Three papers in analysis and general topology.

Ph.D. Thesis

by

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Introduction to thesis.

During my time at UCL I have been working in several different areas. My thesis will reflect this fact in that it will consist of three papers I have written on three different subjects. The two first papers have been submitted for publication and the third will be.

The first paper is on measurable selection in usco-maps. The basis of the paper is an observation that a method used by Sion for obtaining a selection can by a slight alteration be made to work in more general circumstances. The rest of the paper explores the possibilities of this more general method.

The second paper deals with the question of when the sets of first Baire- and Borel-class functions coincide. Here the crucial point is a new technique, singled out as Lemma 1, that allows me to approximate a Borel-1 function with continuous functions. In a sense, the approximations get simultaneously better and worse, and this is the crucial idea.

The third paper contains a generalisation of a theorem by Namioka, Phelps and Preiss. Ribarska obtained a strengthening of the conclusion to the theorem and Deville pointed out that the original conclusion could be obtained with a weaker hypothesis. I have weakened the hypothesis a bit more and obtain the same conclusion as did Ribarska.

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Table of contents.

	page
Introduction to thesis	2
Acknowledgements	3
Table of contents	4
Selection from upper semi-continuous compact-valued mappings.	5
When are Borel functions Baire functions?	22
A Banach-space with a Lipschitz Gateaux-smooth bump has w^*-fragmentable dual.	52

5

Selection from upper semi-continuous compact-valued mappings.

Abstract.

The aim of this paper is to show that if axiom M (or the continuum hypothesis) is assumed, then every upper semi-continuous compact-valued map from the space of irrationals to a compact (not necessarily metric) space has a selection, which is measurable in the sense that pre-images of Baire measurable sets are universally measurable. The methods used will yield generalisations and easier proofs of well-known theorems, namely of a selection theorem by Sion [1], and a representation theorem by Ioffe [3].

0. Introduction.

It was conjectured by Jørgen Hoffmann-Jørgensen that all upper semi-continuous compact-valued maps of the irrationals into a compact Hausdorff space, K, have a selection, which is measurable in the sense that pre-images of Baire sets are universally measurable. A result of this kind would have implications in asymptotic likelihood theory and in the theory for continuity of stochastic processes. This note shows that such selections indeed do exist, if a special axiom called axiom M is assumed. Axiom M says that, on the unit interval with the Lebesgue measure, the union of strictly less than continuum many Lebesgue null-sets is a Lebesgue null-set. Axiom M is clearly implied by continuum hypothesis and also by Martin's axiom, see [5]. First, a general characterisation of minimal usco-maps is given, showing that images of hereditarily separable spaces by such maps are separable. Next, a number of selection results are proved, using a method which is a modification of that used by Sion in [1], leading to the answer to the original question. Among these results is a generalisation of Sion's selection result for set-valued maps with a simpler proof. Finally, a new proof of a representation theorem for set-valued maps by Ioffe in [3] is given. Again, the new proof is simpler than Ioffe's and allows a more general conclusion.

1. Definitions.

All spaces used here will be assumed to be Hausdorff. An usco-map ϕ of X into Y is a set-valued correspondence which is upper semi-continuous and compact-valued. For a set-valued correspondence we define the kernel: $\ker\phi=\{x:\phi(x)\neq\emptyset\}$; if ϕ is an usco-map then $\ker\phi$ is closed. The space of all usco-maps of X into Y are given a partial order as follows: $\psi\leq\phi$ if $\psi(x)$ is a subset of $\phi(x)$ for all $x\in X$ and $\ker\psi=\ker\phi$. An usco-map is said to be minimal if it is minimal in this partial ordering. A selection from a set-valued correspondence, $\phi:X\to Y$, is a function, $f:\ker\phi\to Y$ such that $f(x)\in\phi(x)$ for all $x\in\ker\phi$.

A function is $\mathcal{A}\to\mathcal{B}$ measurable if $f^{-1}(B)\in\mathcal{A}$ for all $B\in\mathcal{B}$. On any space, X, the families of sets $\mathcal{F}(X)$,$\mathcal{G}(X)$,$\mathcal{B}o(X)$,$\mathcal{B}a(X)$ and $\mathcal{M}u(X)$ are the families of closed, open, Borel, Baire and universally measurable subsets of X, respectively. A subset of X is universally measurable if it is measurable with respect to any σ-finite Radon measure on X. If $\mathcal{A}$ and $\mathcal{B}$ are families of sets we say that a function f is $\mathcal{A}\to\mathcal{B}$ measurable if $f^{-1}(B)\in\mathcal{A}$ for all $B\in\mathcal{B}$.

The space of irrationals will be identified with $\mathbb{N}^{\mathbb{N}}$ equipped with the product topology. A space is said to be K-analytic if it is the image of $\mathbb{N}^{\mathbb{N}}$ by an usco-map. A Souslin scheme is a map, A, of $\mathbb{N}^{(\mathbb{N})}$, the set of all finite sequences of integers, into 2^X, the set of subsets of X. Performing the Souslin-operation on A yields the set

$$S(A)=\bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n\in\mathbb{N}}A(\sigma|n).$$

The paving $S(\mathcal{F}(X))$ consists of all subsets of X of the form S(A) where A is a closed-valued Souslin scheme. We denote by $\mathcal{B}l(X)$ the σ-algebra $S(\mathcal{F}(X))\cap CS(\mathcal{F}(X))$ of biSouslin-sets. The paving $\mathcal{S}_1(X)$ is the least σ-algebra containing $S(\mathcal{F}(X))$.

A pair consisting of a space, X, and a σ-algebra, $\mathcal{A}$, on X is said to be Blackwell if

$$\ker A = \{\sigma : \bigcap_n A(\sigma|n) \neq \emptyset\} \in S(\mathcal{F}(\mathbb{N}^{\mathbb{N}}))$$

for all $\mathcal{A}$-valued Souslin-schemes, A. The pair $(X,\mathcal{A})$ is Blackwell with the selection property if it is Blackwell and, for all $\mathcal{A}$-valued Souslin schemes, there exists a $\mathcal{M}u(\mathbb{N}^{\mathbb{N}}) \to \mathcal{A}$ measurable selection from the correspondence $\sigma \to \bigcap_n A(\sigma|n)$.

The weight of a space is the least cardinal, τ, such that the space has a base of cardinality τ. A space, X, is said to be injective if there exists a universally measurable injection of X into the real line. If A is a subset of a space X, then A^c is the complement of A in X.

2. Minimal usco-maps.

The main result of this section is Proposition 2, which gives a necessary and sufficient condition for a usco-map to be minimal. We start with a little lemma.

Lemma 1.

