Closure and Completeness Problems in Approximation Theory

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Abstract

Much of the work in the theory of approximation has crystallised around two classical problems; the first is the Tchebyshev problem of finding a polynomial of degree n which gives the best uniform approximation to a given continuous function on an interval. The second is that of Weierstrass, to show that every continuous function on a closed bounded interval can be uniformly approximated with arbitrarily small error by a polynomial.

Closure and completeness problems fall in the second category. In this thesis, we look at some sequences of rational functions in certain Hilbert spaces of bounded analytic functions, and ask when every function in these Hilbert spaces can be approximated arbitrarily closely (in the Hilbert space norm) by finite linear combinations of the rational functions. Furthermore, we provide a characterisation of the closed subspaces that the rational functions generate in the case when this is not the whole space. Finally, we obtain necessary and sufficient conditions in order that the rational functions will constitute Schauder bases in the closed subspaces which they generate.

Contents

List of frequently used symbols

Δ	Open unit disc, p. 11
P^+	Right half-plane, p. 12
$H^2(\Delta)$	A space of bounded analytic functions in Δ, p. 58
$H^2(P^+)$	A space of bounded analytic functions in P^+, p. 12
D_α	Weighted Dirichlet spaces, p. 12
$\mathcal{T}(f, \alpha)$	Laguerre transform, p. 13
$\overline{V}$	Closed span of the exponentials in $L_2(0, \infty)$, p. 17–18
$\overline{X^{(n)}}$	Closed span of the system $\{x_i\}_{i=1;i\neq n}^{\infty}$, p. 20
$P_{(n)}$	Span of the sequence $\{x_i\}_{i=1}^{n}$, p. 20
$P^{(n)}$	Closed span of the system $\{x_i\}_{i=n+1}^{\infty}$, p. 20
$\sigma_{(n)}$	Unit sphere in $P_{(n)}$, p. 20
$\sigma^{(n)}$	Unit sphere in $P^{(n)}$, p. 20
$\varphi(x)$, $\psi(x)$	p. 32
$\overline{\Phi}$	p. 41
$\overline{\Psi}$	p. 43

...Of all the triangles, the one whose sides are 3, 4, and 5 cubits respectively, is sacred.

From Problem 6, Moscow Papyrus

$\approx$ 2000 B.P. (Before Pythagoras)

...M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phenomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde.

C.G.J. Jacobi

Letter to Legendre, 1830

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Introduction

This thesis concerns the closure and completeness of various systems in some classical Hilbert function spaces. The archetypal problems discussed proceed as follows.

Let $\{f_n\}$ denote a sequence of functions in a given Hilbert function space H. We say that $\{f_n\}$ is closed in H if every element in H can be approximated arbitrarily closely by finite linear combinations of $f_1, f_2, \ldots$. In the most general case, there is a well-known necessary and sufficient condition for a given sequence $\{f_n\}$ to be closed in H. We consider certain special Hilbert spaces of analytic functions and give a characterisation of the closed linear manifold $\overline{span\{f_n\}}$ say, that $\{f_n\}$ generates if the closure condition fails to hold. Finally, in these special cases, we provide a necessary and sufficient condition for $\{f_n\}$ to constitute a Schauder basis for $\overline{span\{f_n\}}$, still assuming that the closure condition is not satisfied.

We collect a few definitions in the first chapter, and provide the motivation for this line of research.

In Chapter Two, we study the closure (completeness) of the normalised sequences $\{\varphi_n(x)\} = \{2x(Rez_n/\pi)^{1/2}(x^2 + z_n^2)^{-1}\}$ and $\{\psi_n(x)\} = \{2|z_n|(Rez_n/\pi)^{1/2}(x^2 + z_n^2)^{-1}\}$ in the Lebesgue space $L_2(0,\infty)$, where $z_1, z_2, \ldots$ are complex numbers in the right half-plane with separation $|z_{n+1}| - |z_n| \geq \theta > 0$, and $Rez_n \to \infty$ as $n \to \infty$. It is shown that $\{\varphi_n\}$ is closed in $L_2(0,\infty)$ if, and only if $\sum Rez_n(1+|z_n|^2)^{-1} = \infty$. This condition is also shown to be necessary and sufficient for $\{\psi_n\}$ to be closed in $L_2(0,\infty)$. Now, suppose $\sum Rez_n(1+|z_n|^2)^{-1} < \infty$. We show that $\overline{span\{\varphi_n\}}$ consists of the Laplace sine transforms of certain functions analytic in the right half-plane $Rez > 0$, restricted to the positive real axis. The closed span of the "sister sequence" $\{\psi_n\}$ is shown to consist of the Laplace cosine transforms of these analytic functions. The chapter concludes with an outline of a proof that $\{\varphi_n\}$ and $\{\psi_n\}$ will each be a basis for the respective closed linear manifold of $L_2(0,\infty)$ which they generate if, and only if

$$\inf_n \prod_{k=1;k\neq n}^{\infty} \left|\frac{z_k - z_n}{\overline{z_k} + z_n}\right| = \delta > 0.$$

Chapter Three is concerned with analogous results for the sequence of normalised Cauchy kernels $\{k_n(z)\} = \{(1-|z_n|^2)^{1/2}(1-\overline{z_n}z)^{-1}\}$ in the Hardy space $H^2(\Delta)$, $(\Delta = \{z : |z| < 1\})$. We prove, following Otto Szász, that $\{k_n(z)\}$ is closed in $H^2(\Delta)$ if, and only if $\sum(1-|z_n|) = \infty$. When this condition is not satisfied, we show the closed span of this system to consist of the Laguerre transform of some analytic functions in the right half-plane, restricted to the positive real axis. A necessary and sufficient condition for $\{k_n(z)\}$ to be a Schauder basis for $\overline{span\{k_n\}}$ is again that the infinite Blaschke product

$$\inf_n \prod_{k=1;k\neq n}^{\infty} \left|\frac{z_k - z_n}{1 - \overline{z_n}z_k}\right|$$

be bounded away from zero.

In Chapter Four, we investigate the closure of the kernels $\{\ell_n(\alpha, z)\} = \{(1-\overline{z_n}z)^{\alpha-1}\}$ in the space D_α $(0 \leq \alpha < 1)$ of functions analytic in Δ and for which the norm $\|f\|^2_{D_\alpha} = \sum |a_n|^2/n^\alpha$ is finite. The results obtained here are almost identical with those in Chapter 3. For example, a necessary condition for $\{\ell_n(\alpha, z)\}$ to be closed in D_α is that $\sum(1 - |z_n|) = \infty$. For $0 < z_1 \leq z_2 \leq \cdots < 1$, the necessary condition is also sufficient. The main point of the chapter is to emphasise the uniformity and strength of the methods used throughout this thesis, viz., the computation of distances using the theory of reproducing kernel, characterisation of function spaces with the help of the hyperbolic metric, and the use of interpolation theory to solve bases problems.

Chapter 1

Preliminaries

We define here, some of the function spaces in which our investigations will be carried out, and provide the motivation for the research contained in this thesis. The definition of the Laguerre transform is somewhat new, but all the results in this chapter are known.

1.1 H^p-Spaces

Let Δ be the open unit disc $\{z : |z| < 1\}$. $H^p(\Delta)$ denotes the Hardy space of functions f analytic in Δ such that the integral means

$$M_p\{f,r\} = \left(\frac{1}{2\pi}\int_0^{2\pi} \mid f(re^{i\theta}) \mid^p d\theta\right)^{1/p}, \quad 0 < p < \infty;$$

$$M_\infty\{f,r\} = \max_{0\leq\theta\leq 2\pi} \mid f(re^{i\theta}) \mid$$

remain bounded as $r \to 1$. Each of the means M_p is a non-decreasing function of r. If $f \in H^p$, then it has radial limit $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ for almost all θ.

Consider the half-plane $P^+ = \{w : Re w > 0\}$. If f is analytic in P^+, then f is said to be in the class $H^2(P^+)$ provided that the integrals

$$\int_{-\infty}^{\infty} | f(x+iy) |^2 \, dy$$

are bounded for each $x > 0$.

1.2 D_α-Spaces

Let $\phi(z) = \sum c_n z^n$ be an analytic function in Δ with $c_0 = 1, c_n > 0$ and

$$c_n^2 \le c_{n-1} c_{n+1}, \quad n = 1, 2, \ldots.$$

(For example, $c_n = 1/n^\alpha, \alpha \ge 0$.) Let D_ϕ be the Hilbert space of functions $f(z) = \sum a_n z^n$ analytic in Δ with the weighted norm

$$\|f\|^2 = \sum \frac{1}{c_n} |a_n|^2 < \infty.$$

If, as in the above example, $c_n = 1/n^\alpha$, $f(0) = 0$ and $\alpha = 1$, then

$$\|f\|^2 = \sum_1^\infty n|a_n|^2 = \frac{1}{\pi} \int\int_{|z|<1} |f'(z)|^2 \, dxdy < \infty,$$

so D_ϕ corresponds to the classical space of analytic functions in Δ with finite Dirichlet integral. These spaces, though of Dirichlet type, are sometimes called

Bergman spaces.

If $g(z) = \sum b_n z^n \in D_\phi$, the inner product (f, g) is given by

$$(f,g) = \sum \frac{1}{c_n} a_n \overline{b_n}.$$

As an example (and one which will be very useful for our purposes), consider the functions

$$\phi_\alpha(z) = \frac{1}{(1-\overline{\zeta}z)^{1-\alpha}}, \qquad 0 \leq \alpha < 1, \qquad \zeta \in \Delta.$$

We denote by D_α, the Hilbert spaces corresponding to these functions, and say that the D_α are weighted Dirichlet spaces. In ([22], p. 227) Shapiro and Shields established the following isometric correspondence between D_α and $L_2(0, \infty)$;

$$z^n \left(\frac{\Gamma(\alpha+n+1)}{n!}\right)^{1/2} \longleftrightarrow t^{\alpha/2} e^{-t/2} \left(\frac{n!}{\Gamma(\alpha+n+1)}\right)^{1/2} L_n^\alpha(t),$$

where

$$L_n^\alpha(t) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-t)^\nu}{\nu!}$$

are the associated Laguerre Polynomials. The left-to-right arrow of this correspondence yields the isometry $\mathcal{T} : L_2(0, \infty) \to D_\alpha$

$$\mathcal{T}\left\{(1-\overline{\zeta})^{-1-\alpha} \exp\left(-t\overline{\zeta}/(1-\overline{\zeta})\right)\right\} = \frac{1}{(1-\overline{\zeta}z)^{1-\alpha}}, \tag{1.1}$$

where $\zeta \in \Delta$ is such that $Re\overline{\zeta}(1-\overline{\zeta})^{-1} > 0$. We refer to $\mathcal{T}$ as the *Laguerre transform.*

1.3 Closure and Completeness

The sequence of functions $\{f_n\}$ is complete in a Hilbert function space H if for all bounded linear functionals $L \in H^* = H$, $L(f_n) = 0$, $n = 1, 2, ...$ imply that $L \equiv 0$. Thus, for example, the sequence $\{f_n\}$ is complete in $H^2(\Delta)$ if the only function $g \in H^2(\Delta)$ which is orthogonal to all of $f_1, f_2, \ldots$ is the zero function; that is, the vanishing inner products

$$(f_n\,,\,g) = \int_0^\infty f_n(t)\overline{g(t)}dt = 0, \quad n = 1, 2, ...$$

imply that $g(t) \equiv 0$ in Δ.

The sequence of functions $\{f_n\}$ is called *total* or is said to be *closed* in H if the subspace that it generates is everywhere dense in H; that is, if every element $f \in H$ can be approximated arbitrarily closely by finite linear combinations of the f_n, so that given an $f \in H$ and an $\varepsilon > 0$, we can find constants $c_{n1}, \ldots, c_{nn}$ such that

$$\left\| f - \sum_{\nu=1}^{n} c_{n\nu} f_\nu \right\|_H \leq \varepsilon.$$

The fundamental theorem of closure and completeness says that a sequence is closed in a normed linear space if and only if it is complete. This is a consequence of the Hahn-Banach Extension Theorem. We will use both terms in this thesis so as to draw attention to the appropriate defining property. For closure, the defining property is transitivity. This means that if $\{f_n\}$ is closed in H, then a second system $\{g_n\}$ is closed in H if and only if "$\{g_n\}$ is closed in $\{f_n\}$". We say that $\{g_n\}$ is closed in $\{f_n\}$ if each f_n can be approximated arbitrarily closely by finite linear combinations of the g_n. Lauricella gave an elegant proof of the transitivity of closure (see [9], p. 264–265).

Theorem 1.3.1 (Müntz Closure Theorem) *The set of functions* $\{1, x^{\lambda_1}, x^{\lambda_2}, \cdots\}$, *where* $1 \leq \lambda_n \to \infty$ *is closed in* $C[0,1]$ *in the uniform norm if, and only if* $\sum(1/\lambda_n) = \infty$.

This is a generalisation of the classical theorem of Weierstrass which asserts that every continuous function $f(x)$ on the finite interval $a \leq x \leq b$ can be approximated by a polynomial to within an ε, for each $\varepsilon > 0$. In the terminology of Functional Analysis, this says that the system $\{x^n\}$ $(n = 1, 2, \ldots)$ is closed (total) in $C[a, b]$. The proof of the Müntz Closure Theorem can be found in elementary texts on Approximation Theory. See for example, Akhieser [1], Cheney [7] or Davis [9].

On making the change of variable $x = e^{-y}$, we can transform the Müntz Closure Theorem into a result concerning the closure of the sequence $\left\{e^{-\lambda_n y}\right\}$ in $L_2(0, \infty)$. In fact, it is well-known ([20], p. 99) that if $\{\lambda_n\}$ is a sequence of complex numbers in P^+ satisfying $|\lambda_n| \to \infty$, then $\left\{e^{-\lambda_n x}\right\}$ is closed in $L_2(0, \infty)$ if, and only if $\sum Re\lambda_n(1 + |\lambda_n|^2)^{-1} = \infty$. In [24], the closure and completeness of various sequences of rational functions are studied. Using a direct and elementary procedure, the following theorem is proved.

Theorem 1.3.2 (Szász) *Suppose that z_n is a sequence of points in Δ. Then the system $\{(1-z_n z)^{-1}\}_1^\infty$ is closed in $H^2(\Delta)$ if, and only if $\sum_{n=1}^\infty (1-|z_n|) = \infty$.*

Akhieser [1] gave a proof of Szász's theorem from which some information can be obtained about the closed span of the system $\{(1-z_n z)^{-1}\}_1^\infty$. We shall elaborate on this in §3.4.

1.4 Free systems

A sequence of functions $\{f_n\}$ in a normed linear space is called *free* if it satisfies any one of the following equivalent conditions:

(1) No function f_n belongs to the closed linear manifold generated by the remaining functions, i.e., $f_n \notin \overline{span\{f_\nu\}}_{\nu\neq n}$.

(2) If

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} a_{n\nu} f_\nu = 0,$$

then $\lim_{n\to\infty} a_{n\nu} = 0$ for all ν.

(3) For every representation

$$f = \lim_{n\to\infty} \sum_{\nu=1}^{n} a_{n\nu} f_\nu\,, \tag{1.2}$$

the limit $\lim_{n\to\infty} a_{n\nu}$ exists for all ν. If we set

$$\lim_{n\to\infty} a_{n\nu} = a_\nu\,, \tag{1.3}$$

then $\{a_\nu\}$ is determined for a given f independent of the relation (1.2) as long as this makes sense; that is, as long as f belongs to the closure of the subspace generated by $\{f_n\}$. The symbolic representation is

$$f \sim \sum_{\nu=1}^{\infty} a_\nu f_\nu. \tag{1.4}$$

We use $\sim$ and not $=$ to emphasise that we are not exactly dealing with a sum, but with the limit of sums. (In general, the series $\sum a_\nu f_\nu$ diverges in norm.) Furthermore, the representation $\sum a_\nu f_\nu$ does not necessarily characterise f, since many functions can have the same representation. If $\{f_\nu\}$ is a free system and $f \in \overline{span\{f_\nu\}}$ has the associated expansion $f \sim \sum a_\nu f_\nu$, then $a_\nu = L_\nu(f)$ can be seen as a linear functional on $\overline{span\{f_\nu\}}$. The significance of (1.3) will become clear in §1.5 when we define what it means for a free system to be a basis for the closure of the subspace that it generates.

Let λ_n be a set of distinct positive integers with $\sum(1/\lambda_n) < \infty$. Denote by $\overline{\Lambda(X)}$, the closure of the subspace of $C[0,1]$ that the free system $\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ now generates. Suppose further that λ_n satisfies $\inf_n(\lambda_{n+1}/\lambda_n) \geq c > 1$. Clarkson and Erdös showed in [8] that $\overline{\Lambda(X)}$ consists of the restriction to the unit interval, of functions analytic in Δ and of the form $\sum a_n z^{\lambda_n}$. Korevaar [16] extended the result to one where $\{\lambda_n\}$ is any sequence of integers. Schwartz [19] obtained the same result with milder restrictions on the λ_n. We will say more on this later.

Consider again the system $\left\{e^{-\lambda_n x}\right\}$ in $L_2(0,\infty)$. Denote by $\overline{V}$, the closure in the L_2-norm, of the linear manifold generated by this system. We follow Binmore [5] in calling the inequality

$$\sum_{n=1}^{\infty} \frac{Re\lambda_n}{1+|\lambda_n|^2} < \infty$$

the *Müntz-Szász condition.* If this condition is satisfied, then $\left\{e^{-\lambda_n x}\right\}$ is not total, i.e. $\overline{V} \neq L_2(0,\infty)$ (see [9] or [20]). Moreover $\left\{e^{-\lambda_n x}\right\}$ is a free system ([19], p. 29). Thus, by (1.4) we can associate with each $f \in \overline{V}$ a unique representation of the form

$$f(x) \sim \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x). \tag{1.5}$$

Schwartz [19] obtained upper bounds for $|a_n|$ when the sequence $\{\lambda_n\}$ is real and satisfies certain conditions of regularity (for example, $\lambda_{k+1} - \lambda_k \geq c > 0$ for $k = 1, 2, \ldots$) and showed that the formal series in (1.5) does in fact converge to $f(x)$ "normally" for all $x \geq \varepsilon > 0$. (A series is normally convergent if the series of absolute values is locally uniformly convergent.) Furthermore, the function $f(z)$, for $z \in P^+$ admits the "Dirichlet expansion" $f(z) = \sum a_n \exp(-\lambda_n z)$ for $Rez > 0$. Reciprocity results are also proved; for example:

Theorem 1.4.1 (Schwartz, 1943) *Let $\{\lambda_n\}$ be a sequence of real numbers with $0 < \lambda_1 < \lambda_2 < \cdots$ and $\lambda_{k+1} - \lambda_k \geq c > 0$. Suppose that $\sum(1/\lambda_n) < \infty$. If $f(x) \in L_2(0,\infty)$ has the following properties;*

(1) $f(x)$ is analytic in $(0,\infty)$ and can be extended by the holomorphic function $f(z)$ to P^+,

(2) $f(z)$ has the formal Dirichlet representation

$$f(z) \sim \sum a_n \exp(-\lambda_n z);$$

the series being normally convergent, then $f(x) \in \overline{V}$, and the a_n are the coefficients of f with respect to the free system $\{\exp(-\lambda_n x)\}$.

This theorem demonstrates that we have here a method of characterising functions in the closed span of the exponentials when these constitute a free system. For the characterisation problems studied in this thesis, we find order-of-growth estimates for the coefficients of functions in the closed span of certain free systems, and prove results that are similar to those of Schwartz.

1.5 Schauder Bases

Let X denote an infinite dimensional Banach space. A sequence $\{x_n\}$ in X is called a *basic sequence* if, for all $x \in \overline{span\{x_n\}}$, there exists a unique sequence of real or complex numbers $\{\alpha_n\}$ such that

$$\lim_{n \longrightarrow \infty} \left\| x - \sum_{\nu=1}^{n} \alpha_\nu x_\nu \right\|_X = 0.$$

If in addition, $\overline{span\{x_n\}} = X$, then we say that $\{x_n\}$ is a *Schauder basis* for X.

All the important function spaces that we study are infinite dimensional and we will not have occasion to refer to other type of bases, so the word Schauder will sometimes be omitted.

We introduce a few notations.

$\overline{X^{(n)}} = [x_1, \ldots, x_{n-1}, x_{n+1}, \ldots]$ is the closed linear manifold generated by the sequence $x_1, \ldots, x_{n-1}, x_{n+1}, \ldots$.

$P_{(n)} = [x_1, \ldots, x_n] \quad (n = 1, 2, \ldots)$,

$P^{(n)} = [x_{n+1}, x_{n+2}, \ldots] \quad (n = 1, 2, \ldots)$,

$\sigma_{(n)} = \{x \in P_{(n)} : \|x\| = 1\} \quad (n = 1, 2, \ldots)$,

$\sigma^{(n)} = \{x \in P^{(n)} : \|x\| = 1\} \quad (n = 1, 2, \ldots)$.

