Euler systems and their applications

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In memory of Maria Bordin,

who left right before the start of this journey,

but will always be part of it.

I, Giada Grossi, confirm that the work presented in this thesis is my own, except the contents of Chapter 5, which is part of a joint work with F. Castella, J. Lee and C. Skinner. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

The main theme of this thesis is the theory of Euler and Kolyvagin systems. Such systems are norm compatible classes in the Galois cohomology of p-adic representations. We focus on two aspects of this theory: how to prove these norm compatibilities in the case of the Asai representation attached to a quadratic Hilbert modular form on one hand and how to use norm compatible classes to bound Selmer groups in the case of elliptic curves with a rational p-isogeny on the other.

More precisely, in the first part of this thesis we study certain classes in motivic cohomology of Hilbert modular surfaces, first constructed by Lei–Loeffler–Zerbes. We prove norm relations for the Euler system built from these classes for the Asai representation attached to a Hilbert modular form over a quadratic real field F. Under a strong condition on the underlying real quadratic field, we give a proof of the norm relations for primes that split in F, using the technique introduced by the authors. We then redefine the classes in the language used by Loeffler–Skinner–Zerbes in the $GSp(4)$ case and prove norm relations using local representation theory. With this technique we are able to remove the above mentioned assumption and prove tame norm relations for all inert and split primes.

In the second part, we present part of a joint work with F. Castella, J. Lee and C. Skinner in which we use the Heegner point Kolyvagin system to prove a bound on the Selmer group attached to a rational elliptic curve with a rational p-isogeny, extending a result by Howard. This result is crucial in the proof of the anticyclotomic Iwasawa main conjecture, which is used in the above mentioned work to prove new cases of the p-part of the Birch and Swinnerton-Dyer conjecture.

Impact Statement

We expect that the results of this thesis will have impact in various areas of Mathematics. Indeed, we have been using tools from Number Theory, Geometry, and Representation Theory. Thus, we believe that our work will influence and generate new research in these areas. To achieve this impact, we submitted our work for publication in peer-reviewed journals. In particular the content of Chapter 4 was accepted for publication in International Journal of Number Theory. As evidence of interest in our work, we have been recently invited to present our research in various occasions around Europe; the list of hosting institutions includes the ENS of Lyon (France), University of Oxford (UK), University College London (UK), University of Nottingham (UK).

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Contents

Chapter 1

Introduction

One of the most challenging open problems in Number Theory is the conjecture of Birch and Swinnerton-Dyer for elliptic curves and its generalisation to motives, as given by Bloch and Kato [BK90] and later refined by Fontaine and Perrin-Riou [FPR94]. Roughly speaking, in order to study a motive M (for example the one attached to an elliptic curve E over $\mathbb{Q}$), one can look at two objects of very different nature: an L-function, which is an analytic object "collecting local data at all primes", and an algebraic object encoding "global aspects" of the motive, for example the set of all rational points of E or, more generally, a Selmer group. The Birch and Swinnerton-Dyer and the Bloch-Kato conjectures predict that there is a surprisingly strict relation between these apparently different sides. The simplest form of the conjecture can be stated as follows: the L-function $L(M,s)$ attached to the motive M is a priori defined on some half plane, but it is expected to have analytic continuation for all $s \in \mathbb{C}$. Denote by $X(M)$ the algebraic object attached to M; under some assumptions, these conjectures relate $X(M)$ and the value at some integer s_0 of the L-function $L(M,s)$. In particular they predict

$$L(M,s_0) \neq 0 \Rightarrow X(M) \text{ is finite,} \tag{rank 0}$$

$$\mathrm{ord}_{s=s_0} L(M,s) \geq 1 \Rightarrow X(M) \text{ is infinite.} \tag{rank ≥ 1}$$

The theory of Euler and Kolyvagin systems is a powerful tool that can be used to attack these conjectures. In this thesis we present some results in two different settings: in Chapter 3 and 4 the motive[1] considered is attached to a quadratic Hilbert modular form, in Chapter 5 it is the one given by elliptic curves over $\mathbb{Q}$. The results we present are motivated by their application in proving the statement (rank 0) in the former case and a refinement of (rank ≥ 1) in rank one in the latter case.

1.1 What is an Euler system?

The notion of Euler system has been developed in the last 30 years, starting with the work of Thaine [Tha88], where he introduced a remarkable new method for studying ideal class groups of real abelian number fields using cyclotomic units, the simplest example of an Euler system. Thaine used such units to construct explicitly a large collection of principal ideals of real abelian number fields to bound the exponent of the different Galois eigencomponents of the ideal class group of the field. Soon after this work, Kolyvagin independently discovered a similar method [Kol88, Kol90], using Heegner points on (modular) rational elliptic curves to bound their Selmer group. Another important construction has been made by Kato in [Kat04] using Siegel units to prove bounds on Selmer groups of cuspidal eigenforms. What these works have in common is the idea of producing a large collection of classes in the Galois cohomology of certain p-adic Galois representations. If these classes are *compatible* in some sense, then they can be used to bound some Selmer groups.

A formalisation of this theory appeared in [Rub00] and [MR04] and we refer the reader to such books for further details. We recall here the definition in the simplest setting. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$ and p be a prime and E a finite extension of $\mathbb{Q}_p$. We consider the case of representations of $G_{\mathbb{Q}}$ on E-vector spaces V of finite dimension d, where we assume that

$$\rho : G_{\mathbb{Q}} \to \mathrm{Aut}(V) \cong \mathrm{GL}_d(E)$$

is continuous with respect to the profinite topology of $G_{\mathbb{Q}}$ and the p-adic topology on $\mathrm{GL}_d(E)$. We also assume that such representation is unramified outside a finite

[1]Here by motive we simply mean a compatible system of Galois representations.

set Σ of places including p and the archimedean places, i.e. for all $\ell \notin \Sigma$, we have $\rho(I_\ell) = \{1\}$, where I_ℓ is the inertia group at ℓ.

The first example is the representation $\mathbb{Z}_p(1)$. Let $\mu_{p^n} = \left\{x \in \bar{\mathbb{Q}}^\times : x^{p^n} = 1\right\}$. Then μ_{p^n} is finite cyclic of order p^n and $G_\mathbb{Q}$ acts on it. The p-power map sends $\mu_{p^{n+1}} \to \mu_{p^n}$ and we define

$$\mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}, \quad \mathbb{Q}_p(1) := \mathbb{Z}_p(1) \otimes \mathbb{Q}_p.$$

This is a 1-dimensional continuous $\mathbb{Q}_p$-representation, unramified outside $\{p\}$. Moreover $G_\mathbb{Q}$ acts by the cyclotomic character $\chi_{\mathrm{cyc}} : G_\mathbb{Q} \to \mathbb{Z}_p^\times$. For any given representation V as above, we denote by $V(n)$ the representation

$$V(n) := V \otimes \mathbb{Q}_p(1)^{\otimes n}.$$

Finally, let $\mathbb{Q}(\mu_m)$ be the cyclotomic extension of $\mathbb{Q}$, obtained by adding to $\mathbb{Q}$ the m-th roots of unity $\mu_m = \left\{x \in \bar{\mathbb{Q}}^\times : x^m = 1\right\}$ and denote by V^* the dual representation of V.

We are finally ready to give the definition of an Euler system for a representation V as above. Recall that the inclusion $G_{\mathbb{Q}(\mu_m)} \supset G_{\mathbb{Q}(\mu_n)}$ for $m \mid n$ induces a corestriction map in Galois cohomology

$$\mathrm{cores}^{\mathbb{Q}(\mu_n)}_{\mathbb{Q}(\mu_m)} : H^1(\mathbb{Q}(\mu_n), V) \to H^1(\mathbb{Q}(\mu_m), V).$$

An Euler system for V is a collection of Galois cohomology-classes $(z_m)_{m\geq 1}$ with $z_m \in H^1(\mathbb{Q}(\mu_m), V^*(1))$ satisfying the following norm relations:

$$\mathrm{cores}^{\mathbb{Q}(\mu_{m\ell})}_{\mathbb{Q}(\mu_m)} z_{m\ell} = \begin{cases} z_m & \ell \mid m \quad \text{or} \quad \ell \in \Sigma \\ P_\ell(\mathrm{Frob}_\ell^{-1}) z_m & \text{otherwise}, \end{cases} \tag{NR}$$

where Frob_ℓ^{-1} is the geometric Frobenius and $P_\ell(x) = \det(1 - \mathrm{Frob}_\ell^{-1} x | V)$ is the characteristic polynomial[2].

[2] Note that in [Rub00], an Euler system is defined by *bounded* classes, i.e. classes taking values

Remark 1.1.1. We make a few comments about the more general case, where $\mathbb{Q}$ is replaced by a number field K.

1. Following [Rub00, II.1], the definition of Euler system can be adapted to p-adic Galois representations of the absolute Galois group of a number field K by requiring that we have Galois cohomology classes "in every ray class field extension of K" and modifying the norm relations accordingly. Notice that the ray class field extensions of $\mathbb{Q}$ are precisely the cyclotomic fields.

2. The above-mentioned definition does not cover the case of Kolyvagin's Heegner points. The classes obtained using those special points will not be defined over abelian extensions of $\mathbb{Q}$, but rather over abelian extensions of an imaginary quadratic field K which are not abelian over $\mathbb{Q}$. On the other hand, if one tries to make the definition work for K, the problem is that the classes are defined only over abelian extension of K which are anticyclotomic over $\mathbb{Q}$. However, the process of "taking the Kolyvagin derivatives" of these classes works also in that setting and the *Kolyvagin system* can be used to prove the desired bound of the Selmer group of elliptic curves over K.

1.2 Selmer group bounds

Selmer groups are the *algebraic objects* attached to a p-adic Galois representation V that are conjecturally linked to the analytic L-functions attached to V. If V is an E-vector space and $\mathscr{O}$ denotes the ring of integers of E, we fix a $G_{\mathbb{Q}}$-stable lattice $T \subset V$. The idea is that the Galois cohomology group $H^1(\mathbb{Q},T)$ carries a lot of information about the representation, but it is too big, often of infinite rank. Hence instead of considering $H^1(\mathbb{Q},T)$, one defines a Selmer group by taking a subspace cut out by imposing local conditions. For every place v of $\mathbb{Q}$ we have the natural restriction map

$$\mathrm{loc}_v : H^1(\mathbb{Q},T) \to H^1(\mathbb{Q}_v,T)$$

in the cohomology of a fixed lattice $T \subset V$ independent of m. Even though that is the right setting for applications, we decided to define the classes with values in V, since this is the setting of Chapters 3 and 4. However, one can show that the classes there defined can be suitably modified to be integral. See for example Remark 4.8.14.

given by the inclusion $G_{\mathbb{Q}_v} \subset G_{\mathbb{Q}}$. There are different *Selmer structures* one can consider, one example is the following (which is usually referred to as *strict* Selmer group):

$$\mathrm{Sel}(\mathbb{Q},T) = \left\{ c \in H^1(\mathbb{Q},T) : \begin{array}{c} \mathrm{loc}_\ell(c) \in \ker\left(H^1(\mathbb{Q}_\ell,T) \to H^1(I_\ell,T)\right) \quad \text{if } \ell \nmid p\infty \\ \mathrm{loc}_p(c) = \mathrm{loc}_\infty(c) = 0. \end{array} \right\}$$

Selmer structures are always defined by imposing the unramified condition outside a finite set of places and one can show (see for example [MR04, Proposition 2.1.5]) that this implies that the corresponding Selmer group has finite rank over $\mathscr{O}$.

Now assume that we have an *integral* Euler system for V, i.e. classes $z_m \in H^1(\mathbb{Q}(\mu_m),T^*(1))$ satisfying (NR). Under some technical assumptions on T, we have the following result.

Theorem 1.2.1 (Cfr. II.2.2 [Rub00]). *If the bottom class z_1 of the Euler system is not zero, then the Selmer group* $\mathrm{Sel}(\mathbb{Q},T)$ *is finite.*

Actually, this statement can be made more precise and one has a bound on the size of the Selmer group in terms of the index of the class z_1, see *op. cit.* for more details. We also remark that, even if the statement only involves the bottom class, one really needs the full Euler system to produce such bound.

It may look surprising that classes in Galois cohomology over cyclotomic extensions are able to give information about Galois cohomology over $\mathbb{Q}$, but this is precisely where the Kolyvagin derivative process comes into play. Let ℓ be an odd prime and Γ_ℓ be the Galois group of $\mathbb{Q}(\mu_\ell)/\mathbb{Q}$, which is cyclic of order $\ell-1$ and generated by an element σ_ℓ. For $\ell \neq p$, one considers the *derivative operator*

$$D_\ell := \sum_{i=0}^{\ell-2} i\sigma_\ell^i \in \mathbb{Z}[\Gamma_\ell]. \qquad \text{(Kolyvagin derivative)}$$

An easy computation verifies the following equality

$$(\sigma_\ell - 1)D_\ell = (\ell-1) - \mathrm{Nm}_{\mathbb{Q}(\mu_\ell)/\mathbb{Q}}. \qquad (1.2.1)$$

One considers similarly, for any square-free integer m coprime to p and letting again Γ_m be the Galois group of $\mathbb{Q}(\mu_m)/\mathbb{Q}$, the operator

$$D_m := \prod_{\ell|m} D_\ell \in \mathbb{Z}[\Gamma_m].$$

Let ϖ be a uniformiser of $\mathscr{O}$. The idea is to consider the classes z_m seen as elements in $H^1(\mathbb{Q}(\mu_m), T^*(1)/\varpi^M)$ for some integer M and an infinite set of integers m chosen to satisfy certain congruence conditions with respect to ϖ^M (for example one requires that all the primes dividing m are congruent to 1 modulo ϖ^M). Using (1.2.1) and the norm relations (NR) one shows that

$$D_m \cdot z_m \in H^1(\mathbb{Q}(\mu_m), T^*(1)/\varpi^M)^{\Gamma_m}.$$

The next step is to prove that we can find a well-defined preimage, denoted by $\kappa_{M,m}$, of $D_m \cdot z_m$ under the restriction map $H^1(\mathbb{Q}, T^*(1)/\varpi^M) \to H^1(\mathbb{Q}(\mu_m), T^*(1)/\varpi^M)^{\Gamma_m}$. The localisation of these classes will satisfy certain conditions, for example (NR) is used to relate $\mathrm{loc}_\ell(\kappa_{M,\ell m})$ and $\mathrm{loc}_\ell(\kappa_{M,m})$ for $\ell \nmid m$. Classes satisfying such relations are the so-called *Kolyvagin systems* and they are used to prove Theorem 1.2.1.

More details about how one can obtain such results are presented in Chapter 5, where the classes obtained by Heegner points, which indeed form a *Kolyvagin system*, are used to bound a Selmer group over a quadratic imaginary field attached to a rational elliptic curve.

Remark 1.2.2. In this type of argument, one does not need an Euler/Kolyvagin system for all integers m. It suffices to have classes for a "large enough" infinite set of integers. More precisely, the argument of [Rub00, Chapter V] applies Čebotarev density theorem to find primes whose Frobenius is in the same conjugacy class of a certain fixed element in the absolute Galois group of $\mathbb{Q}$. For example, in the proof of [LLZ18, Theorem 9.5.3], the authors verify that this condition forces such primes to be inert in the real quadratic field.

Finally, we briefly mention that the strategy one would like to apply to use Theorem 1.2.1 for proving results like (rank 0) is to relate the bottom class z_1 of the Euler system to the L-function $L(V^*(1),s)$. Proving for example that the critical value $L(V^*(1),s_0)$ vanishes if and only if $z_1 = 0$, combined with Theorem 1.2.1, would give one implication of (rank 0).

1.3 A method for constructing Euler systems

Even though Euler systems are expected to exist for "representations coming from geometry" (see [PR95, PR98]), it is very difficult to construct them. Until recently the only known non-trivial constructions were cyclotomic units, elliptic units (see for example [Rub91]) and Kato classes ([Kat04]). In the last few years some new Euler systems have been constructed, e.g. an Euler system for the p-adic representation attached to the Rankin-Selberg convolution of two modular forms [LLZ14], for the Asai representation of a quadratic Hilbert modular form [LLZ18] and for the spin representation of a genus 2 Siegel modular form [LSZ20a]. Some progress on the construction of an Euler system for a genus 3 Siegel modular form has been made in [CRJ18]. The common input of these works, following the ideas of [Kat04], are Siegel units, which are invertible elements in $\mathscr{O}(Y_{\mathrm{GL}_2})$, where Y_{GL_2} denotes the modular curve. More generally one considers Eisenstein classes, which are elements in the first motivic cohomology group of Y_{GL_2} with coefficients in some specific motivic sheaves.

The idea of the aforementioned papers is then to consider embeddings $\mathrm{GL}_2 \hookrightarrow G$ (or $\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \hookrightarrow G$ in [LSZ20a] and $\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \hookrightarrow G$ in [CRJ18]), where G is a suitable algebraic group. These embeddings are chosen to be such that they induce a closed embedding of Shimura varieties. Pushing forward the Siegel units via such embedding, one gets classes in a motivic cohomology group of the Shimura variety Y_G. Such embeddings are then suitably "perturbed" in order to define classes in the motivic cohomology of the base change over cyclotomic extensions $Y_G \times \mu_m$. Via the étale regulator one obtains classes in the continuous étale cohomology of $Y_G \times \mu_m$. The group G is chosen using some *numerology* (see

[LZ18, 2.4]), so that, after applying the Hochschild-Serre spectral sequence, one obtains classes in the middle degree étale cohomology of Y_G. More precisely, in the simplest case, one finds classes

$$z_m \in H^1(\mathbb{Q}(\mu_m), H^{\dim Y_G}_{\text{ét}}(Y_{G,\bar{\mathbb{Q}}}, \mathbb{Q}_p)).$$

The middle degree étale cohomology of Shimura varieties is the natural place where Galois representations V_Π attached to automorphic representations Π of the corresponding group G appear. Hence projecting to the Π-isotypic component of the middle degree étale cohomology, one finds classes in $H^1(\mathbb{Q}(\mu_m), V_\Pi)$ giving rise to an Euler system. For a more detailed overview of the circle of ideas of these works see the lecture notes [LZ18].

1.4 Norm relations and Asai–Flach classes

The main difficulty in proving that the classes constructed as above form an Euler system is the proof of the tame Euler system norm relations, i.e. comparing classes $z_{m\ell}$ and z_m when $\ell \nmid m$. In the Rankin–Selberg [LLZ14] and in the Asai case [LLZ18], these relations are proved via some explicit computations in the Hecke algebra. This approach would have been much more difficult (or even impossible) for the Euler system attached to a genus 2 Siegel modular form, as the structure of the group GSp_4 is too complicated. In [LSZ20a], indeed, the technique used was different: the norm relations were obtained using results from smooth representation theory.

The classes appearing in Chapters 3 and 4 are the Asai-Flach classes, originally constructed in [LLZ18]. Let us give more details about this case. Let $F/\mathbb{Q}$ be a real quadratic field and let $\{\sigma_1, \sigma_2\}$ be the set of embeddings of F into $\mathbb{R}$. We let G be the $\mathbb{Q}$-algebraic group obtained as the Weil restriction of GL_2 from F to $\mathbb{Q}$. The reflex field of the Hilbert modular surface Y_G is $\mathbb{Q}$.

Let p be a prime and let f be a Hilbert cuspidal eigenform over F of level coprime to p. One has a 2 dimensional p-adic Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ associated to f. From that one can obtain, via tensor induction, a 4 dimensional

p-adic Galois representation of $G_{\mathbb{Q}}$; it is called Asai representation attached to f and we denote it by V_f^{As}. This representation appears in the middle degree étale cohomology of the Hilbert modular surface Y_G.

In [LLZ18], the authors constructed the Asai–Flach classes in the cohomology of the Hilbert modular surface and were able to build Galois cohomology classes satisfying (NR) assuming that F has trivial narrow class group. More precisely, they prove (NR) in the case where ℓ is inert and sketch the proof of the case where ℓ splits and the primes above ℓ are trivial in the narrow class group. In Chapter 3 we give the details of the latter for classes in motivic cohomology with trivial coefficients. The technique used involves some explicit computations in the Hecke algebra, which, combined with the properties of Siegel units, allow to prove the desired relations.

In Chapter 4, we redefine such classes and prove norm relations with the smooth representation theory technique introduced in [LSZ20a], which allows to remove the above assumptions and to prove (NR) for every unramified prime. We therefore obtain the following result, with no need of assuming the triviality of the narrow class group of F.

Theorem 1.4.1 ([LLZ18, Gro20])**.** *Suppose f has level $\mathfrak{N} \neq 1$ and is of weights $(k+2, k'+2)$, for $k, k' \geq 0$ and $\mathfrak{N}$ sufficiently large*[3] *coprime to $6p$ and the discriminant of F. Let j be an integer such that $0 \leq j \leq \min(k, k')$. Assume f is not a base change lift of a modular form of $\mathrm{GL}_2/\mathbb{Q}$. Then there exists an Euler system $({z_m}^{[f,j]})_{m\geq 1}$ for $V_f^{\mathrm{As}}(1+j)$, satisfying (NR).*

In order to (re)define the Euler system constructed in [LLZ18], we construct a special map $\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}$ for k, k', j as above with values in degree 3 motivic cohomology groups of some motivic sheaf $\mathscr{D}(2)$ over Y_G. Such map will be of "global nature", more precisely it is a map

$$\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}} : \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q}) \otimes \mathscr{H}(G(\mathbb{A}_f), \mathbb{Q}) \longrightarrow H^3_{\mathrm{mot}}(Y_G, \mathscr{D}(2))$$

satisfying some conditions of $H(\mathbb{A}_f) \times G(\mathbb{A}_f)$-equivariance. Here we let $H = \mathrm{GL}_2$,

[3] in the sense of Remark 2.2.7.

$\mathbb{A}_f$ is the group of finite adèles of $\mathbb{Q}$, $\mathscr{H}(G(\mathbb{A}_f),\mathbb{Q})$ is the Hecke algebra over $G(\mathbb{A}_f)$ and $\mathscr{S}(\mathbb{A}_f^2,\mathbb{Q})$ denotes the space of Schwartz functions on $\mathbb{A}_f^2$, which parametrises Eisenstein classes. The Asai–Flach classes are defined as images of explicit elements under $\mathscr{AF}^{\text{k,k',j}}_{\text{mot}}$. Proving norm relations (in motivic cohomology) turns out to be equivalent to proving relations locally at a certain prime ℓ. In order to do this, after recalling some standard tools of local representation theory, we study local zeta integrals attached to principal series representations, using them to characterise the local Euler factor appearing in (NR). The key result we need to use such zeta integrals to prove tame norm relations is then a multiplicity one result (see Theorem 4.5.1). It will follow from [Pra90, Theorem 1.1] in some cases and we prove it in the remaining needed cases, using tools of Mackey theory following the strategy used by Prasad in *op. cit.* and a result of [KMS03] in some degenerate cases.

1.4.1 Future work and applications to Bloch–Kato conjecture

A priori, the construction above could give a system of trivial classes, namely we do not know whether ${z_m}^{[f,j]} = 0$ for every m and every f. However, applying the complex regulator to the bottom class in motivic cohomology and computing the pairing with some differential form associated to a Hilbert modular form f, the authors of [LLZ18] can prove (see [LLZ18, Corollary 5.4.9, Proposition 5.1.3]) that if $|k-k'| \geq 3$ then the motivic class in non-zero. Assuming the conjectured injectivity of the étale regulator, one has that the classes obtained in the étale cohomology of the Hilbert modular surface are non-zero. A second piece of evidence of the non-triviality of this construction is provided in [LSZ20b], where the authors express the localisations at p of the étale classes in terms of overconvergent p-adic modular forms.

Therefore, one can aim to find a sufficient condition for the class ${z_1}^{[f,j]}$ to be different from zero and hence, applying Theorem 1.2.1, for the strict Selmer group of $T_f^{\mathrm{As}}(1+j)$ to be finite. In particular if s_0 is the central critical point for

$L((V_f^{\mathrm{As}})^*(-j),s)$, proving the implication

$$L((V_f^{\mathrm{As}})^*(-j),s_0) \neq 0 \Rightarrow {z_1}^{[f,j]} \neq 0 \tag{$\star$}$$

could give new cases of the conjecture (rank 0). Usually, these types of results are proved relating the bottom class of the Euler system to values of a suitable p-adic L-functions. In the case of the Asai representation attached to Hilbert modular forms, there is no known construction of a p-adic L-function interpolating the values of the complex Asai L-function.

In current work in progress, we first plan to construct such an L-function, using methods recently developed in [LPSZ19], where the authors construct a p-adic L-function for the spin representation of genus 2 Siegel modular forms, relying crucially on Pilloni's recent work on higher Hida theory [Pil20]. The idea of their construction is as follows: in [Har04] Harris shows that critical values of the spin L-function can be expressed as cup products of classes in coherent cohomology, and Pilloni's results can be used to show that these coherent cohomology classes vary in p-adic families and hence give rise to a p-adic L-function. We plan to adapt this strategy to quadratic Hilbert modular forms.

Hida's theory of ordinary p-adic families of modular forms has been used to construct p-adic Rankin–Selberg L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_2$ (by Hida [Hid85] and Panchishkin [Pan83]), and triple product L-functions for the group $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ (by Harris–Tilouine [HT01]). Classical Hida theory is sufficient for those cases since one works with products of the modular curve and it suffices to vary p-adically the degree zero cohomology group. In the case of the Siegel threefold or of the Hilbert modular surfaces, the classes one needs to vary are in the degree one cohomology group and that is why higher Hida theory comes into play.

We are currently developing in [Gro] higher Hida theory for Hilbert modular varieties (in the case where the prime p is totally split). We plan to use it to construct a p-adic Asai L-function and then, in the quadratic case, aim to relate it to the Asai–Flach Euler system via the so-called "explicit reciprocity laws". They should relate the image of the bottom Euler system class under the syntomic regulator to a (non-critical) value of the p-adic L-function. This is the key result needed for a proof of ($\star$).

1.5 Heegner points and Selmer groups of elliptic curves

As already mentioned, the classes constructed from Heegner points do not fit in the definition given above. However, with the Kolyvagin classes obtained from them one still obtains interesting Selmer group bounds. Consider E an elliptic curve over $\mathbb{Q}$ and L a number field. Fix p a rational prime; we write $\mathrm{Sel}_{p^\infty}(E/L), S_p(E/L)$ for the usual p-Selmer groups sitting into the following exact sequences

$$0 \to E(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E/L) \to Ш[p^\infty] \to 0 \tag{1.5.1}$$
$$0 \to E(L) \otimes \mathbb{Z}_p \to S_p(E/L) \to \varprojlim_n Ш[p^n] \to 0,$$

where Ш is the Tate-Shafarevic group of E/L. We have $\mathrm{Sel}_{p^\infty}(E/L) \subset H^1(L, E[p^\infty]), S_p(E/L) \subset H^1(L, T_p(E))$, where $E[p^\infty]$ is the p^∞-torsion of E and $T_p(E)$ denotes the p-adic Tate module of E.

Consider a quadratic imaginary field K satisfying the Heegner hypothesis, i.e. such that all primes dividing the conductor of E split in K. Furthermore, we also assume that the prime p splits in K. The theory of complex multiplication gives a family of points on the modular curve of level equal to the conductor which are rational over abelian extensions of K. More precisely, for every squarefree product n of rational primes inert in K, one constructs a point defined over $K[n]$, the ring class field of K of conductor n. Fixing a modular parametrisation of E yields a family of points $P[n] \in E(K[n])$ which satisfy *Euler system-like* norm relations. After applying the Kummer map and the Kolyvagin derivative operator to the points $P[n]$, one finds classes

$$\kappa_n \in \mathrm{Sel}_{\mathscr{F}(n)}(K, T_p(E)/I_nT_p(E)) \subset H^1(K, T_p(E)/I_nT_p(E))$$

for some ideals $I_n \subset \mathbb{Z}_p$, with $I_1 = \{0\}$. The classes κ_n lie in Selmer groups defined using some *transverse condition* at primes dividing n and so that $\mathrm{Sel}_{\mathscr{F}(1)}(K, T_p(E)) = S_p(E/K)$. Such classes form a *Kolyvagin system* and can be used to prove interesting Selmer group bounds.

In Chapter 5, we prove the following theorem, generalising a result by Howard [How04] in the case where the G_K-representation on $E[p]$ is irreducible.

Theorem 1.5.1. *Assume that p is a prime of good reduction for E and that $E[p](K) = 0$. If $\kappa_1 \neq 0$ then $S_p(E/K)$ is a free $\mathbb{Z}_p$-module of rank one and there is a finite $\mathbb{Z}_p$-module M such that $\mathrm{Sel}_{p^\infty}(E/K) \cong (\mathbb{Q}_p/\mathbb{Z}_p) \oplus M \oplus M$ and*

$$\mathrm{length}_{\mathbb{Z}_p}(M) \leq \mathrm{length}_{\mathbb{Z}_p}(S_p(E/K)/\kappa_1 \cdot \mathbb{Z}_p) + t,$$

where t is a non-negative integer depending only on $\mathrm{Im}(G_K \to \mathrm{GL}(T_p(E)) \simeq \mathrm{GL}_2(\mathbb{Z}_p))$.

Remark 1.5.2. One can characterise the *error term t* and, in particular, we prove that $t = 0$ if $E[p]$ is irreducible, recovering Howard's result. In [CGLS20], we prove the result by similar methods for the twist of the representation $T_p(E)$ by certain anticyclotomic characters. This allows us to prove the Heegner point Iwasawa main conjecture, originally formulated by Perrin-Riou in [PR87] and proved in the irreducible case by Howard [How04].

1.5.1 Applications to the Birch and Swinnerton-Dyer conjecture

This type of result has interesting applications in terms of the conjecture (rank ≥ 1). More preciely, consider $E/\mathbb{Q}$ an elliptic curve. The L-function $L(E,s)$ attached to it is known to have analytic continuation to the whole complex plane thanks to the work of Wiles, Taylor–Wiles and Breuil–Conrad–Diamond–Taylor [Wil95, TW95, BCDT01]. Its central critical value is at $s = 1$. Moreover the Mordell–Weil theorem asserts that the group of rational points of E is isomorphic to $\mathbb{Z}^r \oplus T$, where T is a finite abelian group and $r \geq 0$ is an integer, called the algebraic rank of E. The Birch–Swinnerton-Dyer conjecture predicts that the Tate–Shafarevich group of E is finite and

$$\operatorname{ord}_{s=1} L(E,s) = r.$$

Notice that, assuming that the p-part of the Tate–Shafarevich group of E is finite, r is also equal to the corank of the p^∞-Selmer group $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ using (1.5.1).

The celebrated work of Gross–Zagier [GZ86] gives the following characterisation of the class κ_1:

$$\frac{d}{ds}L(E/K,s)_{|s=1} \neq 0 \Leftrightarrow \kappa_1 \neq 0. \tag{1.5.2}$$

The combination of this result and the mentioned work of Kolyvagin, yields the following remarkable case of the conjecture.

Theorem 1.5.3 (Gross–Zagier, Kolyvagin)**.** *Let E be an elliptic curve over $\mathbb{Q}$, then (rank ≥ 1) holds true in the rank one case. More precisely*

$$\operatorname{ord}_{s=1} L(E,s) = 1 \Rightarrow \operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1 \textit{ and } \#Ш(E/\mathbb{Q}) < \infty.$$

In [CGLS20] we consider $E/\mathbb{Q}$ an elliptic curve and p an odd prime of good ordinary reduction. Assume that E admits a p-isogeny over $\mathbb{Q}$. Recall that by the work of Mazur [Maz78] this implies $p \leq 37$; however, by the same work, we also have that for $p \leq 13$ we have infinitely many isomorphism classes of elliptic curves with a rational p-isogeny. Under some assumptions, we prove the anticyclotomic Iwasawa main conjecture for E/K using the generalisation of Theorem 1.5.1 mentioned in Remark 1.5.2. Choosing the field K carefully and following a strategy first introduced by Skinner in [Ski20] in the irreducible case, we are then able to prove the *p-converse* to Theorem 1.5.3.

Theorem 1.5.4 ([CGLS20])**.** *Let $E/\mathbb{Q}$ be an elliptic curve and p an odd prime of good ordinary reduction. Assume that E has a rational p-isogeny with the character giving the action on its kernel being different from the trivial character or the cyclotomic character when restricted to the decomposition group at p. We have that "the p-part of the converse implication of (rank ≥ 1) in the rank one case" holds* *true. More precisely*

$$\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1 \textit{ and } \#\text{Ш}(E/\mathbb{Q})[p^{\infty}] < \infty \Rightarrow \mathrm{ord}_{s=1} L(E,s) = 1.$$

Finally, we briefly mention that, as explained for example in [Wil06], the Birch–Swinnerton-Dyer conjecture also predicts an exact formula for the leading term of the Taylor expansion of $L(E,s)$ at $s=1$. The p-part of the formula for elliptic curves of rank one has been established in some cases ([JSW17, Cas18]), always using the irreducibility of the representation $E[p]$ as an important assumption. In the work [CGLS20], as another application of the anticyclotomic Iwasawa main conjecture, we also deduce the p-part of the formula for elliptic curves of rank one with a rational p-isogeny such that the character giving the action on its kernel is either ramified at p and odd or unramified at p and even. In a future project, we also plan to work on proving the p-part of the formula in the complementary case, i.e. when the character is either unramified at p and odd or ramified at p and even. A key input in the strategy will be again the anticyclotomic Iwasawa main conjecture for E/K.

Chapter 2

Preliminaries

In this chapter we recall some background material, that will be useful in Chapter 3 and 4. In particular, we give the definition of modular curves, Siegel units, which are invertible functions on some modular curves, and Hilbert modular surfaces. We then talk about motivic cohomology and define some *motivic sheaves*. Siegel units (and their generalisations) can be seen as elements in the motivic cohomology of modular curves; they will be used to define Asai–Flach classes, which are elements in the motivic cohomology of Hilbert modular surfaces. Finally, we recall the definition of Hilbert modular forms and of certain Galois representations attached to them.

2.1 Modular curves and Siegel units

2.1.1 Modular curves

We start by recalling some definitions and properties of modular curves; the notation is the same of [LLZ14] and [Kat04]. As general references to modular curves, we refer to [DS05] or [DDT97].

We write $E[N]$ for the N-torsion of an elliptic curve E and $\langle -,- \rangle_{E[N]} : E[N] \times E[N] \to \mu_N$ for its Weil pairing.

Definition 2.1.1 (See [DR73])**.** For $N \geq 5$, let $Y(N)$ the smooth affine curve over $\mathbb{Q}$

representing the functor from the category of $\mathbb{Q}$-schemes sending

$$S \mapsto \left\{ \begin{matrix} \text{isomorphism classes of triples } (E, e_1, e_2), \\ E \text{ elliptic curve over } S \text{ and} \\ e_1, e_2 \text{ sections of } E/S \text{ generating } E[N] \end{matrix} \right\}.$$

Remark 2.1.2. The curve $Y(N)$ comes with a universal elliptic curve $\mathscr{E} \to Y(N)$, which represents the functor $S \mapsto \{$isomorphism classes of $((E, e_1, e_2), s)$, with $(E, e_1, e_2) \in Y(N)(S)$ and $s \in E(S)\}$. Moreover, there is a surjective morphism $Y(N) \to \mu_N^\circ$, where μ_N° is the scheme of primitive N-th roots of unity, given by

$$(E, e_1, e_2) \mapsto \langle e_1, e_2 \rangle_{E[N]},$$

where $\langle -, - \rangle_{E[N]}$ denotes the Weil pairing on $E[N]$. The fibre of $Y(N)(\mathbb{C})$ over the point $e^{2\pi i/N} \in \mu_N^\circ(\mathbb{C})$ is canonically identified with $\Gamma(N)\backslash\mathscr{H}$, where $\mathscr{H}$ is the upper half-plane and $\Gamma(N)$ the principal congruence subgroup of level N in $\mathrm{SL}_2(\mathbb{Z})$, via the map

$$\tau \mapsto (\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \tau/N, 1/N)$$

The group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on $Y(N)$ in the following way

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (E, e_1, e_2) = (E, ae_1 + be_2, ce_1 + de_2).$$

Taking quotients of $Y(N)$ by subgroups of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ gives the other modular curves we are interested in.

Definition 2.1.3. For $M, N \geq 1$ and $L \geq 5$ divisible by M and N, let $Y(M, N)$ be the quotient of $Y(L)$ by the group

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) : a - 1 \equiv b \equiv 0 \pmod{M}, c \equiv d - 1 \equiv 0 \pmod{N} \right\}.$$

The curve $Y(M, N)$ represents the functor of triples (E, e_1, e_2) where e_1 has order M, e_2 has order N and e_1, e_2 generate a subgroup of E of order MN.

Definition 2.1.4. Let $Y_1(N)$ be the smooth affine curve over $\mathbb{Q}$ representing the functor

$$S \mapsto \left\{ \begin{matrix} \text{isomorphism classes of pairs } (E,e), \\ E \text{ elliptic curve over } S \text{ and} \\ e \text{ section of } E/S \text{ of exact order } N \end{matrix} \right\}.$$

One has that $Y_1(N) = Y(1,N)$; moreover the following proposition identifies $Y_1(N) \times \mu_m^\circ$ with the quotient of $Y(L)$ for a suitable L.

Proposition 2.1.5. *[LLZ14, Proposition 2.1.5] If $N \geq 5$, $m \geq 1$ and $L \geq 5$ is divisible by both N and m, then we have a map*

$$\begin{aligned} Y(L) &\longrightarrow Y_1(N) \times \mu_m^\circ \\ (E,e_1,e_2) &\mapsto \left((E, \tfrac{L}{N}e_2), \langle \tfrac{L}{m}e_1, \tfrac{L}{m}e_2\rangle_{E[m]}\right), \end{aligned}$$

where $\langle -,- \rangle_{E[m]}$ denotes the Weil pairing and μ_m° is the scheme of primitive m-th roots of unity. It identifies the target with the quotient of $Y(L)$ by the subgroup of $\mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z})$ given by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) : c \equiv d-1 \equiv 0 \ (mod\, N), ad-bc \equiv 1 \ (mod\, m) \right\}.$$

Remark 2.1.6. We have $\mu_m^\circ = \mathrm{Spec}(\mathbb{Q}(\mu_m))$, where $\mathbb{Q}(\mu_m)$ is the extension of $\mathbb{Q}$ obtained adding all m-th roots of unity. If X is a variety over $\mathbb{Q}$, then $X \times \mu_m^\circ$ is the image of the base change of X over $\mathbb{Q}(\mu_m)$ under the forgetful functor from $\mathbb{Q}(\mu_m)$-varieties to $\mathbb{Q}$-varieties.

We also define a map between certain modular curves, using again the Weil pairing on elliptic curves.

Definition 2.1.7. Let $m,N \geq 1$, we define the morphism $t_m : Y(m,mN) \to Y_1(N) \times \mu_m^\circ$ given by

$$(E,e_1,e_2) \mapsto \left((E/\langle e_1\rangle, [me_2]), \langle e_1, Ne_2\rangle_{E[m]}\right).$$

We notice that, writing explicitly this morphism on the complex points we find

$$\left(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau, \frac{\tau}{m}, \frac{1}{mN}\right) \mapsto \left(\left(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\frac{\tau}{m}, \frac{1}{N}\right), \zeta_m\right)$$

and hence t_m is given by $\tau \mapsto \tau/m$ on the upper half plane.

Using the morphism t_m one is able to define a morphism as in the following lemma.

Lemma 2.1.8. *[LLZ14, Lemma 2.7.1] Let $m,N \geq 1$ with $m^2N \geq 5$ and $j \in \mathbb{Z}$. There is a unique morphism of algebraic varieties over $\mathbb{C}$*

$$\kappa_j : Y_1(m^2N)_{\mathbb{C}} \to Y_1(N)_{\mathbb{C}}$$

such that the diagram of morphism of complex-analytic manifolds

$$\begin{tikzcd} \mathscr{H} \arrow[r, "\tau\mapsto\tau+j/m"] \arrow[d] & \mathscr{H} \arrow[d] \\ Y_1(m^2N)(\mathbb{C}) \arrow[r, "k_j"] & Y_1(N)(\mathbb{C}) \end{tikzcd}$$

commutes. The morphism is defined over $\mathbb{Q}(\mu_m)$ and depends only on the residue class of j modulo m.

Proof. The morphism κ_j is obtained via the composition of three maps. The first one is the morphism $f : Y_1(m^2N) \times \mu_m \to Y(m,mN)$ coming from the map $Y(m^2N) \to Y(m,mN)$ sending $(E,e_1,e_2) \mapsto (E/\langle mNe_2\rangle, [mNe_1], [e_2])$ which factors through the quotient of $Y(m^2N)$ by the subgroup of matrices $\left(\begin{smallmatrix} u & * \\ 0 & 1 \end{smallmatrix}\right) : u \equiv 1 \bmod m$. Indeed we have

$$\begin{aligned} \left(\begin{smallmatrix} u & b \\ 0 & 1 \end{smallmatrix}\right) \cdot (E,e_1,e_2) = (E, ue_1+be_2, e_2) &\mapsto (E/\langle mNe_2\rangle, [mNue_1+mNbe_2], [e_2]) \\ &= (E/\langle mNe_2\rangle, [mNe_1], [e_2]), \end{aligned}$$

where we used the fact that $u = 1+km$ for some integer k and that $m^2Ne_1 = 0$. Such a quotient is identified, thanks to Proposition 2.1.5, with $Y_1(m^2N) \times \mu_m$. The map is given on $\mathscr{H}$ by the multiplication by m. One then considers the map induced by the action of $\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)$ on $Y(m,mN)$, i.e. $(E,e_1,e_2)\mapsto(E,e_1+jNe_2,e_2)$, given on $\mathscr{H}$ by $\tau\mapsto\tau+j$. Finally we get κ_j as the composition

$$Y_1(m^2N)\times\mu_m\to Y(m,mN)\xrightarrow{\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)}Y(m,mN)\xrightarrow{t_m}Y_1(N)\times\mu_m.$$

This map is given on $\mathscr{H}$ by $\tau\mapsto\tau+j/m$, depends only on the class of j modulo m and is defined over $\mathbb{Q}(\mu_m)$ since all the three maps above commute with the projections to μ_m. $\square$

2.1.2 Siegel units

We now want to define some special elements in $\mathscr{O}(Y(N))^\times$, following [Kat04, §1.1].

Let E be an elliptic curve over a scheme S and c an integer, we denote by E_c the kernel of the multiplication by c on E, viewed as Cartier divisor on E. Similarly we write (0) for the zero section of E, viewed as Cartier divisor. We denote by c^* the pullback by the multiplication by c. Moreover if a is an integer coprime with c, the multiplication by a restricts to a morphism $a : E\setminus E_{ac}\to E\setminus E_c$. We then denote by N_a the norm map $N_a:\mathscr{O}(E\setminus E_{ac})^\times\to\mathscr{O}(E\setminus E_c)^\times$.

The key proposition used for defining Kato's Siegel units is then the following.

Proposition 2.1.9. *[Kat04, Proposition 1.3] Let E be an elliptic curve over a scheme S and c an integer such that $(6,c)=1$. Then*

(1) there exists a unique ${}_c\theta_E\in\mathscr{O}(E\setminus E_c)^\times$ satisfying:

(i) the divisor of ${}_c\theta_E$ is $c^2(0)-E_c$;

(ii) $N_a({}_c\theta_E)={}_c\theta_E$ for any integer a coprime with c.

(2) If d is another integer such that $(6,d)=1$ then

$$({}_d\theta_E)^{c^2}(c^*({}_d\theta_E))^{-1}=({}_c\theta_E)^{d^2}(d^*({}_c\theta_E))^{-1},$$

as elements in $\mathscr{O}(E\setminus E_{cd})^\times$.

(3) *For $\tau \in \mathscr{H}$, consider the elliptic curve $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. Writing ${}_c\theta_\tau := {}_c\theta_E$, for $z \in E \setminus E_c$, one has*

$$ {}_c\theta_\tau(z) = q^{\frac{1}{12}(c^2-1)}(-t)^{\frac{1}{2}(c-c^2)}\gamma_q(t)^{c^2}\gamma_q(t^c)^{-1}, $$

where $q = e^{2\pi i\tau}$, $t = e^{2\pi i z}$ and

$$ \gamma_q(t) = \prod_{n\geq 0}(1-q^n t)\prod_{n\geq 1}(1-q^n t^{-1}). $$

(4) *If $h : E \to E'$ is an isogeny of elliptic curves over S of degree coprime to c and if we denote by h_* the norm map, then $h_*({}_c\theta_E) = {}_c\theta_{E'}$.*

(5) *If $T \to S$ is a morphism, $E_T := E \times_S T$ and $pr : E_T \to E$ is the base change morphism, then $pr^*({}_c\theta_E) = {}_c\theta_{E_T}$.*

Consider now $\mathscr{E} \to Y(N)$ the universal elliptic curve over $Y(N)$ and $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2 \setminus \{(0,0)\}$ of order dividing by N and coprime with the integer c, so that we can write $(\alpha,\beta) = (a/N, b/N)$ for $a,b \in \mathbb{Z}$. We also define the morphism

$$ \iota_{\alpha,\beta} = ae_1 + be_2 : Y(N) \to \mathscr{E} \setminus \mathscr{E}_c, $$

i.e. the morphism that sends an S-point (E,e_1,e_2) of $Y(N)$ to $((E,e_1,e_2), ae_1 + be_2) \in \mathscr{E}(S)$, for S a $\mathbb{Q}$-scheme.

Remark 2.1.10. The image of $\iota_{\alpha,\beta}$ is in $\mathscr{E} \setminus \mathscr{E}_c$ since c is coprime with the order of $(\alpha,\beta) = (a/N, b/N)$ and e_1, e_2 have order N.

We can finally define Siegel units.

Definition 2.1.11. If $(\alpha,\beta), c, N$ are as above we define ${}_cg_{\alpha,\beta} := \iota_{\alpha,\beta}^*({}_c\theta_{\mathscr{E}}) \in \mathscr{O}(Y(N))^\times$. Furthermore, if $c \equiv 1 \bmod N$ and $c \neq \pm 1$ we let $g_{\alpha,\beta} := {}_cg_{\alpha,\beta} \otimes (c^2-1)^{-1} \in \mathscr{O}(Y(N))^\times \otimes \mathbb{Q}$.

Remark 2.1.12. We will see in § 2.3 that Siegel units can be seen as elements in the motivic cohomology of the modular curve (see Example 2.3.10).

We observe that $g_{\alpha,\beta}$ is independent on the choice of the integer c used in the definition. Indeed we have the following.

Lemma 2.1.13. *The element $g_{\alpha,\beta}$ is well defined; in other words, if $c,d \equiv 1 \bmod N$ and $c,d \neq \pm 1$ then*

$$ {}_c g_{\alpha,\beta} \otimes (c^2-1)^{-1} = {}_d g_{\alpha,\beta} \otimes (d^2-1)^{-1} $$

Proof. Using the definition of the Siegel units and (2) of Proposition 2.1.9 we get

$$ \begin{aligned} {}_c g_{\alpha,\beta} \otimes (c^2-1)^{-1} &= {}_c g_{\alpha,\beta}^{(d^2-1)} \otimes \frac{1}{(c^2-1)(d^2-1)} \\ &= \frac{({}_d g_{\alpha,\beta})^{c^2} \iota_{\alpha,\beta}^* d^* ({}_c\theta_{\mathscr{E}})}{\iota_{\alpha,\beta}^* c^* ({}_d\theta_{\mathscr{E}}) {}_c g_{\alpha,\beta}} \otimes \frac{1}{(c^2-1)(d^2-1)}. \end{aligned} $$

Using then the fact that $c,d \equiv 1$, one has that $\iota_{\alpha,\beta} = c \cdot \iota_{\alpha,\beta} = d \cdot \iota_{\alpha,\beta}$ and hence we get

$$ {}_c g_{\alpha,\beta} \otimes (c^2-1)^{-1} = \frac{({}_d g_{\alpha,\beta})^{c^2} {}_c g_{\alpha,\beta}}{({}_d g_{\alpha,\beta})({}_c g_{\alpha,\beta})} \otimes \frac{1}{(c^2-1)(d^2-1)} = {}_d g_{\alpha,\beta} \otimes (d^2-1)^{-1}. $$

□

We will also need some properties of the Siegel units, that we collect in the following proposition.

Proposition 2.1.14. *(i) Let $\sigma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Recall that this group acts on $Y(N)$; moreover for (α,β) as above we write $(\alpha',\beta') = (\alpha,\beta)\sigma$. Then we have*

$$ \sigma^*({}_c g_{\alpha,\beta}) = {}_c g_{\alpha',\beta'} \text{ and } \sigma^*(g_{\alpha,\beta}) = g_{\alpha',\beta'}. $$

(ii) Let $m \geq 1$ be a nonzero integer coprime with 6 and the orders of α,β, then

$$ g_{\alpha,\beta}(mz) = \prod_{\beta'} g_{\alpha,\beta'}(z), $$

where the product is over all $\beta' \in \mathbb{Q}/\mathbb{Z}$ such that $m\beta' = \beta$.

(iii) We can write $g_{\alpha,\beta}$, with $(\alpha,\beta) = (a/N, b/N)$, as function on $\mathscr{H}$, via the pullback along the map $\mathscr{H} \to Y(N)(\mathbb{C})$, then we find

$$g_{\alpha,\beta}(\tau) = q^{1/12 - a/2N + (1/2)(a/N^2)} \cdot \prod_{n \geq 0}(1 - q^n q^{a/N} \zeta_N^b) \cdot \prod_{n \geq 1}(1 - q^n q^{-a/N} \zeta_N^{-b}),$$

where $q = e^{2\pi i \tau}$.

Proof. (i) The universal property of the elliptic curve $\mathscr{E}/Y(N)$ says that for any triple (E, e_1, e_2), where E is an elliptic curve over S and e_1, e_2 are sections of E over S generating $E[N]$, there exists a unique morphism $S \to Y(N)$ such that E is isomorphic to the pullback $\mathscr{E} \times_{Y(N)} S$, i.e. we have the following commutative diagram

$$\begin{tikzcd} E \simeq \mathscr{E} \times_{Y(N)} S \arrow[r, "pr"] \arrow[d] & \mathscr{E} \arrow[d] \\ S \arrow[r] & Y(N). \end{tikzcd}$$

Moreover an S-section of $\mathscr{E}$ is given by $x = (E, e_1, e_2, P)$ where $(E, e_1, e_2) \in Y(N)(S)$ and $P \in E(S)$, i.e. we have the following diagram

$$\begin{tikzcd} S \arrow[dr, dashed, "P"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "id"'] & & \\ & E \arrow[r, "pr"] \arrow[d] & \mathscr{E} \arrow[d] \\ & S \arrow[r] & Y(N). \end{tikzcd}$$

Using Proposition 2.1.9 (5), we get that ${}_c\theta_{\mathscr{E}}(x) = x^*({}_c\theta_{\mathscr{E}}) = P^* pr^*({}_c\theta_{\mathscr{E}}) = {}_c\,\theta_E(P)$. Writing $(\alpha, \beta) = (m/N, n/N)$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and considering $(E, e_1, e_2) \in Y(N)(S)$ we find

$$(\iota_{\alpha,\beta} \circ \sigma)((E, e_1, e_2)) = (E, ae_1 + be_2, ce_1 + de_2, (ma + nc)e_1 + (mb + nd)e_2),$$

$$(\iota_{\alpha',\beta'})(E, e_1, e_2) = (E, e_1, e_2, (ma + nc)e_1 + (mb + nd)e_2).$$

Hence we obtain, considering the two $Y(N)$-sections of $\mathscr{E}$ $\iota_{\alpha,\beta}\circ\sigma$ and $\iota_{\alpha',\beta'}$,

$$\sigma^*(\iota_{\alpha,\beta})^*{}_c\theta_{\mathscr{E}}((E,e_1,e_2)) = {}_c\theta_E((ma+nc)e_1+(mb+nd)e_2) = {}_c\theta_{\mathscr{E}}((\iota_{\alpha',\beta'})(E,e_1,e_2)),$$

for any S-section (E,e_1,e_2), and therefore the equality $\sigma^*({}_cg_{\alpha,\beta}) = {}_cg_{\alpha',\beta'}$. The second equality descends from this, by definition of the elements $g_{\alpha,\beta}, g_{\alpha',\beta'}$.

(iii) The formula is obtained via direct computation using the analytic description of theta elements (Proposition 2.1.9 (3)), see [Kat04, 1.9].

(ii) We show how, using point (iii), one can deduce (ii). First of all, writing $(\alpha,\beta) = (a/N, b/N)$, we have that the product on the RHS of the equality runs through the elements $\beta' = b/mN + i/m$ for $0 \le i \le m-1$. Then we get

$$\prod_{\beta'} g_{\alpha,\beta'}(\tau) = q^{m(1/12-a/2N+(1/2)(a/N^2))}\cdot\prod_{n\ge 0}\prod_{i=1}^{m-1}(1-q^nq^{a/N}\zeta_{Nm}^b\zeta_m^i)$$
$$\cdot\prod_{n\ge 1}\prod_{i=1}^{m-1}(1-q^nq^{-a/N}\zeta_{Nm}^{-b}\zeta_m^i).$$

We now use the equality $x^m-\alpha^m = \prod_{i=1}^{m-1}(x-\alpha\zeta_m^i)$, which gives, for $x=1$ and $\alpha = q^nq^{a/N}\zeta_{Nm}^b$ ($\alpha = q^nq^{-a/N}\zeta_{Nm}^{-b}$ respectively),

$$\prod_{i=1}^{m-1}(1-q^nq^{a/N}\zeta_{Nm}^b\zeta_m^i) = (1-q^{nm}q^{ma/N}\zeta_N^b),$$

$$\prod_{i=1}^{m-1}(1-q^nq^{-a/N}\zeta_{Nm}^{-b}\zeta_m^i) = (1-q^{nm}q^{-ma/N}\zeta_N^{-b}).$$

So we obtained $g_{a/N,b/N}(m\tau) = \prod_{\beta'} g_{\alpha,\beta'}(\tau)$. □

Lemma 2.1.15. *Let m and (α,β) be as in (ii) of the previous Proposition. We have the equality*

$$g_{\alpha,\beta} = \prod_{\alpha',\beta'} g_{\alpha',\beta'},$$

where the product runs over (α',β') such that $(m\alpha', m\beta') = (\alpha,\beta)$.

Proof. The formula is obtained using Proposition 2.1.9(1.ii). See [Kat04, Lemma 1.7 (2)]. □

Remark 2.1.16. Viewing $Y_1(N)$ as quotient of $Y(N)$ by the subgroup $U' = \{ \left(\begin{smallmatrix} u & * \\ 0 & 1 \end{smallmatrix}\right) : u \in (\mathbb{Z}/N\mathbb{Z})^\times \}$ of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ as in Definition 2.1.3, one can use point (i) of the previous proposition to see that $g_{0,\frac{b}{N}} \in \mathscr{O}(Y_1(N))^\times \otimes \mathbb{Q}$. Indeed for any $\sigma' \in U'$, one gets $(\sigma')^* g_{0,\frac{b}{N}} = g_{0,\frac{b}{N}}$. In particular we have $g_{0,\frac{b}{m^2N}} \in \mathscr{O}(Y_1(m^2N))^\times \otimes \mathbb{Q}$ and $g_{0,\frac{b}{mN}} \in \mathscr{O}(Y_1(mN))^\times \otimes \mathbb{Q}$. We can then regard these Siegel units via pullback along $Y_1(m^2N) \times \mu_m \to Y_1(m^2N)$ and $Y(m,mN) \to Y_1(mN)$ as elements $g_{0,\frac{b}{m^2N}} \in \mathscr{O}(Y_1(m^2N) \times \mu_m)^\times \otimes \mathbb{Q}$ and $g_{0,\frac{b}{mN}} \in \mathscr{O}(Y(m,mN))^\times \otimes \mathbb{Q}$.

We conclude this discussion about Siegel units by proving a lemma that will be useful later. First recall the definition of f, the morphism used in the proof of Lemma 2.1.8. It came from the morphism

$$\begin{aligned} &Y(m^2N) \to Y(m,mN) \\ &(E,e_1,e_2) \mapsto (E/\langle mNe_2 \rangle, [mNe_1], [e_2]). \end{aligned}$$

On complex points it is defined by $\tau \mapsto m\tau$. Moreover it factors through the quotient by the subgroup U defining $Y_1(m^2N) \times \mu_m$ as quotient of $Y(m^2N)$ as in the previous remark. Hence f defines a map

$$f : Y_1(m^2N) \times \mu_m \to Y(m,mN). \tag{2.1.1}$$

Lemma 2.1.17. *Viewing $g_{0,1/mN} \in \mathscr{O}(Y(m,mN))^\times$ and $g_{0,1/m^2N} \in \mathscr{O}(Y_1(m^2N) \times \mu_m)^\times$ as in remark 2.1.16, we have*

$$g_{0,1/mN} = f_* g_{0,1/m^2N}.$$

Proof. We write U and U' for the two subgroups of $\mathrm{GL}_2(\mathbb{Z}/m^2N\mathbb{Z})$ given by

$$U = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : c \equiv d-1 \equiv 0 \ \ (m^2N), a \equiv 1 \ \ (m) \right\}.$$

$$U' = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : c \equiv 0 \ \ (m^2N), a \equiv 1 \ \ (m), d \equiv 1 \ \ (mN) \right\}.$$

We have $U \subset U'$ and we identify the quotient of $Y(m^2N)$ by U with $Y_1(m^2N) \times \mu_m$.

We also consider the set of representatives of U/U' given by the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1+mNt \end{smallmatrix}\right)$, where $0 \leq t < m$. We then find that the pushforward of $g_{0,1/m^2N}$ via the natural map $h : Y_1(m^2N) \times \mu_m \to U' \backslash Y(m^2N)$ is given by

$$\prod_{t=0}^{m-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1+mNt \end{smallmatrix}\right) g_{0,1/m^2N}(\tau) = \prod_{t=0}^{m-1} g_{0,1/m^2N+t/m}(\tau) = g_{0,1/mN}(m\tau),$$

where we used Proposition 2.1.14 (i) and (ii) in the first and second equality respectively. To conclude we notice that conjugation by $\left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$ sends U' to

$$U'' = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : c \equiv d-1 \equiv 0 \ \ (mN), a-1 \equiv b \equiv 0 \ \ (m) \right\},$$

which is the subgroup defining $Y(m,mN)$, i.e. $U'' \backslash Y(m^2N) = Y(m,mN)$. Writing σ_m for the map $U' \backslash Y(m^2N) \to Y(m,mN)$ defined by the multiplication by $\left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and using $(\sigma_m)_* = (\sigma_{m^{-1}})^*$ we get

$$(\sigma_m \circ h)_* g_{0,1/m^2N}(\tau) = (\sigma_{m^{-1}})^* g_{0,1/mN}(m\tau) = g_{0,1/mN}(\tau).$$

Since $\sigma_m \circ h = f$, we are done. $\square$

2.2 Hilbert modular surfaces

In this section we recall some definitions and properties of Hilbert modular surfaces. We then define embeddings of modular curves into these surfaces. These can be thought as *degenerate case* of the diagonal embedding (and its *perturbations*)

$$Y_1(N) \times \mu_m \to Y_1(N)^2 \times \mu_m,$$

used in [LLZ14] to define Beilinson–Flach elements.

2.2.1 Definitions and the closed embedding of the modular curve

We start by fixing some notation. We let F be a real quadratic field, we denote by $\mathscr{O}_F$ its ring of integers, by $\mathfrak{d}$ its different ideal and we fix a set $\{\sigma_1, \sigma_2\}$ of real embeddings of F; if $\lambda \in F$, we write $\lambda_i = \sigma_i(\lambda)$. Let $\mathbb{G}_m$ be the multiplicative group. We then define following algebraic groups.

Definition 2.2.1. We define the algebraic groups $D := \mathrm{Res}^F_{\mathbb{Q}}\mathbb{G}_m$, $G := \mathrm{Res}^F_{\mathbb{Q}}\mathrm{GL}_2$ and $G^* := G \times_{D,\det} \mathbb{G}_m$.

Remark 2.2.2. Notice that we have a natural embedding $\mathrm{GL}_2 \hookrightarrow G^*$. In the degenerate case where $F = \mathbb{Q} \oplus \mathbb{Q}$, one recovers the embedding $\mathrm{GL}_2 \hookrightarrow \mathrm{GL}_2 \times \mathrm{GL}_2$. We now define a Shimura variety associated to it (corresponding to the product of two modular curves in the degenerate case) and then get a closed embedding of the modular curve in it.

Let $\mathscr{H}_F$ be the set of the elements of $F \otimes \mathbb{C}$ of totally positive imaginary part; it can be identified with two copies of the upper half plane $\mathscr{H} \times \mathscr{H}$. We have a natural action of $G(\mathbb{R})^+$ (the elements of totally positive determinant) given by the two real embeddings, namely for $(\tau_1, \tau_2) \in \mathscr{H}_F$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{R})^+$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (\tau_1, \tau_2) = \left(\tfrac{a_1\tau_1+b_1}{c_1\tau_1+d_1}, \tfrac{a_2\tau_2+b_2}{c_2\tau_2+d_2}\right)$$

We will denote by $\mathbb{A}_f$ the finite adeles of $\mathbb{Q}$.

Definition 2.2.3. Let $H \in \{G, G^*, \mathrm{GL}_2\}$. An open compact subgroup $U \subset H\left(\mathbb{A}_f\right)$ is sufficiently small if for every $h \in H\left(\mathbb{A}_f\right)$ the quotient

$$\frac{H(\mathbb{Q})^+ \cap hUh^{-1}}{U \cap \{\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right) : u \in \mathscr{O}_F^\times\}}$$

acts without fixed points on $\mathscr{H}_F$ (or on $\mathscr{H}$ if $H = \mathrm{GL}_2$).

Definition 2.2.4 (Cfr. [Del79])**.** For U^* (respectively U, $U_{\mathbb{Q}}$) an open compact subgroup of $G^*(\mathbb{A}_f)$ (respectively of $G(\mathbb{A}_f)$, $\mathrm{GL}_2(\mathbb{A}_f)$), we denote by $Y_{G^*}(U^*)$ (respectively $Y_G(U), Y_{\mathrm{GL}_2}(U_{\mathbb{Q}})$) the complex manifold of dimension 2 (respectively 2 and 1) given by

$$Y_{G^*}(U^*) = G^*(\mathbb{Q})^+ \backslash [G^*(\mathbb{A}_f) \times \mathscr{H}_F] \,/\, U^*,$$

$$Y_G(U) = G(\mathbb{Q})^+ \backslash [G(\mathbb{A}_f) \times \mathscr{H}_F] \,/\, U,$$

$$Y_{\mathrm{GL}_2}(U_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})^+ \setminus [\mathrm{GL}_2(\mathbb{A}_f) \times \mathscr{H}] \;/\; U_{\mathbb{Q}}.$$

It is known that, when the considered subgroup is sufficiently small, such complex manifolds admit a unique structure of smooth quasi-projective variety defined over $\mathbb{Q}$. By abuse of notation we write $Y_{G^*}(U^*), Y_G(U), Y_{\mathrm{GL}_2}(U_{\mathbb{Q}})$ also to denote such $\mathbb{Q}$-varieties. The analytification of their complex points is given by the above quotients.

Remark 2.2.5. These are instances of a more general class of varieties, called Shimura varieties, that descend to varieties defined over a number field, called reflex field, see [Del79, Mil05]. For more details about this point of view in the Hilbert modular variety case (i.e. when considering the groups G, G^* for a totally real field) see for example [Gor02, Edi01, BG20].

Let $\hat{\mathscr{O}}_F = \mathscr{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$. We now define specific level subgroups U.