For an usco-map $\phi: X \to Y$ and an open set $G \subseteq Y$ define for each $x \in X$

$$\phi(G)(x) = \begin{cases} \phi(x) \setminus G, \ \underline{\text{if}} \ x \in \operatorname{int}(\{x : \phi(x) \setminus G \neq \emptyset\} \cup \ker\phi^c), \\ \phi(x), \ \underline{\text{if}} \ x \in \operatorname{cl}(\{x : \phi(x) \subseteq G\} \cap \ker\phi). \end{cases}$$

Then $\phi(G)$ is usco and $\phi(G) \leq \phi$.

Proof.

Let F be a closed subset of Y, then

$$\{x:\phi(G)(x)\cap F\neq\varnothing\}$$

$$=[\{x:\phi(x)\cap F\cap G^c\neq\varnothing\}\cap \mathrm{int}(\{x:\phi(x)\setminus G\neq\varnothing\}\cup \ker\phi^c)]$$

$$\cup[\{x:\phi(x)\cap F\neq\varnothing\}\cap \mathrm{cl}(\{x:\phi(x)\subseteq G\}\cap\ker\phi)]$$

$$=\{x:\phi(x)\cap F\cap G^c\neq\varnothing\}\cup[\{x:\phi(x)\cap F\neq\varnothing\}\cap \mathrm{cl}(\{x:\phi(x)\subseteq G\}\cap\ker\phi)],$$

since

$$\{x:\phi(x)\cap F\cap G^c\neq\varnothing\}\subseteq$$

$$[\{x:\phi(x)\cap F\cap G^c\neq\varnothing\}\cap \mathrm{int}(\{x:\phi(x)\setminus G\neq\varnothing\}\cup \ker\phi^c)]$$

$$\cup[\{x:\phi(x)\cap F\neq\varnothing\}\cap \mathrm{cl}(\{x:\phi(x)\subseteq G\cap\ker\phi)]$$

We conclude that $\phi(G)$ is usco and the rest of the lemma is immediate. □

Proposition 2.

<u>An usco-map,</u> $\phi:X\to Y$, <u>is minimal, if and only if,</u>

(*) $\{x:\phi(x)\cap G\neq\varnothing\}\subseteq \mathrm{cl}(\{x:\phi(x)\subseteq G\}\cap\ker\phi)$

<u>for each open set</u> $G\subseteq Y$.

Proof.

Assume (*) holds for all open subsets of Y and let $\psi:X\to Y$ be an usco-map such that $\psi\leq\phi$ and $\psi(y)\neq\phi(y)$. Since $\psi(y)$ is compact we can find an open set U such that $\psi(y)\subseteq U$ and $\phi(y)\cap[\mathrm{cl}(U)]^c\neq\varnothing$. Then

$$y\in\{x:\phi(x)\cap[\mathrm{cl}(U)]^c\neq\varnothing\}\cap\{x:\psi(x)\subseteq U\}$$

$\subseteq cl(\{x:\phi(x)\subseteq[cl(U)]^c\}\cap \ker\phi)\cap\{x:\psi(x)\subseteq U\}$, by (*),

$\subseteq cl(\{x:\psi(x)\subseteq[cl(U)]^c\}\cap \ker\psi)\cap\{x:\psi(x)\subseteq U\}$, since $\psi\leq\phi$.

Hence $\{x:\psi(x)\subseteq[cl(U)]^c\}\cap \ker\psi\cap\{x:\psi(x)\subseteq U\}$ is nonempty which is a contradiction, and we conclude that ϕ is minimal.

Now assume that (*) does not hold for the open subset G of Y. That is, there exists $y\in \mathrm{int}(\{x:\phi(x)\setminus G\neq\emptyset\}\cup \ker\phi^c)$ such that $\phi(y)\setminus G$ is a proper non-empty subset of $\phi(y)$. Using Lemma 1 we conclude that ϕ is not minimal. □

We shall now use this characterization to give some properties of minimal usco-maps. Recall that a function is said to have the Baire property if the pre-image of every open set is an open set modulo a set of the first category. Also recall that a family of sets is said to be T_O-separating if there, for any pair of distinct points, exists a set from the family that contains one of the points but not the other. We do not require that the separating set can be chosen such that it contains, say, the first of the points of the pair.

Corollary 3.

Let $\phi:X\to Y$ be a minimal usco-map.

(i) If $\ker\phi$ is separable, then $\phi(X)$ is separable.

(ii) Any selection from ϕ has the Baire-property.

(iii) If there exists a countable T_O-separating family of open sets in Y, then the set $\{x:\#\phi(x)>1\}$ is of the first category in X.

Proof.

(i) Let $cl\{x_n\}=\ker\phi$ and choose points $y_n\in\phi(x_n)$. If $G\cap\phi(X)\neq\emptyset$ for an open set $G\subseteq Y$, then $\emptyset\neq\{x:\phi(x)\cap G\neq\emptyset\}\subseteq cl(\{x:\phi(x)\subseteq G\}\cap\ker\phi)$ by Proposition 2. Hence the open set $\{x:\phi(x)\subseteq G\}$ has nonempty intersection with $\ker\phi$ and we find x_n such that $y_n\in\phi(x_n)\subseteq G$.

(ii) Let f be a selection from ϕ. Then

$$\{x:\phi(x)\subseteq G\}\cap\ker\phi\subseteq f^{-1}(G)\subseteq\{x:\phi(x)\cap G\neq\emptyset\}\subseteq cl(\{x:\phi(x)\subseteq G\})\cap\ker\phi$$

by Proposition 2, and hence $f^{-1}(G)$ has the Baire-property for any open subset G of Y.

(iii) Let $\{G_n\}$ be a countable T_0-separating family of open subsets of Y. Then

$$\{x:\#\phi(x)>1\}=\bigcup_n(\{x:\phi(x)\cap G_n\neq\emptyset\}\setminus\{x:\phi(x)\subseteq G_n\})$$
$$\subseteq\bigcup_n(cl(\{x:\phi(x)\subseteq G_n\})\setminus\{x:\phi(x)\subseteq G_n\}).$$

The latter set, as a countable union of sets of the first category, is of the first category. □

Let an usco-map, $\phi:X\to Y$, be given. Given a well-ordering of a base for the open subsets of Y we explicitly construct a minimal usco-map $\psi:X\to Y$ such that $\psi\leq\phi$.

Let $\{G_\gamma:\gamma<\Omega\}$ be a wellordering of a base for the open subsets of Y and use Lemma 1 to define $\phi_1=\phi(G_1)$ and, still using the lemma, define inductively for $\gamma<\Omega$:

$\phi_\gamma = (\bigcap_{\beta<\gamma} \phi_\beta)(G_\gamma)$. Let $\psi = \bigcap_{\gamma<\Omega} \phi_\gamma$.

