathbb{H}^1(\overline{S_\Gamma}, i_{!*}\mathrm{End}^0(\mathbb{V}))$ appearing in [64, Proposition 2.3], more precisely see page 4284 line 8 and Remark 2.4 in *op. cit.*. □

Eventually we have checked all the conditions of Theorem 7.3.2 and therefore concluded the proof of Theorem 7.3.1. Indeed we proved that the traces of Γ, seen in $\mathrm{GL}(V_\mathbb{C})$ via the standard representation, are integral, which implies the same result for the adjoint representation. □

7.4 Weil restriction, after Deligne–Simpson

The following is the main result of the section and concludes to proof of the theorem that appeared at the end of Chapter 1. As in the previous section, we let Γ be a lattice in $\mathrm{SU}(1,n)$, for some $n > 1$ and S_Γ be the associated ball quotient.

Theorem 7.4.1. *The trace field K of Γ is totally real and for each embedding $\sigma : K \to \mathbb{R}$ the representation*

$$\Gamma \subset \mathbf{G}(K) \to \mathbf{G} \times_{K,\sigma} \mathbb{R}$$

is induced by a $\mathcal{O}_K$-VHS, denoted by $\mathbb{V}^\sigma$. Moreover the $\mathcal{O}_K$-VHS

$$\widehat{\mathbb{V}} := \bigoplus_{\sigma : K \to \mathbb{R}} \mathbb{V}^\sigma \tag{7.4.1}$$

has a natural structure of $\mathbb{Z}$-VHS of weight zero.

The proof is inspired by the arguments of [162, Theorem 5] and [46]. For completeness and related discussions we refer also to [43, Section 10] and [105, Proposition 7.1].

Proposition 7.4.2 (Simpson)**.** *If for each embedding $\sigma : K \to \mathbb{C}$, the local system $\mathbb{V}^\sigma$ associated to the representation*

$$\Gamma \subset \mathbf{G} \to \mathbf{G} \times_{K,\sigma} \mathbb{C}$$

underlies a polarised complex VHS of weight zero, then the direct sum of

$$\widehat{\mathbb{V}} = \bigoplus_{\sigma : K \to \mathbb{C}} \mathbb{V}^\sigma$$

has a natural structure of $\mathbb{Z}$-VHS.

To be more precise, in the proof of Proposition 7.4.2 one has to use all the embeddings of E into $\mathbb{C}$, where, as in the proof of Theorem 7.3.1, we let $E \subset \mathbb{C}$ be the field generated by the traces of Γ in $\mathrm{GL}(V_\mathbb{C})$. In our exposition a choice of CM type of E is implicitly made. For more details we refer to the proof of [105, Proposition 7.1].

Proof of Theorem 7.4.1. The fact that $\widehat{\mathbb{V}}$ has a natrual strucutre of $\mathbb{Z}$-VHS, rather than of $\mathbb{Q}$-VHS, is the content of Bass-Serre theory [16]. For each embedding $\sigma : K \to \mathbb{C}$, let $G_\sigma(\mathbb{C})$ be complex group $\mathbf{G} \times_{K,\sigma} \mathrm{Spec}(\mathbb{C})$. Consider the complex representation

$$\rho_\sigma : \pi_1(S_\Gamma) \to \mathbf{G}(\mathcal{O}_K) \to G_\sigma(\mathbb{C}).$$

Here we prove that ρ_σ is induced by an $\mathcal{O}_K$-VHS of weight zero $\mathbb{V}^\sigma$. Since

$$H^1(\Gamma, \mathrm{Ad} \circ \rho_\sigma) = H^1(\Gamma, \mathrm{Ad}) = 0,$$

the twisted representation ρ_σ is again cohomologically rigid (see also [43, Lemma 6.6]). Moreover $\mathbb{V}^\sigma$ has also quasi-unipotent monodromy at infinity by construction. Indeed, with the notation of the proof of (2) of Theorem 7.3.1, let P_i be a Γ-rational parabolic corresponding to an irreducible divisor B_i of the boundary and denote by T_i the local monodromy for $\mathbb{V}$ and by T_i^σ the local monodromy for $\mathbb{V}^\sigma$. By construction of $\mathbb{V}^\sigma$, the element T_i^σ is obtained by $T_i \in \Gamma_{U_{P_i}} \subset \Gamma \subset \mathbf{G}(\mathcal{O}_K)$ by applying σ to its entries. Since being unipotent is a geometric condition, it is enough to check that $T_i^\sigma \in G_\sigma \otimes \mathbb{C} \cong G \otimes \mathbb{C}$ is unipotent, which holds true because T_i is unipotent.