Definition 2.2.6. Let $\mathfrak{M}, \mathfrak{N}$ be non-zero ideals of $\mathscr{O}_F$. We define

$$U(\mathfrak{M},\mathfrak{N}) := \left\{ \gamma \in \mathrm{GL}_2(\hat{\mathscr{O}}_F) : \gamma \equiv 1 \bmod \left(\begin{smallmatrix} \mathfrak{M} & \mathfrak{M} \\ \mathfrak{N} & \mathfrak{N} \end{smallmatrix}\right) \right\}$$

and $U^*(\mathfrak{M},\mathfrak{N}) := U(\mathfrak{M},\mathfrak{N}) \cap G^*$. We also write $U_1(\mathfrak{N}) := U(1,\mathfrak{N})$ and $U_1^*(\mathfrak{N}) = U^*(1,\mathfrak{N})$. One defines similarly groups $U_{\mathbb{Q}}(M,N) \subset \mathrm{GL}_2(\mathbb{A}_f)$ for integers M,N. Considering a third ideal $\mathfrak{L}$, let

$$U(\mathfrak{M},\mathfrak{N}(\mathfrak{L})) := \left\{ \gamma \in \mathrm{GL}_2(\hat{\mathscr{O}}_F) : \gamma \equiv 1 \bmod \left(\begin{smallmatrix} \mathfrak{M} & \mathfrak{M} \\ \mathfrak{N}\mathfrak{L} & \mathfrak{N} \end{smallmatrix}\right) \right\}$$

$$U(\mathfrak{M}(\mathfrak{L}),\mathfrak{N}) := \left\{ \gamma \in \mathrm{GL}_2(\hat{\mathscr{O}}_F) : \gamma \equiv 1 \bmod \left(\begin{smallmatrix} \mathfrak{M} & \mathfrak{M}\mathfrak{L} \\ \mathfrak{N} & \mathfrak{N} \end{smallmatrix}\right) \right\}.$$

One defines similarly groups $U_{\mathbb{Q}}(M,N(\ell)), U_{\mathbb{Q}}(M(\ell),N) \subset \mathrm{GL}_2(\mathbb{A}_f)$ for integers M,N,ℓ.

Remark 2.2.7. If $\mathfrak{N}$ is sufficiently large, then the subgroups $U_1^*(\mathfrak{N}), U_1(\mathfrak{N})$ are sufficiently small. More precisely, this follows from ([Dim09, Lemma 2.1(iii)-(iv)], by assuming that $\mathfrak{N}$ is coprime to 6 and the discriminant of F and that $\mathfrak{N}$ is divisible by a prime satisfying the conditions of Lemma 2.1(iii) of *op. cit.* We will always assume this to be the case. Moreover, working with GL_2, we recover the modular curves of the previous sections, namely $Y_{\mathbb{Q}}(U_{\mathbb{Q}}(M,N)) = Y(M,N)$.

We write $Y_1^*(\mathfrak{N}) = Y_{G^*}(U_1^*(\mathfrak{N}))$ and $Y_1(N) = Y_{\mathrm{GL}_2}(U_{\mathbb{Q}}(1,N))$. In particular the first is a smooth surface and latter is the modular curve of Definition 2.1.4. If $\mathfrak{N}$ is an ideal such that $\mathfrak{N} \cap \mathbb{Z} = (N)$, we have $U_{\mathbb{Q}}(1,N) = U_1^*(\mathfrak{N}) \cap \mathrm{GL}_2(\mathbb{A}_f)$.

The action of $\mathrm{GL}_2(F)$ on $\mathscr{H}_F$ uses the two embedding of F, so that we find that the action of $\gamma \in \Gamma_1(\mathfrak{N})$ on $\mathscr{H} \times \mathscr{H}$ restricted to the subgroup $\Gamma_1(N)$ coincides with the usual action of $\Gamma_1(N)$ on each component, since σ_1, σ_2 fix $\mathbb{Z}$.

So the embedding of algebraic groups $\mathrm{GL}_2 \hookrightarrow G^*$ induces a closed embedding

$$\iota : Y_1(N) \hookrightarrow Y_1^*(\mathfrak{N}),$$

which is precisely the one we were aiming for.

We now describe the embedding ι on complex points. First of all, notice that ι commutes with the natural determinant maps

$$\det : Y_1(N) \to (\mathbb{Q}^\times)^+ \backslash \mathbb{A}_f^\times / 1 + N\hat{\mathbb{Z}} \simeq \mu_N,$$

$$\det : Y_1^*(\mathfrak{N}) \to (\mathbb{Q}^\times)^+ \backslash \mathbb{A}_f^\times / (1 + \mathfrak{N}\hat{\mathscr{O}}_F) \cap \hat{\mathbb{Z}} \simeq \mu_N.$$

Moreover, fixing a primitive complex N-th root of unity ζ as in Remark 2.1.2, taking fibres over ζ of $Y_1(N)(\mathbb{C})$ and $Y_1^*(\mathfrak{N})(\mathbb{C})$, yields a map

$$\iota : \mathrm{GL}_2(\mathbb{Q})^+ \cap U_1(N) \backslash \mathscr{H} \hookrightarrow G^*(\mathbb{Q})^+ \cap U_1^*(\mathfrak{N}) \backslash \mathscr{H} \times \mathscr{H}.$$

Recalling that for a $\mathbb{Q}$ scheme S we have $\mathrm{Res}_{\mathbb{Q}}^F G(S) = G(S \times_{\mathbb{Q}} F)$, we get $D(\mathbb{Q}) = \mathbb{G}_m(\mathbb{Q} \otimes F) = F^\times$ and $G(\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q} \otimes F) = \mathrm{GL}_2(F)$. So we find

$$G^*(\mathbb{Q}) = \mathrm{GL}_2(F) \otimes_{F^\times} \mathbb{Q}^\times = \{\gamma \in \mathrm{GL}_2(F) : \det \gamma \in \mathbb{Q}\},$$

$$G^*(\mathbb{Q})^+ = \{\gamma \in \mathrm{GL}_2(F) : \det \gamma \in \mathbb{Q}_{\geq 0}\}.$$

Moreover, using the fact that $\hat{\mathscr{O}}_F \cap F = \mathscr{O}_F$ and $(\hat{\mathscr{O}}_F)^\times \cap \mathbb{Q}_{\geq 0} = \{+1\}$, we get

$$G(\mathbb{Q})^+ \cap U_1(\mathfrak{N}) = \left\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathscr{O}_F)^+ : a \equiv d \equiv 1, c \equiv 0 \bmod \mathfrak{N}\right\};$$

$$\Gamma_1(\mathfrak{N}) := G^*(\mathbb{Q})^+ \cap U_1^*(\mathfrak{N}) = \left\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathscr{O}_F) : a \equiv d \equiv 1, c \equiv 0 \bmod \mathfrak{N}\right\}.$$

Similarly, using, $\hat{\mathbb{Z}} \cap \mathbb{Q}_{\geq 0} = \{+1\}$, as we were anticipating, we get

$$\begin{aligned} \mathrm{GL}_2(\mathbb{Q})^+ \cap U_1(N) &= \left\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1, c \equiv 0 \bmod N\right\} \\ &= \Gamma_1(N) = \Gamma_1(\mathfrak{N}) \cap \mathrm{GL}_2(\mathbb{Q})^+. \end{aligned}$$

Hence, on complex points, the embedding ι is given by the diagonal map $\mathscr{H} \to \mathscr{H} \times \mathscr{H}$

$$\begin{aligned} \iota : \Gamma_1(N)\backslash\mathscr{H} &\longrightarrow \Gamma_1(\mathfrak{N}) \backslash \mathscr{H} \times \mathscr{H} \\ \tau &\longmapsto (\tau, \tau). \end{aligned}$$

One can similarly describe closed embeddings $\iota : Y(M,N) \hookrightarrow Y^*(\mathfrak{M},\mathfrak{N})$ for ideal $\mathfrak{M},\mathfrak{N}$ such that $(M) = \mathfrak{M} \cap \mathbb{Z}$ and $(N) = \mathfrak{N} \cap \mathbb{Z}$.

We now fix $a \in \mathscr{O}_F$ and consider the matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$. If the ideal $\mathfrak{M}$ divides the ideal $\mathfrak{N}$, the subgroup $U^*(\mathfrak{M},\mathfrak{N})$ is normalized by this matrix, i.e. for $\gamma \in U^*(\mathfrak{M},\mathfrak{N})$

$$\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \cdot \gamma \cdot \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)^{-1} \in U^*(\mathfrak{M},\mathfrak{N}).$$

Definition 2.2.8. For $a \in \mathscr{O}_F$, $\mathfrak{M}$ dividing $\mathfrak{N}$, we define $\mathbf{u}_a$ to be the endomorphism of $Y^*(\mathfrak{M},\mathfrak{N})$ induced by multiplication by the matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$, i.e. given on the complex points by $\underline{\tau} = (\tau, \tau') \mapsto \underline{\tau} + a = (\tau + a_1, \tau' + a_2)$, for $\underline{\tau} \in \mathscr{H}_F$.

Remark 2.2.9. Notice that the map $\mathbf{u}_a$ is well defined because

$$\mathbf{u}_a(\gamma \cdot \underline{\tau}) = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\gamma \cdot \underline{\tau} = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\gamma\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \cdot \underline{\tau} = \gamma' \cdot \mathbf{u}_a(\underline{\tau}),$$

for $\gamma' \in \Gamma(\mathfrak{M},\mathfrak{N})$.

2.2.2 Interpretation as moduli space

Hilbert modular surfaces of the form $Y^*(\mathfrak{M},\mathfrak{N})$ are known to be fine moduli space of polarised abelian surfaces with some level structure, in complete analogy with modular curves.

2.2.2.1 Preliminaries on polarized abelian varieties with real multiplication

We start with some general definitions and results about the objects that the Hilbert modular surfaces will parametrize. We keep the discussion more general, considering F a totally real field of degree g over $\mathbb{Q}$. We will moreover stick to the case of complex abelian varieties, since we will only describe the complex points of the Hilbert modular varieties. We will follow closely [Gor02, § 2].

Definition 2.2.10. A complex abelian variety with real multiplication (also denoted by RM) by $\mathscr{O}_F$ is a g-dimensional abelian variety A over $\mathbb{C}$ with a fixed embedding

$$i : \mathscr{O}_F \hookrightarrow \mathrm{End}(A).$$

Example 2.2.11. The easiest example is the one of the (iso-simple) abelian variety obtained by taking an elliptic curve E over $\mathbb{C}$ and considering the abelian variety

$$E \otimes_{\mathbb{Z}} \mathscr{O}_F \cong E^g,$$

whose complex points are given by $E(\mathbb{C}) \otimes \mathscr{O}_F$ and where the isomorphism is obtained by fixing a $\mathbb{Z}$-basis of $\mathscr{O}_F$. The action of $\mathscr{O}_F$ on the abelian variety is the canonical right $\mathscr{O}_F$ action. In the case $g = 2$, writing $F = \mathbb{Q}(\sqrt{D})$ and choosing the $\mathbb{Z}$-basis of $\mathscr{O}_F$ $\{1, \sqrt{D}\}$ if $D \equiv 2, 3 \pmod 4$ and $\{1, (1+\sqrt{D})/2\}$ if $D \equiv 1 \pmod 4$, some easy computations show that the endomorphism obtained by the action of $\sqrt{D}$ and $(1+\sqrt{D})/2$ respectively on E^2 is given by the matrix

$$\begin{pmatrix} 0 & D \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & (D-1)/4 \\ 1 & 1 \end{pmatrix} \text{ respectively.}$$

Remark 2.2.12. If (A,i) is a complex abelian variety with RM by $\mathscr{O}_F$, so is its dual. This can be seen by defining an embedding $\mathscr{O}_F \hookrightarrow \mathrm{End}(A^\vee)$ simply by taking for each $\lambda \in \mathscr{O}_F$ the dual map of $i(\lambda)$.

We now proceed with the construction of some complex abelian variety with RM. Take $\underline{\tau} \in \mathscr{H}^g$ and define $\Lambda_{\underline{\tau}} := \mathscr{O}_F \underline{\tau} + \mathscr{O}_F$. First of all we notice that Dedekind's lemma yields the $\mathbb{R}$-linear independence of

$$(\sigma_1(\alpha_1), \dots, \sigma_g(\alpha_1)), \cdots, (\sigma_1(\alpha_g), \dots, \sigma_g(\alpha_g))$$

in $\mathbb{R}^g$, for α_i a basis of $\mathscr{O}_F$. Hence $\Lambda_{\underline{\tau}}$ is a lattice in $\mathbb{C}^g$. Then one can show that for any $r \in (\mathfrak{d}^{-1})^+$

$$H_{r,\underline{\tau}}(x,y) := \sum_{i=1}^{g} \mathrm{Im}(\tau_i)^{-1} \sigma_i(r) x_i \overline{y_i}, \tag{2.2.1}$$

where $z, z' \in \mathbb{C}^g$, is a Riemann form for $\Lambda_{\underline{\tau}}$ (see [Gor02, Lemma 2.8,2.9] for a proof). Using the standard connection between Riemann forms and polarizations, we get that the complex torus $A_{\underline{\tau}} := \mathbb{C}^g/\Lambda$ is polarized and hence is a complex abelian variety with RM by $\mathscr{O}_F$.

Now we give the last definitions needed in order to state the final result.

Definition 2.2.13. Let (A,i) an abelian variety with RM by $\mathscr{O}_F$. We define

$$\mathscr{M}_A := \{\lambda : A \to A^\vee : \lambda = \lambda^\vee \text{ and } \lambda \text{ is } \mathscr{O}_F\text{-linear}\},$$

$$\mathscr{M}_A^+ := \{\lambda \in \mathscr{M}_A : \ \lambda \text{ is a polarization}\}.$$

In the case of $A = A_{\underline{\tau}}$ we have that the map sending $r \mapsto H_{r,\underline{\tau}}$ gives an isomorphism $m_{\underline{\tau}} : (\mathfrak{d}^{-1}, (\mathfrak{d}^{-1})^+) \xrightarrow{\simeq} (\mathscr{M}_A, \mathscr{M}_A^+)$. We have the following result, see for example [Gor02, Theorem II.2.17]].

Theorem 2.2.14. *(1) The isomorphism classes of complex abelian varieties (A,i) with RM by $\mathscr{O}_F$ such that there exists an isomorphism $(\mathscr{M}_A, \mathscr{M}_A^+) \xrightarrow{\simeq} (\mathfrak{d}^{-1}, (\mathfrak{d}^{-1})^+)$ are parametrized by the quotient $\mathrm{GL}_2(\mathscr{O}_F)^+ \backslash \mathscr{H}^g$.*

(2) The isomorphism classes of complex abelian varieties (A,i,m) with RM by $\mathscr{O}_F$ with a fixed isomorphism $m : (\mathscr{M}_A, \mathscr{M}_A^+) \xrightarrow{\simeq} (\mathfrak{d}^{-1}, (\mathfrak{d}^{-1})^+)$ are parametrized by the quotient $\mathrm{SL}_2(\mathscr{O}_F) \backslash \mathscr{H}^g$.

2.2.2.2 The universal abelian variety over the Hilbert modular surface

We know go back to the setting of 2.2.1. We can describe explicitly the universal abelian variety over $Y_{G^*}(\mathfrak{M},\mathfrak{N})$. We start with the following definition.

Definition 2.2.15. (i) Let P be the subgroup of $\mathrm{Res}_{\mathbb{Q}}^F \mathrm{GL}_3$ consisting of the matrices of the form

$$\begin{pmatrix} 1 & r & s \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

and let P^* the subgroup with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G^*$.

(ii) Let $\mathbb{C}_F := F \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}^2$ and $\mathscr{J}_F := \mathscr{H}_F \times \mathbb{C}_F$. We define an action of $P(\mathbb{R})^+$ on $\mathscr{J}_F$ via

$$\begin{pmatrix} 1 & r & s \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + r\tau + s}{c\tau + d} \right).$$

We then write, for $V^* \subset P^*(\mathbb{A}_f)$ an open compact,

$$A(V^*) := P^*(\mathbb{Q})^+ \backslash [P^*(\mathbb{A}_f) \times \mathscr{J}_F] \,/\, V^*.$$

We first of all notice that the action of $P^*(\mathbb{Q})^+$ on $\mathscr{H}_F$ is compatible with the action of $G^*(\mathbb{Q})^+$ on it. Moreover if U^* is the image of V^* in G^*, the natural map $\mathscr{J}_F \to \mathscr{H}_F$ induces

$$A(V^*) \to Y_{G^*}(U^*).$$

The following result is [LLZ18, Proposition 2.5.2].

Theorem 2.2.16. *IF V^* is given by the elements of P^* such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U^* \subset G^*(\hat{\mathbb{Z}})$ and $r,s \in \hat{\mathscr{O}}_F$, then $A(U^*) := A(V^*)$ is an abelian variety over $Y_{G^*}(U^*)$ with endomorphisms by $\mathscr{O}_F$.*

Remark 2.2.17. The above definition makes sense also for a totally real field F of degree greater than 2. Similarly the stated theorem holds in general.

We will not go into the proof of this theorem, but we are going to look in details what happens when we take one of the congruence subgroups U^* defined above. The easiest case is the case $U^* = \mathrm{GL}_2(\hat{\mathscr{O}}_F) \cap G^* = U^*(1,1)$. Proceeding as we have done before we get

$$Y^* := Y_{G^*}(U^*)(\mathbb{C}) = \mathrm{SL}_2(\mathscr{O}_F) \setminus \mathscr{H}_F.$$

Taking V^* as in the theorem, one similarly finds $A := A(U^*)(\mathbb{C}) = V \setminus (\mathscr{H}_F \times \mathbb{C}_F)$, where

$$V = \left\{ \begin{pmatrix} 1 & r & s \\ 0 & a & b \\ 0 & c & d \end{pmatrix} : r,s \in \mathscr{O}_F, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathscr{O}_F) \right\}.$$

We want to view Y^* as moduli space of certain abelian varieties. To do so we consider the map $A \to Y^*$ and look at its fibres. Take $\underline{\tau} = (\tau, \tau') \in \mathscr{H}_F$, we write $A_{\underline{\tau}}$ for the fibre over $[\underline{\tau}] \in Y^*$.

Claim. $A \to Y^*$ parametrizes isomorphism classes of complex abelian surfaces with RM by $\mathscr{O}_F$.

Proof of Claim. The fibres $A_{\underline{\tau}}$, for every $\underline{\tau} \in \mathscr{H}_F$, are isomorphic as complex abelian variety with RM to the 2-dimensional complex tori $\mathbb{C}^2 \,/\, \mathscr{O}_F \underline{\tau} \oplus \mathscr{O}_F$. We then essentially use Theorem 2.2.14. All complex tori isomorphic as abelian variety with fixed polarization and with RM by $\mathscr{O}_F$ to $\mathbb{C}^2 \,/\, \mathscr{O}_F \underline{\tau} \oplus \mathscr{O}_F = \mathbb{C}^2 / \Lambda_{\underline{\tau}}$ are of the form $\mathbb{C}^2 \,/\, \mathscr{O}_F \frac{a\underline{\tau}+b}{c\underline{\tau}+d} \oplus \mathscr{O}_F$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathscr{O}_F)$, where the isomorphism is given by $z \mapsto \left(\begin{smallmatrix} c_1\tau + d_1 & 0 \\ 0 & c_2\tau' + d_2 \end{smallmatrix}\right)^{-1} \cdot z = \frac{z}{c\underline{\tau}+d}$. Now, the fibre over $\underline{\tau}$ is the quotient of $\mathbb{C}^2$ by $\mathscr{O}_F^2$, were the action is given by $(r,s) \cdot z = z + r\underline{\tau} + s$, hence precisely the complex torus

$\mathbb{C}^2/\Lambda_{\underline{\tau}}$. Changing the representative for the class $[\underline{\tau}]$, i.e. considering $\frac{a\underline{\tau}+b}{c\underline{\tau}+d}$, we find $\mathbb{C}^2$ modulo the action of $\mathscr{O}_F^2$ given by $(r,s)\cdot z = \frac{z+r\underline{\tau}+s}{c\underline{\tau}+d}$, hence we get precisely $(c\underline{\tau}+d)^{-1}\cdot\mathbb{C}^2/\Lambda_{\underline{\tau}} = \mathbb{C}^2 \;/\; \mathscr{O}_F\frac{a\underline{\tau}+b}{c\underline{\tau}+d}\oplus\mathscr{O}_F$. □

Remark 2.2.18. The same can be done by replacing P by an analogous subgroup $P_{\mathbb{Q}}$ of $\mathrm{GL}_3/\mathbb{Q}$ and $\mathscr{J}_F$ by $\mathscr{H}\times\mathbb{C}$. For $U_{\mathbb{Q}} = \mathrm{GL}_2(\hat{\mathbb{Z}})$ one then finds the universal elliptic curve over $Y(1)$.

Again as in the case of modular curves, one can rearrange the above reasoning to describe the Hilbert modular surfaces as moduli space parametrizing triples $(A, \iota_{\mathfrak{M}}, \iota_{\mathfrak{N}})$ with A polarized abelian surface with RM by $\mathscr{O}_F$ as above and $\iota_{\mathfrak{M}}, \iota_{\mathfrak{N}}$ embeddings of the form

$$\iota_{\mathfrak{M}} : \mathfrak{M}^{-1}\mathscr{O}_F/\mathscr{O}_F \hookrightarrow A_{tors},$$

$$\iota_{\mathfrak{N}} : \mathfrak{N}^{-1}\mathscr{O}_F/\mathscr{O}_F \hookrightarrow A_{tors}.$$

In particular if for example $\mathfrak{M} = (M)$ where M is an integer, this gives a point P of order M of A.

Now recall that for an abelian variety A there is a non-degenerate alternating bilinear pairing

$$\langle\ ,\ \rangle'_{A[M]} : A[M]\times A^\vee[M] \to \mu_M,$$

where $A[M]$ and $A^\vee[M]$ denote the M-torsion of A and $A^\vee$ respectively. If A is polarized and has endomorphism by $\mathscr{O}_F$, this induces an $\mathscr{O}_F$-linear pairing

$$\langle\ ,\ \rangle_{A[M]} : A[M]\times A[M] \to \mu_M\otimes\mathscr{O}_F.$$

If we consider the Shimura variety corresponding to G^*, we get that, considering for example $Y^*(M, M\mathfrak{N})$, i.e. the one corresponding to the subgroup $U^*(M, M\mathfrak{N})$, for M integer, it parametrizes triples (A, P, ι) where A is as above, P is a M-torsion point of A and ι is an embedding $\iota : (M\mathfrak{N})^{-1}\mathscr{O}_F/\mathscr{O}_F \hookrightarrow A_{tors}$ such that

(i) the pairing considered above gives $\langle P, \iota(1/M)\rangle \in \mu_M$,

(ii) P and $\iota(1/M)$ are independent over $\mathscr{O}_F$.

In particular condition (i) reflects the fact that the determinant map is defined with target in $\mathbb{G}_m$ and not in $\mathrm{Res}^F_{\mathbb{Q}}\mathbb{G}_m$. Hence it induces

$$\begin{aligned} Y^*(M, M\mathfrak{N}) &\to \mu_M^\circ \\ (A, P, \iota) &\mapsto \langle P, \iota(1/M)\rangle_{A[M]}. \end{aligned}$$

Using this, we are now able to define a map corresponding to the one in Definition 2.1.7 for modular curves.

Definition 2.2.19. For $\mathfrak{N}$ ideal of $\mathscr{O}_F$ and $M \geq 1$ integer, we define

$$\begin{aligned} \mathbf{t}_M : Y^*(M, M\mathfrak{N}) &\longrightarrow Y_1^*(\mathfrak{N}) \times \mu_M^\circ \\ (A, P, \iota) &\longmapsto ((A/\langle P\rangle, M\cdot\iota)\ ,\ \langle P, \iota(1/M)\rangle_{A[M]}) \end{aligned}$$

Remark 2.2.20. On complex points the map $\mathbf{t}_M$ is given by $\underline{\tau} \mapsto \frac{\underline{\tau}}{M}$, for $\underline{\tau} \in \mathscr{H}_F$.

Proposition 2.2.21. *For $b \in (\mathbb{Z}/M\mathbb{Z})^\times$, the map $\mathbf{t}_M$ intertwines the action of $\left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $Y^*(M, M\mathfrak{N})$ with the automorphism $1 \times \sigma_b$ on $Y_1^*(\mathfrak{N}) \times \mu_M^\circ$, where $\sigma_b : \zeta \mapsto \zeta^b$.*

Proof. We need to prove the commutativity of the following diagram

$$\begin{tikzcd} Y^*(M, M\mathfrak{N}) \arrow[r] \arrow[d, "\mathbf{t}_M"'] & Y^*(M, M\mathfrak{N}) \arrow[d, "\mathbf{t}_M"] \\ Y_1^*(\mathfrak{N}) \times \mu_M^\circ \arrow[r, "1\times\sigma_b"] & Y_1^*(\mathfrak{N}) \times \mu_M^\circ, \end{tikzcd}$$

where the top arrow is the morphism given by the action of $\left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right)$, sending $(A, P, \iota) \mapsto (A, bP, \iota)$. Since $b \in (\mathbb{Z}/M\mathbb{Z})^\times$ and P has order M, we have $\langle P\rangle = \langle bP\rangle$. So we find

$$\mathbf{t}_M((A, bP, \iota)) = [(A/\langle P\rangle, M\cdot\iota), \langle bP, \iota(1/M)\rangle] = [(A/\langle P\rangle, M\cdot\iota), \langle P, \iota(1/M)\rangle^b],$$

using the bilinearity of the pairing. On the other side we get

$$\begin{aligned}(1\times\sigma_b)(\mathbf{t}_M((A,P,\iota))) &= (1\times\sigma_b)([(A/\langle P\rangle, M\cdot\iota),\langle P,\iota(1/M)\rangle]) \\ &= [(A/\langle P\rangle, M\cdot\iota),\langle P,\iota(1/M)\rangle^b].\end{aligned}$$

□

2.2.2.3 The action of $\mathscr{G}$ on the abelian surface

Let $V^* = \hat{\mathscr{O}_F}^2 \rtimes U^*$, where $U^* \subset G^*(\mathbb{A}_f)$ is sufficiently small. In other words, V^* is as in Theorem 2.2.16. By abuse of notation, we will denote by $A(U^*)$ the abelian variety $A(V^*) \to Y^*(U^*)$. Consider

$$g \in \mathscr{G} := G(\mathbb{Q})^+ G^*(\mathbb{A}_f) \subset G(\mathbb{A}_f) \quad \text{such that } \; g^{-1} \text{ has entries in } \hat{\mathscr{O}}_F.$$

Assuming that both U^* and gU^*g^{-1} are contained in $G(\hat{\mathbb{Z}})$ we can define (see [LLZ18, Definition 2.5.4]) an $\mathscr{O}_F$-isogeny of abelian varieties over $Y^*(U^*)$. Firstly note that by [LLZ18, Remark 2.5.3], we can write

$$A(U^*) = P^*(\mathbb{Q})^+ \backslash [\mathscr{P} \times \mathscr{J}_F] \,/\, V^*,$$

where $\mathscr{P}$ is the subgroup of $P(\mathbb{A}_f)$ with $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathscr{G}$. So we have a left action of g as above on $A(U^*)$. We then define the $\mathscr{O}_F$-isogeny

$$\Phi_g : A\left(U^*\right) \to g^* A\left(gU^*g^{-1}\right).$$

given by

$$A(\hat{\mathscr{O}_F}^2 \rtimes U^*) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & g\end{smallmatrix}\right)} A((\hat{\mathscr{O}_F}^2 \cdot g^{-1}) \rtimes gU^*g^{-1}) \to A(\hat{\mathscr{O}_F}^2 \rtimes gU^*g^{-1})$$

where the second map is given by the inclusion $(\hat{\mathscr{O}_F}^2 \cdot g^{-1}) \subset \hat{\mathscr{O}_F}^2$. Such isogenies satisfy the relation

$$\Phi_{g_1 g_2} = g_2^*(\Phi_{g_1}) \circ \Phi_{g_2}, \tag{2.2.2}$$

when all the terms are defined. Moreover if $g = \left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$ for $x \in \mathscr{O}_F$, then $gU^*g^{-1} = U^*$ and Φ_g is simply the multiplication by x on $A(U^*)$. Therefore, using this, we can extend the definition of Φ_g for any $g \in \mathscr{G}$ as elements of $\mathrm{Hom}(A(U^*), g^*A(gU^*g^{-1})) \otimes \mathbb{Q}$.

2.3 Motivic cohomology

In this section we recall some properties of motivic cohomology and recall the definition of motivic cohomology *with coefficients* (i.e. with coefficients over relative Chow motives). We then define the *coefficients sheaves* we will be working with.

2.3.1 Motivic cohomology

Let X be an object in the category Sm of smooth variety over a field $k \subset \mathbb{C}$. Then Voevodsky defined motivic cohomology as homomorphisms in the triangulated category $\mathscr{DM}_-(k)$ of motivic complexes. For a construction of this category see [Voe02]; he equips it with a functor $M : Sm \to \mathscr{DM}_-(k)$ and with a Tate object $\mathbf{Q}(1)$.

Definition 2.3.1. The *motivic cohomology* of X as above is defined by

$$H^i_{\mathrm{mot}}(X, \mathbf{Q}(j)) := \mathrm{Hom}_{\mathscr{DM}_-(k)}(M(X), \mathbf{Q}(j)[i]).$$

Voevodsky shows that this motivic cohomology group can be identified with hypercohomology with respect to the Zariski topology, more precisely

$$H^i_{\mathrm{mot}}(X, \mathbf{Q}(j)) \simeq \mathbb{H}^i_{Zar}(X, C_\bullet(\mathbb{Z}(j))),$$

where $C_\bullet(\mathbb{Z}(j))$ is the Suslin complex of sheaves in the Zariski topology (see [Voe02] for more details).

The idea of motives and motivic cohomology is in some sense to collect together the information coming from all Weil cohomology theories $\mathscr{T}$. Among the others we can consider $\mathscr{T} \in \{\text{ét}, \mathrm{dR}, \mathrm{B}\}$, continuous étale, de Rham and Betti cohomology theories. We write $\mathbf{Q}_{\mathscr{T}}$ for the trivial coefficient sheaf of the cohomology theory $\mathscr{T}$ and $\mathbf{Q}_{\mathscr{T}}(n)$ for the n-th power of $\mathbf{Q}_{\mathscr{T}}$, where in our examples $\mathbf{Q}_{\text{ét}}(1) = \mathbb{Q}_p(1), \mathbf{Q}_{\text{dR}}(1) = k, \mathbf{Q}_{\text{B}}(1) = 2\pi i\mathbb{Q}$. All these cohomology theories are then related by natural maps, the comparison isomorphisms, that give

$$H^i_{\text{dR}}(X,k) \otimes_k \mathbb{C} \simeq H^i_{\text{B}}(X(\mathbb{C}),\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

$$H^i_{\text{ét}}(X,\mathbb{Q}_p) \simeq H^i_{\text{B}}(X(\mathbb{C}),\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

The first isomorphism is a standard result, for the proof of the second one see for example [Mil, I.21].

To relate then motivic cohomology groups to the "more traditional" ones, there are regulator maps

$$\text{reg}_{\mathscr{T}} : H^i_{\text{mot}}(X,\mathbf{Q}(n)) \to H^i_{\mathscr{T}}(X,\mathbf{Q}_{\mathscr{T}}(n)),$$

all compatible with comparison isomorphisms. For this see [Hub00].

We can similarly construct motivic cohomology with "non trivial coefficient sheaves", using the formalism of relative Chow motives of [DM91]. Consider S a smooth, connected, quasiprojective k-variety and the category of relative Chow motives over S, denoted by $\text{CHM}(S)_{\mathbb{Q}}$. It is a pseudo-abelian tensor category. For any field of characteristic zero one can similarly consider $\text{CHM}(S)_L$, defined as the pseudo-abelian envelope of $\text{CHM}(S)_{\mathbb{Q}} \otimes L$. The objects of such category are triples (X,p,n), where X is a smooth projective S-variety of relative dimension m, p is an idempotent element of $\text{CH}^m(X \times_S X)$ and $n \in \mathbb{Z}$. The Tate object $(S,\text{id},1)$ will be denoted by $\mathbb{Q}(1)$ or $L(1)$.

This category comes equipped with a contravariant functor from the category $SmPr(S)$ of smooth projective S-schemes

$$M : SmPr(S) \to \text{CHM}(S)_{\mathbb{Q}}.$$

One can take $\mathscr{F}_{\mathscr{T}}$ the realisation of an object in $\mathscr{F} \in \text{CHM}(S)_L$ in a cohomology theory $\mathscr{T}$ as above. This takes value in the category of sheaves on S with extra structure depending on $\mathscr{T}$. This is why we will often, by abuse of notation, refer to an object $\mathscr{F}$ as *motivic sheaf*. In particular, if $\mathscr{T} = \text{ét}$ is the p-adic étale cohomology and L is a p-adic field, $\mathscr{F}_{\text{ét}}$ is a lisse étale L-sheaf over S. The sheaves $\mathscr{F}_{\mathscr{T}}$ are naturally graded objects, in particular $\mathscr{F}_{\text{ét}} = \oplus_j \mathrm{Gr}^j \mathscr{F}_{\text{ét}}$ and, if $\mathscr{F} = M(X)$, then $\mathrm{Gr}^j \mathscr{F}_{\text{ét}} = \mathscr{H}^j_{\text{ét}}(X/S)$ the relative étale cohomology sheaf of X/S of degree j.

Theorem 2.3.2 ([DM91]). *Let A/S be an abelian variety. Then there is a canonical decomposition in the category of relative Chow motives over S*

$$M(A) = \bigoplus_{i=0}^{2\dim A} M^i(A),$$

such that for all the realisations $\mathrm{Gr}^j M^i(A)_{\mathscr{T}} = 0$ *if $i \neq j$.*

We are now ready to define motivic cohomology groups with *coefficients in* $\mathscr{F} = (X, p, n)$, an object in the category $\mathrm{CHM}(S)_Q$ as above. We assume that the realisations of $\mathscr{F}$ are non-zero only in one degree r and let

$$H^i_{\mathrm{mot}}(S, \mathscr{F}(j)) := p^* H^{i+r+2n}_{\mathrm{mot}}(X, \mathbf{Q}(j+n)),$$

where recall that p^* is the endomorphism on the cohomology of X given by $p \in \mathrm{CH}^m(X \times_S X)$.

As above, we find regulator maps

$$\mathrm{reg}_{\mathscr{T}} : H^i_{\mathrm{mot}}(S, \mathscr{F}(j)) \to H^i_{\mathscr{T}}(S, \mathscr{F}_{\mathscr{T}}(j)),$$

and similarly when extending to a field extension L.

If $\iota : S \hookrightarrow T$ is a closed immersion of codimension d, there exists a pullback functor

$$\iota^* : \mathrm{CHM}(T)_L \to \mathrm{CHM}(S)_L$$

and a Gysin map

$$\iota_* : H^i_{\mathrm{mot}}(S, \iota^* \mathscr{F}(j)) \to H^{i+2d}_{\mathrm{mot}}(T, \mathscr{F}(j+d)), \tag{2.3.1}$$

For more details see [LLZ18, § 3.1d], [MVW06, Theorem 15.15].

2.3.2 Relative Chow motives over modular curves and Hilbert modular surfaces

We will be interested in sheaves over the modular curve and over the Hilbert modular surface arising from universal modular abelian varieties over them. We first fix some notation. Given a Shimura datum $(\mathbf{G},X)$, we write $Y_{\mathbf{G}}$ for the inverse limit over K, compact open subgroups of $\mathbf{G}(\mathbb{A}_f)$, of the varieties $Y_{\mathbf{G}}(K) := \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)\times X/K$. Similarly every time we consider a cohomology group for $Y_{\mathbf{G}}$ we mean the limit of the cohomology groups of $Y_{\mathbf{G}}(K)$. Throughout this section we will write $H = \mathrm{GL}_2$. We work with $\mathbf{G} = H, G^*, G$, for which X is the Siegel plane $\mathbb{C}\setminus\mathbb{R}$ in the first case and two copies of the Siegel plane in the second and third ones (with action of $G^*(\mathbb{Q}), G(\mathbb{Q})$ given by the two real embeddings σ_1,σ_2). We obtain a smooth curve Y_H, which is the infinite level modular curve, and smooth surfaces Y_{G^*}, Y_G which are infinite level Hilbert modular surfaces. They are defined over $\mathbb{Q}$.

If $\mathbf{G} = H, G^*$, then the corresponding finite level Shimura varieties are of PEL type and, using the functor of [Anc15, Theorem 8.6], one can associate to representations of $\mathbf{G}$ a relative Chow motive over $Y_{\mathbf{G}}(K)$ for any sufficiently small level K. The following result is due to Ancona and is a special case of [Anc15, Theorem 8.6], which applies more generally to PEL type Shimura varieties.

Theorem 2.3.3 ([Anc15])**.** *Let $\mathbf{G} = H, G^*$ and K a sufficiently small subgroup of $\mathbf{G}$. There is a functor*

$$\mathscr{F}_{\mathbf{G}} : \mathrm{Rep}_{\mathbb{Q}}(\mathbf{G}) \to \mathrm{CHM}(Y_{\mathbf{G}}(K))$$

from the category of representations of $\mathbf{G}$ over $\mathbb{Q}$ to the category of relative Chow motives over $Y_{\mathbf{G}}(K)$ such that

- *$\mathscr{F}_{\mathbf{G}}$ preserves tensor products and duals;*
- *$\mathscr{F}_{\mathbf{G}}(\det)$ is the motive $\mathbf{Q}(1)$, where $\det : \mathbf{G} \to \mathbb{G}_m$;*

- *if V is the dual of the standard representation of $\mathbf{G}$, then $\mathscr{F}_{\mathbf{G}}(V) = M^1(\mathscr{A})$, where $\mathscr{A}$ is the universal abelian variety over $Y_{\mathbf{G}}(K)$;*

- *for any prime p, the p-adic étale realisation of $\mathscr{F}_{\mathbf{G}}(V)$ is the étale lisse sheaf*[1] *associated to $V \otimes \mathbb{Q}_p$ seen as a left K-representation via $K \hookrightarrow \mathbf{G}(\mathbb{A}_f) \twoheadrightarrow \mathbf{G}(\mathbb{Q}_p)$.*

One can clearly extend the functors of the above theorem replacing $\mathbb{Q}$ by a larger extension L.

The following result (see [Tor19, Theorem 9.7], where it is proved more generally for admissible pairs of Shimura data) explains how the functors behave with respect to the Gysin map (2.3.1) induced by the closed immersion

$$\iota : Y_H(K \cap H(\mathbb{A}_f)) \to Y_{G^*}(K),$$

for K a sufficiently small subgroup of $G^*(\mathbb{A}_f)$.

Theorem 2.3.4 ([Tor19])**.** *The following diagram is commutative up to natural isomorphism*

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}}(G^*) & \xrightarrow{\mathscr{F}_{G^*}} & \mathrm{CHM}(Y_{G^*}(K)) \\ \big\downarrow{\scriptstyle |_H} & & \big\downarrow{\scriptstyle \iota^*} \\ \mathrm{Rep}_{\mathbb{Q}}(H) & \xrightarrow{\mathscr{F}_H} & \mathrm{CHM}(Y_H(K \cap H(\mathbb{A}_f))), \end{array}$$

where the first vertical arrow denotes the restriction to H and ι^* is the pullback functor.

This theorem, combined with the existence of the Gysin map (2.3.1), tells us that whenever $W \in \mathrm{Rep}_{\mathbb{Q}}(H)$ is a direct summand of the restriction to H of a representation $V \in \mathrm{Rep}_{\mathbb{Q}}(G^*)$, the pushforward by ι defines a map

$$\begin{aligned} \iota_* : H^i_{\mathrm{mot}}(Y_H(K \cap H(\mathbb{A}_f)), \mathscr{F}_H(W)) &\to H^i_{\mathrm{mot}}(Y_H(K \cap H(\mathbb{A}_f)), \iota^*(\mathscr{F}_{G^*}(V)) \\ &\to H^{i+2}_{\mathrm{mot}}(Y_{G^*}(K), \mathscr{F}_{G^*}(V)(1)). \end{aligned} \tag{2.3.2}$$

[1] See for example [Nek18, 0.4].

This kind of maps will be used in § 4.7.1 for the definition of Asai–Flach classes. In the easiest case, if we take V to be the determinant representation the map is simply

$$\iota_* : H^i_{\mathrm{mot}}(Y_H(K \cap H(\mathbb{A}_f)), \mathbf{Q}(1)) \to H^{i+2}_{\mathrm{mot}}(Y_{G^*}(K), \mathbf{Q}(2)) \tag{2.3.3}$$

and taking $i = 1$ it will be used in § 3.1 to define Asai–Flach classes in the trivial coefficients case.

We now briefly go over the definitions of the relative Chow motives we will be working with.

2.3.2.1 Modular curves

We consider $\mathscr{E} \to Y$, where $Y = Y_H(K)$ is the modular curve of level K, K is a sufficiently small open compact of $H(\mathbb{A}_f)$ and $\mathscr{E}$ is the universal elliptic curve over Y.

Definition 2.3.5. Let $k \geq 0$ be an integer. We define $\mathrm{TSym}^k \mathscr{H}_L(\mathscr{E})$ to be the object of $\mathrm{CHM}(Y)_L$ given by the image of the functor $\mathscr{F}_H$ of the TSym^k power[2] of the standard representation of GL_2.

By the properties of Ancona's functor, $\mathrm{TSym}^k \mathscr{H}_L(\mathscr{E})$ is the k-th symmetric power of $M^1(\mathscr{E})(1) = M^1(\mathscr{E})^\vee$, where $M^1(\mathscr{E})$ is given by the decomposition of Theorem 2.3.2 and (1) denotes the twist by $L(1)$.

2.3.2.2 Hilbert modular surfaces

Similarly, consider $\mathscr{A} \to Y^*$, where $Y^* = Y_{G^*}(K^*)$ is the Hilbert modular surface of level K^*, K^* is a sufficiently small open compact of $G^*(\mathbb{A}_f)$ and $\mathscr{A}$ is the universal abelian surface over Y^*. Recall that $\mathscr{O}_F$ acts on $\mathscr{A}$ by endomorphisms. Consider the following object of $\mathrm{CHM}(Y^*)_L$

$$\mathscr{H}_L(\mathscr{A}) = M^3(\mathscr{A})(2) = M^1(\mathscr{A})^\vee.$$

[2] $\mathrm{TSym}^k(V)$ is defined to be the submodule of invariants under the permutation actions of the symmetric group $\mathfrak{S}_k$ in the k-fold tensor product $V \otimes \cdots \otimes V$. It is the dual of the more familiar module $\mathrm{Sym}^k(V^\vee)$ of $\mathfrak{S}_k$-coinvariants of $V^\vee \otimes \cdots \otimes V^\vee$.

By enlarging L if necessary, we assume we have two non-zero embeddings $\theta_i : F \hookrightarrow L$. Then the object considered above decomposes as

$$\mathscr{H}_L(\mathscr{A}) = \mathscr{H}_L(\mathscr{A})^{(1)} \oplus \mathscr{H}_L(\mathscr{A})^{(2)},$$

where $\mathscr{H}_L(\mathscr{A})^{(i)}$ is the direct summand where, for $x \in \mathscr{O}_F$, we have $[x]_* = \sigma_i(x)$ (see [LLZ18, § 3.2b] for more details).

Definition 2.3.6. Let $k, k' \geq 0$ be integers. We define $\mathrm{TSym}^{[k,k']} \mathscr{H}_L(\mathscr{A})$ to be the object of $\mathrm{CHM}(Y^*)_L$ given by the image under Ancona's functor of the tensor product of the TSym^k power and $\mathrm{TSym}^{k'}$ power of the standard representation of each copy of GL_2.

Explicitly, using the above decomposition, $\mathrm{TSym}^{[k,k']} \mathscr{H}_L(\mathscr{A})$ is

$$\mathrm{TSym}^k(\mathscr{H}_L(\mathscr{A})^{(1)}) \otimes \mathrm{TSym}^{k'}(\mathscr{H}_L(\mathscr{A})^{(2)})$$

One can similarly define $\mathrm{Sym}^{(k,k')} \mathscr{H}_L(\mathscr{A})$ and we have that its dual is $\mathrm{TSym}^{[k,k']} \mathscr{H}_L(\mathscr{A})$.

The Shimura variety for the larger group G is not of PEL type, so we cannot directly apply Ancona's functor. However, from the relative Chow motives defined above, one constructs motivic sheaves for Hilbert modular surfaces with respect to G. The étale cohomology of the étale realisation of these sheaves will be the natural place where the Galois representations we are interested in will show up. Let us start by considering integers k, k', t, t' such that $k, k' \geq 0$ and $k + 2t = k' + 2t'$. Write λ for the quadruple (k, k', t, t'). Fix an open compact subgroup $U \subset G(\mathbb{A}_f)$ and consider $Y_G(U)$ and $Y_{G^*}(U \cap G^*(\mathbb{A}_f))$. One considers, with notation as above, the sheaf $\tilde{\mathscr{H}}_L^{[\lambda]}$ over $Y_{G^*}(U \cap G^*(\mathbb{A}_f))$ defined by

$$\left[\mathrm{TSym}^k\left(\mathscr{H}_L(\mathscr{A})^{(1)}\right) \otimes \det\left(\mathscr{H}_L(\mathscr{A})^{(1)}\right)^t\right] \otimes \left[\mathrm{TSym}^{k'}\left(\mathscr{H}_L(\mathscr{A})^{(2)}\right) \otimes \det\left(\mathscr{H}_L(\mathscr{A})^{(2)}\right)^{t'}\right]. \tag{2.3.4}$$

Let, as in 2.2.2.3,

$$\mathscr{G} = G(\mathbb{Q})^{+} G^{*}(\mathbb{A}_f).$$

One defines a relative Chow motive $\mathscr{H}_L^{[\lambda]}$ over $Y_G(U)$, using the map $Y_{G^*}(U \cap G^*(\mathbb{A}_f)) \to Y_G(U)$, whose fibres are given (see [LLZ18, Proposition 2.2.5]) by the orbits of the finite group

$$\Delta(U) = \frac{\mathscr{G} \cap U}{(U \cap G^*(\mathbb{A}_f)) \cdot (Z(\mathscr{G}) \cap U)}$$

In particular one can write $Y_G(U)$ as disjoint union over a finite set of elements $g \in G(\mathbb{A}_f)$, whose determinant are representatives of the finite set $\mathbb{A}_{F,f}^{\times}/(F^{\times})^{+}\mathbb{A}_f^{\times} \det(U)$, of the following varieties

$$Y_{G^*}(gUg^{-1} \cap G^*(\mathbb{A}_f))/\Delta(gUg^{-1}) = \mathrm{Im}(Y_{G^*}(gUg^{-1} \cap G^*(\mathbb{A}_f)) \xrightarrow{q_g} Y_G(gUg^{-1})).$$

The authors construct in [LLZ18, § 3.2c] a relative Chow motive on each of these components considering the pushforward of (2.3.4) under the projection map q_g; they then take its the image under a projector with respect to the action of $\Delta(gUg^{-1})$ (see [LLZ18, Proposition 3.2.8, Definition 3.2.9]. One then gets a relative Chow motive $\mathscr{H}_L^{[\lambda]}$ over $Y_G(U)$, independent (up to a canonical isomorphism) of the choice of the representatives defining the components of $Y_G(U)$.

Remark 2.3.7. One has similarly the dual sheaf $\mathscr{H}_L^{(\lambda)}$ over Y_G. The ℓ-adic étale realisation of $\mathscr{H}_L^{(\lambda)}$ is the lisse $\bar{\mathbb{Q}}_\ell$-sheaf associated to the representation of $\mathrm{GL}_2^{\mathrm{Hom}(F,\mathbb{R})}$ given by

$$\left(\mathrm{Sym}^k(\mathrm{Std}^\vee) \otimes \det^t(\mathrm{Std}^\vee)\right) \otimes \left(\mathrm{Sym}^{k'}(\mathrm{Std}^\vee) \otimes \det^{t'}(\mathrm{Std}^\vee)\right),$$

where $\mathrm{Std}^\vee$ is the dual of the standard two-dimensional representation of GL_2. This is the sheaf $\mathscr{L}_{\xi,\ell}$ considered in [Nek07, §5.5].

2.3.2.3 Infinite level sheaves

The sheaves constructed above on finite level modular curves and Shimura varieties for G^* and G give rise to a $H(\mathbb{A}_f)$-equivariant (respectively $\mathscr{G}$-equivariant and $G(\mathbb{A}_f)$-equivariant) relative Chow motive over the infinite level varieties Y_H, Y_{G^*}, Y_G, in the sense of [LSZ20a, § 6.2].

More precisely, recall that we showed in 2.2.2.3 that the abelian varieties $\mathscr{A}/Y_{G^*}(U^*)$ for varying $U^* \subset G^*(\mathbb{A}_F)$ have natural $\mathscr{G}$-equivariant structure up to isogenies. It induces a $\mathscr{G}$-equivariant structure on the relative Chow motive $\mathscr{H}_L(\mathscr{A})$ via pullback and hence also on $\mathrm{TSym}^{[k,k']}\mathscr{H}_L(\mathscr{A})$. In other words, the functor of Theorem 2.3.3, gives rise to a functor

$$\mathscr{F}_{G^*} : \mathrm{Rep}_{\mathbb{Q}}(G^*) \to \mathrm{CHM}(Y_{G^*})^{\mathscr{G}},$$

where the target is the category of $\mathscr{G}$-equivariant relative Chow motives on the pro-variety Y_{G^*}.

The case of modular curves is completely analogous, since $H(\mathbb{A}_f)$ acts by isogenies as in 2.2.2.3 on the universal elliptic curve $\mathscr{E}/Y_H(U)$ for varying $U \subset H(\mathbb{A}_f)$. So we obtain a functor

$$\mathscr{F}_H : \mathrm{Rep}_{\mathbb{Q}}(H) \to \mathrm{CHM}(Y_H)^{H(\mathbb{A}_f)}.$$

Finally, the action of $G(\mathbb{A}_f)$ on the relative Chow motive $\mathscr{H}_L^{[\lambda]}$ comes from the construction given above and the $\mathscr{G}$-action defined in 2.2.2.3. Indeed if we take $g \in G(\mathbb{A}_f)$, a sufficiently small subgroup U and consider the natural map

$$Y_G(U) \to Y_G(gUg^{-1}),$$

the relative Chow motives constructed on those surfaces are built from relative Chow motives over varieties of the form $Y_{G^*}(g_iUg_i^{-1} \cap G^*)$ and $Y_{G^*}(g_igUg^{-1}g_i^{-1} \cap G^*)$ respectively, where the finite set of g_i can be chosen to be the same (since $\det(U) = \det(gUg^{-1})$). Since whenever the determinant of two matrices is in the same class in the finite quotient $\mathbb{A}_{F,f}^{\times}/(F^{\times})^{+}\mathbb{A}_f^{\times}\det(U)$ we can assume they differ by an element in $\mathscr{G}$, for every i we can construct maps

$$Y_{G^*}(g_iUg_i^{-1}\cap G^*)\to Y_{G^*}(g_jgUg^{-1}g_j^{-1}\cap G^*)$$

for some j, which are given by multiplication by an element in $\mathscr{G}$ and hence induce a map between the corresponding relative Chow motives as in 2.2.2.3.

From now on, we will write $\mathrm{TSym}^k\,\mathscr{H}_L(\mathscr{E}),\mathrm{TSym}^{[k,k']}\,\mathscr{H}_L(\mathscr{A}),\mathscr{H}_L^{[\lambda]}$ to denote by abuse of notation both the *finite level* Chow motives and the *infinite level* ones. No confusion will arise, since it will be clear from the context which is the object we are working with. Moreover, when we write motivic cohomology of the pro-varieties Y_H,Y_{G^*},Y_G with coefficients in *infinite level* relative Chow motives, we mean the limit of the cohomology group of the finite level Shimura varieties with coefficients in the corresponding *finite level* Chow motives.

Finally, we remark that the Hecke operators we define later (as double cosets in § 3.2 or as locally constant compactly supported functions on the adelic points of the group in Chapter 4) act naturally on the universal abelian varieties we considered (similarly as in 2.2.2.3, see also [LLZ18, Remark 2.6.1]) and hence on the relative Chow motives we constructed. In particular they act on the motivic cohomology groups with coefficients in such relative Chow motives.

2.3.2.4 Clebsch–Gordan map

Write $Y=Y_H(U^*\cap\mathrm{GL}_2)$ and $Y^*=Y_{G^*}(U^*)$ for U^* a sufficiently small subgroup of $G^*(\mathbb{A}_f)$. Write $\mathscr{E},\mathscr{A}$ for the elliptic curve over Y and the abelian surface over Y^* respectively. We have a closed embedding

$$\iota:Y\hookrightarrow Y^*.$$

One has that the abelian variety $\iota^*(\mathscr{A})$ is canonically isomorphic to $\mathscr{O}_F\otimes_{\mathbb{Z}}\mathscr{E}$, compatibly with the $\mathscr{O}_F$ action. In particular both $\iota^*(\mathscr{H}_L(\mathscr{A})^{(1)})$ and $\iota^*(\mathscr{H}_L(\mathscr{A})^{(2)})$ can be identified with $\mathscr{H}_L(\mathscr{E})$. Hence we obtain two maps

$$\mathrm{TSym}^{k+k'}\mathscr{H}_L(\mathscr{E}) \to \mathrm{TSym}^{k}\mathscr{H}_L(\mathscr{E}) \otimes \mathrm{TSym}^{k'}\mathscr{H}_L(\mathscr{E}) = \iota^*\left(\mathrm{TSym}^{[k,k']}\mathscr{H}_L(\mathscr{A})\right),$$

$$L(1) = \bigwedge_L^2 \mathscr{H}_L(\mathscr{E}) \to \mathscr{H}_L(\mathscr{E}) \otimes \mathscr{H}_L(\mathscr{E}) = \iota^*\left(\mathrm{TSym}^{[1,1]}\mathscr{H}_L(\mathscr{A})\right).$$

Combining these two maps using multiplication in the symmetric tensor algebra, we find

Proposition 2.3.8. *([LLZ18, Proposition 3.3.1]). For any integers k, k', j satisfying $0 \leq j \leq \min(k, k')$, we have a morphism*

$$CG_{\mathrm{mot}}^{[k,k',j]} : \mathrm{TSym}^{k+k'-2j}\mathscr{H}_L(\mathscr{E}) \to \iota^*(\mathrm{TSym}^{[k,k']}\mathscr{H}_L(\mathscr{A}))(-j).$$

This is analogous to the map defined in [KLZ15] (see Corollary 5.2.2) for the $\mathrm{GL}_2 \times \mathrm{GL}_2$ case. It is an instance of the maps produced using Theorem 2.3.4.

Moreover, consider $Y_G(U)$ for $U \subset G(\mathbb{A}_f)$ such that $U \cap G^*(\mathbb{A}_f) = U^*$, one can use the fact that by construction the pullback to Y^* of the sheaf $\mathscr{H}_L^{[\lambda]}$ over $Y_G(U)$ is $\mathrm{TSym}^{[k,k']}\mathscr{H}_L(\mathscr{A})(t+t')$ to find

$$CG_{\mathrm{mot}}^{[k,k',j]} : \mathrm{TSym}^{k+k'-2j}\mathscr{H}_L(\mathscr{E}) \to \iota_G^*(\mathscr{H}_L^{[\lambda]}(-j-t-t')), \tag{2.3.5}$$

where ι_G denotes the natural embedding $Y \hookrightarrow Y_G(U)$.

2.3.3 Chow groups

In this section, we briefly mention the relation between motivic cohomology with trivial coefficients and Chow groups. Bloch defined the higher Chow groups $\mathrm{CH}^i(X, j)$ in [Blo86]. Without going into the definition, we cite the theorem which shows the relationship with motivic cohomology.

Theorem 2.3.9. *For any X smooth variety over a field $k \subset \mathbb{C}$ and $p, q \geq 0$, there is a natural isomorphism*

$$H^p_{\mathrm{mot}}(X, \boldsymbol{Q}(q)) \cong \mathrm{CH}^q(X, 2q-p) \otimes \mathbb{Q}.$$

Proof. See [Voe02, Corollary 2]. □

Example 2.3.10. The theorem implies for example that

$$H^1_{\text{mot}}(X, \mathbf{Q}(1)) \cong \mathrm{CH}^1(X, 1) \otimes \mathbb{Q} = \mathscr{O}(X)^\times \otimes \mathbb{Q}.$$

where for the last identification we refer to [MVW06, Corollary 4.2]. In particular, if X is the modular curve $Y(N)$, the Siegel units of Definition 2.1.11 define elements in $H^1_{\text{mot}}(Y(N), \mathbf{Q}(1))$. There is a generalisation to *non-trivial coefficients*, namely there are elements called Eisenstein classes, which will be recalled in § 4.6, defined in $H^1_{\text{mot}}(Y(N), \mathrm{TSym}^k \mathscr{H}_{\mathbf{Q}}(\mathscr{E})(1))$.

Using Quillen K-theory and the Gersten complex (see [Qui73]), one gets the following useful theorem. For more details see [LLZ14, Corollary 2.5.7, Proposition 2.5.8].

Theorem 2.3.11. *If X is a smooth variety of finite type over a field k, then $\mathrm{CH}^2(X, 1) \cong Z^2(X, 1)/T$, where $Z^2(X, 1)$ is the kernel of a boundary map in the named Gersten complex and T is some subgroup. More explicitly*

$$Z^2(X,1) = \left\{ \sum_i (C_i, \phi_i) : C_i \textit{ subvariety of codimension 1}, \phi_i \in k(C_i)^\times \textit{ s.t. } \sum_i \mathrm{div}(\phi_i) = 0 \right\}.$$

Remark 2.3.12. We stated the previous theorem very vaguely, avoiding precise definitions. We justify this saying that we will only have to deal with $Z^2(X, 1)$ and, moreover, this is where the Euler systems classes of [LLZ14, LLZ18] are defined. The compatibilities properties already hold in that group, with no need to get to the quotient.

2.4 Hilbert modular forms and the Asai L-function

In this section, we recall the simplest definition of Hilbert modular forms and discuss some Galois representations and L-functions attached to them.

2.4.1 Hilbert modular forms

With the same notation as in the previous sections, we now want to define Hilbert modular forms for F. We first of all fix δ^{-1} a totally positive generator of the fractional ideal $\mathfrak{d}^{-1}$, which is principal in the case of quadratic fields. We will consider subgroups $\Gamma \subset \mathrm{GL}_2(\mathscr{O}_F)$ of the form $U(1,\mathfrak{N}) \cap G(\mathbb{Q})^+$. Moreover for $\lambda \in F$ and $\underline{r} = (r_1, r_2) \in \mathbb{Z}^2$, we will write $\lambda^{\underline{r}} = \lambda_1^{r_1}\lambda_2^{r_2}$ and extend this to $\mathscr{H}_F$.

Definition 2.4.1. A Hilbert modular form of level $\mathfrak{N}$ and weight $\underline{r} = (r_1, r_2) \in \mathbb{Z}^2$, with $r_1 + 2t_1 = r_2 + 2t_2$ is a function $f : \mathscr{H}_F \times G(\mathbb{A}_f) \to \mathbb{C}$ such that

(i) for every $g \in G(\mathbb{A}_f)$, $f(g,-)$ is holomorphic on $\mathscr{H}_F = \mathscr{H} \times \mathscr{H}$;

(ii) for every $u \in U(1,\mathfrak{N}), g \in G(\mathbb{A}_f), \underline{\tau} \in \mathscr{H}_F$, $f(gu,\underline{\tau}) = f(g,\underline{\tau})$;

(iii) for every $\gamma^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{Q})^+$ and for every $\underline{\tau} = (\tau,\tau') \in \mathscr{H}_F$, $f(\gamma g,\underline{\tau}) = f(g,\underline{\tau})_{|_{(\underline{r},\underline{t})}}\gamma^{-1}$, where

$$f(g,\underline{\tau})_{|_{(\underline{r},\underline{t})}}\gamma^{-1} = (\det\gamma)^{\underline{t}}\mathrm{Norm}_{F/\mathbb{Q}}(\det\gamma)^{-1}(c\underline{\tau}+d)^{-\underline{r}}f(g,\gamma^{-1}\underline{\tau})$$

Remark 2.4.2 (Fourier expansion)**.** If f is a Hilbert cusp form, then it has a Fourier-Whittaker expansion of the form

$$f\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right),\underline{\tau}\right) = \| x \|_{\mathbb{A}_{F,f}} \sum_{\alpha \in F^{\times,+}} \alpha^{-\underline{t}} c(\alpha x, f) e^{2\pi i(\alpha_1\tau + \alpha_2\tau')},$$

where $c(-,f)$ is a locally constant $\mathbb{C}$-valued function on $\mathbb{A}_{F,f}^{\times}$ and $c(x,f)$ depends only on the fractional ideal generated by x and is zero unless it is contained in δ^{-1}.

2.4.1.1 Hecke operators and Hilbert eigenforms

On the space of Hilbert modular forms of level $U(1,\mathfrak{N})$ one has Hecke operators $T(\mathfrak{n})$ for every integral ideal of $\mathscr{O}_F$ coprime with $\mathfrak{N}$. The definition is analogous to the one for classical modular forms. In the next chapters we will give the definitions of Hecke operators: one can see them as double cosets as in § 3.2 or as locally constant compactly supported functions on $G(\mathbb{A}_f)$ as in Chapter 4. We then can give the following definition.

Definition 2.4.3. A cuspidal Hilbert modular form is an eigenform if it is an eigenvector for every Hecke operator $T(\mathfrak{n})$. We say that it is normalised if $c(\delta^{-1}, f) = \delta^{-(t_1+t_2)/2}$.

One has that, applying [Shi78, 2.20 et seq.], the Hecke operators map the set of Hilbert modular forms with algebraic Fourier coefficients to itself and this implies that the eigenvalues are algebraic numbers. For more details see [Shi78, Proposition 2.2].

Remark 2.4.4 (Automorphic representations)**.** It is a standard result (explained for example in [Kud03] and [vdG88, I.7]) that there exists a correspondence between holomorphic Hilbert modular forms and the associated automorphic functions, which are slowly decreasing functions on the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ satisfying certain conditions. The space of such (cuspidal) functions decomposes as direct sum of irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ modules, where $\mathfrak{g}$ is the Lie algebra of G and K_∞ is the maximal compact subgroup. Such modules are the irreducible cuspidal automorphic representations of G. They can be written as

$$\pi = \pi_\infty \otimes (\otimes' \pi_\ell).$$

Fixing the $(\mathfrak{g}, K_\infty)$-module π_∞ to be a discrete series of weight (k, k'), these representations appear in the middle degree parabolic étale cohomology of Y_G with coefficients in the étale realisation of the sheaves defined above (see Remark 2.3.7, Theorem 4.8.6). One associates to each of these representations π a Hilbert modular eigenform of weight (k, k'). The action of the Hecke algebra on the $G(\mathbb{A}_f)$-representation, as described in Chapter 4, corresponds to the action of Hecke operators on the eigenform. Moreover, for almost every ℓ, the $G(\mathbb{Q}_\ell)$-representation π_ℓ is spherical (see § 4.2.4) and we study such representations in more details in §4.3-§4.4.

2.4.2 Asai L-function

As in the case of classical modular forms, one can attach to a Hilbert eigenform f of level $\mathfrak{N}$, nebentype ε and weight $(k+2, k'+2)$, with $k, k' \geq 0$, a Galois representation. More precisely, let L be the number field generated by the Hecke eigenvalues $\lambda_{\mathfrak{m}}$ of f with respect to $T(\mathfrak{m})$. Let w an integer such that $w \equiv k \equiv k' \mod 2$.