By Lemma 1 each ϕ_γ is usco, contained in ϕ and has the same kernel so this is also true for ψ. By Proposition 2 the map ψ is minimal, for

$$\{x:\psi(x)\cap G\neq\emptyset\} = \bigcup_{\{\gamma:G_\gamma\subseteq G\}} \{x:\psi(x)\cap G_\gamma\neq\emptyset\}$$

$$\subseteq \bigcup_{\{\gamma:G_\gamma\subseteq G\}} \{x:\phi_\gamma(x)\cap G_\gamma\neq\emptyset\} \subseteq \bigcup_{\{\gamma:G_\gamma\subseteq G\}} \mathrm{cl}(\{x: \bigcap_{\beta<\gamma} \phi_\beta(x)\subseteq G_\gamma\}\cap \ker\psi)$$

$$\subseteq \mathrm{cl}(\bigcup_{\{\gamma:G_\gamma\subseteq G\}} \{x:\psi(x)\subseteq G_\gamma\}\cap \ker\psi) \subseteq \mathrm{cl}(\{x:\psi(x)\subseteq G\}\cap \ker\psi)$$

by the construction and the fact that $\psi\subseteq\phi_\gamma\subseteq \bigcap_{\beta<\gamma} \phi_\beta$. □

3. Selection.

We shall now consider another way of cutting compact-valued (not necessarily usco) correspondances down. The approach used here will be very much like that of Sion in [1], but the results we shall obtain will be more general. The proofs in the rest of this note will depend on the properties of the following construction.

For a compact-valued correspondence $\phi:X\to Y$ and an open set $G\subseteq Y$ we define for each x in X

$$\phi_G(x)=\begin{cases}\phi(x)\setminus G, & \text{if } \phi(x)\setminus G\neq\emptyset,\\ \phi(x), & \text{otherwise.}\end{cases}$$

Let $\{G_\gamma:\gamma<\Omega\}$ be a T_0-separating family of open subsets of Y, and define $\phi_1=\phi_{G_1}$ and, for each $\gamma<\Omega$, define inductively: $\phi_\gamma=(\bigcap_{\beta<\gamma}\phi_\beta)_{G_\gamma}$. Let $\psi=\bigcap_{\gamma<\Omega}\phi_\gamma$, then we have the following consequences.

(i) $\ker\phi=\ker\psi$.

(ii) $\#\psi(x)\leq 1 \ \forall x\in X$.

(iii) $\{x:\psi(x)\subseteq G_\gamma\}=\{x:\bigcap_{\beta<\gamma}\phi_\beta(x)\subseteq G_\gamma\}=\bigcup_{\beta<\gamma}\{x:\phi_\beta(x)\subseteq G_\gamma\}$ for all $x\in X$.

(iv) For any open set $G\subseteq Y$ we have

$$\{x:\phi_\gamma(x)\subseteq G\}$$
$$=\{x:\bigcap_{\beta<\gamma}\phi_\beta(x)\subseteq G\}\cup(\{x:\bigcap_{\beta<\gamma}\phi_\beta(x)\subseteq G\cup G_\gamma\}\cap\{x:\bigcap_{\beta<\gamma}\phi_\beta(x)\subseteq G_\gamma\}^c)$$
$$=\bigcup_{\beta<\gamma}\{x:\phi_\beta(x)\subseteq G\}\cup(\bigcap_{\beta<\gamma}\{x:\phi_\beta(x)\subseteq G\cup G_\gamma\}\cap\bigcap_{\beta<\gamma}\{x:\phi_\beta(x)\subseteq G_\gamma\}^c).$$

Define $f:\ker\phi\to Y$ by $\{f(x)\}=\psi(x)$ for all x in $\ker\phi$. From the construction we immediately get the following generalisation of Sion's result.

Proposition 4.

Let $\phi:X\to Y$ be a compact-valued correspondence, $\{G_\gamma:\gamma<\omega_1\}$ be a T_0-separating family of open subsets of Y such that every open subset of Y is a countable union of sets from this family, and let $\mathcal{H}$ be a σ-algebra on X such that $\{x:\phi(x)\subseteq G_\gamma\}\in\mathcal{H}$ for all $\gamma<\omega_1$. Then ϕ has a $\mathcal{H}\to\mathcal{B}o$ measurable selection.

Proof.

Let ψ be a selection from ϕ given by the construction above and let $\{f(x)\}=\psi(x)$ for all $x\in\ker\phi$. It is sufficient to prove that $f^{-1}(G_\gamma)\in\mathcal{H}$ for every $\gamma<\omega_1$. By (iii) we have

$$f^{-1}(G_\gamma)=\bigcup_{\beta<\gamma}\{x:\phi_\beta(x)\subseteq G_\gamma\}.$$

The result follows by (iv) and transfinite induction. □

In [1] Sion requires that Y be regular. Following [6], Proposition 1-6-2 we find that on a measurable space (X,Σ), where Σ is countably generated, the universally Σ-measurable sets are stable under the union of strictly less than continuum sets when axiom M is assumed. The proof of the next proposition is similar to the proof of Proposition 4.

Proposition 5.

Let X be of countable weight and assume that the weight of Y is strictly less than continuum. Let $\phi:X\to Y$ be a compact-valued correspondence such that $\{x:\phi(x)\subseteq G\}$ is universally measurable for all open subsets G of Y. Assume axiom M. Then ϕ has a $\mathcal{M}u(X)\to\mathcal{B}o(Y)$ measurable selection. □

Using Propositions 4 and 5 we obtain the next two propositions.

Proposition 6.

Let Y be K-analytic, hereditarily Lindelöf and of weight less than or equal to $\aleph_1$. Then $(Y,\mathcal{B}\ell(Y))$ is Blackwell with the selection property.

Remark: If the K-analytic space Y is regular and hereditarily Lindelöf, then all open subsets of Y are Souslin-$\mathcal{F}$ sets. This again implies that $\mathcal{B}o(Y)=\mathcal{B}\ell(Y)$ and Y is hereditarily Lindelöf.

Proof.

Write $Y=\phi(\mathbb{N}^{\mathbb{N}})$ where ϕ is usco and let F be an $\mathcal{F}(Y)$-Souslin scheme. The map $\psi:\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}\to Y$, defined by $\psi(\sigma,\tau)=\phi(\sigma)\cap\bigcap_n F(\tau|n)$, is usco and $\ker F=\pi_2(\ker\psi)$, where π_2 is the projection of $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ onto the second coordinate. Let f be a Borel measurable selection from ψ and let $g:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ be $\mathcal{S}_1\to\mathcal{S}_1$ measurable such that $(\tau,g(\tau))\in\ker\psi$ for all τ in $\ker F$ (see [2]). Then $f\circ g$ is an $\mathcal{S}_1\to\mathcal{S}_1$ measurable selection from F. □

The proof of the next proposition is similar to the proof of Proposition 6.

Proposition 7.

Let Y be K-analytic of weigth strictly less than continuum and assume axiom M. Then $(Y,\mathcal{B}\ell(Y))$ is Blackwell with the selection property.

If we are willing to accept weaker measurability properties of selections, this allows us to weaken the conditions of Propositions 4 and 5.

Proposition 8.