The last two notations denote the unit spheres in the respective sub-manifolds.

The following is part of Theorem 7.1 in (Singer [23]). We shall make repeated use of it.

Theorem 1.5.1 *Let X be an infinite dimensional Banach space and $\{x_n\}$ a total sequence in X with $x_n \neq 0$, $n = 1, 2, \ldots$. Then the following statements are equivalent*

(a) $\{x_n\}$ is a Schauder basis for X.

(b) There exists a constant C with $1 \leq C < \infty$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq C \left\| \sum_{i=1}^{n+m} \alpha_i x_i \right\|,$$

for all integers m, n and all complex numbers $\alpha_1, \ldots \alpha_{m+n}$.

(c1)

$$\inf_{1<n<\infty} \text{dist} \left(\frac{x_n}{\|x_n\|}, \overline{X^{(n)}} \right) > 0,$$

and

(c2)

$$\inf_{1\le n,k<\infty} \operatorname{dist}\left(\sigma_{(n)}\,,\,\sigma^{(n+k)}\right) > 0.$$

Let $\{\lambda_n\}$ be a sequence of points in the right half-plane, P^+ with $Re\lambda_n \to \infty$ as $n \to \infty$. For the space $\overline{V} \neq L_2(0,\infty)$, the functions $\left\{e^{-\lambda_n x}\right\}$ form a basis for $\overline{V}$ if for each $f \in \overline{V}$, there exist a unique sequence $a_1, a_2, \ldots$ of complex numbers such that

$$\lim_{n\to\infty} \left\| f(x) - \sum_{\nu=1}^{n} a_\nu \exp(-\lambda_\nu x) \right\| = 0. \tag{1.6}$$

Now if the λ_n are real, the Müntz-Szász condition is equivalent to $\sum(1/\lambda_n) < \infty$. In this case, Gurariĭ and Macaev [13] showed that a necessary and sufficient condition for (1.6) to hold is that $(\lambda_{n+1}/\lambda_n) \geq c > 1$, c being independent of n. Anderson [2] proved the case when the λ_n are complex numbers.

Theorem 1.5.2 (Anderson, 1975) *Suppose that $\{\lambda_n\}$ is a sequence in P^+ with $Re\lambda_n \to \infty$ as $n \to \infty$ and $\sum Re\lambda_n(1+|\lambda_n|^2)^{-1} < \infty$. Set $\overline{V} \subset L_2(0,\infty)$, $\overline{V} \neq L_2(0,\infty)$. Then the system $\left\{e^{-\lambda_n x}\right\}$ constitutes a Schauder basis for $\overline{V}$ if, and only if*

$$\inf_n \prod_{\nu=1;\nu\neq n}^{\infty} \left| \frac{\lambda_\nu - \lambda_n}{\overline{\lambda_\nu} + \lambda_n} \right| = \delta > 0.$$

To prove Theorem 1.5.2, Anderson uses some results of Shapiro and Shields concerning Bessel sequences and interpolation in $H^2(\Delta)$. (See §1.7 for the definition of Bessel and interpolating sequences.) We shall have occasion to expand on this theorem when attacking bases problems in the closed subspaces studied in this thesis.

1.6 Reproducing Kernel

H will continue to denote a Hilbert function space and the elements of H will be defined on some set Q endowed with the notion of measure (for example, the Euclidean n-space or the set of complex numbers). If the value of f for each $y \in Q$ is a bounded (continuous) linear functional of f, then, by virtue of the Riesz representation theorem, there is a unique element k_y such that

$$(f\,,\,k_y) = f(y).$$

k_y is called the *reproducing element* for the point y. The totality of reproducing elements for all the points in Q is called the *reproducing kernel* (r.k.) of H. Another way of viewing the r.k. is as the map

$$(t\,,\,y) \longrightarrow k_y(t),$$

from $Q \times Q$ to the set of complex numbers. $k(t, y)$ will sometimes be used in place of $k_y(t)$ and k_y will be loosely referred to as the reproducing kernel. Much of the material in this section is contained in (Aronszajn [3], [4]) and (Shapiro [20], pp. 82-85). The following are some basic properties of the reproducing kernel.

(a) $k(x,y) = (k_y\,,\,k_x)$

(b) $k(x,y) = \overline{k(y,x)}$

(c) $k(x,x) \geq 0$. Since $k(x,x) = (k_x\,,\,k_x)$, then $k(x,x) = 0 \Longrightarrow k_x = 0$ and all the elements of H vanish.

(d) $|k(x,y)| \leq k(x,x)^{1/2}k(y,y)^{1/2}$. This follows from (a), (c) and Schwarz's inequality.

(e) k is *positive definite* on Q, i.e., for every $n \geq 1$ and every set of points $x_1, \ldots, x_n \in Q$, the quadratic hermitian matrix

$$\|k(x_i, x_j)\|, \qquad i,j = 1,2,\ldots,n$$

is positive semi-definite, or more precisely, the quadratic hermitian form

$$\sum_{i,j=1}^{n} k(x_i, x_j)\lambda_i\overline{\lambda_j} \geq 0$$

for all complex numbers $\lambda_1, \ldots, \lambda_n$ and all $x_1, \ldots, x_n \in Q$. From (a), the determinant of $\|k(x_i, x_j)\|$ is the Gramian of $k_{x_1}, \ldots, k_{x_n}$, or what is the same thing, the hermitian form associated with this matrix equals $\|\sum_{i=1}^{n} \lambda_i k_{x_i}\|^2$, which is clearly positive semi-definite.

If H is a Hilbert function space, then so is every closed subspace J of H. J therefore has its own r.k., since the bounded evaluation functionals of H will be bounded for every subspace thereof. The following simple result is of vital importance in the application of the theory of reproducing kernels to closure problems.

Lemma 1.6.1 *Let H be a Hilbert space with reproducing kernel k_w and let H^* be a closed subspace of H with reproducing kernel k_w^*. Then the function f^* whose value at $w \in Q$ is given by the inner product*

$$f^*(w) = (f\,,\,k_w^*),\ \ f \in H$$

is an element of H^*, and f^* is the orthogonal projection of f on H^*. Also, k_w^* is the orthogonal projection of k_w on H^*.

Proof: Let $f \in H$ and let f_o be the orthogonal projection of f on H^*, i.e., $f_o \in H^*$ and $f - f_o \perp H^*$. Then, since $k^* \in H^*$, we have for $w \in Q$

$$f^*(w) = (f\,,\,k_w^*) = (f - f_o\,,\,k_w^*) + (f_o\,,\,k_w^*) = f_o(w),$$

proving the first assertion.

At $z \in Q$, the orthogonal projection of k_w on H^* takes the value

$$(k_w\,,\,k_z^*) = \overline{(k_z^*\,,\,k_w)} = \overline{k_z^*(w)} = k_w^*(z).$$

This proves the second assertion. □

Corollary 1 *If H^* and H^{**} are complementary orthogonal subspaces of H with reproducing kernels k^* and k^{**} respectively, then*

$$k(x,y) = k^*(x,y) + k^{**}(x,y).$$

A useful consequence of the corollary is that if J is a subspace of H, their reproducing kernels are related by the inequality $k^J(x,x) \leq k^H(x,x)$. In order to compute the orthogonal projection of H on H^* by the formula $f^*(w) = (f\ ,\ k_w^*)$, it is not necessary that H possess a reproducing kernel, but that H^* does. This is important in applications where for example, H may be $L_2(0, \infty)$, which has no r.k., as any talk of bounded evaluation functionals for classes of functions is meaningless. Our investigations in $L_2(0, \infty)$ will not however, be using the theory of reproducing kernels, but work in all the other Hilbert function spaces will draw on the theory.

1.7 Bessel and Interpolating Sequences

A sequence of functions $\{f_n\}$ in a Hilbert function H is called a *Bessel sequence* with bound M if

$$\sum_{n=1}^{\infty} |(f, f_n)|^2 \leq M\|f\|^2$$

for all $f \in H$.

Let A_n be an $n \times n$ matrix $(f_i\ ,\ f_j)$ $(i, j = 1, 2, \ldots, n)$ and a an n-tuple

$$a = (a_1, \ldots, a_n).$$

Let A be the inner product matrix $(f_i\ ,\ f_j)$ $(i, j = 1, 2, \ldots)$. Then the matrix A is said to be bounded below by m if $\|a\| \leq m\|A_n a\|$ for all n-tuples a, and all n. Here the norms are the l_2-norm, where l_2 is the space of all complex number sequences that are square-summable. The following theorem is given in ([21], p. 525). We provide the proof here because this is what we will need later.

Theorem 1.7.1 *The sequence $\{f_n\}$ is a Bessel sequence with bound M if and only if the matrix A is a bounded operator on l_2 with bound M.*

Proof: Suppose the following statements hold: $\{f_n\}$ is a Bessel sequence, $\{c_n\}$ is in l_2, and $f = c_1 f_1 + \cdots + c_p f_p$ for a fixed p. Then from the hypotheses,

$$\begin{aligned} \|f\|^4 = |(f\,,\,f)|^2 &= \left|\sum_{n=1}^{p} \overline{c_n}(f\,,\,f_n)\right|^2 \\ &\leq \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} |(f\,,\,f_n)|^2 \\ &\leq M\|f\|^2 \sum_{n=1}^{p} |c_n|^2. \end{aligned}$$

Thus,

$$\|f\|^2 \leq M \sum_{n=1}^{p} |c_n|^2.$$

On the other hand, $\|f\|^2 = (f\,,\,f) = \sum_{i,j=1}^{p} c_i\overline{c_j}(f_i\,,\,f_j)$, and so

$$\sum_{i,j=1}^{p} c_i\overline{c_j}(f_i\,,\,f_j) \leq M \sum_{n=1}^{p} |c_n|^2,$$

giving the boundedness of the matrix A.

To prove the converse, suppose that A be a bounded operator with bound M and let $\{e_n\}$ be a complete orthonormal sequence in H. First, we show that there is a bounded operator $T : H \to H$, such that $Te_n = f_n$. Then we establish the Bessel property.

Let $f_i = \sum b_{ni}e_n$, and let T be the infinite matrix (b_{ni}). Then $T^*T = A$, and so $\|T\|^2 = \|A\| = M$, (where T^* is the conjugate transform of T). Also, $Te_n = f_n$.

Now, the Bessel property. Let $f \in H$ be given. Then $(f\,,\,f_n) = (f\,,\,Te_n) = (T^*f\,,\,e_n)$, and so $\sum |(f\,,\,f_n)|^2 = \|T^*f\|^2 \leq M\|f\|^2$. □

The sequence of points $\{z_n\}$ in Δ is called an *interpolating sequence* if, given an arbitrary bounded sequence of complex numbers $\{w_n\}$, there is a bounded analytic function f such that

$$f(z_n) = w_n \quad (n = 1, 2, \ldots).$$

As is well-known, (see [21], p. 514) the set of points $\{z_n\}$ in Δ is the set of zeros of an $f \in H^2(\Delta)$ $(f \not\equiv 0)$ precisely when $\sum(1 - |z_n|) < \infty$.

Theorem 1.7.2 (Carleson) *A necessary and sufficient condition for $\{z_n\}$ to be an interpolating sequence is that there exist a positive δ such that*

$$\prod_{j \neq k} \left| \frac{z_k - z_j}{1 - \overline{z_j} z_k} \right| \geq \delta \;\; (k = 1, 2, \ldots). \tag{1.7}$$

Lemma 1.7.1 *Suppose that the sequence $\{z_n\}$ is in Δ and satisfies condition (1.7). Then there exists a constant M such that*

$$\sum_{k=1}^{\infty} |g(z_k)|^2 (1 - |z_k|^2) \leq M\|g\|^2$$

for all $g \in H^2(\Delta)$.

Let $r_{mn} = |z_m - z_n| / |1 - \overline{z_m} z_n|$ and

$$P_n(\lambda) = \prod_{m \neq n} \left\{ 1 - (1 - r_{mn})^{\lambda} \right\}.$$

Hayman [14] showed that $P_n(1) \geq \delta > 0$ (i.e., condition (1.7)) is necessary for $\{z_n\}$ to be an interpolating sequence and that $P_n(\lambda) \geq \delta > 0$ for some $\lambda < 1$ is sufficient. The necessity and sufficiency of (1.7) was settled by Carleson [6]; see also Newman [17], and Shapiro and Shields [21] (where we refer the reader for the proofs of the theorem and the lemma. Theorem 1.7.2 is Theorem 1, p. 517, and Lemma 1.7.1 is the first part of Lemma 1, p. 519).

1.8 Parseval's Identity

The following theorem establishes an isometry between the L_2 functions. The proof can be found in Titchmarsh ([25], pp. 70-76).

Theorem 1.8.1 *Let $f(x)$ be a function in $L_2(0,\infty)$, and let*

$$F_c(x,a) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a f(y) \cos xy\, dy.$$

Then, as $a \to \infty$, $F_c(x,a)$ converges in the mean over $(0,\infty)$ to a function $F_c(x)$ of $L_2(0,\infty)$; and reciprocally

$$f_c(x,a) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a F_c(y) \cos xy\, dy$$

converges to $f(x)$. We have, almost everywhere,

$$F_c(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty f(y) \frac{\sin xy}{y} dy,$$

and

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty F_c(y) \frac{\sin xy}{y} dy.$$

Moreover,

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(x)|^2\,dx.$$

It is this last equality that we refer to as *Parseval's identity.* The analogue of Theorem 1.8.1 for sine transforms holds, with $\cos xy$ replaced by $\sin xy$ and $\sin xy$ replaced by $1 - \cos xy$.

1.9 Paley-Wiener Isometry

Theorem 1.9.1 (Paley and Wiener) *A complex-valued function f in the right half-plane belongs to the class $H^2(P^+)$ if and only if f has the form*

$$f(w) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(t) \exp(-wt)\,dt \tag{1.8}$$

for some unique $\hat{f}$ in $L_2(0,\infty)$.

Theorem 1.9.2 *If f is in $H^2(P^+)$, then the function*

$$g(z) = f\left(\frac{1+z}{1-z}\right)$$

is in $H^2(\Delta)$.

Theorem 1.9.2 follows from the fact that the fractional linear transformation $(1+z)/(1-z)$ maps the disc conformally onto the right half-plane. For more details on Theorems 1.9.1 and 1.9.2, (see Hoffman[15], pp. 103-106 and pp. 127-129). Moreover, it is a consequence of Parseval's identity that, when (1.8) holds, we have

$$\|f\|_{H^2(P^+)} = \sqrt{2\pi}\|\hat{f}\|_{L^2(0,\infty)}.$$

This equality is sometimes called the *Paley-Wiener isometry,* notwithstanding the incidental factor of $\sqrt{2\pi}$.

Chapter 2

Closure and Completeness in $L_2(0, \infty)$

2.1 Introduction

$L_2(0, \infty)$ is the Hilbert space of complex measurable functions $f(x)$ in the interval $0 < x < \infty$ such that

$$\|f\|_{L_2} = \int_0^\infty |f(x)|^2 \, dx < \infty.$$

A metric $d(. \, , .)$ in $L_2(0, \infty)$ is defined by

$$d(f_1 \, , f_2) = \|f_1 - f_2\|_{L_2} \, ,$$

for $f_1 \, , f_2 \in L_2(0, \infty)$, and an inner product $(. \, , .)$ is defined by

$$(f_1 \, , f_2) = \int_0^\infty f_1(x)\overline{f_2(x)} \, dx.$$

In this chapter, we study the closure and completeness in the L_2-norm of the sequences

$$\{\varphi_n(x)\} = \left\{2x(Rez_n/\pi)^{1/2}(x^2+z_n^2)^{-1}\right\}$$

and

$$\{\psi_n(x)\} = \left\{2|z_n|(Rez_n/\pi)^{1/2}(x^2+z_n^2)^{-1}\right\}.$$

These are unit vectors (functions) in $L_2(0,\infty)$, since for $Rez_n > 0$, $n = 1, 2, \ldots$,

$$\begin{aligned}
\left\|\frac{1}{x^2+z_n^2}\right\|_{L_2} &= \left(\int_0^\infty \left|\frac{1}{x^2+z_n^2}\right|^2 dx\right)^{1/2} \\
&= \left(\int_0^\infty \frac{dx}{(x^2+z_n^2)(x^2+\overline{z_n}^2)}\right)^{1/2} \\
&= \left\{(\overline{z_n}^2 - z_n^2)^{-1}\int_0^\infty \left(\frac{1}{x^2+z_n^2} - \frac{1}{x^2+\overline{z_n}^2}\right) dx\right\}^{1/2} \\
&= \left\{\frac{\pi}{2z_n\overline{z_n}}\left(\frac{\overline{z_n}-z_n}{\overline{z_n}^2 - z_n^2}\right)\right\}^{1/2} \\
&= \frac{1}{2|z_n|}\left(\frac{\pi}{Rez_n}\right)^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
\left\|\frac{x}{x^2+z_n^2}\right\|_{L_2} &= \left(\int_0^\infty \left|\frac{x}{x^2+z_n^2}\right|^2 dx\right)^{1/2} \\
&= \left(\int_0^\infty \frac{x^2dx}{(x^2+z_n^2)(x^2+\overline{z_n}^2)}\right)^{1/2} \\
&= \left\{(\overline{z_n}^2 - z_n^2)^{-1}\int_0^\infty \left(\frac{\overline{z_n}^2}{x^2+\overline{z_n}^2} - \frac{z_n^2}{x^2+z_n^2}\right) dx\right\}^{1/2} \\
&= \left\{\frac{\pi}{2}\left(\frac{\overline{z_n}-z_n}{\overline{z_n}^2 - z_n^2}\right)\right\}^{1/2} \\
&= \frac{1}{2}\left(\frac{\pi}{Rez_n}\right)^{1/2}.
\end{aligned}$$

The closure and completeness of the sequences $\{x(x^2+z_n^2)^{-1}\}$ and $\{(x^2+z_n^2)^{-1}\}$ were first considered by O. Szász. In [24], Szász showed that the above sequences are closed in $L_2(0,\infty)$ if and only if $\sum Rez_k(1+|z_k|^2)^{-1}=\infty$. In §§2.2 and 2.3, we reprove Szász's Theorem for the normalised sequences $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$. Suppose now that the closed span $\overline{\Phi}$ of $\{\varphi_n(x)\}$ does not contain all L_2 functions. §2.4 is concerned with describing some properties of functions in $\overline{\Phi}$ when the Müntz-Szász condition is satisfied, i.e., when the inequality $\sum Rez_k(1+|z_k|^2)^{-1}<\infty$ holds. In §2.5, we provide the necessary and sufficient condition in order that $\{\varphi_n(x)\}$ will constitute a Schauder basis for $\overline{\Phi}$. In fact, with the help of Parseval's identity, our problem is almost solved by Theorem 1.5.2.

2.2 Closure and Completeness of the sequence $\{\varphi_n(x)\}$ in $L_2(0,\infty)$

Theorem 2.2.1 (Szász) *Let $\{z_n\}$ be a sequence of distinct complex numbers in P^+, $Rez_n>0$. The sequence*

$$\varphi_n(x)=\left\{2x(Rez_n/\pi)^{1/2}(x^2+z_n^2)^{-1}\right\}\quad (n=1,2,\ldots)$$

is closed in $L_2(0,\infty)$ if, and only if

$$\sum_{k=1}^{\infty}\frac{Rez_k}{1+|z_k|^2}=\infty.$$

To prove this, we will need the following lemmata.