Theorem 2.4.5 (Blasius, Rogawski, Taylor)**.** *For every finite place v of L, one has a Galois representation*

$$\rho_{f,v} : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(L_v)$$

such that for all primes $\mathfrak{p} \nmid \mathfrak{N} Nm_{L/\mathbb{Q}}(v)$, the representation $\rho_{f,v}$ is unramified at $\mathfrak{p}$ and

$$\det\left(1 - X\rho_{f,v}\left(\mathrm{Frob}_{\mathfrak{p}}^{-1}\right)\right) = 1 - \lambda_{\mathfrak{p}} X + \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{p})^{w-1}\varepsilon(\mathfrak{p})X^2.$$

One can then consider the *classical L-function attached to f*, i.e. the one attached to the system of Galois representations $(\rho_{f,v})_v$. In [FLHS15], it is proved that all elliptic curves over real quadratic fields are modular and hence the L-function of one of those elliptic curve is equal to the L-function of f for some Hilbert eigenform f of parallel weight 2.

The motivic Asai–Flach classes we work with are not related to this representation, but rather to a representation of $G_{\mathbb{Q}}$ obtained from $\rho_{f,v}$. This representation appears in the parabolic étale cohomology of the Hilbert modular variety Y_G, as we will recall in § 4.8.2 (see Theorem 4.8.6).

Definition 2.4.6. Writing as before $k + 2t = k' + 2t'$ one defines the *Asai Galois representation attached to f* by

$$\rho^{\mathrm{As}}{}_{f,v} := \left(\bigotimes - \mathrm{Ind}\right)_F^{\mathbb{Q}} (\rho_{f,v}) \otimes L_v(t+t') : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(L_v),$$

where $\bigotimes - \mathrm{Ind}$ denotes the tensor induction[3]. It is called the Asai Galois representation attached to f because it was first considered by Asai in [Asa77].

Remark 2.4.7. The tensor induction of a representation W from a subgroup $H < G$ of index n to G is given considering $W^{\otimes n}$ with the action of $H^n \rtimes S_n$ on it and viewing G inside $H^n \rtimes S_n$ via the Frobenius embedding. Fixing cosets representatives

[3] See for example [Pac05, Definition 1.1].

$\{Hg_1,\dots,Hg_n\}$ we call π the permutation representation of G on them, obtained by right multiplication. For every $x \in G$, there is then a unique $h(i,x) \in H$ such that $g_i \cdot x = h(i,x) \cdot g_{i \cdot \pi(x)}$. The Frobenius embedding is given by

$$x \mapsto ((h(1,x),\dots,h(n,x)),\pi(x)).$$

In our situation $H = G_F$, $G = G_{\mathbb{Q}}$ and $n = 2$. In the degenerate case where we replace F by $\mathbb{Q} \oplus \mathbb{Q}$ we have $G = H$ and can think about the set of coset representatives as $\{H \cdot 1, H \cdot 1\}$. Hence $h(1,x) = h(2,x) = x$ and $\pi(x) = id$ for every $x \in G$, hence the tensor induction of a representation W is simply given by the representation $W \otimes W$. So if we start with a classical eigenform f and view ${\rho^{\mathrm{As}}}_{f,v}$ as degenerate case as above, we obtain the Rankin-Selberg convolution $\rho_{f,v} \otimes \rho_{f,v}$.

One can then consider the corresponding L-function, called *Asai L-function* and denoted by $\mathrm{L}_{\mathrm{As}}(f,s)$. It is defined as a product of local Euler factors as follows.

Definition 2.4.8. For f as above, we define the local Euler factor for $\ell \neq p$ to be

$$P_\ell^{\mathrm{As}}(f,X) := \det(1 - X\,\mathrm{Frob}_\ell^{-1}\,|(V_f^{\mathrm{As}})^{I_\ell}),$$

where Frob_ℓ is the arithmetic Frobenius at ℓ and I_ℓ is the inertia subgroup at ℓ. The local Euler factor at p is defined by the same polynomial acting on the Galois representation ${\rho^{\mathrm{As}}}_{f,w}$ for some auxiliary w such that $p \nmid w$.

Then the *Asai L-function* is defined by

$$\mathrm{L}_{\mathrm{As}}(f,s) := \prod_\ell P_\ell^{\mathrm{As}}(f,\ell^{-s})$$

This product converges for $\mathrm{Re}(s) > \frac{k+k'}{2}$ and it admits an analytic continuation to the whole complex plane. It also satisfies a functional equation relating the value at s with the value at $k+k'-1-s$. The Euler factors at good primes are explicitly characterised as follows

Proposition 2.4.9. *([Asa77, Theorem 2], [LLZ18, Proposition 4.3.4]) If f is of level*

$\mathfrak{N}$ *and* $\ell \nmid \mathrm{Nm}(\mathfrak{N})p$ *then the polynomial* $P_\ell^{\mathrm{As}}(f, \ell^{t+t'}X)$ *is equal to*

$$\begin{cases} (1-\alpha_1\alpha_2X)(1-\alpha_1\beta_2X)(1-\beta_1\alpha_2X)(1-\beta_1\beta_2X), & \text{if } \ell = \mathfrak{l}_1 \cdot \mathfrak{l}_2 \text{ splits in } F \\ (1-\alpha X)(1-\beta X)(1-\alpha\beta X^2) & \text{if } \ell \text{ is inert in } F. \end{cases}$$

where α_i, β_i *and* α, β *are the roots of* $X^2 - a_{\mathfrak{l}_i}(f)X + \ell^{w-1}\varepsilon(\mathfrak{l}_i)$ *and of* $X^2 - a_\ell(f)X + \ell^{2(w-1)}\varepsilon(\ell)$ *respectively, where* $w = k+2+2t = k'+2+2t'$.

Remark 2.4.10. In §4.3-§4.4, the polynomials of the previous proposition will be studied and described in terms of local zeta integrals associated to the $G(\mathbb{Q}_\ell)$-representations π_ℓ of Remark 2.4.4.

Chapter 3

Asai–Flach classes tame norm relations by means of Hecke algebra congruences

In this chapter we define Asai–Flach classes, following [LLZ18], as elements in the motivic cohomology of the Hilbert modular surface. We give the definition only in the case of *trivial coefficients*, namely working with classes defined as pushforward of Siegel units in some higher Chow group of the Hilbert modular surface. We also prove that these classes satisfy tame norm relations at primes which split in F and are narrowly principal. The method is the one used in [LLZ18, Section 7], where it is used to prove tame norm relations for inert primes and is only sketched for the above mentioned primes. We give the details of such method in this case. It relies on very explicit computations in the Hecke algebra acting on the cohomology of the Hilbert modular surface.

3.1 Definition of Asai–Flach classes (trivial coefficients case)

Recall that F is a real quadratic field of discriminant D, $\mathcal{O}_F$ its ring of integers, $\mathfrak{d}$ its different. Recall that the embedding $\mathrm{GL}_2 \subset G^*$ defines a closed immersion

$$\iota : Y_1(N) \hookrightarrow Y_1^*(\mathfrak{N}),$$

for $\mathfrak{N}$ integral ideal of F such that $\mathfrak{N}\cap\mathbb{Z}=(N)$. We finally define Asai–Flach classes for $Y_1^*(\mathfrak{N})$. We have the pushforward map defined in (2.3.3),

$$\iota_*: H^1_{\mathrm{mot}}(Y_1(N),\mathbf{Q}(1)) = \mathscr{O}(Y_1(N))^\times\otimes\mathbb{Q} \longrightarrow H^3_{\mathrm{mot}}(Y_1^*(\mathfrak{N}),\mathbf{Q}(2)) = \mathrm{CH}^2(Y_1^*(\mathfrak{N}),1)\otimes\mathbb{Q}$$
$$g\longmapsto (\iota(Y_1(N)),\iota_*(g)),$$

where we applied Theorem 2.3.9 and used the identifications given in Example 2.3.10 for the modular curve $Y_1(N)$ and Theorem 2.3.11 for the surface $Y_1^*(\mathfrak{N})$.

Definition 3.1.1 (Asai–Flach classes for $M=1$)**.** We define the Asai–Flach class $\mathrm{AF}_{1,\mathfrak{N}}$ to be the image of the Siegel unit $g_{0,1/N}$ under ι_*, i.e.

$$\mathrm{AF}_{1,\mathfrak{N}} = (\iota(Y_1(N)),\iota_*(g_{0,1/N}))\in\mathrm{CH}^2(Y^*(\mathfrak{N}),1).$$

Remark 3.1.2 (Beilinson–Flach classes)**.** As we already said many times, Beilinson–Flach classes (for $M=1$) are obtained precisely in the same way, but using the closed embedding

$$\Delta: Y_1(N)\to Y_1(N)^2,$$

and getting an element in $\mathrm{CH}^2(Y_1(N)^2,1)\otimes\mathbb{Q}$.

One then defines more general Asai–Flach classes on the base extension of $Y_1^*(\mathfrak{N})$, which is a smooth surface over $\mathbb{Q}$, to cyclotomic fields, using Remark 2.1.6. Consider $M\geq 1$ integer. Via pullback under the natural map

$$h: Y^*(M,M\mathfrak{N})\to Y_1^*(M\mathfrak{N})$$

coming from the inclusion $U^*(M,M\mathfrak{N})\subset U^*(1,M\mathfrak{N})$, one can see the Asai–Flach class $\mathrm{AF}_{1,M\mathfrak{N}}$ as an element in $\mathrm{CH}^2(Y^*(M,M\mathfrak{N}),1)\otimes\mathbb{Q}$. Recall from definition 2.2.8 that for $a\in\mathscr{O}_F$ we have an endomorphism $\mathbf{u}_a$ of $Y^*(M,M\mathfrak{N})$ and from definition 2.2.19 a map $\mathbf{t}_M: Y^*(M,M\mathfrak{N})\to Y_1^*(\mathfrak{N})\times\mu_M$, where, for the ease of notation, we denote by μ_m the group scheme of primitive m-th roots of unity.

Definition 3.1.3 (translated Asai–Flach classes)**.** Let M be an integer such that $\mathfrak{N}$ is divisible by M. Let $a \in \mathscr{O}_F/(M\mathscr{O}_F + \mathbb{Z})$, we then define

$$\widetilde{\mathrm{AF}}_{M,\mathfrak{N},a} := (\mathbf{u}_a)_* \mathrm{AF}_{1,M\mathfrak{N}} \in \mathrm{CH}^2(Y^*(M,\mathfrak{N}),1).$$

Definition 3.1.4 (Asai–Flach classes for $M > 1$)**.** We define the Asai–Flach class $\mathrm{AF}_{M,\mathfrak{N},a}$ to be the image of $\mathrm{AF}_{1,M\mathfrak{N}} \in \mathrm{CH}^2(Y^*(M,M\mathfrak{N}),1)\otimes\mathbb{Q}$ under $(\mathbf{t}_M \circ \mathbf{u}_a)_*$, i.e.

$$\mathrm{AF}_{M,\mathfrak{N},a} = (\mathbf{t}_M)_*(\widetilde{\mathrm{AF}}_{M,\mathfrak{N},a}) = (\mathbf{t}_M \circ \mathbf{u}_a)_*(\mathrm{AF}_{1,M\mathfrak{N}}) \in \mathrm{CH}^2(Y_1^*(\mathfrak{N}) \times \mu_M, 1)\otimes\mathbb{Q}.$$

If $a = 0$, we will simply write $\mathrm{AF}_{M,\mathfrak{N}}$ for $\mathrm{AF}_{M,\mathfrak{N},0}$.

Lemma 3.1.5. *Consider the map $\iota_{M,\mathfrak{N},a}$ given by the composition*

$$\iota_{M,\mathfrak{N},a} : Y_1(M^2N)\times\mu_M \xrightarrow{f} Y(M,MN) \xrightarrow{\iota'} Y^*(M,M\mathfrak{N}) \xrightarrow{\mathbf{u}_a} Y^*(M,M\mathfrak{N}) \xrightarrow{\mathbf{t}_M} Y_1^*(\mathfrak{N})\times\mu_M,$$

where f is defined in (2.1.1). We then have $\mathrm{AF}_{M,\mathfrak{N},a} = (\iota_{M,\mathfrak{N},a})_*(g_{0,1/M^2N})$.

Proof. Recall that the map f is the one obtained by the map $Y(M^2N) \to Y(M,MN)$ sending $(E,e_1,e_2) \mapsto (E/\langle Me_2\rangle, [MNe_1], [e_2])$, which factors through the quotient, and hence defines a morphism

$$f : Y_1(M^2N)\times\mu_M \to Y(M,MN).$$

Since the following diagram commutes

$$\begin{array}{ccc} Y(M,MN) & \longrightarrow & Y_1(MN) \\ \iota' \big\downarrow & & \big\downarrow \iota \\ Y^*(M,M\mathfrak{N}) & \longrightarrow & Y_1^*(M\mathfrak{N}), \end{array}$$

where the horizontal arrow are the natural projection maps, the result follows from the definition of $\mathrm{AF}_{1,M\mathfrak{N}}$ and the equality $g_{0,1/MN} = f_* g_{0,1/M^2N}$, proved in Lemma 2.1.17. □

Remark 3.1.6. Notice that, using Remark 2.2.20, the description of f and $\mathbf{u}_a$, we get that the map $\iota_{M,\mathfrak{N},a}$ is given on complex points by

$$\tau \mapsto M\tau \mapsto (M\tau, M\tau) \mapsto (M\tau + a_1, M\tau + a_2) \mapsto \left(\tau + \frac{a_1}{M}, \tau + \frac{a_2}{M}\right).$$

Remark 3.1.7. Thanks to the previous lemma, we can also show that the element $\mathrm{AF}_{M,\mathfrak{N},a}$ depends only on the class of a in $\mathscr{O}_F/(M\mathscr{O}_F + \mathbb{Z})$. We want to show that $AF_{M,\mathfrak{N},a}$ is equal to (the pullback via $Y_1^*(\mathfrak{N}) \times \mu_M \to Y_1^*(\mathfrak{N})$ of) $AF_{1,\mathfrak{N}}$ if $a \in M\mathscr{O}_F + \mathbb{Z}$. Indeed if $a \in M\mathscr{O}_F$, we have that $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \in U^*(M, M\mathfrak{N})$ and hence $\mathbf{u}_a = \mathbf{u}_0 = id_{Y^*(M,M\mathfrak{N})}$. Moreover if $a \in \mathbb{Z}$, then we can consider the automorphism $u_a : Y(M,MN) \to Y(M,MN)$ given by the action of the matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and get the following commutative diagram

$$\begin{array}{ccc} Y(M,MN) & \xrightarrow{u_a} & Y(M,MN) \\ {\scriptstyle \iota'}\downarrow & & \downarrow{\scriptstyle \iota'} \\ Y^*(M,M\mathfrak{N}) & \xrightarrow{\mathbf{u}_a} & Y^*(M,M\mathfrak{N}). \end{array}$$

We know that $(u_a)_* = (u_{-a})^*$ and that $(u_{-a})^*(g_{0,1/MN}) = g_{0,1/MN}$, thanks to Proposition 2.1.14 (i). Hence we find that, if $a \in \mathbb{Z}$,

$$(\iota_{M,\mathfrak{N},a})_*(g_{0,1/M^2N}) = (\mathbf{t}_M)_*(\iota')_*((u_a)_*(g_{0,1/MN})) = (\iota_{M,\mathfrak{N},0})_*(g_{0,1/M^2N}).$$

Hence we want to show that $(\iota_{M,\mathfrak{N},0})_*(g_{0,1/M^2N}) = \mathrm{AF}_{1,\mathfrak{N}}$, denoting by $\mathrm{AF}_{1,\mathfrak{N}}$, by abuse of notation, the pullback of $\mathrm{AF}_{1,\mathfrak{N}}$ from $Y_1^*(\mathfrak{N})$ to $Y_1^*(\mathfrak{N}) \times \mu_M$. We can write $\iota_{M,n,0} = (\iota \times id) \circ t_M \circ f$, where $\iota : Y_1(N) \hookrightarrow Y_1^*(\mathfrak{N})$. Now, the composition $t_M \circ f$ is equal to the projection $\pi : Y_1(M^2N) \times \mu_m \to Y_1(N) \times \mu_m$. Viewing both the target and the source as quotients of $Y(M^2N)$ as in Proposition 2.1.5, we find the explicit description of the pushforward via π, i.e.

$$\pi_*(g_{0,1/M^2N}) = \prod_{0 \le i,j \le M^2-1} \left(\begin{smallmatrix} 1 & 0 \\ iN & 1+jN \end{smallmatrix}\right)^*(g_{0,1/M^2N}) = \prod_{0 \le i,j \le M^2-1} g_{i/M^2, 1/M^2N + j/M^2} = g_{0,1/N},$$

where in the second equality we used Proposition 2.1.14 (i) and in the last one

Lemma 2.1.15, for $m = M^2$. Moreover $g_{0,1/N}$ denotes the pullback of $g_{0,1/N}$ from $Y_1(N)$ to $Y_1(N) \times \mu_M$. Therefore we proved the desired equality.

We conclude this section proving a useful property of the Asai–Flach classes.

Proposition 3.1.8. *For $b \in (\mathbb{Z}/M\mathbb{Z})^\times$, denote by σ_b the automorphism of μ_M given by $\zeta \mapsto \zeta^b$. We then have*

$$\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}^* \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a} = \widetilde{\mathrm{AF}}_{M,\mathfrak{N},b^{-1}a} \quad \textit{and} \quad \sigma_b \cdot \mathrm{AF}_{M,\mathfrak{N},a} = \mathrm{AF}_{M,\mathfrak{N},b^{-1}a}.$$

Proof. We will prove only the second equality, the first one follows similarly, thanks to Proposition 2.2.21. Using the description given in the previous lemma, we need to check that (i) $\sigma_b^*(\iota_{M,\mathfrak{N},a})_*(g_{0,1/M^2N}) = (\iota_{M,\mathfrak{N},b^{-1}a})_*(g_{0,1/M^2N})$ and that (ii) $\sigma_b^{-1}(\iota_{M,\mathfrak{N},a})(Y) = (\iota_{M,\mathfrak{N},b^{-1}a})(Y)$, where we wrote $Y = Y_1(M^2N) \times \mu_M$. First of all, using the properties of the pushforward and the fact that σ_b is an automorphism with inverse $\sigma_{b^{-1}}$, we get that $\sigma_b^* = (\sigma_{b^{-1}})_*$. Using Proposition 2.2.21 and Proposition 2.1.14 (i) (together with the computation $(0, 1/M^2N) \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix} = (0, 1/M^2N)$) one gets

$$\begin{aligned} \sigma_b^*(\mathbf{t}_M)_*(\mathbf{u}_a \circ \iota \circ f)_*(g_{0,1/M^2N}) &= \mathbf{t}_M \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}_* \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}_* (\iota \circ f)_*(g_{0,1/M^2N}) \\ &= \mathbf{t}_M \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}_* \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}_* \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}^* (\iota \circ f)_*(g_{0,1/M^2N}) \\ &= \mathbf{t}_M \circ \mathbf{u}_{b^{-1}a} \circ \iota \circ f(g_{0,1/M^2N}) = \mathrm{AF}_{M,\mathfrak{N},b^{-1}a}, \end{aligned}$$

where in the third equality we used the relation

$$\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b^{-1}a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}^{-1}. \tag{3.1.1}$$

To prove (ii) we proceed similarly. We have that the elements in $(\mathbf{u}_a \circ \iota \circ f)(Y)$ are in the form $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)\cdot(u,u)$, for $u\in f(Y)$. Hence, using again (3.1.1), we can write

$$\begin{aligned}\sigma_b^{-1}(\iota_{M,\mathfrak{N},a})(Y) &= \mathbf{t}_M\left(\left(\begin{smallmatrix}b^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)\{(u,u):u\in f(Y)\}\right)\\ &= \mathbf{t}_M\left(\left(\begin{smallmatrix}1 & b^{-1}a\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}b^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)\{(u,u):u\in f(Y)\}\right)\\ &= \mathbf{t}_M\left(\left(\begin{smallmatrix}1 & b^{-1}a\\ 0 & 1\end{smallmatrix}\right)\{(u,u):u\in f(Y)\}\right)\\ &= (\mathbf{t}_M\circ\mathbf{u}_{b^{-1}a}\circ\iota\circ f)(Y),\end{aligned}$$

where in the last equality we used the fact that the image of $\{(u,u):u\in f(Y)\}$ via $\left(\begin{smallmatrix}b^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)$ is again $\{(u,u):u\in f(Y)\}$: indeed, clearly the matrix sends the image of the diagonal embedding to itself; one gets the other inclusion using the inverse matrix. Hence we get (ii). $\square$

3.2 Hecke operators as double cosets

We let $\mathscr{G}$ denote the subgroup $G(\mathbb{Q})^+G^*\left(\mathbb{A}_f\right)\subseteq G\left(\mathbb{A}_f\right)$ as in 2.2.2.3. Then there are bijections $Y^*\left(U^*\right)=G(\mathbb{Q})^+\backslash\left[\mathscr{G}\times\mathscr{H}_F\right]/U^*$ for each U^*, and we obtain maps of $\mathbb{Q}$-varieties for any $g\in\mathscr{G}$

$$Y^*\left(U^*\right)\to Y^*\left(gU^*g^{-1}\right).$$

We define the Hecke algebra on $Y^*(U^*)$ as the $\mathbb{Z}$-algebra generated by double cosets U^*gU^*, for all elements $g\in\mathscr{G}$. One recovers the action given by one such operator on the cohomology of the corresponding Hilbert modular surface as follows. Let $H,K\subset G^*(\mathbb{A}_f)$ level subgroups and consider the double coset HgK. It defines the following maps

$$\begin{tikzcd} Y^*(g^{-1}Hg\cap K) \arrow[r, "\cdot g"] \arrow[d, "p_1"] & Y^*(H\cap gKg^{-1}) \arrow[d, "p_2"] \\ Y(K) & Y^*(H) \end{tikzcd}$$

We choose to look at the operator HgK acting on cohomology via $(p_2)_*\circ(g)_*\circ p_1^*$.

We can rewrite the double coset as disjoint union of right cosets

$$HgK = \bigsqcup_i H \cdot g\alpha_i,$$

where the α_i's are coset representatives of the quotient $(g^{-1}Hg \cap K)\setminus K$. In the case where $H = K = U^*(\mathfrak{M},\mathfrak{N})$ where $\mathfrak{M} \mid \mathfrak{N}$ we recall the definition of the standard Hecke operators

- The diamond operators: for $x \in (\mathscr{O}_F/\mathfrak{N})^\times$, consider any lift of x in $\hat{\mathscr{O}}_F^\times$ and let $\langle x \rangle$ be the double coset of $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$[1];
- The Frobenius maps: for $x \in (\mathbb{Z}/\mathbb{Z} \cap \mathfrak{M})^\times$, consider any lift of x in $\hat{\mathbb{Z}}^\times$ and let σ_x be the double coset of $\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}$;
- The operator $R'(x)$: for $x \in F^\times$, we write $R'(x)$ for the double coset of the scalar matrix $\begin{pmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}$.
- The operators $T'(x), U'(x)$: for $x \in \mathscr{O}_F$ which is totally positive and square-free, we define $T'(x), U'(x)$ as the double coset of $\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ in the case where x is coprime to $\mathfrak{N}$, respectively divides $\mathfrak{N}$.

We denote by $T(x), U(x), R(x)$ the operators given by the same double cosets of $T'(x), U'(x), R'(x)$, but acting on cohomology via $(p_1)_* \circ (g)^* \circ p_2^*$.

Remark 3.2.1. These Hecke operators act on the universal abelian variety over the Hilbert modular surface as explained in [LLZ18, Remark 2.6.1], using the construction given in 2.2.2.3. In particular the operator $R'(x)$ acts as pushforward of the multiplication by x.

Finally, if $a \in \mathscr{O}_F$, we can consider the inclusion $H' = U^*(\mathfrak{M},\mathfrak{N}a) \subset H = U^*(\mathfrak{M},\mathfrak{N})$. It induces a canonical projection map $pr_{1,a} : Y^*(\mathfrak{M},\mathfrak{N}a) \to Y^*(\mathfrak{M},\mathfrak{N})$. Moreover, we write $pr_{2,a} : Y^*(\mathfrak{M},\mathfrak{N}a) \to Y^*(\mathfrak{M},\mathfrak{N})$ for the map induced by the

[1] Note that with our convention, the action on cohomology of $\langle x \rangle$ is given by the pullback of the map induced by multiplication by $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$

multiplication by $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We can rewrite the pushforward via these maps as double cosets as follows:

$$(pr_{1,a})_* = H \text{ id } H',$$

$$(pr_{2,a})_* = H \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) H'.$$

3.2.1 Some computations in the Hecke algebra

From now on we assume we are in the case where the chosen prime ℓ splits in F and we can write $\ell = \lambda\bar{\lambda}$, where $\lambda, \bar{\lambda}$ are totally positive integers of F, that we identify with the two prime ideals in $\mathscr{O}_F$ above ℓ that they generate.

We now focus on the operators we are interested in. First we rename the subgroups consider above as follows. Consider two integral ideals $M \mid \mathfrak{N}$ coprime to λ and $\bar{\lambda}$, with $M \in \mathbb{Z}$. Let

- $G_0 := U^*(M, \mathfrak{N})$
- $G_\lambda := U^*(M, \mathfrak{N}\lambda) = \{\gamma \in G_0 : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\lambda}\} = \{ \left(\begin{smallmatrix} (a_v) & (b_v) \\ (c_v) & (d_v) \end{smallmatrix}\right) \in G_0 : c_\lambda, d_\lambda - 1 \in \lambda \cdot \mathscr{O}_{F,\lambda}\}$
- $G_\ell := U^*(M, \mathfrak{N}\ell) = \{\gamma \in G_0 : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\ell}\} = \{ \left(\begin{smallmatrix} (a_v) & (b_v) \\ (c_v) & (d_v) \end{smallmatrix}\right) \in G_\lambda : c_{\bar{\lambda}}, d_{\bar{\lambda}} - 1 \in \bar{\lambda} \cdot \mathscr{O}_{F,\bar{\lambda}}\}$

We had to define the "intermediate" subgroup G_λ since we want to decompose the operators we are interested in using the formula $\ell = \lambda\bar{\lambda}$, as we will see in a moment. Firstly we decompose the following operators as disjoint union of right cosets. In order to choose the coset representatives for $U'(\lambda), U'(\bar{\lambda}), T'(\lambda), T'(\bar{\lambda})$, we fix isomorphisms

$$\mathscr{O}_{F,\lambda}/\lambda \cdot \mathscr{O}_{F,\lambda} \simeq \mathbb{Z}/\ell\mathbb{Z}, \;\; \mathscr{O}_{F,\bar{\lambda}}/\bar{\lambda} \cdot \mathscr{O}_{F,\bar{\lambda}} \simeq \mathbb{Z}/\ell\mathbb{Z}.$$

We then have

$$(pr_{1,\bar{\lambda}})_* = G_\lambda \text{ id } G_\ell = G_\lambda \cdot \text{ id}$$

$$(pr_{1,\lambda})_* = G_0 \text{ id } G_\lambda = G_0 \cdot \text{ id}$$

$$(pr_{1,\ell})_* = G_0 \text{ id } G_\ell = (pr_{1,\lambda})_*(pr_{1,\bar{\lambda}})_* = G_0 \cdot \text{ id}$$

$$(pr_{2,\bar{\lambda}})_* = G_\lambda \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} G_\ell = G_\lambda \cdot \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix}$$

$$(pr_{2,\lambda})_* = G_0 \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} G_\lambda = G_0 \cdot \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

$$(pr_{2,\ell})_* = G_0 \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} G_\ell = (pr_{2,\lambda})_*(pr_{2,\bar{\lambda}})_* = G_0 \cdot \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}$$

$$U'(\lambda) = G_\ell \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} G_\ell = \bigsqcup_{i=0}^{\ell-1} G_\ell \cdot \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & iM \\ 0 & 1 \end{pmatrix}$$

$$T'(\lambda) = G_0 \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} G_0 = \bigsqcup_{i=0}^{\ell-1} G_0 \cdot \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & iM \\ 0 & 1 \end{pmatrix} \sqcup G_0 \cdot \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} X,$$

where $X \in G_0$ is given as follows: if v is a prime of F not dividing $\lambda\mathfrak{N}$, $X_v \in \mathrm{GL}_2(\mathscr{O}_{F_v})$ is the identity matrix. Let N be the norm of $\mathfrak{N}$, fix $x \in \mathbb{Z}$ such that $x\ell = 1 + Nk \in \mathbb{Z}$ and $\ell \nmid x$. Let $t = \bar{\lambda}x \in \mathscr{O}_F$. We then take the component of X at all primes dividing $\lambda\mathfrak{N}$ to be $\begin{pmatrix} \lambda t & N \\ N & 1 \end{pmatrix}$. Note that we defined an element in G_0 because $\lambda t - N^2$ is unit at places dividing $\lambda\mathfrak{N}$ and it is equal to $1 + Nk - N^2 \in \mathbb{Z}$.

One has a similar decomposition for $U'(\bar{\lambda})$ and $T'(\bar{\lambda})$.

Lemma 3.2.2. *With the above assumptions and conventions for $\ell, \lambda, \bar{\lambda}$, we have the following equality of double cosets*

$$\begin{aligned}(pr_{1,\ell})_* U'(\ell) = & T'(\ell)(pr_{1,\ell})_* - \langle \lambda^{-1} \rangle T'(\bar{\lambda})(pr_{2,\lambda})_*(pr_{1,\bar{\lambda}})_* \\ & - \langle \bar{\lambda}^{-1} \rangle T'(\lambda)(pr_{1,\lambda})_*(pr_{2,\bar{\lambda}})_* + \langle \ell^{-1} \rangle (pr_{2,\ell})_*.\end{aligned}$$

Proof. We will only prove the following equality

$$T'(\lambda)(pr_{1,\lambda})_* - (pr_{1,\lambda})_* U'(\lambda) = \langle \lambda^{-1} \rangle (pr_{2,\lambda})_*.$$

The analogous equality holds replacing λ with $\bar{\lambda}$ and the proof is exactly the same. Combining such equalities we get the one in the statement of the lemma.

By the above decompositions of $U'(\lambda), T'(\lambda)$, we find

$$T'(\lambda)(pr_{1,\lambda})_* - (pr_{1,\lambda})_* U'(\lambda) = G_0 \cdot \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} X.$$

Moreover, the action on cohomology of pushforward by $pr_{2,\lambda} : Y^*(M, \mathfrak{N}\lambda) \to Y^*(M, \mathfrak{N})$ is equal to the action of the pushforward of the map induced by multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Fix $t = \bar{\lambda} x \in \mathscr{O}_F$ as above such that t is not divisible by λ and let $y \in \hat{\mathscr{O}}_F$ defined by $y_v = \lambda^{-1}$ for all places $v \nmid \lambda\mathfrak{N}$ and $y_v = t$ for all places dividing $\lambda\mathfrak{N}$. Note that y^{-1} is a lift of $\lambda \in (\mathscr{O}_F/\mathfrak{N}\mathscr{O}_F)^\times$ in $\hat{\mathscr{O}}_F^\times$. We hence find

$$\langle \lambda^{-1} \rangle (pr_{2,\lambda})_* = G_0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}.$$

Let $Y = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \in G(\mathbb{A}_f)$, $W = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} X \in G(\mathbb{A}_f)$. We now verify that $WY^{-1} \in G^*(\mathbb{A}_f)$ and that furthermore it is an element of $G_0 = U^*(M, \mathfrak{N})$. Hence $G_0 \cdot Y = G_0 \cdot W$ and this will conclude the proof of the claimed equality. At places v not dividing $\lambda\mathfrak{N}$, we have

$$W_v Y_v^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_v^{-1} & 0 \\ 0 & \lambda y_v \end{pmatrix} = \mathrm{id}.$$

At places $v \mid \lambda\mathfrak{N}$, we find

$$W_v Y_v^{-1} = \begin{pmatrix} t & \lambda^{-1} N \\ N & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} 1 & tN \\ t^{-1} N & \lambda t \end{pmatrix}.$$

Since $\lambda t \equiv 1 \mod N$ and the determinant is a unit both in the ring of integers of F_λ and of F_v for $v \mid \mathfrak{N}$, we have proved $WY^{-1} \in G_0$. □

3.2.2 The Asai Euler factor

We now define the Asai Euler factor as a polynomial with coefficients in the Hecke algebra of level $U^*(\mathfrak{M}, \mathfrak{N})$, with the assumption $\mathfrak{M} \mid \mathfrak{N}$. The reason of the term "Asai Euler factor" is explained by the fact that its action on a Hilbert modular eigenform gives the local factor at ℓ of the Asai L-function attached to it, as in 2.4.8.

Definition 3.2.3. The Asai Euler factor $P_\ell(X)$ at a rational prime $\ell \nmid \mathfrak{N}$ unramified in F is defined as follows:

i) If ℓ is inert, we let

$$P_\ell(X) = (1 - T(\ell)X + \ell^2\langle\ell\rangle R(\ell)X^2)(1 - \ell^2\langle\ell\rangle R(\ell)X^2).$$

ii) If ℓ is split, we let

$$\begin{aligned} P_\ell(X) = & 1 - T(\ell)X + \left(T(\ell)^2 - T(\ell^2) - \ell^2\langle\ell\rangle R(\ell)\right) X^2 - \\ & - \ell^2\langle\ell\rangle R(\ell)T(\ell)X^3 + \ell^4\langle\ell^2\rangle R(\ell)^2 X^4 \end{aligned}$$

We define similarly the Asai Euler factor P'_ℓ for the corresponding polynomial with $\langle x\rangle, T(x), R(x)$ replaced with $\langle x^{-1}\rangle, T'(x), R'(x)$ respectively.

Remark 3.2.4. In the case where ℓ splits and the primes above it are narrowly principal, so that we can write $\ell = \lambda\bar{\lambda}$, with $\lambda \in \mathscr{O}_F^+$, the coefficient of X^2 in $P_\ell(X)$ can be rewritten as

$$\ell\langle\lambda\rangle R(\lambda)T(\bar{\lambda})^2 + \ell\langle\bar{\lambda}\rangle R(\bar{\lambda})T(\lambda)^2 - 2\ell^2\langle\ell\rangle R(\ell),$$

and similarly for $P'_\ell(X)$.

3.3 Tame norm relations for split primes (narrowly principal)

We give the details of the proof of tame norm relations, following the strategy sketched in [LLZ18], in the cases where ℓ splits in F and the two primes above it are narrowly principal.

We will state the compatibility relation under suitable projection maps for the translated Asai–Flach classes $\widetilde{\mathrm{AF}}_{M,\mathfrak{N},a}$ and obtain the norm compatibility relations for $\mathrm{AF}_{M,\mathfrak{N},a}$, since the chosen projections at the level $Y^*(M,\mathfrak{N})$ will translate in norm maps at the level of the surfaces $Y^*(\mathfrak{N}) \times \mu_M$ (see the observation after

[LLZ18, Theorem 7.1.2a]). More precisely, we consider the following morphisms: for any $m \in \mathbb{Z}_{\geq 1}$, we denote by $\hat{pr}_{2,m}$ the degeneracy map given by the action of the matrix $\begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix}$

$$\hat{pr}_{2,m} : Y^*(Mm, \mathfrak{N}) \to Y^*(M, \mathfrak{N});$$

we also consider the canonical projection $pr_{1,\ell}$ coming from the inclusion $U^*(M, \ell\mathfrak{N}) \subset U^*(M, \mathfrak{N})$

$$pr_{1,\ell} : Y^*(M, \ell\mathfrak{N}) \to Y^*(M, \mathfrak{N}).$$

We will prove the following result.

Theorem 3.3.1 (Cyclotomic compatibility for $\widetilde{\mathrm{AF}}$, tame case)**.** *Let $M \geq 1$, $\mathfrak{N}$ an integral ideal of F divisible by M and ℓ a rational prime which does not divide $Nm(\mathfrak{N})$. Let $a \in \mathscr{O}_F/(\ell M \mathscr{O}_F + \mathbb{Z})$ such that it is a unit at ℓ and suppose that ℓ is split in F and the primes $\mathfrak{l}, \bar{\mathfrak{l}}$ above it are narrowly principal. Then the push-forward via the composition $pr_{1,\ell} \circ \hat{pr}_{2,\ell} : Y^*(\ell M, \ell\mathfrak{N}) \to Y^*(M, \ell\mathfrak{N}) \to Y^*(M, \mathfrak{N})$ maps $\widetilde{\mathrm{AF}}_{\ell M, \ell\mathfrak{N}, a}$ to*

$$\sigma_\ell[(\ell-1)(1-\langle \ell^{-1} \rangle \sigma_\ell^{-2}) - \ell P'(\ell^{-1}\sigma_\ell^{-1})] \cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a}.$$

Corollary 3.3.2. *Let $M \geq 1$, $\mathfrak{N}$ an integral ideal of F and ℓ a rational prime which does not divide $\mathrm{Nm}(\mathfrak{N})M$. Let $a \in \mathscr{O}_F/(\ell M \mathscr{O}_F + \mathbb{Z})$ such that it is a unit at ℓ and suppose that ℓ satisfies the splitting assumption as above. Then the norm map $Y^*(\mathfrak{N}) \times \mu_{M\ell} \to Y^*(\mathfrak{N}) \times \mu_M$ sends the class $\mathrm{AF}_{M\ell,\mathfrak{N},a}$ to*

$$\sigma_\ell[(\ell-1)(1-\langle \ell^{-1} \rangle \sigma_\ell^{-2}) - \ell P'(\ell^{-1}\sigma_\ell^{-1})] \cdot \mathrm{AF}_{M,\mathfrak{N},a}.$$

Proof. The result follows directly from the theorem applied to the classes $\widetilde{\mathrm{AF}}_{\ell M, \ell M\mathfrak{N}, a}$, $\widetilde{\mathrm{AF}}_{M, M\mathfrak{N}, a}$, noticing that the pushforward map $(\mathbf{t}_M)_*$ used in Definition 3.1.4 commutes with the Hecke operators. □

3.3.1 Euler system norm relations

Before getting into the proof of the theorem, we briefly explain how to deduce the Euler system norm relations in Galois cohomology using such result.

Consider a Hilbert modular eigenform f, we can construct classes in Galois cohomology taking a "projection" to $H^1\left(\mathbb{Q},(V_f^{\mathrm{As}})^*\right)$. The construction for motivic coefficient sheaves is very similar and is presented in § 4.8.2.

- We have (see [Hub00]) a realisation functor for continuous étale cohomology (as defined in [Jan88]) for smooth varieties Y defined over $\mathbb{Q}$

$$r_{\text{ét}}: H^3_{\text{mot}}(Y,\mathbf{Q}(2)) \longrightarrow H^3_{\text{ét}}(Y,\mathbb{Q}_p(2)).$$

- There is an Hochschild–Serre spectral sequence (see again [Jan88]) relating continuous étale cohomology for varieties Y over $\mathbb{Q}$ with étale cohomology of the base change over $\bar{\mathbb{Q}}$

$$E_2^{p,q} = H^p(\mathbb{Q}, H^q_{\text{ét}}(Y,L_v(n)) \Rightarrow H^{p+q}_{\text{ét}}(Y_{\bar{\mathbb{Q}}},L_v(n)),$$

  for any finite extension $L_v/\mathbb{Q}_p$. From this, one gets a map from the kernel of the map $H^i_{\text{ét}}(Y,L_v(n)) \to H^i_{\text{ét}}(Y_{\bar{\mathbb{Q}}},L_v(n))^{G_{\mathbb{Q}}}$ to $H^1\left(\mathbb{Q},H^{i-1}_{\text{ét}}(Y_{\bar{\mathbb{Q}}},L_v(n))\right)$. In particular, for $i=3$ and if Y is a surface, since Artin vanishing theorem tells us that $H^i_{\text{ét}}(Y_G(K)_{\bar{\mathbb{Q}}},L_v(n))=0$ being $i>\dim(Y)=2$, we obtain a map, for any finite extension $L_v/\mathbb{Q}_p$

$$HS: H^3_{\text{ét}}(Y,L_v(2)) \longrightarrow H^1\left(\mathbb{Q},H^2_{\text{ét}}(Y_{\bar{\mathbb{Q}}},L_v(2))\right).$$

  Applying the étale regulator and the map obtained via Hochschild–Serre for $Y=Y^*(\mathfrak{N})$ to the classes $\mathrm{AF}_{M,\mathfrak{N},a}$ we obtain elements in

$$H^1\left(\mathbb{Q},H^2_{\text{ét}}(Y^*(\mathfrak{N})\times_{\mathbb{Q}}\mu_M))_{\bar{\mathbb{Q}}},L_v(2))\right).$$

- We also recall (see [LLZ18, Corollary 4.4.4]) that if f is an Hilbert eigenform of level $\mathfrak{N}$ and weight $(2,2)$, there is a canonical $G_{\mathbb{Q}}$-equivariant map

$$pr_f : H^2_{\text{ét}}(Y^*(\mathfrak{N})_{\bar{\mathbb{Q}}}, L_v(2)) \to (V_f^{\text{As}})^*, \tag{3.3.1}$$

where $v \mid p$ is a place of the number field generated by the Hecke eigenvalues of f and V_f^{As} is the $G_{\mathbb{Q}}$ representation of Definition 2.4.6. Moreover for each prime $\ell \nmid p\mathrm{Disc}(F/\mathbb{Q})\mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{N})$, this intertwines the dual operator-valued Asai Euler factor $P'_\ell(X)$ of Definition 3.2.3 on the left-hand side with the polynomial $P_\ell^{\text{As}}(f,X)$ of Definition 2.4.8 (see [LLZ18, Corollary 4.4.4]).

- Recall that, by Remark 2.1.6, for any variety X over $\mathbb{Q}$ we naturally have the following isomorphism of $G_{\mathbb{Q}}$-modules

$$H^i_{\text{ét}}((X \times_{\mathbb{Q}} \mu_N)_{\bar{\mathbb{Q}}}, L_v(n)) \simeq \mathrm{Ind}_{G_{\mathbb{Q}(\mu_N)}}^{G_{\mathbb{Q}}} H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}}, L_v(n)).$$

  Moreover, by Shapiro's lemma we have

$$H^1(\mathbb{Q}, \mathrm{Ind}_{G_{\mathbb{Q}(\mu_N)}}^{G_{\mathbb{Q}}} V) = H^1(\mathbb{Q}(\mu_N), V).$$

Applying all the steps mentioned above we find, for every integer $M \geq 1$ and f as above, a map

$$\pi_{M,f} : H^3_{\text{mot}}(Y^*(\mathfrak{N}) \times \mu_M, \mathbf{Q}(2)) \otimes L \to H^1(\mathbb{Q}(\mu_M), (V_f^{\text{As}})^*).$$

We are then finally able to produce a collection of classes in Galois cohomology.

Definition 3.3.3. Let $M \geq 1$ be an integer and $a \in \mathcal{O}_F/(M\mathcal{O}_F + \mathbb{Z})$. We define

$$z^f_{M,a} := \tfrac{1}{M}\pi_{M,f}(\mathrm{AF}_{M,\mathfrak{N},a}) \in H^1(\mathbb{Q}(\mu_M), (V_f^{\text{As}})^*).$$

Corollary 3.3.2, combined with the properties of the map (3.3.1), implies the following result.

Corollary 3.3.4. *If $M \geq 1$ is an integer coprime to to $p\,\mathrm{Disc}(F/\mathbb{Q})\,\mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{N})$ and ℓ is a rational prime coprime to $Mp\,\mathrm{Disc}(F/\mathbb{Q})\,\mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{N})$ which splits in F and such that the primes above it are trivial in the narrow class group of F, then the following relation holds true*

$$\mathrm{cores}^{\mathbb{Q}(\mu_{M\ell})}_{\mathbb{Q}(\mu_M)}(z^f_{M\ell,a}) = -\sigma_\ell Q(\sigma_\ell^{-1}) z^f_{M,a},$$

where $Q(X) \in \mathscr{O}_L[X]$ is a polynomial congruent to $\det(1 - X\,\mathrm{Frob}_\ell^{-1}\,|(V_f^{\mathrm{As}}(1))^{I_\ell})$ modulo $(\ell - 1)$.

In other words, we have proved that modulo $(\ell - 1)$ and multiplication by σ_ℓ, the classes satisfy the tame norm relations at primes splitting in F and such that the primes above them are trivial in the narrow class group. In fact, this is essentially all one needs. Working more carefully and producing similarly *integral* classes, one can then get rid of the σ_ℓ using [KLZ15, Lemma 7.3.2] and then lift the classes "removing the $(\ell - 1)$-error term" using [KLZ15, Lemma 7.3.4], [Rub00, Lemma IX.6.1]. All of these procedures do not modify the bottom class, i.e. one gets an Euler system $(\tilde{z}^f_{M,a})_M$ for $V_f^{\mathrm{As}}(1)$ with $\tilde{z}^f_{1,a} = z^f_{1,a}$.

3.3.2 The other compatibilities in motivic cohomology

Theorem 3.3.1 shows that the classes $\widetilde{\mathrm{AF}}_{M,\mathfrak{N},a}$ in the motivic cohomology of Hilbert modular surfaces satisfy some relation when changing the *cyclotomic variable M*. For the sake of completeness and since it will be useful for the proof of such theorem, we state some relations satisfied by the classes when changing the *level variable $\mathfrak{N}$*.

Theorem 3.3.5. *Let $M \geq 1, \mathfrak{N}$ an ideal divisible by M, $\mathfrak{l}$ a prime ideal of $\mathscr{O}_F$ and ℓ the rational prime lying below $\mathfrak{l}$.*

1. *Then the image of $\widetilde{\mathrm{AF}}_{M,\mathfrak{l}\mathfrak{N},a}$ under pushforward along the natural projection $pr_{1,\mathfrak{l}} : Y^*(M, \mathfrak{l}\mathfrak{N}) \to Y^*(M, \mathfrak{N})$ is given by*

$$\begin{cases} \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a} & \text{if } \ell \mid \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{N}) \\ \left(1 - \langle \ell^{-1} \rangle\, \sigma_\ell^{-2}\right) \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a} & \text{otherwise.} \end{cases}$$

2. *The image of the class* $\widetilde{\mathrm{AF}}_{M,\ell\mathfrak{N},a}$ *under pushforward along the* twisted *projection map* $pr_{2,\ell} : Y^*(M,\ell\mathfrak{N}) \to Y^*(M,\mathfrak{N})$ *is given by*

$$\begin{cases} \ell \cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},\ell a} & \text{if } \ell \mid \mathfrak{N} \\ \ell\sigma_\ell^{-1}\left(1-\left\langle \ell^{-1}\right\rangle \sigma_\ell^{-2}\right)\cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a} & \text{if } \ell \nmid \mathfrak{N} \end{cases}$$

Proof. This is the trivial coefficient case of [LLZ18, Theorem 7.1.1a,Corollary 7.4.2], noticing that, thanks to Remark 3.2.1, the action of the operator $R'(\ell)$ is trivial on motivic cohomology with trivial coefficients. □

Moreover, one also has the *wild cyclotomic* compatibility (which will imply the wild Euler system norm relations), i.e. the classes satisfy some relation also when changing the level by a prime that divides the conductor $\mathfrak{N}$.

Theorem 3.3.6. *Let* $M \geq 1$, *let* ℓ *be prime, and let* $\mathfrak{N}$ *be an ideal of divisible by* ℓM. *Let* $a \in \mathscr{O}_F/(\ell M\mathscr{O}_F + \mathbf{Z})$ *and assume* a *is a unit at* ℓ. *Recall the map* $\hat{pr}_{2,\ell}$: $Y^*(\ell M,\mathfrak{N}) \to Y^*(M,\mathfrak{N})$. *Then*

$$\left(\hat{pr}_{2,\ell}\right)_*\left(\widetilde{\mathrm{AF}}_{\ell M,\mathfrak{N},a}\right) = \begin{cases} U'(\ell)\cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a} & \text{if } \ell \mid M \\ (U'(\ell)-\sigma_\ell)\cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a} & \text{if } \ell \nmid M \end{cases}$$

Proof. See [LLZ18, Theorem 7.1.2a], again in the trivial coefficient case. □

3.3.3 Proof of Theorem 3.3.1

Let ℓ be as in Theorem 3.3.1 and write $\mathfrak{l} = (\lambda), \bar{\mathfrak{l}} = (\bar{\lambda})$ for $\lambda, \bar{\lambda}$ totally positive elements such that $\lambda\bar{\lambda} = \ell$.

In order to prove Theorem 3.3.1 we will need the following result, which is the translation of Theorem 7.5.1 of [LLZ18] for motivic classes and $j = 0$.

Theorem 3.3.7. *Let* $a \in \mathscr{O}_F/(M\mathscr{O}_F + \mathbb{Z})$ *and* $\ell, \lambda, \bar{\lambda}$ *as above. Assume that* $\bar{\lambda} \nmid \mathfrak{N}, \lambda \nmid \mathfrak{N}$. *Then we have*

$$(pr_{2,\lambda})_*\widetilde{\mathrm{AF}}_{M,\lambda\mathfrak{N},a} = \sigma_\ell^{-1}[T'(\bar{\lambda}) - \sigma_\ell^{-1}\langle\bar{\lambda}^{-1}\rangle T'(\lambda)]\cdot\widetilde{\mathrm{AF}}_{M,\mathfrak{N},a}.$$

One also has the analogous equality for $(pr_{2,\bar\lambda})_*\widetilde{\mathrm{AF}}_{M,\bar\lambda\mathfrak{N},a}$, switching the roles of λ and $\bar\lambda$.

Proof. Let $a'\in\mathscr{O}_F$. We denote by $\iota_{a',\lambda}$ and $\iota_{a',1}$ the maps obtained by the composition

$$\iota_{a',\lambda}: Y(M,N\ell)\hookrightarrow Y^*(M,\mathfrak{N}\lambda)\xrightarrow{\mathbf{u}_{a'}} Y^*(M,\mathfrak{N}\lambda),$$

$$\iota_{a',1}: Y(M,N)\hookrightarrow Y^*(M,\mathfrak{N})\xrightarrow{\mathbf{u}_{a'}} Y^*(M,\mathfrak{N}).$$

The proof follows the line of [LLZ15, Lemma A.2.1]. We will divide it in three steps. Recall the congruence subgroups $U^*(M,\mathfrak{N}(\lambda)),U^*(M(\lambda),\mathfrak{N})$ and $U_{\mathbb{Q}}(M,N(\ell)),U_{\mathbb{Q}}(M(\ell),N)$ introduced in Definition 2.2.6. The idea, following [Kat04], is to consider

$$Y(M,N\ell)\to Y(M,N(\ell))\xrightarrow{\varphi_\ell} Y(M(\ell),N)\to Y(M,N),$$

where $\varphi_\ell: Y(M,N(\ell))\xrightarrow{\simeq} Y(M(\ell),N)$ is the isomorphism of [Kat04, § 2.8], given on complex points by multiplication by ℓ. We write $\iota_{a',(\lambda)},\hat\iota_{a',(\bar\lambda)}$ for the maps obtained as follows

$$\iota_{a',(\lambda)}: Y(M,N(\ell))\hookrightarrow Y^*(M,\mathfrak{N}(\lambda))\xrightarrow{\mathbf{u}_{a'}} Y^*(M,\mathfrak{N}(\lambda)),$$

$$\hat\iota_{a',(\bar\lambda)}: Y(M(\ell),N)\hookrightarrow Y^*(M(\bar\lambda),\mathfrak{N})\xrightarrow{\mathbf{u}_{a'}} Y^*(M(\bar\lambda),\mathfrak{N}).$$

Finally, recall that, following our convention, if $\bar\lambda\nmid\mathfrak{N}$ the action of the Hecke operator $T'(\bar\lambda)$ is given by $(\hat{pr}_{2,\bar\lambda})_*\circ(\hat{pr}_{1,\bar\lambda})^*$, where $\hat{pr}_{1,\bar\lambda}$ is the natural projection map in the following diagram

$$\begin{tikzcd}
Y^*(M(\bar\lambda),\mathfrak{N}) \arrow[r, "\simeq"] \arrow[d, "\hat{pr}_{1,\bar\lambda}"] \arrow[dr, "\hat{pr}_{2,\bar\lambda}"] & Y^*(M,\mathfrak{N}(\bar\lambda)) \arrow[d, "\pi_{1,\bar\lambda}"] \\
Y^*(M,\mathfrak{N}) & Y^*(M,\mathfrak{N})
\end{tikzcd}$$

and $\hat{pr}_{2,\bar\lambda}$ is the composition of the horizontal isomorphism given by multiplication by $\begin{pmatrix}\bar\lambda^{-1} & 0\\ 0 & 1\end{pmatrix}$ and the natural projection map on the right. We will denote by $\pi_{2,\bar\lambda}$ the following map

$$\pi_{2,\bar{\lambda}} : Y^*(M,\mathfrak{N}(\bar{\lambda})) \xrightarrow{\cdot\begin{pmatrix}\bar{\lambda} & 0\\ 0 & 1\end{pmatrix}} Y^*(M(\bar{\lambda}),\mathfrak{N}) \xrightarrow{\hat{pr}_{1,\bar{\lambda}}} Y^*(M,\mathfrak{N}),$$

where the first map is the inverse of the isomorphism above. We will use the analogous notation for the prime λ.

Step 1. First of all we consider the commutative diagrams

$$\begin{tikzcd} Y(M,N\ell) \arrow[r, "\iota_{a,\lambda}"] \arrow[d, "pr"'] & Y^*(M,\mathfrak{N}\lambda) \arrow[d, "\tilde{pr}"] \\ Y(M,N(\ell)) \arrow[r, "\iota_{a,(\lambda)}"] & Y^*(M,\mathfrak{N}(\lambda)) \end{tikzcd} \qquad \begin{tikzcd} Y^*(M,\mathfrak{N}\lambda) \arrow[r, "pr_{2,\lambda}"] \arrow[dr, "\tilde{pr}"] & Y^*(M,\mathfrak{N}) \\ & Y^*(M,\mathfrak{N}(\lambda)), \arrow[u, "\pi_{2,\lambda}"'] \end{tikzcd}$$

where pr and $\tilde{pr}$ are the natural projection maps. Hence we have $(pr_{2,\lambda})_*\widetilde{\mathrm{AF}}_{M,\lambda\mathfrak{N},a} = (\pi_{2,\lambda})_*(\iota_{a,(\lambda)})_*pr_*(Y(M,N\ell),g_{0,1/N\ell})$. Following [Kat04, Lemma 2.12 and p.132], one can show that

$$pr_*g_{0,1/N\ell} = \begin{cases} \varphi_\ell^*(g_{0,1/N}) & \text{if } \ell \mid N \\ \varphi_\ell^*(g_{0,1/N}) \cdot (g_{0,\ell^{-1}/N})^{-1} & \text{if } \ell \nmid N, \end{cases}$$

where we denoted by $g_{0,\ell^{-1}/N}$ the Siegel unit $g_{0,\beta}$, pulled back via the natural map $Y(M,N(\ell)) \to Y(M,N)$, where β is the unique element of $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ such that $\ell\beta = \frac{1}{N}$. Similarly, the Siegel unit $g_{0,1/N}$ is seen as an element of $\mathscr{O}(Y(M(\ell),N))^\times$ via the pullback of $Y(M(\ell),N) \to Y(M,N)$.

By our assumptions, $\ell \nmid \mathfrak{N}$, hence we find $(pr_{2,\lambda})_*\widetilde{\mathrm{AF}}_{M,\lambda\mathfrak{N},a} =$

$$(\pi_{2,\lambda})_*(\iota_{a,(\lambda)})_*(Y(M,N(\ell)),\varphi_\ell^*(g_{0,1/N})) - (\pi_{2,\lambda})_*(\iota_{a,(\lambda)})_*(Y(M,N(\ell)),g_{0,\ell^{-1}/N}).$$

Step 2. We now want to compute $(\bigstar) := (\pi_{2,\lambda})_*(\iota_{a,(\lambda)})_*(Y(M,N(\ell)),\varphi_\ell^*(g_{0,1/N}))$. We consider the map $f : Y^*(M,\mathfrak{N}(\lambda)) \to Y^*(M(\bar{\lambda}),\mathfrak{N})$ induced by multiplication by $\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have the following commutative diagram

$$\begin{tikzcd}
Y(M,N(\ell)) \arrow[r, "\iota_{a,(\lambda)}"] \arrow[d, "\varphi_\ell"'] & Y^*(M,\mathfrak{N}(\lambda)) \arrow[d, "f"'] \arrow[dr, "\pi_{2,\lambda}"] & \\
Y(M(\ell),N) \arrow[r, "\hat{\iota}_{\ell a,(\bar{\lambda})}"] & Y^*(M(\bar{\lambda}),\mathfrak{N}) \arrow[r, "\hat{pr}_{2,\bar{\lambda}}"] & Y^*(M,\mathfrak{N}),
\end{tikzcd}$$

where $\hat{pr}_{2,\bar{\lambda}}$ is given by the action of $\left(\begin{smallmatrix} \bar{\lambda}^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Hence we have, combining this with Step 1 and using the fact that φ_ℓ is an isomorphism,

$$(\bigstar) = (\hat{pr}_{2,\bar{\lambda}})_*(\hat{\iota}_{\ell a,(\bar{\lambda})})_*(Y(M(\ell),N), g_{0,1/N}).$$

Next we consider the commutative diagram

$$\begin{tikzcd}
Y(M(\ell),N) \arrow[r, "\hat{\iota}_{\ell a,(\bar{\lambda})}"] \arrow[d] & Y^*(M(\bar{\lambda}),\mathfrak{N}) \arrow[d, "\hat{pr}_{1,\bar{\lambda}}"] \\
Y(M,N) \arrow[r, "\iota_{\ell a,1}"] & Y^*(M,\mathfrak{N}),
\end{tikzcd}$$

where the vertical maps are the natural projections. Reasoning as in [LLZ14, Lemma 2.4.5], one can show that $(\hat{pr}_{1,\bar{\lambda}})^* \circ (\iota_{\ell a,1})_*$ is equal to the pullback under the first vertical map composed with $(\hat{\iota}_{\ell a,(\bar{\lambda})})_*$. Using this and the above equality, we get

$$\begin{aligned}
(\bigstar) &= (\hat{pr}_{2,\bar{\lambda}})_*(\hat{pr}_{1,\bar{\lambda}})^*(\iota_{\ell a,1}(Y(M,N)), (\iota_{\ell a,1})_* g_{0,1/N}) \\
&= (\hat{pr}_{2,\bar{\lambda}})_*(\hat{pr}_{1,\bar{\lambda}})^* \cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},\ell a} = T'(\bar{\lambda}) \cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},\ell a}.
\end{aligned}$$

Since $\ell \nmid N$, we have that $\widetilde{\mathrm{AF}}_{M,\mathfrak{N},\ell a} = \sigma_\ell^{-1} \cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a}$, thanks to Proposition 3.1.8. Hence we get the first term in the claimed equation.

Step 3. We are now left with computing $(\blacklozenge) := (\pi_{2,\lambda})_*(\iota_{a,(\lambda)})_*(Y(M,N(\ell)), g_{0,\ell^{-1}/N})$.

The commutative diagram we are using this time is the following

$$\begin{tikzcd} Y(M,N(\ell)) \arrow[r, "\iota_{a,(\lambda)}"] \arrow[d] & Y^*(M,\mathfrak{N}(\lambda)) \arrow[d, "\pi_{1,\lambda}"] \\ Y(M,N) \arrow[r, "\iota_{a,1}"] & Y^*(M,\mathfrak{N}), \end{tikzcd}$$

where again the vertical arrows are the natural projections. We use this commutative square as above, applying a result like [LLZ14, Lemma 2.4.5]. Hence we get

$$(\blacklozenge) = (\pi_{2,\lambda})_*(\pi_{1,\lambda})^*(\iota_{a,1}(Y(M,N)),(\iota_{a,1})_* g_{0,\ell^{-1}/N})$$

We have, using Proposition 2.1.14(i), $(\iota_{a,1}(Y(M,N)),(\iota_{a,1})_* g_{0,\ell^{-1}/N}) = \left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x^{-1}\end{smallmatrix}\right)^* \cdot (\iota_{a,1}(Y(M,N)),(\iota_{a,1})_* g_{0,1/N})$, where we chose $x \in \hat{\mathbb{Z}}^\times$ to be a lift of $\ell \in (\mathbb{Z}/N\mathbb{Z})^\times$. We have

$$(\iota_{a,1}(Y(M,N)),(\iota_{a,1})_* g_{0,\ell^{-1}/N}) = \left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x^{-1}\end{smallmatrix}\right)^* \cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a}, \tag{3.3.2}$$

Using the equalities $(\pi_{2,\lambda})_*(\pi_{1,\lambda})^* \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right)^* = T'(\lambda)$, where $t \in \hat{\mathscr{O}}_F^\times$ is a lift of $\lambda \in (\mathscr{O}_F/\mathfrak{N})^\times$ and $\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x^{-1}\end{smallmatrix}\right)^* = \langle \ell^{-1}\rangle \sigma_\ell^{-2}$, we find

$$(\pi_{2,\lambda})_*(\pi_{1,\lambda})^* \left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x^{-1}\end{smallmatrix}\right)^* = \sigma_\ell^{-2}\langle \bar{\lambda}^{-1}\rangle T'(\lambda), \tag{3.3.3}$$

Combining (3.3.2) and (3.3.3), we find

$$(\blacklozenge) = \sigma_\ell^{-2}\langle \bar{\lambda}^{-1}\rangle T'(\lambda) \cdot \widetilde{\mathrm{AF}}_{M,\mathfrak{N},a}$$

Combining this result with the ones of the previous steps we proved the statement. □

We now have all the ingredients for proving Theorem 3.3.1.

Proof of Theorem 3.3.1. We need to compute $(pr_{1,\ell})_*((\hat{pr}_{2,\ell})_*(\widetilde{\mathrm{AF}}_{\ell M,\ell\mathfrak{N},a}))$. By Theorem 3.3.6, this class is equal to

$$(pr_{1,\ell})_*(U'(\ell)-\sigma_\ell)(\widetilde{\mathrm{AF}}_{M,\ell\mathfrak{N},a}).$$

Applying Lemma 3.2.2, we can rewrite the operator acting on $\widetilde{\mathrm{AF}}_{M,\ell\mathfrak{N},a}$ as

$$(T'(\ell)-\sigma_\ell)(pr_{1,\ell})_* - \langle\lambda^{-1}\rangle T'(\bar{\lambda})(pr_{2,\lambda})_*(pr_{1,\bar{\lambda}})_*$$
$$-\langle\bar{\lambda}^{-1}\rangle T'(\lambda)(pr_{1,\lambda})_*(pr_{2,\bar{\lambda}})_* + \langle\ell^{-1}\rangle(pr_{2,\ell})_*.$$

In order to compute the image of $\widetilde{\mathrm{AF}}_{M,\ell\mathfrak{N},a}$ under $(pr_{1,\ell})_*$ and $(pr_{2,\ell})_*$ we apply Theorem 3.3.5.1 and 3.3.5.2 respectively. The remaining terms can be computed applying Theorem 3.3.7 combined again with Theorem 3.3.5.1. One then finds that the Hecke polynomial obtained is equal to the one claimed in the theorem, using the explicit description of $P'_\ell(X)$ provided in Definition 3.2.3 and Remark 3.2.4. $\square$

Chapter 4

Asai–Flach classes norm relations by means of local smooth representation theory

This chapter is devoted to the proof of norm relations for Asai–Flach classes, removing the assumption for split primes introduced in Chapter 3. We will work with classes in the motivic cohomology of the Hilbert modular surface with coefficients in the sheaves of § 2.3.2. In the case of trivial coefficients we recover the classes of Chapter 3. To achieve a proof of norm relations for all inert and split primes, one needs to change completely the strategy: following ideas of [LSZ20a], we re-define the motivic classes using a *representation theoretic* language and prove some result using local smooth representation theory to deduce norm relations. The content of this chapter appeared in [Gro20].