Let X be Lindelöf and let Y be of weight less than or equal to the first uncountable ordinal. Then every usco-map of X into Y has a $\mathcal{B}o(X)\to\mathcal{B}a(Y)$ *measurable selection.*

Proof.

Let $\phi:X\to Y$ be usco and let $\{G_\gamma:\gamma<\omega_1\}$ be a base for Y. By our construction we have a selection f from ϕ such that $f^{-1}(G_\gamma)\in\mathcal{B}o(X)$ for all γ. Let $F=\bigcap_n G_n$ be a closed G_δ in Y. Then $F\cap\phi(X)$ is Lindelöf, and we can, for each n, find basic open sets such that $F\cap\phi(X)\subseteq\bigcup_m G_{mn}\cap\phi(X)\subseteq G_n\cap\phi(X)$, implying $F\cap\phi(X)=\bigcap_n\bigcup_m G_{mn}\cap\phi(X)$. Now $f^{-1}(F)=\bigcap_n\bigcup_m f^{-1}(G_{mn})\in\mathcal{B}o(X)$. Finally note that $\mathcal{B}a(Y)$ is generated by a family of closed G_δ-sets. □

Proposition 9.

Let X be of countable weight and let Y be of weight less than or equal to continuum. Assume axiom M. Then every usco-map of X into Y has a $\mathcal{M}u(X)\to\mathcal{B}a(Y)$ *measurable selection.*

Proof.

Substitute $\mathcal{M}u(X)$ for $\mathcal{B}o(X)$ and 2^{ω_0} for ω_1 in the proof for Proposition 8. □

Theorem 10.

Let X be separable and Lindelöf and let Y be regular. Then every usco-map of X into Y with nonempty values has a selection, f, with the following measurability properties.

(i) (CH) f is $\mathcal{B}o(X)\to\mathcal{B}a(Y)$ measurable.

(ii) (CH) If Y also is hereditarily Lindelöf, then f is $\mathcal{B}o(X)\to\mathcal{B}o(Y)$ measurable.

Let, in addition, X be of countable weight.

(iii) (M) f is $\mathcal{M}u(X)\to\mathcal{B}a(Y)$ measurable.

(iv) (M) If Y also is hereditarily Lindelöf, then f is $\mathcal{M}u(X)\to\mathcal{B}o(Y)$ measurable.

Proof.

Let $\phi:X\to Y$ be a minimal usco-map with nonempty values. Then $cl(\phi(X))$ is separable and regular, hence, by [4], Theorem 1.5.6., the weight of $cl(\phi(X))$ is less than or equal to continuum, and so (i) and (iii) follow from Propositions 8 and 9. If Y is hereditarily Lindelöf, then (ii) and (iv) follow from Propositions 4 and 5. □

Theorem 10 (iii) implies that, under axiom M, all compact Hausdorff spaces with the Baire σ-algebra are Blackwell with the selection property. But in the case where the range space is compact we can obtain conclusions (i) and (iii) of Theorem

10 with weaker conditions on the domain space.

Theorem 11.

Let Y be compact of weight τ and let ϕ:X→Y be usco. Then ϕ has a selection, f, with the following measurability properties.

(i) If $\tau \leq \aleph_1$, then f is $\mathcal{B}o(X) \to \mathcal{B}a(Y)$ measurable.

(ii) If $\tau \leq 2^{\aleph_0}$, then f is $\mathcal{M}u(X) \to \mathcal{B}a(Y)$ measurable.

Remark.

If X is separable then the weight of cl(ϕ(X)) is less than or equal to continuum.

Proof.

Let C(Y) be the space of continuous functions from Y to $\mathbb{R}$ equipped with the topology of uniform convergence. Then there is a dense subset, $\mathcal{C}$, of C(Y) of cardinality τ. (Use the Stone-Weierstraß Theorem. It can be found in [7].) The family of sets $\mathcal{U}_2 = \{ \{f<a\}: a \in \mathbb{Q}, f \in \mathcal{C}\}$ generates the Baire σ-algebra on Y and is of cardinality τ. Let $\mathcal{U}_1$ be a base for Y of cardinality τ. Then $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ is also a base for Y of cardinality τ, and $\mathcal{B}a(Y)$ is contained in the σ-algebra generated by $\mathcal{U}$.

If $\tau \leq \aleph_1$, then by property (iii) of the construction of the selection we have $f^{-1}(\mathcal{U}) \subseteq \mathcal{B}o(X)$ and (i) follows. If $\tau \leq 2^{\aleph_0}$, then by axiom M we have $f^{-1}(\mathcal{U}) \subseteq \mathcal{M}u(X)$ and (ii) follows. □

4. Representation.

Finally, we shall prove a representation theorem analoguous to that of [3]. In comparison to Theorem 2 and Corollary 2.1 in [3], Theorem 13 below gives only a bit more information about the measurability properties we can require selections to have. The main reason for including Theorem 13 in the present paper is that it shows how the method of selection that we have employed here can be applied to obtain representations. Furthermore, we can avoid using $\cup$-homomorphisms. We shall first prove a set-theoretical lemma.

Lemma 12.

Let Y *be regular of weight* τ, *let* $\mathcal{U}_1$ *be a base for* Y *of cardinality* τ *and let*

$$\mathcal{U}=\mathcal{U}_1\cup\{\,[\mathrm{cl}(U)]^c:\ U\in\mathcal{U}_1\}=\{U(\gamma):\gamma<\tau\}.$$

Let $\Sigma\subseteq\tau^\tau$ *be the set of bijections from* τ *to* τ.

For any $y\in Y$ *there exists* $\sigma\in\Sigma$ *such that*

$$y\in U(\sigma(\gamma))\ \Rightarrow\ \exists\beta<\gamma:\ y\in U(\sigma(\beta))^c\subseteq U(\sigma(\gamma)).$$

Proof.

Let $\gamma<\tau$ and assume $\{\sigma^\beta:\beta<\gamma\}\subseteq\Sigma$ have been defined so that:

(i) $y\in U(\sigma^\beta(\eta)),\ \eta\leq\beta\ \Rightarrow\exists\xi<\eta: y\in U(\sigma^\beta(\xi))^c\subseteq U(\sigma^\beta(\eta))$; and

(ii) $\forall\eta<\tau\ \exists\xi<\gamma:\quad \sigma^\alpha(\eta)=\sigma^\beta(\eta)\qquad\forall\alpha,\beta<\xi$

and $\quad \sigma^{\beta}(\eta)=\sigma^{\xi}(\eta) \qquad \forall \xi \leq \beta < \gamma.$

According to (ii) we can define $\psi \in \Sigma$ by $\psi(\eta)=\lim_{\beta<\gamma} \sigma^{\beta}(\eta)$.