Lemma 2.2.1 *Let $\{z_k\}$, $k = 1,\ldots,n$ be any sequence of complex numbers. Then the determinant*

$$A_n = \left[\frac{(Rez_j Rez_k)^{1/2}}{z_j + \overline{z_k}}\right]_{j,k=1}^{n}$$

has value

$$\prod_{k=1}^{n} Rez_k \prod_{j,k=1}^{n} (z_j + \overline{z_k})^{-1} \prod_{j>k\geq 1} (z_j - z_k)(\overline{z_j} - \overline{z_k}).$$

Proof: The determinant

$$\begin{vmatrix} \frac{1}{2} & \frac{(Rez_1 Rez_2)^{1/2}}{z_1+\overline{z_2}} & \cdots & \frac{(Rez_1 Rez_n)^{1/2}}{z_1+\overline{z_n}} \\ \frac{(Rez_2 Rez_1)^{1/2}}{z_2+\overline{z_1}} & \frac{1}{2} & \cdots & \frac{(Rez_2 Rez_n)^{1/2}}{z_2+\overline{z_n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{(Rez_n Rez_1)^{1/2}}{z_n+\overline{z_1}} & \frac{(Rez_n Rez_2)^{1/2}}{z_n+\overline{z_2}} & \cdots & \frac{1}{2} \end{vmatrix}$$

is just

$$\prod_{\nu=1}^{n} Rez_\nu \begin{vmatrix} \frac{1}{z_1+\overline{z_1}} & \frac{1}{z_1+\overline{z_2}} & \cdots & \frac{1}{z_1+\overline{z_n}} \\ \frac{1}{z_2+\overline{z_1}} & \frac{1}{z_2+\overline{z_2}} & \cdots & \frac{1}{z_2+\overline{z_n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{z_n+\overline{z_1}} & \frac{1}{z_n+\overline{z_2}} & \cdots & \frac{1}{z_n+\overline{z_n}} \end{vmatrix}.$$

By Cauchy's Lemma ([7], p. 195), with $a_i = z_i,\ b_j = \overline{z_j},\ \ i,j = 1,\ldots,n$, the above determinant is given by

$$\prod_{i,j=1}^{n}(z_i+\overline{z_j})^{-1}\prod_{j>i\geq 1}(z_j-z_i)(\overline{z_j}-\overline{z_i}).$$

This proves Lemma 2.2.1. □

A useful consequence of the lemma is the following:

Corollary 2 *For $n \leq m$, where n and m are arbitrary integers, the ratio of the Gramians*

$$A_{m,n}=\frac{\left[(Rez_jRez_k)^{1/2}(z_j+\overline{z_k})^{-1}\right]_{j,k=1}^{m}}{[(Rez_jRez_k)^{1/2}(z_j+\overline{z_k})^{-1}]_{j,k=1:j,k\neq n}^{m}} \tag{2.1}$$

is given by

$$A_{m,n}=\frac{1}{2}\prod_{k=1;k\neq n}^{m}\frac{(z_n-z_k)(\overline{z_n}-\overline{z_k})}{(z_n+\overline{z_k})(\overline{z_n}+z_k)}.$$

Proof: Consider the numerator $Num(A_{m,n})$ of $A_{m,n}$. From Lemma 2.2.1,

$$\begin{aligned}
Num(A_{m,n}) &= \left[(Rez_jRez_k)^{1/2}(z_j+\overline{z_k})^{-1}\right]_{j,k=1}^{m}\\
&= \prod_{j=1}^{m}Rez_j\prod_{j,k=1}^{m}(z_j+\overline{z_k})^{-1}\prod_{j>k\geq 1}(z_j-z_k)(\overline{z_j}-\overline{z_k})\\
&= \prod_{j=1}^{m}Rez_j\prod_{j=1}^{m}(z_j+\overline{z_1})^{-1}\cdots(z_j+\overline{z_m})^{-1}\\
&\times \prod_{n>k}(z_n-z_k)(\overline{z_n}-\overline{z_k})\prod_{j>n}(z_j-z_n)(\overline{z_j}-\overline{z_n})\\
&\times \prod_{j>k\geq 1;j,k\neq n}(z_j-z_k)(\overline{z_j}-\overline{z_k}).
\end{aligned}$$

Rearrangement with emphasis on z_n and $\overline{z_n}$ gives

$$
\begin{aligned}
Num(A_{m,n}) &= \prod_{j=1}^{m} Rez_j(z_n+\overline{z_n})^{-1} \prod_{j,k=1;j,k\neq n}^{m} (z_j+\overline{z_k})^{-1} \\
&\times \prod_{k=1;k\neq n}^{m} (z_n+\overline{z_k})^{-1}(z_k+\overline{z_n})^{-1} \\
&\times \prod_{n>k}(z_n-z_k)(\overline{z_n}-\overline{z_k}) \prod_{j>n}(z_j-z_n)(\overline{z_j}-\overline{z_n}) \\
&\times \prod_{j>k\geq 1;j,k\neq n} (z_j-z_k)(\overline{z_j}-\overline{z_k}) \\
&= \frac{\prod_{j=1}^{m} Rez_j}{(z_n+\overline{z_n})} \prod_{k=1;k\neq n}^{m} \frac{(z_n-z_k)(\overline{z_n}-\overline{z_k})}{(z_n+\overline{z_k})(\overline{z_n}+z_k)} \\
&\times \prod_{j,k=1;j,k\neq n}^{m} (z_j+\overline{z_k})^{-1} \prod_{j>k\geq 1;j,k\neq n}^{m} (z_j-z_k)(\overline{z_j}-\overline{z_k}).
\end{aligned}
$$

The denominator of (2.1) reduces to

$$
\prod_{j=1;j\neq n}^{m} Rez_j \prod_{j,k=1;j,k\neq n}^{m} (z_j+\overline{z_k})^{-1} \prod_{j>k\geq 1;j,k\neq n} (z_j-z_k)(\overline{z_j}-\overline{z_k}),
$$

and the corollary follows immediately. □

Lemma 2.2.2 *Let $z_1, z_2, z_3, \ldots$ be a sequence in P^+, $z_\nu \neq z_k$ for $\nu \neq k$. If $\{z_n\}$ has a limit point in P^+, then the sequence*

$$
\{\varphi_n(x)\} = \left\{ \frac{2x(Rez_n/\pi)^{1/2}}{x^2+z_n^2} \right\} \quad (n = 1,2,3,\ldots)
$$

is complete in $L_2(0,\infty)$.

Proof: Since normalisation does not alter the closed span of $\{\varphi_n\}$ in $L_2(0,\infty)$, it is enough to show that the sequence $\{x(x^2+z_n^2)^{-1}\}$ is complete in $L_2(0,\infty)$; that is, if $g(x) \in L_2(0,\infty)$ is such that

$$
\int_0^\infty \frac{xg(x)}{x^2+z_n^2}\,dx = 0, \quad (n = 1,2,\ldots),
$$

then $g(x)$ vanishes almost everywhere. Define the function

$$F(z) = \int_0^\infty \frac{xg(x)}{x^2+z^2}\,dx$$

analytic in P^+ (actually, $F \in H^2(P^+)$, although we do not use this fact). Now suppose that $F(z_n) = 0$ for all n, and $\{z_n\}$ has a limit point inside P^+. Then $F(z) \equiv 0$. In particular, for z real and positive,

$$\begin{aligned} F(z) &= \int_0^\infty \int_0^\infty xg(x)\exp\{-(x^2+z^2)t\}\,dtdx \\ &= \int_0^\infty \exp(-z^2 t)G(t)\,dt \equiv 0, \end{aligned}$$

where

$$G(t) = \int_0^\infty xg(x)\exp(-x^2 t)\,dx.$$

Let $z^2 = x$. Then $\int e^{-xt}G(t)dt \equiv 0$ for $x \in (0,\infty)$. Since the inverse Laplace transform is an isometry, we obtain $G(t) \equiv 0$. Now set $x^2 = u$. Then $\int g(\sqrt{u})e^{-ut}du \equiv 0$ for $t \in (0,\infty)$. By an earlier argument, $g(x) \equiv 0$ for $x \in (0,\infty)$. □

It is known that closure and completeness are equivalent concepts in $L_2(0,\infty)$. Hence under the assumption of Lemma 2.2.2, given each $\varepsilon > 0$ and each $f \in L_2(0,\infty)$, there exists an integer n and real or complex numbers $\{a_{\nu n}\}$, $\nu = 1,\ldots,n$ such that

$$\int_0^\infty \left| f(x) - \sum_{\nu=1}^n \frac{a_{\nu n}x}{x^2+z_\nu^2} \right|^2 dx < \varepsilon. \tag{2.2}$$

Lemma 2.2.3 *Let z_0 be an arbitrary point in P^+, and φ_0 the corresponding kernel. The minimum distance, d_m, in $L_2(0,\infty)$, from $\varphi_0(x)$ to the closed subspace generated by the functions $\varphi_1(x),\ldots,\varphi_m(x)$ is given by the Blaschke product*

$$d_m = \prod_{k=1}^{m}\left|\frac{z_k - z_0}{\overline{z_k} + z_0}\right|. \tag{2.3}$$

Proof:

$$\begin{aligned} d_m &= \min_{b_k}\frac{2}{\sqrt{\pi}}\left\|\frac{x(Rez_n)^{1/2}}{x^2+z_0^2} - \sum_{k=1}^{m}\frac{b_k x(Rez_k)^{1/2}}{x^2+z_k^2}\right\|_{L_2} \\ &= \frac{2}{\sqrt{\pi}}\left\|\sum_{k=0}^{m}\frac{a_k x(Rez_k)^{1/2}}{x^2+z_k^2}\right\|_{L_2}, \end{aligned}$$

with $a_0 = 1$ and the a_k chosen to minimise d_m. Thus,

$$\begin{aligned} d_m^2 &= \frac{4}{\pi}\int_0^\infty\left|\sum_{k=0}^{m}\frac{a_k x(Rez_k)^{1/2}}{x^2+z_k^2}\right|^2 dx \\ &= \frac{4}{\pi}\sum_{j,k=0}^{m}\int_0^\infty\frac{a_j\overline{a_k}(Rez_jRez_k)^{1/2}x^2\,dx}{(x^2+z_j^2)(x^2+\overline{z_k}^2)} \\ &= \frac{4}{\pi}\sum_{j,k=0}^{m}a_j\overline{a_k}(Rez_jRez_k)^{1/2}\int_0^\infty\frac{x^2\,dx}{(x^2+z_j^2)(x^2+\overline{z_k}^2)} \\ &= 2\sum_{j,k=0}^{m}\frac{a_j\overline{a_k}(Rez_jRez_k)^{1/2}}{(z_j+\overline{z_k})}. \end{aligned}$$

But from Gram's Lemma ([9], Theorem 8.7.4), the minimum of this bilinear quadratic hermitian form is given by

$$\frac{\left[(Rez_jRez_k)^{1/2}(z_j+\overline{z_k})^{-1}\right]_{j,k=0}^{m}}{\left[(Rez_jRez_k)^{1/2}(z_j+\overline{z_k})^{-1}\right]_{j,k=1}^{m}}.$$

Hence from the corollary of Lemma 2.2.1,

$$d_m^2 = 2\left\{\frac{1}{2}\prod_{k=1}^{m}\frac{(z_0-z_k)(\overline{z_0}-\overline{z_k})}{(z_0+\overline{z_k})(\overline{z_0}+z_k)}\right\}$$

$$= \prod_{k=1}^{m} \left| \frac{z_k - z_0}{\overline{z_k} + z_0} \right|^2$$

which is (2.3). □

Proof of Theorem 2.2.1: Necessity

A necessary condition for $\{\varphi_\nu\}_{\nu=1}^{\infty}$ to be closed in $L_2(0,\infty)$ is that

$$\lim_{m\to\infty} \prod_{k=1}^{m} \left| \frac{z_k - z_0}{\overline{z_k} + z_0} \right|^2 = 0; \tag{2.4}$$

that is, the function

$$\varphi_0(x) = \frac{2x(Rez_0/\pi)^{1/2}}{x^2 + z_0^2}, \quad x \in (0,\infty)$$

can be approximated by $\varphi_1(x), \ldots, \varphi_m(x)$ in the sense of (2.2). Now,

$$\begin{aligned} \lim_{m\to\infty} \prod_{k=1}^{m} \left| \frac{z_k - z_0}{\overline{z_k} + z_0} \right|^2 &= \prod_{k=1}^{\infty} \left\{ 1 - \left(1 - \left| \frac{z_k - z_0}{\overline{z_k} + z_0} \right|^2 \right) \right\} \\ &= \prod_{k=1}^{\infty} \left(1 - \frac{4Rez_k Rez_0}{|\overline{z_k} + z_0|^2} \right). \end{aligned}$$

Since $|\overline{z_k} + z_0|^2 - |z_k - z_0|^2 = 4Rez_k Rez_0$, we have

$$0 < \frac{4Rez_k Rez_0}{|\overline{z_k} + z_0|^2} \le 1.$$

Hence

$$\prod_{k=1}^{\infty} \left(1 - \frac{4Rez_k Rez_0}{|\overline{z_k} + z_0|^2} \right) = 0 \implies \exp\left(-\sum_{k=1}^{\infty} \frac{4Rez_k Rez_0}{|\overline{z_k} + z_0|^2} \right) = 0,$$

i.e.,

$$\sum_{k=1}^{\infty} \frac{4Rez_k Rez_0}{|\overline{z_k} + z_0|^2} = \infty,$$

or,

$$\sum_{k=1}^{\infty} \frac{Rez_k}{|\overline{z_k} + z_0|^2} = \infty.$$

Now, the above equality holds for any choice of $z_0 \in P^+$; in particular, for $z_0 = x > 0$, we have

$$\sum_{k=1}^{\infty} \frac{Rez_k}{(Rez_k + x)^2 + Imz_k^2} = \infty.$$

But $|z_k|^2 + x^2 \leq (Rez_k + x)^2 + Imz_k^2 \leq 2(|z_k|^2 + x^2)$. Thus,

$$\sum_{k=1}^{\infty} \frac{Rez_k}{|z_k|^2 + x^2} = \infty.$$

Since x is at our disposal, there is no loss of generality in putting $x = 1$. What we have just proved then amounts to the following: Condition (2.4) is equivalent to

$$\sum_{k=1}^{\infty} \frac{Rez_k}{1 + |z_k|^2} = \infty,$$

and so this condition is necessary for completeness.

Proof of Theorem 2.2.1: Sufficiency

Suppose now that $\sum_{k=1}^{\infty} Rez_k(1 + |z_k|^2)^{-1} = \infty$, and hence (2.4) holds for all $z = x > 0$, and $x \neq z_k$ for all k; that is, (2.4) holds for $z = x_0 \neq z_k$ for all k, and the function

$$\varphi(x) = \frac{2x(x_0/\pi)^{1/2}}{x^2 + x_0^2}$$

is in the closed span of the sequence $\{\varphi_n\}$. Now, for $g \in L_2(0,\infty)$ such that

$$\int_0^{\infty} g(x)\overline{\varphi_n(x)}dx = 0 \quad \text{for} \quad n = 1, 2, \ldots,$$

the continuity of the inner product yields

$$\int_0^\infty g(x)\overline{\varphi(x)}dx = 0 \quad \text{for} \quad n = 1, 2, \ldots.$$

Thus the analytic function

$$F(z) = \int_0^\infty \frac{2xg(x)}{x^2+z^2}dx$$

vanishes for all $z = x_0 \neq z_k$ for any k. But such z have a point of accumulation in P^+ and $F(z) \equiv 0$. As in Lemma 2.2.2, $g \equiv 0$ and we obtain the completeness of $\{\varphi_n\}$. In this case, if $\overline{\Phi}$ denotes the closure of the span of $\{\varphi_n(x)\}$, then $\overline{\Phi} = L_2(0,\infty)$.

2.3 Closure and Completeness of $\{\psi_n(x)\}$ in $L_2(0,\infty)$

We subject the sequence $\{\psi_n(x)\} = \left\{2|z_n|(Rez_n/\pi)^{1/2}(x^2+z_n^2)^{-1}\right\}$ $(Rez_n > 0)$ to the same treatment as we did $\{\varphi_n\}$ in §2.2. To prove Lemma 2.2.2 with $\{\psi_n(x)\}$ in place of $\{\varphi_n(x)\}$, we prove that the "un-normalised" sequence $\{(x^2+z_n^2)^{-1}\}$ is complete in $L_2(0,\infty)$. The proof proceeds in the same manner as that of Lemma 2.2.2 with $F(z)$ replaced by the function $\int g(x)(x^2+z^2)^{-1}dx$ and $G(t)$ by $\int g(x)e^{-x^2t}dx$. For the ψ_n equivalent of Lemma 2.2.3, we consider again the minimum of

$$d_m = \frac{2}{\sqrt{\pi}}\left(\int_0^\infty \left|\sum_{k=1}^m \frac{a_k|z_k|(Rez_k)^{1/2}}{x^2+z_k^2}\right|^2 dx\right)^{1/2}.$$

Now, for $Rez_j > 0$, $Rez_k > 0$,

$$\begin{aligned}
d_m^2 &= \frac{4}{\pi}\sum_{j,k=1}^{m} a_j\overline{a_k}|z_j||z_k|(Rez_jRez_k)^{1/2}\int_0^\infty \frac{dx}{(x^2+z_j^2)(x^2+\overline{z_k}^2)} \\
&= \frac{4}{\pi}\sum_{j,k=1}^{m} \frac{a_j\overline{a_k}|z_jz_k|(Rez_jRez_k)^{1/2}}{\overline{z_k}^2 - z_j^2}\int_0^\infty \left(\frac{1}{x^2+z_j^2} - \frac{1}{x^2+\overline{z_k}^2}\right)dx \\
&= 2\sum_{j,k=1}^{m} \frac{a_j\overline{a_k}|z_jz_k|(Rez_jRez_k)^{1/2}}{\overline{z_k}z_j(z_j+\overline{z_k})}.
\end{aligned}$$

As before, the minimum of the hermitian form above is given by the ratio of the Gramians

$$\frac{\left[|z_jz_k|(Rez_jRez_k)^{1/2}(z_k\overline{z_j}(z_j+\overline{z_k}))^{-1}\right]_{j,k=1}^{m}}{\left[|z_jz_k|(Rez_jRez_k)^{1/2}(z_k\overline{z_j}(z_j+\overline{z_k}))^{-1}\right]_{j,k=1:j,k\neq n}^{m}}.$$

The numerator of this ratio is the determinant

$$\begin{vmatrix}
\frac{1}{2} & \frac{|z_1z_2|(Rez_1Rez_2)^{1/2}}{z_1\overline{z_2}(z_1+\overline{z_2})} & \cdots & \frac{|z_1z_m|(Rez_1Rez_m)^{1/2}}{z_1\overline{z_m}(z_1+\overline{z_m})} \\
\frac{|z_2z_1|(Rez_2Rez_1)^{1/2}}{z_2\overline{z_1}(z_2+\overline{z_1})} & \frac{1}{2} & \cdots & \frac{|z_2z_m|(Rez_2Rez_n)^{1/2}}{z_2\overline{z_m}(z_2+\overline{z_m})} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{|z_mz_1|(Rez_mRez_1)^{1/2}}{z_m\overline{z_1}(z_m+\overline{z_1})} & \frac{|z_mz_2|(Rez_mRez_2)^{1/2}}{z_m\overline{z_2}(z_m+\overline{z_2})} & \cdots & \frac{1}{2}
\end{vmatrix}.$$

This reduces to

$$\frac{(|z_1|\cdots|z_m|)(|z_1|\cdots|z_m|)\left[(Rez_jRez_k)^{1/2}(z_j+\overline{z_k})^{-1}\right]_{j,k=1}^{m}}{(z_1\cdots z_m)(\overline{z_1}\cdots\overline{z_m})},$$

which is the numerator of the Gramian ratio in (2.1). Hence

$$d_m^2 = \prod_{k=1;k\neq n}^{m}\left|\frac{z_k - z_n}{\overline{z_k}+z_n}\right|^2,$$

and we can apply exactly the same arguments as in §2.2 to conclude that the system $\{\psi_n\}$ is closed in $L_2(0,\infty)$ if and only if $\sum Re z_k(1+|z_k|^2)^{-1} = \infty$. In this case, if $\overline{\Psi}$ denotes the closure of the span of the sequence $\{\psi_n\}$, then $\overline{\Psi} = \overline{\Phi} = L_2(0,\infty)$.

2.4 Approximation of functions by sums of $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$

A Characterisation of $\overline{\Phi}$

For the remainder of the chapter we assume that the Müntz-Szász condition holds, i.e., $\sum Re z_k(1+|z_k|^2)^{-1} < \infty$ and try to say something about the subspaces of $L_2(0,\infty)$ which $\overline{\Phi}$ and $\overline{\Psi}$ now represent. Now, if the Müntz-Szász condition holds, then $\{\varphi_\nu\}$ is not complete in $L_2(0,\infty)$. From Lemmata 2.2.2 and 2.2.3 and the proof of Theorem 2.2.1, for an arbitrary $z_0 \in P^+$, we have $\sum Re z_k(z_0^2+|z_k|^2)^{-1} < \infty$, or what is the same thing,

$$\lim_{m\to\infty} \prod_{k=1}^{m} \left|\frac{z_k - z_0}{\overline{z_k} + z_0}\right|^2 \neq 0.$$

Thus, φ_0 does not belong to the subspace generated by the remaining functions. The system is therefore free, and so we can find a sequence of complex numbers $a_1, a_2, \ldots$ such that each $f \in \overline{\Phi}$ has the formal representation

$$f(x) \sim \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{a_n x (Re z_n)^{1/2}}{x^2 + z_n^2}.$$

This representation means that for every $\varepsilon > 0$, there exists an integer m and a sequence $a_{1m}, \ldots, a_{mm}$ such that

$$\left\| f(x) - \frac{2}{\sqrt{\pi}} \sum_{n=1}^{m} \frac{a_{nm} x (Rez_n)^{1/2}}{x^2 + z_n^2} \right\|_{L_2} < \varepsilon,$$

and $a_{nm} \to a_n$ as $m \to \infty$. This does not, of itself, imply that the series $\sum a_n \varphi_n(x)$ tends to any function in $L_2(0,\infty)$.