4.1 Structure of the chapter

Let F be a real quadratic field as above and consider the embedding of algebraic groups over $\mathbb{Q}$

$$H := \mathrm{GL}_2 \hookrightarrow G := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2 . \tag{4.1.1}$$

We will be working with representations Π of $\mathrm{GL}_2(\mathbb{A}_{F,f})$, where $\mathbb{A}_{F,f}$ denotes the finite adèles over F, which are the finite part of automorphic representations of $\mathrm{GL}_{2,F}$. Equivalently we can view Π as a representation of $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ are the finite adèles over $\mathbb{Q}$. At every place ℓ we have a representation of $G(\mathbb{Q}_\ell)$, which, depending on ℓ has the following shape:

- if ℓ is inert and hence $G(\mathbb{Q}_\ell) = \mathrm{GL}_2(F_\ell)$ where F_ℓ is an unramified quadratic extension of $\mathbb{Q}_\ell$, the representation is Π_ℓ;
- if $\ell = v_1 \cdot v_2$ is split and hence $G(\mathbb{Q}_\ell) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell)$, the representation is $\Pi_{v_1} \otimes \Pi_{v_2}$.

In order to (re)define the Euler system classes, we define a special map $\mathscr{A}\mathscr{F}^{\text{k,k',j}}_{\text{mot}}$ for $k, k' \geq 0$ integers and $0 \leq j \leq \min(k,k')$ with values in $W = H^3_{\text{mot}}(Y_G, \mathscr{D}(2))$, where Y_G is the Shimura variety associated to G and $\mathscr{D}$ is a motivic sheaf depending on k, k', j (cfr. § 2.3.2). Such map will be of "global nature", more precisely it is a map

$$\mathscr{A}\mathscr{F}^{\text{k,k',j}}_{\text{mot}} : \mathscr{S}(\mathbb{A}^2_f, \mathbb{Q}) \otimes \mathscr{H}(G(\mathbb{A}_f), \mathbb{Q}) \longrightarrow H^3_{\text{mot}}(Y_G, \mathscr{D}(2))$$

satisfying some conditions of $H(\mathbb{A}_f) \times G(\mathbb{A}_f)$-equivariance. Here $\mathscr{S}(\mathbb{A}^2_f, \mathbb{Q})$ denotes the space of Schwartz functions on $\mathbb{A}_f$ and $\mathscr{H}(G(\mathbb{A}_f), \mathbb{Q})$ the Hecke algebra over $G(\mathbb{A}_f)$. The Asai–Flach classes will be defined as images via $\mathscr{A}\mathscr{F}^{\text{k,k',j}}_{\text{mot}}$ of certain elements in $\mathscr{S}(\mathbb{A}^2_f, \mathbb{Q}) \otimes \mathscr{H}(G(\mathbb{A}_f), \mathbb{Q})$. In §4.6 we recall the definition of Eisenstein classes as $H(\mathbb{A}_f)$-equivariant maps

$$\mathscr{S}(\mathbb{A}^2_f, \mathbb{Q}) \longrightarrow H^1_{\text{mot}}(Y_H, \mathrm{TSym}^k \mathscr{H}_{\mathbf{Q}}(\mathscr{E})(1)).$$

In particular if $k = 0$ and $\phi = \mathrm{ch}((a,b) + N\hat{\mathbb{Z}})$ for some $N \geq, a, b \in \mathbb{Q}^2 - N\mathbb{Z}^2$, then $g_\phi = g_{a/N, b/N}$, the Siegel unit of Definition 2.1.11.

The global map $\mathscr{A}\mathscr{F}^{\text{k,k',j}}_{\text{mot}}$ is defined in §4.7 using the Eisenstein classes map and the pushforward in motivic cohomology induced by (4.1.1), as in (2.3.2). Proving norm relations (in motivic cohomology) will turn out to be equivalent to proving relations of such classes locally at a certain prime ℓ, i.e. we will be looking at a map

$$(\mathscr{A}\mathscr{F}^{\text{k,k',j}}_{\text{mot}})_\ell : \mathscr{S}(\mathbb{Q}^2_\ell, \mathbb{Q}) \otimes \mathscr{H}(G(\mathbb{Q}_\ell), \mathbb{Q}) \longrightarrow H^3_{\text{mot}}(Y_G, \mathscr{D}(2)).$$

While we will be able to prove p-direction norm relations already in motivic cohomology, in order to prove "tame norm relations" we will have strong assumptions on the target of such map. We will have to apply the étale regulator and Hochschild—Serre spectral sequence to pass to Galois cohomology and finally take the projection to an automorphic representation of $G(\mathbb{A}_f)$ as above (see § 4.8.2). The local components Π_v at a "good prime" v of F will be an irreducible spherical principal series representation $I_{\mathrm{GL}_2(F_v)}(\chi,\psi)$, for χ,ψ unramified characters of $F_v^\times$. Hence we will need to work and prove results for maps

$$\mathfrak{z}:\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})\otimes\mathscr{H}(G)\longrightarrow\sigma=\begin{cases} I_{\mathrm{GL}_2(F_\ell)}(\tilde{\chi},\tilde{\psi}) & \text{if } \ell \text{ is inert,}\\ I_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi_1,\psi_1)\otimes I_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi_2,\psi_2) & \text{if } \ell \text{ splits,}\end{cases}$$

where, by abuse of notation, we denoted by $\mathscr{H}(G)$ the Hecke algebra over $G(\mathbb{Q}_\ell)$.

The first sections of the chapter are devoted to the study of these local representations. First we recall in §4.2 some useful tools to work with representations of GL_2 over a local field. The following sections, §4.3 and §4.4, should be thought in parallel: we move to local representations of G over $\mathbb{Q}_\ell$ proving the same results for both the inert and split case, giving explicit descriptions of local L-factors $L(\mathrm{As}(\sigma),s)$ of principal series representations σ as above as local zeta integrals.

In §4.5, we will relate images under maps as above of elements in $\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})\otimes\mathscr{H}(G)$ given by Definitions 4.2.28, 4.5.16 and 4.5.17. The main results of this section are

Proposition 4.1.1 (Proposition 4.5.19)**.** *For any $\mathfrak{z}:\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})\otimes\mathscr{H}(G)\to W$, where W is a smooth complex representation of $G(\mathbb{Q}_\ell)$ we have*

$$\mathfrak{z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_{m+1}K_{m,n}))=\begin{cases}\frac{1}{\ell}U'(\ell) \\ \frac{1}{\ell-1}(U'(\ell)-1)\end{cases}\cdot\mathfrak{z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_m K_{m,n}))\quad \begin{matrix}\textit{if } m\geq 1\\ \textit{if } m=0.\end{matrix}$$

Corollary 4.1.2 (Corollary 4.5.20)**.** *Let $W=\sigma^\vee$ for σ a principal series representation with central character χ_σ. Let $\chi=|\cdot|^{1/2+k}\tau$, for τ a finite order character* *and $k \geq 0$, and $\psi = |\cdot|^{-1/2}$. Assume*

$$\chi\psi \cdot \chi_\sigma = 1 \tag{4.1.2}$$

and, if ℓ is inert, assume the pair of characters $(\tilde{\chi}, \tilde{\psi})$ is different, when restricted to $\mathbb{Q}_\ell^\times$, from the pair (χ, ψ). Let $\mathfrak{Z} : \mathscr{S}(\mathbb{Q}_\ell^2, \mathbb{C}) \otimes \mathscr{H}(G) \to \sigma^\vee$ such that it factors as $\mathfrak{Z} = \mathfrak{Z}' \circ f$, where f is the Siegel section map defined in §4.2 and

$$\mathfrak{Z}' : I_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi, \psi) \otimes \mathscr{H}(G) \to \sigma^\vee.$$

Then we have

$$\mathfrak{Z}(\phi_{1,\infty} \otimes (\mathrm{ch}(K) - \mathrm{ch}(\eta_1 K))) = \tfrac{\ell}{\ell-1} L(\mathrm{As}(\sigma), h)^{-1} \cdot \mathfrak{Z}(\phi_0, \mathrm{ch}(K)).$$

While the proposition can be proved directly for any such $\mathfrak{Z}$, the corollary follows from Theorem 4.5.8. It states the analogous equality for any function in $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(I_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi, \psi) \otimes \sigma, \mathbb{C})$ (which is in canonical bijection with the space of functions $\mathfrak{Z}'$ as above). The proof of the theorem follows from an explicit proof of the claimed equality for a specific choice of a nonzero element $\mathfrak{z}_{\chi,\psi} \in \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(I_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi, \psi) \otimes \sigma, \mathbb{C})$ built using the local zeta integrals of §§2,3 (see Definition 4.5.6). One then crucially needs the following multiplicity one result in order to prove it for any $\mathfrak{z} \in \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(I_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi, \psi) \otimes \sigma, \mathbb{C})$.

Theorem 4.1.3 (Theorem 4.5.1, Multiplicity one)**.** *Let σ, χ, ψ satisfying (4.1.2) and assume σ satisfies in the inert case the same additional condition of the Corollary. Assume that $\chi\psi^{-1} \neq |\cdot|^{-1}$. We have*

$$\dim \left(\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(I_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi, \psi) \otimes \sigma, \mathbb{C}) \right) \leq 1.$$

This theorem follows from [Pra90, Theorem 1.1] in the case where ℓ splits and $I_{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi, \psi)$ is irreducible and it is proved in Theorem 4.5.1 for the remaining cases. We use tools of Mackey theory following the strategy used by Prasad in *op.*

cit.

The motivic Asai–Flach elements defined in § 4.7 are image of elements in $\mathscr{S}(\mathbb{A}_f^2,\mathbb{Q})\otimes\mathscr{H}(G(\mathbb{A}_f))$ that are described at every place in terms of the elements considered in the above Proposition and Corollary. These classes are closely related to the ones constructed in [LLZ18] and in the previous chapter. In fact, the *bottom class* will be exactly the same. The perturbation of the embedding to obtain the other classes will be encoded by the action of the Hecke algebra, with some modified factors which will allow to prove norm relations without the error term divisible by $\ell-1$ appearing in [LLZ18] (and Corollary 3.3.4). More details about this can be found in 4.7.3.

Finally in §4.8 we prove some pushforward compatibilities of such motivic classes (Theorem 4.8.1 and 4.8.2, corresponding to Theorem 3.3.5 and 3.3.6) using the local result given by the above Proposition. We then use these classes to find elements in Galois cohomology of the Asai representation of a Hilbert cuspidal eigenform and prove Euler system norm relations (Theorem 4.8.11 and 4.8.12); vertical norm relations follow from the p-direction compatibility of motivic classes, while tame norm relations rely on the local result of the above Corollary.

4.2 Local representation theory for GL_2

In this section we recall the standard tools of local representation theory that will be useful later in the proof of norm relations. We follow [Bum97, Chapter 4].

We let E be a non-Archimedean local field and denote by $\mathscr{O},\mathfrak{p},\varpi$ respectively the ring of integers in E, the maximal ideal and a fixed uniformiser of $\mathfrak{p}$. Let $|\cdot|$ be the norm and q such that $|\varpi|=q^{-1}$. We also fix an Haar measure dx on E and $d^\times x$ on $E^\times$ such that $\int_{\mathscr{O}}dx=1$, $\int_{\mathscr{O}^\times}dx^\times=1$. For a smooth character χ of $E^\times$ we define its local $L-$factor

$$L(\chi,s)=L(\chi|\cdot|^s,0)=\begin{cases}(1-\chi(\varpi)q^{-s})^{-1} & \text{if } \chi_{|_{\mathscr{O}^\times}}=1\\ 1 & \text{otherwise.}\end{cases}$$

4.2.1 Induced representations

We recall here some basics about induced representations of totally disconnected topological groups. See for example [BZ76, §§2.21-2.29]. Let X be a group as above with a right Haar measure d_R on X and a left Haar measure d_L.

Definition 4.2.1. The modular quasicharacter δ_X of X is defined by $d_R(x) = \delta_X(x)d_L(x)$. If $\delta_X = 1$, X is said to be unimodular.

A trivial example of unimodular group is any abelian group. A less trivial example is $X = \mathrm{GL}_n(E)$. A group which is not unimodular is the Borel subgroup B of $\mathrm{GL}_n(E)$. For $n = 2$, its modular quasicharacter is given by

$$\delta_B(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) = |\tfrac{a}{d}|.$$

Assume now X is locally compact. Consider Y a closed subgroup of X. We have a restriction functor from the category of smooth representations of X to smooth representations of Y. This functor has a left and a right adjoint, given by induction and compact induction.

Definition 4.2.2. Let (V,τ) be a smooth representation of Y. We denote by $\mathrm{Ind}_Y^X \tau$ the space of smooth functions $f : X \to V$ satisfying the following condition

$$f(yx) = \delta_X^{-1/2}(y)\delta_Y^{1/2}(y)\tau(y)f(x) \text{ for every } x \in X, y \in Y.$$

We denote by $\text{c-}\mathrm{Ind}_Y^X(\tau)$ the subspace of $\mathrm{Ind}_Y^X \tau$ consisting of functions which in addition are compactly supported modulo Y. This coincides with $\mathrm{Ind}_Y^X(\tau)$ when X/Y is compact. These are X-representations with action of X given by right multiplication.

Theorem 4.2.3 (Frobenius reciprocity)**.** *Let (W,σ) be a smooth representation of X and (V,τ) a smooth representation of Y. denote by $(\ )^\vee$ the smooth dual of a representation. We then have the following isomorphisms:*

$$\mathrm{Hom}_X(\sigma, \mathrm{Ind}_Y^X \tau) \simeq \mathrm{Hom}_Y(\sigma_{|Y}, \delta_Y^{1/2}\delta_X^{-1/2}\tau);$$

$$\mathrm{Hom}_X(c\text{-}\mathrm{Ind}_Y^X\,\tau,\sigma^\vee) \simeq \mathrm{Hom}_Y(\delta_Y^{-1/2}\delta_X^{1/2}\tau,(\sigma_{|Y})^\vee).$$

If X is a totally disconnected locally compact algebraic group, Y is a closed subgroup and (V,τ) is a finite dimensional complex smooth representation of Y, we can realise induced representations as sections of a complex vector bundle $\mathscr{B}$ (an ℓ-sheaf in the notation of [BZ76, 1.13]) over X/Y with fibres V. Let $\mathscr{B}$ be the quotient space of $X\times V$ by the equivalence relation given by

$$(x,v)\sim(xy,\delta_X^{1/2}(y)\delta_Y^{-1/2}(y)\tau(y)v) \quad \text{for } x\in X, y\in Y, v\in V.$$

This defines a complex vector bundle over X/Y, with fibres isomorphic to V, in the sense of [BZ76, 1.13, 2.23]. Moreover, writing $\Gamma(X/Y,\mathscr{B})$ for the smooth sections of $\mathscr{B}$ and $\Gamma_c(X/Y,\mathscr{B})$ for the compactly supported smooth sections, we have

$$\Gamma(X/Y,\mathscr{B}) = \mathrm{Ind}_Y^X(\tau) \quad \text{and} \quad \Gamma_c(X/Y,\mathscr{B}) = \text{c-}\mathrm{Ind}_Y^X(\tau).$$

We now state a general lemma about compactly supported smooth sections of line bundles on totally disconnected locally compact algebraic groups (see for example [Pra90, Lemma 5.1]).

Lemma 4.2.4. *Let X be a totally disconnected locally compact algebraic group, Z a closed subgroup and B a line bundle over X. Then we have an exact sequence*

$$0\to\Gamma_c(X-Z,B_{|X-Z})\to\Gamma_c(X,B)\to\Gamma_c(Z,B_{|Z})\to 0.$$

We now apply this lemma in a particular case and find an exact sequence of induced representations that will be useful later. Let H,J be closed subgroups of a totally disconnected locally compact algebraic group G and τ a smooth one-dimensional representation of J. Assume that the quotient $H\backslash G/J$ has two elements. This means that the action of H on the space G/J has two orbits, one open and one closed. We can write these two orbits as $H/H_1, H/H_2$, where $H_1=\mathrm{Stab}_H(1\cdot J)$ and $H_2=\mathrm{Stab}_H(\varepsilon\cdot J)$, where $\varepsilon\in G$ such that $\varepsilon\cdot J$ is in the open orbit. We can compute these subgroups as follows

$$H_1 = H \cap J, \quad H_2 = H \cap \varepsilon^{-1} J \varepsilon.$$

Applying the above lemma for $Z = H/H_1, X - Z = H/H_2$ and normalising appropriately one finds an exact sequence of H-modules

$$0 \to \text{c-Ind}_{H_2}^{H} \tau_2 \to (\text{c-Ind}_J^G \tau)_{|H} \to \text{c-Ind}_{H_1}^{H} \tau_1 \to 0, \tag{4.2.1}$$

where $\tau_1 = \delta_J^{1/2} \cdot \delta_{H_1}^{-1/2} \cdot \tau_{|H_1}$ and τ_2 is a representation of H_2 given by

$$\tau_2(h) = \delta_J^{1/2}(\varepsilon h \varepsilon^{-1}) \delta_{H_2}^{-1/2}(h) \tau(\varepsilon h \varepsilon^{-1}).$$

4.2.2 Principal series representations

Definition 4.2.5. Let $H = \mathrm{GL}_2(E)$ and ξ, ψ two quasicharacters of $E^\times$. We define a space of functions on H as follows

$$I_H(\chi, \psi) := \{ f : H \to \mathbb{C} \text{ smooth } : f(\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right) \cdot h) = \left|\tfrac{a}{d}\right|^{1/2} \chi(a) \psi(d) f(h) \}$$

We will denote $I_H(\chi, \psi)$ simply as $I(\chi, \psi)$; notice that this is the space given by the normalised induction from the Borel subgroup $B(E)$ of $\mathrm{GL}_2(E)$ (consisting of upper triangular matrices). We see $I(\chi, \psi)$ as a $\mathrm{GL}_2(E)$-representation letting $\mathrm{GL}_2(E)$ act by right translation, i.e. for $g \in \mathrm{GL}_2(E)$

$$g \cdot f(h) = f(hg) \text{ for every } f \in I(\chi, \psi), h \in \mathrm{GL}_2(E).$$

In other words letting τ be the one dimensional representation of $B(E)$ given by $\tau(\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right)) = \chi(a)\psi(d)$, we have $I(\chi, \psi) = \mathrm{Ind}_{B(E)}^{\mathrm{GL}_2(E)} \tau$.

Definition 4.2.6. The $\mathrm{GL}_2(E)$-representations $I(\chi, \psi)$, for χ, ψ quasicharacters of $E^\times$, are called *principal series representations*.

To characterise such representations, we recall the definition of the intertwining operator. Fix χ, ψ and write them as

$$\chi = |\cdot|^{s_1}\xi_1, \quad \psi = |\cdot|^{s_2}\xi_2,$$

where $s_i \in \mathbb{C}$ and ξ_i are unitary characters. Let $f \in I(\chi, \psi)$. We write, for $h \in \mathrm{GL}_2(E)$,

$$Mf(h) = \int_F f(w \cdot m_x \cdot h) dx,$$

where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $m_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

The following proposition determines when this integral makes sense and in which space Mf will be defined.

Proposition 4.2.7. *[Bum97, Proposition 4.5.6]. If* $\mathrm{Re}(s_1 - s_2) > 0$ *then the above integral is absolute convergent and it defines a nonzero intertwining map*

$$\begin{aligned} M : I(\chi, \psi) &\to I(\psi, \chi) \\ f &\mapsto Mf. \end{aligned}$$

In the case where $\mathrm{Re}(s_1 - s_2) < 0$ we clearly have an analogous operator M obtained by switching χ and ψ. The procedure for defining such an operator in the case where $\mathrm{Re}(s_1 - s_2) = 0$ uses *flat sections* and the fact that we can write $\mathrm{GL}_2(E) = B(E) \cdot K$, where $K = \mathrm{GL}_2(\mathscr{O})$ (known as *Iwasawa decomposition*, see [Bum97, Proposition 4.5.2]). Indeed one starts with noticing that $f \in I(\chi, \psi)$ is uniquely determined by its restriction to K, which satisfies

$$f(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot k) = \xi_1(a)\xi_2(d)f(k),$$

for $a, d \in \mathscr{O}^\times, b \in \mathscr{O}_F$ and $k \in K$. We denote by V_0 the space of smooth functions on K satisfying this condition, having fixed ξ_1, ξ_2. Then for any $s_1, s_2 \in \mathbb{C}$ and $f_0 \in V_0$ there exists a unique extension of f_0 to an element f_{s_1,s_2} in $V_{s_1,s_2} := I(|\cdot|^{s_1}\xi_1, |\cdot|^{s_2}\xi_2)$. Fixing f_0, the function

$$(s_1, s_2) \mapsto f_{s_1,s_2}, \tag{4.2.2}$$

is called a flat section. We then have

Proposition 4.2.8. *[Bum97, Proposition 4.5.7]. Fix $f_0 \in V_0$. For a fixed $h \in \mathrm{GL}_2(E)$ the integral $Mf_{s_1,s_2}(h)$ defined as above for $\mathrm{Re}(s_1 - s_2) > 0$ has analytic continuation to all s_1, s_2 where $\chi \neq \psi$. We hence have defined an intertwining operator*

$$M : I(|\cdot|^{s_1}\xi_1, |\cdot|^{s_2}\xi_2) \to I(|\cdot|^{s_2}\xi_2, |\cdot|^{s_1}\xi_1).$$

We have the following theorem characterising principal series representations.

Theorem 4.2.9. *[Bum97, Theorem 4.5.1 and 4.5.2]. Let χ, ψ be quasicharacters of $E^\times$. Then $I(\chi, \psi)$ is an irreducible $\mathrm{GL}_2(E)$-representation except in the following two cases*

(i) if $\chi\psi^{-1} = |\cdot|^{-1}$, then $I(\chi, \psi)$ has a one-dimensional invariant subspace and the quotient representation is irreducible;

(ii) if $\chi\psi^{-1} = |\cdot|$, then $I(\chi, \psi)$ has an irreducible codimension one invariant subspace.

If $I(\chi, \psi)$ is irreducible, then it is isomorphic to $I(\psi, \chi)$ via the intertwining operator M. Moreover if we have two such representations, for quasicharacters $\chi_1, \psi_1, \chi_2, \psi_2$, and $\mathrm{Hom}_{\mathrm{GL}_2(E)}(I(\chi_1, \psi_1), I(\chi_2, \psi_2))$ is non zero then either $\chi_1 = \chi_2$ and $\psi_1 = \psi_2$ or $\chi_1 = \psi_2$ and $\psi_1 = \chi_2$.

Another tool we need to introduce is a pairing on $I(\chi, \psi) \times I(\chi^{-1}, \psi^{-1})$ which identifies $I(\chi^{-1}, \psi^{-1})$ with the smooth dual of $I(\chi, \psi)$. See [Bum97, Proposition 4.5.5]. The pairing is defined by an integral as follows

Definition/Proposition 4.2.10. The following integral defines a perfect pairing $\langle\ ,\ \rangle : I(\chi, \psi) \times I(\chi^{-1}, \psi^{-1}) \to \mathbb{C}$

$$\langle f_1, f_2 \rangle := \int_{\mathrm{GL}_2(\mathscr{O})} f_1(h) f_2(h) dh,$$

for every $f_1 \in I(\chi, \psi), f_2 \in I(\chi^{-1}, \psi^{-1})$.

4.2.3 Whittaker models

Let Ψ be a fixed nontrivial additive character of E.

Definition 4.2.11. Let V be a smooth representations of $\mathrm{GL}_2(E)$. A *Whittaker functional* on V is a linear functional $\lambda : V \to \mathbb{C}$ satisfying

$$\lambda(m_x \cdot v) = \Psi(x)\lambda(v),$$

for every $x \in F, v \in V$, where as above $m_x = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$.

Proposition 4.2.12. *[Bum97, Proposition 4.5.4]. The dimension of the space of Whittaker functionals for the representation $I(\chi,\psi)$ is exactly 1.*

We have an explicit example of a Whittaker functional μ for $I(\chi,\psi)$. The above proposition tells us that every other Whittaker functional for this representation is scalar multiple of μ.

Definition 4.2.13. We define $\mu : I(\chi,\psi) \to \mathbb{C}$ by

$$\mu(f) = \int_F f(w \cdot m_x)\Psi(-x)dx,$$

where w is as defined in the previous section.

Remark 4.2.14. With the above notation, this integral converges if $\mathrm{Re}(s_1 - s_2) > 0$, but we can proceed with an analytic continuation to every s_1, s_2 using flat sections as above.

Definition 4.2.15. The Whittaker model of $I(\chi,\psi)$ is defined to be the function

$$W : f \mapsto \left(W_f : h \mapsto \mu(h \cdot f) = \int_F f(w \cdot m_x \cdot h)\Psi(-x)dx \right)$$

It satisfies $W_f(m_x \cdot \left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right) \cdot h) = \Psi(x) \cdot \chi\psi(z) \cdot W_f(h)$.

We can associate to every Whittaker functional a *Whittaker model* as in the previous definition. The dimension one proposition tells us that they differ by a scalar.

Remark 4.2.16. One similarly defines a Whittaker model for every (V,π) smooth representation of $\mathrm{GL}_2(E)$. For any Whittaker functional λ, one lets $W_\lambda : v \mapsto (W_{\lambda,v} : h \mapsto \lambda(h\cdot v))$. The image of W_λ defines a subspace of the space of functions Λ on $\mathrm{GL}_2(E)$, satisfying $\Lambda(m_x \cdot h) = \Psi(x)\cdot\Lambda(h)$. The group $\mathrm{GL}_2(E)$ acts naturally by right translation on this space, and the image of W_λ is invariant for this action. Such image is isomorphic as $\mathrm{GL}_2(E)$-representation to (V,π) and indeed provides a "concrete" model for (V,π).

Lemma 4.2.17. *For the Whittaker model of $I(\chi,\psi)$ we have*

$$W_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\cdot f}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \Psi(d^{-1}by)\cdot\chi\psi(d)\cdot W_f\left(\begin{pmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Proof. This is straightforward rewriting

$$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = m_{d^{-1}by}\cdot\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\cdot\begin{pmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{pmatrix}$$

and using $W_f(m_{d^{-1}by}\cdot\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\cdot\begin{pmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{pmatrix}) = \Psi(d^{-1}by)\cdot\chi\psi(d)\cdot W_f(\begin{pmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{pmatrix})$ □

4.2.4 Spherical representations

Let (V,π) an irreducible admissible representation of $\mathrm{GL}_2(E)$. One can consider the subspace $V^{\mathrm{GL}_2(\mathscr{O})}$ of vectors fixed by the action of $\mathrm{GL}_2(\mathscr{O})$. This is at most one dimensional (see [Bum97, Theorem 4.6.2]).

Definition 4.2.18. An irreducible admissible representation (V,π) of $\mathrm{GL}_2(E)$ is called *spherical* if it contains a $\mathrm{GL}_2(\mathscr{O})-$fixed vector.

Remark 4.2.19. The reason for which we are interested in spherical representations is that automorphic representations of GL_2 decompose into a restricted product of local representations and these are all spherical outside a finite set of places.

Example 4.2.20 (Principal series representations)**.** The representation $I(\chi,\psi)$ with χ,ψ unramified and $\chi\psi \neq |\cdot|^{\pm 1}$ is spherical. To see this, we define the *normalised spherical vector* φ_0 as function on $\mathrm{GL}_2(E)$ by

$$\varphi_0(h) = \varphi_0(b\cdot k) := |a/d|^{1/2}\chi(a)\psi(d), \quad \text{where } b = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix}, k \in K = \mathrm{GL}_2(\mathscr{O}).$$

To write $h \in \mathrm{GL}_2(E)$ as $b \cdot k$, we use again Iwasawa decomposition. It is clear by the definition that this function is fixed by K. We check that it is well defined and that it is an element of $I(\chi, \psi)$. Suppose that $bk = b'k'$ for $b = \left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right), b' = \left(\begin{smallmatrix} a' & * \\ 0 & d' \end{smallmatrix}\right) k \in K = \mathrm{GL}_2(\mathscr{O})$. Then we have $b = b'u$ with $u \in K \cap B(E)$ i.e. $u = \left(\begin{smallmatrix} x & * \\ 0 & y \end{smallmatrix}\right)$ with $x, y \in \mathscr{O}^\times$. Since $\chi(x) = \psi(y) = 1$ and $|x| = |y| = 1$, we find $\varphi_0(bk) = \varphi_0(b'k')$. To check that this defines an element of $I(\chi, \psi)$ we compute $\varphi_0(b'h) = \varphi(b'bk)$, where as before $h = bk$, $b = \left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right), b' = \left(\begin{smallmatrix} a' & * \\ 0 & d' \end{smallmatrix}\right), k \in K = \mathrm{GL}_2(\mathscr{O})$. Hence

$$\varphi_0(b'h) = \varphi_0(\left(\begin{smallmatrix} aa' & * \\ 0 & dd' \end{smallmatrix}\right) \cdot k) = |aa'/dd'|^{1/2} \chi(aa') \psi(dd') = |a'/d'|^{1/2} \chi(a') \psi(d') \varphi_0(h).$$

It turns out that this example is enough to determine every spherical representation (of dimension greater than 1).

Theorem 4.2.21. *[Bum97, Theorem 4.6.4]. Let (V, π) be a spherical representation of $\mathrm{GL}_2(E)$ of dimension greater than 1. Then π is a spherical principal series representation.*

Remark 4.2.22. More precisely, a spherical representation π will be isomorphic to $I_H(\chi, \psi)$, where χ, ψ are the unramified quasicharacters of $E^\times$ determined by

$$\chi(\varpi) = \alpha, \quad \psi(\varpi) = \beta,$$

and α, β are the roots of the polynomial $X^2 - q^{-1/2}\lambda X + \mu$, where λ, μ are the eigenvalues of $T(\mathfrak{p}), R(\mathfrak{p})$ on the one-dimensional space of spherical vectors of V. Indeed the Hecke algebra of locally constant compactly supported complex valued functions on $\mathrm{GL}_2(E)$ acts on (V, π) via the formula $\xi \cdot v = \int_{\mathrm{GL}_2(E)} \xi(g)(\pi(g)v) dg$ (see Definition 4.3.1). The action of the subalgebra of $\mathrm{GL}_2(\mathscr{O})$-biequivariant functions preserves the one-dimensional space $V^{\mathrm{GL}_2(\mathscr{O})}$. In particular we can consider the eigenvalues for the action on spherical vectors of the operators

$$T(\mathfrak{p}) := \mathrm{ch}(\mathrm{GL}_2(\mathscr{O}) \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathrm{GL}_2(\mathscr{O})), \quad R(\mathfrak{p}) := \mathrm{ch}(\mathrm{GL}_2(\mathscr{O}) \left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right) \mathrm{GL}_2(\mathscr{O})).$$

We now want to characterise the Whittaker model of Definition 4.2.13 for $\varphi_0 \in$

$I(\chi,\psi)$ as in the above example. First we let

$$\alpha := \chi(\varpi), \quad \beta := \psi(\varpi).$$

We have the following result, that will be extremely helpful later. We write $W_0 := W_{\varphi_0}$.

Theorem 4.2.23. *[Bum97, Theorem 4.6.5]. Let α,β as above. Then for any $y \in E^\times$, let $m := \mathrm{ord}(y)$. We have*

$$(1-q^{-1}\alpha\beta^{-1})^{-1}W_0\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} 0 & \text{if } m<0 \\ q^{-m/2}\cdot\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} & \text{if } m\geq 0 \end{cases}$$

We want to work with a Whittaker model $\mathscr{W}$ such that for $\mathscr{W}_{\varphi_0}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$ if $y \in \mathscr{O}^\times$.

Definition 4.2.24. The *normalised Whittaker model of $I(\chi,\psi)$* is defined by $(1-q^{-1}\alpha\beta^{-1})^{-1}\cdot W$, for α,β as above.

4.2.5 Siegel sections

This section contains exactly the same results and definitions of [LSZ20a, §3.2]. We report them for the sake of completeness and refer to *loc. cit.* for the proofs.

Definition 4.2.25. Let $\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})$ be the space of Schwartz functions on $\mathbb{Q}_\ell^2$. For $\phi \in \mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})$, we write $\hat{\phi}$ for its Fourier transform, i.e.

$$\hat{\phi}(x,y) = \int_{\mathbb{Q}_\ell}\int_{\mathbb{Q}_\ell} e_\ell(xv-yu)\phi(u,v)du\,dv,$$

where e_ℓ is the standard additive character on $F=\mathbb{Q}_\ell$, mapping ℓ^{-n} to $\exp(2\pi i/\ell^n)$.

In the first part of [LSZ20a, Proposition 3.2.2], the authors define a map from $\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})$ to $I_H(\chi,\psi)$ for χ,ψ characters of $\mathbb{Q}_\ell^\times$ using explicit integrals. With the same notation we write

$$\begin{aligned}\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C}) &\to I_H(\chi,\psi)\\ \phi &\mapsto f_{\phi,\chi,\psi}.\end{aligned}$$

Proposition 4.2.26. *The above mentioned map satisfies*

$$f_{g\cdot\phi,\chi,\psi}(h)=\chi(\det g)^{-1}|\det g|^{-1/2}f_{\phi,\chi,\psi}(hg),$$

$$f_{\widehat{g\cdot\phi},\chi,\psi}(h)=\psi(\det g)^{-1}|\det g|^{-1/2}f_{\hat{\phi},\chi,\psi}(hg).$$

In particular if $\psi=|\cdot|^{-1/2}$ *and* χ *is unramified, then the map*

$$\begin{aligned}\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C}) &\to I_H(\chi,\psi)\\ \phi &\mapsto F_{\phi,\chi,\psi}:=f_{\hat{\phi},\chi,\psi}\end{aligned}$$

is $H(\mathbb{Q}_\ell)$*-equivariant.*

Proposition 4.2.27. *With notation as above, we have*

$$M(f_{\phi,\chi,\psi})=\frac{\varepsilon(\psi/\chi)}{L(\chi\psi^{-1},1)}\cdot f_{\hat{\phi},\psi,\chi},$$

where $\varepsilon(\psi/\chi)$ *is the local* ε*-factor (equal to 1 if* ψ/χ *is unramified).*

We now define some special Schwartz function that will be useful later.

Definition 4.2.28. For integers $t\geq 0$ we define $\phi_t\in\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})$ as follows

- for $t=0$, $\phi_0:=\mathrm{ch}(\mathbb{Z}_\ell)\,\mathrm{ch}(\mathbb{Z}_\ell)$,
- for $t>0$, $\phi_t:=\mathrm{ch}(\ell^t\mathbb{Z}_\ell)\,\mathrm{ch}(\mathbb{Z}_\ell^\times)$.

This functions are preserved by the action of

$$K_{H,0}(\ell^t):=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in H(\mathbb{Z}_\ell):c\equiv 0 \bmod \ell^t\}.$$

Lemma 4.2.29. *Let χ, ψ be unramified characters. The function $f_{\phi_t,\chi,\psi}$ is supported on $B(\mathbb{Q}_\ell)K_{H,0}(\ell^t)$ and*

$$f_{\phi_t,\chi,\psi}(1) = \begin{cases} 1 & \text{if } t = 0 \\ L(\chi\psi^{-1}, 1)^{-1} & \text{if } t \geq 1. \end{cases}$$

Definition 4.2.30. For integers $t \geq 1$ we define $\phi_{1,t} \in \mathscr{S}(\mathbb{Q}_\ell^2, \mathbb{C})$ to be $\mathrm{ch}(\ell^t\mathbb{Z}_\ell)\,\mathrm{ch}(1+\ell^t\mathbb{Z}_\ell)$. This function is preserved by the action of

$$K_{H,1}(\ell^t) := \{\gamma \in H(\mathbb{Z}_\ell) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod \ell^t\}.$$

4.3 Zeta integrals for $G(\mathbb{Q}_\ell) = \mathrm{GL}_2(F_\ell)$ (inert prime case)

Let E be an unramified quadratic extension of $\mathbb{Q}_\ell$. We will work with the representation $\sigma = I_G(\chi, \psi)$ of $G = \mathrm{GL}_2(E)$. We denote by K the subgroup $\mathrm{GL}_2(\mathscr{O})$, where $\mathscr{O}$ is the ring of integers of E.

4.3.1 Action of the Hecke algebra on Whittaker model

First we recall the definition of Hecke algebra acting on σ.

Definition 4.3.1. We denote by $\mathscr{H}(G)$ the Hecke algebra of locally constant compactly supported $\mathbb{C}$-valued functions on $G = \mathrm{GL}_2(E)$. It is an algebra under convolution, defined by

$$\phi_1 \star \phi_2(g) := \int_G \phi_1(gh^{-1})\phi_2(h)dh,$$

for $\phi_1, \phi_2 \in \mathscr{H}(G)$. Moreover we regard σ as left $\mathscr{H}(G)$- module via

$$\phi \cdot f = \int_G \phi(g)(g \cdot f)dg.$$

Lemma 4.3.2. *We have*

$$g_1 \cdot (\phi \cdot (g_2 \cdot f)) = \phi(g_1^{-1}(-)g_2^{-1}) \cdot f.$$

Example 4.3.3 (The operator $U(\ell)$). We define $U(\ell) \in \mathscr{H}(G)$ to be

$$U(\ell) := \tfrac{1}{\mathrm{Vol}(K')} \mathrm{ch}(K' \cdot \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot K'),$$

where K' is any subgroup of K contained in $\{\gamma \in K : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod \ell\}$ and containing the subgroup of unipotent matrices. We can write $K' \cdot \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot K'$ as union of left cosets

$$(K' \cdot \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot K') = \bigsqcup_{\gamma \in J} \gamma \cdot \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot K'$$

where J is a set of representatives for the left quotient $(\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) K' \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \cap K') \backslash K'$. We claim that we can take $J = \{\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\}_{u \in (\mathscr{O}/\ell\mathscr{O})}$. Indeed the subgroup for which we are taking the quotient is the subgroup K'' of matrices of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K'$ such that $b \equiv 0 \bmod \ell$. The matrices considered are clearly in distinct cosets and since for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K'$, $d \not\equiv 0 \bmod \ell$ we can choose $u \in \mathscr{O}/\ell\mathscr{O}$ such that $b \equiv ud \bmod \ell$. In other words

$$\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K''.$$

Hence we can rewrite

$$(K' \cdot \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot K') = \bigsqcup_{u \in (\mathscr{O}/\ell\mathscr{O})} \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot K' = \bigsqcup_{u \in (\mathscr{O}/\ell\mathscr{O})} \left(\begin{smallmatrix} \ell & u \\ 0 & 1 \end{smallmatrix}\right) \cdot K'.$$

We will need to define an appropriate additive character of E and then work with the normalised Whittaker model for σ as in Definition 4.2.24. Consider e_ℓ the standard additive character on $\mathbb{Q}_\ell$. Let us fix $\delta \in \mathscr{O}_E$ such that $E = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell(\delta)$ and such that the trace of δ is zero. We define an additive character Ψ on E letting

$$\Psi : x \to e_\ell(\mathrm{Tr}_{E/\mathbb{Q}_\ell}(\delta^{-1}x)).$$

We can assume $v(\delta) = 0$ since $E/\mathbb{Q}_\ell$ is unramified. This character has conductor $\mathscr{O}_E$ (see for example [RV99, Exercise 3(e), Chapter 7]).

We describe how the action of the operator $U(\ell)$ of Example 4.3.3 modifies the Whittaker model.

Proposition 4.3.4. *Let $\varphi \in \sigma$ a spherical vector. Then for any $y \in E^\times$, we have*

$$\mathscr{W}_{U(\ell)\cdot\varphi}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \begin{cases} 0 & \text{if } v(y) < 0 \\ \ell^2 \mathscr{W}_\varphi\left(\left(\begin{smallmatrix} \ell y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) & \text{if } v(y) \geq 0. \end{cases}$$

Proof. We prove the result for W as in Definition 4.2.15. We can also assume $\phi = \phi_0$ the normalised spherical vector. By definition

$$W_{U(\ell)\cdot\varphi_0}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \mu\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\cdot(U(\ell)\cdot\varphi_0)\right) = \sum_{u\in(\mathscr{O}_E/\ell\mathscr{O}_E)} \mu\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell & u \\ 0 & 1 \end{smallmatrix}\right)\cdot\varphi_0\right),$$

where in the second equality we used the decomposition of $U(\ell)$ as in Example 4.3.3 and the fact that φ_0 is K-invariant. Now we write

$$\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell & u \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & yu \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y\ell & 0 \\ 0 & 1 \end{smallmatrix}\right) = m_{yu}\cdot\left(\begin{smallmatrix} y\ell & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

So we find

$$W_{U(\ell)\cdot\varphi_0}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \sum_{u\in(\mathscr{O}_E/\ell\mathscr{O}_E)} \Psi(yu) W_{\varphi_0}\left(\left(\begin{smallmatrix} \ell y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).$$

If $v(\ell y) < 0$, i.e. $v(y) < -1$, applying Theorem 4.2.23, we find that the above quantity is zero. If $v(\ell y) = 0$, i.e. $v(y) = -1$, the sum is equal to

$$\begin{aligned} \sum_{u\in(\mathscr{O}_E/\ell\mathscr{O}_E)} \Psi(yu) &= \sum_{0\leq i,j\leq \ell-1} \Psi(y(i+\delta j)) = \sum_{0\leq i,j\leq\ell-1} \Psi(iy)\Psi(\delta jy)) \\ &= \sum_{0\leq i,j\leq\ell-1} e_\ell(\mathrm{Tr}(\delta^{-1}y))^i e_\ell(\mathrm{Tr}(y))^j. \end{aligned}$$

Having assumed that $v(\delta) = 0$ and having $v(y) = -1$, we have that at least one of the two terms $e_\ell(\mathrm{Tr}(\delta^{-1}y)), e_\ell(\mathrm{Tr}(y))$ is equal to $\zeta_\ell = \exp^{2\pi i/\ell}$. Assume for example

$e_\ell(\mathrm{Tr}(\delta^{-1}y)) = \zeta_\ell$, we can rewrite the sum as

$$\sum_{0\le j\le \ell-1} e_\ell(\mathrm{Tr}(y))^j \cdot \left(\sum_{0\le i\le \ell-1} \zeta_\ell^i\right) = 0.$$

Finally, if $v(y\ell) > 0$, i.e. $y \in \mathscr{O}_F$, $\Psi(yu) = 1$ and hence

$$W_{U(\ell)\cdot\varphi_0}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \ell^2 W_{\varphi_0}\left(\left(\begin{smallmatrix} \ell y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).$$

Hence the result. $\square$

4.3.2 Zeta integrals

As above fix the irreducible spherical principal series representation $\sigma = I_G(\chi,\psi)$, for χ,ψ quasicharacters of $E^\times$. Let

$$\alpha := \chi(\ell), \quad \beta = \psi(\ell)$$

and let χ_σ the central character of σ, i.e. $\chi_\sigma = \chi\psi$. We define the *local Asai L-factor*[1] of σ to be

$$L(\mathrm{As}(\sigma),s) := [(1-\alpha\ell^{-s})(1-\beta\ell^{-s})(1-\alpha\beta\ell^{-2s})]^{-1},$$

Moreover if η is an unramified character of $\mathbb{Q}_\ell^\times$, we let

$$L(\mathrm{As}(\sigma\otimes\eta),s) = [(1-\alpha\eta(\ell)\ell^{-s})(1-\beta\eta(\ell)\ell^{-s})(1-\alpha\beta\eta(\ell)^2\ell^{-2s})]^{-1}.$$

Definition 4.3.5. Let σ as above and η an unramified character of $\mathbb{Q}_\ell^\times$. For every $f \in \sigma$, we define

$$Z(\sigma,\eta,f,s) := L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}\int_{\mathbb{Q}_\ell^\times} |y|^{s-1}\eta(y)\mathscr{W}_f\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)d^\times y.$$

[1]The standard L-factor of σ is $[(1-\alpha\ell^{-2s})(1-\beta\ell^{-2s})]^{-1}$ and can be obtained by the same integral we consider here, but integrating over $y \in E^\times$ with norm and measure on E rather than on $\mathbb{Q}_\ell$. It will be clear later the reason of the name Asai L-factor.

The following three lemmas will be very useful.

Lemma 4.3.6 (Zeta integral at the spherical vector)**.** *There exist $r(\sigma,\eta) \in \mathbb{R}$ such that for ever $f \in \sigma$ and $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > r(\sigma,\eta)$, the above integral is absolutely convergent and, as function of s, lies in $\mathbb{C}[\ell^s, \ell^{-s}]$; in particular it has analytic continuation for all $s \in \mathbb{C}$. Moreover, if φ_0 is the normalised spherical vector as above, we have*

$$Z(\sigma,\eta,\varphi_0,s) = L(\eta^2\chi_\sigma, 2s)^{-1}.$$

Proof. It is enough to check convergence and analytic continuation for $f = g \cdot \varphi_0$, where $g \in G$. The validity of these statements for such f depends only on the class of g in $N\backslash G/G(\mathscr{O}_E)$. Since representatives of this quotient are elements of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$, Lemma 4.2.17 implies that it suffices to look at the integral for $f = \varphi_0$.

Applying Theorem 4.2.23 (notice that in our case $q = \ell^2$), we find

$$\begin{aligned}\int_{\mathbb{Q}_\ell^\times} |y|^{s-1}\eta(y)\mathscr{W}_{\varphi_0}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) d^\times y &= \sum_{m\geq 0} (\ell^{s-1})^{-m}\ell^{-m}\eta(\ell)^m \cdot \frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} \\ &= \sum_{m\geq 0} X^m \cdot \frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta},\end{aligned}$$

where $X = \ell^{-s}\eta(\ell)$. We can manipulate the latter series and obtain

$$\begin{aligned}\sum_{m\geq 0} X^m \cdot \frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} &= \frac{1}{\alpha-\beta}\sum_{m\geq 0} (\alpha\cdot(X\alpha)^m - \beta(X\beta)^m) \\ &= \frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha X} - \frac{\beta}{1-\beta X}\right) = \frac{1}{(1-\alpha X)(1-\beta X)}.\end{aligned}$$

The series converges for $|\alpha X|_{\mathbb{C}} < 1, |\beta X|_{\mathbb{C}} < 1$, that is for $\mathrm{Re}(s) > r(\sigma,\eta)$, for some real number depending on σ and η. Substituting $X = \ell^{-s}\eta(\ell)$, for s in this region, we find

$$\begin{aligned}\int_{\mathbb{Q}_\ell^\times} |y|^{s-1}\eta(y)\mathscr{W}_{\varphi_0}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) d^\times y &= [(1-\alpha\eta(\ell)\ell^{-s})(1-\beta\eta(\ell)\ell^{-s})]^{-1} \\ &= L(\mathrm{As}(\sigma\otimes\eta),s)\cdot(1-\alpha\beta\eta(\ell)^2\ell^{-2s})\end{aligned}$$

and $(1-\alpha\beta\eta(\ell)^2\ell^{-2s}) = L(\chi_\sigma\eta^2, 2s)^{-1}$. $\square$

Lemma 4.3.7 (Action of $U(\ell)$ on the zeta integral)**.** *If φ_0 is the normalised spherical vector as above, we have*

$$\begin{aligned} Z(\sigma,\eta,U(\ell)\varphi_0,s) &= \tfrac{\ell^{s+1}}{\eta(\ell)}[Z(\sigma,\eta,\varphi_0,s) - L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}] \\ &= \tfrac{\ell^{s+1}}{\eta(\ell)}[L(\eta^2\chi_\sigma,2s)^{-1} - L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}] \end{aligned}$$

Proof. First we apply Proposition 4.3.4 and find

$$\begin{aligned} Z(\sigma,\eta,U(\ell)\varphi_0,s) &= \ell^2 L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}\int_{|y|<\ell}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}\ell y & 0\\ 0 & 1\end{smallmatrix}\right)\right)d^\times y \\ &= \ell^{s+1}\eta(\ell)^{-1}L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}\int_{|y|<1}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)\right)d^\times y, \end{aligned}$$

where in the second equality we used the change of variables $y \rightsquigarrow \ell y$. We then rewrite the integral in the last term as

$$\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)\right)d^\times y - \int_{|y|\geq 1}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)\right)d^\times y.$$

Then we apply Theorem 4.2.23 and obtain

$$\int_{|y|\geq 1}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)\right)d^\times y = \int_{\mathbb{Z}_\ell^\times}\mathscr{W}_0\left(\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)\right)d^\times y = \int_{\mathbb{Z}_\ell^\times}d^\times y = 1.$$

Putting everything together we find $Z(\sigma,\eta,U(\ell)\varphi_0,s) =$

$$\begin{aligned} &\ell^{s+1}\eta(\ell)^{-1}L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}\left(\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)\right)d^\times y - 1\right) \\ &= \ell^{s+1}\eta(\ell)^{-1}(Z(\sigma,\eta,\varphi_0,s) - L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}). \end{aligned}$$

$\square$

Lemma 4.3.8 (Action of the Borel subgroup of $\mathrm{GL}_2(\mathbb{Q}_\ell)$)**.** *For any $f \in \sigma$, $a,d \in \mathbb{Q}_\ell^\times$, we have*

$$Z(\sigma,\eta,\left(\begin{smallmatrix}a & *\\ 0 & d\end{smallmatrix}\right)\cdot f,s) = \left|\tfrac{d}{a}\right|^{s-1}\chi_\sigma(d)\eta(a^{-1}d)\cdot Z(\sigma,\eta,f,s)$$

Proof. We apply Lemma 4.2.17 together with the fact that, for our choice of Ψ, we have $\Psi(x) = 1$ for every $x \in \mathbb{Q}_\ell$. We find

$$\begin{aligned} Z(\sigma,\eta,\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right)\cdot f,s) &= \chi_\sigma(d)L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\eta(y)\mathscr{W}_f\left(\left(\begin{smallmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)d^\times y \\ &= \chi_\sigma(d)|d/a|^{s-1}\eta(a^{-1}d)L(\mathrm{As}(\sigma\otimes\eta),s)^{-1}\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\eta(y)\mathscr{W}_f\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)d^\times y \\ &= \chi_\sigma(d)|d/a|^{s-1}\eta(a^{-1}d)\cdot Z(\sigma,\eta,f,s) \end{aligned}$$

where in the second equality we used the change of variable $y \rightsquigarrow d^{-1}ay$. □

4.4 Zeta integrals for $G(\mathbb{Q}_\ell) = \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell)$ (split prime case)

4.4.1 Whittaker models for $G = \mathrm{GL}_2 \times \mathrm{GL}_2$

Let $\chi_1, \psi_1, \chi_2, \psi_2$ be quasicharacters of $\mathbb{Q}_\ell^\times$. We now consider a representation of $G = \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell)$.

Definition 4.4.1. For $\chi_1, \psi_1, \chi_2, \psi_2$ as above, let

$$I_G(\underline{\chi},\underline{\psi}) := I_H(\chi_1,\psi_1)\otimes I_H(\chi_2,\psi_2),$$

i.e. $f \in I_G(\underline{\chi},\underline{\psi})$ is $f : G \to \mathbb{C}$ such that

$$f((\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right),\left(\begin{smallmatrix} d' & * \\ 0 & d' \end{smallmatrix}\right))\cdot g) = \left|\tfrac{a}{d}\right|^{1/2}\left|\tfrac{a'}{d'}\right|^{1/2}\chi_1(a)\psi_1(d)\chi_2(a')\psi_2(d')f(g).$$

We see $I_G(\underline{\chi},\underline{\psi})$ as a G-representation letting G act by right translation.

Definition 4.4.2. The G-representations $I_G(\underline{\chi},\underline{\psi})$, for $\chi_1,\chi_2,\psi_1,\psi_2$ quasicharacters of $\mathbb{Q}_\ell^\times$ are called *principal series representations for G*.

We need to define what is a Whittaker functional for a representation V of G, having fixed an additive character Ψ.

Definition 4.4.3. A *Whittaker functional* on V is a linear functional $\lambda : V \to \mathbb{C}$

satisfying

$$\lambda(m_{x,x'} \cdot v) = \Psi(x - x')\lambda(v),$$

for every $x, x' \in \mathbb{Q}_\ell, v \in V$, where $m_{x,x'} = (m_x, m_{x'}) = (\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & x' \\ 0 & 1 \end{smallmatrix}\right))$.

We now define a Whittaker model for $\sigma = I_G(\underline{\chi}, \underline{\psi})$, for $\underline{\chi} = (\chi_1, \chi_2), \underline{\psi} = (\psi_1, \psi_2)$. We will be using Whittaker models for $I_H(\chi_1, \psi_1)$ and $I_H(\chi_2, \psi_2)$ as constructed above. Recall that everything depends on the choice of the additive character. We will consider the functionals as in Definition 4.2.13, but with different choices of the additive character. Fix such an additive character Ψ for which we want to obtain a Whittaker functional for σ. We then let $\Psi_1 = \Psi$ and $\Psi_2 = \Psi(-(\cdot))$. And write $\mu_i : I(\chi_i, \psi_i) \to \mathbb{C}$ where

$$\mu_1(f_1) = \int_F f_1(w \cdot m_x)\Psi_1(-x)dx,$$

$$\mu_2(f_2) = \int_F f_2(w \cdot m_x)\Psi_2(-x)dx.$$

And finally let $\mu : \sigma \to \mathbb{C}$ to be defined by

$$\mu(f_1, f_2) = \mu_1(f_1) \cdot \mu_2(f_2).$$

It is straightforward to see that it is a Whittaker functional for σ.

Definition 4.4.4. We let W be the Whittaker model for σ defined by

$$W : f \mapsto \big(W_f : g = (g_1, g_2) \mapsto \mu(g \cdot f)\big)$$

From the definition we have, for $f = f_1 \otimes f_2$,

$$\begin{aligned} W_f(g_1, g_2) &= \mu_1(g_1 \cdot f_1) \cdot \mu(g_2 \cdot f_2) \\ &= \left(\int_F f_1(w \cdot m_x \cdot g_1)\Psi_1(-x)dx\right) \cdot \left(\int_F f_2(w \cdot m_x \cdot g_2)\Psi_2(-x)dx\right) \\ &= W_{1,f_1}(g_1) \cdot W_{2,f_2}(g_2), \end{aligned}$$

where W_{1,f_1}, W_{2,f_2} are the Whittaker models for $I_H(\chi_1, \psi_1)$ and $I_H(\chi_2, \psi_2)$ obtained from the functionals μ_1, μ_2.

Lemma 4.4.5. *For the Whittaker model of σ we have*

$$W_{\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right) \cdot f}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \chi_\sigma(d) \cdot W_f\left(\left(\begin{smallmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{smallmatrix}\right)\right),$$

where $\chi_\sigma = \chi_1 \psi_1 \chi_2 \psi_2$ will be called the central character of σ.

Proof. This is straightforward from Lemma 4.2.17. Indeed, by definition, the left hand side term is equal to

$$W_{1, \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \cdot f_1}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \cdot W_{2, \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \cdot f_2}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$$

and applying the lemma, this is equal to

$$\Psi(d^{-1}by) \cdot \Psi(-d^{-1}by) \cdot \chi_1 \psi_1(d) \cdot \chi_2 \psi_2(d) \cdot W_{1,f_1}\left(\left(\begin{smallmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \cdot W_{2,f_2}\left(\left(\begin{smallmatrix} d^{-1}ay & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).$$

□

Definition 4.4.6. The *normalised Whittaker model for σ* is defined by

$$\mathscr{W} = (1 - \ell^{-1}\alpha\beta^{-1})^{-1}(1 - \ell^{-1}\gamma\delta^{-1})^{-1} \cdot W$$

where $\alpha = \chi_1(\ell), \beta = \psi_1(\ell), \gamma = \chi_2(\ell), \delta = \psi_2(\ell)$.

The definition and properties of spherical representations of H carry over to representations of G, using the subgroup $\mathrm{GL}_2(\mathbb{Z}_\ell) \times \mathrm{GL}_2(\mathbb{Z}_\ell)$. In particular we define the *normalised spherical vector of σ* to be

$$\varphi_0 = \varphi_{1,0} \otimes \varphi_{2,0},$$

where $\varphi_{i,0}$ is the normalised spherical vector for $I_H(\chi_i, \psi_i)$ as in Example 4.2.20. Let then

$$\mathscr{W}_0 := \mathscr{W}_{\varphi_0}.$$

Theorem 4.4.7. *Let, as above, $\alpha = \chi_1(\ell), \beta = \psi_1(\ell), \gamma = \chi_2(\ell), \delta = \psi_2(\ell)$. Then for any $y \in \mathbb{Q}_\ell^\times$, let $m := \mathrm{ord}(y)$. We have*

$$\mathscr{W}_0\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \begin{cases} 0 & \text{if } m < 0 \\ \ell^{-m} \cdot \frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} \cdot \frac{\gamma^{m+1}-\delta^{m+1}}{\gamma-\delta} & \text{if } m \geq 0 \end{cases}$$

Proof. This is a corollary of Theorem 4.2.23 □

4.4.2 Action of the Hecke algebra on Whittaker model

We will now recall the definition of the Hecke algebra acting on $\sigma = I_G(\underline{\chi}, \underline{\psi})$.

Definition 4.4.8. We denote by $\mathscr{H}(G)$ the Hecke algebra of locally constant compactly supported $\mathbb{C}$-valued functions on G. It is an algebra under convolution, defined by

$$\phi_1 \star \phi_2(g) := \int_G \phi_1(gh^{-1})\phi_2(h)dh,$$

for $\phi_1, \phi_2 \in \mathscr{H}(G)$. Moreover we regard σ as left $\mathscr{H}(G)$- module via

$$\phi \cdot f = \int_G \phi(g)(g \cdot f)dg.$$

Lemma 4.4.9. *We have*

$$g_1 \cdot (\phi \cdot (g_2 \cdot f)) = \phi(g_1^{-1}(-)g_2^{-1}) \cdot f.$$

Example 4.4.10 (The operator $U(\ell)$)**.** We define $U(\ell) \in \mathscr{H}(G)$ to be, essentially $(U(\ell), U(\ell))$, i.e. the usual $U(\ell)$ operator on each of the $\mathrm{GL}_2(\mathbb{Q}_\ell)$. More precisely

$$U(\ell) := \tfrac{1}{\mathrm{Vol}(K')} \mathrm{ch}(K' \cdot \left(\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \cdot K')$$

for $K' = K_1' \times K_2'$ subgroup of $\mathrm{GL}_2(\mathbb{Z}_\ell) \times \mathrm{GL}_2(\mathbb{Z}_\ell)$, with $K_1', K_2' \subset \{\gamma \in K : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ mod $\ell\}$ containing the subgroup of unipotent matrices. Proceeding as in Example

4.3.3, we can rewrite

$$\begin{aligned} K' \cdot \left(\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) \cdot K' &= \bigsqcup_{0 \le u,v \le \ell-1} \left(\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right) \right) \cdot \left(\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) \cdot K' \\ &= \bigsqcup_{0 \le u,v \le \ell-1} \left(\left(\begin{smallmatrix} \ell & u \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \ell & v \\ 0 & 1 \end{smallmatrix}\right) \right) \cdot K'. \end{aligned}$$

From now on we take $\Psi = e_\ell$, the standard additive character of $\mathbb{Q}_\ell$, i.e. the one mapping ℓ^{-n} to $\exp(2\pi i/\ell^n)$.

We describe how the action of the operator $U(\ell)$ of Example 4.4.10 modifies the Whittaker model.

Proposition 4.4.11. *Let $\varphi \in \sigma$ a spherical vector. Then for any $y \in \mathbb{Q}_\ell^\times$, we have*

$$\mathscr{W}_{U(\ell)\cdot\varphi} \left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) = \begin{cases} 0 & \text{if } |y| \ge \ell \\ \ell^2 \mathscr{W}_\varphi \left(\begin{pmatrix} \ell y & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \ell y & 0 \\ 0 & 1 \end{pmatrix} \right) & \text{if } |y| < \ell. \end{cases}$$

Proof. We prove the result for the Whittacker model W. By definition

$$\begin{aligned} W_{U(\ell)\cdot\varphi} \left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) &= \mu \left(\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) \cdot (U(\ell)\cdot\varphi) \right) \\ &= \sum_{0 \le u,v \le \ell-1} \mu \left(\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \ell & u \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \ell & v \\ 0 & 1 \end{smallmatrix}\right) \right) \cdot \varphi \right), \end{aligned}$$

where in the second equality we used the decomposition of $U(\ell)$ as in Example 4.4.10 and the fact that φ is K-invariant. Now we write

$$\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \ell & u \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \ell & v \\ 0 & 1 \end{smallmatrix}\right) \right) = \left(\left(\begin{smallmatrix} 1 & yu \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} y\ell & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & yv \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} y\ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) = m_{yu,yv} \cdot \left(\left(\begin{smallmatrix} y\ell & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y\ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \right).$$

So we find

$$W_{U(\ell)\cdot\varphi} \left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) = \sum_{0 \le u,v \le \ell-1} \Psi(yu)\Psi(-yv) W_\varphi \left(\left(\begin{smallmatrix} \ell y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \ell y & 0 \\ 0 & 1 \end{smallmatrix}\right) \right).$$

If $|y\ell| > 1$, applying Theorem 4.4.7, we find that the above quantity is zero. Similarly if $|y\ell| = 1$,i.e. $e_\ell(y) = \zeta_\ell := e^{2\pi i/\ell}$, the sum is equal to

$$c \cdot \sum_{0 \le u,v \le \ell-1} e_\ell(yu) e_\ell(-yv) = c \cdot \sum_{0 \le u,v \le \ell-1} \zeta_\ell^u \zeta_\ell^{-v} = 0,$$

where $\varphi = c \cdot \varphi_0$. Finally, if $|y\ell| < 1$, $e_\ell(yu) = e_\ell(-yv) = 1$ for every u, v and hence

$$W_{U(\ell)\cdot\varphi}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \ell^2 W_\varphi\left(\left(\begin{smallmatrix} \ell y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \ell y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).$$

Hence the result. □

4.4.3 Zeta integrals

We fix the quasicharacters $\chi_1, \psi_1, \chi_2, \psi_2$ such that $\chi_1\psi_1^{-1} \neq |\cdot|^{\pm 1}, \chi_2\psi_2^{-1} \neq |\cdot|^{\pm 1}$ and then fix the irreducible spherical principal series representation $\sigma = I_G(\underline{\chi}, \underline{\psi})$ as above. We define the *local L-factor* of σ to be

$$L(\sigma, s) := [(1-\alpha\gamma\ell^{-s})(1-\alpha\delta\ell^{-s})(1-\beta\gamma\ell^{-s})(1-\beta\delta\ell^{-s})]^{-1},$$

where, as above, $\alpha = \chi_1(\ell), \beta = \psi_1(\ell), \gamma = \chi_2(\ell), \delta = \psi_2(\ell)$. Moreover if η is an unramified character of $\mathbb{Q}_\ell^\times$, we define

$$L(\sigma \otimes \eta, s) = [(1-\alpha\gamma\eta(\ell)\ell^{-s})(1-\alpha\delta\eta(\ell)\ell^{-s})(1-\beta\gamma\eta(\ell)\ell^{-s})(1-\beta\delta\eta(\ell)\ell^{-s})]^{-1}.$$

Definition 4.4.12. Let σ as above and η an unramified character of $\mathbb{Q}_\ell^\times$. For every $f \in \sigma$, we define

$$Z(\sigma, \eta, f, s) := L(\sigma \otimes \eta, s)^{-1} \int_{\mathbb{Q}_\ell^\times} |y|^{s-1} \eta(y) \mathscr{W}_f\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) d^\times y.$$

The following three useful lemmas are the analogues of Lemmas 4.3.6, 4.3.7, 4.3.8 of the previous section.