If $y \in U(\gamma)$ and for no $\beta<\gamma$ we have $y \in U(\psi(\beta))^{c} \subseteq U(\gamma)$, find ξ such that $y \in cl(U) \subseteq U(\gamma)$, where $U \in \mathfrak{U}_1$ and $U(\xi)=[cl(U)]^{c}$. In this case we let

$$\sigma^{\gamma}(\eta)=\begin{cases}\xi, \text{ if } \eta=\gamma,\\ \gamma, \text{ if } \eta=\xi\\ \psi(\eta), \text{ otherwise.}\end{cases}$$

Otherwise let $\sigma^{\gamma}=\psi$.

Finally put $\sigma=\lim_{\gamma<\tau} \sigma^{\tau}$. □

Theorem 13.

Let ϕ:X→Y be usco with nonempty values, let Y be regular and let τ be the ordinal corresponding to the weight of cl(ϕ(X)). Let the space τ^{τ} have the topology induced by the base consisting of sets of the form $\{\sigma':\sigma'|\gamma=\sigma|\gamma\}$, $\sigma \in \tau^{\tau}, \gamma<\tau$. Let $\Sigma \subseteq \tau^{\tau}$ be the set of bijections from τ to τ.

There exists a function h:XxΣ→Y such that x→h(x,σ) has measurability properties as indicated by Propositions 4 to 11, for all $\sigma \in \Sigma$. Furthermore $\sigma \rightarrow h(x,\sigma)$ is continuous from Σ to Y for all $x \in X$ and $h(x,\Sigma)=\phi(x)$ for all $x \in X$.

Proof.

Let $\mathfrak{U}_1$ be a base of cardinality τ for cl(ϕ(X)) and

$$\mathfrak{U}=\mathfrak{U}_1 \cup \{[cl(U)]^{c}: U \in \mathfrak{U}_1\}=\{U(\gamma): \gamma<\tau\}.$$

Define $\phi_1^\sigma = \phi_{U(1)}$, $\phi_\gamma^\sigma = (\bigcap_{\beta<\gamma} \phi_\beta^\sigma)_{U(\sigma(\gamma))}$ and $\{h(x,\sigma)\} = \bigcap_{\gamma<\tau} \phi_\gamma^\sigma(x)$. This produces selections with the desired properties. That any $y \in \phi(X)$ can be targeted follows from Lemma 12. □

5. References.

[1] M.Sion, On uniformization of sets in topological spaces, Trans. Amer. Math. Soc., 96 (1960), pp.237-245.

[2] C.A.Rogers et al., Analytic Sets, Academic Press 1980.

[3] A.D.Ioffe. Representation theorems for measurable multifunctions. Measure theory Oberwolfach 1979. Springer 1980.

[4] R.Engelkind, General Topology, Warszawa 1977.

[5] J.Hoffmann-Jørgensen, How to make a divergent sequence convergent by Martin's axiom, Aarhus Universitet, Preprint series 1977/78.

[6] M.Talagrand, Pettis integral and measure theory, Memoirs of the Amer. Math. Soc., sept 84, vol 51, number 307.

[7] J.Dugundji, Topology, Allyn and Bacon inc., Boston 1966.

When are Borel functions Baire functions ?

Abstract.

The following two theorems give the flavour of what will be proved.

Theorem.

Let Y be a complete metric space. Then the families of first Baire class functions and of first Borel class functions from [0,1] to Y coincide, if and only if, Y is connected and locally connected.

Theorem.

Let Y be a separable metric space. Then the families of second Baire class functions and of second Borel class functions from [0,1] to Y coincide, if and only if, for all finite sequences, $U_1,\ldots,U_q$, of nonempty open subsets of Y there exists a continuous function, $\phi:[0,1]\rightarrow Y$, such that $\phi^{-1}(U_i)\neq\emptyset$ for all $i\leq q$.

0. Introduction.

Given metric spaces X and Y we let $\mathcal{B}a_0(X,Y)$ be the family of all continuous functions from X to Y. For all ordinals $0<\alpha<\omega_1$ we define the Baire class α, denoted by $\mathcal{B}a_\alpha(X,Y)$, to be the family of all limits of pointwise convergent sequences of functions from $\bigcup_{\beta<\alpha} \mathcal{B}a_\beta(X,Y)$. A class α Borel function from X to Y ($0<\alpha<\omega_1$) is a function, f, such that $f^{-1}(G)$ is a Borel set of additive class α whenever G is an open subset of Y. For reference on Borel sets see [1]. We denote the family of all class α Borel functions by $\mathcal{B}o_\alpha(X,Y)$.

The first Baire and Borel classes do not coincide in general. The function $f:[0,1]\to\{0,1\}$ defined by $f(1)=1$ and $f(t)=0$ when $t<1$ is of first Borel class, but clearly is not of first Baire class.

The Lebesgue-Hausdorff Theorem in [1], page 391 tells us that if X is metric and if Y is an n-dimensional cube, $[0,1]^n$, $n\in\mathbb{N}$, or the Hilbert cube, $[0,1]^{\mathbb{N}}$, then the first Baire and Borel classes of functions from X to Y do coincide.

More general theorems of this kind has been proved. Rolewicz showed in [4] that if Y is a separable convex subset of a normal linear space, then the first Baire and Borel classes of functions from X to Y coincide. In [2] Hansell gave an extension of the Lebesgue-Hausdorff Theorem asserting that, if every continuous function from a closed subset of X to Y can be extended continuously to X, then every σ-discrete (see section 2) first Borel class function from X to Y is also of first Baire class. It was pointed out that Hansell's proof was incomplete, and in [5] Rogers gave a corrected version of Hansell's statement, namely that if every continuous function from a closed subset of X to Y can be extended continuously to X, and if, for each point y in Y and each neighbourhood L of y, there is a second neighbourhood N of y such that, for each closed subset F of X and each continuous map f from F to N, there is a continuous extension of f mapping X into L, then every σ-discrete first Borel class function from X to Y is also of first Baire class.

In the present paper we prove that all σ-discrete first Borel class functions from a metric space into a metric, arcwise connected and locally arcwise connected space are of first Baire class. We then look for a converse to this result and prove the following theorem. We write $I=[0,1]$ and $\Sigma(X,Y)$ for the class of σ-discrete functions from X to Y.

Theorem 2. Let Y be complete metric. Then the following three statements are equivalent: (i) Y is connected and locally connected; (ii) $\mathcal{B}a_1(I,Y)=\mathcal{B}o_1(I,Y)$; and (iii) $\mathcal{B}a_1(X,Y)=\mathcal{B}o_1(X,Y)\cap\Sigma(X,Y)$ for all metric spaces X.