Eventually we can apply Theorem 7.4.3, which is recalled below, to conclude that each $\mathbb{V}^\sigma$ is a $\mathcal{O}_K$-VHS. This imply that all infinite places of K are real, that is K is totally real. To compute the weight of $\mathbb{V}^\sigma$, notice that it corresponds to a map

$$\mathbb{S} \to G_\sigma,$$

and each of the G_σ is of adjoint type (since $G = \mathrm{PU}(1,n)$ is of adjoint type and they are geometrically isomorphic). Therefore each $\mathbb{V}^\sigma$ has weight zero. The reason being that the weight homomorphism is a map from $\mathbb{G}_m$ to the centre of G_σ, see [50, 1.1.13.]. □

The following is [43, Theorem 8.1], see also references therein.

Theorem 7.4.3 (Corlette)**.** *Suppose $\mathbb{V}$ is local system with quasi-unipotent monodromy at infinity. If $\mathbb{V}$ is rigid, then it underlines a $\mathbb{C}$-VHS.*

In the next section we make explicit how the $\mathbb{Z}$-VHS $\widehat{\mathbb{V}}$ of Theorem 7.4.1 realises S_Γ inside a period domain (in the sense of section 2.2.1.3).

7.4.1 Generalised modular embeddings

As recalled in section 2.2.1.3 we have a Mumford–Tate domain associated to $(\widehat{\mathbf{G}}, M)$, were $\widehat{\mathbf{G}}$, as in section 7.2.2, is the Weil restriction from K to $\mathbb{Q}$ of $\mathbf{G}$[4] and M a compact subgroup of $\widehat{G}$. If M happens to be a compact maximal subgroup, the Mumford–Tate domain is indeed a Shimura variety.

As in section 7.2.2, we write r for the degree of K over $\mathbb{Q}$ and $\sigma_i : K \to \mathbb{R}$ the real embeddings of K, ordered in such a way that σ_1 is simply the identity on K.

Let $\psi : S_\Gamma{}^{\mathrm{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ be the period map associated to the $\mathbb{Z}$-VHS $\widehat{\mathbb{V}}$ we constructed in 7.4.1. We have a commutative diagram in the analytical category

$$\begin{array}{ccc} X & \overset{\tilde{\psi}}{\hookrightarrow} & D = D_{\widehat{G}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S_\Gamma{}^{\mathrm{an}} & \overset{\psi}{\longrightarrow} & \widehat{\mathbf{G}}(\mathbb{Z})\backslash D \end{array}.$$

By construction $D_{\widehat{G}}$ is the product of X and a homogeneous space under the group $\prod_{i=2,\dots,r} G_{\sigma_i}$. We notice here that, by Theorem 7.2.5, the group $\prod_{i=2,\dots,r} G_{\sigma_i}$ is compact if and only if Γ is an arithmetic lattice. Given such decomposition we can write:

$$\tilde{\psi}(x) = (x, x_{\sigma_2}, \dots, x_{\sigma_r}),$$

where x_{σ_i} is the Hodge structure obtained by the fibre of $\mathbb{V}^{\sigma_i}$, or more precisely of its lift to X, at x. Finally it is important to observe that $\tilde{\psi}$ is holomorphic and Γ-equivariant, in the sense that for each $\gamma \in \Gamma$, we have

$$\tilde{\psi}(\gamma x) = (\gamma x, \rho_{\sigma_2}(\gamma) x_{\sigma_2}, \dots, \rho_{\sigma_r}(\gamma) x_{\sigma_r}),$$

where $\rho_{\sigma_i} : \Gamma \to G_{\sigma_i}$ is obtained by applying $\sigma_i : K \to \mathbb{R}$ to the coefficients of $\Gamma \subset \mathbf{G}(\mathcal{O}_K)$.

Remark 7.4.4. The map ψ generalises the theory of modular embeddings of trian-

[4] To be more precise here we should take the generic Mumford-Tate group of $\widehat{\mathbb{V}}$, which, a priori, could be a strict subgroup of $\widehat{\mathbf{G}}$.

gle groups [181, 37], which are indeed cohomological rigid lattices and Deligne–Mostow lattices [38].