Lemma 4.4.13 (Zeta integral at the spherical vector)**.** *There exist $r(\sigma, \eta) \in \mathbb{R}$ such that for ever $f \in \sigma$ and $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > r(\sigma, \eta)$, the above integral is absolutely convergent and, as function of s, lies in $\mathbb{C}[\ell^s, \ell^{-s}]$; in particular it has* *analytic continuation for all $s \in \mathbb{C}$. Moreover, if φ_0 is the normalised spherical vector as above, we have*

$$Z(\sigma, \eta, \varphi_0, s) = L(\eta^2\chi_\sigma, 2s)^{-1}.$$

Proof. In order to prove the first statements, we reduce to compute the integral for $f = \varphi_0$, arguing as in the proof of Lemma 4.3.6. Applying Theorem 4.4.7, we find

$$\begin{aligned}
&\int_{\mathbb{Q}_\ell^\times} |y|^{s-1}\eta(y)\mathscr{W}_{\varphi_0}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) d^\times y \\
&\quad = \sum_{m \geq 0} (\ell^{s-1})^{-m}\ell^{-m}\eta(\ell)^m \cdot \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} \cdot \frac{\gamma^{m+1} - \delta^{m+1}}{\gamma - \delta} \\
&\quad = \sum_{m \geq 0} X^m \cdot \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} \cdot \frac{\gamma^{m+1} - \delta^{m+1}}{\gamma - \delta},
\end{aligned}$$

where $X = \ell^{-s}\eta(\ell)$. We can manipulate the latter series and obtain

$$\begin{aligned}
&\sum_{m \geq 0} X^m \cdot \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} \cdot \frac{\gamma^{m+1} - \delta^{m+1}}{\gamma - \delta} \\
&\quad = \frac{1}{(\alpha - \beta)(\gamma - \delta)}\left(\frac{\alpha\gamma}{1 - \alpha\gamma X} - \frac{\alpha\delta}{1 - \alpha\delta X} - \frac{\beta\gamma}{1 - \beta\gamma X} + \frac{\beta\delta}{1 - \beta\delta X}\right) \\
&\quad = \frac{1}{\alpha - \beta}\left(\frac{\alpha + \alpha\beta^2\gamma\delta X^2 - \beta - \alpha^2\beta\gamma\delta X^2}{(1 - \alpha\gamma X)(1 - \alpha\delta X)(1 - \beta\gamma X)(1 - \beta\delta X)}\right) \\
&\quad = \frac{1 - \alpha\beta\gamma\delta X^2}{(1 - \alpha\gamma X)(1 - \alpha\delta X)(1 - \beta\gamma X)(1 - \beta\delta X)}.
\end{aligned}$$

This is a standard computation, see for example Jacquet's refreshing exercise [Jac72, Lemma 15.9.4]. We have conditions on the convergence giving the condition $\mathrm{Re}(s) > r(\sigma, \eta)$. Substituting $X = \ell^{-s}\eta(\ell)$, we find

$$\begin{aligned}
&\int_{\mathbb{Q}_\ell^\times} |y|^{s-1}\eta(y)\mathscr{W}_{\varphi_0}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) d^\times y \\
&\quad = (1 - \chi_\sigma(\ell)\eta^2(\ell)\ell^{-2s})L(\sigma \otimes \eta, s) = L(\eta^2\chi_\sigma, 2s)^{-1}L(\sigma \otimes \eta, s).
\end{aligned}$$

□

Lemma 4.4.14 (Action of $U(\ell)$ on the zeta integral)**.** *If φ_0 is the normalised spher-*

ical vector as above, we have

$$\begin{aligned} Z(\sigma,\eta,U(\ell)\varphi_0,s) &= \tfrac{\ell^{s+1}}{\eta(\ell)}[Z(\sigma,\eta,\varphi_0,s) - L(\sigma\otimes\eta,s)^{-1}] \\ &= \tfrac{\ell^{s+1}}{\eta(\ell)}[L(\eta^2\chi_\sigma,2s)^{-1} - L(\sigma\otimes\eta,s)^{-1}] \end{aligned}$$

Proof. First we apply Proposition 4.4.11 and find

$$\begin{aligned} Z(\sigma,\eta,U(\ell)\varphi_0,s) &= \ell^2 L(\sigma\otimes\eta,s)^{-1}\int_{|y|<\ell}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}\ell y&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}\ell y&0\\0&1\end{smallmatrix}\right)\right)d^\times y \\ &= \ell^{s+1}\eta(\ell)^{-1}L(\sigma\otimes\eta,s)^{-1}\int_{|y|<1}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)\right)d^\times y, \end{aligned}$$

where in the second equality we used the change of variables $y \rightsquigarrow \ell y$. We then rewrite the integral in the last term as

$$\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)\right)d^\times y - \int_{|y|\geq 1}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)\right)d^\times y.$$

Then we apply Theorem 4.4.7 and obtain

$$\int_{|y|\geq 1}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)\right)d^\times y = \int_{\mathbb{Z}_\ell^\times}\mathscr{W}_0\left(\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)\right)d^\times y = \int_{\mathbb{Z}_\ell^\times}d^\times y = 1.$$

Putting everything together we find

$$\begin{aligned} &Z(\sigma,\eta,U(\ell)\varphi_0,s) \\ &\quad= \ell^{s+1}\eta(\ell)^{-1}L(\sigma\otimes\eta,s)^{-1}\left(\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\eta(y)\mathscr{W}_0\left(\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)\right)d^\times y - 1\right) \\ &\quad= \ell^{s+1}\eta(\ell)^{-1}(Z(\sigma,\eta,\varphi_0,s) - L(\sigma\otimes\eta,s)^{-1}). \end{aligned}$$

□

Lemma 4.4.15 (Action of the Borel subgroup of $\mathrm{GL}_2(\mathbb{Q}_\ell)$)**.** *For any $f \in \sigma$, $a,d \in \mathbb{Q}_\ell^\times$, we have*

$$Z(\sigma,\eta,\left(\left(\begin{smallmatrix}a&*\\0&d\end{smallmatrix}\right),\left(\begin{smallmatrix}a&*\\0&d\end{smallmatrix}\right)\right)\cdot f,s) = \left|\tfrac{d}{a}\right|^{s-1}\chi_\sigma(d)\eta(a^{-1}d)\cdot Z(\sigma,\eta,f,s)$$

Proof. We apply Lemma 4.4.5 and find

$$\begin{aligned}
&Z\left(\sigma,\eta,\left(\begin{pmatrix} a & * \\ 0 & d\end{pmatrix},\begin{pmatrix} a & * \\ 0 & d\end{pmatrix}\right)\cdot f,s\right)\\
&\qquad=\chi_\sigma(d)L(\sigma\otimes\eta,s)^{-1}\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\eta(y)\mathscr{W}_f\left(\begin{pmatrix} d^{-1}ay & 0 \\ 0 & 1\end{pmatrix},\begin{pmatrix} d^{-1}ay & 0 \\ 0 & 1\end{pmatrix}\right)d^\times y\\
&\qquad=\chi_\sigma(d)|d/a|^{s-1}\eta(a^{-1}d)L(\sigma\otimes\eta,s)^{-1}\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\eta(y)\mathscr{W}_f\left(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix},\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\right)d^\times y\\
&\qquad=\chi_\sigma(d)|d/a|^{s-1}\eta(a^{-1}d)\cdot Z(\sigma,\eta,f,s)
\end{aligned}$$

where in the second equality we used the change of variable $y\rightsquigarrow d^{-1}ay$. □

4.5 Towards norm relations

Let G be the algebraic group over $\mathbb{Q}$ defined in the introduction, i.e. $G=\mathrm{Res}_F^{\mathbb{Q}}\mathrm{GL}_2$, for F real quadratic field. We will now prove some results using the zeta integrals of the two previous sections. We will denote by σ an unramified irreducible principal series representation of $G(\mathbb{Q}_\ell)$, i.e. $\sigma=I_{\mathrm{GL}_2(\mathbb{Q}_\ell)\times\mathrm{GL}_2(\mathbb{Q}_\ell)}(\underline{\chi},\underline{\psi})$ if ℓ splits and $\sigma=I_{\mathrm{GL}_2(F_\ell)}(\tilde{\chi},\tilde{\psi})$ for F_ℓ the unramified quadratic extension of $\mathbb{Q}_\ell$ if ℓ is inert. We will denote by χ_σ the following characters: $\chi_\sigma=\chi_1\psi_1\chi_2\psi_2$ in the first case and $\chi_\sigma=\tilde{\chi}\tilde{\psi}$ in the second one. By abuse of notation, we will often write H for $H(\mathbb{Q}_\ell)=\mathrm{GL}_2(\mathbb{Q}_\ell)$ and denote by $L(\mathrm{As}(\sigma),s)$ both the local L-factor we considered in §4.3.2 and §4.4.3, i.e.

$$L(\mathrm{As}(\sigma),s)=\begin{cases} L(\sigma,s) & \text{if } \ell \text{ splits and } \sigma=I_{\mathrm{GL}_2(\mathbb{Q}_\ell)\times\mathrm{GL}_2(\mathbb{Q}_\ell)}(\underline{\chi},\underline{\psi})\\ L(\mathrm{As}(\sigma),s) & \text{if } \ell \text{ is inert and } \sigma=I_{\mathrm{GL}_2(F)}(\tilde{\chi},\tilde{\psi}).\end{cases}$$

We also let

$$\alpha_i=\chi_i(\ell),\beta_i=\psi_i(\ell)\quad\text{if } \ell \text{ splits,}$$

$$\alpha=\tilde{\chi}(\ell),\beta=\tilde{\psi}(\ell)\quad\text{if } \ell \text{ splits,}$$

4.5.1 Multiplicity one

We will fix σ as above and another pair of unramified characters χ, ψ satisfying

$$\chi\psi \cdot \chi_\sigma = 1.$$

We will moreover assume that $I_H(\chi, \psi)$ is either irreducible or it has an infinite dimensional subrepresentation, i.e. $\chi\psi^{-1} \neq |\cdot|^{-1}$. We will be considering the embedding

$$\iota : H(\mathbb{Q}_\ell) \hookrightarrow G(\mathbb{Q}_\ell).$$

In the split case $\iota(h) := (h,h) \in \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell)$, while in the inert case $\iota(h) := h \in \mathrm{GL}_2(\mathbb{Q}_\ell) \subset \mathrm{GL}_2(F)$.

Theorem 4.5.1 (Multiplicity one)**.** *Let σ, χ, ψ as above. We assume that*

$$(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}) \neq (\chi, \psi), (\psi, \chi) \ \text{ if } \ell \text{ is inert} \tag{$\star$}$$

We then have

$$\dim\left(\mathrm{Hom}_H(I_H(\chi, \psi) \otimes \sigma, \mathbb{C})\right) \leq 1.$$

Proof. If ℓ splits and $I_H(\chi, \psi)$ is irreducible, i.e. $\chi\psi^{-1} \neq |\cdot|$, this is Theorem 1.1 of [Pra90]. We apply it for $V_1 = I_H(\chi, \psi), V_2 = I_H(\chi_1, \psi_1), V_3 = I_H(\chi_2, \psi_2)$ (i.e. $V_2 \otimes V_3 = \sigma$). To deal with the case when $\chi\psi^{-1} = |\cdot|$, which was already treated in [HS01], we will make use of the exact sequence (4.2.1), following, as the authors of *op. cit.*, the strategy of [Pra90, Proof of Theorem 1.2 Case 2]. Firstly, recall that, by Theorem 4.2.9, the representation $I_H(\chi, \psi)$ has an irreducible codimension one subrepresentation, that we denote by π. The quotient is the one dimensional representation of H given by a character γ, where $\chi = |\cdot|^{1/2}\gamma, \psi = |\cdot|^{-1/2}\gamma$. We find an exact sequence

$$0 \to \mathrm{Hom}_H(V_2 \otimes \gamma, V_3^\vee) \to \mathrm{Hom}_H(V_2 \otimes I_H(\chi, \psi), V_3^\vee) \to \mathrm{Hom}_H(V_2 \otimes \pi, V_3^\vee).$$

The last term in the sequence is at most one dimensional, again by Theorem 1.1 of

[Pra90], applied for $V_1 = \pi$. The term in the middle is the one we are interested in and, by Theorem 4.2.9, the first one is zero if $V_2 \otimes \gamma$ is not isomorphic to $V_3^\vee$. In this case we obtain $\mathrm{Hom}_H(V_2 \otimes I_H(\chi, \psi), V_3^\vee)$ is one-dimensional, as desired. Let us now treat the case $V_2 \otimes \gamma \simeq V_3^\vee$. We write for simplicity $V = V_2 = I_H(\chi_1, \psi_1)$. Recall the exact sequence (4.2.1). Let $G = \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell)$, $J = B(\mathbb{Q}_\ell) \times B(\mathbb{Q}_\ell)$, $H = \mathrm{GL}_2(\mathbb{Q}_\ell)$ and τ is given by

$$\tau\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)\right) = \chi_1(a)\psi_1(d)\chi(a')\psi(d').$$

In this case $H_1 = B(\mathbb{Q}_\ell)$ and $H_2 = T = \{\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right), \lambda_i \in \mathbb{Q}_\ell^\times\}$ (the maximal split torus). To show that, one can take $\varepsilon = (\mathrm{Id}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$. Using the fact that T is unimodular and

$$\delta_{B\times B}\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)\right) = |\tfrac{a}{d}||\tfrac{a'}{d'}|,$$

$$\delta_B\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right) = |\tfrac{a}{d}|,$$

$$\delta_{B\times B}\left(\varepsilon(\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right))\varepsilon^{-1}\right) = 1,$$

we find the exact sequence of $\mathrm{GL}_2(\mathbb{Q}_\ell)$-modules

$$0 \to \text{c-Ind}_{H_2}^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \tilde{\tau} \to (V \otimes I_H(\chi, \psi))_{|H} \to I_H(\chi_1\gamma|\cdot|, \psi_1\gamma|\cdot|^{-1}) \to 0,$$

where $\tilde{\tau}(\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)) = \chi_1\psi(\lambda_1)\psi_1\chi(\lambda_2)$. Applying $\mathrm{Hom}_H(-, V_3^\vee)$ to the above exact sequence, we find

$$\begin{aligned} 0 &\to \mathrm{Hom}_H(I_H(\chi_1\gamma|\cdot|, \psi_1\gamma|\cdot|^{-1}), V_3^\vee) \to \mathrm{Hom}_H(V \otimes I_H(\chi, \psi), V_3^\vee) \\ &\to \mathrm{Hom}_H(\text{c-Ind}_{H_2}^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \tilde{\tau}, V_3^\vee) \to \mathrm{Ext}^1_H(I_H(\chi_1\gamma|\cdot|, \psi_1\gamma|\cdot|^{-1}), V_3^\vee)) \to \dots \end{aligned}$$

Since $V_3^\vee \simeq V \otimes \gamma = I_H(\chi_1\gamma, \psi_1\gamma)$, we have, from the second part of Theorem 4.2.9, that

$$\mathrm{Hom}_H(I_H(\chi_1\gamma|\cdot|, \psi_1\gamma|\cdot|^{-1}), V_3^\vee) \neq 0,$$

if and only if $\chi_1|\cdot| = \chi_1, \psi_1|\cdot|^{-1} = \psi_1$ or $\chi_1|\cdot| = \psi_1, \psi_1|\cdot|^{-1} = \chi_1$. The only possible case is the latter and it would imply $\chi_1\psi_1^{-1} = |\cdot|^{-1}$, contradicting the irreducibility of V_2. Hence, the first space in the sequence is zero which implies, by [Pra90, Corollary 5.9], that also the Ext^1 is zero. We hence find

$$\mathrm{Hom}_H(\sigma_{|H}, (I_H(\chi,\psi))^\vee) \simeq \mathrm{Hom}_H(\text{c-Ind}_{H_2}^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \tilde{\tau}, V_3^\vee) \simeq \mathrm{Hom}_{H_2}(\tilde{\tau}, ((V_3)_{|H_2})^\vee).$$

By assumption we have $\chi_2^{-1}\psi_2^{-1} = \chi_1\psi\psi_1\chi$, hence the central characters of $\tilde{\tau}$ and $V_3^\vee$ agree. We can apply [Wal85, Lemma 9], saying that this space is at most one dimensional.

We now prove the inert case, applying again the exact sequence (4.2.1). Let $G = GL_2(F_\ell)$, where F_ℓ is the quadratic unramified extension of $\mathbb{Q}_\ell$ and choose a $\mathbb{Q}_\ell$ basis $\{1,\alpha\}$ such that $\alpha^2 \in \mathbb{Q}_\ell$. Then let $J = B(F_\ell)$, $H = \mathrm{GL}_2(\mathbb{Q}_\ell)$ and τ be the smooth representation of J given by

$$\tau\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \tilde{\chi}(a)\tilde{\psi}(d).$$

The two orbits of the action of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ on $\mathrm{GL}_2(F_\ell)/B(F_\ell) \simeq \mathbb{P}^1_{F_\ell}$ are the $\mathrm{GL}_2(\mathbb{Q}_\ell)$ orbit of $(1:0)$ (essentially $\mathbb{P}^1_{\mathbb{Q}_\ell}$) and the $\mathrm{GL}_2(\mathbb{Q}_\ell)$ orbit of $(1:\alpha)$, which is given by $(a:b) \in \mathbb{P}^1_{F_\ell}$ such that $a/b \notin \mathbb{Q}_\ell, ab \neq 0$. Writing $a = x_0 + y_0\alpha, b = x_1 + y_1\alpha$, we have

$$\begin{pmatrix} y_1 & -y_0 \\ -x_1 & x_0 \end{pmatrix} \cdot (a:b) = (x_0y_1 - x_1y_0 : \alpha(x_0y_1 - x_1y_0)) = (1:\alpha)$$

and $x_0y_1 - x_1y_0 \neq 0$, otherwise if $y_1 \neq 0$, $a/b = y_0/y_1 \in \mathbb{Q}_\ell$. If $y_1 = 0$, then $x_1 \neq 0$ and $y_0 \neq 0$. We find that the stabiliser of the closed orbit is $H_1 = B(\mathbb{Q}_\ell)$ and, taking $\varepsilon = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$, the stabiliser of the open one is $H_2 = \{\begin{pmatrix} a & b \\ \alpha^2 b & a \end{pmatrix}, a,b \in \mathbb{Q}_\ell^2 - (0,0)\}$. To see that H_2 can be written of this form, we compute the conjugate of $B(F_\ell)$ by ε.

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} a - b\alpha & b \\ a\alpha - b\alpha^2 - d\alpha & b\alpha - d \end{pmatrix}.$$

Requiring that such matrices lie in $\mathrm{GL}_2(\mathbb{Q}_\ell)$ implies that $b \in \mathbb{Q}_\ell$ and $a = a_1 + b\alpha, d = a_1 - b\alpha$, for $a_1 \in \mathbb{Q}_\ell$. The group H_2 is a (non-split) maximal torus in

$\mathrm{GL}_2(\mathbb{Q}_\ell)$. This is again unimodular and

$$\delta_{B(F_\ell)}(\varepsilon \left(\begin{smallmatrix} a & b \\ \alpha^2 b & a \end{smallmatrix}\right) \varepsilon^{-1}) = \delta_{B(F_\ell)}(\left(\begin{smallmatrix} a-b\alpha & b \\ 0 & a+b\alpha \end{smallmatrix}\right)) = |a-b\alpha||a+b\alpha|^{-1} = 1.$$

We find the exact sequence of $GL_2(\mathbb{Q}_\ell)$-modules

$$0 \to \text{c-Ind}_{H_2}^{GL_2(\mathbb{Q}_\ell)} \tilde{\tau} \to \sigma_{|H} \to I_H(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}) \to 0,$$

where $\tilde{\tau}(\left(\begin{smallmatrix} a & b \\ \alpha^2 b & a \end{smallmatrix}\right)) = \tilde{\chi}(a-bt) \cdot \tilde{\psi}(a+bt)$.

Let $V = I_H(\chi, \psi)$, applying $\mathrm{Hom}_H(-, V^\vee)$ to the above exact sequence we find

$$\begin{aligned} 0 \to &\mathrm{Hom}_H(I_H(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}), V^\vee) \to \mathrm{Hom}_H(\sigma_{|H}, V^\vee) \\ &\to \mathrm{Hom}_H(\text{c-Ind}_{H_2}^{GL_2(\mathbb{Q}_\ell)} \tilde{\tau}, V^\vee) \to \mathrm{Ext}^1_H(I_H(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}), V^\vee)) \to \dots \end{aligned}$$

Since the smooth dual of V is $V^\vee = I_H(\chi^{-1}, \psi^{-1})$, we have, arguing as above, that

$$\mathrm{Hom}_H(I_H(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}), V^\vee) \neq 0,$$

if and only if $\tilde{\chi}_{|\mathbb{Q}_\ell^\times} = \chi^{-1}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times} = \psi^{-1}$ or $\tilde{\chi}_{|\mathbb{Q}_\ell^\times} = \psi^{-1}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times} = \chi^{-1}$. Assumption ($\star$) implies that this is not the case, hence the above space is zero and so is Ext^1. We hence find

$$\mathrm{Hom}_H(\sigma_{|H}, V^\vee) \simeq \mathrm{Hom}_H(\text{c-Ind}_{H_2}^{GL_2(\mathbb{Q}_\ell)} \tilde{\tau}, V^\vee) \simeq \mathrm{Hom}_{H_2}(\tilde{\tau}, (V_{|H_2})^\vee).$$

By assumption we have $\chi^{-1}\psi^{-1} = (\tilde{\chi} \cdot \tilde{\psi})_{|\mathbb{Q}_\ell^\times}$, and we can again apply [Wal85, Lemma 9], saying that this space is at most one dimensional. □

4.5.2 A basis for $\mathrm{Hom}_H(I_H(\chi, \psi) \otimes \sigma, \mathbb{C})$

Using the zeta integral defined above, we now want to construct an explicit nonzero element of $\mathrm{Hom}_H(I_H(\chi, \psi) \otimes \sigma, \mathbb{C})$, which by the above theorem will be a basis.

Definition 4.5.2. Let $\eta = \psi$, for ψ as above. For any $\varphi \in \sigma, s \in \mathbb{C}$, we define a function $z_{s,\varphi}$ on $H(\mathbb{Q}_\ell)$ by

$$z_{s,\varphi}(h) := Z(\sigma,\eta,\iota(h)\cdot\varphi,s+\tfrac{1}{2}),$$

for any $h \in H(\mathbb{Q}_\ell)$.

We now let, for $s \in \mathbb{C}$, $\psi_s := \psi|\cdot|^s$, $\chi_s := \chi|\cdot|^{-s}$.

Proposition 4.5.3. *The above function defines an element $z_\varphi \in I_H(\psi_s^{-1},\chi_s^{-1})$ for every $\varphi \in \sigma$. Moreover*

$$z_{s,\varphi_0}(1) = L(\tfrac{\psi}{\chi},2s+1)^{-1},$$

$$z_{s,U(\ell)\varphi_0}(1) = \tfrac{\ell^{s+3/2}}{\eta(\ell)}[L(\tfrac{\psi}{\chi},2s+1)^{-1} - L(\mathrm{As}(\sigma\otimes\eta),s+\tfrac{1}{2})^{-1}]$$

Proof. The first assertion is a straightforward corollary of Lemma 4.3.8 and Lemma 4.4.15. Indeed

$$\begin{aligned} z_\varphi(\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right)\cdot h) &= Z(\sigma,\eta,\iota(\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right))\iota(h)\cdot\varphi,s+\tfrac{1}{2}) = \left|\tfrac{a}{d}\right|^{s+1/2}\chi_\sigma(d)\eta(a^{-1}d)\cdot Z(\sigma,\eta,\iota(h)\cdot\varphi,s+\tfrac{1}{2}) \\ &= \left|\tfrac{a}{d}\right|^{1/2}\psi^{-1}(a)|a|^s\chi^{-1}(d)|d|^{-s}\cdot z_\varphi(h), \end{aligned}$$

using $\chi_\sigma = \psi^{-1}\chi^{-1}$ and $\eta = \psi$. The formula for the value at φ_0 ($U(\ell)\varphi_0$ respectively) follows from Lemma 4.3.6 and Lemma 4.4.13 (Lemma 4.3.7 and Lemma 4.4.14 respectively). $\square$

By definition, the map

$$\begin{aligned} z_s : \sigma &\to I_H(\psi_s^{-1},\chi_s^{-1}) \\ \varphi &\mapsto z_{s,\varphi} \end{aligned}$$

is H-equivariant. Moreover it follows from Proposition 4.5.3 that z_0 is different from zero if $L(\frac{\psi}{\chi},2s+1)^{-1}$ and $L(\mathrm{As}(\sigma\otimes\eta),s+\frac{1}{2})^{-1}$ do not both vanish at $s=0$. Notice that if ℓ is inert then $L(\frac{\psi}{\chi},2s+1)^{-1}$ divides $L(\mathrm{As}(\sigma\otimes\eta),s+\frac{1}{2})^{-1}$.

Lemma 4.5.4. *If ℓ splits, assume that $L(\frac{\psi}{\chi},2s+1)^{-1}$ and $L(\mathrm{As}(\sigma\otimes\eta),s+\frac{1}{2})^{-1}$ do not both vanish at $s=0$. If ℓ is inert, assume that $L(\frac{\psi}{\chi},2s+1)^{-1}$ and*

$L(\frac{\psi}{\chi}, 2s+1)L(\mathrm{As}(\sigma\otimes\eta), s+\frac{1}{2})^{-1}$ do not both vanish at $s=0$. Then the image of the homomorphism z_0 is contained in the unique irreducible subrepresentation of $I_H(\psi^{-1},\chi^{-1})$.

Proof. If $L(\frac{\psi}{\chi}, 2s+1)^{-1}$ does not vanish, then $I_H(\psi^{-1},\chi^{-1})$ is irreducible and there is nothing to prove. Otherwise, $\chi\psi^{-1} = |\cdot|$ and $I_H(\psi^{-1},\chi^{-1})$ has a unique infinite dimensional irreducible subrepresentation $\mathrm{St}(\gamma)$ and one dimensional quotient with action given by $\gamma(\det)$. In the split case, we claim that if $L(\mathrm{As}(\sigma\otimes\eta), s+\frac{1}{2})^{-1}$ does not vanish at $s=0$, the space $\mathrm{Hom}_H(\sigma_{|H}, \gamma(\det))$ is zero, and, consequently, the image of z_0 is contained in $\mathrm{St}(\gamma)$. The proof uses the same methods as the one of Theorem 4.5.1. With the same notation, in the split case one finds an exact sequence

$$\begin{aligned}0 \to &\mathrm{Hom}_H(I_H(\chi_1\chi_2|\cdot|^{1/2}, \psi_1\psi_2|\cdot|^{-1/2}), \gamma(\det)) \to \mathrm{Hom}_H(\sigma_{|H}, \gamma(\det))\\ &\to \mathrm{Hom}_H(\text{c-Ind}_{H_2}^{\mathrm{GL}_2(\mathbb{Q}_\ell)}\tilde{\tau}, \gamma(\det)) \to \mathrm{Ext}^1_H(I_H(\chi_1\chi_2|\cdot|^{1/2}, \psi_1\psi_2|\cdot|^{-1/2}), \gamma(\det)) \to \dots\end{aligned}$$

Since $L(\mathrm{As}(\sigma\otimes\psi), \frac{1}{2})^{-1}$ is not zero, then none of the characters $\chi_1\psi_2, \chi_1\chi_2, \psi_1\chi_2, \psi_1\psi_2$ is equal to $\psi^{-1}|\cdot|^{-1/2} = \gamma$. On the other hand, applying Frobenius reciprocity one finds that the first space in the sequence is non zero if and only if $\chi_1\chi_2 = \gamma$ and $\psi_1\psi_2 = \gamma$, while the third one is non zero if and only if $\chi_1\psi_2 = \gamma$ and $\psi_1\chi_2 = \gamma$. This proves the claim.

Similarly in the inert case, we have

$$\begin{aligned}0 \to &\mathrm{Hom}_H(I_H(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}), \gamma(\det)) \to \mathrm{Hom}_H(\sigma_{|H}, \gamma(\det))\\ &\to \mathrm{Hom}_H(\text{c-Ind}_{H_2}^{GL_2(\mathbb{Q}_\ell)}\tilde{\tau}, \gamma(\det)) \to \mathrm{Ext}^1_H(I_H(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}), \gamma(\det)) \to \dots\end{aligned}$$

The assumption $L(\frac{\psi}{\chi}, 2s+1)L(\mathrm{As}(\sigma\otimes\eta), s+\frac{1}{2})^{-1} = (1-\tilde{\chi}\psi(\ell)\ell^{-s-1/2})(1-\tilde{\psi}\psi(\ell)\ell^{-s-1/2}) \neq 0$ at $s=0$ and Frobenius reciprocity again imply that the first space in the sequence is zero. The third space is at most one dimensional. If it is zero, then we conclude as above. Otherwise, we find that also $\mathrm{Hom}_H(\sigma_{|H}, \gamma(\det))$ is at most one dimensional. Consider the exact sequence

$$0 \to \mathrm{Hom}_H(\sigma_{|H}, \mathrm{St}(\gamma)) \to \mathrm{Hom}_H(\sigma_{|H}, I_H(\psi^{-1},\chi^{-1})) \to \mathrm{Hom}_H(\sigma_{|H}, \gamma(\det)).$$

We deduce that if the last map is not zero, which in particular would imply that the statement of the Lemma is false, then $\mathrm{Hom}_H(\sigma_{|H}, \mathrm{St}(\gamma)) = 0$. This yields a contradiction. Indeed, consider the sequence

$$\begin{aligned}0 \to &\mathrm{Hom}_H(I_H(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}), \mathrm{St}(\gamma)) \to \mathrm{Hom}_H(\sigma_{|H}, \mathrm{St}(\gamma)) \\ &\to \mathrm{Hom}_H(\text{c-Ind}_{H_2}^{GL_2(\mathbb{Q}_\ell)} \tilde{\tau}, \mathrm{St}(\gamma)) \to \mathrm{Ext}^1_H(I_H(\tilde{\chi}_{|\mathbb{Q}_\ell^\times}, \tilde{\psi}_{|\mathbb{Q}_\ell^\times}), \mathrm{St}(\gamma)) \to \dots\end{aligned}$$

By [Wal85, Lemme 9] the third space has dimension one and, as in the proof of 4.5.1, we have that the first and fourth terms are zero. □

Unlike in the GSp_4 case where temperedness considerations allow to assume the non-vanishing of both the abelian L-factor and the one where the principal series appears, in our setting it can actually happen that they both are zero, e.g. it is possible that at some split primes $\sigma = I_H(\chi_1, \psi_1) \otimes I_H(\gamma\chi_1^{-1}, \gamma\psi_1^{-1})$. The following lemma shows that in this case z_0 is identically zero.

Lemma 4.5.5. *If the assumptions of the previous lemma are not satisfied, then z_0 is identically zero.*

Proof. This follows from the explicit description of functions in the Kirillov model of principal series representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ as recalled for example in [Jac72, Lemma 14.3]. In the split case, the functions $y \mapsto W_{i,\varphi}(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right))$ are indeed in the Kirillov model of $I_H(\chi_i, \psi_i)$. By definition

$$L(\mathrm{As}(\sigma \otimes \psi), s + 1/2) = L(\chi_1\chi_2\gamma^{-1}, s)L(\chi_1\psi_2\gamma^{-1}, s)L(\psi_1\chi_2\gamma^{-1}, s)L(\psi_1\psi_2\gamma^{-1}, s)$$

Firstly we assume that the order of vanishing of $L(\mathrm{As}(\sigma \otimes \psi), s + 1/2)^{-1}$ is 2 and, without loss of generality, we can assume $\chi_1\chi_2 = \psi_1\psi_2 = \gamma$, where $\psi^{-1} = |\cdot|^{1/2}\gamma$. Since the order is 2, we have $\chi_i \neq \psi_i$, and $W_{i,\varphi}(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right))$ can be written as

$$f_i(y)\chi_i(y)|y|^{1/2} + g_i(y)\psi_i(y)|y|^{1/2},$$

for some $f_i, g_i \in \mathscr{S}(\mathbb{Q}_\ell)$. Hence the function $Z(\sigma, \psi, \varphi_1 \otimes \varphi_2, s + 1/2)$ is equal to

$L(\mathrm{As}(\sigma\otimes\psi),s+1/2)^{-1}$ multiplied by the integral

$$\begin{aligned}&\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\gamma^{-1}(y)(f_1(y)\chi_1(y)|y|^{1/2}+g_1(y)\psi_1(y)|y|^{1/2})\\&\quad\cdot(f_2(y)\chi_2(y)|y|^{1/2}+g_2(y)\psi_2(y)|y|^{1/2})d^\times y\\&=P_1(s)L(\mathbf{1},s)+P_2(s)L(\chi_1\psi_2\gamma^{-1},s)+P_3(s)L(\psi_1\chi_2\gamma^{-1},s),\end{aligned}$$

where $P_i(s)$ are polynomials in ℓ^{-s},ℓ^s. The equality follows from the description of the L-factor $L(\mu,s)$ for any quasicharacter of $\mathbb{Q}_\ell^\times$ (see [Jac72], the discussion after Lemma 14.3). Since in our situation we have

$$L(\mathrm{As}(\sigma\otimes\psi),s+1/2)^{-1}=L(\mathbf{1},s)^{-2}L(\chi_1\psi_2\gamma^{-1},s)^{-1}L(\psi_1\chi_2\gamma^{-1},s)^{-1},$$

the result follows. If the order of vanishing of $L(\mathrm{As}(\sigma\otimes\psi),s+1/2)^{-1}$ is 4, we have $\chi_1=\psi_1$ and $\chi_2=\psi_2=\gamma\chi_1^{-1}$. In this case

$$\begin{aligned}&\int_{\mathbb{Q}_\ell^\times}|y|^{s-1}\gamma^{-1}(y)(f_1(y)\chi_1(y)|y|^{1/2}+g_1(y)\chi_1(y)v(y)|y|^{1/2})\\&\quad\cdot(f_2(y)\gamma\chi_1^{-1}(y)|y|^{1/2}+g_2(y)\gamma\chi_1^{-1}(y)v(y)|y|^{1/2})d^\times y\\&=P_1(s)L(\mathbf{1},s)+P_2(s)L(\mathbf{1},s)^2+P_3(s)L(\mathbf{1},s)^3.\end{aligned}$$

Here v is the valuation of $\mathbb{Q}_\ell$ and the equality follows from what said above and [Jac72, (14.2.1)] in the case where v or v^2 appears in the integral. Being $L(\mathrm{As}(\sigma\otimes\psi),s+1/2)^{-1}$ equal to $L(\mathbf{1},s)^{-4}$, again the result follows.

Similarly, in the inert case we have that the Kirillov function $y\mapsto W_\varphi(\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right))$ can be written as

$$|y|_{F_\ell}^{1/2}(f(y)\gamma(y)+g(y)\gamma(y)v(y)).$$

for $f,g\in\mathscr{S}(F_\ell)$ and $|\cdot|_{F_\ell}$ is equal to $|\cdot|^2$ when restricted to $\mathbb{Q}_\ell$. Indeed the fact that both the L factor vanish implies that $\tilde{\chi}=\tilde{\psi}=\gamma$, where $\psi^{-1}=|\cdot|^{1/2}\gamma$. Hence the integral in the definition of $z_{s,\varphi}$ is

$$\int_{\mathbb{Q}_\ell^\times} |y|^s \gamma^{-1}(y)(f(y)\gamma(y) + g(y)\gamma(y)v(y))d^\times y = P_1(s)L(\mathbf{1},s) + P_2(s)L(\mathbf{1},s)^2.$$

In this case $L(\mathrm{As}(\sigma \otimes \psi), s+1/2) = L(\mathbf{1},s)^{-4}$ and the result follows. □

We recall then the intertwining operator defined thanks to Proposition 4.2.8 and the pairing of definition 4.2.10

$$M : I_H(\chi_s, \psi_s) \to I_H(\psi_s, \chi_s) \quad \text{and} \quad \langle -,- \rangle : I_H(\psi_s, \chi_s) \times I_H(\psi_s^{-1}, \chi_s^{-1}) \to \mathbb{C}.$$

Definition/Proposition 4.5.6. For every $f \in I_H(\chi, \psi), \varphi \in \sigma$, we let

$$\mathfrak{z}_{\chi,\psi}(f \otimes \varphi) := \lim_{s \to 0} L(\tfrac{\psi}{\chi}, 2s+1)\langle Mf_s, z_{s,\varphi} \rangle,$$

where $f_s \in I_H(\chi_s, \psi_s)$ is any polynomial section passing through f.[2] This gives a well defined element $\mathfrak{z}_{\chi,\psi} \in \mathrm{Hom}_H(I_H(\chi, \psi) \otimes \sigma, \mathbb{C})$ which is not zero.

Proof. First of all one notices that $\langle Mf, z_{s,\varphi} \rangle = \langle f_s, Mz_{s,\varphi} \rangle$. If χ, ψ, σ are as in Lemma 4.5.5, then z_s vanishes for $s \to 0$. If χ, ψ, σ are as in Lemma 4.5.4, then z_s is an element of the non-generic irreducible subrepresentation of $I_H(\psi^{-1}, \chi^{-1})$, which is the kernel of the M operator if $L(\frac{\psi}{\chi}, 2s+1)$ has a pole at $s = 0$. Hence in both cases the limit is well defined and depends only on f. Moreover, the first formula of Proposition 4.5.3 implies that $\mathfrak{z}_{\chi,\psi}(f \otimes \varphi_0) \neq 0$, for some nice choice of f, e.g. for $f = F_{\phi_0}$ one can see this from the computation in the proof of Theorem 4.5.8, where we show

$$\mathfrak{z}_{\chi,\psi}(F_{\phi_0} \otimes \varphi_0) = L(\chi\psi^{-1}, 1)^{-1} \mathrm{Vol}(H(\mathbb{Z}_\ell)),$$

which is different from zero since we assumed $\chi\psi^{-1} \neq |\cdot|^{-1}$. □

[2] Here by polynomial section passing through f we mean a function on $H \times \mathbb{C}$, sending $(g,s) \mapsto f_s(g)$ such that $g \mapsto f_s(g)$ is in $I_H(\chi_s, \psi_s)$ for each $s \in \mathbb{C}$, $s \mapsto f_s(g)$ lies in $\mathbb{C}[\ell^s, \ell^{-s}]$ for every g and $f_0 = f$. One constructs it as in (4.2.2).

The following corollary is then a straightforward consequence of the above proposition and of multiplicity one (Theorem 4.5.1).

Corollary 4.5.7. *Take σ, ψ, χ as above and assume that condition $(\star)$ holds and that $|\chi(\ell)|_{\mathbb{C}} \neq |\psi(\ell)|_{\mathbb{C}}$. Then $\mathfrak{z}_{\chi,\psi}$ is a basis for* $\mathrm{Hom}_H(I_H(\chi,\psi)\otimes\sigma,\mathbb{C})$.

Using this specific element of $\mathrm{Hom}_H(I_H(\chi,\psi)\otimes\sigma,\mathbb{C})$, we now prove a theorem that will play a key role in the proof of norm relations. Having fixed χ, ψ, for every $\phi \in \mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})$ we let, as in Proposition 4.2.26,

$$F_\phi := F_{\phi,\chi,\psi} = f_{\hat{\phi},\chi,\psi} \in I_H(\chi,\psi).$$

We also recall the special elements $\phi_0, \phi_1 \in \mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})$ as in Definition 4.2.28.

Theorem 4.5.8. *With notation as above, we assume that $(\star)$ holds and that the characters χ, ψ are as follows*

- $\chi = |\cdot|^{1/2+k}\cdot\tau$, *for τ a finite order unramified character and $k \geq 0$ integers;*
- $\psi = |\cdot|^{-1/2}$.

Then for any $\mathfrak{z} \in \mathrm{Hom}_H(I_H(\chi,\psi)\otimes\sigma,\mathbb{C})$ we have

(i) $\mathfrak{z}(F_{\phi_1}\otimes\varphi_0) = \frac{1}{(\ell+1)}\cdot\left(1-\frac{\ell^k}{\tau(\ell)}\right)\cdot\mathfrak{z}(F_{\phi_0}\otimes\varphi_0)$;

(ii) $\mathfrak{z}(F_{\phi_1}\otimes U(\ell)\cdot\varphi_0) = \frac{\ell}{(\ell+1)}\cdot\left[\left(1-\frac{\ell^k}{\tau(\ell)}\right) - L(\mathrm{As}(\sigma),0)^{-1}\right]\cdot\mathfrak{z}(F_{\phi_0}\otimes\varphi_0)$,

where in (ii) the Hecke operator $U(\ell)$ is the one of Examples 4.3.3 and 4.4.10, in the inert and split prime case respectively.

Proof. We will prove both the statements for the specific function $\mathfrak{z}_{\chi,\psi}$, which is a basis of $\mathrm{Hom}_H(I_H(\chi,\psi)\otimes\sigma,\mathbb{C})$.

First we notice that F_{ϕ_t} is the value at $s=0$ of the Siegel section $f_{\hat{\phi}_t,\chi_s,\psi_s}$. We apply Proposition 4.2.27 and find

$$M(f_{\hat{\phi}_t,\chi_s,\psi_s}) = L(\chi\psi^{-1},1-2s)^{-1} f_{\phi_t,\psi_s,\chi_s}.$$

We then apply the definition of the pairing $\langle -,-\rangle$ and get

$$\begin{aligned}\langle M(f_{\hat{\phi}_t,\chi_s,\psi_s}), z_{\varphi_0}\rangle &= L(\chi\psi^{-1}, 1-2s)^{-1}\cdot\int_{H(\mathbb{Z}_\ell)} f_{\phi_t,\psi_s,\chi_s}(h) z_{\varphi_0}(h)dh\\ &= L(\chi\psi^{-1}, 1-2s)^{-1} f_{\phi_t,\psi_s,\chi_s}(1)\cdot\int_{K_0(\ell^t)} z_{s,\varphi_0}(h)dh.\end{aligned}$$

For the last equality we used that f_{ϕ_t,ψ_s,χ_s} restricted to $H(\mathbb{Z}_\ell)$ is a scalar multiple of $\mathrm{ch}(K_0(\ell^t))$. This follows from the fact that, by Lemma 4.2.29, f_{ϕ_t,ψ_s,χ_s} restricted to $H(\mathbb{Z}_\ell)$ is supported on $K_0(\ell^t)$ and ϕ_t is invariant by the action of $K_0(\ell^t)$. Recall that $g\cdot\varphi_0 = \varphi_0$ for any $g\in G(\mathbb{Z}_\ell)$. Hence $z_{s,\varphi_0}(h) = z_{s,\varphi_0}(1)$ for any $h\in H(\mathbb{Z}_\ell)$ and we can continue the chain of equality writing

$$\begin{aligned}&\mathfrak{z}(F_{\phi_t}\otimes\varphi_0) = L(\chi\psi^{-1},1)^{-1} f_{\phi_t,\psi,\chi}(1)\,\mathrm{Vol}(K_0(\ell^t))\cdot\lim_{s\to 0}\left(L(\tfrac{\psi}{\chi},1+2s) z_{s,\varphi_0}(1)\right)\\ &= \begin{cases} L(\chi\psi^{-1},1)^{-1}L(\psi\chi^{-1},1)^{-1}\,\mathrm{Vol}(K_0(\ell)) & \text{if } t=1\\ L(\chi\psi^{-1},1)^{-1}\,\mathrm{Vol}(H(\mathbb{Z}_\ell)) & \text{if } t=0,\end{cases}\end{aligned}$$

where we applied Lemma 4.2.29 for the value $f_{\phi_t,\psi,\chi}(1)$ and the first formula of Proposition 4.5.3 to show that the limiting value is exactly equal to 1. Since $\frac{\mathrm{Vol}(K_0(\ell))}{\mathrm{Vol}(H(\mathbb{Z}_\ell))} = [H(\mathbb{Z}_\ell):K_0(\ell)]^{-1} = \frac{1}{\ell+1}$ and

$$L(\psi\chi^{-1},1) = L(|\cdot|^{-1-k}\tau^{-1},1) = (1-\tfrac{\ell^k}{\tau(\ell)})^{-1}$$

we obtain (i). We proceed similarly to get (ii), using in addition the second formula of Proposition 4.5.3. We find

$$\begin{aligned}\mathfrak{z}_{\chi,\psi}(F_{\phi_1},U(\ell)\varphi_0) &= \mathrm{Vol}(K_0(\ell))L(\chi\psi^{-1},1)^{-1}L(\psi\chi^{-1},1)^{-1}\cdot\lim_{s\to 0} L(\tfrac{\psi}{\chi},1+2s) z_{U(\ell)\varphi_0}(1)\\ &= \tfrac{\ell^{3/2}}{\eta(\ell)}\mathrm{Vol}(K_0(\ell))L(\chi\psi^{-1},1)^{-1}\lim_{s\to 0} L(\tfrac{\psi}{\chi},1+2s)^{-1}L(\tfrac{\psi}{\chi},1+2s)\\ &\qquad\cdot\left(z_{\varphi_0}(1) - L(\mathrm{As}(\sigma\otimes\eta),s+\tfrac12)^{-1}\right)\\ &= \ell\,\mathrm{Vol}(K_0(\ell))L(\chi\psi^{-1},1)^{-1}\left[L(\tfrac{\psi}{\chi},1)^{-1} - L(\mathrm{As}(\sigma\otimes\eta),\tfrac12)^{-1}\right].\end{aligned}$$

Using the formula proved above for the value $\mathfrak{z}(F_{\phi_0}\otimes\varphi_0)$ and noticing that

$L(\mathrm{As}(\sigma\otimes\eta),\frac{1}{2}) = L(\mathrm{As}(\sigma\otimes|\cdot|^{-1/2}),\frac{1}{2}) = L(\mathrm{As}(\sigma),0)$, we obtain (ii). □

Remark 4.5.9. We emphasise that, in order to prove this theorem for any $\mathfrak{z} \in \mathrm{Hom}_H(I_H(\chi,\psi)\otimes\sigma,\mathbb{C})$, we used

- σ is a principal series representation for G with central character such that $\chi\psi\cdot\chi_\sigma = 1$ and ($\star$) holds;
- χ,ψ are in the form $\chi = |\cdot|^{1/2+k}\cdot\tau$, $\psi = |\cdot|^{-1/2}$;
- $\dim(\mathrm{Hom}_H(I_H(\chi,\psi)\otimes\sigma,\mathbb{C})) = 1$.

4.5.3 From $\mathrm{Hom}_H(\tau\otimes\sigma,\mathbb{C})$ to $\mathfrak{X}(\tau,\sigma^\vee)$

Let τ,σ be smooth representations of $H(\mathbb{Q}_\ell)$ and $G(\mathbb{Q}_\ell)$ respectively. We will now establish a bijection from the space $\mathrm{Hom}_H(\tau\otimes\sigma,\mathbb{C})$ and the space $\mathfrak{X}(\tau,\sigma^\vee)$ of linear maps $\mathfrak{Z} : \tau\otimes_{\mathbb{C}}\mathscr{H}(G) \to \sigma^\vee$ satisfying certain properties. For the specific choice $\tau = \mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})$, we will prove results that are essential in the proof of the norm relations (in motivic cohomology). In particular for σ an unramified principal series representation as above, we will use the above mentioned bijection and will be able to combine these results with Theorem 4.5.8, obtaining a result that is a key point in the proof of tame norm relations (in Galois cohomology).

Definition 4.5.10. Let τ,σ be smooth representations of $H(\mathbb{Q}_\ell)$ and $G(\mathbb{Q}_\ell)$ respectively. We define $\mathfrak{X}(\tau,\sigma^\vee)$ to be the space of linear maps $\mathfrak{Z} : \tau\otimes_{\mathbb{C}}\mathscr{H}(G) \to \sigma^\vee$, which are $H(\mathbb{Q}_\ell)\times G(\mathbb{Q}_\ell)$-equivariant, with the actions defined as follows:

- $H(\mathbb{Q}_\ell)$ acts trivially on $\sigma^\vee$ and on $\tau\otimes\mathscr{H}(G)$ via
$$h\cdot(v\otimes\xi) = (h\cdot v)\otimes\xi(h^{-1}(-)).$$
- $G(\mathbb{Q}_\ell)$ acts naturally on $\sigma^\vee$ (which is a $G(\mathbb{Q}_\ell)$-representation) and on $\tau\otimes\mathscr{H}(G)$ via
$$g\cdot(v\otimes\xi) = v\otimes\xi((-)g).$$

We now state explicitly the bijection we were mentioning above.

Proposition 4.5.11. *There is a canonical bijection between* $\mathrm{Hom}_H(\tau\otimes\sigma,\mathbb{C})$ *and* $\mathfrak{X}(\tau,\sigma^\vee)$ *characterised as follows*

$$\begin{aligned}\mathrm{Hom}_H(\tau\otimes\sigma,\mathbb{C}) &\longrightarrow \mathfrak{X}(\tau,\sigma^\vee)\\ \mathfrak{z} &\longmapsto \mathfrak{Z},\end{aligned}$$

where $\mathfrak{Z}(f\otimes\xi)(F)=\mathfrak{z}(f\otimes(\xi\cdot F))$*, for every* $f\in\tau,\xi\in\mathscr{H}(G)$ *and* $F\in\sigma$.

Proof. We start by rewriting Lemma 4.4.9 as

(1) $g\cdot(\xi\cdot F)=\xi(g^{-1}(-))\cdot F$;

(2) $\xi\cdot(g\cdot F)=\xi((-)g^{-1})\cdot F$;

for every $\xi\in\mathscr{H}(G),F\in\sigma,g\in G(\mathbb{Q}_\ell)$.

Firstly we check that $\mathfrak{Z}$ is $G(\mathbb{Q}_\ell)$-equivariant. By definition of the action on the smooth dual of σ, for every $g\in G(\mathbb{Q}_\ell)$ and $\Phi\in\sigma^\vee$, $g\cdot\Phi(-)=\Phi(g^{-1}\cdot(-))$. We have

$$\begin{aligned}[g\cdot\mathfrak{Z}(f\otimes\xi)](F)=\mathfrak{Z}(f\otimes\xi)(g^{-1}\cdot F)=\mathfrak{z}(f\otimes(\xi\cdot(g^{-1}\cdot F)))&\overset{(2)}{=}\mathfrak{z}(f\otimes(\xi((-)g)\cdot F))\\ &=\mathfrak{Z}(f\otimes(\xi((-)g))(F)=\mathfrak{Z}(g\cdot(f\otimes\xi))(F).\end{aligned}$$

Then we check that $\mathfrak{Z}$ is $H(\mathbb{Q}_\ell)$-equivariant, recalling that $H(\mathbb{Q}_\ell)$ acts trivially on $\sigma^\vee$. For $h\in H(\mathbb{Q}_\ell)$ we have

$$\begin{aligned}\mathfrak{Z}(h\cdot(f\otimes\xi))(F)=\mathfrak{Z}((h\cdot f)\otimes\xi(h^{-1}(-)))(F)&=\mathfrak{z}((h\cdot f)\otimes(\xi(h^{-1}(-))\cdot F))\\ &\overset{(1)}{=}\mathfrak{z}((h\cdot f)\otimes(h\cdot(\xi\cdot F)))\overset{\mathfrak{z}\in\mathrm{Hom}_H}{=}\mathfrak{z}(f\otimes\xi\cdot F)=\mathfrak{Z}(f\otimes\xi)(F).\end{aligned}$$

Hence $\mathfrak{Z}\in\mathfrak{X}(\tau,\sigma^\vee)$.

The fact that this defines a bijection follows from the isomorphism

$$\mathrm{Hom}_G(\text{c-Ind}_H^G(\tau),\sigma^\vee)\simeq\mathrm{Hom}_H(\tau\otimes\sigma,\mathbb{C}),$$

which is essentially given by Frobenius reciprocity (see [LSZ20a, Proposition

3.8.1]). Here we denoted by $\text{c-Ind}_H^G(\tau)$ the compact induction. Using $\tau \otimes \mathscr{H}(G) = \text{c-Ind}_H^G \tau$, one finds

$$\mathrm{Hom}_G(\text{c-Ind}_H^G(\tau), \sigma^\vee) \simeq \mathfrak{X}(\tau, \sigma^\vee).$$

$\square$

Definition 4.5.12. Let $R \in \mathscr{H}(G)$. we define $R' \in \mathscr{H}(G)$ by $R'(g) := R(g^{-1})$.

Remark 4.5.13. It is an easy computation to check that for every $\Phi \in \sigma^\vee, F \in \sigma$, we have

$$\Phi(R \cdot F) = R' \cdot \Phi(F).$$

Indeed one one side we have, $\Phi(R \cdot F) = \Phi\left(\int_G R(g) g \cdot F dg\right) = \int_G R(g) \Phi(g \cdot F) dg$, using linearity of Φ. On the other we find $R' \cdot \Phi(F) = \int_G R(g^{-1}) g \Phi(F) dg = \int_G R(g^{-1}) \Phi(g^{-1} \cdot F) dg$. This integrals are equal since G is unimodular.

Corollary 4.5.14. *Let $\mathfrak{z} \leftrightarrow \mathfrak{Z}$ as in the above Proposition. Let $U_1 \leq U_0$ be subgroups of G, $f_0, f_1 \in \tau$ and $g_0, g_1 \in G$ such that*

$$\mathfrak{z}(f_1, g_1 \cdot F) = \mathfrak{z}(f_0, g_0 \cdot (R \cdot F))$$

for some $R \in \mathscr{H}(U_0 \backslash G / U_0)$ and for every $F \in \sigma^{U_0}$. Then the elements $\mathfrak{Z}_i := \mathfrak{Z}(f_i \otimes \mathrm{ch}(g_i U_i)) \in (\sigma^\vee)^{U_i}$ satisfy

$$\sum_{u \in U_0/U_1} u \cdot \mathfrak{Z}_1 = R' \cdot \mathfrak{Z}_0 \in (\sigma^\vee)^{U_0}.$$

Proof. It is clear by the definition of the action of G that $\mathfrak{Z}_i \in (\sigma^\vee)^{U_i}$, moreover summing over quotient representatives gives also $\sum_{u \in U_0/U_1} u \cdot \mathfrak{Z}_1 \in (\sigma^\vee)^{U_0}$. Writing $(\sigma^\vee)^{U_0} = (\sigma^{U_0})^\vee$, we are then left to check that both the L.H.S. and the R.H.S. take the same value at every $F \in \sigma^{U_0}$. Applying the Lemma above, we find

$$\begin{aligned} R' \cdot \mathfrak{Z}_0(F) &= \mathfrak{Z}_0(R \cdot F) = \mathfrak{Z}(f_0 \otimes \mathrm{ch}(g_0 U_0))(R \cdot F) \\ &= \mathfrak{z}(f_0 \otimes (\mathrm{ch}(g_0 U_0) R) \cdot F) = \mathrm{Vol}(U_0) \mathfrak{z}(f_1, g_1 \cdot F). \end{aligned}$$

In the last equality we used the assumption $\mathfrak{z}(f_1, g_1 \cdot F) = \mathfrak{z}(f_0, g_0 \cdot (R \cdot F))$ together with the fact that $g_0 \cdot (R \cdot F) = R(g_0^{-1}(-)) \cdot F$ and

$$\mathrm{ch}(g_0U_0) \star R(g) = \int_G \mathrm{ch}(g_0U_0)(gh)R(h^{-1})dh = \int_{g^{-1}g_0U_0} R(h^{-1})dh = \mathrm{Vol}(U_0)R(g_0^{-1}g),$$

where we obtained the last equality from the fact that R is in $\mathscr{H}(U_0 \backslash G/U_0)$. Moreover for every $u \in U_0/U_1$ we have

$$u \cdot \mathfrak{Z}_1(F) = u \cdot \mathfrak{Z}(f_1 \otimes \mathrm{ch}(g_1U_1))(F) = \mathfrak{Z}(f_1 \otimes \mathrm{ch}(g_1U_1))(u^{-1}F) = \mathfrak{z}(f_1 \otimes \mathrm{ch}(g_1U_1)((-)u) \cdot F).$$

We also find that

$$\begin{aligned}\sum_{u \in U_0/U_1} \mathrm{ch}(g_1U_1)((-)u) \cdot F &= \sum_u \int_G \mathrm{ch}(g_1U_1)(gu)\; g \cdot F dg = \sum_u \int_{g_1U_1} gu^{-1} \cdot F dg \\ &= \sum_u \int_{U_1} g_1 \cdot F dg = \mathrm{Vol}(U_1)[U_0 : U_1]\; g_1 \cdot F = \mathrm{Vol}(U_0)\; g_1 \cdot F,\end{aligned}$$

where we used the fact that F is invariant by $U_0 \geq U_1$. The result follows using linearity and the above expression for $u \cdot \mathfrak{Z}_1(F)$. $\square$

We now work in the setting where:

- We take $\tau = \mathscr{S}(\mathbb{Q}_\ell^2, \mathbb{C})$;
- We replace $\sigma^\vee$ by an arbitrary smooth complex representation W of $G(\mathbb{Q}_\ell)$.

We consider $\mathfrak{X}(W)$ to be, similarly as above, the space of functions

$$\mathfrak{Z} : \mathscr{S}(\mathbb{Q}_\ell^2, \mathbb{C}) \otimes \mathscr{H}(G) \to W$$

satisfying the $H(\mathbb{Q}_\ell) \times G(\mathbb{Q}_\ell)$ equivariance property with actions defined as above.

Lemma 4.5.15. *Let $\xi \in \mathscr{H}(G)$ be invariant by left translation of the principal congruence subgroup of level ℓ^T in $H(\mathbb{Z}_\ell)$ for some $T \geq 0$. Then for any $\mathfrak{Z} \in \mathfrak{X}(W)$ the expression*

$$\frac{1}{\mathrm{Vol}(K_{H,1}(\ell^t))} \mathfrak{Z}(\phi_{1,t} \otimes \xi)$$

is independent of $t \geq T$*, where* $K_{H,1}(\ell^t), \phi_{1,t}$ *are as in Definition 4.2.30.*

Proof. This is the analogous of [LSZ20a, Lemma 3.9.2]. The proof carries over, we sketch it for the sake of completeness. For any $t \geq T$ we fix J a set of coset representatives for the quotient $K_{H,1}(\ell^T)/K_{H,1}(\ell^t)$ such that J is contained in the principal congruence subgroup of level ℓ^T. We can write $\phi_{1,T} = \sum_{k \in J} k \cdot \phi_{1,t}$. From that, using $H(\mathbb{Q}_\ell)$-equivariance of $\mathfrak{z}$ and the fact that ξ is invariant by the action of the principal congruence subgroup of H of level ℓ^T, we obtain

$$\mathfrak{z}(\phi_{1,T} \otimes \xi) = \sum_{k \in J} \mathfrak{z}(k \cdot (\phi_{1,t} \otimes (k^{-1} \cdot \xi)) = \tfrac{\mathrm{Vol}(K_{H,1}(\ell^T))}{\mathrm{Vol}(K_{H,1}(\ell^t))} \mathfrak{z}(\phi_{1,t} \otimes \xi).$$

□

Definition 4.5.16. We define $\mathfrak{z}(\phi_{1,\infty} \otimes \xi)$ to be the limiting value defined by the above lemma.

We now define a precise choice for ξ, that will be used for the definition of the Euler system classes.

Definition 4.5.17. Let $m \geq 0$ integer and $a \in \mathbb{Z}_\ell^\times$, we define $\eta_m^{(a)} \in G(\mathbb{Q}_\ell)$ by

$$\eta_m^{(a)} := \begin{cases} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{a}{\ell^m} \\ 0 & 1 \end{pmatrix} \right) \in \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell) & \text{if } \ell \text{ splits} \\ \begin{pmatrix} 1 & \delta \cdot \frac{a}{\ell^m} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(F_\ell) & \text{if } \ell \text{ is inert.} \end{cases}$$

In the second case we fix $\delta \in \mathscr{O}_{F_\ell}$ such that $F_\ell = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell(\delta)$ as in §4.3. We will write $\eta_m = \eta_m^{(1)}$.

For $n \geq \max(m,1)$ we also let $K_{m,n}^{(a)}$ be the subgroup given by

$$\begin{cases} \{(g_1, g_2) \in \mathrm{GL}_2(\mathbb{Z}_\ell) \times \mathrm{GL}_2(\mathbb{Z}_\ell) : g_1, g_2 \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}) \bmod \ell^n, \det g_1, \det g_2 \equiv a \bmod \ell^m\} \\ \{g \in G(\mathscr{O}_{F_\ell}) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod \ell^n, \det g \equiv a \bmod \ell^m\} \end{cases}$$

in the split and inert case respectively. We denote by $K_{m,n}$ the subgroup $K_{m,n}^{(1)}$.

Remark 4.5.18 (On the choice of η_m)**.** The choice of these elements in $G(\mathbb{Q}_\ell)$ corresponds to the choice of the "embedding twist" in the original definition of the Asai-Flach classes of [LLZ18] (and of Beilinson-Flach classes in [LLZ14]). The choice of the matrices is given by something of the form $\iota(\left(\begin{smallmatrix} 1 & \ell^{-m} \\ 0 & 1 \end{smallmatrix}\right))$ "twisted" by some upper triangular matrix of $G(\mathbb{Z}_\ell)$ not coming from $H(\mathbb{Z}_\ell)$, i.e. something of the form

$$\left(\left(\begin{smallmatrix} 1 & a_1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & a_2 \\ 0 & 1 \end{smallmatrix}\right)\right) \quad \text{and} \quad \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \text{ respectively}$$

for $a_1, a_2 \in \mathbb{Z}_\ell/\ell\mathbb{Z}_\ell, a_1 \neq a_2$ and $a \in \mathscr{O}_{F_\ell}/(\ell\mathscr{O}_{F_\ell} + \mathbb{Z}_\ell)$ respectively.

Recall then the Hecke operator $R = U(\ell)$ in Example 4.3.3 and 4.4.10. Taking $K' = K_{m,n}$ we have a decomposition as left cosets as in the mentioned examples. We now denote by $U'(\ell)$ the element R' (see definition 4.5.12) of the Hecke algebra invariant (on the left and on the right) by $K_{m,n}$, explicitly it is

$$U'(\ell) = \frac{1}{\mathrm{Vol}(K_{m,n})} \mathrm{ch}(K_{m,n}\iota(\left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right))K_{m,n}) \in \mathscr{H}(K_{m,n}\backslash G/K_{m,n}).$$

Proposition 4.5.19. *For any $\mathfrak{Z} \in \mathfrak{X}(W)$, we have*

$$\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1}K_{m,n})) = \begin{cases} \frac{1}{\ell}U'(\ell) & \\ & \cdot\, \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_m K_{m,n})) \\ \frac{1}{\ell-1}(U'(\ell)-1) & \end{cases} \quad \begin{matrix} \text{if } m \geq 1 \\ \text{if } m = 0. \end{matrix}$$

Proof. First of all we notice that (similarly as in Remark 4.5.13) we have $U'(\ell) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_m K_{m,n})) = \mathfrak{Z}(\phi_{1,\infty} \otimes (\mathrm{ch}(\eta_m K_{m,n}) \star U(\ell)))$. We moreover apply the decomposition of Examples 4.3.3 and 4.4.10 to find

$$U'(\ell) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_m K_{m,n})) = \begin{matrix} \text{(S) } \sum_{0 \leq u,v \leq \ell-1} \mathfrak{Z}\left(\phi_{1,\infty} \otimes \mathrm{ch}\left(\eta_m \left(\left(\begin{smallmatrix} \ell & u \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \ell & v \\ 0 & 1 \end{smallmatrix}\right)\right) K_{m,n}\right)\right) \\ \text{(I) } \sum_{0 \leq i,j \leq \ell-1} \mathfrak{Z}\left(\phi_{1,\infty} \otimes \mathrm{ch}\left(\eta_m \left(\begin{smallmatrix} \ell & i+\alpha j \\ 0 & 1 \end{smallmatrix}\right) K_{m,n}\right)\right), \end{matrix}$$

where (S) denotes the split case and (I) the inert one. In both cases we are going to rewrite the Hecke algebra element using the invariance of $K_{m,n}$ by $\mathbb{Z}_\ell$ translation.