Having considered the case for the first Baire and Borel classes, we then turn our attention to the higher classes. Here the classic theorem is the Banach Theorem which can be found in [1] or in [10]. This theorem uses the concept of an analytically representable function. The analytically representable functions of class one are the first Borel class functions and the analytically representable functions of class α are the functions which are pointwise limits of analytically representable functions of classes lower than α. The Banach Theorem tells us that if X is metric and if Y is separable and metric then the set of analytically representable functions of class α coincide with the set of Borel class α $(\alpha+1)$

functions when α is finite (infinite). Another theorem by Banach in [10] states that if Y is also arcwise connected then the set of Baire class α functions can replace the analytically representable functions of class α in the Banach Theorem for $\alpha \geq 2$. In [6] Brown showed in particular that if X is compact metric and Y is arcwise connected, separable and metric, then the class of all Borel functions and the class of all Baire functions coincide. In [2] Hansell generalised the Banach Theorem to the case where Y need not be separable, using the notion of a σ-discrete function. Hansell's result states that if Y is metric and $f:X \rightarrow Y$ is σ-discrete and of Borel class α $(\alpha+1)$ then f is analytically representable of class α, when α is finite (infinite).

Again we find that a very simple 'connectedness' condition tells us when the Baire and Borel classes coincide. We shall prove the following theorem.

Theorem 3. Let Y be a separable metric space. The following statements are equivalent: (i) For all finite sequences, $U_1,\ldots,U_q$, of nonempty open subsets of Y, there exists a continuous function $\phi:I \rightarrow Y$ with $\phi^{-1}(U_i) \neq \varnothing$ for all $i \leq q$; (ii) $\mathcal{B}a_2(X,Y)=\mathcal{B}o_2(X,Y)$ for all metric spaces X; (iii) For each metric space X and for each finite (countable, infinite) ordinal $\alpha \geq 2$, $\mathcal{B}a_\alpha(X,Y)$ coincides with $\mathcal{B}o_\alpha(X,Y)$ $(\mathcal{B}o_{\alpha+1}(X,Y))$.

In [6] Brown makes a remark that implies that when Y is a separable metric space the condition (i) implies that $\bigcup_{\alpha \geq 0}\mathcal{B}a_\alpha(X,Y)=\bigcup_{\alpha \geq 0}\mathcal{B}o_\alpha(X,Y)$ for all metric spaces X. His remark seems to imply the converse. He gives proof of neither of these results.

1. Definitions.

A space is said to be locally arcwise connected, if each point of the space has an arbitrarily small (not necessarily open) arcwise connected neighbourhood. This is equivalent to requiring that, for all ϵ, every point of the space must have an open neighbourhood, such that any two points of the neighbourhood, can be joined with an arc of diameter less than ϵ.

We say that a family of sets $\mathcal{A}$ refines a second family of sets $\mathcal{B}$, if each set of $\mathcal{A}$ is contained in a set of $\mathcal{B}$, and $\cup\mathcal{A}=\cup\mathcal{B}$. We write this $\mathcal{A}\prec\mathcal{B}$. Given any set A in a metric space X and $\epsilon>0$, we denote the generalised open ball with 'centre' A and radius ϵ as $B(A,\epsilon)=\{x\in X : d(x,A)<\epsilon\}$.

2. σ-discrete functions.

We shall make use of the notion of a σ-discrete function, as developed by A.H.Stone and R.W.Hansell, to allow us to consider general metric spaces, using techniques normally used for separable metric spaces. In this section the necessary definitions and results concerning σ-discrete functions will be given. The reader is refered to [2], [3] and [8] for further information. All spaces are assumed to be metric.

A family of sets in a topological space is said to be discrete, if each point of the space has a neighbourhood that meets at most one of the sets of the family. The family is said to be uniformly discrete, if there exists $\epsilon>0$, such that the distance between any two sets of the family is greater than ϵ. A family of sets is said to be σ-discrete, if the family can be decomposed into countably many subfamilies, each of which are discrete. By [8], Lemma 2, and its proof, if $\mathcal{A}$ is a σ-discrete family of $\mathcal{F}_\sigma$-sets, then there exists a uniformly σ-discrete family, $\mathcal{B}$, of $\mathcal{F}_\sigma$-sets such that $\mathcal{B} \prec \mathcal{A}$.

A family of sets is a base for a function from one topological space into another, if the pre-image of any open set is the union of sets from the family. A function is said to be σ-discrete, if it has a σ-discrete base. The family of all σ-discrete functions from X to Y is denoted by $\Sigma(X,Y)$. In any metric space there exists a σ-discrete family of open sets, forming a base for the topology, (see [1],p.235.) Using this it can be shown that any continuous map with metric range is σ-discrete. The family $\Sigma(X,Y)$ is closed under pointwise limits ([3]), so all Baire class α functions, $\alpha<\omega_1$, are σ-discrete. In [1], page 386, it is shown that functions of

Baire class α are of Borel class α, respectively $\alpha+1$, according as α is finite or infinite. In [2] it is shown that a σ-discrete Borel class α function, where $\alpha \geq 2$, is the pointwise limit of a sequence of σ-discrete Borel functions, all of which are of classes strictly lower than α. Hence, if for some $\alpha \geq 1$ $\mathcal{B}a_{\alpha}(X,Y)$ is equal to $\mathcal{B}o_{\alpha}(X,Y) \cap \Sigma(X,Y)$, respectively $\mathcal{B}o_{\alpha+1}(X,Y) \cap \Sigma(X,Y)$, according as α is finite or infinite, then we have that $\mathcal{B}a_{\beta}(X,Y)$ is equal to $\mathcal{B}o_{\beta}(X,Y) \cap \Sigma(X,Y)$ ($\mathcal{B}o_{\beta+1}(X,Y) \cap \Sigma(X,Y)$) for all finite (infinite) ordinals β greater than α. A σ-discrete function of the first Borel class from one metric space into another, has a σ-discrete closed base ([2]). Every function from a metric space to a separable metric space is σ-discrete, and every Borel function from a space, that is a Souslin-$\mathcal{F}$ set in some complete metric space, to a metric space is σ-discrete ([2]). It is consistent with and independent of ZFC to assume that all Borel functions from a metric space to a metric space are σ-discrete, (see [9].)

3. First Baire class functions.

We shall start with two lemmas. The first is purely technical.

Lemma 1.

Let $\mathcal{A}$ be a σ-discrete family of closed sets covering the mètric space X. Then there exists families, $\mathcal{B}^p$, p=1,2,..., of closed sets such that:

$\bigcup_p \mathcal{B}^p \prec \mathcal{A}$;

each member of $\mathcal{B}^p$ is contained in some member of $\mathcal{B}^{p+1}$, for all p;

$\mathcal{B}^p$ is uniformly discrete for all p; and

$\bigcup \bigcup_p \mathcal{B}^p = X$.

Proof.