It is interesting to compare the above diagram with a famous result of Corlette. Let S be a closed connected Kähler manifold, with universal covering $\tilde{S}$ and fundamental group Γ. Let

$$\rho : \Gamma \to \widehat{G}$$

be a group homomorphism form Γ into a real reductive group. If the Zariski closure of Γ in $\widehat{G}$ is reductive, then Corlette [41] proved that there exists a ρ-equivariant harmonic map from $\tilde{S}$ to the quotient of $\widehat{G}$ by a compact maximal subgroup $\widehat{K_0}$. If S is a ball quotient S_Γ, we have constructed a holomorphic map from X to $\widehat{G}/M$, where M is a compact. We can compose the map $\tilde{\psi}$ with $\widehat{G}/M \to \widehat{G}/\widehat{K_0}$, but the result wont be holomorphic.

Chapter 8

General conclusions and furhter directions

We conclude by presenting some related results, conjectures and future directions in our research. Part of this thesis was indeed motivated by the Zilber–Pink conjecture for (mixed) Shimura varieties. We explain here how the results of Chapter 7 bring non-arithmetic ball quotients in the realm of a Zilber–Pink conjecture for $\mathbb{Z}$-VHS and we discuss both geometric and arithmetic aspects of such conjecture. This Chapter is an extract of the preprint [14], which is a joint work with E. Ullmo.

8.1 Klingler's generalised Zilber-Pink conjecture

Recently B. Klingler [102] has proposed a far reaching generalisation of the Zilber–Pink Conjecture, as discussed in section 3.4.1, for arbitrary irreducible smooth quasi-projective complex varieties supporting a (mixed) $\mathbb{Z}$-VHS. See [102, Section 5.1.] and references therein. In this section we explain his conjecture and discuss its consequences for the pair $(S_\Gamma, \widehat{\mathbb{V}})$, where S_Γ is a ball quotient as in Chapter 7, associated to a non-arithmetic lattice $\Gamma \subset \mathrm{PU}(1,n)$ (for $n > 1$) and $\widehat{\mathbb{V}}$ is the $\mathbb{Z}$-VHS constructed in Theorem 7.4.1.

We start with some definitions from [102]. As in section 2.2.1.3, let S be a smooth quasi-projective complex variety, with a period map

$$\psi : S^{\mathrm{an}} \to R(\mathbb{Z})\backslash D$$

associated to a $\mathbb{Z}$-VHS $\mathcal{V}$ on S with generic Mumford-Tate group $R/\mathbb{Q}$.

Definition 8.1.1. Let $W \subset S$ be an irreducible subvariety. The *Hodge codimension* of W is the codimension of the tangent space at a Hodge-generic smooth point of $\psi(W)$ in the corresponding horizontal tangent space of $R(\mathbb{Z})\backslash D$:

$$\text{H-cd}(W) := \operatorname{rk} T_h(R(\mathbb{Z})\backslash D) - \dim W.$$

Let $\mathfrak{r}$ be the Lie algebra of R. Notice that the rank of $T_h(R(\mathbb{Z})\backslash D)$ is the dimension, over $\mathbb{C}$, of the -1 part of $\mathfrak{r}$, with respect to the Hodge-filtration induced from $\mathcal{V}$. Moreover

$$T_h(R(\mathbb{Z})\backslash D) = T(R(\mathbb{Z})\backslash D)$$

holds if and only if $R(\mathbb{Z})\backslash D$ is a Shimura variety. If this is the case, the Hodge codimension of W is just the codimension of W in its special closure (that is the smallest sub-Shimura variety of $R(\mathbb{Z})\backslash D$ containing W).

Definition 8.1.2. An irreducible subvariety $W \subset S$ is said to be *Hodge-optimal* if for any irreducible subvariety $W \subsetneq Y \subset S$ we have

$$\text{H-cd}(W) < \text{H-cd}(Y).$$

The following is [102, Conjecture 1.9].

Conjecture 8.1.3 (Klingler)**.** *There are only finitely many Hodge optimal subvarieties of S.*

Consider now the pair $(S_\Gamma, \widehat{\mathbb{V}})$, as in the previous chapter. Conjecture 8.1.3 for $(S_\Gamma, \widehat{\mathbb{V}})$ is meaningful only if Γ is non-arithmetic. Otherwise the period map $\psi : S_\Gamma{}^{\text{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$ is an isomorphism, the Hodge codimension of S_Γ is zero and therefore there are no Hodge-optimal subvarieties.