(S) In the first case

$$\begin{aligned}\eta_m \cdot \left(\begin{pmatrix} \ell & u \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \ell & v \\ 0 & 1 \end{pmatrix} \right) &= \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \ell & (v-u)+\ell^{-m} \\ 0 & 1 \end{pmatrix} \right) \\ &= \iota(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}) \cdot \eta_{m+1}^{(1+\ell^m(v-u))}.\end{aligned}$$

In this case we denote by $x_{u,v}$ the integer $v-u$.

(I) Similarly in the second case, we write

$$\eta_m \begin{pmatrix} \ell & i+\delta j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & \delta(j+\ell^{-m}) \\ 0 & 1 \end{pmatrix} = \iota(\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}) \cdot \eta_{m+1}^{(1+\ell^m j)}.$$

In this case, we write $x_{i,j} = j$ for all $0 \le i \le \ell - 1$.

Now we write the above sum (both in the (S) and (I) case) as

$$\begin{aligned}&\sum_{0 \le u,v \le \ell-1} \mathfrak{Z} \left(\phi_{1,\infty} \otimes \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathrm{ch}(\eta_{m+1}^{(1+\ell^m x_{u,v})} K_{m,n}) \right) \\ &\quad = \tfrac{1}{\mathrm{Vol}(K_{H,1}(\ell^n))} \sum_{0 \le u,v \le \ell-1} \mathfrak{Z} \left(\phi_{1,n} \otimes \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathrm{ch}(\eta_{m+1}^{(1+\ell^m x_{u,v})} K_{m,n}) \right) \\ &\quad = \tfrac{1}{\mathrm{Vol}(K_{H,1}(\ell^n))} \sum_{0 \le u,v \le \ell-1} \mathfrak{Z} \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \phi_{1,n} \otimes \mathrm{ch}(\eta_{m+1}^{(1+\ell^m x_{u,v})} K_{m,n}) \right) \right) \\ &\quad = \tfrac{1}{\mathrm{Vol}(K_{H,1}(\ell^n))} \sum_{0 \le u,v \le \ell-1} \mathfrak{Z} \left(\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \phi_{1,n} \otimes \mathrm{ch}(\eta_{m+1}^{(1+\ell^m x_{u,v})} K_{m,n}) \right) \\ &\quad = \tfrac{1}{\mathrm{Vol}(K_{H,1}(\ell^n))} \sum_{0 \le u,v \le \ell-1} \mathfrak{Z} \left(\mathrm{ch}(\ell^{n+1}\mathbb{Z}_\ell \times (1+\ell^n\mathbb{Z}_\ell)) \otimes \mathrm{ch}(\eta_{m+1}^{(1+\ell^m x_{u,v})} K_{m,n}) \right) \\ &\quad = \ell^{-1} \sum_{0 \le u,v \le \ell-1} \mathfrak{Z} \left(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1}^{(1+\ell^m x_{u,v})} K_{m,n}) \right)\end{aligned}$$

In the second equality we used that $\phi_{1,n}$ is fixed by $\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}$, in the third equality we used the fact that $\mathfrak{Z}$ is $H(\mathbb{Q}_\ell)$-equivariant and the action on the target is trivial. The fourth one is a consequence of the definition of $\phi_{1,n}$ and the action of $H(\mathbb{Q}_\ell)$ on Schwartz functions. For the last one one reasons as follows. Write $S' = \ell^{n+1}\mathbb{Z}_\ell \times (1+\ell^n\mathbb{Z}_\ell)$ and $S = \ell^{n+1}\mathbb{Z}_\ell \times (1+\ell^{n+1}\mathbb{Z}_\ell)$; in particular $\mathrm{ch}(S) = \phi_{1,n+1}$. Now we write

$$\mathrm{Stab}(S) = K_{H,1}(\ell^{n+1}) \subset \mathrm{Stab}(S') = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(\mathbb{Z}_\ell) : c \equiv 0(\ell^{n+1}), d \equiv 1(\ell^n) \}.$$

We also write $\Sigma = \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1+\ell^n k \end{smallmatrix}\right) \}_{0 \le k \le \ell-1}$, which is a set of representatives for the quotient $\mathrm{Stab}(S')/\,\mathrm{Stab}(S)$. We can then write

$$\mathrm{ch}(S') = \sum_{\sigma \in \Sigma} \sigma \cdot \mathrm{ch}(S) = \sum_{\sigma \in \Sigma} \sigma \cdot \phi_{1,n+1}.$$

It is easy to check that for every $\sigma \in \Sigma$, letting $\xi_{u,v} := \mathrm{ch}(\eta_{m+1}^{(1+\ell^m x_{u,v})} K_{m,n})$ we have $\sigma \cdot \xi_{u,v} = \xi_{u,v}$. Hence we find

$$\begin{aligned}
\mathfrak{Z}(\mathrm{ch}(S') \otimes \xi_{u,v}) &= \sum_{\sigma} \mathfrak{Z}(\sigma \cdot \phi_{1,n+1} \otimes \xi_{u,v}) \\
&= \sum_{\sigma} \mathfrak{Z}(\sigma \cdot (\phi_{1,n+1} \otimes \sigma^{-1} \cdot \xi_{u,v})) = \sum_{\sigma} \mathfrak{Z}(\phi_{1,n+1} \otimes \xi_{u,v}) \\
&= \ell \cdot \mathfrak{Z}(\phi_{1,n+1} \otimes \xi_{u,v}).
\end{aligned}$$

Hence we can write

$$\begin{aligned}
\tfrac{1}{\mathrm{Vol}(K_{H,1}(\ell^n))} \sum_{0 \le u,v \le \ell-1} \mathfrak{Z}\left(\mathrm{ch}(S') \otimes \xi_{u,v}\right) &= \tfrac{\ell}{\mathrm{Vol}(K_{H,1}(\ell^n))} \sum_{0 \le v \le \ell-1} \mathfrak{Z}\left(\phi_{1,n+1} \otimes \xi_{u,v}\right) \\
&= \tfrac{\ell^{-1}}{\mathrm{Vol}(K_{H,1}(\ell^{n+1}))} \sum_{0 \le u,v \le \ell-1} \mathfrak{Z}\left(\phi_{1,n+1} \otimes \xi_{u,v}\right) \\
&= \ell^{-1} \sum_{0 \le u,v \le \ell-1} \mathfrak{Z}\left(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1}^{(1+\ell^m v)} K_{m,n})\right),
\end{aligned}$$

where for the second equality we used $[K_{H,1}(\ell^n) : K_{H,1}(\ell^{n+1})] = \ell^2$.

Now we notice that

$$\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \eta_{m+1} \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) = \eta_{m+1}^{(a)}.$$

Moreover for $a \equiv 1$ modulo $\ell^m \mathbb{Z}_\ell$, $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in K_{m,n}$ and hence

$$\mathrm{ch}(\eta_{m+1} K_{m,n}) \left(\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) (-) \right) = \mathrm{ch}(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \eta_{m+1} \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) K_{m,n}) = \mathrm{ch}(\eta_{m+1}^{(a)} K_{m,n}).$$

Moreover, for such a's, we have $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \phi_{1,t} = \phi_{1,t}$ hence we can write

$$\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1}^{(a)} K_{m,n})) = \mathfrak{Z}\left(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1} K_{m,n}) \left(\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)(-)\right)\right) = \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1} K_{m,n}))$$

$\boxed{m \geq 1}$ Applying what we wrote above, we get that all the terms in the sum are equal to $\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1} K_{m,n}))$ and hence

$$U'(\ell) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_m K_{m,n})) = \ell \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1} K_{m,n}))$$

$\boxed{m = 0}$ We can apply the same reasoning for all $x_{u,v}$ but for $x_{u,v} \equiv -1$ modulo ℓ. For such $x_{u,v}$ we find $\mathrm{ch}(\eta_1^{(1+x_{u,v})} K_{0,n}) = \mathrm{ch}(K_{0,n})$. We have exactly ℓ pairs (u,v) such that $x_{u,v} \equiv -1$; for the remaining $\ell^2 - \ell = \ell(\ell - 1)$ terms we have $\left(\begin{smallmatrix} 1+x_{u,v} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in K_{0,n}$ and we find as above $\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1^{(1+x_{u,v})} K_{0,n})) = \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1 K_{0,n}))$. We therefore obtain

$$U'(\ell) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(K_{0,n})) = (\ell - 1) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1 K_{0,n})) + \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(K_{0,n}))$$

$\square$

We now want to go back to the case where W is the smooth dual of principal series representation and $\tau = I_H(\chi, \psi)$ in order to use the bijection of Proposition 4.5.11 and Theorem 4.5.8. First of all let $K = G(\mathbb{Z}_\ell)$. We assume that the Haar measures on $G(\mathbb{Q}_\ell)$ and on $H(\mathbb{Q}_\ell)$ are normalised to that $\mathrm{Vol}(G(\mathbb{Z}_\ell)) = \mathrm{Vol}(H(\mathbb{Z}_\ell)) = 1$. We also recall the Siegel section map used above

$$\begin{aligned} \mathscr{S}(\mathbb{Q}_\ell^2, \mathbb{C}) &\to I_H(\chi, \psi) \\ \phi &\mapsto F_{\phi,\chi,\psi} := f_{\hat{\phi},\chi,\psi}, \end{aligned}$$

that is $H(\mathbb{Q}_\ell)$ equivariant if χ, ψ are unramified.

Corollary 4.5.20. *Let $W = \sigma^\vee$ for σ a principal series representation with central character χ_σ. Let χ, ψ unramified characters such that*

- $\chi = |\cdot|^{1/2+k}\tau$*, for τ a finite order character (that may be ramified) and $k \geq 0$*

integers;

- $\psi = |\cdot|^{-1/2}$*;*

- *we assume that* σ *satisfies* $\chi\psi \cdot \chi_\sigma = 1$ *and* $(\star)$ *holds.*

Let $\mathfrak{Z} \in \mathfrak{X}(\sigma^\vee)$ *and assume that it* *factors through the Siegel section map* *for the above* χ, ψ*, i.e.*

$$\begin{tikzcd} \mathscr{S}(\mathbb{Q}_\ell^2, \mathbb{C}) \otimes \mathscr{H}(G) \arrow[rr, "\mathfrak{Z}"] \arrow[dr] & & \sigma^\vee \\ & I_H(\chi,\psi) \otimes \mathscr{H}(G) \arrow[ur, dotted] & \end{tikzcd}$$

Then we have

$$\mathfrak{Z}(\phi_{1,\infty} \otimes (\mathrm{ch}(K) - \mathrm{ch}(\eta_1 K))) = \tfrac{\ell}{\ell-1} L(\mathrm{As}(\sigma), 0)^{-1} \cdot \mathfrak{Z}(\phi_0 \otimes \mathrm{ch}(K)),$$

where ϕ_0 *is as in Definition 4.2.28.*

Proof. We write $\phi_{0,1} := \mathrm{ch}(\ell\mathbb{Z}_\ell \times \mathbb{Z}_\ell^\times)$ and

$$K_0 := \mathrm{Stab}(\ell\mathbb{Z}_\ell \times \mathbb{Z}_\ell^\times) = \{\gamma \in H(\mathbb{Z}_\ell) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod \ell\}.$$

We also recall that $\phi_{1,1} = \mathrm{ch}(\ell\mathbb{Z}_\ell \times (1 + \ell\mathbb{Z}_\ell))$ and write $K_1 := K_{H,1}(\ell) = \mathrm{Stab}(\ell\mathbb{Z}_\ell \times (1 + \ell\mathbb{Z}_\ell))$. For every $\sigma \in H(\mathbb{Z}_\ell) \subset K = G(\mathbb{Z}_\ell)$, we have $\sigma \cdot \mathrm{ch}(K) = \mathrm{ch}(K)$ and hence

$$\mathfrak{Z}(\sigma \cdot \phi_{1,1} \times \mathrm{ch}(K)) = \mathfrak{Z}(\sigma \cdot (\phi_{1,1} \times \mathrm{ch}(K))) = \mathfrak{Z}(\phi_{1,1} \times \mathrm{ch}(K)).$$

Applying this and writing $\phi_{0,1} = \sum_{\sigma \in K_0/K_1} \sigma \cdot \phi_{1,1}$, we obtain

$$\begin{aligned} \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(K)) &= \tfrac{1}{\mathrm{Vol}(K_1)} \mathfrak{Z}(\phi_{1,1} \otimes \mathrm{ch}(K)) = \tfrac{1}{[K_0:K_1]\,\mathrm{Vol}(K_1)} \mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K)) \\ &= \tfrac{1}{\mathrm{Vol}(K_0)} \mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K)) = (\ell+1)\mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K)), \end{aligned} \tag{4.5.1}$$

where in the last step we used the fact that $\mathrm{Vol}(H(\mathbb{Z}_\ell)) = 1$ and $[H(\mathbb{Z}_\ell) : K_0] = \ell + 1$.

Then applying the previous proposition we find, writing $K_{G,1} := K_{0,1}$,

$$\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1 K_{G,1})) = \tfrac{1}{\ell-1}(U'(\ell)-1) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(K_{G,1})).$$

Let $K_{G,0}$ be the subgroup of G given by matrices congruent to $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ modulo ℓ. Next we sum on both side of the last equality over representatives of $K/K_{G,1}$. On the left hand side we obtain $\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1 K))$. On the right hand side, writing $K/K_{G,1} = K/K_{G,0} \cdot K_{G,0}/K_{G,1}$, and using the fact that $K_{G,0}/K_{G,1}$ commutes with the Hecke operator $U'(\ell)$, we obtain

$$\frac{1}{\ell-1} \sum_{\gamma \in K/K_{G,0}} \gamma \cdot (U'(\ell)-1)\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(K_{G,0})).$$

Moreover we can argue as before, using the fact that $\sigma \cdot \mathrm{ch}(K_{G,0}) = \mathrm{ch}(K_{G,0})$ for $\sigma \in K_0 \subset K_{G,0}$, we can rewrite $\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(K_{G,0}))$ as $(\ell+1)\mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K_{G,0}))$. Overall we have obtained

$$\begin{aligned} \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1 K)) &= \frac{\ell+1}{\ell-1} \sum_{\gamma \in K/K_{G,0}} \gamma \cdot (U'(\ell)-1)\mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K_{G,0})) \\ &= \frac{\ell+1}{\ell-1} \sum_{\gamma \in K/K_{G,0}} \gamma \cdot U'(\ell)\mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K_{G,0})) - \frac{\ell+1}{\ell-1}\mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K)). \end{aligned} \tag{4.5.2}$$

Combining (4.5.1) and (4.5.2), one obtains

$$\begin{aligned} \mathfrak{Z}(\phi_{1,\infty} \otimes (\mathrm{ch}(K) - \mathrm{ch}(\eta_1 K))) &= (\ell+1)(1+\tfrac{1}{\ell-1})\mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K)) \\ &\quad - \tfrac{\ell+1}{\ell-1} \sum_{\gamma \in K/K_{G,0}} \gamma \cdot U'(\ell)\mathfrak{Z}(\phi_{0,1} \otimes \mathrm{ch}(K_{G,0})). \end{aligned} \tag{4.5.3}$$

We finally use the assumption that $\mathfrak{Z}$ factors through the Siegel section. First we suppose that τ is ramified. Since both ϕ_0 and $\phi_{0,1}$ are invariant under the action of matrices of the form $\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right)$ for $a, d \in \mathbb{Z}_\ell^\times$, $* \in \mathbb{Z}_\ell$, we get

$$F_{\phi_0,\chi,\psi} = \chi(a) \cdot F_{\phi_0,\chi,\psi},$$

and being χ ramified, this implies that $F_{\phi_0,\chi,\psi} = 0$. Similarly $F_{\phi_{0,1},\chi,\psi} = 0$ and the claimed equality reads $0=0$. So we can suppose τ unramified, so that we are able to apply Theorem 4.5.8 (where ϕ_1 is our $\phi_{0,1}$). Using Remark 4.5.13, the two equalities of the theorem give us

$$\mathfrak{Z}(\phi_{0,1}\otimes \mathrm{ch}(K)) = \tfrac{1}{(\ell+1)}\cdot\left(1-\tfrac{\ell^k}{\tau(\ell)}\right)\cdot\mathfrak{Z}(\phi_0\otimes\mathrm{ch}(K)),$$

$$U'(\ell)\cdot\mathfrak{Z}(\phi_{0,1}\otimes\mathrm{ch}(K_{G,0})) = \tfrac{\ell}{(\ell+1)}\cdot\left[\left(1-\tfrac{\ell^k}{\tau(\ell)}\right)-L(\mathrm{As}(\sigma),0)^{-1}\right]\cdot\mathfrak{Z}(\phi_0\otimes\mathrm{ch}(K_{G,0})).$$

Hence we rewrite the two terms on the right hand side of (4.5.3) as

$$(\ell+1)(1+\tfrac{1}{\ell-1})\mathfrak{Z}(\phi_{0,1}\otimes\mathrm{ch}(K)) = \tfrac{\ell}{\ell-1}\left(1-\tfrac{\ell^k}{\tau(\ell)}\right)\cdot\mathfrak{Z}(\phi_0\otimes\mathrm{ch}(K)),$$

$$\begin{aligned}
&\tfrac{\ell+1}{\ell-1}\sum_{\gamma\in K/K_{G,0}}\gamma\cdot U'(\ell)\mathfrak{Z}(\phi_{0,1}\otimes\mathrm{ch}(K_{G,0}))\\
&\quad=\tfrac{\ell}{(\ell-1)}\cdot\left[\left(1-\tfrac{\ell^k}{\tau(\ell)}\right)-L(\mathrm{As}(\sigma),0)^{-1}\right]\sum_{\gamma\in K/K_{G,0}}\gamma\cdot\mathfrak{Z}(\phi_0\otimes\mathrm{ch}(K_{G,0}))\\
&\quad=\tfrac{\ell}{(\ell-1)}\cdot\left[\left(1-\tfrac{\ell^k}{\tau(\ell)}\right)-L(\mathrm{As}(\sigma),0)^{-1}\right]\mathfrak{Z}(\phi_0\otimes\mathrm{ch}(K))
\end{aligned}$$

and get the claimed equality. $\square$

Remark 4.5.21. An easy adaptation of the arguments allows to prove analogous results for $\mathfrak{Z}\in\mathfrak{X}(\sigma^\vee)[h]$, where we denote by $\mathfrak{X}(\sigma^\vee)[h]$ the space of functions $\mathfrak{Z}:(\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})\otimes|\cdot|^h)\otimes\mathscr{H}(G)\to\sigma^\vee$, for some positive integer h. Hence we are twisting the action of $H(\mathbb{Q}_\ell)$ on $\mathscr{S}(\mathbb{Q}_\ell^2,\mathbb{C})$ by a power of the determinant. Note that in the proof of Proposition 4.5.19, the the invariance of $\mathfrak{Z}$ by matrices of non-invertible determinant is used only to *pull out* $\left(\begin{smallmatrix}\ell^{-1}&0\\0&1\end{smallmatrix}\right)$ and hence the statement becomes

$$\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_{m+1}K_{m,n})) = \begin{cases}\tfrac{\ell^{-h}}{\ell}U'(\ell) & \\ \tfrac{1}{\ell-1}(\ell^{-h}U'(\ell)-1) & \end{cases}\cdot\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_m K_{m,n}))\quad\begin{matrix}\text{if } m\geq 1\\ \text{if } m=0.\end{matrix}$$

Moreover, the space of maps in $\mathfrak{X}(\sigma^\vee)[h]$ factoring through the Siegel section for $\chi\psi\chi_\sigma=1$ will now be isomorphic, via the bijection of Proposition 4.5.11, to a space of the form $\mathrm{Hom}_H(I_H(\chi|\cdot|^h, \psi|\cdot|^h)\otimes\sigma, |\det|^h)$. Theorem 4.5.1 implies that this space is again one dimensional, and the construction of a basis carries through as in Section § 4.5, where in the choice of the auxiliary character in Definition 4.5.2 ψ is replaced by $\psi|\cdot|^h$. The multiplying factor on the RHS of (ii) in Theorem 4.5.8 becomes

$$\frac{\ell^{1+h}}{(\ell+1)}\cdot\left[\left(1-\frac{\ell^k}{\tau(\ell)}\right)-L(\mathrm{As}(\sigma),h)^{-1}\right]$$

and the statement of Corollary 4.5.20 hence becomes

$$\mathfrak{Z}(\phi_{1,\infty}\otimes(\mathrm{ch}(K)-\mathrm{ch}(\eta_1 K)))=\tfrac{\ell}{\ell-1}L(\mathrm{As}(\sigma),h)^{-1}\cdot\mathfrak{Z}(\phi_0\otimes\mathrm{ch}(K)).$$

Remark 4.5.22 (Towards Asai–Flach Euler system)**.** As anticipated in the introduction, in order to (re)define the Euler system constructed in [LLZ18], we will define a special map $\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}$ for $k,k'\geq 0$ integers and $0\leq j\leq \min(k,k')$ with values in $W=H^3_{\mathrm{mot}}(Y_G,\mathscr{D}(2))$, where Y_G is the Shimura variety associated to G and $\mathscr{D}$ is a motivic sheaf depending on k,k',j. Such map will be of "global nature", more precisely it is a map

$$\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}:\mathscr{S}(\mathbb{A}^2_f,\mathbb{Q})\otimes\mathscr{H}(G(\mathbb{A}_f),\mathbb{Q})\longrightarrow H^3_{\mathrm{mot}}(Y_G,\mathscr{D}(2))$$

satisfying conditions of $H(\mathbb{A}_f)\times G(\mathbb{A}_f)$-equivariance with actions defined as in Definition 4.5.10. The Asai–Flach classes will be defined by images via $\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}$ of very precise elements in $\mathscr{S}(\mathbb{A}^2_f,\mathbb{Q})\otimes\mathscr{H}(G(\mathbb{A}_f),\mathbb{Q})$, whose local components will be the one we considered in this section. Proving norm relations (in motivic cohomology) will turn out to be equivalent to prove relations of such classes locally at a certain prime q, i.e. we will be looking at a map

$$\mathfrak{Z}:=(\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}})_q:\mathscr{S}(\mathbb{Q}^2_q,\mathbb{Q})\otimes\mathscr{H}(G(\mathbb{Q}_q),\mathbb{Q})\longrightarrow W=H^3_{\mathrm{mot}}(Y_G,\mathscr{D}(2))\in\mathfrak{X}(W).$$

In order to prove norm relations of vertical type, we will be able to apply Proposition 4.5.19. While for proving "tame norm relatitons" the input local data will be essentially the one in Corollary 4.5.20, but we have the strong assumption on W.

We will have to apply the étale regulator and Hochschild—Serre spectral sequence to pass to Galois cohomology and finally take the projection to an automorphic representation of G associated to an Hilbert modular form of weight $(k+2, k'+2)$. As anticipated in Remark 4.2.19, the local component at a "good prime" ℓ of this representation will be a spherical principal series representation, so we will finally be able to apply Corollary 4.5.20.

4.6 Eisenstein classes for $H = \mathrm{GL}_2$

We now recall which are the elements in motivic cohomology that we are actually going to consider. This is [LSZ20a, § 7].

Write $\mathscr{S}_0(\mathbb{A}_f^2, \mathbb{Q}) \subset \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q})$ for the subspace of functions ϕ satisfying $\phi(0,0) = 0$. Recall the notation of § 2.3.2, where we denoted by Y_H the infinite level modular curve and we defined the relative Chow motives $\mathrm{TSym}^k \mathscr{H}_{\mathbf{Q}}(\mathscr{E})$.

Theorem 4.6.1 (Eisenstein symbol maps)**.**

1. *([Col04, Théorème 1.8]) There is a canonical $H(\mathbb{A}_f)$-equivariant map*

$$\begin{aligned} \mathscr{S}_0(\mathbb{A}_f^2, \mathbb{Q}) &\longrightarrow H^1_{\mathrm{mot}}(Y_H, \mathbf{Q}(1)) = \mathscr{O}(Y_H)^\times \otimes \mathbb{Q} \\ \phi &\longmapsto g_\phi \end{aligned}$$

characterised by the following: if $\phi = \mathrm{ch}((a,b) + N\hat{\mathbb{Z}})$ for some $N \geq, a, b \in \mathbb{Q}^2 - N\mathbb{Z}^2$, then $g_\phi = g_{a/N, b/N}$, the Siegel unit of Definition 2.1.11.

2. *([BL94, §2]) Let $k \geq 1$. There is a $H(\mathbb{A}_f)$-equivariant map*

$$\begin{aligned} \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q}) &\longrightarrow H^1_{\mathrm{mot}}(Y_H, \mathrm{TSym}^k \mathscr{H}_{\mathbf{Q}}(\mathscr{E})(1)) \\ \phi &\longmapsto \mathrm{Eis}^k_\phi, \end{aligned}$$

characterised by the following: the pullback of its de Rham realisation is the $\mathrm{TSym}^k \mathscr{H}(\mathscr{E})$-valued differential 1-form $-F_\phi^{(k+2)}(\tau)(2\pi i dz)^k(2\pi i d\tau)$, where $F_\phi^{(k+2)}$ is the Eisenstein series defined as in [LSZ20a, Theorem 7.2.2].

Remark 4.6.2. If $\phi = \mathrm{ch}((0,b)+N\hat{\mathbb{Z}})$, then Eis^k_ϕ is the class defined in [KLZ15, Theorem 4.1.1]. Moreover, it is a consequence of Kronecker limit formula that if $\phi \in \mathscr{S}_0(\mathbb{A}^2_f,\mathbb{Q})$, $d\log g_\phi$, which is the de Rham realisation of g_ϕ, is equal to $-F^{(2)}_\phi(2\pi i d\tau)$.

Remark 4.6.3. The $H(\mathbb{A}_f)$-equivariance of the map in (1) is equivalent to some of the properties of Siegel units we stated in Chapter 2 (see Proposition 2.1.14).

We will need a description of the target of these maps in terms of "adelic induced representations". The reader should have in mind, for the following discussion, that we are going to define classes using Eisenstein elements and, in order to apply the local results of the previous sections, it will be helpful to identify motivic cohomology with $H(\mathbb{A}_f)$ representations that locally look like $I_{H(\mathbb{Q}_\ell)}(\chi,\psi)$. More precisely, we have the following.

Definition 4.6.4. For $k \geq 0$ and η a finite order character of $\mathbb{A}_f^\times/\mathbb{Q}^{\times +}$ such that $\eta(-1) = (-1)^k$, we define $I_k(\eta)$ to be the space of functions $f : H(\mathbb{A}_f) \to \mathbb{C}$ such that

$$f\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)g\right) = \| a \|^{k+1} \| d \|^{-1} \eta(a)f(g), \quad \text{for every } g \in H(\mathbb{A}_f), a,d \in \mathbb{A}_f^\times, b \in \mathbb{A}_f.$$

We view it as a $H(\mathbb{A}_f)$ representation by right translation. For $k = 0$ and $\eta = 1$, we define $I^0_0(1)$ to be the subrepresentation which is the kernel of the integration over $H(A_f)/B(\mathbb{A}_f)$ on $I_0(1)$.

Remark 4.6.5. Notice that restricting $f \in I_k(\eta)$ to $H(\mathbb{Q}_\ell)$, we find an element f_ℓ in the space

$$I_{H(\mathbb{Q}_\ell)}(|\cdot|^{1/2+k}\eta_\ell, |\cdot|^{-1/2}),$$

with notation as in §4.2.

We finally relate motivic cohomology to these representations.

Theorem 4.6.6. *With notation as above,*

1. *([Sch89, Theorem 3]) there is a $H(\mathbb{A}_f)$-equivariant isomorphism*

$$\partial_0 : \frac{\mathscr{O}^\times(Y_H)}{(\mathbb{Q}^{ab})^\times} \otimes \mathbb{C} \longrightarrow I_0^0(1) \oplus \bigoplus_{\eta \neq 1} I_0(\eta),$$

characterised by the fact that $\partial_0(g)(1)$ is the order of vanishing of g at the cusp ∞.

2. *For $k \geq 1$, there is a surjective $H(\mathbb{A}_f)$-equivariant map*

$$\partial_k : H^1_{\mathrm{mot}}(Y_H, \mathrm{TSym}^k \mathscr{H}_{\mathbf{Q}}(\mathscr{E})(1)) \otimes \mathbb{C} \longrightarrow \bigoplus_{\eta} I_k(\eta),$$

such that $\partial_k(x)(1)$ is the residue at ∞ of the de Rham realisation of x. Moreover this map is an isomorphism on the image of the Eisenstein symbol. (See [SS91, Theorem 7.4] and [Lem17, Lemma 4.3])

Moreover, we have an explicit description of the image of the Eisenstein symbols via these maps. Write $\mathscr{S}(\mathbb{A}_f^2, \mathbb{C})^\eta$ for the subspace of $\mathscr{S}(\mathbb{A}_f^2, \mathbb{C})$ on which $\hat{\mathbb{Z}}^\times$ acts via the character η.

Proposition 4.6.7. *([LSZ20a, Proposition 7.3.4]) Let $\phi \in \mathscr{S}(\mathbb{A}_f^2, \mathbb{C})^\eta$ and write $\phi = \prod_\ell \phi_\ell$. If $k = 0$ and $\eta = 1$, assume that $\phi(0,0) = 0$. Then we have*

$$\partial_k(\mathrm{Eis}^k_{\mathrm{mot},\phi}) = \frac{2(k+1)!L(k+2,\eta)}{(-2\pi i)^{k+2}} \prod_\ell f_{\widehat{\phi_\ell}, |\cdot|^{1/2+k}\eta_\ell, |\cdot|^{-1/2}},$$

where the functions in the product are the Siegel sections of Proposition 4.2.26.

4.7 Definition of Asai–Flach map and classes

4.7.1 Definition of the map

We fix integers $k, k' \geq 0$ such that $k + 2t = k' + 2t'$ and write $\mathscr{D}^{k,k'} := \mathscr{H}_L^{[\lambda]}(-t-t')$, for the relative Chow motive over the infinite level Hilbert modular surface Y_G as defined in 2.3.2.2. We will fix j such that $0 \leq j \leq \min(k, k')$. The goal of this section is to define a map

$$\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}} : \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q}) \otimes \mathscr{H}(G(\mathbb{A}_f), \mathbb{Q}) \longrightarrow H^3_{\mathrm{mot}}(Y_G, \mathscr{D}^{k,k'}(2-j))$$

that is $H(\mathbb{A}_f) \times G(\mathbb{A}_f)$ equivariant, with actions given as follows

- $H(\mathbb{A}_f)$ acts trivially on the target and it acts on $\mathscr{S}(\mathbb{A}_f^2, \mathbb{Q}) \otimes \mathscr{H}(G(\mathbb{A}_f), \mathbb{Q})$ via
$$h \cdot (\phi \otimes \xi) = (h \cdot \phi) \otimes \xi(h^{-1}(-)).$$
- $G(\mathbb{A}_f)$ acts via the natural action on $H^3_{\mathrm{mot}}(Y_G, \mathscr{D}^{k,k'}(2-j))$ and on the source via
$$g \cdot (\phi \otimes \xi) = \phi \otimes \xi((-)g).$$

We consider open compact subgroups $U \subset G(\mathbb{A}_f)$ such that the natural map

$$\iota_U : Y_H(U \cap H) \to Y_G(U)$$

is a closed embedding. It is easy to check that this holds for U sufficiently small. We then have that the Hecke algebra $\mathscr{H}(G(\mathbb{A}_f), \mathbb{Z})$ is generated as a $\mathbb{Z}$-module by the functions of the form $\mathrm{ch}(gU)$ where $g \in G(\mathbb{A}_f)$ and U is as above.

Definition 4.7.1. Fix an Haar measure on $H(\mathbb{A}_f)$ and let $V \subset H(\mathbb{A}_f)$ an open compact subgroup. We define a map $A_V : \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q}) \to \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q})^V$ by

$$A_V(\phi) := \int_V h \cdot \phi dh = \mathrm{Vol}(W) \cdot \sum_{v \in V/W} v \cdot \phi,$$

where W is an open compact subgroup of V fixing ϕ.

The following lemma is an immediate consequence of the definition.

Lemma 4.7.2. *If $V' \subset V$, we have $A_V(\phi) = \sum v \cdot A_{V'}(\phi)$, where $V = \bigsqcup vV'$.*

Now let $x \in G(\mathbb{A}_f)$ and U such that xUx^{-1} is sufficiently small. Let

$$\xi = \mathrm{ch}(xU) \in \mathscr{H}(G(\mathbb{A}_f), \mathbb{Z}), \quad V = H \cap xUx^{-1}.$$

We denote by ι_{xU} the closed embedding obtained by

$$\iota_{xU} : Y_H(V) \xhookrightarrow{\iota_{xUx^{-1}}} Y_G(xUx^{-1}) \xrightarrow[\cdot x]{\simeq} Y_G(U).$$

Moreover (2.3.5) gives a map $CG^{[k,k',j]}_{\mathrm{mot}}$

$$H^i_{\mathrm{mot}}(Y_H(V), \mathrm{TSym}^{k+k'-2j}\,\mathscr{H}_L(n))[j+t+t'] \to H^i_{\mathrm{mot}}(Y_H(V), \iota^*(\mathscr{H}_L^{[\lambda]})(n-j-(t+t'))),$$

where we added the twist by the $(j+t+t')$-th power of the determinant, meaning tensoring with the one dimensional representation on which $H(\mathbb{A}_f)$ acts as $(j+t+t')$-th power of the determinant. One also has, as in (2.3.1), a pushforward map

$$(\iota_{xUx^{-1}})_* : H^i_{\mathrm{mot}}(Y_H(V), \iota^*_{xUx^{-1}}(\mathscr{H}_L^{[\lambda]})(n)) \to H^{i+2}_{\mathrm{mot}}(Y_G(xUx^{-1}), \mathscr{H}_L^{[\lambda]}(n+1)).$$

Composing such morphisms for $i = 1, n = 1$ with the isomorphism in cohomology[3] induced by multiplication by x, we obtain a map

$$H^1_{\mathrm{mot}}(Y_H(V), \mathrm{TSym}^{k+k'-2j}\,\mathscr{H}_{\mathbb{Q}}(1))[j+t+t'] \xrightarrow{\iota^{[k,k',j]}_{xU,*}} H^3_{\mathrm{mot}}(Y_G(U), \mathscr{H}_L^{[\lambda]}(2-j-(t+t'))).$$

We also have, from the previous chapter, a $H(\mathbb{A}_f)$-equivariant map

$$\begin{aligned}\mathscr{S}(\mathbb{A}_f^2, \mathbb{Q}) &\to H^1_{\mathrm{mot}}(Y_H, \mathrm{TSym}^{k+k'-2j}\,\mathscr{H}_{\mathbb{Q}}(1))\\ \phi &\longmapsto \mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},\phi}.\end{aligned}$$

In particular if $\phi \in \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q})^V$ for some $V \subset H(\mathbb{A}_f)$, we have $\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},\phi} \in H^1_{\mathrm{mot}}(Y_H(V), \mathrm{TSym}^{k+k'-2j}\,\mathscr{H}_{\mathbb{Q}}(1))$. We can finally make the following definition:

Definition 4.7.3. The level U motivic Asai–Flach map $\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}\ ,U}$ for k, k', j and U

[3] Note that by abuse of notation we denote by $\mathscr{H}_L^{[\lambda]}$ both the relative Chow motive on $Y_G(U)$ and on $Y_G(xUx^{-1})$; the map in cohomology is the one described in 2.3.2.3.

as above is defined by

$$\mathscr{S}(\mathbb{A}_f^2,\mathbb{Q})[j+t+t']\otimes\mathscr{H}(G(\mathbb{A}_f),\mathbb{Z})\longrightarrow H^3_{\mathrm{mot}}(Y_G(U),\mathscr{H}_L^{[\lambda]}(2-j-(t+t')))$$
$$\phi\otimes\xi\longmapsto \iota_{xU,*}^{[k,k',j]}(\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},A_V(\phi)}),$$

where $\xi=\mathrm{ch}(xU)$ as above and $V=H\cap xUx^{-1}$. Since the Hecke algebra is spanned by functions of this form, $\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}$ is defined extending by $\mathbb{Z}$-linearity.

Proposition 4.7.4. *The above defined map satisfies*

(a) If $\xi'=g\cdot\xi$ for $g\in G(\mathbb{A}_f)$, then

$$\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}{}_{,gUg^{-1}}(\phi\otimes\xi')=g\cdot\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}{}_{,U}(\phi\otimes\xi);$$

(b) For every $h\in H(\mathbb{A}_f)$, one has

$$\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}{}_{,U}(h\cdot(\phi\otimes\xi))=\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}{}_{,U}(\phi\otimes\xi);$$

(c) If $U'\subset U$, writing $\pi:Y_G(U')\to Y_G(U)$ for the natural projection map, we find

$$\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}{}_{,U'}(\phi\otimes\xi)=\pi^*\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}{}_{,U}(\phi\otimes\xi).$$

Proof. We prove all the statements for $\xi=\mathrm{ch}(xU)$, which is enough because functions of this form span the Hecke algebra.

(a) We find that $\xi'=\mathrm{ch}(xg^{-1}(gUg))$. Then the statement follows from the commutativity of the following diagram

$$\begin{array}{ccc} Y_G(xUx^{-1}) & \xrightarrow{\cdot x} & Y_G(U) \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \cdot g^{-1}} \\ Y_G(xUx^{-1}) & \xrightarrow{\cdot xg^{-1}} & Y_G(gUg^{-1}). \end{array}$$

together with the fact that the action of g on cohomology is precisely given by the pushforward of the right vertical map.

(b) We have, by definition of the action, $h\cdot(\phi\otimes\xi)=h\cdot\phi\otimes\mathrm{ch}(hxU)$. Writing $V=xUx^{-1}\cap H$ and using $A_{hVh^{-1}}(h\cdot\phi)=h\cdot A_V(\phi)$ we get the desired equality.

(c) It suffices to prove the statement in the case where $U'\trianglelefteq U$, since otherwise we can compare both U and U' with a third open compact normal in both of them). We can write $\xi=\mathrm{ch}(xU)=\sum_{u\in U/U'}\mathrm{ch}(xuU')$. We then use the commutativity of the following diagram

$$\begin{array}{ccc} Y_H(V') & \xrightarrow{\iota_{xU'}} & Y_G(U') \\ \Big\downarrow{\scriptstyle \pi_{V/V'}} & & \Big\downarrow{\scriptstyle \pi} \\ Y_H(V) & \xrightarrow{\iota_{xU}} & Y_G(U), \end{array}$$

where $V=H(\mathbb{A}_f)\cap xUx^{-1}$, $V'=H(\mathbb{A}_f)\cap xU'x^{-1}$ and the vertical arrows are the natural projection maps. We find

$$\begin{aligned}\pi^*\pi_*\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}\ ,U'}(\phi\otimes\mathrm{ch}(xU'))&=\pi^*\pi_*(\iota_{xU'})_*(\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},A_{V'}(\phi)})\\ &=\pi^*(\iota_{xU})_*(\pi_{V/V'})_*(\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},A_{V'}(\phi)})=\pi^*(\iota_{xU})_*(\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},A_V(\phi)})=\pi^*\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}\ ,U'}(\phi\otimes\mathrm{ch}(xU)),\end{aligned}$$

where in the second to last equality we applied the $H(\mathbb{A}_f)$-equivariance of the Eisenstein map and Lemma 4.7.2. Since $\pi^*\circ\pi_*=\sum_{u\in U/U'}u$ and $\sum_{u\in U/U'}u\cdot\mathrm{ch}(xU')=\mathrm{ch}(xU)$, we can apply (a) and the assumption on U' being normal in U to conclude. □

Definition 4.7.5. We define

$$\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}:\mathscr{S}(\mathbb{A}_f^2,\mathbb{Q})[j+t+t']\otimes\mathscr{H}(G(\mathbb{A}_f),\mathbb{Z})\longrightarrow H^3_{\mathrm{mot}}(Y_G,\mathscr{H}_L^{[\lambda]}(2-j-(t+t')))$$

to be the direct limit $\varinjlim_U\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}\ ,U}$. This is well defined thanks to (c) in the above Proposition and is $H(\mathbb{A}_f)\times G(\mathbb{A}_f)$-equivariant with respect to the action given above thanks to (a)-(b) in the above Proposition.

4.7.2 Definition of the classes in motivic cohomology

In order to define the Asai–Flach elements in motivic cohomology, we will specify the choice of an element in $\mathscr{S}(\mathbb{A}_f^2,\mathbb{Q})\otimes\mathscr{H}(G(\mathbb{A}_f),\mathbb{Z})$ to which we will apply $\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}$.

We start by fixing a prime p, a finite set of primes S not containing p. Our choice will also depend on integers $m, M \geq 1$ with M coprime to S and p. We now will define $K_{M,m,n} \subset G(\mathbb{A}_f), W \subset H(\mathbb{A}_f)$ and $\phi_{M,m,n} \in \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q}), \xi_{M,m,n} \in \mathscr{H}(G(\mathbb{A}_f), \mathbb{Z})$ satisfying certain properties and apply $\mathscr{AF}_{\mathrm{mot}}^{\mathrm{k,k',j}}$ in order to define an element

$$z_{M,m,n}^{[k,k',j]} := \frac{1}{\mathrm{Vol}(W)} \mathscr{AF}_{\mathrm{mot}}^{\mathrm{k,k',j}}(\phi_{M,m,n} \otimes \xi_{M,m,n}) \in H^3_{\mathrm{mot}}(Y_G(K_{M,m,n}), \mathscr{H}_L^{[\lambda]}(2-j-(t+t'))).$$

Every definition of such data will be given in term of local data. Writing $\mathbb{Q}_S = \prod_{\ell \in S} \mathbb{Q}_\ell$ we will define

- subgroups $K_S \subset G(\mathbb{Q}_S), K_{p,n} \subset G(\mathbb{Q}_p)$ and let

$$K_n := K_S \times K_{p,n} \times \prod_{\ell \notin S \cup \{p\}} G(\mathbb{Z}_\ell) \subset G(\mathbb{A}_f);$$

- A subgroup $K_{M,m,n} \subset K_n$, defined by $K_n \cap \det^{-1}(1 + Mp^m \hat{\mathscr{O}}_F)$;
- functions $\phi_S \in \mathscr{S}(\mathbb{Q}_S^2, \mathbb{Z}), \phi_\ell \in \mathscr{S}(\mathbb{Q}_\ell^2, \mathbb{Z})$ for $\ell \notin S$ and let

$$\phi_{M,m,n} := \phi_S \otimes \bigotimes_{\ell \notin S} \phi_\ell;$$

- elements $\xi_\ell \in \mathscr{H}(G(\mathbb{Q}_\ell), \mathbb{Z})$ for $\ell \notin S$ and let

$$\xi_{M,m,n} := \mathrm{ch}(K_S) \otimes \bigotimes_{\ell \notin S} \xi_\ell;$$

- an open compact subgroup $W \subset H(\mathbb{A}_f)$ defined choosing $W_S \subset H(\mathbb{Q}_S) \cap K_S$ acting trivially on ϕ_S and $W_\ell \subset H(\mathbb{Q}_\ell)$ for $\ell \notin S$ and letting

$$W := W_S \times \prod_{\ell \notin S} W_\ell.$$

We consider fixed the choices at S and require that the global elements satisfy the following

(i) $\xi_{M,m,n}$ is fixed by right translation of $K_{M,m,n}$,

(ii) $\xi_{M,m,n}$ is fixed by left translation of W,

(iii) $\phi_{M,m,n}$ is stable under the action of W.

We first define the level subgroup K_n. We are only left with saying what is the choice at p. We let

$$K_{p,n} := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{Z}_p) : c \equiv d-1 \equiv 0 \bmod p^n \}.$$

The desired subgroup $K_{M,m,n}$ will then be given at p by

$$\{g \in K_{p,n} : \det g \equiv 1 \bmod Mp^m\hat{\mathbb{Z}}\}.$$

Write $K^*_{M,m,n} := K_{M,m,n} \cap G^*(\mathbb{A}_f) \subset K^*_n := K_n \cap G^*(\mathbb{A}_f)$. We then have (*cf.* [LSZ20a, Proposition 5.4.2]) that the determinant map induces an isomorphism

$$Y_{G^*}(K^*_{M,m,n}) \simeq Y_{G^*}(K^*_n) \times_{\mathbb{Q}} \mu_{Mp^m}, \tag{4.7.1}$$

where as in the previous Chapter μ_{Mp^m} denotes the group scheme of primitive Mp^m-th roots of unity. We now define the local terms of $\xi_{M,m,n}, \phi_{M,m,n}, W$ at places $\ell \notin S$, dividing the three cases $\ell \nmid Mp, \ell \mid M, \ell = p$. First of all we write, for $r \geq 0$,

$$\eta_{\ell,r} := \begin{cases} \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \frac{1}{\ell^r} \\ 0 & 1 \end{smallmatrix}\right) \right) \in \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell) & \text{if } \ell \text{ splits} \\ \left(\begin{smallmatrix} 1 & \delta\frac{1}{\ell^r} \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(F_\ell) & \text{if } \ell \text{ is inert.} \end{cases}$$

This is the element η_r defined in Definition 4.5.17.

$\underline{\ell \nmid Mp}$: We let

$$\xi_\ell = \mathrm{ch}(G(\mathbb{Z}_\ell)), \quad W_\ell = H(\mathbb{Z}_\ell), \quad \phi_\ell = \mathrm{ch}(\mathbb{Z}_\ell^2).$$

<u>$\ell \mid M$</u>: First we define $K_{\ell,1} = \{g \in G(\mathbb{Z}_\ell) : \det g \equiv 1 \bmod \ell\}$. We then let

$$\xi_\ell = \mathrm{ch}(K_{\ell,1}) - \mathrm{ch}(\eta_{\ell,1}K_{\ell,1}),$$

$$W_\ell = \{h \in H(\mathbb{Z}_\ell) : \det h \equiv 1 \bmod \ell, h \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod \ell^2\},$$

$$\phi_\ell = \mathrm{ch}(\ell^2\mathbb{Z}_\ell \times (1+\ell^2\mathbb{Z}_\ell)).$$

<u>$\ell = p$</u>: We define $K_{p,m,n} = \{g \in G(\mathbb{Z}_p) : \det g \equiv 1 \bmod p^m, g \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p^n\}$. Let

$$\xi_p = \mathrm{ch}(\eta_{p,m}K_{p,m,n}).$$

We then choose an integer $t \geq 1$ big enough such that $W_p \subset \eta_{p,m}K_{p,m,n}\eta_{p,m}^{-1}$, where

$$W_p = \{h \in H(\mathbb{Z}_\ell) : \det h \equiv 1 \bmod p^m, h \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p^t\}.$$

Finally for such choice of t we let

$$\phi_p = \mathrm{ch}(p^t\mathbb{Z}_p \times (1+p^t\mathbb{Z}_p)).$$

It follows easily from the definitions that conditions (i),(ii),(iii) above are satisfied. We finally can, as anticipated above, make the following definition:

Definition 4.7.6. For M,m,n and $W, \phi_{M,m,n}, \xi_{M,m,n}$ as above, we define

$$z_{M,m,n}^{[k,k',j]} := \tfrac{1}{\mathrm{Vol}(W)}\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}}(\phi_{M,m,n}\otimes\xi_{M,m,n}) \in H^3_{\mathrm{mot}}(Y_G, \mathscr{H}^{[\lambda]}(2-j-(t+t')))$$

Lemma 4.7.7. *The above definition is independent on the choice of the Haar measure on $H(\mathbb{A}_f)$ and on the choice of t at the place p.*

Proof. Writing $U := K_{M,m,n}$, we have, from (i) and (ii) that $\xi_{M,m,n} \in \mathscr{H}(W\backslash G/U)$. We rewrite it as

$$\xi_{M,m,n} = \sum \mathrm{ch}(x_iU),$$

where $\mathrm{ch}(x_iU)$ is left invariant under W, i.e. $W \subset V_i := H(\mathbb{A}_f) \cap x_iUx_i^{-1}$. Hence writing $\iota_i := \iota_{x_iU}$ we have that by definition our classes are

$$\frac{1}{\mathrm{Vol}(W)}\sum_i (\iota_i)_* \circ CG^{[k,k',j]}_{\mathrm{mot}}(\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},A_{V_i}(\phi)}),$$

where $\phi = \phi_{M,m,n}$. Using (iii), the definition of the averaging map and the fact that $\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},-}$ is $H(\mathbb{A}_f)$-equivariant, we can write $\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},A_{V_i}(\phi)} = \mathrm{Vol}(W)\sum_{v\in V_i/W} v\cdot \mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},\phi}$, from which the independence on the Haar measure becomes clear.

Write now W for the subgroup defined by the condition at p with a fixed choice of t and W^0 for the subgroup defined with a different choice, say $t_0 > t$. We similarly write $\phi_{M,m,n}$ and $\phi^0_{M,m,n}$. We can write

$$\phi_{M,m,n} = \sum_{w\in W/W^0} w\cdot \phi^0_{M,m,n}.$$

We obtain

$$\begin{aligned}
\frac{1}{\mathrm{Vol}(W)}\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}(\phi_{M,m,n}\otimes\xi_{M,m,n}) &= \frac{1}{\mathrm{Vol}(W)}\sum_w \mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}(w\cdot\phi^0_{M,m,n}\otimes\xi_{M,m,n})\\
&= \frac{1}{\mathrm{Vol}(W)}\sum_w \mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}(w\cdot(\phi^0_{M,m,n}\otimes\xi_{M,m,n}))\\
&= \frac{[W:W^0]}{\mathrm{Vol}(W)}\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}(\phi^0_{M,m,n}\otimes\xi_{M,m,n})\\
&= \frac{1}{\mathrm{Vol}(W^0)}\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}(\phi^0_{M,m,n}\otimes\xi_{M,m,n}).
\end{aligned}$$

In the second equality we used (ii), and in the third the fact that $\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}$ is $H(\mathbb{A}_f)$-equivariant. $\square$

Condition (i) together with the fact that $\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}}$ is $G(\mathbb{A}_f)$-equivariant implies

$$z^{[k,k',j]}_{M,m,n} \in H^3_{\mathrm{mot}}(Y_G(K_{M,m,n}),\mathscr{H}^{[\lambda]}(2-j-(t+t'))),$$

as wanted.

4.7.3 Comparison with Chapter 3 and [LLZ18]

In order to recover the definition of Asai–Flach classes given in the previous chapter for $k = k' = 0$ (and more in general in [LLZ18]), we fix $\mathfrak{N}$ an ideal of $\mathscr{O}_F$ coprime to p. Write $(N) = \mathfrak{N} \cap \mathbb{Z}$ and choose the set S to be given by primes $\ell \mid N$. For simplicity we will assume N is square-free, in particular $\mathfrak{N} = \prod_{\ell \mid N} \mathfrak{L}_\ell$, where $\mathfrak{L}_\ell$ is a prime ideal of $\mathscr{O}_F$ above ℓ. We then let $K_S = \prod_{\ell \mid N} K_\ell \subset G(\mathbb{Q}_S)$, where

$$K_\ell := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{Z}_\ell) : c \equiv d-1 \equiv 0 \bmod \mathfrak{L}_\ell \}.$$

We also choose $\phi_S \in \mathscr{S}(\mathbb{Q}_S^2, \mathbb{Z})$ to be $\otimes_{\ell \mid N} \mathrm{ch}(\ell\mathbb{Z}_\ell \times (1+\ell\mathbb{Z}_\ell))$. We finally let $W_S = \prod_{\ell \mid N} W_\ell$, where

$$W_\ell := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H(\mathbb{Z}_\ell) : c \equiv d-1 \equiv 0 \bmod \ell \}.$$

Pulling back $z_{M,m,n}^{[k,k',j]}$ to Y_{G^*}, we find elements in

$$H^3_{\mathrm{mot}}(Y_1(\mathfrak{N}p^n) \times \mu_{Mp^m}, \mathrm{TSym}^{[k,k']} \mathscr{H}(\mathscr{A})(2-j)).$$

If $M = 1, m = 0$, these are the classes $\mathrm{AF}^{[k,k',j]}_{\mathrm{mot},p^n}$ of [LLZ18, Definition 3.4.2]. In particular, if $k = k' = j = 0$, we recover the classes $\mathrm{AF}_{1,\mathfrak{N}p^n}$ of Definition 3.1.1. To see this, notice that, from the choice of the local data, the averaging map sends $\phi_{M,m,n}$ to $\mathrm{Vol}(W)\phi_{M,m,n}$. So what our map does is simply sending $\phi_{M,m,n} \otimes \xi_{M,m,n}$ to $\iota_*(\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},\phi_{M,m,n}})$, where ι is the closed embedding

$$Y_1(Np^n) \hookrightarrow Y_1^*(\mathfrak{N}p^n)$$

and $\phi_{M,m,n} = \mathrm{ch}(Np^n\hat{\mathbb{Z}} \times (1+Np^n\hat{\mathbb{Z}}))$. In particular, if $k = k' = j = 0$, Theorem 4.6.1(1) tells us that the class constructed is precisely $\iota_*(g_{0,1/N})$ as in Definition 3.1.1.

The classes $\mathrm{AF}^{[k,k',j]}_{\mathrm{mot},Mp^m,p^n,a}$ are defined in [LLZ18, Definition 3.5.1] (where the trivial coefficient case is Definition 3.1.4/Lemma 3.1.5) using a twist by the matrix $\left(\begin{smallmatrix} 1 & \frac{a}{Mp^m} \\ 0 & 1 \end{smallmatrix}\right)$, for any $a \in \mathscr{O}_F/(Mp^m\mathscr{O}_F + \mathbb{Z})$. The role played by the Hecke algebra $\mathscr{H}(G(\mathbb{A}_f))$ in the definition of our map is exactly to produce such perturbation of the embedding ι. Moreover, the input elements $\xi_{M,m,n} \in \mathscr{H}(G(\mathbb{A}_f))$ we are considering involve matrices of the same form for a specific choice of $a \in \mathscr{O}_F \otimes$

$\hat{\mathbb{Z}}/(Mp^m+\hat{\mathbb{Z}})$. More precisely we would get the same classes of [LLZ18] if we took the ℓ-component of $\xi_{M,m,n}$ for $\ell \mid M$ to be $\mathrm{ch}(\eta_{\ell,1}K_{\ell,1})$. The term $\mathrm{ch}(K_{\ell,1})$ appearing in our definition can be thought as a *correction term*. Indeed the tame norm relations we will prove are not the same as the ones obtained in Corollary 3.3.4 (and more generally in [LLZ18, Theorem 3.5.3, Corollary 4.3.8]), where the factor appearing is the local Euler factor $P(\sigma_\ell^{-1}\ell^{-1-j})$ summed with a term divisible by $\ell-1$. Thanks to the correction term at the primes dividing M, we in fact do not get this extra factor and we are able to show that our classes form an Euler system, with no need to lift the classes using [LLZ14, Lemma 7.3.4] ([Rub00, Lemma IX.6.1]).

4.8 The Asai–Flach Euler systems norm relations

4.8.1 Pushforward compatibilities in motivic cohomology

We now prove that the classes just defined satisfy compatibility properties if we vary the level K_n and if we vary the cyclotomic field in the p-direction.

Theorem 4.8.1. *For $n \geq 1$ we have, writing $\pi_n : Y_G(K_{M,m,n+1}) \to Y_G(K_{M,m,n})$ for the natural projection,*

$$(\pi_n)_*(z^{[k,k',j]}_{M,m,n+1}) = z^{[k,k',j]}_{M,m,n}.$$

Proof. Going back to the definition of the local data in §4.7.2, we see that the only place where these differ is p, where

$$\xi_p = \mathrm{ch}(\eta_{p,m}K_{p,m,n}),$$

while we can choose the same t sufficiently large, so that we have the same W and the same ϕ_p in the definition of $z^{[k,k',j]}_{M,m,n+1}$ and $z^{[k,k',j]}_{M,m,n}$. So locally at p, we need to check that

$$(\pi_n)_*((\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}})_p(\phi\otimes \mathrm{ch}(xK_{p,m,n+1}))) = (\mathscr{AF}^{\mathrm{k,k',j}}_{\mathrm{mot}})_p(\phi\otimes \mathrm{ch}(xK_{p,m,n})).$$

But this is true, since we can write $\mathrm{ch}(xK_{p,m,n}) = \sum_{k\in K_{p,m,n}/K_{p,m,n+1}} \mathrm{ch}(xkK_{p,m,n+1})$ and the pushforward act on cohomology by multiplication of coset representatives

$k \in K_{p,m,n}/K_{p,m,n+1}$. $\square$

The following theorem is essentially the proof of the vertical type Euler system norm relation for the classes we will obtain in Galois cohomology in the next section starting with the motivic input $z^{[k,k',j]}_{M,m,n}$.

Theorem 4.8.2. *For $m \geq 1$ we have, writing $\pi_m : Y_G(K_{M,m+1,n}) \to Y_G(K_{M,m,n})$ for the natural projection,*

$$(\pi_m)_*(z^{[k,k',j]}_{M,m+1,n}) = \begin{cases} \frac{U'(p)}{p^{j+t+t'}} & \\ \left(\frac{U'(p)}{p^{j+t+t'}} - 1\right) & \end{cases} \cdot z^{[k,k',j]}_{M,m,n} \quad \begin{matrix} \text{if } m \geq 1 \\ \text{if } m = 0, \end{matrix}$$

where $U'(p)$ is the Hecke operator in $\mathscr{H}(K_{p,m,n}\backslash G(\mathbb{Q}_p)/K_{p,m,n})$ given by the double coset of $\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Proof. This theorem follows from the choice of the local data and from Proposition 4.5.19. As in the previous theorem, the elements $\phi_{M,m+1,n} \otimes \xi_{M,m+1,n}$ and $\phi_{M,m,n} \otimes \xi_{M,m,n}$ are the same at places different from p. Hence we are comparing two values of the p-part map

$$(\mathscr{AF}^{\text{k,k',j}}_{\text{mot}})_p : \mathscr{S}(\mathbb{Q}_p)[j+t+t'] \otimes \mathscr{H}(G(\mathbb{Q}_p)) \to H^3_{\text{mot}}(Y_G, \mathscr{H}^{[\lambda]}_L(2-j-(t+t'))),$$

which is $H(\mathbb{Q}_p) \times G(\mathbb{Q}_p)$-equivariant. Since it is enough to check the equality after tensoring with $\mathbb{C}$, we can apply Proposition 4.5.19. Indeed, reasoning as in the proof of the previous theorem, we have, that on the left hand side we have $\frac{1}{\text{Vol}(W_{p,m+1})}(\mathscr{AF}^{\text{k,k',j}}_{\text{mot}})_p(\phi_p \otimes \text{ch}(\eta_{p,m+1}K_{p,m,n}))$, where $\phi_p = \text{ch}(p^t\mathbb{Z}_p \times (1+p^t\mathbb{Z}_p))$ and $W_{p,m+1} = \{h \in H(\mathbb{Z}_\ell) : \det h \equiv 1 \bmod p^{m+1}, h \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p^t\}$. Hence the classes we need to compare are

$$[K_{H,1}(p^t) : W_{p,m+1}](\mathscr{AF}^{\text{k,k',j}}_{\text{mot}})_p(\phi_{1,\infty} \otimes \text{ch}(\eta_{p,m+1}K_{p,m,n})),$$

$$[K_{H,1}(p^t) : W_{p,m}](\mathscr{AF}^{\text{k,k',j}}_{\text{mot}})_p(\phi_{1,\infty} \otimes \text{ch}(\eta_{p,m}K_{p,m,n})).$$

The proposition tells us, together with Remark 4.5.21, that $(\mathscr{AF}^{\text{k,k',j}}_{\text{mot}})_p(\phi_{1,\infty} \otimes$

$\mathrm{ch}(\eta_{p,m+1}K_{p,m,n})) =$

$$\begin{cases} \frac{p^{-(j+t+t')}}{p}U'(p) & \\ \frac{1}{p-1}(U'(p)p^{-(j+t+t')}-1) & \end{cases} \cdot (\mathscr{A}\mathscr{F}^{\mathrm{k,k',j}}_{\mathrm{mot}})_p(\phi_{1,\infty}\otimes \mathrm{ch}(\eta_{p,m}K_{p,m,n})) \quad \begin{matrix} \text{if } m \geq 1 \\ \text{if } m = 0. \end{matrix}$$

The factors $\frac{1}{p}, \frac{1}{p-1}$ cancels out since $[K_{H,1}(p^t):W_{p,m}]/[K_{H,1}(p^t):W_{p,m+1}] = [W_{p,m}:W_{p,m+1}]$ is equal to $p, p-1$ respectively. □

4.8.2 Hilbert cuspforms and Galois representations

In the previous chapter, we constructed some classes

$$z^{[k,k',j]}_{M,m,n} \in H^3_{\mathrm{mot}}(Y_G(K_{M,m,n}), \mathscr{H}^{[\lambda]}(2-j-(t+t'))).$$

We now will realize these classes in étale cohomology and use the Hochschild–Serre spectral sequence to find elements in Galois cohomology of the representation attached to a weight $(k+2,k'+2)$ Hilbert cuspform. We will then show that they satisfy the Euler system norm relations.

We consider f a cuspidal Hilbert newform of weight $(k+2,k'+2)$ and of level $K_f \subset G(\mathbb{A}_f)$. We assume $k \equiv k' \bmod 2$ and we write $w = k+2+2t = k'+2+2t'$. Denote by L the number field generated by the Hecke eigenvalues $\{\lambda_{\mathfrak{m}}\}_{\mathfrak{m}\subset\mathscr{O}_F}$ and fix a prime p. We fix an arbitrary place v of L dividing p.

Consider the Asai representation attached to f as in Definition 2.4.6 and the Asai L-function, which was defined by an Euler product of factors $P^{\mathrm{As}}_{\ell}(f,\ell^{-s})$ as in Definition 2.4.8.

Recall that from the action of $G(\mathbb{A}_f)$ on $H^2_{\text{ét}}\left((Y_G)_{\bar{\mathbb{Q}}}, \mathscr{H}^{(\lambda)}_{L_v}(t+t')\right)$ we obtain the finite part of the automorphic representation corresponding to f. We will denote it by $\Pi_f = \otimes'_{\ell}\Pi_{\ell}$, where Π_{ℓ} is a $G(\mathbb{Q}_{\ell})$-representation and it is spherical for all but finitely many primes ℓ. We can describe these Π_{ℓ} and relate the local L-factor with the Asai Euler factor at ℓ using Proposition 2.4.9.

Proposition 4.8.3. *For ℓ as above, let*

$$\sigma = \begin{cases} I_{G(\mathbb{Q}_\ell)}(\underline{\chi}, \underline{\psi}) & \textit{if } \ell \textit{ splits} \\ I_{G(\mathbb{Q}_\ell)}(\chi, \psi) & \textit{if } \ell \textit{ is inert}, \end{cases}$$

where

$$\begin{aligned} \chi_1(\ell) &= \alpha_1 \ell^{-1/2}, \quad \psi_1(\ell) = \beta_1 \ell^{-1/2}, \\ \chi_2(\ell) &= \alpha_2 \ell^{-1/2}, \quad \psi_2(\ell) = \beta_2 \ell^{-1/2}. \end{aligned} \quad \textit{and} \quad \chi(\ell) = \alpha \ell^{-1}, \psi(\ell) = \beta \ell^{-1}.$$

We then find that $\Pi_\ell \simeq \sigma$ and

$$P_\ell^{\mathrm{As}}(f, \ell^{-1-s+t+t'}) = \begin{cases} L(\sigma, s) \\ L(\mathrm{As}(\sigma), s) \end{cases}$$

Proof. This follows from Proposition 2.4.9 and by applying Theorem 4.2.21 and Remark 4.2.22. First we deal with the split prime case. At a place ℓ as above the spherical representation is determined by the values $\chi_i(\ell), \psi_i(\ell)$ being the roots of $X^2 - \ell^{-1/2}\lambda_i X + \mu_i$, where λ_i, μ_i are the eigenvalues of $T(\mathfrak{l}_i), R(\mathfrak{l}_i)$. Since f is a newform we have $\lambda_i = a_{\mathfrak{l}_i}(f)$ and $\mu_i = \ell^{w-2}\varepsilon_{\ell_i}(f)$. Hence we need to solve

$$\begin{aligned} a_{\mathfrak{l}_1}(f) &= \alpha_1 + \beta_1 = \ell^{1/2}(\chi_1(\ell) + \psi_1(\ell)), \quad \ell^{w-2}\varepsilon_{\mathfrak{l}_1}(f) = \ell^{-1}\alpha_1\beta_1 = \chi_1(\ell)\psi_1(\ell); \\ a_{\mathfrak{l}_2}(f) &= \alpha_2 + \beta_2 = \ell^{1/2}(\chi_2(\ell) + \psi_2(\ell)), \quad \ell^{w-2}\varepsilon_{\mathfrak{l}_2}(f) = \ell^{-1}\alpha_2\beta_2 = \chi_2(\ell)\psi_2(\ell). \end{aligned}$$

From where we find the claimed values of $\chi_i(\ell), \psi_i(\ell)$.