Write $\mathcal{A}$ as the countable union of discrete families of closed sets $\mathcal{A}_k$, and, for each j, let $\mathcal{C}_j = \{A \setminus \bigcup_{k<j} \bigcup \mathcal{A}_k : A \in \mathcal{A}_j\}$ and let $\mathcal{C} = \bigcup_j \mathcal{C}_j$. Then $\mathcal{C}$ is a disjoint, σ-discrete family of $\mathcal{F}_\sigma$-sets such that $\mathcal{C} \prec \mathcal{A}$. By [8], Lemma 2, and its proof, we can write each $C \in \mathcal{C}$ as an increasing union of $\mathcal{F}_\sigma$-sets, $C = \bigcup_n D_C^n$, where the families $\{D_C^n : C \in \mathcal{C}\}$ are uniformly discrete for all n. Write each $D_C^n = \bigcup_m F_C^{nm}$, where the sets F_C^{nm} are closed. Then the families $\mathcal{B}^p = \{\bigcup_{n,m \leq p} F_C^{nm} : C \in \mathcal{C}\}$, $p \in \mathbb{N}$, satisfy the conclusion of the lemma. □

Our second lemma provides us with approximating functions to a given function.

Lemma 2.

Let X be a metric space and let Y be a metric and arcwise connected space. Let $f:X\to Y$ be given and let $\mathfrak{D}_n$, $n=1,\dots,p$ be families of non-empty closed sets in X such that :

> *$\mathfrak{D}_n$ is uniformly discrete for all n;*
>
> *each member of $\mathfrak{D}_{n+1}$ is contained in some member of $\mathfrak{D}_n$ for all $n<p$; and*
>
> *$x_1,x_2\in A\in\mathfrak{D}_n$ implies that $f(x_1)$ and $f(x_2)$ can be joined with an arc of diameter less than 2^{-n}.*

Then there exists a continuous function $g:X\to Y$ such that when $x\in\cup\mathfrak{D}_n$, $n\leq p$, then $d(f(x),g(x))\leq 2^{-n+2}$.

Proof.

Since the families $\mathfrak{D}_n$, $n\leq p$, are uniformly discrete, we can find $\epsilon_1>2\epsilon_2>\dots>2^{p-1}\epsilon_p>0$ so that $\{B(A,\epsilon_n):A\in\mathfrak{D}_n\}$ is a discrete family of open sets for each $n\leq p$. Observe that, for $A\subset C$ with $A\in\mathfrak{D}_{n+1}$ and $C\in\mathfrak{D}_n$, we have $B(A,\epsilon_{n+1})\subset B(C,\epsilon_n/2)$. Pick $y'\in f(X)$ and $y_A\in f(A)$ for all $A\in\bigcup_{n\leq p}\mathfrak{D}_n$. For each $A\in\mathfrak{D}_1$ let $\phi_A^1:I\to Y$ be an arc with $\phi_A^1(0)=y'$ and $\phi_A^1(1)=y_A$. For each $A\in\mathfrak{D}_n$, $1<n\leq p$ there is a unique $C\in\mathfrak{D}_{n-1}$ with $A\subset C$, since the family $\mathfrak{D}_{n-1}$ is disjoint. Let $\phi_A^n:I\to Y$ be an arc of diameter at most 2^{-n+1}, with $\phi_A^n(0)=y_C$ and $\phi_A^n(1)=y_A$.

We shall define a sequence $g_0,g_1,g_2,\dots,g_p$ of continuous functions from X to Y and arrange that the function $g=g_p$ satisfies the requirements of the lemma. We start the inductive process by taking

$$g_0(x)=y'$$

for all x in X.

Write

$$D_1=\bigcup\{\overline{B(A,\epsilon_1/2)}:A\in\mathfrak{D}_1\},$$

$$E_1=X\setminus\bigcup\{B(A,\epsilon_1):A\in\mathfrak{D}_1\}.$$

Since the family

$$\{B(A,\epsilon_1):A\in\mathfrak{D}_1\}$$

is discrete, the sets D_1 and E_1 are disjoint closed sets. Write

$$h_1(x)=\frac{d(x,E_1)}{d(x,D_1)+d(x,E_1)},$$

so that h_1 is a continuous function on X taking the value 1 on D_1 and the value 0 on E_1. Take

$$g_1(x)=g_0(x)=y' \text{ on } E_1,$$

$$g_1(x)=y_A \text{ on } \overline{B(A,\epsilon_1/2)} \text{ for each } A\in\mathfrak{D}_1, \text{ and}$$

$$g_1(x)=\phi_A^1\circ h_1(x) \text{ on } B(A,\epsilon_1)\setminus\overline{B(A,\epsilon_1/2)} \text{ for each } A\in\mathfrak{D}_1.$$

Since the sets E_1, $\overline{B(A,\epsilon_1/2)}$, $B(A,\epsilon_1)\setminus\overline{B(A,\epsilon_1/2)}$, for $A\in\mathfrak{D}_1$, are disjoint with union X, the function g_1 is well-defined. We verify that g_1 is continuous on X. For $x_0\in X$ we can choose a neighbourhood N of X that meets at most one set of the family

$$\{\overline{B(A,\epsilon_1)}:A\in\mathfrak{D}_1\}.$$

If N meets none of these sets $g_1(x)=g_0(x)$ for x in N, and g_1 is continuous at x_0. Suppose that N meets $\overline{B(A,\epsilon_1)}$ for some A in $\mathfrak{D}_1$, but meets no set $\overline{B(A',\epsilon_1)}$ with $A'\in\mathfrak{D}_1$, $A'\neq A$. Then, on

$$N\setminus\overline{B(A,\epsilon_1)},$$

h_1 takes the value 0 and g_1 takes the value y'. Thus

$$g_1(x)=\phi_A^1 \circ h_1(x),$$

both on $\overline{B(A,\epsilon_1)}$ and on $N\backslash\overline{B(A,\epsilon_1)}$ and so on N. Hence g_1 is again continuous at x_0.

In particular, g_1 is a continuous function on X, with

$$g_1(x)=g_0(x) \text{ on } E_1=X\backslash \bigcup\{B(A,\epsilon_1):A\in\mathfrak{D}_1\}$$

and

$$g_1(x)=y_A \quad \text{on } \overline{B(A,\epsilon_1/2)} \text{ for each } A\in\mathfrak{D}_1.$$

Now suppose that for some n, $1\leq n<p$, we have defined continuous functions $g_1,g_2,\ldots,g_n$ on X so that

$$g_n(x)=g_{n-1}(x) \text{ on } E_n=X\backslash \bigcup\{B(A,\epsilon_n):A\in\mathfrak{D}_n\}$$

and

$$g_n(x)=y_A \quad \text{on } \overline{B(A,\epsilon_n/2)} \text{ for each } A\in\mathfrak{D}_n.$$

Write

$$E_{n+1}=X\backslash\bigcup\{B(A,\epsilon_{n+1}):\ A\in\mathfrak{D}_{n+1}\},$$

$$D_{n+1}=\bigcup\{\overline{B(A,\epsilon_{n+1}/2)}:A\in\mathfrak{D}_{n+1}\}.$$

Since the family of sets

$$\{B(A,\epsilon_{n+1}):A\in\mathfrak{D}_{n+1}\}$$

is discrete, the sets E_{n+1} and D_{n+1} are disjoint closed sets. Hence the function

$$h_{n+1}(x)=\frac{d(x,E_{n+1})}{d(x,D_{n+1})+d(x,E_{n+1})}$$

is a continuous function on X taking the value 1 on D_{n+1} and the value O on E_{n+1}.