The following is [14, Proposition 6.1.4]. Here we say that an algebraic subvariety W of S_Γ is *totally geodesic* if its smooth locus is totally geodesic with respect to the Kähler metric coming from the universal covering of S_Γ. We say that W is a *maximal totally geodesic subvariety* if the only totally geodesic subvariety of S_Γ strictly containing W is S_Γ itself.

Theorem 8.1.4 (Baldi-Ullmo)**.** *If Γ is non-arithmetic, maximal totally geodesic subvarieties of S_Γ are Hodge optimal.*

In [14, Theorem 1.2.1], we proved the following instance of Conjecture 8.1.3. Interestingly, one of the main difficulties in proving the following theorem is that the André-Deligne monodromy theorem of section 6.2.1 can not be applied to study the natural VHS on S_Γ (since it is not a $\mathbb{Z}$-VHS).

Theorem 8.1.5 (Baldi-Ullmo)**.** *If $\Gamma \subset \mathrm{PU}(1,n)$, for $n > 1$, is non-arithmetic, then S_Γ contains only finitely many maximal totally geodesic subvarieties.*

The idea behind the proof of Theorem 8.1.5 is to interpret the problem as a phenomenon of *unlikely intersections* inside the period domain for polarised $\mathbb{Z}$-VHS, thanks to the discussions of section 7.4.1 and deduce the finiteness from the Ax-Schanuel theorem established by Bakker and Tsimerman [8]. For a complete discussion about the role of the Ax-Schanuel towards the Zilber-Pink conjecture, we refer to [44].

Remark 8.1.6. It is interesting to point out that a similar statement holds true also for a locally symmetric space associated to a lattice $\Gamma \subset \mathrm{SO}(1,n)$. Indeed the problem was originally proposed informally by Reid and, independently, by McMullen for real hyperbolic lattices [55, Question 7.6], [118, Question 8.2]. For $\mathrm{SO}(1,n)$ this was recently proven by Bader, Fisher, Miller and Stover [5] and, for closed hyperbolic 3-manifolds, by Margulis and Mohammadi [111]. Such approaches build on superrigidity theorems and use results on equidistribution from homogeneous dynamics. Finally around the same time a similar result has also been proven by Bader, Fisher, Miller and Stover [6] (they consider both real and complex geodesics). It would certainly be worth investigating more the link between homogeneous dynamics and Zilber–Pink like problems.

Since this thesis presented results about both the geometry and the arithmetic of Shimura varieties, in the next section, we discuss arithmetic aspects of Conjecture 8.1.3 that may be approached in the future.

8.2 André–Oort type conjectures

It is worth noticing that non-arithmetic ball quotients still have models over number fields. We discuss some conjectures that appeared in [14, Section 8]. They mix geometry and arithmetic in a similar fashion as the problems described so far.

8.2.1 Models

Let S be a smooth complex quasi-projective variety. We say that S admits a $\overline{\mathbb{Q}}$-*model* if there exists $Y/\overline{\mathbb{Q}}$ such that $Y \times_{\overline{\mathbb{Q}}} \mathbb{C} \cong S$, with respect to some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

For example if Γ is cocompact, [158, Theorem 1] and [32, Theorem 1] imply the following.

Theorem 8.2.1 (Shimura, Calabi, Visentini)**.** *Let Γ be a lattice in $\mathrm{PU}(1,n)$ such that $S_\Gamma = \Gamma \backslash \mathbb{B}^n$ is projective. Let Θ be the sheaf of germs of holomorphic sections of the tangent bundle of S_Γ. If the dimension of S_Γ is greater or equal to two, then*

$$H^1(S_\Gamma, \Theta) = 0.$$

It follows that S_Γ admits a $\overline{\mathbb{Q}}$-model.

If Γ is arithmetic, the above proof can be generalised to cover the case when S_Γ is not compact, by using Mumford's theory [131]. See indeed [66, 140]. It should be possible to generalise such arguments to the case of non-arithmetic lattices. It is also possible to give a different argument which uses only the fact that all lattices in $\mathrm{PU}(1,n)$, for $n > 1$, have entries in some number field, as explained in section 7.2.1. Both approaches give only the existence of a $\overline{\mathbb{Q}}$-model, rather than a canonical model in the sense of section 2.2.2. The following is [14, Theorem 8.4.4].