For the inert prime case we proceed similarly, finding $\chi(\ell), \psi(\ell)$ to be roots of $X^2 - (\ell^2)^{-1/2}\lambda X + \mu$, where λ, μ are the eigenvalues of $T(\ell), R(\ell)$. Now $\lambda = a_\ell(f)$ and $\mu = \ell^{2(w-2)}\varepsilon_\ell(f)$. Hence from

$$a_\ell(f) = \alpha + \beta = \ell(\chi(\ell) + \psi(\ell)), \quad \ell^{2(w-2)}\varepsilon_\ell(f) = \ell^{-2}\alpha\beta = \chi(\ell)\psi(\ell)$$

we find the claimed values of $\chi(\ell), \psi(\ell)$. $\square$

We use the characterisation of the local components of $\Pi = \Pi_f$ obtained in the previous corollary to prove that if the Hilbert modular form is not a base change lift of a modular form of $\mathrm{GL}_2/\mathbb{Q}$, then a certain Hom-space is zero. We denote by ω_Π the Hecke character of F given by the central character of Π and we let χ_{Π_ℓ} be the character of $\mathbb{Q}_\ell^\times$ given by the restriction of Π_ℓ to the center of $H(\mathbb{Q}_\ell)$.

Proposition 4.8.4. *Let τ be the representation of $H(\mathbb{A}_f)$ given by $\gamma(\mathrm{det})$, where γ is a character of the idèles of $\mathbb{Q}$ such that γ_ℓ^2 is equal to χ_{Π_ℓ} for every ℓ. If Π is not a twist of a base change lift of a cuspidal representation of $H(\mathbb{A}_f)$, then*

$$\mathrm{Hom}_{H(\mathbb{A}_f)}(\Pi,\tau)=0.$$

Proof. We will assume for simplicity that γ is trivial. If $\mathrm{Hom}_{H(\mathbb{A}_f)}(\Pi,\tau)\neq 0$ then $\mathrm{Hom}_{H(\mathbb{Q}_\ell)}(\Pi_\ell,\mathbf{1})\neq 0$ for every ℓ. In particular for all primes ℓ as above which split in F, we have

$$\mathrm{Hom}_{H(\mathbb{Q}_\ell)}(I(\chi_1,\psi_1)\otimes I(\chi_2,\psi_2),\mathbf{1})=\mathrm{Hom}_{H(\mathbb{Q}_\ell)}(I(\chi_1,\psi_1),I(\chi_2^{-1},\psi_2^{-1}))\neq 0.$$

Hence $\Pi_\ell=\Pi_\lambda\otimes\Pi_{\bar\lambda}$ is of the form

$$I(\chi_1,\psi_1)\otimes I(\chi_1^{-1},\psi_1^{-1})\quad\text{or}\quad I(\chi_1,\psi_1)\otimes I(\psi_1^{-1},\chi_1^{-1}).$$

Hence $\Pi_\lambda\simeq\Pi_{\bar\lambda}\otimes\chi_1\psi_1$. Letting σ be the non trivial automorphism of $F/\mathbb{Q}$ and $\sigma(\Pi)_\lambda=\Pi_{\sigma(\lambda)}$, the representations Π and $\sigma(\Pi)\otimes\omega_\Pi$ are isomorphic at all but finitely many primes. This follows from the above reasoning for all but finitely many split primes; for inert primes we have $\sigma(\Pi)_\lambda=\Pi_\lambda$ and $\mathrm{Hom}_{H(\mathbb{Q}_\ell)}(\Pi_\ell,\mathbf{1})\neq 0$ forces the central character of Π_λ to be trivial. Moreover ω_Π restricted to the idèles of $\mathbb{Q}$ is trivial. We can then apply [LR98, Theorem 2(a)], which implies that Π is a twist of a base change lift of a cuspidal representation of $\mathrm{GL}_2/\mathbb{Q}$ and reach the desired contradiction.

If γ is not trivial, we proceed as above and obtain that for all but finitely many split primes $\Pi_\lambda\simeq\Pi_{\bar\lambda}\otimes\chi_1\psi_1\gamma_\ell^{-1}$; for inert primes we have $\Pi_\ell\simeq\Pi_\ell\otimes\omega_{\Pi_\ell}\cdot(\gamma_\ell^{-1}\circ\mathrm{Nm}_{F_\ell/\mathbb{Q}_\ell})$. We find as above that Π is a twist of a base change lift of a cuspidal representation of $\mathrm{GL}_2/\mathbb{Q}$, since it is isomorphic to a twist of $\sigma(\Pi)$ by a Hecke character trivial on the idèles of $\mathbb{Q}$. $\square$

We now see that the Asai representation appears in the parabolic étale cohomology of Y_G. Write $\lambda = (k,k',t,t')$.

Definition 4.8.5. We define $\mathscr{H}_{L_v}^{[\lambda]}$ to be the étale sheaf of L_v-vector spaces on Y_G, for U sufficiently small, which is the étale realisation of the motivic sheaf $\mathscr{H}_L^{[\lambda]}$ of §2.3.2.2. We denote by $\mathscr{H}_{L_v}^{(\lambda)}$ its dual.

For simplicity let $\mathscr{L} := \mathscr{H}_{L_v}^{(\lambda)}(t+t')$. We consider parabolic étale cohomology: let Y_G^{BB} be the Bailey-Borel compactification of Y_G and write $j : Y_G \to Y_G^{BB}$ for the natural open embedding. Then *parabolic cohomology* is defined by

$$H^i_{\text{ét},!}(Y_{G,\bar{\mathbb{Q}}},\mathscr{L}) = \lim_{\to K} H^i_{\text{ét}}((Y_G(K)^{BB})_{\bar{\mathbb{Q}}}, j_{!*}\mathscr{L}).$$

These cohomology groups have both a $G_{\mathbb{Q}}$ and a $G(\mathbb{A}_f)$ action.

Theorem 4.8.6 ([Nek18],[BL84])**.** *Let $\mathscr{L}$ be as above, with $\lambda = (k,k',t,t')$ where $k+2t = k'+2t'$. There is a $G_{\mathbb{Q}} \times G(\mathbb{A}_f)$-equivariant decomposition*

$$H^2_{\text{ét},!}(Y_{G,\bar{\mathbb{Q}}},\mathscr{L}) = \bigoplus_{\Pi} V_\Pi \otimes \Pi^\vee,$$

where Π runs over the finite part of cuspidal automorphic representations $\Pi\otimes\Pi_\infty$ of G where Π_∞ is a discrete series of weight $(k+2,k'+2)$. We denote by $\Pi^\vee$ its dual $G(\mathbb{A}_f)$-representation and V_Π is the $G_{\mathbb{Q}}$-representation defined by the tensor induction of ρ_Π twisted by $t+t'$, where $JL(\rho_\Pi) = \Pi$. In other words, if Π is the automorphic representation generated by a Hilbert cuspidal eigenform f, $\rho_\Pi = \rho_{f,v}$ and $V_\Pi = V_f^{\mathrm{As}}$.

Taking the dual (as $G_{\mathbb{Q}}$-module) of the cohomology group in the theorem, we get a $G_{\mathbb{Q}} \times G(\mathbb{A}_f)$-equivariant decomposition

$$H^2_{\text{ét},!}(Y_{G,\bar{\mathbb{Q}}},\mathscr{H}_{L_v}^{[\lambda]}(2-(t+t'))) = \bigoplus_{\Pi} V_\Pi^* \otimes \Pi^\vee.$$

Let us now fix an automorphic cuspidal representation Π. We have the following

Proposition 4.8.7. *Let $K \subset G(\mathbb{A}_f)$ be a level such that $\Pi^K \neq 0$ and T a set of primes including the ones at which K is ramified. Let I be the maximal ideal of the Hecke algebra away from T given by the kernel of the action on Π^K. Then the localisation at I of $H^i_{\text{ét}}(Y_G(K)_{\bar{\mathbb{Q}}}, \mathscr{L})$ is zero for $i \neq 2$ is 0 and is equal to the localisation of parabolic cohomology for $i = 2$.*

Moreover such localisation is given by

$$\left(H^2_{\text{ét}}(Y_G(K)_{\bar{\mathbb{Q}}}, \mathscr{L})\right)_I = \left(H^2_{\text{ét},!}(Y_G(K)_{\bar{\mathbb{Q}}}, \mathscr{L})\right)_I = V_\Pi \otimes ((\Pi)^\vee[t+t'])^K,$$

In particular the localisation is independent on T.

Proof. For the fact that cuspidal representations contribute only to the degree 2 parabolic cohomology, see [Nek18, (5.9)]. The fact that the canonical map from parabolic cohomology to étale cohomology in an isomorphism when localising at I follows from example from the exact sequence in [Nek18, A6.17]. Finally the Π-component is the only one appearing in the decomposition thanks to strong multiplicity one (see [PS79, Sha74]). □

Now recall that the target of our map $\mathscr{A}\mathscr{F}^{\text{k,k',j}}_{\text{mot}}$ is $H^3_{\text{mot}}(Y_G, \mathscr{H}_L^{[\lambda]}(2-j-(t+t')))$. Let f be a fixed Hilbert eigenform of weight $(k+2, k'+2)$ and Π the corresponding $G(\mathbb{A}_f)$-representation, so that $V_\Pi^* = (V_f^{\text{As}})^*$. In order to find classes in Galois cohomology of $(V_f^{\text{As}})^*$ we will, roughly, use the continuous étale realisation map and then apply the above proposition together with the Hochschild–Serre spectral sequence. We will find a $G(\mathbb{A}_f)$-equivariant map

$$pr_\Pi : H^3_{\text{mot}}(Y_G, \mathscr{H}_L^{[\lambda]}(2-j-(t+t'))) \longrightarrow H^1(\mathbb{Q}, (V_\Pi)^*(-j)) \otimes \Pi^\vee.$$

We work for any K level subgroup of $G(\mathbb{A}_f)$.

- We have (see [Hub00]) a realisation functor for continuous étale cohomology (as defined in [Jan88]) for varieties defined over $\mathbb{Q}$

$$r_{\text{ét}}: H^3_{\text{mot}}(Y_G(K), \mathscr{H}_L^{[\lambda]}(2-j-(t+t'))) \longrightarrow H^3_{\text{ét}}(Y_G(K), \mathscr{H}_{L_v}^{[\lambda]}(2-j-(t+t'))).$$

- There is an Hochschild–Serre spectral sequence (see again [Jan88]) relating continuous étale cohomology for varieties over $\mathbb{Q}$ with étale cohomology of the base change over $\bar{\mathbb{Q}}$

$$E_2^{p,q} = H^p(\mathbb{Q}, H^q_{\text{ét}}(Y_G(K), \mathscr{D})) \Rightarrow H^{p+q}_{\text{ét}}(Y_G(K)_{\bar{\mathbb{Q}}}, \mathscr{D}).$$

From this, one gets a map from the kernel of the map $H^i_{\text{ét}}(Y_G(K), \mathscr{D}) \to H^i_{\text{ét}}(Y_G(K)_{\bar{\mathbb{Q}}}, \mathscr{D})^{G_{\mathbb{Q}}}$ to $H^1\left(\mathbb{Q}, H^{i-1}_{\text{ét}}(Y_G(K)_{\bar{\mathbb{Q}}}, \mathscr{D})\right)$. In particular, for $i = 3$, since Artin vanishing theorem tells us that $H^i_{\text{ét}}(Y_G(K)_{\bar{\mathbb{Q}}}, \mathscr{D}) = 0$ being $i > \dim(Y_G(K)) = 2$, we obtain a map

$$HS: H^3_{\text{ét}}(Y_G(K), \mathscr{H}_{L_v}^{[\lambda]}(n)) \longrightarrow H^1\left(\mathbb{Q}, H^2_{\text{ét}}(Y_G(K)_{\bar{\mathbb{Q}}}, \mathscr{H}_{L_v}^{[\lambda]}(n))\right),$$

where in particular we can take $n = 2 - j - (t+t')$.

- We now localise at the maximal ideal I given by the kernel of the Hecke algebra acting on Π^K as in Proposition 4.8.7. Applying such proposition and projecting to the Π-isotypic part we find

$$(H^3_{\text{ét}}(Y_G(K), \mathscr{H}_{L_v}^{[\lambda]}(2-j-(t+t'))))_I \longrightarrow H^1\left(\mathbb{Q}, (V_f^{\text{As}})^*(-j)\right) \otimes (\Pi^\vee)^K.$$

Since all these maps are compatible with respect to changing K and since, by Proposition 4.8.7, the localisation is independent on the choice of the set of primes T (which may vary changing K), we can construct a map of $G(\mathbb{A}_f)$-representation.

Definition 4.8.8. We define pr_Π to be the $G(\mathbb{A}_f)$-equivariant map

$$pr_\Pi: H^3_{\text{mot}}(Y_G, \mathscr{H}_L^{[\lambda]}(2-j-(t+t'))) \longrightarrow H^1\left(\mathbb{Q}, (V_f^{\text{As}})^*(-j)\right) \otimes \Pi^\vee$$

obtained by the previous steps and taking the limit with respect to K.

In order to define classes in Galois cohomology, we need to take a "projection" to $H^1\left(\mathbb{Q},(V_f^{\mathrm{As}})^*(-j)\right)$ from the target of the map in the previous definition. To do that we assume that Π is unramified at p. We can consider $\Pi_p^{G(\mathbb{Z}_p)} \neq 0$. Let

$$K_0(p)=\{\gamma\in G(\mathbb{Z}_p):\gamma\equiv\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p\}.$$

We choose α to be one of the eigenvalues of the Hecke operator $U(p)$ acting on $\Pi_p^{K_0(p)}$. We fix a finite set of primes S to be set of primes outside which Π_ℓ is a spherical representation. We now fix the local data as in §4.7.2. Write

$$K' := K_S\times\prod_{\ell\nmid pS} G(\mathbb{Z}_\ell)\times K_0(p),$$

where K_S is chosen so that $\Pi^{K'}\neq 0$ and K' is sufficiently small. In particular we assume that the conductor of Π is not trivial and it is coprime to $2,3$ and the discriminant of F. We fix and choose an arbitrary vector $v_\alpha\in\Pi^{K'}$ in the $U(p)=\alpha$ eigenspace. This gives a homomorphism

$$v_\alpha:(\Pi^\vee)^{K'}\longrightarrow L_v.$$

Note that the choice of this line in $(\Pi^\vee)^{K'}$ is arbitrary, however if one wants to work with integral classes and apply these results in the setting of Iwasawa theory, then the assumption of Π being ordinary at p is added and one chooses α to be the unique eigenvalue of $U(p)$ which is a p-adic unit.

What we are going to do is to consider the image of the $K_{M,m,n}$-invariant classes defined in §4.7.2, take the image via the $G(\mathbb{A}_f)$-equivariant map pr_Π and then apply v_α. For $W,\phi_{M,m,n},\xi_{M,m,n}$ as in §4.7.2, we consider $z_{M,m,n}^{[k,k',j]}$ as in Definition 4.7.6,

$$z_{M,m,n}^{[k,k',j]}=\tfrac{1}{\mathrm{Vol}(W)}\mathscr{A}\mathscr{F}_{\mathrm{mot}}^{\mathrm{k,k',j}}(\phi_{M,m,n}\otimes\xi_{M,m,n})\in H^3_{\mathrm{mot}}(Y_G,\mathscr{H}^{[\lambda]}(2-j-(t+t'))).$$

Since these elements actually lied in the $K_{M,m,n}$-invariant subspace of the motivic cohomology group, when we apply the étale regulator and the map obtained via Hochschild–Serre we obtain classes in

$$H^1\left(\mathbb{Q}, H^2_{\text{ét}}(Y_G(K_{M,m,n})_{\bar{\mathbb{Q}}}, \mathscr{H}^{[\lambda]}_{L_v}(2-j-(t+t')))\right).$$

Recall from (4.7.1) that, restricting to G^*, we find

$$Y_{G^*}(K^*_{M,m,n}) \simeq Y_{G^*}(K^*_n) \times_{\mathbb{Q}} \mu_{Mp^m}.$$

We now recall a result that will be useful to use the above isomorphism to land in Galois cohomology over cyclotomic extensions.

Proposition 4.8.9. *[Nek18, Corollary 5.8]. Let $U \subset G(\mathbb{A}_f)$ be the stabiliser of Y_{G^*}. We have a $G_{\mathbb{Q}} \times G(\mathbb{A}_f)$ isomorphism*

$$H^i_{\text{ét}}(Y_{G,\bar{\mathbb{Q}}}, \mathscr{L}) \simeq \operatorname{Ind}_U^{G(\mathbb{A}_f)} H^i_{\text{ét}}(Y_{G^*,\bar{\mathbb{Q}}}, \iota^*\mathscr{L}),$$

where the natural embedding $\iota : Y_{G^*} \hookrightarrow Y_G$ is an open immersion.

We also recall, as in § 3.3.1, that for any variety X over $\mathbb{Q}$ we naturally have, applying Remark 2.1.6, the following isomorphism of $G_{\mathbb{Q}}$-modules

$$H^i_{\text{ét}}((X \times_{\mathbb{Q}} \mu_N)_{\bar{\mathbb{Q}}}, \mathscr{L}) \simeq \operatorname{Ind}_{G_{\mathbb{Q}(\mu_N)}}^{G_{\mathbb{Q}}} H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathscr{L}).$$

Moreover, by Shapiro's lemma we have

$$H^1(\mathbb{Q}, \operatorname{Ind}_{G_{\mathbb{Q}(\mu_N)}}^{G_{\mathbb{Q}}} V) = H^1(\mathbb{Q}(\mu_N), V).$$

Applying the above proposition and these isomorphisms for $N = Mp^m$ and for the $G_{\mathbb{Q}}$-module $H^2_{\text{ét}}(Y_{G^*}(K^*_{M,m,n})_{\bar{\mathbb{Q}}}, \operatorname{TSym}^{[k,k']} \mathscr{H}_{L_v}(\mathscr{A})(2-j))$, we can give the following

Definition 4.8.10. For $m \geq 0$ we define a class

$$z^{\Pi,j}_{Mp^m,\alpha} \in H^1(\mathbb{Q}(\mu_{Mp^m}), (V_f^{\text{As}})^*(-j))$$

by letting

$$\frac{1}{M}\cdot\begin{cases}\left(\frac{p^{j+t+t'}\sigma_p}{\alpha}\right)^m & \\ \left(1-\frac{p^{j+t+t'}\sigma_p}{\alpha}\right) & \end{cases}\cdot(v_\alpha\circ\mathrm{pr}_{\Pi^\vee})\left(z^{[k,k',j]}_{M,m,n}\right)\quad\begin{matrix}\text{if } m\geq 1\\ \text{if } m=0,\end{matrix}$$

where σ_p is the arithmetic Frobenius at p in $\mathrm{Gal}(\mathbb{Q}(\mu_M)/\mathbb{Q})$.

4.8.3 Norm relations in Galois cohomology

These are the classes that, as we are going to show, form an Euler system for $V_f^{\mathrm{As}}(j+1)$.

Theorem 4.8.11 (Vertical norm relations)**.** *Let $j\leq\min(k,k')$ and $k,k'\geq 0$. We have*

$$\mathrm{cores}^{\mathbb{Q}(\mu_{Mp^{m+1}})}_{\mathbb{Q}(\mu_{Mp^m})}\left(z^{\Pi,j}_{Mp^{m+1},\alpha}\right)=z^{\Pi,j}_{Mp^m,\alpha}.$$

Proof. Since the pushforward by

$$H^2_{\text{ét}}(Y_{G^*}(K_n^*)\times\mu_{Mp^{m+1}},\mathrm{TSym}^{[k,k']}\mathscr{H}_{L_v}(\mathscr{A}))\xrightarrow{\pi_m}H^2_{\text{ét}}(Y_{G^*}(K_n^*)\times\mu_{Mp^m},\mathrm{TSym}^{[k,k']}\mathscr{H}_{L_v}(\mathscr{A}))$$

induces corestriction in Galois cohomology, the result is an immediate corollary of Theorem 4.8.2. This indeed can be rewritten as

$$(\pi_m)_*(z^{[k,k',j]}_{M,m+1,n})=\begin{cases}\frac{U'(p)}{p^{j+t+t'}} & \\ (\frac{U'(p)}{p^{j+t+t'}}-1) & \end{cases}\cdot z^{[k,k',j]}_{M,m,n}\quad\begin{matrix}\text{if } m\geq 1\\ \text{if } m=0,\end{matrix}$$

seen as elements in $H^2_{\text{ét}}(Y_{G^*}(K^*_{M,m,n}),\mathrm{TSym}^{[k,k']}\mathscr{H}_{L_v}(\mathscr{A}))$. Here $U'(p)$ is the Hecke operator given by the double coset of $\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}$ in $\mathscr{H}(K^*_{M,m,n}\backslash G^*(\mathbb{A}_f)/K^*_{M,m,n})$, and we used the fact that, as explained in the proof [LLZ18, Proposition 4.3.4][4], the Hecke operator of Theorem 4.8.2, acts on $H^2_{\text{ét}}(Y_{G^*}(K^*_{M,m,n})_{\bar{\mathbb{Q}}},\mathrm{TSym}^{[k,k']}(\mathscr{H}_{L_v}(\mathscr{A})))$ as $U'(p)$. The isomorphism (4.7.1) intertwines $U'(p)$ with $U'(p)\times\sigma_p^{-1}$, where

[4] The pullback of the projection from Y_{G^*} to Y_G intertwines $U'(p)$ on the cohomology of Y_{G^*} with $p^{-(t+t')}\mathscr{U}'(p)$ on the cohomology of Y_G, where $\mathscr{U}'(p)$ is the normalised Hecke operator given by $p^{t+t'}U'(p)$.

$U'(p) \in \mathscr{H}(K_n^* \backslash G^*(\mathbb{A}_f)/K_n^*)$ and σ_p^{-1} is the arithmetic Frobenius at p in $\mathrm{Gal}(\mathbb{Q}(\mu_M)/\mathbb{Q})$. Since v_α projects to the $U'(p) = \alpha$ eigenspace, the theorem follows. □

Theorem 4.8.12 (Tame norm relations)**.** *Let $j \leq \min(k,k')$ and $k,k' \geq 0$. We assume that Π is not a twist of a base change lift of a cuspidal representation of $\mathrm{GL}_2/\mathbb{Q}$. For any $\ell \nmid Mp$, $\ell \notin S$, we have*

$$\mathrm{cores}_{\mathbb{Q}(\mu_{Mp^m})}^{\mathbb{Q}(\mu_{\ell Mp^m})}\left(z_{\ell Mp^m,\alpha}^{\Pi,j}\right) = Q(\sigma_\ell^{-1}) z_{Mp^m,\alpha}^{\Pi,j},$$

where $Q(X) = \det(1 - X\,\mathrm{Frob}_\ell^{-1} \,|V_f^{\mathrm{As}}(1+j))$, i.e. $Q(\sigma_\ell^{-1}) = P_\ell(\ell^{-1-j}\sigma_\ell^{-1})$ for $P_\ell(X) = \det(1 - X\,\mathrm{Frob}_\ell^{-1} \,|V_f^{\mathrm{As}})$ as in Definition 2.4.8.

Proof. First of all we notice as above that the corestriction map is induced by push-forward under the projection $\pi : Y_G(K_{\ell M,m,n}) \to Y_G(K_{M,m,n})$. The class on the left is then obtained in motivic cohomology by applying

$$\mathscr{S}(\mathbb{A}_f^2,\mathbb{Z})[j+t+t'] \otimes \mathscr{H}(G(\mathbb{A}_f)) \xrightarrow{\mathscr{A}\mathscr{F}_{\mathrm{mot}}^{\mathrm{k,k',j}}} H^3_{\mathrm{mot}}(Y_G(K_{\ell M,m,n}), \mathscr{D}(2-j))$$
$$\xrightarrow{\pi_*} H^3_{\mathrm{mot}}(Y_G(K_{M,m,n}), \mathscr{D}(2-j))$$

We have $\pi_* \circ \mathscr{A}\mathscr{F}_{\mathrm{mot}}^{\mathrm{k,k',j}}(\phi \otimes \xi) = \sum_k k \cdot \mathscr{A}\mathscr{F}_{\mathrm{mot}}^{\mathrm{k,k',j}}(\phi \otimes \xi) = \sum_k \mathscr{A}\mathscr{F}_{\mathrm{mot}}^{\mathrm{k,k',j}}(\phi \otimes (k \cdot \xi))$, where k runs over coset representatives of $K_{\ell M,m,n}/K_{M,m,n}$. In particular we find that

$$\sum_k k \cdot \xi_{\ell M,m,n} = \xi'_{\ell M,m,n},$$

where $\xi'_{\ell M,m,n}$ is equal to $\xi_{\ell M,m,n}$ at every component but at ℓ where we find

$$\sum_k k_\ell \cdot \left(\mathrm{ch}(K_{\ell,1}) - \mathrm{ch}(\eta_{\ell,1}K_{\ell,1})\right) = \mathrm{ch}(G(\mathbb{Z}_\ell)) - \mathrm{ch}(\eta_{\ell,1}G(\mathbb{Z}_\ell)).$$

Hence both the left hand side and the right hand side of the claimed equality are obtained as image of the same map $v_\alpha \circ \mathrm{pr}_{\Pi^\vee} \circ \mathscr{A}\mathscr{F}_{\mathrm{mot}}^{\mathrm{k,k',j}}$

$$\mathscr{S}(\mathbb{A}_f^2,\mathbb{Z})[j+t+t'] \otimes \mathscr{H}(G(A_f),\mathbb{Z}) \longrightarrow H^1(\mathbb{Q}(\mu_{Mp^m}), (V_f^{\mathrm{As}})^*(-j)).$$

They are obtained as image of elements that are the same at every component different from ℓ, where the right hand side is the image of

$$\tfrac{\ell-1}{\ell}\left((\mathrm{ch}(\ell^2\mathbb{Z}_\ell \times (1+\ell^2\mathbb{Z}_\ell)) \otimes (\mathrm{ch}(G(\mathbb{Z}_\ell)) - \mathrm{ch}(\eta_{\ell,1}G(\mathbb{Z}_\ell)))\right)$$

and the left hand side the image of $\mathrm{ch}(\mathbb{Z}_\ell^2) \otimes \mathrm{ch}(G(\mathbb{Z}_\ell))$. The factor $\ell - 1$ appears comparing $\mathrm{Vol}(W)$ for the two different motivic classes, while $\frac{1}{\ell}$ comes from the $\frac{1}{M}$-factor in the definition of the Galois cohomology classes. So it is enough to compare the image of these two elements via the component at ℓ of the above map. We will first compare the images through each of the the maps

$$\mathfrak{z} : \mathscr{S}(\mathbb{Q}_\ell^2, \mathbb{Z})[j+t+t'] \otimes \mathscr{H}(G(\mathbb{Q}_\ell), \mathbb{Z}) \longrightarrow H^1(\mathbb{Q}(\mu_{Mp^m}), (V_f^{\mathrm{As}})^*(-j)) \otimes \Pi_\ell^\vee \longrightarrow \Pi_\ell^\vee, \tag{4.8.1}$$

where the last map is a $G(\mathbb{A}_f)$-equivariant projection to $\Pi_\ell^\vee$, obtained by choosing a basis element of $H^1(\mathbb{Q}(\mu_{Mp^m}), (V_f^{\mathrm{As}})^*(-j))$. Note that this Galois cohomology group is a priori infinite dimensional, but since it is actually equal to the Galois cohomology of some maximal unramified (outside a finite set of places) extension, we are reduced to take this projection map for a finite number of basis elements. First we assume that $k + k' - 2j \neq 0$. By definition and by Theorem 4.6.6, Proposition 4.6.7 and Theorem 4.8.6, we find that $\mathfrak{z}$ satisfies the condition of Corollary 4.5.20, for $k = k + k' - 2j$. Condition ($\star$) follows from purity.

If $M = 1, m = 0$ we can apply then Corollary 4.5.20 with $\sigma = \Pi_\ell$, together with Remark 4.5.21 for $h = j + t + t'$. The factor of discrepancy is then $L(\sigma, h)^{-1}$. We then apply Corollary 4.8.3 to get

$$L(\sigma, h)^{-1} = L(\Pi_\ell, j + t + t')^{-1} = P_\ell(\ell^{-1-j}).$$

The multiplication by such scalar is carried when we take the projection via v_α into Galois cohomology and this is precisely what we were looking for (since σ_ℓ is trivial in this situation).

If $M > 1, m > 0$, we apply this to every twist by Dirichlet characters modulo Mp^m and apply Shapiro's lemma. We are now comparing classes in $H^1(\mathbb{Q}(\mu_{Mp^m}),(V_f^{\mathrm{As}})^*(-j))$. Since $\rho := (V_f^{\mathrm{As}})^*(-j)$ is a $G_{\mathbb{Q}}$-module, we have $\mathrm{Ind}_{G_{\mathbb{Q}}}^{G_{\mathbb{Q}(\mu_{Mp^m})}}(\rho) = \bigoplus_\eta \rho \otimes \eta$, where η varies over all characters of the quotient $G_{\mathbb{Q}}/G_{\mathbb{Q}(\mu_{Mp^m})} = \mathrm{Gal}(\mathbb{Q}(\mu_{Mp^m})/\mathbb{Q}) \simeq (\mathbb{Z}/Mp^m\mathbb{Z})^\times$. We hence find

$$H^1(\mathbb{Q}(\mu_{Mp^m}),(V_f^{\mathrm{As}})^*) = \bigoplus_\eta H^1(\mathbb{Q},(V_f^{\mathrm{As}})^* \otimes \eta).$$

Since σ_ℓ is the image of ℓ^{-1} in $\mathrm{Gal}(\mathbb{Q}(\mu_{Mp^m})/\mathbb{Q})$, if we write $z \in H^1(\mathbb{Q}(\mu_{Mp^m}),(V_f^{\mathrm{As}})^*(-j))$ as $(z_\eta)_\eta$, we have that $\sigma_\ell^{-1} \cdot z = (\eta(\ell)\cdot z_\eta)_\eta$. Hence we have reduced to prove that the η-components of the classes we are considering differ by the factor $P_\ell(\ell^{-1-j}\eta(\ell))$. We are then again in the case $M=1, m=0$. The character η can be seen as an unramified character of $\mathbb{Q}_\ell^\times$ for $\ell \nmid Mp$ via class field theory and it then defines a one dimensional representation of $G^*(\mathbb{Q}_\ell)$ and of $H(\mathbb{Q}_\ell)$ via the determinant map. Hence the classes z_η we are considering are locally at ℓ images of the map (4.8.1) with the action of $H(\mathbb{Q}_\ell)$ twisted by η. The space of such maps factoring through the Siegel section will now be isomorphic, via the bijection of Proposition 4.5.11, to a space of the form $\mathrm{Hom}_H(I_H(\chi\eta,\psi\eta)\otimes\Pi_\ell,\eta)$, where $\chi\psi\cdot\chi_{\Pi_\ell} = 1$. Theorem 4.5.1 implies that this space is again one dimensional, and the construction of a basis carries through as in Section § 4.5, where in the choice of the auxiliary character in Definition 4.5.2 ψ is replaced by $\psi\eta$. We obtain the same results, but with $L(\sigma\otimes\eta,h)$ in place of $L(\sigma,h)$. We then find, as we wanted,

$$L(\Pi_\ell \otimes \eta, j+t+t')^{-1} = P_\ell(\ell^{-1-j}\eta(\ell)).$$

We are left with the case $k+k'-2j=0$. The issue here is that the divisor map from $\mathscr{O}^\times(Y)\otimes\mathbb{C}$ in (1) of Theorem 4.6.6 has a kernel. It consists of non-generic representations of $H(\mathbb{A}_f)$. For any such representation τ we have that $\mathrm{Hom}_{H(\mathbb{A}_f)}(\tau\otimes\Pi,\mathbb{C}) = 0$ thanks to the assumption that Π is not a base change lift from $\mathrm{GL}_2/\mathbb{Q}$ and Proposition 4.8.4. Hence the local map factors through the Siegel section also in this case and the proof follows as above. □

Remark 4.8.13. These classes hence satisfy the Euler system norm relations (NR) as stated in the Introduction. In particular we proved the tame norm relations for all primes $\ell \notin S$. In [LLZ18] these were proved only for ℓ inert in F or ℓ split with the condition of the two primes ideal in F above it being narrowly principal.

Remark 4.8.14 (Integral classes)**.** In fact, one is interested in "integral classes": fixing a $G_{\mathbb{Q}}$-stable lattice $T \subset (V_f^{\mathrm{As}})^*(-j)$, we would like to have classes in $H^1(\mathbb{Q}(\mu_m), T)$ satisfying the same norm relations. To do that one works with integral Eisenstein classes, applies the map $\mathscr{AF}_{\mathrm{mot}}^{\mathrm{k,k',j}}$ and slightly modifies the projection map $v_\alpha \circ pr_\Pi$ by choosing an appropriate Hecke operator that will define a lattice as above. This is explained in details for the case $G = \mathrm{GSp}_4$ in [LSZ20a, § 8.4.6] and in the discussion following [LSZ20a, Proposition 10.5.2].

4.8.4 A remark on Beilison–Flach Euler system

It should now be clear to the reader that, proceeding in a completely analogous way, one can reprove Euler system norm relations for Beilinson–Flach classes. These elements were constructed in [LLZ14] and [KLZ15] and lay in Galois cohomology of the representation attached to the Rankin–Selberg convolution of two modular forms f, g of weight $k+2, k'+2$ respectively. This means that in this case one works with $\Pi = \Pi_f \otimes \Pi_g$, where Π_f, Π_g are automorphic representations of $\mathrm{GL}_2(\mathbb{A}_f)$. Hence we have, at all but finitely many places, a spherical representation Π_ℓ of $G(\mathbb{Q}_\ell)$ as in Definition 4.4.2, where now $G = \mathrm{GL}_2 \times \mathrm{GL}_2$. Using §4.4, one can restate all the results of §4.5 for G; everything is already there, since we are in the degenerate case where all primes split. One then defines a map

$$\mathscr{BF}_{\mathrm{mot}}^{\mathrm{k,k',j}} : \mathscr{S}(\mathbb{A}_f^2, \mathbb{Q})[j] \otimes \mathscr{H}(G(\mathbb{A}_f), \mathbb{Z}) \longrightarrow H^3_{\mathrm{mot}}(Y_G, \mathrm{TSym}^{[k,k']} \mathscr{H}(\mathscr{E})(2-j)),$$

similarly as in §4.7.1, where now $\mathrm{TSym}^{[k,k']} \mathscr{H}_L(\mathscr{E})$ is a motivic sheaf over the $\mathrm{GL}_2 \times \mathrm{GL}_2$ Shimura variety and the considered embedding ι at the level of algebraic groups is the diagonal embedding $\mathrm{GL}_2 \hookrightarrow \mathrm{GL}_2 \times \mathrm{GL}_2$. In this case one uses

$$CG_{\mathrm{mot}}^{[k,k',j]} : \mathrm{TSym}^{k+k'-2j} \mathscr{H}(\mathscr{E}) \to \iota^*(\mathrm{TSym}^{[k,k']} \mathscr{H}(\mathscr{E}))(-j),$$

as in [KLZ15, Corollary 5.2.2]. The local input is then the same as in §4.7.2, again in the "all split primes" case. The proofs of all results in §4.8 carry over, where in this setting the Galois representation $(V_f \otimes V_g)^*$ appears in $H^2_{\text{ét}}(Y_{G,\bar{\mathbb{Q}}}, \mathrm{TSym}^{[k,k']}\mathscr{H}(\mathscr{E})(2))$.

Both in this case and in the Asai–Flach one, the obtained classes are not exactly the ones obtained pushing forward Eisenstein classes via "perturbed embeddings". The classes explicitly defined this way in [LLZ14] and [LLZ18] satisfy the expected tame norm relations at ℓ only modulo $(\ell-1)$; one obtains an Euler system thanks to a result by Rubin stating that these relations are enough to "lift" such classes to an Euler system. This error term does not appear in this setting because at primes $\ell \mid M$ we already add a correction term in the definition of the local Hecke algebra element $\xi_{M,m,n}$ (see 4.7.3). This can be seen to be the right choice from the local computation of Corollary 4.5.20.

Chapter 5

Kolyvagin systems and Selmer groups of elliptic curves

In this chapter we study an application of the existence of Kolyvagin systems for rational elliptic curves. More precisely, we generalise a result of [How04] which gives a bound on the torsion part of a Selmer group attached to a rational elliptic curve and a quadratic imaginary field K, subject to the non-vanishing of the bottom class of such Kolyvagin system.

5.1 Main result and Heegner points

Let $E/\mathbb{Q}$ be an elliptic curve and p an odd prime of good ordinary reduction such that $E[p]^{G_{\mathbb{Q}}} = \{0\}$. We then work over an auxiliary quadratic imaginary field K satisfying the following (slight generalisation of the) Heegner assumption

$$\text{every prime of bad reduction of } E \text{ and } p \text{ split in } K. \qquad \text{(Heegner hypothesis)}$$

We also assume that $E[p]^{G_K} = \{0\}$ and that E does not have CM by K. Notice that we can produce infinitely many K satisfying these conditions.

In this chapter we prove the following result

Theorem 5.1.1. *Consider $T = T_p(E)$ as above and let $\mathscr{F}$ be a Selmer structure. Suppose there is a Kolyvagin system for $(T, \mathscr{F})$ (see Definition 5.2.8) such that $\kappa_1 \neq 0$. Then $H^1_{\mathscr{F}}(K, T)$ is a free rank one module over $\mathbb{Z}_p$ and there exists a finite*

$\mathbb{Z}_p$-module M such that

(i) $H^1_{\mathscr{F}}(K, E[p^\infty]) \simeq \mathbb{Q}_p/\mathbb{Z}_p \oplus M \oplus M$,

(ii) $\mathrm{length}_{\mathbb{Z}_p}(M) \leq \mathrm{length}_{\mathbb{Z}_p}(H^1_{\mathscr{F}}(K,T)/\mathbb{Z}_p \cdot \kappa_1) + t$,

where t is a non-negative integer depending only on $\mathrm{Im}(G_K \to \mathrm{GL}(T_p(E)) \simeq \mathrm{GL}_2(\mathbb{Z}_p))$ *and is equal to zero when such representation is surjective.*

Remark 5.1.2. Notice that the theorem applies also in the case not covered in [How04] where the elliptic curve E admits a rational p-isogeny with non-cyclic kernel. Moreover the constant t can be thought as *measuring how much the representation* $G_K \to \mathrm{GL}(T_p(E)) \simeq \mathrm{GL}_2(\mathbb{Z}_p)$ *fails to be surjective*.

5.1.1 Heegner points

We now recall the existence of a Kolyvagin system for T and a certain choice of $\mathscr{F}$, such that the bottom class is non-zero if and only if the analytic rank of E/K is one. Such Kolyvagin system is built using Heegner points.

The construction of these classes is carefully explained in [How04, § 1.7]. We only sketch the construction for the sake of completeness and we refer the interested reader to *op. cit.* for more details. We let $P[m]$ be the Heegner point of conductor m. It is constructed as follows. Thanks to the Heegner hypothesis we can fix an integral ideal $\mathfrak{N}$ of $\mathscr{O}_K$ such that $\mathscr{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$, where N is the conductor of E. For ℓ a prime which is inert in K we denote by $a_\ell \in \mathbb{Z}$ the trace of the Frobenius at ℓ on $T_p(E)$. We define an ideal $I_\ell \subset \mathbb{Z}_p$ to be the smallest ideal containing $\ell+1$ for which Frob_ℓ acts with characteristic polynomial X^2-1. Therefore $I_\ell = (a_\ell, \ell+1)$. For every integer n square-free product of primes as above, we let h_n to be a point on the modular curve $X_0(N)$ corresponding to the cyclic N-isogeny

$$h_n = \left[\mathbb{C}/\mathscr{O}_n \to \mathbb{C}/(\mathscr{O}_n \cap \mathfrak{N})^{-1}\right],$$

where $\mathscr{O}_n$ is the order of conductor n in $\mathscr{O}_K$. The point h_n is defined over $K[n]$, the ring class field of K of conductor n. Fix a modular parametrisation

$$\phi : X_0(N) \to E$$

and let $P[n] := \phi(h_n) \in E(K[n])$. These points satisfy the following norm relations for $\ell \nmid n$:

$$\mathrm{Nm}_{K[n\ell]/K[n]} P[n\ell] = a_\ell \cdot P[n].$$

In order to produce the Kolyvagin system, one applies a derivative operator very similar to the one considered in the Introduction (see (Kolyvagin derivative)). If n is as above we let $\mathscr{G}(n) := \mathrm{Gal}(K[n]/K)$ and

$$G(n) := \mathrm{Gal}(K[n]/K[1]).$$

For every prime ℓ, fix σ_ℓ a generator of $G(\ell)$ and define the derivative operator $D_\ell \in \mathbb{Z}_p[G(\ell)]$ as

$$D_\ell = \sum_{i=1}^{\ell} i\sigma_\ell^i,$$

We then let $D_n = \prod_{\ell \mid n} D_\ell \in \mathbb{Z}_p[G(n)]$. Similarly as in (1.2.1), one finds

$$(\sigma_\ell - 1) D_\ell = \ell + 1 - \mathrm{Norm}.$$

Let S be a set of representatives for $G(n) \subset \mathscr{G}(n)$, and let

$$\tilde{\kappa}_n = \sum_{s \in S} s D_n(P[n]) \in E(K[n]).$$

Let $I_n = \sum_{\ell \mid n} I_\ell \subset \mathbb{Z}_p$, where I_ℓ is the ideal $(\ell + 1, a_\ell(E)) \subset \mathbb{Z}_p$. One then shows, using the above congruence relations, that the image of the class in $E(K[n])/I_n E(K[n])$ is fixed by $\mathscr{G}(n)$ and hence, also its image under the Kummer map in $H^1(K[n], T/I_nT)$. Now consider the restriction map

$$H^1(K, T/I_nT) \xrightarrow{\mathrm{res}} H^1(K[n], T/I_nT)^{\mathscr{G}(n)}.$$

In [How04], the class κ_n is defined to be the preimage under this map of the class built above in $H^1(K[n], T/I_nT)^{\mathscr{G}(n)}$. In *op. cit.* the fact that this map is an isomorphism follows from the representation T being irreducible and it is the only point in the construction where this assumption is invoked. However, this holds true also in our setting because

$$H^0(K[n], T/I_nT) = H^0(K[n], E[I_n]) = 0.$$

This follows from the fact that $E[p](K[n]) = 0$. This is a consequence of $K[n]$ being disjoint over $\mathbb{Q}$ from extensions generated by p-torsion points of E. Such extensions are indeed unramified outside p and the places dividing N, the conductor of E. Since n is coprime to Np and divisible only by inert primes, the extension $K[n]/\mathbb{Q}$ is unramified at places dividing Np.

Now consider the Selmer structure $\mathscr{F}$ on $V = T_p(E) \otimes \mathbb{Q}_p$ given by the unramified local condition (see § 5.2.1) at places of K not dividing p and at $v|p$ take the image of the local Kummer map

$$E(K_v) \otimes \mathbb{Q}_p \to H^1(K, V)$$

Define the local conditions on $T_p(E)$ and $E[p^\infty] \cong V/T_p(E)$ by propagating $\mathscr{F}$. As shown for example in [Rub00, Proposition I.6.8], this Selmer structure gives rise to the usual p-Selmer group considered in 1.5.1, namely

$$H^1_{\mathscr{F}}(K, T_p(E)) = \mathrm{Sel}_p(E/K), \quad H^1_{\mathscr{F}}(K, E[p^\infty]) = \mathrm{Sel}_{p^\infty}(E/K).$$

Finally, one considers some *modified* Selmer structures $\mathscr{F}(n)$ (see Definition 5.2.7 below). It can be shown (see [How04, Lemma 1.7.3 *et seq.*]) that the classes κ_n actually lie in the corresponding Selmer groups, i.e.

$$\kappa_n \in H^1_{\mathscr{F}(n)}(K, T/I_nT)$$

and they satisfy the Kolyvagin relations, defined in (K) below. In other words, applying Gross–Zagier's result (1.5.2) and Theorem 5.1.1 we find

Corollary 5.1.3. *Let $E/\mathbb{Q}$ be an elliptic curve and p a prime of good ordinary reduction such that $E[p]^{G_\mathbb{Q}} = \{0\}$. Let K be a quadratic imaginary field chosen as above. If the analytic rank of E/K is one, then $\mathrm{Sel}_p(E/K)$ is a free rank one module over $\mathbb{Z}_p$ and there exists a finite $\mathbb{Z}_p$-module M such that*

(i) $\mathrm{Sel}_{p^\infty}(E/K) \simeq \mathbb{Q}_p/\mathbb{Z}_p \oplus M \oplus M$*,*

(ii) $\mathrm{length}_{\mathbb{Z}_p}(M) \leq \mathrm{length}_{\mathbb{Z}_p}(\mathrm{Sel}_p(E/K)/\mathbb{Z}_p \cdot \kappa_1) + t$ *for some* $t \in \mathbb{Z}_{\geq 0}$ *as in Theorem 5.1.1.*

5.1.2 Iwasawa theoretic applications

As mentioned in Remark 1.5.2, Theorem 5.1.1 is proved in [CGLS20] for twists of $T_p(E)$ obtained as follows. Let $\Gamma :=\mathrm{Gal}(K_\infty/K)$ be the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K, let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ be the anticyclotomic Iwasawa algebra and $\mathfrak{P} \neq p\Lambda$ a height one prime ideal of Λ. Consider R the integral closure of the ring $\Lambda/\mathfrak{P}$ and the G_K-representation $T = T_p(E) \otimes_{\mathbb{Z}_p} R(\alpha_\mathfrak{P})$, where the character $\alpha_\mathfrak{P}$ is given as follows:

$$\alpha_\mathfrak{P} : G_K \twoheadrightarrow \Gamma \to \Lambda \to R.$$

The existence of a Λ-adic Heegner point Kolyvagin system, implies the existence of a Kolyvagin system (for suitable Selmer structures) also for such T. Let us denote the bottom class with $\kappa_\mathfrak{P}$. One then proves that the torsion part of the Selmer group for V/T is $M_\mathfrak{P} \oplus M_\mathfrak{P}$, where

$$\mathrm{length}_{\mathbb{Z}_p}(M_\mathfrak{P}) \leq \mathrm{Ind}(\kappa_\mathfrak{P}) + (t + e_\mathfrak{P})(\mathrm{rk}_{\mathbb{Z}_p} R), \tag{5.1.1}$$

for some $e_\mathfrak{P} \geq 0$, which depends on $\mathfrak{P}$.

In order to prove one divisibility in the Heegner point Iwasawa main conjecture, one needs to prove inequalities for height one prime ideals $\mathfrak{P}$ dividing the Λ-ideals involved in the main conjecture. However, one cannot apply Theorem 5.1.1 directly to $T = T_p(E) \otimes_{\mathbb{Z}_p} R(\alpha_\mathfrak{P})$. Taking $\mathfrak{P} = (g)$ such a prime, one proves the desired inequality by taking the auxiliary ideals $\mathfrak{P}_m = (g + p^m)$ and considering a limit for $m \to \infty$. If $m \gg 0$, $\mathfrak{P}_m$ is also a height one prime ideal and it satisfies $\mathrm{rk}_{\mathbb{Z}_p}(\Lambda/\mathfrak{P}) = \mathrm{rk}_{\mathbb{Z}_p}(\Lambda/\mathfrak{P}_m)$ and $e_{\mathfrak{P}} = e_{\mathfrak{P}_m}$. Hence the error term appearing in the inequalities (5.1.1) for $T = T_p(E) \otimes_{\mathbb{Z}_p} R(\alpha_{\mathfrak{P}_m})$ is independent on m and it disappears when one divides by m and takes the limit for $m \to \infty$.

5.2 Selmer groups and Kolyvagin systems

5.2.1 Selmer structures and Selmer groups

Let F be a number field. Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers R and uniformiser ϖ and consider M an R-module with continuous action of G_F, the absolute Galois group of F. We consider also a triple (V,T,W) where V is a finite dimensional L-vector space with continuous G_F-action, $T \subset V$ is a G_F-stable R-lattice and $W = V/T$.

Definition 5.2.1. A Selmer structure $\mathscr{F}$ on M is a choice of local conditions $H^1_{\mathscr{F}}(F_w, M) \leq H^1(F_w, M)$ for every place w in a fixed finite set of places $\Sigma(\mathscr{F})$ containing p, all archimedean places and all the places at which M is ramified.

Examples of such local conditions are

$$H^1_{\mathscr{F}}(F_w,M) = \begin{cases} H^1_{un}(F_w,M) := \ker\left(H^1(F_w,M) \to H^1(I_w,M)\right) & \text{unramified at } w, \\ \qquad\qquad\qquad \simeq H^1(G_{F_w}/I_w, M^{I_w}) & \\ H^1(F_w,M) & \text{relaxed at } w, \\ 0 & \text{strict at } w. \end{cases}$$

Given a Selmer structure one defines the associated Selmer group

$$H^1_{\mathscr{F}}(F,M) := \ker(H^1(F^{\Sigma(\mathscr{F})}/F, M) \to \prod_{w \in \Sigma(\mathscr{F})} H^1(F_w,M)/H^1_{\mathscr{F}}(F_w,M)),$$

where $F^{\Sigma(\mathscr{F})}$ is the maximal extension of F unramified outside $\Sigma(\mathscr{F})$.

A Selmer structure $\mathscr{F}$ on V defines Selmer structures on T and W taking respectively the preimage and image of $H^1_{\mathscr{F}}(F_w,V)$ via $0 \to T \to V \to W \to 0$.

Definition 5.2.2. Given a Selmer structure $\mathscr{F}$ on M and a finite set of primes S we define the Selmer structure $\mathscr{F}^S$ on M where $\Sigma(\mathscr{F}^S)$ is given by $\Sigma(\mathscr{F}) \cup S$, the local condition at primes not in S is unchanged and the one at primes in S is the relaxed condition, namely

$$H^1_{\mathscr{F}^S}(F_w, T) = \begin{cases} H^1(F_w, T) & \text{if } w \in S \\ H^1_{\mathscr{F}}(F_w, T) & \text{if } w \notin S, \end{cases}$$

Example 5.2.3. The Bloch-Kato Selmer group, for a p-adic representation V as above satisfying some assumptions, is defined with the finite local conditions $H^1_f(F_w, V)$, where

$$H^1_f(F_w, V) = \begin{cases} H^1_{un}(F_w, V) & \text{if } w \nmid p\infty \\ \ker\left(H^1(F_w, V) \to H^1(I_w, V \otimes_{\mathbb{Q}_p} B_{cris})\right) & \text{if } w \mid p \\ 0 & \text{if } w \mid \infty. \end{cases}$$

If we have a Selmer structure $\mathscr{F}$ on M, one can define a dual Selmer structure $\mathscr{F}^*$ on the Pontryagin dual $M^* = \mathrm{Hom}_{cont}(M, \mathbb{Q}_p/\mathbb{Z}_p(1))$[1] using local duality and letting

$$H^1_{\mathscr{F}^*}(F_w, M^*) := \text{ the annihilator of } H^1_{\mathscr{F}}(F_w, M) \text{ via local duality.}$$

If we have $\mathscr{F}, \mathscr{G}$ two Selmer structures on M, we write $\mathscr{F} \leq \mathscr{G}$ if $H^1_{\mathscr{F}}(F_w, M) \subseteq H^1_{\mathscr{G}}(F_w, M)$ for every w. Local duality gives a perfect bilinear pairing

$$\langle -, - \rangle_w : H^1_{\mathscr{G}}(F_w, M)/H^1_{\mathscr{F}}(F_w, M) \times H^1_{\mathscr{F}^*}(F_w, M^*)/H^1_{\mathscr{G}^*}(F_w, M^*) \to \mathbb{Q}/\mathbb{Z}.$$

Theorem 5.2.4 (Poitou-Tate global duality)**.** *Given $\mathscr{F} \leq \mathscr{G}$ two Selmer structures*

[1]The notation $(-)^*$ for the Pontryagin dual of $(-)$ should not be confused with the same one used in the previous Chapters to denote the standard dual of a representation.

on M, there are exact sequences

$$0 \to H^1_{\mathscr{F}}(F,M) \to H^1_{\mathscr{G}}(F,M) \xrightarrow{\text{loc}} \bigoplus_w H^1_{\mathscr{G}}(F_w,M)/H^1_{\mathscr{F}}(F_w,M)$$

$$0 \to H^1_{\mathscr{G}^*}(F,M^*) \to H^1_{\mathscr{F}^*}(F,M^*) \xrightarrow{\text{loc}} \bigoplus_w H^1_{\mathscr{F}^*}(F_w,M^*)/H^1_{\mathscr{G}^*}(F_w,M^*)$$

and the images on the localisations maps are orthogonal complements with respect to the pairing $\sum_w \langle -,- \rangle_w$, where $\langle -,- \rangle_w$ are the local Tate pairings. This yields the duality exact sequence

$$\begin{aligned} 0 \to & H^1_{\mathscr{F}}(F,M) \to H^1_{\mathscr{G}}(F,M) \xrightarrow{\text{loc}} \bigoplus_w H^1_{\mathscr{G}}(F_w,M)/H^1_{\mathscr{F}}(F_w,M) \xrightarrow{\text{loc}^\vee} H^1_{\mathscr{F}^*}(F,M^*)^\vee \to \\ & \to H^1_{\mathscr{G}^*}(F,M^*)^\vee \to 0, \end{aligned} \tag{LES}$$

where loc$^\vee$ *is the dual of the localisation map in the second short exact sequence above, identifying $H^1_{\mathscr{G}^*}(F_w,M^*)$ with $(H^1(F_w,M)/H^1_{\mathscr{G}}(F_w,M))^\vee$ via local Tate duality (and similarly for $H^1_{\mathscr{F}^*}(F_w,M^*)$).*

5.2.2 Kolyvagin systems

Let F be a number field and T be an R-module with a continuous G_F-action. Take w a finite prime. Recall the singular quotient $H^1_s(F_w,T)$ which is given by $H^1(F_w,T)/H^1_f(F_w,T)$. We have the following result.

Proposition 5.2.5. *Assume w does not divide p and T is unramified at w. Letting k_w be the residue field of F_w, if $|k_w^\times| \cdot T = 0$, then there are canonical isomorphisms*

$$H^1_f(F_w,T) \simeq T/(\text{Frob}_w - 1)T \qquad H^1_s(F_w,T) \otimes k_w^\times \simeq T^{\text{Frob}_w = 1}$$

Proof. See [MR04, Lemma 1.2.1]. □

Definition 5.2.6. For w as in the previous Proposition, if G_{F_w} acts trivially on T we define the finite-singular comparison map to be the isomorphism given by the canonical isomorphisms above

$$\phi_w^{\mathrm{fs}} : H^1_f(F_w,T) \simeq T \simeq H^1_s(F_w,T) \otimes k_w^\times.$$

More precisely ϕ_w^{fs} is given by the composition of

$$\begin{gathered} H^1_f(F_w,T) \xrightarrow{\mathrm{ev}_{\mathrm{Frob}_w}} T, \quad T \xleftarrow{f_w} H^1_s(F_w,T) \otimes k_w^\times \\ \kappa \mapsto \kappa(\mathrm{Frob}_w), \quad \kappa(\sigma_\alpha) \leftarrow\!\shortmid \kappa \otimes \alpha, \end{gathered}$$

where $\sigma_\alpha \in \mathrm{Gal}(\bar{F}_w, F_w^{un}) = I_w$, the inertia subgroup of G_{F_w}, denotes the Artin symbol of any lift of α to F_w.

Definition 5.2.7. Given a Selmer structure $\mathscr{F}$ on T and a triple of positive integers a,b,c, we define a Selmer structure $\mathscr{F}_b^a(c)$ on T where $\Sigma(\mathscr{F}_b^a(c))$ is given by $\Sigma(\mathscr{F})$ together with all primes dividing abc and

$$H^1_{\mathscr{F}_b^a(c)}(F_w,T) = \begin{cases} H^1(F_w,T) & \text{if } w \mid a \\ 0 & \text{if } v \mid b \\ H^1_{\mathrm{tr}}(F_w,T) & \text{if } w \mid c \\ H^1_{\mathscr{F}}(F_w,T) & \text{if } w \nmid abc, \end{cases}$$

where $H^1_{\mathrm{tr}}(F_w,T)$ denotes the transverse condition submodule (see [How04, §1.1]). To simplify the notation, we denote by $\mathscr{F}(n)$ the Selmer structure $\mathscr{F}_1^1(n)$.

We now recall that, under the assumptions of Definition 5.2.6, the singular quotient projects isomorphically to the transverse condition submodule. This gives a splitting

$$H^1(F_w,T) = H^1_f(F_w,T) \oplus H^1_{\mathrm{tr}}(F_w,T).$$

Moreover $H^1_f(F_w,T)$ and $H^1_f(F_w,T^*)$ (respectively $H^1_{\mathrm{tr}}(F_w,T)$ and $H^1_{\mathrm{tr}}(F_w,T^*)$) are exact orthogonal complements under the local duality pairing. See [MR04, Lemma 1.2.4, Proposition 1.3.2].

From now on, we go back to $F = K$, a quadratic imaginary field as in the previous sections.

As in [How04, § 1.2, Definition 1.2.1], we denote by $\mathscr{L}_0 = \mathscr{L}_0(T)$ the set of inert primes of K which do not divide p or any prime at which T is ramified. For any $\ell \in \mathscr{L}_0$ we let:

- I_ℓ be the smallest ideal of R containing $\ell+1$ for which the Frobenius at the prime $\lambda \mid \ell$ in K acts trivially on $T/I_\ell T$;
- G_ℓ be the quotient $k_\ell^\times/\mathbb{F}_\ell^\times$, where k_ℓ is the residue field of $\lambda \mid \ell$;
- $\mathscr{N}_0$ be the set of squarefree products of primes in $\mathscr{L}_0$ and for $n \in \mathscr{N}_0$, let

$$I_n = \sum_{\ell \mid n} I_\ell \subset R, \quad G_n = \bigotimes_{\ell \mid n} G_\ell.$$

By convention, let $I_1 = 0$ and $G_1 = \mathbb{Z}$.

Definition 5.2.8. By a Selmer triple $(T, \mathscr{F}, \mathscr{L})$ we mean a choice of Selmer structure $\mathscr{F}$ on T and a subset $\mathscr{L} \subset \mathscr{L}_0$ disjoint from $\Sigma(\mathscr{F})$. We let $\mathscr{N} = \mathscr{N}(\mathscr{L})$ be the set of squarefree products of primes in $\mathscr{L}$, with the convention of $1 \in \mathscr{N}$. A Kolyvagin system for $(T, \mathscr{F}, \mathscr{L})$ is a collection of classes $\kappa_n \in H^1_{\mathscr{F}(n)}(K, T/I_n T) \otimes G_n$ such that

$$\phi_\ell^{\mathrm{fs}}(\mathrm{loc}_\ell(\kappa_n)) = \mathrm{loc}_\ell(\kappa_{n\ell}). \tag{K}$$

Remark 5.2.9. As explained in [How04, § 1.6], a Kolyvagin system for $(T, \mathscr{F}, \mathscr{L})$ gives a Kolyvagin system for $(T/\varpi^N T, \mathscr{F}, \mathscr{L}^{(N)})$, where

$$\mathscr{L}^{(N)} := \{\ell \in \mathscr{L} : I_\ell \subset p^N R\}.$$

5.3 Structure theorem and error terms

5.3.1 Howard's results on the structure of Selmer groups

Consider an elliptic curve $E/\mathbb{Q}$ as in the introduction, K a quadratic imaginary field satisfying (Heegner hypothesis). Let

$$T = T_p(E) \text{ where } \bar{T}^{G_K} = 0.$$

Consider the Selmer structure $\mathscr{F}$ on $V = T_p(E) \otimes \mathbf{Q}_p$ given by the unramified local condition at places of K not dividing p and at $v|p$ take the image of the local Kummer map

$$E(K_v) \otimes \mathbf{Q}_p \to H^1(K,V)$$

Define the local conditions on $T_p(E)$ and $E[p^\infty] \cong V/T_p(E)$ by propagating $\mathscr{F}$.

We now specify the set of primes $\mathscr{L}$ we will consider. For $n \geq 1$, we let $K(T/p^N)$ the field extension of K such that $G_{K(T/p^N)}$ acts trivially on $T/p^N T$. We let

$$\begin{aligned}\mathscr{L}_N := \{\ell \ :\ & T \text{ is unramified at } \ell \text{ and the conjugacy class of } \mathrm{Frob}_\ell \\ & \text{in } \mathrm{Gal}(K(T/p^N)/\mathbb{Q}) \text{ is equal to the class of the complex conjugation } \tau\}.\end{aligned}$$

Čebotarev density theorem implies that this set has positive density. Notice that if ℓ is in $\mathscr{L}_N$, it is inert in K. Moreover since the Frobenius at the prime $\lambda \mid \ell$ of K is the square of Frob_ℓ, the congruence of characteristic polynomials means that it acts trivially on $T/p^N T$. We hence have that for every $\ell \in \mathscr{L}_N$, the ideal I_ℓ is contained in $p^N \mathbb{Z}_p$. In other words, letting $\mathscr{L} := \bigcup_{N \geq 1} \mathscr{L}_N$,

$$\mathscr{L}_N \subseteq \mathscr{L}^{(N)}. \tag{5.3.1}$$

Notice that the Selmer triple $(T, \mathscr{F}, \mathscr{L})$ satisfies all the hypotheses in [How04, §1.3], but H.1 and H.2. We instead only have that $\bar{T}^{G_K} = 0$, but allow $\bar{T}$ to be a reducible G_K-representation, where $\bar{T} = T/pT$. Because of these properties, everything Howard proves in sections 1.3-1.4-1.5 holds true also in our setting. We recall what we will need for the proof of the bound on the Selmer group.

Lemma 5.3.1. *We have isomorphisms for every* $0 \leq i \leq N$

$$H^1_{\mathscr{F}}(K, T/p^N T)[p^i] \simeq H^1_{\mathscr{F}}(K, T/p^N T[p^i]) \simeq H^1_{\mathscr{F}}(K, T/p^i T).$$

$$H^1_{\mathscr{F}}(K, T/p^N T)[p] \simeq H^1_{\mathscr{F}}(K, \bar{T}).$$

Proof. This follows as in [MR04, Lemma 3.5.4], where only the assumption $\bar{T}^{G_K} = 0$ is needed. □

Proposition 5.3.2. *For every $N \geq 0$ and $n \in \mathcal{N}$, there exists a finite $\mathbb{Z}_p/p^N$-module $M_N(n)$ such that*

$$H^1_{\mathscr{F}(n)}(K, T/p^N T) \simeq (\mathbb{Z}_p/p^N)^{\varepsilon} \oplus M_N(n) \oplus M_N(n),$$

where $\varepsilon \in \{0,1\}$ is independent on both N and n.