We will concern ourselves now with obtaining some order-of-growth estimates for $|a_n|$ under special conditions on the z_n, and from this, we will deduce that $\sum a_n \varphi_n(x)$ is the Laplace sine transform of a series which converges uniformly in the right half-plane to an analytic function.

Lemma 2.4.1 *Let $\{z_n\}$ be a sequence in P^+ satisfying $|z_{n+1}| - |z_n| \geq \theta > 0$ for all n, and suppose that $\sum Rez_n(1+|z_n|^2)^{-1} < \infty$, $|\arg z_n| \leq \lambda < \pi/2$ for all n, $Rez_n \to \infty$ as $n \to \infty$. Then for each $f(x) \in \overline{\Phi}$ with the formal representation $f(x) \sim \sum a_n \varphi_n(x)$, we have*

$$|a_n| = O\left(e^{\varepsilon Rez_n}\right) \quad as \quad n \longrightarrow \infty$$

for all $\varepsilon > 0$.

Since $f \in \overline{\Phi}$, then for arbitrarily small $\varepsilon > 0$ and for some constants $a_{1m}, \ldots, a_{mm}$, we have for sufficiently large m, $\|\sum_{k=1}^{m} a_{km}\varphi_k - f\| < \varepsilon$. Therefore $\|\sum_{k=1}^{m} a_{km}\varphi_k\| \leq \|f\| + \varepsilon \leq (1+\delta)\|f\|$, where δ is an arbitrarily small number. Thus,

$$(1+\delta)\,\|f\|_{L_2} \;\geq\; \left\| \sum_{k=1}^{m} a_{km}\varphi_k \right\|_{L_2}$$

$$
\begin{aligned}
&= \left\| \frac{2}{\sqrt{\pi}} \sum_{k=1}^{m} \frac{a_{km} x (Re z_k)^{1/2}}{x^2 + z_k^2} \right\|_{L_2} \\
&= \frac{2}{\sqrt{\pi}} \left\| \frac{a_{nm} x (Re z_n)^{1/2}}{x^2 + z_n^2} - \sum_{k=1;k\neq n}^{m} \frac{b_{km} x (Re z_k)^{1/2}}{x^2 + z_k^2} \right\|_{L_2} \\
&= \frac{2}{\sqrt{\pi}} |a_{nm}| \left\| \frac{(Re z_n)^{1/2} x}{x^2 + z_n^2} - \sum_{k=1;k\neq n}^{m} \frac{c_{km} x (Re z_k)^{1/2}}{x^2 + z_k^2} \right\|_{L_2}
\end{aligned}
$$

with $c_{km} = b_{km}/a_{nm} = -a_{km}/a_{nm}$. Since the above norm is at least the distance given in equation (2.3), Lemma 2.2.3 and the hypothesis that the Müntz-Szász condition is satisfied jointly imply that

$$
(1+\delta)\, \|f\|_{L_2} \geq |a_{nm}| \prod_{k=1;\neq n}^{m} \left| \frac{z_k - z_n}{\overline{z_k} + z_n} \right| .
$$

Thus,

$$
|a_{nm}| \leq (1+\delta) \|f\|_{L_2} \prod_{k=1;\neq n}^{m} \left| \frac{\overline{z_k} + z_n}{z_k - z_n} \right| .
$$

From (1.2) and (1.3), $a_{nm} \to a_n$ as $m \to \infty$, and since δ is arbitrarily small, we have

$$
|a_n| \leq \|f\|_{L_2} P_n = \|f\|_{L_2} \prod_{k=1;k\neq n}^{\infty} \left| \frac{\overline{z_k} + z_n}{z_k - z_n} \right| . \tag{2.5}
$$

Now, let r denote a rational function which is a finite linear combination of the φ_n, and R the set of all such linear combinations. From (2.5), $|a_n| \leq \|r\| P_n$. Thus,

$$
\frac{\sup_{r\in R} |a_n|}{\|r\|_{L_2}} = P_n .
$$

Since $\overline{R} = \overline{\Phi}$, the coefficients a_n can therefore be considered as bounded linear functionals on the closed subspace of L_2 spanned by the functions $\varphi_n(x)$, $n = 1, 2, 3, \ldots$.

Proof of Lemma 2.4.1: We employ a technique by Gaier ([12], p. 252) to show that $\log P_n = o(Rez_n)$, i.e., that

$$P_n = O\left(e^{\varepsilon Re z_n}\right) \quad \text{as} \quad n \to \infty$$

for all $\varepsilon > 0$, and the result will follow from (2.5).

Write $P_n = \Pi_1\Pi_2\Pi_3$, where Π_1 contains those factors for which $|z_k| < |z_n|$, Π_2 those with $|z_n| < |z_k| \le 2|z_n|$, and Π_3 those with $|z_k| > 2|z_n|$.

For Π_1 we have, by virtue of the separation condition $|z_{k+1}| - |z_k| \ge \theta > 0$,

$$\begin{aligned}
\Pi_1 &= \prod_{|z_k|<|z_n|} \left|\frac{\overline{z_k} + z_n}{z_k - z_n}\right| \\
&\le \prod_{|z_k|<|z_n|} \frac{|z_n| + |z_k|}{|z_k - z_n|} \\
&\le \prod_{|z_k|<|z_n|} \frac{2|z_n|}{|z_n - z_k|}
\end{aligned}$$

For $k < n$, $\quad |z_n| - |z_k| \ge (n-k)\theta$. Thus,

$$\begin{aligned}
\Pi_1 &\le \prod_{|z_k|<|z_n|} \frac{2|z_n|}{(n-k)\theta} \\
&= \frac{(2|z_n|)^{n-1}}{(n-1)!(\theta)^{n-1}} \\
&= \left(\frac{2|z_n|}{\theta}\right)^{n-1} \frac{1}{[(n-1)!]}.
\end{aligned}$$

Stirling's theorem (Titchmarsh [26], p. 58) yields $n^n/n! \le e^n$ for $n = 1, 2, \ldots$. This in turn yields

$$\frac{1}{[(n-1)!]} \le \left(\frac{e}{n-1}\right)^{(n-1)}.$$

Hence,

$$\Pi_1 \le \left(\frac{2|z_n|e}{(n-1)\theta}\right)^{(n-1)},$$

and

$$\log \Pi_1 \leq Rez_n \frac{(n-1)}{Rez_n}\left[\log \frac{|z_n|}{n-1} + A\right], \tag{2.6}$$

where $A = \log(2e/\theta)$.

The Müntz-Szász condition is equivalent to $\sum(1/z_n) < \infty$ for z_n real if, as in the hypothesis, we assume that $z_n \to \infty$ as $n \to \infty$, so that $z_n/n \to \infty$ as $n \to \infty$, and consequently $(n-1)/z_n \to 0$. The restriction to real z_n can be removed by assuming (in addition to the Müntz-Szász condition and the separation condition $|z_{n+1}| - |z_n| \geq \theta$) that the z_n lie in some angle with $|\arg z| \leq \lambda < \pi/2$. Now, from $|z_{n+1}| - |z_n| \geq \theta$, it follows that $|z_{n+1}| \geq (n+1)\theta$, and from $|\arg z| \leq \lambda < \pi/2$, we get $Rez_n \geq |z_n| \cos\lambda$. Thus $Rez_n(1+|z_n|^2)^{-1} > (|z_n|\cos\lambda)(2|z_n|^2)^{-1} = \cos\lambda/2|z_n|$ for large n. Hence,

$$\sum \frac{Rez_n}{1+|z_n|^2} < \infty \Longrightarrow \sum \frac{1}{|z_n|} < \infty,$$

so that, since $Rez_n \to \infty$ as $n \to \infty$, and therefore $|z_n| \to \infty$, we have that

$$\frac{2(n-1)}{|z_n|} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, from (2.6), we conclude that

$$\log \Pi_1 = o(Rez_n) \quad \text{as} \quad n \to \infty,$$

or what is the same thing,

$$\Pi_1 = O\left(e^{\varepsilon Rez_n}\right) \quad \text{as} \quad n \to \infty \quad \text{for} \quad \text{all} \quad \varepsilon > 0.$$

For Π_2, assume that there are N factors such that $|z_n| < |z_k| \leq 2|z_n|$. ($\prod_{|z_n|<|z_k|\leq 2|z_n|} = 1$ if $N = 0$.) Then, as above,

$$\begin{aligned}
\Pi_2 &= \prod_{|z_n|<|z_k|\leq 2|z_n|} \left|\frac{\overline{z_k}+z_n}{z_k-z_n}\right| \\
&\leq \prod_{|z_n|<|z_k|\leq 2|z_n|} \frac{|z_k|+|z_n|}{|z_k-z_n|} \\
&\leq \prod_{|z_n|<|z_k|\leq 2|z_n|} \frac{3|z_n|}{|z_k-z_n|} \\
&\leq \left(\frac{3|z_n|}{\theta}\right)^N \frac{1}{N!} \\
&\leq \left(\frac{3|z_n|e}{N\theta}\right)^N .
\end{aligned}$$

Thus,

$$\log \Pi_2 \leq Rez_n \frac{N}{Rez_n}\left[\log\frac{|z_n|}{N} + B\right] = o(Rez_n) \quad \text{as} \quad n\to\infty,$$

since $n + N = o(Rez_{n+N}) = o(Rez_n)$, hence $N = o(Rez_n)$ or $Rez_n/N \to \infty$.

For Π_3, we have

$$\begin{aligned}
\Pi_3^2 &= \prod_{|z_k|>2|z_n|} \left|\frac{\overline{z_k}+z_n}{z_k-z_n}\right|^2 \\
&= \prod_{|z_k|>2|z_n|} \left\{1-\left(1-\left|\frac{\overline{z_k}+z_n}{z_k-z_n}\right|^2\right)\right\} \\
&= \prod_{|z_k|>2|z_n|} \left(1+\frac{4Rez_nRez_k}{|z_k-z_n|^2}\right).
\end{aligned}$$

Thus,

$$2\log\Pi_3 = \sum_{|z_k|>2|z_n|} \log\left(1+\frac{4Rez_nRez_k}{|z_k-z_n|^2}\right)$$

$$< \ 4Rez_n \sum_{|z_k|>2|z_n|} \frac{Rez_k}{|\ z_k - z_n\ |^2}$$
$$\leq \ 16Rez_n \sum_{|z_k|>2|z_n|} \frac{Rez_k}{|z_k|^2},$$

since $|z_k - z_n| \geq |z_k| - |z_n| \geq |z_k|/2$. Therefore

$$\frac{\log \Pi_3}{Rez_n} < 8 \sum_{|z_k|>2|z_n|} \frac{Rez_k}{|z_k|^2} \to 0 \quad as \quad n \to \infty$$

from the hypothesis that the Müntz-Szász condition holds. This gives

$$\Pi_3 = O\left(e^{\varepsilon Rez_n}\right) \quad \text{as} \quad n \to \infty$$

for all $\varepsilon > 0$.

Combining the above results, we obtain

$$P_n = O\left(e^{\varepsilon Rez_n}\right) \quad \text{as} \quad n \to \infty,$$

and hence

$$|a_n| \leq \|f\| P_n = O\left(e^{\varepsilon Rez_n}\right) \quad \text{as} \quad n \to \infty$$

for all $\varepsilon > 0$. $\square$

We are now ready to prove

Theorem 2.4.1 *Let $z_1, z_2, z_3 \ldots$ be a sequence in P^+ such that $|z_{n+1}| - |z_n| \geq \theta > 0$ for all n, and suppose that $\sum Rez_n(1 + |z_n|^2)^{-1} < \infty$, $|\arg z_n| \leq \lambda < \pi/2$ for all n. Let F be a function defined on $(0,\infty)$ with*

$$\lim_{m\to\infty} \left\| F(x) - 2\sum_{n=1}^{m} a_{nm}(Rez_n/\pi)^{1/2} e^{-z_n x} \right\|_{L_2} = 0, \quad a_{nm} \to a_n \quad \text{as} \quad m \to \infty.$$

Then $F \in L_2(0,\infty)$.

Furthermore, if $z_n > 0$ *for all* n, $z_n \to \infty$ *as* $n \to \infty$, *then, for some* $f \in \overline{\Phi}$, *the series* $2\sum a_n\sqrt{z_n/\pi}\exp(-z_n z)$ *converges locally uniformly in* P^+ *to the function* $F(z)$, *and* $f(x)$ *is the Laplace sine transform of* $F(z)$, *restricted to the half-line* $x > 0$.

Proof: From Lemma 2.4.1, if the Müntz-Szász condition is satisfied and $|z_{n+1}| - |z_n| \geq \theta > 0$ for all n, then the coefficients a_n in the formal representation of f are such that $|a_n| = O\left(e^{\varepsilon Re z_n}\right)$ as $n \to \infty$ for all $\varepsilon > 0$.

Let $\overline{V}$ be the closed span of the sequence $\{e^{-z_n}\}$. Since

$$\left\| F(x) - 2\sum_{n=1}^{m} a_{nm}(Re z_n/\pi)^{1/2}e^{-z_n x} \right\|_{L_2} < \varepsilon$$

for large m and for sufficiently small ε, then $F \in \overline{V}$, thus $F \in L_2(0,\infty)$.

If z_n is real, with $z_n \to \infty$ as $n \to \infty$, then, using the estimates for a_n and applying the root test to the series (for $z \in P^+$)

$$2\sum_{n=1}^{\infty} a_n\sqrt{z_n/\pi}\exp(-z_n z) = \sum_{n=1}^{\infty} b_n \exp(-z_n z),$$

we obtain

$$\begin{aligned}
\limsup_{z_n\to\infty}\left|b_n e^{-z_n z}\right|^{1/z_n} &= \limsup_{z_n\to\infty}\left|a_n\sqrt{z_n/\pi}e^{-z_n z}\right|^{1/z_n} \\
&= \limsup_{z_n\to\infty}\left|e^{\varepsilon z_n}e^{\frac{1}{2}\log z_n/\pi}e^{-z_n z}\right|^{1/z_n} \\
&= |\exp(\varepsilon - z)|.
\end{aligned}$$

But $|\exp(\varepsilon - z)| < 1$ for all $\varepsilon > 0$ if, and only if $Re z > 0$, thus $F(z) = 2\sum a_n\sqrt{z_n/\pi}\exp(-z_n z)$ is analytic in P^+. Restricting $F(z)$ to the half-line $x \geq \delta > 0$ and taking the Laplace sine transform yields

$$\begin{aligned}\frac{2}{\sqrt{\pi}}\int_0^\infty F(t)\sin xt\,dt &= 2\sum_{n=1}^\infty a_n\sqrt{z_n/\pi}\int_0^\infty \frac{e^{ixt}-e^{-ixt}}{2i}e^{-z_n t}\,dt \\ &= 2\sum_{n=1}^\infty \frac{a_n x\sqrt{z_n/\pi}}{x^2+z_n^2} \sim f(x),\end{aligned}$$

where the convergence of the above series to f is such that

$$\left\| f(x) - 2\sum_{n=1}^m \frac{a_{nm}x(z_n/\pi)^{1/2}}{x^2+z_n^2}\right\|_{L_2} < \varepsilon$$

for arbitrarily small ε and for large m. $\square$

A Characterisation of $\overline{\Psi}$

The inequality (2.5) also holds for each $f \in \overline{\Psi}$, since for a given function $\psi_n \in \Psi$, the norm

$$\left\|\psi_n(x) - \sum_{k=1;k\neq n}^m a_k\psi_k(x)\right\|_{L_2}$$

is at least

$$\prod_{k=1;k\neq n}^m \left|\frac{z_k - z_n}{\overline{z_k}+z_n}\right|.$$

Thus, by a previous argument, the system $\{\psi_n\}$ is free when the Müntz-Szász condition holds, hence each $f \in \overline{\Psi}$ has the formal representation $f(x) \sim \sum a_n\psi_n(x)$. We now state the $\overline{\Psi}$ analogue of Theorem 2.4.1.

Theorem 2.4.2 *Let $z_1, z_2, z_3\dots$ be a sequence in P^+ such that*

$|z_{n+1}| - |z_n| \geq \theta > 0$ for all n, and suppose $\sum Re z_n(1+|z_n|^2)^{-1} < \infty$,

$|\arg z_n| \le \lambda < \pi/2$ *for all* n. *Suppose that there is a function* $F \in L_2(0,\infty)$ *with*

$$\lim_{m\to\infty} \left\| F(x) - 2\sum_{n=1}^{m} a_{nm}(Rez_n/\pi)^{1/2}e^{-z_n x} \right\|_{L_2} = 0, \quad a_{nm} \to a_n \quad \text{as} \quad m \to \infty.$$

If $z_n > 0$ *for all* n, $z_n \to \infty$ *as* $n \to \infty$, *then, for some* $f \in \overline{\Psi} \neq L_2(0,\infty)$, *the series* $2\sum a_n\sqrt{z_n/\pi}\exp(-z_n z)$ *converges locally uniformly in* P^+ *to the function* $F(z)$, *and* $f(x)$ *is the Laplace cosine transform of* $F(z)$, *restricted to the half-line* $x > 0$.

Proof: The proof is the same as that of Theorem 2.4.1, with sine replaced by cosine. □

2.5 A Basis Problem for the closed subspaces $\overline{\Phi}$ and $\overline{\Psi}$

If a "lacunary trigonometric series"

$$\frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos n_k t + b_k \sin n_k t)$$

is the Fourier series of a continuous function in the interval $[0, 2\pi]$, it is well-known (see, for example, Gurarĭı and Macaev [13]) that the series converges to this function uniformly. (The series is lacunary if $\{n_k\}$ is a lacunary sequence, i.e., $\inf_k(n_{k+1}/n_k) \ge c > 1$.) This is equivalent to saying that the lacunary sequence from the trigonometric system is a basic sequence in $C[0,1]$, say.

(See §1.5 for the definition of a basic sequence.) The trigonometric system will then constitute a Schauder basis for $C[0,1]$ if it is total in this space.

The main purpose of this section is to obtain a necessary and sufficient condition for the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ to constitute a Schauder basis for their linear spans in $L_2(0,\infty)$. The sequence $\{\varphi_n\}$ forms a basis for $\overline{\Phi}$ if, for each $f \in \overline{\Phi}$, there exists a unique sequence of complex numbers $a_1, a_2, \ldots, a_n$ such that

$$\lim_{n\longrightarrow\infty} \left\| f(x) - 2\sum_{\nu=1}^{n} \frac{a_\nu (Re z_\nu/\pi)^{1/2} x}{x^2+z_\nu^2} \right\|_{L_2} = 0.$$

Note the difference between this condition and that of Theorem 2.4.1, where the coefficients a_ν may depend on n. We introduce some notations along the lines of §1.5, and then invoke Theorems 1.5.1 and 1.5.2.

$\overline{\Phi^{(n)}} = [\varphi_1, \ldots, \varphi_{n-1}, \varphi_{n+1}, \ldots]$ is the closure of the subspace generated by the functions $\varphi_1, \ldots, \varphi_{n-1}, \varphi_{n+1}, \ldots$.

$P_1(n) = [\varphi_1, \ldots, \varphi_n] \quad (n = 1, 2, \ldots)$,

$P_1^{(n)} = [\varphi_{n+1}, \varphi_{n+2}, \ldots] \quad (n = 1, 2, \ldots)$,

$\sigma_{1(n)} = \{f \in P_1(n) : \|f\|_{L_2} = 1\} \quad (n = 1, 2, \ldots)$,

$\sigma_1^{(n)} = \{f \in P_1^{(n)} : \|f\|_{L_2} = 1\} \quad (n = 1, 2, \ldots)$.

The main result here is the following:

Theorem 2.5.1 *Suppose that the sequence $\{z_n\}$ is in P^+, $Re z_n \to \infty$ as $n \to \infty$, and $\sum Re z_n(1+|z_n|^2)^{-1} < \infty$. Let $\overline{\Phi} \subset L_2(0,\infty)$ be the closed span in $L_2(0,\infty)$ of the system $\{\varphi_n(x)\}$, $\overline{\Phi} \neq L_2(0,\infty)$. Then a necessary and* *sufficient condition for the system $\{\varphi_n(x)\}$ to be a Schauder basis for $\overline{\Phi}$ is that*

$$\inf_n \prod_{k\neq n} \left|\frac{z_k - z_n}{\overline{z_k} + z_n}\right| = \delta > 0. \tag{2.7}$$

Consider just the Banach space properties of $\overline{\Phi}$. Since $\{\varphi_n(x)\}$ is total in the space $\overline{\Phi}$, Theorem 1.5.1 gives rise to

Theorem 2.5.2 *The following statements are equivalent*

(a) $\{\varphi_n\}$ is a Schauder basis for $\overline{\Phi}$.

(b) There exists a constant C_1 with $1 \leq C_1 < \infty$ such that

$$\left\|\sum_{k=1}^{n} \frac{\alpha_k (Rez_k/\pi)^{1/2} x}{x^2 + z_k^2}\right\|_{L_2} \leq C_1 \left\|\sum_{k=1}^{m+n} \frac{\alpha_k (Rez_k/\pi)^{1/2} x}{x^2 + z_k^2}\right\|_{L_2}$$

for all integers m, n and all complex numbers $\alpha_1, \alpha_2, ..., \alpha_{m+n}$.