Theorem 8.2.2 (Baldi-Ullmo)**.** *Let S_Γ be a ball quotient of dimension at least two. Then S_Γ admits a model over a number field.*

Now that we have highlighted another similarity between non-arithmetic ball quotients and Shimura varieties, we can discuss conjectures inspired by the André–Oort conjecture for Shimura varieties.

8.2.2 Γ-Special points

Let $G = \mathrm{PU}(1,n)$, for some $n > 1$ and X the associated Hermitian space. Let Γ be a lattice in G and S_Γ the associated quasi-projective variety. Recall that Γ determines a K-form $\mathbf{G}$ of G.

Definition 8.2.3. A point $x \in X$ is *pre-*Γ*-special* if the K-Zariski closure of the image of $x : \mathbb{S} \to G$ in $\mathbf{G}$ is commutative. A point $s \in S_\Gamma$ is Γ*-special* if it is the image along $\pi : X \to S_\Gamma$ of a pre-Γ-special point $x \in X$.

The following can be proven as in the classical case of Shimura varieties (see [46, Section 5]).

Proposition 8.2.4. *Pre-*Γ*-special point are defined over* $\overline{\mathbb{Q}}$*, with respect to the* $\overline{\mathbb{Q}}$*-structure given by looking at* $X \subset X^\vee$ *inside its associated flag variety. Moreover* Γ*-special points are Zariski dense in* S_Γ*.*

A first possibility one could investigate is the following.

Conjecture 8.2.5 (Γ-André–Oort)**.** *An irreducible subvariety* $W \subset S_\Gamma$ *is totally geodesic if it contains a Zariski dense set of* Γ*-special points.*

8.2.3 ℤ-special points

Another natural possibility is to look at zero dimensional intersections between $\psi({S_\Gamma}^{\mathrm{an}})$ and Mumford–Tate sub-domains of $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$.

Definition 8.2.6. A point $s \in S_\Gamma$ is $\mathbb{Z}$*-special* if it is a zero dimensional $\mathbb{Z}$-special subvariety. That is there exists $\mathbf{R}$ a $\mathbb{Q}$-subgroup of $\widehat{\mathbf{G}}$ inducing a Mumford–Tate sub-domain D_R of $D_{\widehat{G}}$ and $\mathbf{R}(\mathbb{Z})\backslash D_R$ intersects $\psi({S_\Gamma}^{\mathrm{an}})$ in a finite number of points containing s.

A first relation between $\mathbb{Z}$-special points and Γ-special is given by the following. For the proof see [14, Proposition 8.2.3.].

Proposition 8.2.7. *Let $\pi(x) \in S_\Gamma$ be a $\mathbb{Z}$-special point, then $\pi(x)$ is Γ-special.*

We propose the following conjecture.

Conjecture 8.2.8. *Let $W \subset S_\Gamma$ an algebraic subvariety. W contains a Zariski dense set of $\mathbb{Z}$-special points if and only if is special and arithmetic. That is W is totally geodesic and the period map ψ restricted to W^{an} is an isomorphism.*

8.2.4 Complex multiplication points

As in Theorem 7.4.1, we denote by $\widehat{\mathbb{V}}$ the $\mathbb{Z}$-VHS on S_Γ induced by $\mathbb{V}$. Another interesting class of points is given by the following (see also Definition 2.2.6).

Definition 8.2.9. A point $s \in S_\Gamma$ is called a *CM-point* if the Mumford–Tate group of $\widehat{\mathbb{V}}$ at s is commutative.

Such points are both $\mathbb{Z}$ and Γ-special, but they are even more special. The following is a special case of [102, Conjecture 5.6] and it is indeed predicted by, the more difficult, Conjecture 8.2.8.

Conjecture 8.2.10. *Let $W \subset S_\Gamma$ be an algebraic irreducible subvariety. W contains a Zariski dense set of CM-points if and only if is special and arithmetic.*

If Γ is an arithmetic lattice, this is the classical André-Oort conjecture. If Γ is non-arithmetic, but $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$ is a Shimura variety, then the above conjecture follows from André–Oort conjecture for Shimura varieties of abelian type, which is now a theorem [171]. With our definition of $\overline{\mathbb{Q}}$-models of S_Γ it is not clear that CM-points are defined over $\overline{\mathbb{Q}}$. This could be a starting point for applying the Pila-Zannier strategy in such context.

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