Proof. By [How04, Theorem 1.4.2, Lemma 1.5.1], for every $n \in \mathcal{N}$ there exist a $\mathbb{Z}_p$-module $M_N(n)$ and $\varepsilon(n,N) \in \{0,1\}$ such that

$$H^1_{\mathscr{F}(n)}(K, T/p^N T) \simeq (\mathbb{Z}_p/p^N)^{\varepsilon(n,N)} \oplus M_N(n) \oplus M_N(n).$$

In order to show that $\varepsilon(n,N)$ does not depend on n and N, one proves that the parity of $\dim_{\mathbb{F}_p}(H^1_{\mathscr{F}(n)}(K, T/p^N T)[p])$ is constant and independent on n and N. We now need the following result.

Lemma 5.3.3. *([How04, Lemma 1.5.3]). Let $\rho(n)^{\pm} := \dim_{\mathbb{F}_p}(H^1_{\mathscr{F}(n)}(K, \bar{T})^{\pm})$. Then:*

(i) If $\mathrm{loc}_\ell(H^1_{\mathscr{F}(n)}(K, \bar{T})^{\pm}) \neq 0$, then $\rho(n\ell)^{\pm} = \rho(n)^{\pm} - 1$.

(ii) If $\mathrm{loc}_\ell(H^1_{\mathscr{F}(n)}(K, \bar{T})^{\pm}) = 0$, then $\rho(n\ell)^{\pm} = \rho(n)^{\pm} + 1$.

Proof. We briefly sketch Howard's proof. Consider a prime $\ell \in \mathscr{L}$ coprime to n and the exact sequences

$$0 \to H^1_{\mathscr{F}(n)_\ell}(K, \bar{T}) \to H^1_{\mathscr{F}(n)}(K, \bar{T}) \to H^1_f(K_\ell, \bar{T})$$

$$0 \to H^1_{\mathscr{F}(n)}(K, \bar{T}) \to H^1_{\mathscr{F}(n)^\ell}(K, \bar{T}) \to H^1_s(K_\ell, \bar{T}).$$

The image of the last arrows are exact orthogonal under local Tate pairing by global duality and the complex conjugation splits $H^1_{\mathrm{f}}(K_\ell, \bar{T})$ and $H^1_s(K_\ell, \bar{T})$ into one-dimensional eigenspaces for the action of the complex conjugation. Consider also the exact sequences

$$0 \to H^1_{\mathscr{F}(n)_\ell}(K,\bar{T}) \to H^1_{\mathscr{F}(n\ell)}(K,\bar{T}) \to H^1_{tr}(K_\ell,\bar{T})$$

$$0 \to H^1_{\mathscr{F}(n\ell)}(K,\bar{T}) \to H^1_{\mathscr{F}(n)^\ell}(K,\bar{T}) \to H^1_f(K_\ell,\bar{T}),$$

where again by global duality the images of the last arrows are exact orthogonal. We find that if the image of $H^1_{\mathscr{F}(n)}(K,\bar{T})^\pm$ under the localisation map is not zero, then $H^1_{\mathscr{F}(n)}(K,\bar{T})^\pm = H^1_{\mathscr{F}(n)^\ell}(K,\bar{T})^\pm$. So we have $H^1_{\mathscr{F}(n)_\ell}(K,\bar{T})^\pm = H^1_{\mathscr{F}(\ell n)}(K,\bar{T})^\pm$, which proves (i).

For the second statement, Howard shows that the localisation of $H^1_{\mathscr{F}(n)^\ell}(K,\bar{T})^\pm$ at ℓ is a maximal isotropic subspace of $H^1\left(K_\ell,\bar{T}\right)^\pm$ and that the only two such subspaces are $H^1_f\left(K_\ell,\bar{T}\right)^\pm$ and $H^1_{tr}\left(K_\ell,\bar{T}\right)^\pm$. This tells us that $H^1_{\mathscr{F}(n)^\ell}(K,\bar{T})^\pm$ is equal either to $H^1_{\mathscr{F}(n)}(K,\bar{T})^\pm$ or to $H^1_{\mathscr{F}(n\ell)}(K,\bar{T})^\pm$. The above argument implies that $H^1_{\mathscr{F}(n)^\ell}(K,\bar{T})^\pm = H^1_{\mathscr{F}(n)}(K,\bar{T})^\pm$ contradicts the assumption $\mathrm{loc}_\ell(H^1_{\mathscr{F}(n)}(K,\bar{T})^\pm) = 0$. So we have $H^1_{\mathscr{F}(n)^\ell}(K,\bar{T})^\pm = H^1_{\mathscr{F}(n\ell)}(K,\bar{T})^\pm$, which implies the result using global duality as above. □

Since by Lemma 5.3.1, $H^1_{\mathscr{F}(n)}(K,T/p^NT)[p] \simeq H^1_{\mathscr{F}(n)}(K,\bar{T})$, the lemma implies that the parity of $\dim_{\mathbb{F}_p}(H^1_{\mathscr{F}(n)}(K,T/p^NT)[p])$ does not depend on n. □

5.3.2 Čebotarev density theorem argument

In order to use the Kolyvagin classes to bound the Selmer group, we exploit the action of the complex conjugation on the Selmer group. Since we have a natural action of complex conjugation τ on $T_p(E)$, we have an action of it on $H^1(K,T)$ and we write

$$H^1_{\mathscr{F}(n)}(K,T) = H^1_{\mathscr{F}(n)}(K,T)^+ \oplus H^1_{\mathscr{F}(n)}(K,T)^-,$$

for the subspaces where the complex conjugation acts as $+1$ and -1 respectively. Similarly, for any finite $\mathbb{Z}_p$-module M with an action of τ, we will write $M = M^+ \oplus$

M^-. From now on, for any $c \in M$ we will also write

$$\mathrm{ord}(c) = \min\{m \geq 0 : p^m \cdot c = 0\}.$$

The following result is due to Nekovář (see [Nek07, § 7.5.1]).

Proposition 5.3.4. *For any pair of classes $c^{\pm} \in H^1(K, T/p^N T)^{\pm}$ laying in different eigenspaces with respect to the action of complex conjugation, there exist infinitely many primes $\ell \in \mathscr{L}_N$, such that we have*

$$\mathrm{ord}(\mathrm{loc}_\ell(c^{\pm})) \geq \mathrm{ord}(c^{\pm}) - e,$$

where e is a constant which depends only on the image of $\mathbb{Z}_p[G_K]$ in $GL_2(\mathbb{Z}_p)$ and is independent of N. Moreover if T is residually irreducible, $e = 0$ and we therefore have an equality $\mathrm{ord}(\mathrm{loc}_\ell(c^{\pm})) = \mathrm{ord}(c^{\pm})$.

Proof. The error term e is the sum of the constants C_2 and C_3 defined in [Nek07, § 6]. We briefly recall how these are defined and how to use Čebotarev density theorem to find the primes satisfying the condition in the statement.

It is well known that $\mathbb{Z}_p^\times \cap \mathrm{Im}(G_K \to \mathrm{GL}_2(\mathbb{Z}_p))$ is open in $\mathbb{Z}_p^\times$. This implies that there is $u \in \mathbb{Z}_p^\times - \{1\}$ such that for every N, u mod p^N lies in the center of U_N, where

$$U_N = \mathrm{Gal}(K(E[p^N])/K) \subset \mathrm{Aut}_{\mathbb{Z}_p}(T/p^N).$$

We let $C_2 := v_p(u-1)$. We have, as in the proof of [Nek07, Proposition 6.1.2], that

$$p^{C_2} \cdot \ker\left(H^1(K, T/p^N) \to H^1(K(E[p^N]), T/p^N)^{U_N}\right) = 0 \quad \text{for every } N. \tag{5.3.2}$$

Notice that, if we have $\mathrm{Im}(G_K \to \mathrm{GL}_2(\mathbb{Z}_p)) = \mathrm{GL}_2(\mathbb{Z}_p)$, then we can take $u \neq 1$ mod p, giving $C_2 = 0$.

If V is an absolutely irreducible representation of G_K, Nekovář defines in [Nek07, Proposition 6.2.2], the constant C_3 to be such that

$$\mathrm{Im}\left(\mathbb{Z}_p[G_K] \to \mathrm{End}_{\mathbb{Z}_p}(T)\right) \supseteq p^{C_3}\mathrm{End}_{\mathbb{Z}_p}(T). \tag{5.3.3}$$

Again, if we have surjectivity, then $C_3 = 0$.

What Nekovář shows at the beginning of [Nek07, (7.5.1)], using [Nek07, Corollary 6.3.4] and Čebotarev density theorem, is that, given $c^+, c^- \in H^1(K, T/p^N T)$, there exist infinitely many primes ℓ, where ℓ is a Kolyvagin prime, such that

$$\mathrm{ord}(\mathrm{loc}_\ell(c^\pm)) \geq \mathrm{ord}(c^\pm) - C_2 - C_3.$$

We briefly recall how the proof goes. Let $d^\pm := \mathrm{ord}(c^\pm) - C_2 - C_3$. If $d^+ = d^- = 0$, then there is nothing to prove. So assume at least one of them is not zero. Let $L = K(E[p^N])$. By (5.3.2), the kernel of the restriction map

$$H^1(K, T/p^N T) \overset{res}{\to} H^1(L, T/p^N T) = \mathrm{Hom}_{G_K}(G_L, T/p^N T).$$

is annihilated by p^{C_2}. Denote by $f^\pm$ the image under this map of $c^\pm$. We hence have

$$\mathrm{ord}(f^\pm) \geq \mathrm{ord}(c^\pm) - C_2.$$

Moreover by (5.3.3) the $\mathbb{Z}_p$-span of the image of $f^\pm$ contains $p^{C_3}\mathrm{End}(T) \cdot f^\pm(G_L)$. Since $\mathrm{ord}(f^\pm) \geq \mathrm{ord}(c^\pm) - C_2$, we have that the $\mathbb{Z}_p$-span of the image of $f^\pm$ contains $p^{N - \mathrm{ord}(c^\pm) + C_2 + C_3} T/p^N T$. In particular, if $g^\pm$ is the projection of $f^\pm$ to the summand $(T/p^N T)^\pm \cong \mathbb{Z}_p/p^N$, then the $\mathbb{Z}_p$-span of the image of $g^\pm$ contains a submodule isomorphic to $\mathbb{Z}_p/p^{d^\pm}$. Since at least one among d^+ and d^- is not zero, then we cannot have that both f^+ and f^- are trivial.

Let $H \subset G_L$ be the intersection of the kernels of f^+ and of f^-, and let $Z = G_L/H$. Note that $H \neq G_L$ since some $f^\pm$ is non-trivial, so Z is a non-trivial torsion $\mathbb{Z}_p$-module. Note also that Z is stable under the action of complex conjugation since each $f^\pm$ is. In particular, $Z = Z^+ \oplus Z^-$.

We have $g^\pm(Z^-) = 0$, since $f^\pm \in \mathrm{Hom}(G_L, T/p^N T)^\pm$. So we find $g^\pm(Z) = g^\pm(Z^+)$ and the $\mathbb{Z}_p$-span of $g^\pm(Z^+)$ contains a submodule isomorphic to $\mathbb{Z}_p/p^{d^\pm}$. It follows that Z^+ is non-trivial.

If $d^\pm > 0$, let $W_\pm \subset Z^+$ be the proper subgroup such that $g^\pm(W_\pm) = p^{N-(d^\pm - 1)}(T/p^N T)^\pm$. If $d^\pm = 0$, let $W_\pm = 0$. Then both W_+ and W_- are proper subgroups of Z^+ (since there exists some $z \in Z^+$ such that $g^\pm(z) \in p^{N-d^\pm}(T/p^NT)^\pm$). It follows that $W_+ \cup W_- \neq Z^+$. Let $z \in Z^+$, $z \notin W_+ \cup W_-$. By definition, we have

$$\operatorname{ord}(g^\pm(z)) \geq d^\pm. \tag{5.3.4}$$

Let $M = \overline{\mathbb{Q}}^H$, so that $\operatorname{Gal}(M/L) = Z$. Let $g = \tau z \in G_\mathbb{Q}$, and let $\ell \nmid Np$ be any prime such that both c^+ and c^- are unramified at ℓ and $\operatorname{Frob}_\ell = g$ in $\operatorname{Gal}(M/\mathbb{Q})$. Čebotarev density theorem implies there are infinitely many such primes. Since Z fixes $E[p^N]$ and K, Frob_ℓ acts as τ on both $E[p^N]$ and K. This means that $a_\ell(E) \equiv \ell + 1 \equiv 0 \mod p^N$ and ℓ is inert in K. That is, $\ell \in \mathscr{L}_N$.

Since ℓ is inert in K, the Frobenius element at ℓ in $\operatorname{Gal}(\bar{\mathbb{Q}}/K)$ is $\operatorname{Frob}_\ell^2$. Consider the restriction of $c^\pm$ to K_ℓ. Since $c^\pm$ is unramified at ℓ, $\operatorname{loc}_\ell(c^\pm)$ is completely determined by the image $c^\pm(\operatorname{Frob}_\ell^2)$ in $E[p^N]/(\operatorname{Frob}_\ell^2 - 1)E[p^N]$. By the choice of ℓ, $\operatorname{Frob}_\ell^2$ acts trivially on $E[p^N]$, so $E[p^N]/(\operatorname{Frob}_\ell^2 - 1)E[p^N] = E[p^N]$. Moreover, $\operatorname{Frob}_\ell^2 = g^2 = z^2 \in \operatorname{Gal}(M/L)$, so $c^\pm(\operatorname{Frob}_\ell^2) = f^\pm(z^2) = 2g^\pm(z) \in E[p^N]^\pm$, where the second equality follows as above from the fact that the projection of $f^\pm$ to $E[p^N]^\mp$ maps $z \in Z^+$ to zero. Since p is odd, (5.3.4) yields

$$\operatorname{ord}(\operatorname{loc}_\ell(c^\pm)) \geq d^\pm.$$

Letting $e = C_2 + C_3$, we have proved the desired result. □

5.4 Bounding the Selmer group

In this section we prove Theorem 5.1.1. We assume from now on that there exists a Kolyvagin system $(\kappa_n)_n$ for our triple $(T, \mathscr{F}, \mathscr{L})$ and that $\kappa_1 \neq 0$. Denoting by $\kappa_1^{(N)}$ the image of κ_1 in $H^1_\mathscr{F}(K, T/p^NT)$, we have that for N big enough $\kappa_1^{(N)}$ is different from zero. For a finitely generated $\mathbb{Z}_p$-module M and $x \in M$, we define the index of x in M by

$$\operatorname{Ind}(x, M) := \max\{m \geq 0 : x \in p^m M\}.$$

Notice that

$$H^1_\mathscr{F}(K, T) = \varprojlim H^1_\mathscr{F}(K, T/p^NT)$$

and the index of κ_1 in $H^1_{\mathscr{F}}(K,T)$ is equal to the index of $\kappa_1^{(N)}$ in $H^1_{\mathscr{F}}(K,T/p^NT)$ for N big enough. Since κ_1 is not zero, the $\mathbb{Z}_p$-rank of $H^1_{\mathscr{F}}(K,T)$ is at least one.

Recall that from Remark 5.2.9 and (5.3.1) we get a Kolyvagin system for $(T/p^NT,\mathscr{F},\mathscr{L}_N)$, which by abuse of notation we will still write as $(\kappa_n)_n$. The class κ_n is an element of $H^1_{\mathscr{F}(n)}(K,T/I_nT)$, but thanks to (5.3.1), we can view it as a class in $H^1_{\mathscr{F}(n)}(K,T/p^NT)$ for every n squarefree product of primes in $\mathscr{L}_N$. We now write any class c in $H^1_{\mathscr{F}(n)}(K,T/p^NT)$ as

$$c=(c^+,c^-)$$

where we denoted by $(-)^{\pm}$ the component of the image of the class lying in the $\pm$-eigenspace with respect to complex conjugation.

Moreover, for every prime $\ell\in\mathscr{L}_N$, we fix a generator of $\alpha_\ell\in G_\ell$, so that we have isomorphisms $H^1_{\mathscr{F}(n)}(K,T/p^NT)\otimes G_n\simeq H^1_{\mathscr{F}(n)}(K,T/p^NT)$ for every n square-free product of primes in $\mathscr{L}_N$. Under this identification, we can view κ_n as an element of $H^1_{\mathscr{F}(n)}(K,T/p^NT)$. We rewrite the map ϕ_ℓ^{fs} as follows

$$\begin{aligned}\phi_\ell^{\mathrm{fs}}: H^1_f(K_w,T/p^NT) &\xrightarrow{\mathrm{ev}_{\mathrm{Frob}_w}} T/p^NT \xleftarrow{f_w} H^1_s(K_w,T/p^NT)\otimes k_w^\times\simeq H^1_s(K_w,T/p^NT)\\ \kappa &\mapsto \kappa(\mathrm{Frob}_w),\quad \kappa'(\sigma_\ell)\leftarrow\!\shortmid \kappa'\otimes\alpha_\ell\leftarrow\!\shortmid\kappa',\end{aligned}$$

where we denoted by σ_ℓ the Artin symbol σ_{α_ℓ}. Then the Kolyvagin relation (K) can be rewritten as

$$\mathrm{loc}_\ell(\kappa_n)(\mathrm{Frob}_w)=\mathrm{loc}_\ell(\kappa_{n\ell})(\sigma_\ell).$$

Finally we show that ϕ_ℓ^{fs} switches the eigenspaces of the complex conjugation, i.e. if $\kappa\in H^1_f(K_w,T/p^NT)^{\pm}$, then $\phi_\ell^{\mathrm{fs}}(\kappa)\in H^1_s(K_w,T/p^NT)^{\mp}$, so that we have

$$\phi_\ell^{\mathrm{fs}}(\mathrm{loc}_\ell(\kappa_n^{\pm}))=\mathrm{loc}_\ell(\kappa_{n\ell}^{\mp})\ \text{for}\ \ell\nmid n. \tag{sign}$$

More precisely the above relation follows from the following

Lemma 5.4.1. *Denote by τ the automorphism defined by the action of complex* *conjugation on cocycles. For primes $\ell \in \mathscr{L}_N$, using the above identifications, we have*

$$\phi_\ell^{fs} \circ \tau(\kappa) = -\tau \circ \phi_\ell^{fs}(\kappa),$$

for every $\kappa \in H^1(K_v, T/p^N T)$, where K_v is the completion of K at $v \mid \ell$.

Proof. Recall that ϕ_ℓ^{fs} is defined as $ev_{\sigma_\ell}^{-1} \circ ev_{\mathrm{Frob}_v}$. We will show that $ev_{\mathrm{Frob}_v} \circ \tau = ev_{\mathrm{Frob}_v}$ and $ev_{\sigma_\ell}^{-1} = -\tau \circ ev_{\sigma_\ell}^{-1}$.

First, we show that $\kappa(\tau \mathrm{Frob}_v \tau) = \kappa(\mathrm{Frob}_v)$. From the fact that $\ell \in \mathscr{L}_N$, we have that τ and Frob_ℓ are in the same conjugacy class in $\mathrm{Gal}(K_v/\mathbb{Q}_\ell)$, hence they are equal and we can write $\tau = \mathrm{Frob}_\ell h$ for some $h \in G_{K_v}$. We also have $\mathrm{Frob}_\ell^{-1} \mathrm{Frob}_v \mathrm{Frob}_\ell = \mathrm{Frob}_v$. We hence find that for every $\kappa \in H^1_f(K_v, T/p^N T)$, $\kappa(\tau^{-1} \mathrm{Frob}_v \tau) = \kappa(h^{-1} \mathrm{Frob}_v h) = \kappa(\mathrm{Frob}_v)$, where in the last equality we used the fact that the cocycle κ is actually a homomorphism since G_{K_v} acts trivially on $T/p^N T$.

The second relation will follow from the fact that, since $\ell \equiv -1 \bmod p^N$, the action of conjugation of Frob_ℓ on the inertia subgroup $\mathrm{Gal}(\bar{K}_v/K_v^{un})$ is given by -1. Hence, if $\kappa \in H^1_s(K_v, T/p^N T)$ we find $\kappa(\tau \sigma_\ell \tau) = \kappa(h^{-1} \mathrm{Frob}_\ell^{-1} \sigma_\ell \mathrm{Frob}_\ell h) = \kappa(h^{-1}(\sigma_\ell)^{-1} h) = -\kappa(\sigma_\ell)$, proving the second relation. □

Proposition 5.4.2. *Let s_1 be the index of κ_1 in $H^1_{\mathscr{F}}(K, T)$ and $N \gg 0$. Consider two classes $c^\pm \in H^1_{\mathscr{F}}(K, T/p^N T)^\pm$. Assume that the order of κ_1^+ is the order of $(\kappa_1^+, \kappa_1^-) \in H^1_{\mathscr{F}}(K, T/p^N T)^\pm$. Then $p^{s_1+2e} \cdot c^- = 0$ and $p^{s_1+4e} \cdot c^+ \in \langle \kappa_1^+ \rangle$, where e is the constant of Proposition 5.3.4.*

Proof. First of all we notice that, since κ_1 is divisible by s_1 and $N \gg 0$, we have

$$\mathrm{ord}(\kappa_1^+, H^1_{\mathscr{F}}(K, T/p^N T)) = N - s_1. \tag{5.4.1}$$

Let us first consider c^-. We can then apply Proposition 5.3.4 to find a prime ℓ such that the inequalities on the orders of the localisations at ℓ hold for κ_1^+ and c^-. Applying (LES) for the pair $\mathscr{F}_\ell \leq \mathscr{F}$ and noticing that $(T/p^N T)^*$ is identified with

T/p^NT via the Weil pairing, we have an exact sequence

$$H^1_{\mathscr{F}}(K,T/p^NT)^- \to H^1_f(K_\ell,T/p^NT)^- \to (H^1_{\mathscr{F}^\ell}(K,T/p^NT)^-)^\vee$$

and hence the image of the first map is isomorphic to the cokernel of $H^1_{\mathscr{F}^\ell}(K,T/p^NT)^- \to H^1_s(K_\ell,T/p^NT)^-$. Notice that $\kappa_\ell^- \in H^1_{\mathscr{F}^\ell}(K,T/p^NT)^-$ and the order of $\mathrm{loc}_\ell(\kappa_\ell^-)$ is equal to the order of $\mathrm{loc}_\ell(\kappa_1^+)$ because of (K), (sign) and the fact that the finite singular homomorphism is an isomorphism. Hence $\mathrm{ord}(\mathrm{loc}_\ell(\kappa_\ell^-)) = \mathrm{ord}(\kappa_1) - e + t =: x$ for some $t \geq 0$. So we have that the image of $H^1_{\mathscr{F}^\ell}(K,T/p^NT)^- \to H^1_s(K_\ell,T/p^NT)^-$ contains $p^{N-x}\mathbb{Z}_p/p^N$. Since $H^1_s(K_\ell,T/p^NT)^- \simeq \mathbb{Z}_p/p^N$, the cokernel of the map is isomorphic to $\mathbb{Z}_p/p^{N-y}$ for $y \geq x$. In particular, since $N - (\mathrm{ord}(\kappa_1^+) - e) \geq N - x \geq N - y$, we find that $p^{N-(\mathrm{ord}(\kappa_1^+)-e)} \cdot \mathrm{loc}_\ell(c^-) = 0$. Moreover, using (5.4.1), we get $p^{s_1+e} \cdot \mathrm{loc}_\ell(c^-) = 0$. This implies, from our choice of ℓ, that $p^{s_1+2e} \cdot c^- = 0$.

We now consider $\kappa_1^+, c^+ \in H^1_{\mathscr{F}}(K,T/p^NT)^+$. Let us write $\kappa := p^{-s_1}\kappa_1^+$. The order of κ in $H^1_{\mathscr{F}}(K,T/p^NT)$ is equal to N. We can take a prime ℓ such that

$$\mathrm{ord}(\mathrm{loc}_\ell(\kappa)) \geq N - e.$$

We let z be such that the image of the localisation map from $H^1_{\mathscr{F}}(K,T/p^NT)^+$ is isomorphic to $\mathbb{Z}_p/p^z \cdot c_0$. We can then write $\mathrm{loc}_\ell(\kappa) = p^x \cdot c_0$ and $\mathrm{loc}_\ell(p^e c^+) = p^y \cdot c_0$ for some $x, y \leq z$. We claim that $y \geq x$, so that

$$\mathrm{loc}_\ell(p^e c^+ - p^{y-x}\kappa) = 0. \tag{5.4.2}$$

This holds since if $y \lneq x$, then $\mathrm{ord}(\mathrm{loc}_\ell(p^e c^+)) = z - y \gneq z - x = \mathrm{ord}(\mathrm{loc}_\ell(\kappa)) \geq N - e$, implying that $p^{N-e} \cdot \mathrm{loc}_\ell(p^e c^+) = p^N \cdot \mathrm{loc}_\ell(c^+) \neq 0$ which is a contradiction. Let $c' := p^e c^+ - p^{y-x}\kappa$. Condition (5.4.2) means that $c' \in H^1_{\mathscr{F}_\ell}(K,T/p^NT)^+ \subset H^1_{\mathscr{F}(\ell)}(K,T/p^NT)^+$. We apply Proposition 5.3.4 to find another prime ℓ' such that the inequalities on the orders of the localisations at ℓ' hold for κ_ℓ^- and c'. Proceeding as above, we find that $p^{N-(\mathrm{ord}(\kappa_\ell)-e)} \cdot \mathrm{loc}_{\ell'}(c') = 0$. Since $\mathrm{ord}(\kappa_\ell) \geq \mathrm{ord}(\mathrm{loc}_\ell(\kappa_\ell)) =$

$\operatorname{ord}(\operatorname{loc}_\ell(\kappa_1))$

$$N-(\operatorname{ord}(\kappa_\ell)-e)\leq N-(\operatorname{ord}(\operatorname{loc}_\ell(\kappa_1))-e)\leq N-(\operatorname{ord}(\kappa_1)-2e).$$

Hence we obtained, $p^{N-(\operatorname{ord}(\kappa_1)-2e)}\cdot\operatorname{loc}_{\ell'}(c')=0$. We proceed again as above, finding that $p^{s_1+3e}\cdot c'=0$, which in turn yields $p^{s_1+4e}\cdot c^+\in\langle\kappa_1^+\rangle$. $\square$

Corollary 5.4.3. *For $N\gneq s_1+4e$ we have*

$$H^1_{\mathscr{F}}(K,T/p^NT)\simeq\mathbb{Z}_p/p^N\oplus\left(\bigoplus_{i=1}^{<\infty}\mathbb{Z}_p/p^{m_i}\right),$$

where $N\gneq s_1+4e\geq m_i$ for every i. In particular, $\varepsilon=1$ and for N big enough $p^tM_N(1)=0$ for $t=s_1+4e\lneq N$.

Proof. Let $e'=4e$. The proposition tells us, in particular, that for N big enough $p^{s_1+e'}H^1_{\mathscr{F}}(K,T/p^NT)$ is cyclic and non-zero. Writing $H^1_{\mathscr{F}}(K,T/p^NT)\simeq\bigoplus_{i=0}^{<\infty}\mathbb{Z}_p/p^{m_i}$ we have that $m_0-s_1-e'\gneq 0$ and $m_i\leq s_1+e'$ for every $i\geq 1$. From the fact that the rank of $H^1_{\mathscr{F}}(K,T)$ is at least one, we know that we must have an element of order N, we have $m_0=N$ (having taken $N\gneq s_1+4e$).

In order to deduce $\varepsilon=1$ and that the maximal order of an element in $M_N(1)$ is strictly less than N we use again the fact that $p^{s_1+e'}H^1_{\mathscr{F}}(K,T/p^NT)$ is cyclic. If we had $\varepsilon=0$ or an element of order N in $M_N(1)$ then we would have a subgroup

$$\mathbb{Z}_p/p^N\cdot c_0\oplus\mathbb{Z}_p/p^N\cdot c_1\subset H^1_{\mathscr{F}}(K,T/p^NT).$$

This is not possible because we have just shown that there is only one cyclic subgroup of $H^1_{\mathscr{F}}(K,T/p^NT)$ of order N. $\square$

Remark 5.4.4. Notice that the corollary in particular implies $H^1_{\mathscr{F}}(K,T)\cong\mathbb{Z}_p$, since the assumption $\bar{T}^{G_K}=0$ implies that $H^1_{\mathscr{F}}(K,T)$ is torsion-free, by [MR04, Proposition 2.1.5]. So we have proved the first statement of Theorem 5.1.1.

We proved that the order of every element in $M_N(1)$ is at most s_1+4e, but this is not enough, since we want the same kind of bound on the length of $M_N(1)$, which can be greater than the maximal order of its elements.

Let us assume without loss of generality that the order of (κ_1^+, κ_1^-) is the order of κ_1^+. If that is not the case, then the order of (κ_1^+, κ_1^-) is the order of κ_1^- and we can proceed analogously swapping the signs. Since for $N \gg 0$, $\kappa_1 \neq 0$ in $H^1_{\mathscr{F}}(K, T/p^N T)$, we can write

$$H^1_{\mathscr{F}}(K,T/p^NT)^+ \simeq \mathbb{Z}_p/p^N \oplus X_1, \;\; H^1_{\mathscr{F}}(K,T/p^NT)^- \simeq Y_1, \;\; \text{where } X_1 \oplus Y_1 = \bigoplus_{i=1}^{r+s} \mathbb{Z}_p/p^{m_i},$$

where we renamed the cyclic summands as

$$\begin{aligned} X_1 &= \mathbb{Z}_p/p^{e_1} \oplus \mathbb{Z}_p/p^{e_2} \oplus \cdots \oplus \mathbb{Z}_p/p^{e_r}, \text{ with } N \gneq e_1 \geq e_2 \geq \cdots \geq e_r \\ Y_1 &= \mathbb{Z}_p/p^{d_1} \oplus \mathbb{Z}_p/p^{d_2} \oplus \cdots \oplus \mathbb{Z}_p/p^{d_s}, \text{ with } N \gneq d_1 \geq d_2 \geq \cdots \geq d_s. \end{aligned} \tag{5.4.3}$$

Remark 5.4.5. Notice that $r+s$ is independent on N. Indeed, using the fact that $H^1_{\mathscr{F}}(K,T/p^NT)[p] \simeq H^1_{\mathscr{F}}(K,\bar{T})$ (which follows from $\bar{T}^{G_K} = 0$ and Lemma 5.3.1), we find that the number of direct summands of $H^1_{\mathscr{F}}(K,T/p^NT)$ is equal to x where $H^1_{\mathscr{F}}(K,\bar{T}) \simeq (\mathbb{Z}_p/p)^x$.

We now fix $N \gg 0$ to be such that $N \gneq (r+s)s_1 + (r+s+4)e$. Our goal is to prove the following

$$s_1 + t \geq \tfrac{1}{2}(e_1 + \cdots + e_r + d_1 + \cdots + d_s), \text{ for some } t \geq 0 \text{ depending only on } T_p(E). \tag{claim}$$

Let us assume this for a moment. We can then prove Theorem 5.1.1.

Proof of Theorem 5.1.1. We have already proved in Corollary 5.4.3 that $H^1_{\mathscr{F}}(K,T)$ is a free $\mathbb{Z}_p$-module of rank one. The claim implies

$$s_1 + t \geq \tfrac{1}{2}\mathrm{length}_R(X_1 \oplus Y_1) = \mathrm{length}_R(M_N(1)).$$

Write $H^1_{\mathscr{F}}(K,W) = (\mathbb{Q}_p/\mathbb{Z}_p)^n \oplus Z$, where Z is a torsion $\mathbb{Z}_p$-module. Consider the N fixed above. Since we have

$$(\mathbb{Z}_p/p^N)^n \oplus Z[p^N] = H^1_{\mathscr{F}}(K,W)[p^N] \simeq H^1_{\mathscr{F}}(K,T/p^NT) = \mathbb{Z}_p/p^N \oplus M_N(1) \oplus M_N(1),$$

where $M_N(1)$ is of maximal order strictly less than N, one deduces that $n=1$ and $Z[p^N] = Z = M_N(1) \oplus M_N(1)$. Applying the inequality above, we hence have proved that

$$H^1_{\mathscr{F}}(K,W) \simeq \mathbb{Q}_p/\mathbb{Z}_p \oplus M \oplus M, \quad \text{where } \mathrm{length}_{\mathbb{Z}_p}(M) \leq s_1 + t.$$

$\square$

We require two lemmas. Before getting into the statements and proofs of these lemmas, let us briefly sketch what is their role in the proof of the (claim): it relies on applying Lemma 5.4.6 and Lemma 5.4.7 inductively in order to get the desired inequality. For simplicity we will write $H^{\pm}_{\mathscr{F}'}$ for the groups $H^1_{\mathscr{F}'}(K,T/p^NT)^{\pm}$, where $\mathscr{F}'$ is some Selmer structure. One considers a Selmer group $H_{\mathscr{F}(n)}$ such that

$$\kappa_n^{\varepsilon} \in H^{\varepsilon}_{\mathscr{F}(n)} = \mathbb{Z}_p/p^N \oplus X_n, \quad H^{-\varepsilon}_{\mathscr{F}(n)} = Y_n,$$

where $\varepsilon \in \{\pm 1\}$ is such that the order of κ_n is equal to the order of κ_n^{ε}. One chooses then a prime $\ell \nmid n$ using Proposition 5.3.4 and Lemma 5.4.6 tells us what is the structure of $H_{\mathscr{F}(n\ell)}$. We will show that $H^{\varepsilon}_{\mathscr{F}(n\ell)}$ is given by something closely related to X_n plus some error terms bounded in terms of e. One then studies the structure of $H^{-\varepsilon}_{\mathscr{F}(n\ell)}$. We have $H^{-\varepsilon}_{\mathscr{F}(n\ell)} \simeq \mathbb{Z}_p/p^N \oplus Y_{n\ell}$ and we will characterise $Y_{n\ell}$ in terms of Y_n. Depending on what happens when we localise at ℓ, we can either bound the size d of the maximal order component of Y_n in terms of e or have $Y_{n\ell}$ being equal to Y_n with the components of the order d (and possibly one other component) removed, plus again some bounded extra factors. One finally uses the Kolyvagin relations (K) and the classes $\kappa_n = \kappa_n^{\varepsilon}$ and $\kappa_{n\ell} = \kappa_{n\ell}^{-\varepsilon}$, to which we can apply Lemma 5.4.7, to prove inequalities

$$\begin{aligned} s_n + 2e &\geq s_{n\ell} && \text{in the first case of step 2} \\ s_n + 2e - \tfrac{1}{2}(\text{length of the removed part}) &\geq s_{n\ell} && \text{in the second one,} \end{aligned}$$

where $s_n, s_{n\ell}$ are the indexes of $\kappa_n, \kappa_{n\ell}$ respectively. One then repeats these steps again, with sign swapped every time. We will of course start with $n = 1$, so that at each step we will have one of the inequalities above for the size of all the components in X_1 and Y_1.

Finally, let us remark that even though the strategy is completely analogous to the one adopted for [CGLS20, Theorem 3.2.1], the proof in *op. cit.* cannot be applied directly in this context, where the character α is the trivial one. Indeed, even though the presence of a character in [CGLS20] forces us to introduce another error term, once we go to an extension trivialising the character, we have that both the $+$ and $-$ component of the Kolyvagin class have big order. So one does not have an asymmetric situation like the one presented here. Compare for example (the proofs of) Proposition 5.3.4 and [CGLS20, Proposition 3.3.6] and (the proofs of) Proposition 5.4.2 and [CGLS20, Theorem 3.3.8].

Lemma 5.4.6. *Let $\mathscr{F}'$ be a Selmer structure of the form $\mathscr{F}(n)$ for some n and such that $H^{\varepsilon}_{\mathscr{F}'} \simeq \mathbb{Z}_p/p^N \oplus X_1$, $H^{-\varepsilon}_{\mathscr{F}'} \simeq Y_1$, where X_1, Y_1 can be written as in (5.4.3) and $d_1, e_1 \leq S$ for some $S \ll N$. Then there exist infinitely many $\ell_1 \in \mathscr{L}_N$, $\ell_1 \nmid n$, such that $H^{-\varepsilon}_{\mathscr{F}'(\ell_1)} = \mathbb{Z}_p/p^N \oplus X_{\ell_1}$, $H^{\varepsilon}_{\mathscr{F}'(\ell_1)} = Y_{\ell_1}$, with*

$$Y_{\ell_1} \simeq \left(\bigoplus_{\substack{i=1 \\ i \neq k_1, k_2}}^{r} \mathbb{Z}_p/p^{e_i} \right) \oplus \mathbb{Z}_p/p^{e'_{k_1}} \oplus \mathbb{Z}_p/p^{e'_{k_2}} \oplus (\mathbb{Z}_p/p^{x_1} \oplus \mathbb{Z}_p/p^{x_2}),$$

$$\text{where } e_{k_i} \leq e'_{k_i} \leq e_{k_i} + 2e, 0 \leq x_1, x_2 \leq 2e$$

and, if $d_1 \leq 2e$, the exponent of X_{ℓ_1} is less or equal then p^{2e}; if $d_1 \gneq 2e$, then

$$X_{\ell_1} \simeq \bigoplus_{\substack{j=2 \\ j \neq j(1)}}^{s} \mathbb{Z}_p/p^{d_j} \oplus \begin{cases} \mathbb{Z}_p/p^{d_{j(1)}} \oplus \mathbb{Z}_p/p^{x}, & 0 \leq x \leq e \\ \mathbb{Z}_p/p^{x_1} \oplus \mathbb{Z}_p/p^{x_2}, & 0 \leq x_1, x_2 \leq e. \end{cases}$$

Proof. Let us assume without loss of generality that $\varepsilon = +$. Take $c_0 \in H^{+}_{\mathscr{F}'}$ to be a generator of $\mathbb{Z}_p/p^N$ and $c_1 \in Y_1 = H^{-}_{\mathscr{F}'}$ to be a generator of $\mathbb{Z}_p/p^{d_1}$, the first component of Y_1. We take ℓ_1 as in Proposition 5.3.4. Since $\mathrm{ord}(\mathrm{loc}_{\ell_1}(c_0)) \geq N - e$, we find that the image of the map

$$H^+_{\mathscr{F}'} \to H^1_f(K_{\ell_1}, T/p^N)^+ \simeq \mathbb{Z}_p/p^N$$

is isomorphic to $\mathbb{Z}_p/p^c$ for some $c \geq N-e$. We also have that $\mathrm{loc}_{\ell_1}(p^S c_0) \neq 0$ since its order is $\mathrm{ord}(\mathrm{loc}_{\ell_1}(c_0)) - S \geq N-S-e \gneq 0$. In particular this tells us that

$$c \gneq S.$$

Let u be a generator of the image. If there existed $x \in X_1$ such that $\mathrm{loc}_{\ell_1}(x) = u$, then we would have $\mathrm{ord}(x) \geq c \gneq S \geq \mathrm{ord}(x)$, giving a contradiction. Hence we proved that the image is generated by $\mathrm{loc}_{\ell_1}(c_0)$. Moreover, using as before (LES), the image of

$$H^+_{\mathscr{F}'(\ell_1)} \to H^1_{\mathrm{tr}}(K_{\ell_1}, T/p^N T)^+ \simeq \mathbb{Z}_p/p^N$$

is $\mathbb{Z}_p/p^{N-c'}$, with $N-c' \leq N-c \leq e$. Hence we find an exact sequence

$$0 \to H^+_{\mathscr{F}'_{\ell_1}} \to H^+_{\mathscr{F}'(\ell_1)} \twoheadrightarrow \mathbb{Z}_p/p^{N-c'} \to 0.$$

We now use again Lemma 5.3.1 to count the number of summands of $H^+_{\mathscr{F}'(\ell_1)}$. Reasoning as in the proof of Proposition 5.3.2, we apply Lemma 5.3.3. If the image of the localisation of the p-torsion of $H^+_{\mathscr{F}'}$ is zero, then $H^+_{\mathscr{F}'_{\ell_1}}[p] \simeq (\mathbb{Z}_p/p)^{r+1}$ and $H^+_{\mathscr{F}'(\ell_1)}[p] \simeq (\mathbb{Z}_p/p)^{r+2}$. Otherwise $H^+_{\mathscr{F}'_{\ell_1}}[p] \simeq (\mathbb{Z}_p/p)^r \simeq H^+_{\mathscr{F}'(\ell_1)}[p]$.

If the image of the localisation of the p-torsion of $H^+_{\mathscr{F}'}$ is not zero, we have that $H^+_{\mathscr{F}'(\ell_1)}$ and $H^+_{\mathscr{F}'_{\ell_1}}$ have r summands. We find either $H^+_{\mathscr{F}'_{\ell_1}} \simeq X_1$ or $H^+_{\mathscr{F}'_{\ell_1}} \simeq (\oplus_{i=1, i\neq k'}^r \mathbb{Z}_p/p^{e_i}) \oplus \mathbb{Z}_p/p^{e_{k'}+N-c}$. We therefore have

$$H^+_{\mathscr{F}'(\ell_1)} \simeq \left(\bigoplus_{\substack{i=1 \\ i\neq k,k'}}^{r} \mathbb{Z}_p/p^{e_i} \right) \oplus \mathbb{Z}_p/p^{e_{k'}+x'} \oplus \mathbb{Z}_p/p^{e_k+x}, 0 \leq x, x' \leq 2e \quad (\text{case } 1)$$

If the image of the localisation of the p-torsion of $H^+_{\mathscr{F}'}$ is zero, the number of summands of $H^+_{\mathscr{F}'_{\ell_1}}$ is $r+1$. So we must have that it is isomorphic to $\mathbb{Z}_p/p^{N-c} \oplus X_1$.

Moreover the Selmer group $H^+_{\mathscr{F}'(\ell_1)}$ has $r+2$ summands. We assume without loss of generality that $c = c'$. Indeed we will be using only the fact that $N - c' \leq e$. We have the following cases

$$H^+_{\mathscr{F}'(\ell_1)} \simeq \begin{cases} X_1 \oplus \mathbb{Z}_p/p^{N-c} \oplus \mathbb{Z}_p/p^{N-c} & \text{(case 2)} \\ X_1 \oplus \mathbb{Z}_p/p^{m_1} \oplus \mathbb{Z}_p/p^{m_2}, \;\; m_1 + m_2 = 2N - 2c & \text{(case 3)} \\ \bigoplus_{\substack{i=1 \\ i \neq k}}^{r} \mathbb{Z}_p/p^{e_i} \oplus \mathbb{Z}_p/p^{N-c} \oplus \mathbb{Z}_p/p^{n_1} \oplus \mathbb{Z}_p/p^{n_2}, \;\; n_1 + n_2 = N - c + e_k & \text{(case 4)}, \end{cases}$$

where we remark that in the last case $\mathbb{Z}_p/p^{e_k}$ is mapped diagonally in $\mathbb{Z}_p/p^{n_1} \oplus \mathbb{Z}_p/p^{n_2}$. Using the inequality $c \geq N - e$ we deduce that in cases 2 and 3, we are adding to X_1 two cyclic groups each of order at most $2e$. In case 4, we replace $\mathbb{Z}_p/p^{e_k}$ by some $\mathbb{Z}_p/p^{e'_k}$ where $e_k \leq e'_k \leq e_k + e$ and the other two summands added in case 4 have order at most $2e$. All in all, we proved

$$H^+_{\mathscr{F}'(\ell_1)} \simeq \left(\bigoplus_{\substack{i=1 \\ i \neq k_1, k_2}}^{r} \mathbb{Z}_p/p^{e_i} \right) \oplus \mathbb{Z}_p/p^{e'_{k_1}} \oplus \mathbb{Z}_p/p^{e'_{k_2}} \oplus (\mathbb{Z}_p/p^{x_1} \oplus \mathbb{Z}_p/p^{x_2}),$$

$$\text{where } \begin{array}{c} e_{k_i} \leq e'_{k_i} \leq e_{k_i} + 2e \\ 0 \leq x_1, x_2 \leq 2e \end{array} \qquad (+)$$

Let us now work with the $-$ eigenspace. First of all we notice that, since by Corollary 5.4.3 $\varepsilon = 1$, we must have an element of order N in $H_{\mathscr{F}'(\ell_1)}$. We claim that this must be in $H^-_{\mathscr{F}'(\ell_1)}$. This holds because every element in $H^+_{\mathscr{F}'(\ell_1)}$ has order strictly less than N. We already know that all the e_i's are strictly less than N by assumption, and for our choice of N for the other summands we may add in (+) we have

$$e_k + 2e \leq S + 2e \lneq N, \quad x_i \leq 2e \lneq N.$$

We now consider the exact sequence

$$0 \to H^-_{\mathscr{F}'_{\ell_1}} \to H^-_{\mathscr{F}'} \simeq Y_1 \to H^1_f(K_{\ell_1}, T/p^N T)^- \simeq \mathbb{Z}_p/p^N.$$

Let us write $\mathbb{Z}_p/p^y$ for the image of the last map. If $y=0$, i.e. the localisation at ℓ_1 is the zero map, we have in particular that $0=\operatorname{ord}(\operatorname{loc}_{\ell_1}(c_1))\geq d_1-e$. Hence all the d_i's are less or equal than e. Using duality again, we have the exact sequence

$$0\to H^-_{\mathscr{F}'_{\ell_1}}\simeq Y_1\to H^-_{\mathscr{F}'(\ell_1)}\to \mathbb{Z}_p/p^N$$

and since all d_i's are strictly less than N, we proved

$$d_1\leq e \text{ and } H^-_{\mathscr{F}'(\ell_1)}\simeq \mathbb{Z}_p/p^N\oplus Y_{\ell_1} \text{ where } Y_{\ell_1}=Y_1. \qquad (-_0)$$

We now assume $y\neq 0$. We use again Lemma 5.3.3 to deduce the following: if the image of the localisation of the p-torsion is zero, the number of summands of $H^-_{\mathscr{F}'(\ell_1)}$ is $s+1$; if the image of the localisation of the p-torsion is not zero, then there exists an element of order d_j such that the localisation has again order d_j and the number of summands of $H^-_{\mathscr{F}'(\ell_1)}$ is $s-1$.

In the first case the kernel of the localisation is isomorphic to $\oplus_{j\neq j_1}\mathbb{Z}_p/p^{d_j}\oplus \mathbb{Z}_p/p^{d_{j_1}-y}$, for some $d_{j_1}\gneq y$. Using duality as above we find an exact sequence

$$0\to H^-_{\mathscr{F}'_{\ell_1}}\simeq \oplus_{j\neq j_1}\mathbb{Z}_p/p^{d_j}\oplus\mathbb{Z}_p/p^{d_{j_1}-y}\to H^-_{\mathscr{F}'(\ell_1)}\twoheadrightarrow \mathbb{Z}_p/p^{N-y'}\to 0,$$

for $y'\geq y$. We have hence two cases: $H^-_{\mathscr{F}'(\ell_1)}\simeq$

$$\begin{cases}\bigoplus_{\substack{j=1\\ j\neq j_1}}^{s}\mathbb{Z}_p/p^{d_j}\oplus\mathbb{Z}_p/p^{m_1}\oplus\mathbb{Z}_p/p^{m_2},\ \ m_1+m_2=N+d_{j_1}-2y' & \text{(case 1)}\\ \bigoplus_{\substack{j=1\\ j\neq j_1,j_2}}^{s}\mathbb{Z}_p/p^{d_j}\oplus\mathbb{Z}_p/p^{d_{j_1}-y'}\oplus\mathbb{Z}_p/p^{n_1}\oplus\mathbb{Z}_p/p^{n_2},\ \ n_1+n_2=N+d_{j_2}-y' & \text{(case 2).}\end{cases}$$

In both cases we must have one of the two m_i,n_i is equal to N. Let us assume $m_1,n_1=N$. This implies $m_2=d_{j_1}-2y'$ in case 1 and $n_2=d_{j_2}-y'$ in case 2. Recall that we choose ℓ_1 such that $\operatorname{ord}(\operatorname{loc}_{\ell_1}(y_1))\geq d_1-e$. Since d_1 is the maximal among the d_j's and $y'\geq y\geq \operatorname{ord}(\operatorname{loc}_{\ell_1})(y_1)$, we find $d_j-y'\leq e$ for every j and hence

$$m_2,n_2\leq e.$$

In the second case, i.e. if the image of the localisation of the p-torsion is not zero, we find

$$H^-_{\mathscr{F}'_{\ell_1}} \simeq \oplus_{j\neq j_1,j_2}\mathbb{Z}_p/p^{d_j}\oplus\mathbb{Z}_p/p^{d_{j_1}+d_{j_2}-y'}.$$

Working as above, we therefore have

$$H^-_{\mathscr{F}'(\ell_1)} \simeq \begin{cases} \bigoplus^s_{\substack{j=1\\ j\neq j_1,j_2}} \mathbb{Z}_p/p^{d_j}\oplus\mathbb{Z}_p/p^{N+d_{j_1}+d_{j_2}-2y'} & \text{(case 3)}\\ \bigoplus^s_{\substack{j=1\\ j\neq j_1,j_2,j_3}} \mathbb{Z}_p/p^{d_j}\oplus\mathbb{Z}_p/p^{d_{j_1}+d_{j_2}-y'}\oplus\mathbb{Z}_p/p^{N+d_{j_3}-y'} & \text{(case 4)}. \end{cases}$$

Since we must have a summand of length N, we must have $\frac{d_{j_1}+d_{j_2}}{2}=y'$ in case 3 and $d_{j_3}=y'$ in case 4.

We can summarise the four cases above as follows

$$H^-_{\mathscr{F}'(\ell_1)} \simeq \mathbb{Z}_p/p^N\oplus\left(\bigoplus_{\substack{j=1\\ j\neq j_1,j_2,j_3}}^{s}\mathbb{Z}_p/p^{d_j}\right)\oplus\begin{cases}\mathbb{Z}_p/p^{d_{j_2}}\oplus\mathbb{Z}_p/p^{d_{j_3}}\oplus\mathbb{Z}_p/p^{x},\\ \mathbb{Z}_p/p^{d_{j_3}}\oplus\mathbb{Z}_p/p^{x_1}\oplus\mathbb{Z}_p/p^{x_2},\\ \mathbb{Z}_p/p^{d_{j_1}+d_{j_2}-d_{j_3}}\oplus\mathbb{Z}_p/p^{x_1}\oplus\mathbb{Z}_p/p^{x_2}, \quad y'=d_{j_3},\end{cases} \tag{5.4.4}$$

with $0\leq x,x_1,x_2\leq e$.

Notice that if $d_1\leq 2e$ (and hence $d_j\leq 2e$ for every j), we have one big cyclic summand of order N and all the remaining summands have again order at most $2e$. On the other hand if $d_1\gneq 2e$, then the localisation of d_1 is different from zero and we must have $j_1=1$. In particular in the last case (which comes from case 4 above), we find $y'=d_{j_3}\geq y\geq d_1$. This implies that $d_1=d_{j_3}$. In other words, if $d_1\gneq 2e$, the possible cases are

$$H^-_{\mathscr{F}'(\ell_1)} \simeq \mathbb{Z}_p/p^N\oplus\left(\bigoplus_{\substack{j=2\\ j\neq j(1)}}^{s}\mathbb{Z}_p/p^{d_j}\right)\oplus\begin{cases}\mathbb{Z}_p/p^{d_{j(1)}}\oplus\mathbb{Z}_p/p^{x}, \quad 0\leq x\leq e & (-,1)\\ \mathbb{Z}_p/p^{x_1}\oplus\mathbb{Z}_p/p^{x_2}, \quad 0\leq x_1,x_2\leq e & (-,2),\end{cases} \tag{$-$}$$

for some index $2\leq j(1)\leq s$. Using again the fact that d_1 is maximal and $y'\geq y\geq$

$d_1 - e$, we also find

$$y' \geq d_1 - e \geq \frac{d_1}{2} - e \;\text{ in case } (-,1), \quad y' \geq \frac{d_1 + d_{j(1)}}{2} - e \;\text{ in case } (-,2). \qquad (5.4.5)$$

So in this case, we have $H^-_{\mathscr{F}'(\ell_1)} \simeq \mathbb{Z}_p/p^N \oplus Y_{\ell_1}$, where Y_{ℓ_1} is Y_1 to which we have removed either only the cyclic factor $\mathbb{Z}_p/p^{d_1}$ or the cyclic factors $\mathbb{Z}_p/p^{d_1} \oplus \mathbb{Z}_p/p^{d_{j(1)}}$. One may also have to add some other summands to this group, but the total number of summands of order strictly greater than $2e$ has decreased by one or two. $\square$

Lemma 5.4.7. *Let $\mathscr{F}'$ be a Selmer structure as in Lemma 5.4.6 and ℓ_1 be a prime produced by such lemma. Assume that there exists a class $c \in H^{-\varepsilon}_{\mathscr{F}'(\ell_1)}$ whose localisation at ℓ_1 has the same order of the localisation of the class $p^{s_1}c_0 \in H^{\varepsilon}_{\mathscr{F}'}$, where c_0 is the generator of the maximal order summand and $s_1 \geq 0$ is such that $N \gg s_1 + S$. Denote by s_{ℓ_1} the index of such class. We then have*

$$s_1 + 2e \geq s_{\ell_1}.$$

Moreover, if $d_1 \gtrsim 2e$, we also have, depending on the two cases of Lemma 5.4.6,

$$s_1 + 2e - \frac{d_1}{2} \geq s_{\ell_1} \quad \textit{or} \quad s_1 + 2e - \frac{d_1 + d_{j(1)}}{2} \geq s_{\ell_1}.$$

Furthermore, c must have a non-trivial component (of maximal order) in $\mathbb{Z}_p/p^N$.

Proof. Assume again without loss of generality that $\varepsilon = +$. We can write $c = p^{s_{\ell_1}} \cdot c'$ for some class $c' \in H^-_{\mathscr{F}'(\ell_1)}$. Since the image of the localisation at ℓ_1 is equal to $p^{y'} H^1_{\mathrm{tr}}(K_{\ell_1}, T/p^N T)^-$ for some $y' \geq 0$, we have that

$$\mathrm{loc}_{\ell_1}(c) \in p^{s_{\ell_1} + y'} H^1_{\mathrm{tr}}(K_{\ell_1}, T/p^N T)^-.$$

The hypothesis shows that

$$\mathrm{loc}_{\ell_1}(p^{s_1} c_0) \in p^{s_{\ell_1} + y'} H^1_f(K_{\ell_1}, T/p^N T)^+.$$

Since the choice of ℓ_1 in Lemma 5.4.6 is such that $\mathrm{ord}(\mathrm{loc}_{\ell_1}(c_0)) = N - e + t$, the index of $\mathrm{loc}_{\ell_1}(p^{s_1}c_0)$ is s_1+e-t and hence $s_1+e-t \geq s_{\ell_1}+y'$ and in particular

$$s_1+e \geq s_{\ell_1}+y'. \tag{5.4.6}$$

Combining this inequality with (5.4.5) in the proof of Lemma 5.4.6, one finds

$$s_1+2e-\tfrac{d_1}{2} \geq s_{\ell_1} \tag{ineq, 1}$$

$$s_1+2e-\tfrac{d_1+d_{j(1)}}{2} \geq s_{\ell_1} \tag{ineq, 2}$$

in case $(-,1)$ and $(-,2)$ respectively. Notice that in the case $d_1 \leq 2e$ (which includes the case $y=0$), it will suffice for our purposes to have the following inequality which is deduced from (5.4.6)

$$s_1+2e \geq s_{\ell_1} \tag{ineq, 3}$$

We also need to show that the class c has a non-trivial component in $\mathbb{Z}_p/p^N$. We have

$$\mathrm{ord}(c) \geq \mathrm{ord}(\mathrm{loc}_{\ell_1}(c)) = \mathrm{ord}(\mathrm{loc}_{\ell_1}(p^{s_1}c_0)) \geq N-s_1-e \gneq 0.$$

Since the order of every element in $X_{\ell_1} \oplus Y_{\ell_1}$ is less or equal than $S+2e$, from our choice of $N \gg 0$, we must have that c has a non-trivial component (of maximal order) in $\mathbb{Z}_p/p^N$. □

We now have all the ingredients to prove the claim.

Proof of the (claim). Recall that N is fixed such that $N \gneq (r+s)s_1+(r+s+4)e$. In order to prove the (claim), we repeatedly apply Lemma 5.4.6 and Lemma 5.4.7, starting with the Selmer structure $\mathscr{F}$. This gives us the following result.

Lemma 5.4.8. *There exist subsets $J \subset \{1,\ldots,s\}$, $I \subset \{1,\ldots,r\}$ such that*

$$s_1+(r+s)(2e) \geq \frac{1}{2}\left(\sum_{j\in J} d_j+\sum_{i\in I} e_i\right)$$

and for all $j \notin J$ (resp. all $i \notin I$) $d_j \leq (r+s)2e$ (resp. $e_i \leq (r+s)2e$).

Proof. We start by considering $H^+_{\mathscr{F}} \simeq \mathbb{Z}_p/p^N \oplus X_1$, $H^-_{\mathscr{F}} \simeq Y_1$. If $d_1 \leq (r+s)2e$ and $e_1 \leq (r+s)2e$, then we can take $I = J = \emptyset$ and there is nothing to prove. Otherwise, we prove that there exists primes $\ell_1, \ldots, \ell_{r+s}$ such that, letting $n = \ell_1 \cdot \cdots \cdot \ell_{r+s}$ and s_n the index of $\kappa_n \in H_{\mathscr{F}(n)}$, we have

$$s_1 + (r+s)(2e) \geq \tfrac{1}{2}\left(\sum_{j\in J} d_j + \sum_{i\in I} e_i\right) + s_n.$$

Notice that thanks to Corollary 5.4.3, we can apply Lemma 5.4.6 to $\mathscr{F}' = \mathscr{F}$, with $S = s_1 + 4e$; moreover taking $c = \kappa_{\ell_1}$, thanks to (K) and (sign), we can also apply Lemma 5.4.7. We then apply these lemmas inductively to $\mathscr{F}' = \mathscr{F}(\ell_1 \cdots \ell_{t-1})$, with $S = s_1 + (2+t-1)2e$ and $c = \kappa_{n_t}$, so that we can find a prime ℓ_t such that, writing $n_t = \ell_1 \cdots \ell_t$, we have

$$H^{\varepsilon}_{\mathscr{F}(n_t)} \simeq \mathbb{Z}_p/p^N \oplus X_{n_t} \;\; H^{-\varepsilon}_{\mathscr{F}(n_t)} \simeq Y_{n_t},$$

where $\varepsilon = (-1)^t$ and

$$X_{n_t} = \bigoplus_{i=1}^{r(n_t)} \mathbb{Z}_p/p^{e_i^{(n_t)}}, \quad r(t) \leq s(t-1), \quad e_1^{(n_t)} \geq e_2^{(n_t)} \geq \cdots \geq e_{r(n_t)}^{(n_t)},$$

$$Y_{n_t} = \bigoplus_{j=1}^{s(n_t)} \mathbb{Z}_p/p^{d_j^{(n_t)}}, \quad s(t) \geq r(t-1), \quad d_1^{(n_t)} \geq d_2^{(n_t)} \geq \cdots \geq d_{s(n_t)}^{(n_t)}.$$

Moreover:

(i) there is an injection $f_t : \{1, \ldots, r(t-1)\} \to \{1, \ldots, s(t)\}$, such that $e_i^{(n_{t-1})} \leq d_{f_t(i)}^{(n_j)} \leq e_i^{(n_{t-1})} + 2e$ and the missing $d_j^{(n_t)}$ are bounded by $2e$;

(ii) if $d_1^{(n_{t-1})} \gneq 2e$, there is an injection $g_t : \{1, \ldots, s(t-1)\} - S_t \to \{1, \ldots, r(t)\}$, such that $e_{g_t(j)}^{(n_t)} = d_j^{(n_{t-1})}$ and the missing $e_i^{(n_t)}$ are bounded by $2e$. Here $S_t \subset \{1, \ldots, s(t-1)\}$ is either a singleton $\{x_t\}$ or it contains two elements $\{x_t, y_t\}$ and we set $d_{y_t}^{(n_{t-1})} = 0$ in the former case; if $d_1^{(n_{t-1})} \leq 2e$, then $d_1^{(n_t)} \leq 2e$ and we set $d_{x_t}^{(n_{t-1})} = d_{y_t}^{(n_{t-1})} = 0$;

(iii) $s_{n_{t-1}} + 2e \geq s_{n_t} + \frac{d_{x_t}^{(n_t-1)} + d_{y_t}^{(n_t-1)}}{2}$, where $s_{n_{t-1}}$ and s_{n_t} are the indexes of $\kappa_{n_{t-1}}$ and κ_{n_t} respectively;

(iv) $s_{n_t} \leq s_{n_{t-1}} + 2e$, the exponent of $X_{n_t} \oplus Y_{n_t}$ is bounded by $s_1 + (2+t)2e$ and κ_{n_t} has a non trivial component of maximal order in $\mathbb{Z}_p/p^N$.

Combining the inequalities of (iii), we find

$$\begin{aligned} s_{n_t} &\leq s_{n_{t-1}} + 2e - \frac{d_{x_t}^{(n_t-1)} + d_{y_t}^{(n_t-1)}}{2} \leq s_{n_{t-2}} + 4e - \frac{d_{x_t}^{(n_t-1)} + d_{y_t}^{(n_t-1)} + d_{x_{t-1}}^{(n_t-2)} + d_{y_{t-1}}^{(n_t-2)}}{2} \\ &\leq s_1 + t(2e) - \tfrac{1}{2} \sum_{i=1}^{t} (d_{x_i}^{(n_i-1)} + d_{y_i}^{(n_i-1)}). \end{aligned}$$

Applying (ii), we find that for $t = r+s$, the exponent of $X_{n_t} \oplus Y_{n_t}$ is bounded by $(r+s)2e$. Moreover, from (i) and (ii) we find that there exist some $J \subset \{1, \ldots, s\}$, $I \subset \{1, \ldots, r\}$, such that there is an injection from $I \cup J$ to $\cup_{i=1}^{t} S_t$ and

$$\sum_{i=1}^{t} d_{x_i}^{(n_i-1)} + d_{y_i}^{(n_i-1)} \geq \sum_{j \in J} d_j + \sum_{i \in I} e_i, \quad \text{and for } j \notin J, i \notin I, d_j \leq t(2e), e_i \leq t(2e).$$

Combining this with the above inequality, we get the desired result. $\square$

In order to conclude we consider the inequality of the previous lemma. Adding $\frac{1}{2}\left(\sum_{j \notin J} d_j + \sum_{i \notin I} e_i\right)$ to both sides, we find

$$\begin{aligned} s_1 + (1 + \tfrac{1}{2}(r+s-\#I-\#J))(r+s)(2e) &\geq s_1 + (r+s)(2e) + \tfrac{1}{2}\left(\sum_{j \notin J} d_j + \sum_{i \notin I} e_i\right) \\ &\geq \tfrac{1}{2}\left(\sum_{j=1}^{s} d_j + \sum_{i=1}^{r} e_i\right). \end{aligned}$$

The left hand side is less or equal than $s_1 + (1+r+s)(r+s)(2e)$, we hence have proved

$$s_1 + (1+r+s)(r+s)(2e) \geq \tfrac{1}{2}\left(\sum_{j=1}^{s} d_j + \sum_{i=1}^{r} e_i\right).$$

Since $x := 1+r+s$ is independent on N by Remark 5.4.5 and so is e by Proposition

5.3.4, we have

$$s_1 + 4x^2(2e) \geq \frac{1}{2}\left(\sum_{j=1}^{s} d_j + \sum_{i=1}^{r} e_i\right),$$

which concludes the proof of the (claim). □

Remark 5.4.9. Let us consider the case where $T = T_p(E)$ is residually irreducible. In this case $e = 0$. Since t is a multiple of $(2e)$, we have proved that

$$\mathrm{length}_{\mathbb{Z}_p}(M) \leq \mathrm{length}_{\mathbb{Z}_p}(H^1_{\mathscr{F}}(K,T)/\mathbb{Z}_p \cdot \kappa_1),$$

giving an alternative proof to Howard's result [How04, Theorem 1.6.1].

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