(c1)

$$\inf_{1<n<\infty} \text{dist} \left(\varphi_n\,,\, \overline{\Phi^{(n)}}\right) > 0,$$

and

(c2)

$$\inf_{1\leq n,k<\infty} \text{dist} \left(\sigma_{1(n)}\,,\, \sigma_1^{(n+k)}\right) > 0.$$

Necessity

The necessity of condition (2.7) follows almost immediately, since we have already established that the minimum distance in the L_2-norm, from the unit vector φ_n to the closed subspace $\overline{\Phi^{(n)}}$ is just

$$\prod_{k=1;k\neq n}^{\infty} \left|\frac{z_k - z_n}{\overline{z_k} + z_n}\right|.$$

It is important always to bear in mind that $\overline{\Phi} \neq L_2(0,\infty)$. Thus,

$$\inf_{1<n<\infty} \text{dist } \left(\varphi_n\,,\,\overline{\Phi^{(n)}}\right) = \prod_{k=1;k\neq n}^{\infty} \left|\frac{z_k - z_n}{\overline{z_k} + z_n}\right| = \delta > 0,$$

which gives $(c1)$ above.

Sufficiency

We suppose (2.7) to be satisfied, prove (b) of Theorem 2.5.2, and then deduce the sufficiency in Theorem 2.5.1 from this. To be more precise, we show that there is a $\beta = \beta(\delta)$ such that

$$\left\|\sum_{k=1}^{m+n} \alpha_k (Rez_k/\pi)^{1/2} x(x^2+z_k^2)^{-1}\right\|_{L_2} \geq \beta(\delta)$$

for every sequence of complex numbers $\{\alpha_k\}$ for which

$$\left\|\sum_{k=1}^{n} \alpha_k (Rez_k/\pi)^{1/2} x(x^2+z_k^2)^{-1}\right\|_{L_2} = 1.$$

Now since

$$\int_0^\infty \sum_{k=1}^{n} \alpha_k (Rez_k/\pi)^{1/2} \exp(-z_k t) \sin xt\, dt = \sum_{k=1}^{n} \alpha_k (Rez_k/\pi)^{1/2} x(x^2+z_k^2)^{-1},$$

Parseval's identity gives

$$\begin{aligned}
&\left\|\sum_{k=1}^{n} a_k (Rez_k/\pi)^{1/2} x(x^2+z_k^2)^{-1}\right\|_{L_2} \\
&= \left(\int_0^\infty \left|\sum_{k=1}^{n} a_k (Rez_k/\pi)^{1/2} \exp(-z_k t)\right|^2 dt\right)^{1/2} \\
&= \left\|\sum_{k=1}^{n} c_k \exp(-z_k t)\right\|_{L_2},
\end{aligned}$$

where $c_k = a_k(Rez_k/\pi)^{1/2}$.

This isometry enables the following restatement: We suppose (2.7) to be satisfied and the sufficiency will follow if we can show that

$$\left\| \sum_{k=1}^{m+n} c_k e^{-z_k t} \right\|_{L_2} \geq \beta(\delta)$$

whenever

$$\left\| \sum_{k=1}^{n} c_k e^{-z_k t} \right\|_{L_2} = 1.$$

This is very close to the problem solved by Anderson in Theorem 1.5.2, except that $\overline{V}$, the closure of the span of the exponentials is not the same as $\overline{\Phi}$. However, it is easy to see that the isometry induced by Parseval's identity will carry $\overline{\Phi}$ onto $\overline{V}$. The concluding part of the proof of Theorem 2.5.1 is similar to that of Theorem 3.7.1, with the half-plane P^+ replacing the disc Δ and the Blaschke products

$$\prod_{k=1;k\neq n}^{\infty} \left| \frac{z_k - z_n}{\overline{z_k} + z_n} \right|$$

replacing the usual Blaschke products for the disc

$$\prod_{k=1;k\neq n}^{\infty} \left| \frac{z_k - z_n}{1 - \overline{z_n} z_k} \right|.$$

Since the full proof of Theorem 3.7.1 is given, we do not give this version of it here. □

In the above discussions, if we replace φ_n by ψ_n, $\overline{\Phi}$ by $\overline{\Psi}$ and observe that

$$\int_0^{\infty} \sum_{k=1}^{n} \alpha_k (Rez_k/\pi)^{1/2} \exp(-z_k t) \cos xt \, dt = \sum_{k=1}^{n} \frac{\alpha_k z_k (Rez_k/\pi)^{1/2}}{x^2 + z_k^2},$$

then we will have no difficulty in concluding that condition (2.7) is necessary and sufficient for $\{\psi_n\}$ to be a basis for $\overline{\Psi}$.

As a closing remark on this chapter, we note that the two subspaces studied here are not identical, since $\overline{\Phi}$ contains those $L_2(-\infty,\infty)$ functions vanishing at the origin while $\overline{\Psi}$ does not.

Chapter 3

Closure and Completeness in the Hardy Space $H^2(\Delta)$

3.1 Introduction

$H^2(\Delta)$ represents the Hilbert space of functions defined in §1.1 for $p = 2$. A metric $d(.\,,.)$ in this space is given by

$$d(f_1\,,f_2) = \|f_1 - f_2\|_{H^2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f_1(e^{i\theta}) - f_2(e^{i\theta})\right|^2 d\theta\right)^{1/2},$$

and an inner product by

$$(f_1\,,f_2) = \int_0^{\infty} f_1(e^{i\theta})\overline{f_2(e^{i\theta})}d\theta, \quad \text{for} \quad f_1, f_2 \in H^2(\Delta).$$

In the first part of this chapter, we consider the closure and completeness of the sequence of normalised Cauchy kernels

$$\{k_n(z)\} = \left\{ \frac{(1-|z_n|^2)^{1/2}}{1-\overline{z_n}z} \right\}_1^\infty$$

in $H^2(\Delta)$, where $\{z_1, z_2, \ldots\}$ is a set of complex numbers satisfying (for the time being) $|z_1| \le |z_2| \le \cdots < 1$.

The motivation comes again from the work of Szász [24] where it is shown that the closed span of the system $\{(1 - z_n z)^{-1}\}_1^\infty$ is dense in $H^2(\Delta)$ if and only $\sum(1 - |z_n|) = \infty$. We again prove Szász's Theorem (1.3.2) for the normalised kernels in §3.2 and §3.3. Following Akhieser, we give another proof in §3.4 that sheds some light on the closed span of this system. In §3.5, a third proof is given, and this makes greater use of the fact that for ζ running through Δ,

$$k(\zeta, z) = \frac{(1-|\zeta|^2)^{1/2}}{1-\overline{\zeta}z}$$

is the (normalised) reproducing kernel for $H^2(\Delta)$. This approach will prove useful in Chapter 4 dealing with weighted Dirichlet spaces and in which distances are otherwise not easy to compute.

In §3.6, we look at some properties of functions in the closed span of $\{k_n(z)\}$ under the assumption that this system is not total. Then, using Bessel and interpolating sequences, we obtain in §3.7, a necessary and sufficient condition for $\{k_n(z)\}$ to be a Schauder basis for the closed subspace of $H^2(\Delta)$ which it now generates. This subspace will be denoted by $\overline{\mathcal{K}}$. The basis problem here has important applications in Systems Control. Dudley Ward and Partington in

[10] investigate rational decompositions of functions in a Hardy-Sobolev class. This is the space of analytic functions on some domain Q whose n-th derivative has bounded p-th integral means, $1 \leq p < \infty$. These functions are decomposed into atoms or molecules which are the Cauchy kernels. In the typical scenario, we have corrupted data from a transfer function and it is desired to model this data using rational wavelets, ($\{k_n(z)\}$ for example). When these wavelets form a Schauder basis for their closed span, some very interesting results can be obtained on convergence rates (see their most recent paper [11] which deals with the decay of wavelet coefficients and error bounds for the approximation of functions in the Hardy-Sobolev class).

3.2 Completeness of $\{k_n(z)\}$ in $H^2(\Delta)$

Let f be an H^2 function, $\{z_n\}$ a sequence of points in Δ, and let

$$\sum_{n=1}^{\infty}(1 - |z_n|) = \infty. \tag{3.1}$$

Suppose that

$$(k_n, f) = \int_0^{2\pi} k_n(e^{i\theta})\overline{f(e^{i\theta})}d\theta = 0 \quad \text{for} \quad n = 1, 2, \ldots.$$

We need to show that $f \equiv 0$. Using the un-normalised system $\{g_n\} = \{(1 - \overline{z_n}z)^{-1}\}$, we suppose that $\sum(1 - |z_n|) = \infty$, and that there is some functional $(.\,, f)$ ($f \in H^2(\Delta)$) which vanishes for all g_n, $n = 1, 2, \ldots$.

Therefore $f(z_n) = (f\,,\,g_n) = \overline{(g_n\,,\,f)} = 0$ for all n, and hence the condition $\sum(1-|z_n|) = \infty$ implies that $f \equiv 0$, by the known properties of the zeros of an H^2 function, and so $\{g_n\}$ is complete.

On the other hand, if

$$\sum_{n=1}^{\infty}(1-|z_n|) < \infty,$$

then there exists an H^2 function f (indeed, a Blaschke product which is an H^∞ function), $f \not\equiv 0$ in Δ whose zeros are $z_1, z_2, \ldots$, so $\{g_n\}$ is not complete.

3.3 Direct Proof of the Closure of $\{k_n(z)\}$ in $H^2(\Delta)$

Lemma 3.3.1 *The minimum distance in the H^2-norm, d_m, from the function $k_n(z)$, $n \leq m$, to the closed subspace spanned by the functions*

$$\{k_i\}_{i=1}^m = \left\{\frac{(1-|z_i|^2)^{1/2}}{1-z_i z}\right\}_{i=1}^m, \quad i \neq n$$

is given by the Blaschke product

$$d_m = \prod_{k=1;k\neq n}^{m}\left|\frac{z_k - z_n}{1-\overline{z_n}z_k}\right|, \tag{3.2}$$

where we are assuming that the z_k are distinct.

Proof: We have,

$$d_m^2 = \min_{a_i}\left\|k_n - \sum_{i=1}^{m} a_i k_i\right\|_{H^2}^2$$

$$= \min_{a_i} \left\| \frac{(1-|z_n|^2)^{1/2}}{1-z_n z} - \sum_{i=1, i\neq n}^{m} \frac{a_i(1-|z_i|^2)^{1/2}}{(1-z_i z)} \right\|_{H^2}^2$$
$$= \left\| \sum_{i=1}^{m} \frac{a_i(1-|z_i|^2)^{1/2}}{1-z_i z} \right\|_{H^2}^2,$$

where $a_n = 1$ and the a_i are chosen to minimise d_m. Thus,

$$\begin{aligned}
d_m^2 &= \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{i=1}^{m} \frac{a_i(1-|z_i|^2)^{1/2}}{1-z_i z} \right|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i,j=1}^{m} a_i\overline{a_j} \frac{(1-|z_i|^2)^{1/2}}{(1-z_i e^{i\theta})} \frac{(1-|z_j|^2)^{1/2}}{(1-\overline{z_j}e^{-i\theta})} d\theta \\
&= \frac{1}{2\pi} \sum_{i,j=1}^{m} a_i\overline{a_j}\{(1-|z_i|^2)(1-|z_j|^2)\}^{1/2} \int_0^{2\pi} \frac{d\theta}{(1-z_i e^{i\theta})(1-\overline{z_j}e^{-i\theta})} \\
&= \sum_{i,j=1}^{m} a_i\overline{a_j}\{(1-|z_i|^2)(1-|z_j|^2)\}^{1/2} \frac{1}{1-\overline{z_j}z_i} \\
&= \sum_{i,j=1}^{m} \frac{a_i\overline{a_j}\{(1-|z_i|^2)(1-|z_j|^2)\}^{1/2}}{z_i(z_i^{-1}-\overline{z_j})}.
\end{aligned}$$

The value of d_m^2 is determined by minimising this bilinear quadratic hermitian form. From Gram's Lemma, we obtain

$$d_m^2 = \frac{\left[\{(1-|z_i|^2)(1-|z_j|^2)\}^{1/2}\{z_i(z_i^{-1}-\overline{z_j})\}^{-1}\right]_{i,j=1}^{m}}{\left[\{(1-|z_i|^2)(1-|z_j|^2)\}^{1/2}\{z_i(z_i^{-1}-\overline{z_j})\}^{-1}\right]_{i,j=1;i,j\neq n}^{m}}.$$

The numerator of d_m^2 is the determinant

$$\begin{vmatrix}
\frac{(1-|z_1|^2)}{z_1(z_1^{-1}-\overline{z_1})} & \frac{\{(1-|z_1|^2)(1-|z_2|^2)\}^{1/2}}{z_1(z_1^{-1}-\overline{z_2})} & \cdots & \frac{\{(1-|z_1|^2)(1-|z_m|^2)\}^{1/2}}{z_1(z_1^{-1}-\overline{z_m})} \\
\frac{\{(1-|z_1|^2)(1-|z_2|^2)\}^{1/2}}{z_2(z_2^{-1}-\overline{z_1})} & \frac{(1-|z_2|^2)}{z_2(z_2^{-1}-\overline{z_2})} & \cdots & \frac{\{(1-|z_2|^2)(1-|z_m|^2)\}^{1/2}}{z_2(z_2^{-1}-\overline{z_m})} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\{(1-|z_m|^2)(1-|z_1|^2)\}^{1/2}}{z_m(z_m^{-1}-\overline{z_1})} & \frac{\{(1-|z_m|^2)(1-|z_2|^2)\}^{1/2}}{z_m(z_m^{-1}-\overline{z_2})} & \cdots & \frac{(1-|z_m|^2)}{z_m(z_m^{-1}-\overline{z_m})}
\end{vmatrix},$$

which reduces to

$$\prod_{k=1}^{m}\frac{1-|z_j|^2}{z_j}\begin{vmatrix}\frac{1}{z_1^{-1}-\overline{z_1}} & \frac{1}{z_1^{-1}-\overline{z_2}} & \cdots & \frac{1}{z_1^{-1}-\overline{z_m}}\\ \frac{1}{z_2^{-1}-\overline{z_1}} & \frac{1}{z_2^{-1}-\overline{z_2}} & \cdots & \frac{1}{z_2^{-1}-\overline{z_m}}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{1}{z_m^{-1}-\overline{z_1}} & \frac{1}{z_m^{-1}-\overline{z_2}} & \cdots & \frac{1}{z_m^{-1}-\overline{z_m}}\end{vmatrix}$$

$$= \left[\frac{1}{z_i^{-1}-\overline{z_j}}\right]_{i,j=1}^{m}\prod_{j=1}^{m}\frac{1-|z_j|^2}{z_j}.$$

Hence

$$d_m^2 = \frac{[(z_i^{-1}-\overline{z_j})^{-1}]_{i,j=1}^{m}\prod_{j=1}^{m}(1-|z_j|^2)z_j^{-1}}{[(z_i^{-1}-\overline{z_j})^{-1}]_{i,j=1;i,j\neq n}^{m}\prod_{i,j=1;i,j\neq n}^{m}(1-|z_j|^2)z_j^{-1}}.$$

On putting $z_i^{-1}=a_i$ and $-\overline{z_j}=b_j$ in the proof of Lemma 2.2.1, one obtains

$$\begin{aligned} d_m^2 &= \frac{1-|z_n|^2}{z_n}\frac{1}{z_n^{-1}-\overline{z_n}}\prod_{j=1;j\neq n}^{m}\frac{(z_j^{-1}-z_n^{-1})(-\overline{z_j}+\overline{z_n})}{(z_j^{-1}-\overline{z_n})(-\overline{z_j}+z_n^{-1})}\\ &= \prod_{j=1;j\neq n}^{m}\frac{(z_n-z_j)(\overline{z_n}-\overline{z_j})}{(1-\overline{z_n}z_j)(1-z_n\overline{z_j})}\\ &= \prod_{j=1;j\neq n}^{m}\left|\frac{z_n-z_j}{1-\overline{z_n}z_j}\right|^2 \end{aligned}$$

which yields (3.2). □

Indeed, Lemma 3.3.1 holds for an arbitrary choice of $z\in\Delta$ (in place of z_n), z distinct from z_i, $i=1,2,\ldots,m$. Thus, for such $z\in\Delta$, we have

$$\begin{aligned}\lim_{m\to\infty}\prod_{j=1;z_j\neq z}^{m}\left|\frac{z-z_j}{1-\overline{z}z_j}\right|^2 &= \prod_{j=1;z_j\neq z}^{\infty}\left\{1-\left(1-\left|\frac{z_j-z}{1-\overline{z}z_j}\right|^2\right)\right\}\\ &= \prod_{j=1;z_j\neq z}^{\infty}\left(1-\frac{(1-|z|^2)(1-|z_j|^2)}{|1-\overline{z}z_j|^2}\right).\end{aligned}$$

So from

$$\lim_{m\to\infty}d_m^2=\prod_{j=1}^{\infty}\left(1-\frac{(1-|z|^2)(1-|z_j|^2)}{|1-\overline{z}z_j|^2}\right)=0,$$

it follows that

$$\sum_{j=1}^{\infty} \frac{(1-|z|^2)(1-|z_j|^2)}{|1-\overline{z}z_j|^2} = \infty,$$

or that

$$4(1-|z|)\sum_{j=1}^{\infty}(1-|z_j|) = \infty,$$

or what is the same thing, $\sum(1-|z_n|) = \infty$. Thus, if $\{k_n\}$ is closed, we do have $\lim_{m\to\infty} d_m^2 = 0$, and so $\sum(1-|z_n|) = \infty$ is a necessary condition for closure.

On the other hand, if $\sum(1-|z_n|) = \infty$, then we have $\lim_{m\to\infty} d_m^2 = 0$ for every choice of $w \in \Delta$, i.e., $(1-\overline{w}z)^{-1} \in \overline{\mathcal{K}}$. From §3.2, if we have a functional $(.\,, f)$ with $(k_n\,, f) = 0$ for $n = 1, 2, \ldots$, then, by continuity of the inner product,

$$\left(\frac{1}{1-\overline{w}z}\,, f\right) = 0 \quad for \quad each \quad w \in \Delta.$$

Hence $f(w) = 0$ for each $w \in \Delta$, or $f \equiv 0$, which in turn implies that $\{k_n\}$ is closed.

3.4 Akhieser's Proof

The following theorem can be used to obtain condition (3.1) as well as contribute slightly to the characterisation of functions in the closed span $\overline{\mathcal{K}}$, of the Cauchy kernels, as we demonstrate below. The complete characterisation will be the object of §3.6.

Theorem 3.4.1 (Akhieser, 1953) *Let $\{\zeta_i\}_{i=1}^{\infty}$ be distinct complex numbers in the exterior of Δ, $|\zeta_i| > 1$. For any natural number n and any $N \geq n$,*

$$\min_{A_j} \max_{|z|=1} \frac{|z^N + A_1 z^{N-1} + \cdots + A_N|}{|(z-\zeta_1)\cdots(z-\zeta_n)|} = \prod_{i=1}^{n} \frac{1}{|\zeta_i|}, \tag{3.3}$$

and for $p > 0$,

$$\min_{A_j} \frac{1}{2\pi} \int_{|z|=1} \left| \frac{z^N + A_1 z^{N-1} + \cdots + A_N}{(z-\zeta_1)\cdots(z-\zeta_n)} \right|^p |dz| = \prod_{i=1}^{n} \frac{1}{|\zeta_i|^p}, \tag{3.4}$$

for arbitrary complex numbers $A_1, \ldots, A_N$.

Theorem 3.4.1 is a slight "adulteration" of a theorem of Akhieser ([1], p. 243). We can reformulate the identities in (3.3) as follows: For $m = 0, 1, \ldots,$

$$\min_{a_j} \max_{|z|=1} \left| z^m - a_1 z^{m-1} - \cdots - a_m - \sum_{i=1}^{n} \frac{a_{m+i}}{z-\zeta_i} \right| = \prod_{i=1}^{n} \frac{1}{|\zeta_i|}, \tag{3.5}$$

where $a_1, \ldots, a_{m+n}$ are arbitrary complex constants. The product on the right-hand side of (3.5) converges to zero as $n \to \infty$ if and only if

$$\sum_{i=1}^{\infty} (1 - |\zeta_i|^{-1}) = \infty. \tag{3.6}$$

Applying (3.5) and (3.6) we have

Corollary 3 *(a) If (3.6) is satisfied then each algebraic polynomial is approximable with arbitrarily prescribed accuracy by linear combinations of $\left\{\frac{1}{z-\zeta_i} : \ i = 1, 2, \ldots\right\}$ in the uniform norm on $\partial\Delta = \{z : |z| = 1\}$.*

(b) If (3.6) is not satisfied then no algebraic polynomial is approximable by linear combinations of $\left\{\frac{1}{z-\zeta_i} : \ i = 1, 2, \ldots\right\}$ in the uniform norm on $\partial\Delta$.

Similarly one derives Corollary 3 for $0 < p < \infty$ from the identities in (3.4).

Now let $\{z_i\}_{i=1}^{\infty}$ be an infinite sequence of distinct complex numbers in $\Delta - \{0\}$. On substituting $\zeta_i = 1/\overline{z_i}$ in Corollary 3, condition (3.6) is replaced by condition (3.1) and the functions $1/(z - \zeta_n)$ by $k_n(z)$, $n = 1, 2, \ldots$. (The factor $(1 - |z_n|^2)^{1/2}$ was introduced purely to obtain d_m^2 in §3.3 as a "neat" Blaschke product.) From Mergelyan's Theorem ([18], p. 390), Corollary 3 and the maximum principle for analytic functions, we obtain

Corollary 4 *(a) If (3.1) is satisfied then each algebraic polynomial and thus each function in $A(\overline{\Delta}) := \{f \in C(\overline{\Delta}),\ f$ analytic in $\Delta\}$ is approximable with arbitrarily prescribed accuracy by linear combinations of $\{k_n : n = 1, 2, \ \ldots\}$ in the uniform norm on $\overline{\Delta} = \{z : |z| \leq 1\}$.*

(b) If (3.1) is not satisfied then no algebraic polynomial is approximable by linear combinations of $\{k_n : n = 1, 2, \ \ldots\}$ with the uniform norm on $\partial\Delta$.

From the last corollary, similar results can be derived for other normed function spaces on the closed unit disc $\overline{\Delta}$.

Now we give a third proof using reproducing kernels. The usefulness of the proof will become apparent when we encounter weighted Dirichlet spaces in Chapter 4. In these spaces, the previous methods become largely redundant.

3.5 Reproducing Kernel Proof

This approach was inspired by Shapiro ([20], p. 97). He used it to obtain the condition for the completeness of the exponentials on a half-line (via the Paley-Wiener Theorem). We now obtain d_m in Lemma 3.3.1 with the knowledge that

$$k(\zeta, z) = \frac{(1-|\zeta|^2)^{1/2}}{1-\overline{\zeta}z}$$

is the normalised reproducing kernel for $H^2(\Delta)$.

Let $\mathcal{K}_m$ be the span of the functions $\{k_{z_n}\}_1^m$ ($k_{z_n} = k_n$). For an arbitrary $z \in \Delta$, we wish to compute the minimum distance in the H^2-norm, d_m, from k_z to $\mathcal{K}_m$.

Now, since $\mathcal{K}_m \subset H^2(\Delta)$, then $\mathcal{K}_m$ is also a Hilbert space in possession of a reproducing kernel (r.k.). Call it $k_z^{(m)}$. This is none other than the orthogonal projection of $k_\zeta(w)$ on $\mathcal{K}_m$ (by Lemma 1.6.1). Also, the space of functions in $H^2(\Delta)$ orthogonal to all the elements of $\mathcal{K}_m$ (the orthocomplement $\mathcal{K}_m^o$ of $\mathcal{K}_m$) is a Hilbert space and has r.k. $r^{(m)}$, say. Thus,

$$d_m^2 = \|k_z - k_z^{(m)}\|^2 = \|r_z^{(m)}\|^2 = \left(r_z^{(m)}, r_z^{(m)}\right).$$

Consider the Blaschke product

$$B_m(w) = \prod_{n=1}^{m} \frac{w - z_n}{1 - \overline{z_n}w} \quad for \quad w \in \Delta.$$

For any fixed $\zeta \in \Delta$, $\overline{B_m(\zeta)}B_m(z)k_\zeta(z)$ is a reproducing element for ζ and lies in $\mathcal{K}_m^o$. Furthermore it is identical with $r^{(m)}$ for ζ. To see this, suppose that

$f \in \mathcal{K}_m^o$. Then

$$\begin{aligned}\left(f(z)\,,\,\overline{B_m(\zeta)}B_m(z)k_\zeta(z)\right) &= \left(B_m(\zeta)\overline{B_m(z)}f(z)\,,\,k_\zeta(z)\right)\\ &= \left(B_m(\zeta)\frac{f(z)}{B_m(z)}\,,\,k_\zeta(z)\right)\\ &= B_m(\zeta)\frac{f(\zeta)}{B_m(\zeta)} = f(\zeta).\end{aligned}$$

Therefore $r^{(m)}$ is identical with $\overline{B_m(\zeta)}B_m(z)k_\zeta(z)$ for all points in Δ, i.e.,

$$\begin{aligned}d_m^2 &= \left(r_z^{(m)}\,,\,r_z^{(m)}\right)\\ &= \left(\overline{B_m(\zeta)}B_m(z)k_\zeta(z)\,,\,\overline{B_m(\zeta)}B_m(z)k_\zeta(z)\right)\\ &= |B_m(z)|^2\,\|B_m(\zeta)k_z(\zeta)\|_{H^2}^2\,.\end{aligned}$$

Since

$$\left(\frac{(1-|z|^2)^{1/2}}{1-\overline{z}\zeta},\frac{(1-|z|^2)^{1/2}}{1-\overline{z}\zeta}\right) = (1-|z|^2)\left(\frac{1}{1-\overline{z}\zeta}\,,\,\frac{1}{1-\overline{z}\zeta}\right) = 1,$$

and $|B_m(\zeta)| = 1$ on $\partial\Delta$, we obtain equation (3.2). Now, from §3.3, $\overline{\mathcal{K}}$ is dense in $H^2(\Delta)$ if and only if $\lim_{m\to\infty}|B_m(z)|^2 = 0$, or equivalently, $\sum(1-|z_n|) = \infty$.

3.6 Characterisation of $\overline{\mathcal{K}}$

If condition (3.1) fails to hold, the system $\{k_n(z)\}$ is not closed in $H^2(\Delta)$ and by the arguments of §3.3, the system is free. Hence we can associate with each $g \in \overline{\mathcal{K}}$, a formal expansion of the form

$$g(z) \sim \sum_{n=0}^{\infty} a_n k_n(z) \tag{3.7}$$

in which a_n is given by $a_n = L_n(g)$, $L_n \in H^2(\Delta)^* = H^2(\Delta)$. The a_n can therefore be seen as the coefficients in the series expansion of g in terms of the system $\{k_n(z)\}$. We give some order-of-growth estimates for $|a_n|$, $n = 1, 2, 3, \ldots$, as we did in §2.4 for the coefficients in the formal series expansion of functions in $\overline{\Phi}$.

Lemma 3.6.1 *Let $\{z_n\}$ be a sequence of real and positive numbers in Δ and set*

$$\gamma_n = \left| \frac{z_{n+1} - z_n}{1 - z_n z_{n+1}} \right|.$$

We suppose that

(i)

$$\sum_{n=1}^{\infty} (1 - z_n) < \infty,$$

(ii)

$$\inf \gamma_n \geq \rho > 0.$$

Set $\overline{K} = \overline{span\{k_n(z)\}} \subset H^2(\Delta)$, $\overline{K} \neq H^2(\Delta)$. Then for each $g \in \overline{K}$ with the formal representation

$$g(z) \sim \sum_{1}^{\infty} \frac{a_n (1 - z_n^2)^{1/2}}{1 - z_n z},$$

we have

$$|a_n| = O\left(\exp\left(\frac{\varepsilon}{1 - z_n}\right)\right) \quad as \quad n \to \infty$$

for all $\varepsilon > 0$.

Proof: Using the arguments in the proof of Lemma 2.4.1, we obtain the $\overline{\mathcal{K}}$ equivalent of the inequality in (2.5); that is,

$$|a_n| \le \|g\| p_n = \|g\| \prod_{k=1, k\neq n}^{\infty} \left| \frac{1 - z_n z_k}{z_k - z_n} \right|. \tag{3.8}$$

The proof now reduces to showing that

$$\log p_n = o\left(\frac{1}{1 - z_n}\right) \quad as \quad n \to \infty.$$

(The condition that the z_n be real can be replaced by the condition that all the points z_n lie with increasing modulus along a ray, $\arg z = \theta$, say.)

Lemma 3.6.1 is the version for the unit disc Δ of Lemma 2.4.1. The function $w = (1+z)/(1-z)$ maps Δ $1-1$ conformally onto P^+. If we set $\lambda_n = (1+z_n)/(1-z_n)$, then the λ_n are real if and only if the z_n are real and

$$\lambda_{n+1} - \lambda_n = \frac{1 + z_{n+1}}{1 - z_{n+1}} - \frac{1 + z_n}{1 - z_n} = \frac{2(z_{n+1} - z_n)}{(1 - z_n)(1 - z_{n+1})}.$$

Now for z_n real, $(1 - z_n)(1 - z_{n+1}) < 1 - z_n z_{n+1}$, and so the condition

$$\left| \frac{z_{n+1} - z_n}{1 - z_n z_{n+1}} \right| > \delta$$

implies that $\lambda_{n+1} - \lambda_n > 2\delta$ which is the condition of Lemma 2.4.1. Also,

$$\frac{1}{1 - z_n} < \lambda_n < \frac{2}{1 - z_n},$$

so the condition $\sum(1 - z_n) < \infty$ corresponds to the condition $\sum(1/\lambda_n) < \infty$. Moreover,

$$\frac{\lambda_n - \lambda_k}{\lambda_n + \lambda_k} = \frac{(1 + z_n)/(1 - z_n) - (1 + z_k)/(1 - z_k)}{(1 + z_n)/(1 - z_n) + (1 + z_k)/(1 - z_k)}$$

$$
\begin{aligned}
&= \frac{(1+z_n-z_k-z_kz_n)-(1+z_k-z_n-z_kz_n)}{(1+z_n-z_k-z_kz_n)-(1+z_k-z_n-z_kz_n)} \\
&= \frac{z_n-z_k}{1-z_nz_k}.
\end{aligned}
$$

Thus the infinite Blaschke product for P^+,

$$\prod_{k=1;k\neq n}^{\infty}\left|\frac{\lambda_n-\lambda_k}{\lambda_n+\lambda_k}\right|$$

corresponds, under the transformation $\lambda_n=(1+z_n)/(1-z_n)$, to the infinite Blaschke product for Δ, namely

$$\prod_{k=1;k\neq n}^{\infty}\left|\frac{z_n-z_k}{1-z_nz_k}\right|.$$

Since we have the estimate $\log P_n=o(\lambda_n)$ in P^+, we obtain the corresponding estimate

$$\log p_n = o\left(\frac{1}{1-z_n}\right) \quad as \quad n\to\infty.$$

This completes the proof. □

Theorem 3.6.1 *Let $z_1, z_2, \dots$ be a sequence in the interval $(0,1)$ with $|z_{k+1}-z_k|/|1-z_kz_{k+1}|\geq\rho>0$ for all k, and suppose that $\sum(1-z_k)<\infty$. For some fixed $\zeta\in\Delta$, $t\in(0,\infty)$, and $\alpha\in[0,1)$, let $\mathcal{T}:L_2(0,\infty)\longrightarrow D_\alpha$ be the Laguerre transform*

$$\mathcal{T}\left\{(1-\overline{\zeta})^{-1-\alpha}\exp\left(-t\overline{\zeta}/(1-\overline{\zeta})\right)\right\}=\frac{1}{(1-\overline{\zeta}z)^{1-\alpha}}.$$

Let $\overline{\mathcal{K}}$ be the closed span of the system $\{k_n(z)\}$ in $H^2(\Delta)$. Then for each $g\in\overline{\mathcal{K}}\subset H^2(\Delta)$, $\overline{\mathcal{K}}\neq H^2(\Delta)$, there exists a function $f\in L_2(0,\infty)$ such that

$$\mathcal{T}\{f(t)\}=g(z).$$

Furthermore, $g(z)$ is the Laguerre transform of the restriction to the half-line $x \geq \delta > 0$, of a function h analytic in P^+, and of the form

$$h(z) = \sum_{n=1}^{\infty} \frac{a_n(1 - z_n^2)^{1/2} \exp(-z_n z/(1 - z_n))}{1 - z_n}.$$

Proof: That $\overline{\mathcal{K}} \neq H^2(\Delta)$ follows from the inequality $\sum(1 - z_k) < \infty$ and the arguments in §3.3. The separation condition $|z_{k+1} - z_k|/|1 - z_k z_{k+1}| \geq \rho > 0$ and Lemma 3.6.1 imply that $|a_n| = O(\exp \varepsilon(1 - z_n)^{-1})$, where the a_n are the coefficients in the formal series representation of g by the normalised Cauchy kernels, and correspond to the coefficients in the series representation of h above.

We prove that the function $h(z)$ is analytic in P^+ by showing that its series representation converges uniformly in P^+. Then we restrict h to the half-line $x \geq \delta > 0$ (call the resulting function f), and show that for $\alpha = 0$, the Laguerre transform of f yields the function $g \in \overline{\mathcal{K}}$.

As in the proof of Theorem 2.4.1, we employ a slight modification of the root test to the series

$$\sum_{n=1}^{\infty} \frac{a_n(1 - z_n^2)^{1/2} \exp(-\lambda_n z)}{1 - z_n} \quad = \quad \sum_{n=1}^{\infty} b_n \exp(-\lambda_n z),$$

where $\lambda_n = z_n(1-z_n)^{-1}$ and $b_n = a_n(1-z_n^2)/(1-z_n)$. Since $|a_n| = O(\exp \varepsilon/(1-z_n))$, and $z_n \to 1$, Lemma 3.6.1 gives

$$\limsup_{\lambda_n \to \infty} \left| b_n e^{-\lambda_n z} \right|^{1/\lambda_n} \quad = \quad \limsup_{\lambda_n \to \infty} \left| a_n(1 - z_n^2)^{1/2} \exp(-\lambda_n z)(1 - z_n)^{-1} \right|^{1/\lambda_n}$$

$$\begin{aligned}
&= \limsup_{\lambda_n\to\infty} \left| e^{\frac{\varepsilon}{1-z_n}} e^{-\log(1-z_n)} e^{\frac{1}{2}\log(1-z_n^2)} e^{-\lambda_n z} \right|^{1/\lambda_n} \\
&= \limsup_{\lambda_n\to\infty} \left| \exp\left(\frac{\varepsilon}{\lambda_n(1-z_n)} - z \right) \right| \\
&= \left| \exp(\varepsilon - z) \right| .
\end{aligned}$$

But $|\exp(\varepsilon - z)| < 1$ for all $\varepsilon > 0$ if, and only if $Re z > 0$.

Now, h restricted to the half-line $x \geq \delta > 0$ yields the function

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n(1-z_n^2)^{1/2} \exp(-\lambda_n x)}{1-z_n}. \tag{3.9}$$

Setting $\alpha = 0$ and applying the Laguerre transform to the series in (3.9), we have

$$\begin{aligned}
\mathcal{T}\{f(x)\} &= \mathcal{T}\left\{ \sum_{n=1}^{\infty} a_n(1-z_n^2)^{1/2}(1-z_n)^{-1} \exp\left(-\frac{z_n x}{1-z_n}\right) \right\} \\
&= \sum_{n=1}^{\infty} a_n(1-z_n^2)^{1/2} \mathcal{T}\left\{ (1-z_n)^{-1} \exp\left(-\frac{z_n x}{1-z_n}\right) \right\} \\
&= \sum_{n=1}^{\infty} \frac{a_n(1-z_n^2)^{1/2}}{1-z_n z} \\
&\sim g(z),
\end{aligned}$$

where the convergence of the above series to g is such that

$$\left\| g(z) - \sum_{\nu=1}^{n} \frac{a_{\nu n}(1-z_\nu^2)^{1/2}}{1-z_\nu z} \right\|_{H^2} < \varepsilon$$

for large n and for sufficiently small ε.□

3.7 Basis Problem for $\overline{\mathcal{K}}$

The main result in this section is the following:

Theorem 3.7.1 *Suppose that $\{z_n\}$ is a sequence in Δ with $\sum(1-|z_n|) < \infty$. Let $\overline{\mathcal{K}}$ be the closure of the span of the system $\{k_n(z)\}_1^\infty$, $\overline{\mathcal{K}} \subset H^2(\Delta)$, $\overline{\mathcal{K}} \neq H^2(\Delta)$. A necessary and sufficient condition for $\{k_n(z)\}_1^\infty$ to be a Schauder basis for $\overline{\mathcal{K}}$ is that*

$$\inf_n \prod_{k=1,k\neq n}^{\infty} \left|\frac{z_k - z_n}{1-\overline{z_n}z_k}\right| = \delta > 0. \tag{3.10}$$

Note that for this particular theorem, we do not require the z_n to be real.

We introduce a few more notations along the lines of §1.5 with a view to invoking Theorem 1.5.1.

$\overline{\mathcal{K}^{(n)}} = [k_1, k_2, \ldots, k_{n-1}, k_{n+1}, \ldots]$ is the closed linear manifold of $H^2(\Delta)$ spanned by the functions $k_1, k_2, \ldots, k_{n-1}, k_{n+1}, \ldots$.

$P_{2(n)} = [k_1, \ldots, k_n] \quad (n = 1, 2, \ldots)$,

$P_2^{(n)} = [k_{n+1}, k_{n+2}, \ldots] \quad (n = 1, 2, \ldots)$,

$\sigma_{2(n)} = \{f \in P_{2(n)} : \|f\| = 1\} \quad (n = 1, 2, \ldots)$,

$\sigma_2^{(n)} = \{f \in P_2^{(n)} : \|f\| = 1\} \quad (n = 1, 2, \ldots)$.

We can therefore restate Theorem 3.7.1 as

Theorem 3.7.2 *The following statements are equivalent:*

(a) $\{k_n(z)\}_1^\infty$ is a Schauder basis for $\overline{\mathcal{K}}$.

(b) There exists a constant C_2 with $1 \leq C_2 < \infty$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i k_i \right\| \leq C_2 \left\| \sum_{i=1}^{m+n} \alpha_i k_i \right\|$$

for all positive integers m,n and all complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_{m+n}$.

(c1)

$$\inf_{1 \leq n < \infty} \text{dist} \left(k_n , \overline{\mathcal{K}^{(n)}} \right) > 0,$$

and (c2)

$$\inf_{1 \leq n,k < \infty} \text{dist} \left(\sigma_{2(n)} , \sigma_2^{(n+k)} \right) > 0.$$

Proof of Theorem 3.7.1

From Lemma 3.3.1, the minimum distance in the H^2-norm from a unit vector k_n for example, to the vector (function) space spanned by all the remaining vectors is just

$$\delta(z_n) = \prod_{k=1, k \neq n}^{\infty} \left| \frac{z_k - z_n}{1 - \overline{z_n} z_k} \right|,$$

so the necessity immediately follows from $(c1)$. Part $(c2)$ is not relevant in this proof, since it is implied by $(c1)$ in the spaces considered above.

To prove the sufficiency, we proceed as follows.

Define the function F by

$$F(z) = \sum_{k=1}^{m+n} a_k \frac{(1 - |z_k|^2)^{1/2}}{1 - \overline{z_k} z}.$$

The H^2-norm of F will not be altered by multiplying by the finite Blaschke product

$$\prod_{k=1}^{m+n} \frac{1 - \overline{z_k} z}{z - z_k}.$$

Thus,

$$Q(z) = F(z) \prod_{k=1}^{m+n} \frac{1 - \overline{z_k} z}{z - z_k} = \sum_{k=1}^{m+n} a_k \frac{(1 - |z_k|^2)^{1/2}}{z - z_k} \left\{ \prod_{\nu=1; \nu \neq k}^{m+n} \frac{1 - \overline{z_\nu} z}{z - z_\nu} \right\}.$$

This new function now belongs to the Hardy class of the exterior of the unit disc, but on replacing z by $1/z$, and setting $G(z) = (1/z)Q(1/z)$, this is equivalent to considering the disc Δ rather than its exterior, so that

$$G(z) = \sum_{k=1}^{m+n} a_k \frac{(1 - |z_k|^2)^{1/2}}{1 - z_k z} \left\{ \prod_{\nu=1; \nu \neq k}^{\infty} \frac{z - \overline{z_\nu}}{1 - z_\nu z} \theta_\nu \right\}, \tag{3.11}$$

where θ_ν are factors of modulus 1 required to make the infinite Blaschke products converge. Thus, $\|F\|_{H^2} = \|G\|_{H^2}$. Now, write $G(z) = G_1(z) + G_2(z)$, where $G_1(z)$ is the sum in (3.11) from $k = 1$ to n and $G_2(z)$ is the corresponding sum from $k = n+1$ to $m+n$. Suppose that $\|G_1(z)\| = 1$. Then we will show that $\|G(z)\| \geq \beta$ for some $\beta = \beta(\delta)$, assuming that the inequality in (3.10) holds; that is, we prove Theorem 3.7.1 by showing that $(c) \Longrightarrow (b) \Longrightarrow (a)$ in Theorem 3.7.2. This method is due to Anderson (loc. cit.).

We now apply some results in the theory of interpolation by Shapiro and Shields (loc. cit.). Lemma 1.7.1 and the first part of the proof of Theorem 1.7.2 give rise to the next theorem. We have replaced $M(\delta)$ by $M(\delta)^2$ for reasons which will soon be clear.

Theorem 3.7.3 *Let $\{z_n\}$ satisfy condition (3.10). Then there is a constant $M(\delta)$ such that* *(i) the inequality*

$$\sum_{k=1}^{\infty} \mid g(z_k) \mid^2 (1 - |z_k|^2) \leq M(\delta)^2 \|g\|^2$$

holds for all $g \in H^2(\Delta)$,

(ii) if

$$R(z) = \sum_{j=s}^{t} \frac{a_j (1 - |z_j|^2)^{1/2}}{1 - \overline{z_j} z} \left(\prod_{\nu=s;\nu\neq j}^{t} \frac{z - z_\nu}{1 - \overline{z_\nu} z} \right),$$

then

$$\|R\|_{H^2}^2 \leq M(\delta)^2 \sum_{j=s}^{t} |a_j|^2$$

for any s and t.

First suppose that $\sum_{k=1}^{m+n} |a_k|^2 \geq A^2$, where A is a constant to be chosen later. Apply Theorem 3.7.3 (i) to the function G in (3.11) to obtain

$$\sum_{k=1}^{m+n} |a_k|^2 \left\{ \prod_{\nu=1;\nu\neq k}^{\infty} \left| \frac{z_k - z_\nu}{1 - \overline{z_\nu} z_k} \right| \right\}^2 \leq M(\delta)^2 \|G\|^2.$$

Hence,

$$\delta^2 \sum_{k=1}^{m+n} |a_k|^2 \leq M(\delta)^2 \|G\|^2,$$

i.e.,

$$\|G\|^2 \geq \frac{A^2 \delta^2}{M(\delta)^2},$$

or

$$\|G\| \geq A\delta / M(\delta). \qquad (3.12)$$

Now suppose that $\sum_{j=1}^{m+n} |a_k|^2 < A^2$. Then $\|G\| \geq \|G_1\| - \|G_2\|$, and by Theorem 3.7.3 (ii),

$$\|G_2\|^2 \leq M(\delta)^2 \sum_{k=n+1}^{m+n} |a_k|^2 \leq A^2 M(\delta)^2.$$

From the hypothesis that $\|G_1\| = 1$, we have

$$\|G\| \geq 1 - AM(\delta). \tag{3.13}$$

But A is at our disposal, so we can take the optimal value

$$A = (M(\delta) + \delta/M(\delta))^{-1}.$$

So in each case, (3.12) or (3.13), we have

$$\|G\| \geq \beta(\delta) = (1 + M(\delta)^2/\delta)^{-1},$$

and the proof of Theorem 3.7.1 is complete. □

Chapter 4

Closure Problems in Weighted Dirichlet Spaces

4.1 Introduction

D_α is the space of functions f analytic in Δ with norm $\|f\|^2 = \sum |a_n|^2/n^\alpha < \infty$, for $0 \leq \alpha < 1$. We shall continue to call these spaces of Dirichlet type *Dirichlet spaces* for economy of presentation. But in fact, for $\alpha > 0$, modern usage calls these spaces *Bergman spaces* rather than weighted Dirichlet spaces.

In §4.2, we employ the methods developed in the previous chapter to show that a necessary (and sometimes sufficient) condition for the sequence of kernels $\{\ell_n(\alpha, z)\} = \{(1 - \overline{z_n}z)^{-\alpha-1}\}$ to be closed in D_α is that $\sum(1 - |z_n|) = \infty$. This is evident for $\alpha = 0$, since $D_0 = H^2(\Delta)$, or equivalently,

$\ell_n(0,z) = \|(1-\overline{z_n}z)^{-1}\|k_n(z)$. For z_n are real and positive, we use some results of Shapiro and Shields to demonstrate that the necessary condition also is sufficient.

Let $\overline{\mathcal{L}_\alpha}$ denote the closed span of the system $\{\ell_n(\alpha,z)\}$. In §4.3, we provide a characterisation of $\overline{\mathcal{L}_\alpha}$ that makes use of the Laguerre transform. Finally, we employ Theorem 1.7.1, Lemma 1.7.1, and a succession of isometries to provide a necessary and sufficient condition for the system $\{\ell_n(\alpha,z)\}$ to be a basis for $\overline{\mathcal{L}_\alpha}$, assuming that the z_n lie in the unit interval and $\sum(1-z_n) < \infty$.

4.2 Closure of $\{\ell_n(\alpha, z)\}$ in D_α

This section is concerned with providing a necessary condition for the closed span $\overline{\mathcal{L}_\alpha}$ of $\{\ell_n(\alpha,z)\}$ to be dense in D_α. Again, the crucial step in the proof is computing the minimum distance (with the D_α topology) from an arbitrary function to the span $\mathcal{L}_m$ of the system $\{\ell_\nu(\alpha,z)\}_1^m$. We employ the fact that $\ell_\zeta(\alpha,z) = (1-\overline{\zeta}z)^{\alpha-1}$ is the reproducing kernel for D_α.

Lemma 4.2.1 *For an arbitrary $z \in \Delta$, the minimum distance d_m in D_α, from $\ell_z(\alpha,w)$ to the span $\mathcal{L}_m$ of the functions $\ell_{z_1}(\alpha,w), \ell_{z_2}(\alpha,w), \ldots, \ell_{z_m}(\alpha,w)$ ($\ell_{z_n}(\alpha,w) = \ell_n(\alpha,w)$, $z \neq z_k$ for any k) satisfies the inequality*

$$d_m \geq (1-|z|^2)^{(\alpha-1)/2} \prod_{n=1}^{m} \left| \frac{z-z_n}{1-\overline{z_n}z} \right|. \tag{4.1}$$

Since $\mathcal{L}_m$ is a closed subspace of D_α, then, from §1.6, $\mathcal{L}_m$ has a reproducing kernel, and this is the orthogonal projection of ℓ_ζ on $\mathcal{L}_m$. By the same argument, the orthocomplement $\mathcal{L}_m^o$ of $\mathcal{L}_m$ has reproducing kernel $t_z^{(m)}$.

The concluding part of the proof is not quite as straightforward as in the H^2 case (corresponding to $\alpha = 0$). In that case, the reproducing kernel $t_z^{(m)}$ for $\mathcal{L}_m^o$ is just $\overline{B_m(\zeta)}B_m(z)\ell_\zeta(z)$, where

$$B_m(w) = \prod_{n=1}^{m} \frac{w - z_n}{1 - \overline{z_n}w} \qquad (w \in \Delta),$$

for

$$\begin{aligned}
\left(f(z), \overline{B_m(\zeta)}B_m(z)\ell_\zeta(z)\right) &= \left(B_m(\zeta)\overline{B_m(z)}f(z), \ell_\zeta(z)\right) \\
&= \left(B_m(\zeta)\frac{f(z)}{B_m(z)}, \ell_\zeta(z)\right) \\
&= B_m(\zeta)\frac{f(\zeta)}{B_m(\zeta)} = f(\zeta).
\end{aligned}$$

In the general case, we must use the fact that $H^2 \subseteq D_\alpha$ for $0 \leq \alpha < 1$. The additional factor $(1 - |z|^2)^{(\alpha-1)/2}$ arises from the fact that

$$\left(\frac{1}{(1 - \overline{z}\zeta)^{1-\alpha}}, \frac{1}{(1 - \overline{z}\zeta)^{1-\alpha}}\right) = (1 - |z|^2)^{\alpha-1}.$$

The conclusion of the lemma is obtained by using the arguments on p. 68 with the appropriate norm.

Now, $\overline{\mathcal{L}_\alpha}$ is dense in D_α if $\lim_{m\to\infty} d_m^2 = 0$. Thus,

$$\lim_{m\to\infty} \frac{1}{(1 - |z|^2)^{1-\alpha}} \prod_{n=1}^{m} \left|\frac{z - z_n}{1 - \overline{z_n}z}\right|^2$$

$$= \frac{1}{(1-|z|^2)^{1-\alpha}} \prod_{n=1}^{\infty} \left\{ 1 - \left(1 - \left| \frac{z - z_n}{1 - \overline{z_n} z} \right|^2 \right) \right\}$$

$$= \frac{1}{(1-|z|^2)^{1-\alpha}} \prod_{n=1}^{\infty} \left(1 - \frac{(1-|z_n|^2)(1-|z|^2)}{|1-\overline{z_n}z|^2} \right) = 0.$$

We use the same arguments as in Chapters 2 and 3, approximating first the kernels $\ell_z(\alpha, z)$, $z \neq z_n$ for a set of values z having a limit point in Δ. From

$$\lim_{m\to\infty} d_m^2 = \frac{1}{(1-|z|^2)^{1-\alpha}} \prod_{n=1}^{\infty} \left(1 - \frac{(1-|z_n|^2)(1-|z|^2)}{|1-\overline{z_n}z|^2} \right) = 0,$$

it follows that

$$\sum_{n=1}^{\infty} \frac{(1-|z_n|^2)(1-|z|^2)}{|1-\overline{z_n}z|^2} = \infty,$$

or that

$$4(1-|z|) \sum_{n=1}^{\infty} (1-|z_n|) = \infty,$$

or what is the same thing,

$$\sum_{n=1}^{\infty} (1-|z_n|) = \infty.$$

Thus the condition $\sum(1-|z_n|) = \infty$, which is condition (3.1), is necessary for completeness.

Sufficiency

We use the following theorem ([22], p. 225) to illustrate that condition (3.1) is sometimes sufficient for $\{\ell_n(\alpha, z)\}$ to be closed in D_α. This is particularly the case for z_n real and positive.

Theorem 4.2.1 (Shapiro and Shields, 1962) *Let f be analytic in the unit disc Δ, and let $0 < z_1 \le z_2 \le \cdots$ be the real zeros of f in Δ. If*

$$|f(z)| \le c \exp\left(\frac{1}{(1-|z|)^\alpha}\right) \tag{4.2}$$

for some $\alpha < 1/2$ and some $c > 0$, then

$$\sum_{n=1}^{\infty}(1-z_n) < \infty.$$

The proof uses elementary results in Nevanlinna Theory to show that f is of bounded characteristic relative to the disc $|z-1/2| < 1/2$. One can replace $\alpha < 1/2$ in Theorem 4.2.1 by $\alpha < 1$, $f(z)$ by the function $F(z) = F(1/2+z/2)$, and the disc $|z-1/2| < 1/2$ by any region making a lower order of contact with the unit circle (e.g. a region enclosed by the two straight lines making an angle τ at $\partial\Delta$ in Figure 4.1). If $f(z)$ satisfies (4.2) for $0 \le \alpha < 1$, then it is a straightforward matter to show that the function $F(z) = F(1/2+z/2)$ is of bounded Nevanlinna characteristic in $|z| < 1$, i.e., loosely speaking, we may say that $f(z)$ has bounded characteristic in the disc $|z-1/2| < 1/2$. If all the z_n are real and positive (or more generally lie in a Stolz angle at $z=1$ as shown in Figure 4.1), then we have that

$$\sum_{n=1}^{\infty}(1-w_n) < \infty \quad \textit{where} \quad F(w_n) = 0.$$

But $w_n = 1/2 + z_n/2$ for z_n real, and so in this case,

$$\sum_{n=1}^{\infty}(1-w_n) < \infty \Longleftrightarrow \sum_{n=1}^{\infty}(1-z_n) < \infty.$$

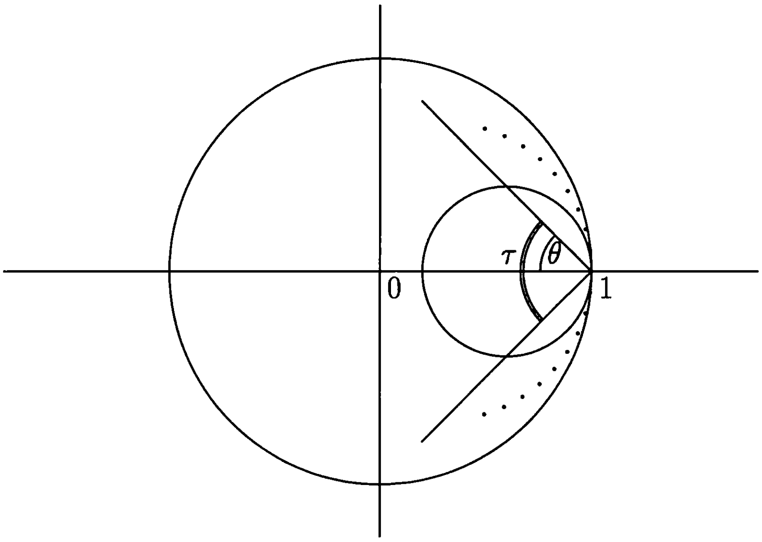

Figure 4.1: Manner in which the z_n approach $\partial\Delta$

Now

$$1 - |z_n| \leq |1 - z_n| \leq \frac{1 - |z_n|}{\cos\theta} \tag{4.3}$$

if $1 - |z_n| \leq |1 - z_n| \cos\theta$ for some $0 \leq \theta < \pi/2$, so if the zeros z_n are not all real, then using (4.3) we can still conclude that $\sum(1 - |z_n|) < \infty$, since

$$\sum_{n=1}^{\infty}(|1 - z_n|) < \infty \Longleftrightarrow \sum_{n=1}^{\infty}(1 - |z_n|) < \infty.$$

Note that it is precisely here that we have used the fact the z_n lie in a Stolz angle at $z = 1$. If the zeros approach $z = 1$ tangentially as shown in Figure 4.1, then the argument breaks down. Thus, it is also here that essential use is made of the fact that the z_n are real. The importance of the requirement that the z_n be real is that it may not be possible to find a function in D_α having zeros $z_1, z_2, \ldots$ if $\{z_n\}$ is any set of points in Δ. In this case, $\{\ell_n(z, \alpha)\}$ will not be complete in D_α, since completeness requires that we find a function

$f \in D_\alpha^* = D_\alpha$ such that the vanishing inner products

$$f(\alpha, z_n) = (f(\alpha, z)\,,\, \ell_{z_n}(\alpha, z)) = 0, \quad n = 1, 2, \ldots$$

imply that $f(z, \alpha) \equiv 0$ in Δ. To be more precise, consider the analytic functions in Δ satisfying

$$|f(z)| \leq \frac{c}{(1 - |z|)^\beta} \tag{4.4}$$

where $z \in \Delta$, c is some constant, and $0 \leq \beta < 1$. The class of functions satisfying (4.4) is identical with the class of Taylor series satisfying $\sum |a_n|^2/n^\epsilon < \infty$, for some suitable $\epsilon > 0$, i.e., those functions in D_ϵ. Now we state a theorem which asserts that functions in D_α cannot vanish at infinitely many points in Δ, chosen arbitrarily.

Theorem 4.2.2 *Let f be analytic in Δ and satisfy condition (4.4). Let $n(r)$ be the number of zeros of f in the disc $|z| < r$ $(r < 1)$. Then*

$$n(r) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right) \quad as \quad r \to 1. \tag{4.5}$$

This is a minor adaptation of Theorem 5 in ([22], p. 225), where we refer the reader for the proof. (We have replaced the constant $k > 0$ there by α for $0 \leq \alpha < 1$.) The relevance of this theorem for our consideration is that the estimate (4.5) is best possible. This shows that the Blaschke condition $\sum(1 - |z_n|) < \infty$ need not be satisfied for the zeros of a function satisfying (4.4). This again emphasises the need to take the z_n to be real or, at least, to lie at a Stolz angle at $z = 1$. Note that (4.4) with any $\beta > 0$ implies (4.2) for any $\alpha > 0$.

For the remainder of the chapter, we will assume that the z_n lie on the unit interval, and study the closed subspaces $\overline{\mathcal{L}_\alpha}$ of D_α that the system $\{\ell_n(\alpha, z)\}$ generates when the Blaschke condition $\sum(1 - z_n) < \infty$ holds.

4.3 A study of the closed subsets $\overline{\mathcal{L}_\alpha}$

Coefficient estimates for functions in $\overline{\mathcal{L}_\alpha}$

Let $\{z_k\}$ be a sequence of numbers in the interval $(0, 1)$. Suppose that Blaschke condition holds. Then from the arguments in §4.2, $\ell_n(\alpha, z) \notin \overline{span\{\ell_\nu(\alpha, z)\}}$, $\nu \neq n$, and so the system $\{\ell_n(\alpha, z)\}$ is free. Thus, each $g \in \overline{\mathcal{L}_\alpha}$ has the formal expansion

$$g(z) \sim \sum_{n=1}^{\infty} a_n \ell_n(\alpha, z), \tag{4.6}$$

where, as before, a_n is a linear functional of g. We employ the methods developed in the last chapter to show that the coefficients a_n in (4.6) are such that $|a_n| = (1 - z_n^2)^{(1-\alpha)/2} O\left(e^{\frac{\varepsilon}{1-z_n}}\right)$, $n = 1, 2, 3, \ldots$ for all $\varepsilon > 0$.

Lemma 4.3.1 *Let $z_1, z_2 \ldots \in (0, 1)$ satisfy*

$$\inf_n \gamma_n = \inf_n \left| \frac{z_{n+1} - z_n}{1 - z_n z_{n+1}} \right| \geq \rho > 0,$$

and suppose that $\sum(1-z_n) < \infty$. *Let* $\overline{\mathcal{L}_\alpha}$ *be the closed span in* D_α *of the system* $\{\ell_n(\alpha, z)\}$. *Then for each* $g \in \overline{\mathcal{L}_\alpha}$ *with the formal representation*

$$g(z) \sim \sum_1^\infty \frac{a_n}{(1-z_n z)^{1-\alpha}},$$

we have

$$\begin{aligned} |a_n| &= \sqrt{(1-z_n^2)^{1-\alpha}}\,O\left(\exp\left(\frac{\varepsilon}{1-z_n}\right)\right) \\ &= O\left(\exp\left(\frac{\varepsilon}{1-z_n}\right)\right), \quad \textit{as} \quad n \longrightarrow \infty \end{aligned}$$

for all $\varepsilon > 0$.

Proof: Since $g \in \overline{\mathcal{L}_\alpha}$, then for arbitrarily small $\varepsilon > 0$ and for some constants $b_{1m}, \ldots, b_{mm}$, we have for sufficiently large m, $\|g(\alpha, z) - \sum_{n=1}^m a_{nm}\ell_n(\alpha, z)\| < \varepsilon$. Therefore $\|\sum_{n=1}^m a_{nm}\ell_n(\alpha, z)\| \le \|g\| + \varepsilon \le (1+\delta)\|g\|$, where δ is an arbitrarily small number. Thus,

$$\begin{aligned} (1+\delta)\|g\|_{D_\alpha} &\ge \left\|\sum_{k=1}^m \frac{a_{km}}{(1-z_k z)^{1-\alpha}}\right\|_{D_\alpha} \\ &= \left\|\frac{a_{nm}}{(1-\overline{z_n}z)^{1-\alpha}} - \sum_{k=1;k\neq n}^m \frac{b_{km}}{(1-z_k z)^{1-\alpha}}\right\|_{D_\alpha} \\ &= |a_{nm}|\left\|\frac{1}{(1-z_n z)^{1-\alpha}} - \sum_{k=1;k\neq n}^m \frac{c_{km}}{(1-z_k z)^{1-\alpha}}\right\|_{D_\alpha}, \end{aligned}$$

with $c_{km} = b_{km}/a_{nm} = -a_{km}/a_{nm}$.

The norm above is at least the distance from $\ell_n(\alpha, z)$ to the closed span of the functions $\ell_1(\alpha, z), \ldots \ell_{n-1}(\alpha, z), \ell_{n+1}(\alpha, z), \ldots, \ell_m(\alpha, z)$. Lemma 4.2.1 and the hypothesis that $\sum(1-z_n) < \infty$ together imply that

$$(1+\delta)\|g\| \ge |a_{nm}|(1-z_n^2)^{(\alpha-1)/2} \prod_{k=1,k\neq n}^m \left|\frac{z_n - z_k}{1-z_n z_k}\right|.$$

Thus,

$$|a_{nm}|(1-z_n^2)^{(\alpha-1)/2} \leq (1+\delta)\|g\| \prod_{k=1,k\neq n}^{m} \left|\frac{1-z_n z_k}{z_n - z_k}\right|.$$

From (1.2) and (1.3), $a_{nm} \to a_n$ as $m \to \infty$, and since δ is arbitrarily small, we have that

$$|a_n|(1-z_n^2)^{(\alpha-1)/2} \leq \|g\| p_n,$$

where

$$p_n = \prod_{k=1,k\neq n}^{\infty} \left|\frac{1-z_n z_k}{z_n - z_k}\right|.$$

Note that we are only using the inequality (4.1) instead of the equality (3.2). The proof now reduces to showing that

$$\log p_n = o\left(\frac{1}{1-z_n}\right), \quad as \quad n \longrightarrow \infty$$

which was the object of Lemma 3.6.1. □

Characterisation of $\overline{\mathcal{L}_\alpha}$

The following theorem provides a characterisation of $\overline{\mathcal{L}_\alpha}$ and emphasises the fact that $\overline{\mathcal{L}_\alpha}$ is a rather tiny subspace of D_α when $\sum(1-z_n) < \infty$. In fact, $\overline{\mathcal{L}_\alpha}$ does not even contain functions of the form $(1-z_i z)^{\alpha-1}$ when $z_i \notin \{z_1, z_2, \ldots\}$, the z_i being real and positive. The theorem is analogous to Theorem 3.6.1 of Chapter 3, where $\alpha = 0$.

Theorem 4.3.1 *Let $z_1, z_2, z_3 \cdots$ be a sequence in (0,1) with $|z_{n+1} - z_n|/|1 - z_n z_{n+1}| \geq \delta > 0$ for $n = 1, 2, \ldots$. Suppose that $\sum(1-z_n) < \infty$. For some fixed*

$\zeta \in \Delta$, $t \in (0,\infty)$, and an α in the interval $[0,1)$, let $\mathcal{T} : L_2(0,\infty) \longrightarrow D_\alpha$ be the Laguerre transform

$$\mathcal{T}\left\{(1-\overline{\zeta})^{-1-\alpha}\exp\left(-t\overline{\zeta}/(1-\overline{\zeta})\right)\right\} = \frac{1}{(1-\overline{\zeta}z)^{1-\alpha}}.$$

Let $\overline{\mathcal{L}_\alpha}$ be the closed span in D_α of the system $\{\ell_n(\alpha,z)\}$, $\overline{\mathcal{L}_\alpha} \neq D_\alpha$. Then for each $g \in \overline{\mathcal{L}_\alpha}$, there exists a function $f \in L_2(0,\infty)$ such that

$$\mathcal{T}\{f(x)\} = g(\alpha,z).$$

Furthermore, g is the Laguerre transform of the restriction to the half-line $x \geq \delta > 0$ of a function h analytic in P^+ and of the form

$$h(z) = \sum_{n=1}^{\infty} \frac{a_n}{(1-z_n)^{1+\alpha}} e^{-\lambda_n z}, \quad \textit{where} \quad \lambda_n = z_n/(1-z_n).$$

Proof: That $\overline{\mathcal{L}}_\alpha \neq D_\alpha$ follows from the inequality $\sum(1-z_k) < \infty$ and the arguments in §4.2. The separation condition $|z_{k+1}-z_k|/|1-z_k z_{k+1}| \geq \rho > 0$ and Lemma 4.3.1 imply that $|a_n| = (1-z_n^2)^{(1-\alpha)/2} O(\exp \varepsilon(1-z_n)^{-1})$, where the a_n are the coefficients in the formal series representation of g by kernels $\ell_n(\alpha,z)$, and correspond to the coefficients in the series representation of h above.

We prove that the function $h(z)$ is analytic in P^+ by showing that its series representation converges uniformly in P^+. Then we restrict h to the half-line $x \geq \delta > 0$ (call the resulting function f), and show that the Laguerre transform of f yields the function $g \in \overline{\mathcal{L}}_\alpha$.

As before, we apply the root test to the power series

$$\begin{aligned} h(z) &= \sum_{n=1}^{\infty} \frac{a_n \exp(-\lambda_n z)}{(1-z_n)^{1-\alpha}} \\ &= \sum_{n=1}^{\infty} b_n \exp(-\lambda_n z), \end{aligned}$$

and show that the function $h(z)$ is analytic for $Re z > 0$. Thus

$$\begin{aligned} \limsup_{\lambda_n \to \infty} \left| b_n e^{-\lambda_n z} \right|^{1/\lambda_n} &= \limsup_{\lambda_n \to \infty} \left| a_n (1-z_n)^{-1+\alpha} \exp(-\lambda_n z) \right|^{1/\lambda_n} \\ &= \limsup_{\lambda_n \to \infty} \left| (1-z_n^2)^{(1-\alpha)/2} e^{\frac{\varepsilon}{1-z_n}} (1-z_n)^{-1+\alpha} e^{-\lambda_n z} \right|^{1/\lambda_n} \\ &\le \limsup_{\lambda_n \to \infty} \left| \sqrt{2^{1-\alpha}} e^{\frac{\varepsilon}{1-z_n}} e^{\frac{(\alpha-1)}{2}\log(1-z_n)} e^{-\lambda_n z} \right|^{1/\lambda_n} \\ &= \limsup_{\lambda_n \to \infty} \left| \exp\left(\frac{\varepsilon}{\lambda_n (1-z_n)} - z \right) \right| \\ &= \left| \exp(\varepsilon - z) \right|. \end{aligned}$$

This proves that $h(z)$ is analytic in P^+, since $|\exp(\varepsilon - z)| < 1$ if, and only if $Re z > 0$ for all $\varepsilon > 0$.

Now, h restricted to the half-line $x \ge \delta > 0$ yields the function

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n \exp(-\lambda_n x)}{(1-z_n)^{1+\alpha}}. \tag{4.7}$$

Applying the Laguerre transform to the series in (4.7), we have

$$\begin{aligned} \mathcal{T}\{f(x)\} &= \mathcal{T}\left\{ \sum_{n=1}^{\infty} a_n (1-z_n)^{-1-\alpha} \exp\left(-\frac{z_n x}{1-z_n} \right) \right\} \\ &= \sum_{n=1}^{\infty} a_n \mathcal{T}\left\{ (1-z_n)^{-1-\alpha} \exp\left(-\frac{z_n x}{1-z_n} \right) \right\} \\ &= \sum_{n=1}^{\infty} \frac{a_n}{(1-z_n z)^{1-\alpha}} \\ &\sim g(\alpha, z), \end{aligned}$$

where the convergence of the above series to g is such that

$$\left\| g(\alpha, z) - \sum_{\nu=1}^{n} \frac{a_{\nu n}}{(1 - z_\nu z)^{1-\alpha}} \right\|_{D_\alpha} < \varepsilon$$

for large n and for sufficiently small ε, i.e., the convergence is in the appropriate Hilbert space norm. □

4.4 Basis Problem for $\overline{\mathcal{L}_\alpha}$

We will continue to assume that the z_n are real. Our objective in this section is to obtain a necessary and sufficient condition for $\{\ell_n(\alpha, z)\}$ to be a basis for $\overline{\mathcal{L}_\alpha}$ in the case when $\overline{\mathcal{L}_\alpha} \neq D_\alpha$.

Theorem 4.4.1 *Suppose that* $0 < z_1 \leq z_2 \leq \cdots < 1$ *and let* $\sum(1 - z_n) < \infty$. *Let* $\overline{\mathcal{L}_\alpha}$ *be the closed linear manifold of* D_α *generated by the system* $\{\ell_n(\alpha, z)\}$, $\overline{\mathcal{L}_\alpha} \subset D_\alpha$, $\overline{\mathcal{L}_\alpha} \neq D_\alpha$. *A necessary and sufficient condition for the functions* $\{\ell_n(\alpha, z)\}$ *to be a Schauder basis for* $\overline{\mathcal{L}_\alpha}$ *is that*

$$\inf_n \prod_{k=1, k\neq n}^{\infty} \left| \frac{z_k - z_n}{1 - z_n z_k} \right| = \kappa > 0. \tag{4.8}$$

As before, we introduce some notations in anticipation of a reformulation of Theorem 1.5.1.

$\overline{\mathcal{L}_\alpha^{(n)}} = [\ell_1, \ell_2, \ldots, \ell_{n-1}, \ell_{n+1}, \ldots]$ is the closed linear manifold of D_α spanned by the functions $\ell_1, \ell_2, \ldots, \ell_{n-1}, \ell_{n+1}, \ldots$.

$P_{3(n)} = [\ell_1, \ldots, \ell_n] \quad (n = 1, 2, \ldots)$,

$P_3^{(n)} = [\ell_{n+1}, \ell_{n+2}, \ldots] \quad (n = 1, 2, \ldots),$

$\sigma_{3(n)} = \{f \in P_{3(n)} : \|f\| = 1\} \quad (n = 1, 2, \ldots),$

$\sigma_3^{(n)} = \{f \in P_3^{(n)} : \|f\| = 1\} \quad (n = 1, 2, \ldots).$

Theorem 4.4.2 *The following statements are equivalent:*

(a) $\{\ell_n(\alpha, z)\}_1^\infty$ *is a Schauder basis for* $\overline{\mathcal{L}_\alpha}$.

(b) There exists a constant C_3 *with* $1 \le C_3 < \infty$ *such that*

$$\left\| \sum_{i=1}^{n} a_i \ell_i \right\|_{D_\alpha} \le C_3 \left\| \sum_{i=1}^{m+n} a_i \ell_i \right\|_{D_\alpha}$$

for all positive integers m,n and all complex numbers $a_1, a_2, \ldots, a_{m+n}$.

(c)

$$\inf_{1 \le n < \infty} \text{dist } \left(\ell_n / \|\ell_n\| \, , \, \overline{\mathcal{L}_\alpha{}^{(n)}} \right) > 0,$$

and

$$\inf_{1 \le n, r < \infty} \text{dist } \left(\sigma_{3(n)} \, , \, \sigma_3^{(n+r)} \right) > 0.$$

For our present purpose, the second part of (c) above is again dysfunctional, since it is implied by the first part.

Proof of Theorem 4.4.1

Recall that the norm of the function $f(z) = \sum a_n z^n \in D_\phi$ is given by

$$\|f\|^2 = \sum \frac{1}{c_i} |a_i|^2$$

where the c_i in this case, are the coefficients in the series expansion of the function $\phi_\alpha(z) = (1 - z_n z)^{\alpha - 1}$, $0 \le \alpha < 1$. Hence $a_i = c_i = \binom{\alpha-1}{i}(-z_n)^i$, and a unit vector $\ell_n(\alpha, z)/\|\ell_n(\alpha, z)\|$ in $\overline{\mathcal{L}}_\alpha$ is

$$\frac{\ell_n(\alpha, z)}{\left(\sum_{i=1}^{\infty} \binom{\alpha-1}{i}(-z_n)^i\right)^{1/2}} = \frac{\ell_n(\alpha, z)}{(1-z_n)^{(1-\alpha)/2}}.$$

The minimum distance from this unit vector $(1-z_n)^{(\alpha-1)/2}\ell_n(\alpha, z)$ to the vector space spanned by all the remaining vectors is such that

$$\kappa(z_n) \geq (1-z_n^2)^{(\alpha-1)/2} \prod_{k=1, k\neq n}^{\infty} \left|\frac{z_k - z_n}{1 - z_n z_k}\right|$$

(from Lemma 4.2.1), so the necessity immediately follows from the first part of (c).

To prove the sufficiency, we apply a succession of isometries to transfer the problem to the Hardy space of the unit disc and then apply the techniques developed in §3.7.

Let

$$g(z) = \sum_{r=1}^{m+n} a_r(1 - z_r z)^{\alpha-1}$$

be a function in $\overline{\mathcal{L}_\alpha}$. From (1.1), the inverse Laguerre transform takes $g \in \overline{\mathcal{L}_\alpha}$ to the function

$$f(t) = \sum_{r=1}^{m+n} a_r(1 - z_r)^{-\alpha-1} e^{-\lambda_r t}$$

in $L_2(0, \infty)$, with $\lambda_r = z_r/(1 - z_r)$.

For $w \in P^+$, the Paley-Wiener isometry now takes f to the function

$$F(w) = \int_0^\infty e^{-wt} \overline{\sum_{r=1}^{m+n} a_r(1 - z_r)^{-\alpha-1} e^{-\lambda_r t}} dt$$

$$= \sum_{r=1}^{m+n} \frac{\overline{a_r}(1-z_r)^{-\alpha}}{w(1-z_r)+z_r}$$

in $H^2(P^+)$.

Now for $z \in \Delta$, we have the fractional linear transformation

$w = (1+z)/(1-z)$. So

$$F(w) = F((1+z)/(1-z)) = \sum_{r=1}^{m+n} \frac{\overline{a_r}(1-z)(1-z_r)^{-\alpha}}{1+z-2z_r z},$$

$F((1+z)/(1-z))$ being a function in $H^2(\Delta)$.

Now, the H^2 norm of $F((1+z)/(1-z))$ will not be altered by multiplying by the finite Blaschke product

$$\prod_{\nu=1}^{m+n} \left(\frac{z-z_\nu}{1-z_\nu z}\right),$$

so

$$\|F\|_{H^2(P^+)} = \|G\|_{H^2(\Delta)},$$

where

$$G(z) = \sum_{r=1}^{m+n} \frac{\overline{a_r}(1-z)(1-z_r)^{-\alpha}}{1+z-2z_r z} \left\{ \prod_{\nu=1;\nu\neq r}^{m+n} \left(\frac{z-z_\nu}{1-z_\nu z}\right) \right\}. \tag{4.9}$$

Write $G(z) = G_1(z) + G_2(z)$, where $G_1(z)$ is the sum in (4.9) from $r = 1$ to n and $G_2(z)$ is the corresponding sum from $r = n+1$ to $m+n$. As in the proof of Theorem 3.7.1, we will show that if $\|G_1(z)\| = 1$, then under condition (4.8), $\|G(z)\| \geq \beta$ for some $\beta = \beta(\kappa)$. From Theorem 1.7.1 and Lemma 1.7.1, we can state

Theorem 4.4.3 *Let $\{z_n\}$ be a sequence of numbers in the interval $(0,1)$ satisfying condition (4.8). Then there is a constant M such that*

(i) the inequality

$$\sum_{r=1}^{\infty} \mid g(z_r) \mid^2 (1-z_r^2) \leq M^2 \|g\|^2$$

holds for all $g \in H^2(\Delta)$,

(ii) if

$$H(z) = \sum_{r=s}^{t} \frac{\overline{a_r}(1-z)(1-z_r)^{-\alpha}}{1+z-2z_r z} \left(\prod_{\nu=s;\nu\neq r}^{t} \frac{z-z_\nu}{1-z_\nu z} \right),$$

then

$$\|H(z)\|^2 \leq M^2 \sum_{r=s}^{t} \frac{|a_r|^2(1-z_r)^{1-\alpha}(1-z_r^2)}{1+z_r-2z_r^2}$$

for any s and t.

Again, we have replaced M in Lemma 1.7.1 by M^2 for convenience.

First suppose that

$$\sum_{r=1}^{m+n} \frac{|a_r|^2(1-z_r)^{1-\alpha}(1-z_r^2)}{1+z_r-2z_r^2} \geq B^2,$$

where B is a constant to be chosen later. Apply Theorem 4.4.3 (i) to the function G in (4.9) to obtain

$$\sum_{r=1}^{m+n} \frac{|a_r|^2(1-z_r)(1-z_r)^{-\alpha}(1-z_r^2)}{1+z_r-2z_r^2} \left\{ \prod_{\nu=1;\nu\neq r}^{\infty} \left| \frac{z_r-z_\nu}{1-z_\nu z_r} \right| \right\}^2 \leq M^2\|G\|^2.$$

Thus,

$$\kappa^2 \sum_{r=1}^{m+n} \frac{|a_r|^2(1-z_r)^{1-\alpha}(1-z_r^2)}{1+z_r-2z_r^2} \leq M^2\|G(z)\|^2,$$

i.e.,

$$\|G(z)\|^2 \geq \frac{B^2\kappa^2}{M^2},$$

or

$$\|G(z)\| \geq B\kappa/M. \tag{4.10}$$

Now suppose that

$$\sum_{r=1}^{m+n} \frac{|a_r|^2(1-z_r)^{1-\alpha}(1-z_r^2)}{1+z_r-2z_r^2} < B^2.$$

Then $\|G(z)\| \geq \|G_1(z)\| - \|G_2(z)\|$, and by Theorem 4.4.3 (ii) (on replacing $H(z)$ in this part of the theorem by $G_2(z)$),

$$\|G_2(z)\|^2 \leq M^2 \sum_{r=n+1}^{m+n} \frac{|a_r|^2(1-z_r)^{1-\alpha}(1-z_r^2)}{1+z_r-2z_r^2} \leq B^2M^2.$$

Thus, since by the hypothesis $\|G_1(z)\| = 1$,

$$\|G(z)\| \geq 1 - BM. \tag{4.11}$$

But B is at our disposal, so we can take, for example, the optimal value

$$B = (M + \kappa/M)^{-1}.$$

Therefore in each case (4.10 or 4.11),

$$\|G(z)\| \geq \beta(\kappa) = (1 + M^2/\kappa)^{-1}$$

which is what we wanted to prove. □

Bibliography

[1] N.I. AKHIESER, "Vorlesungen über Approximationstheorie", Akademie Verlag Berlin (1953) (English Translation: "Theory of Approximation", New York, 1956.)

[2] J.M. ANDERSON, A Note on a Basis Problem, *Proc. Amer. Math. Soc.* **51** (1975), 330-334.

[3] N. ARONSZAJN, La Théorie des Noyaux Reproduisants et ses Applications, *Proc. Cambridge Phil. Soc.* **39** (1943), 133-153.

[4] N. ARONSZAJN, The Theory of Reproducing Kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337-404.

[5] K.G. BINMORE, Interpolation, Approximation and Gap Series, *Proc. London Math. Soc.(3)* **25** (1972) 751-761.

[6] L. CARLESON, An Interpolation Problem for Bounded Analytic Functions, *Amer. J. Math.*, **80** (1958) 921-930.

[7] E.W. CHENEY, "Introduction to Approximation Theory", Mc-Graw-Hill, New York, 1966.

[8] J.A. CLARKSON and P. ERDÖS, Approximation by Polynomials, *Duke Math. J.* **10** (1943) (5-11).

[9] P.J. DAVIS, "Interpolation and Approximation", Blaisdell, 1963.

[10] N.F. DUDLEY WARD and J.R. PARTINGTON, Rational Wavelet Decompositions of Transfer Functions in Hardy-Sobolov Classes. To appear: *Mathematics of Controls, Signals and Systems.*

[11] N.F. DUDLEY WARD and J.R. PARTINGTON, Robust Identification in the Disc Algebra using Rational Wavelets and Orthonormal Basis Functions. (Preprint)

[12] D. GAIER, On the Coefficients and Growth of Gap Power Series, *J. SIAM Numer. Anal.* **3** (1966), 250-265.

[13] V.J. GURARĬI and V.J. MACAEV, Lacunary Power Sequences in the Spaces C and L (in Russian), Izv. Akad. nauk. SSSR. Ser. Mat. **30** (1966), 3-4. (English Translation: *Amer. Math. Soc. Transl.* **2** (1968), 9-21).

[14] W.K. HAYMAN, Interpolation by Bounded Functions, *Université de Grenoble, Annales de l'Institut Fourier,* **XIII** (1958), 277-290.

[15] K. HOFFMAN, "Banach Spaces of Analytic Functions", Prentice Hall, 1962.

[16] J. KOREVAAR, A Characterisation of the Sub-manifold of $C[a,b]$ Spanned by the Sequence x^{n_k}, *Nederl. Akad. Wetensch. Proc. Ser. A.*, **50**(1947)- *Indag. Math.*, **9** (1947), 360-368.

[17] D.J. NEWMAN, Interpolation in H_∞, *Trans. Amer. Math. Soc.*, **92** (1959), 501-507.

[18] W. RUDIN, "Real and Complex Analysis", McGraw Hill, 1992.

[19] L. SCHWARTZ, "Études des Sommes d'Exponentielles Réelles" , 2ième éd. Publ. Inst. Math. Univ. Strasbourg, V, Actualité Sci. Indust., **959**, Paris, 1959.

[20] H.S. SHAPIRO, Topics in Approximation Theory, Lecture Notes in Mathematics, **187**, Springer-Verlag, Berlin, 1971.

[21] H.S. SHAPIRO and A.L. SHIELDS, On some Interpolation Problems for Analytic Functions, *Amer. J. Math.* **83** (1961), 513-532.

[22] H.S. SHAPIRO and A.L. SHIELDS, On the Zeros of Functions with Finite Dirichlet Integral and some related Function Spaces, *Math. Zeitschr.*, **80** (1962), 217-229.

[23] I. SINGER, "Bases in Banach Spaces I", Die Grundlehren der math. Wisenschaften, Band **154**, Springer-Verlag, Berlin and New York, 1970.

[24] O. SZÁSZ, On Closed Sets of Rational Functions, *Ann. Mat. Pura Appl.* **(4) 34** (1953), 195-218.

[25] E.C. TITCHMARSH, "The Theory of Fourier Integrals", Clarendon Press, Oxford, 1948.

[26] E.C. TITCHMARSH, "The Theory of Functions", Clarendon Press, Oxford, 1932.