Take

$$g_{n=1}(x)=g_n(x) \text{ on } E_{n+1},$$

$$g_{n+1}(x)=y_A \text{ on } \overline{B(A,\epsilon_{n+1}/2)} \text{ for each } A\in\mathfrak{D}_{n+1},$$

$$g_{n+1}(x)=\phi_A^{n+1}\circ h_{n+1} \text{ on } B(A,\epsilon_{n+1})\backslash\overline{B(A,\epsilon_{n+1}/2)} \text{ for each } A \text{ in } \mathfrak{D}_{n+1}.$$

Since we have assigned values to g_{n+1} on a family of disjoint sets with union X, the function g_{n+1} is well-defined. We verify that g_{n+1} is continuous on X. Let x_0 be any point of X. If x_0 belongs to none of the sets of the discrete family

$$\{\overline{B(A,\epsilon_{n+1})}: A\in\mathfrak{D}_{n+1}\}$$

of closed sets we can choose a neighbourhood N of x_0 that meets none of these sets. Then $g_{n+1}(x)=g_n(x)$ on N and so g_{n+1} is continuous at x_0. Suppose that

$$x_0\in\overline{B(A,\epsilon_{n+1})}$$

for some $A\in\mathfrak{D}_{n+1}$. Then

$$x_0\in B(C,\epsilon_n/2)$$

for just one C in $\mathfrak{D}_n$. Now we can take N to be a neighbourhood of x_0 contained in $B(C,\epsilon_n/2)$ that necessarily meets $\overline{B(A,\epsilon_{n+1})}$ but meets no set $\overline{B(A',\epsilon_{n+1})}$ with $A'\in\mathfrak{D}_{n+1}$, $A'\neq A$. Now on

$$N\backslash B(A,\epsilon_{n+1})\subset E_{n+1},$$

h_{n+1} takes the value 0 and $g_{n+1}(x)=g_n(x)=y_C$, since $N\subset B(C,\epsilon_n/2)$, and so

$$g_{n+1}(x)=\phi_A^{n+1}\circ h_{n+1}(x).$$

We also have

$$g_{n+1}(x)=\phi_A^{n+1}\circ h_{n+1}(x)$$

on

$$N\cap B(A,\epsilon_{n+1})\backslash\overline{B(A,\epsilon_{n+1}/2)}.$$

Further, on $\overline{B(A,\epsilon_{n+1}/2)}$, the function $h_{n+1}(x)$ takes the 1 and $g_{n+1}(x)$ takes the value y_A so that again

$$g_{n+1}(x)=\phi_A^{n+1}\circ h_{n+1}(x).$$

Thus $g_{n+1}(x)=\phi_A^{n+1}\circ h_{n+1}(x)$ on N and g_{n+1} is continuous at x_0. It follows, in particular, from these considerations that when $g_{n+1}(x) \neq g_n(x)$ we have

$$g_{n+1}(x) = \phi_A^{n+1} \circ h_{n+1}(x)$$

and

$$g_n(x) = y_C.$$

Since the arc in Y given by $\phi_A^{n+1}(t)$, $0 \leq t \leq 1$, is of diameter at most 2^{-n}, we have

$$d(g_{n+1}(x), g_n(x)) \leq 2^{-n},$$

for all x in X.

Proceeding inductively in this way we define continuous functions $g_0, g_1, \ldots, g_p = g$ on X satisfying

$$g_n(x) = y_A \text{ for } x \in A \in \mathfrak{D}_n,$$

for $1 \leq n \leq p$, and

$$d(g_n(x), g_{n+1}(x)) \leq 2^{-n},$$

for $x \in X$ and $1 \leq n < p$. Thus

$$d(g(x), g_n(x)) \leq 2^{-n} + 2^{-n-1} + \ldots + 2^{-p+1} \leq 2^{-n+1}$$

for $x \in X$ and $1 \leq n \leq p$. Now, if $x \in A \in \mathfrak{D}_n$, $1 \leq n \leq p$, we have

$$d(g(x), y_A) = d(g(x), g_n(x)) \leq 2^{-n+1}.$$

Since $y_A \in f(A)$ and any two points of $f(A)$ can be joined by an arc of diameter less than 2^{-n}, we have

$$d(f(x), y_A) \leq 2^{-n}.$$

Hence

$$d(f(x), g(x)) \leq 2^{-n+2}$$

as required. □

Theorem 1.

Let X be a metric space and let Y be an, arcwise connected and locally arcwise connected metric space. Then $\mathcal{B}o_1(X,Y)\cap\Sigma(X,Y)=\mathcal{B}a_1(X,Y)$.

Proof .

Every first Baire class function is σ-discrete and of first Borel class, so let $f\in\mathcal{B}o_1(X,Y)\cap\Sigma(X,Y)$. We shall then define a sequence of continuous functions, converging pointwise to f.

Using locally arcwise connectedness find, for each point $y\in Y$ and each n, an open neighbourhood $U_n(y)$ of y such that any two points of $U_n(y)$ can be joined with an arc of diameter at most 2^{-n}.

Since f is σ-discrete and of first Borel class, f has a σ-discrete closed base, which we will denote by $\mathcal{A}$. For each n define $\mathcal{A}_n=\{A\in\mathcal{A}:\exists y\in Y\ f(A)\subseteq U_n(y)\}$. Note that $\cup\mathcal{A}_n=X$ for each n.

For each n apply Lemma 1 to the family $\mathcal{A}_n$ to find families of closed sets $\mathcal{B}_n^p$ satisfying:

each member of $\mathcal{B}_n^p$ is contained in some member of $\{f^{-1}(U_n(y)):y\in Y\}$;

each member of $\mathcal{B}_n^p$ is contained in some member of $\mathcal{B}_n^{p+1}$;

$\mathcal{B}_n^p$ is uniformly discrete for all p; and

$\bigcup_p\mathcal{B}_n^p$ is a cover of X (for all n).

We shall use these families to construct a new set of families of closed sets by defining, for all p and n,

$$\mathfrak{D}_n^p=\{A_1\cap A_2\cap\ldots\cap A_n : A_m\in\mathfrak{B}_m^p \text{ for all } m\leq n\}.$$

Note that $\mathfrak{D}_n^p$ is uniformly discrete, that each member of $\mathfrak{D}_{n+1}^p$ is contained in some member of $\mathfrak{D}_n^p$ and that $\bigcup_p \mathfrak{D}_n^p$ covers X for all n. For each p apply Lemma 2 to f and the families $\mathfrak{D}_n^p$, $n\leq p$, to yield continuous functions $g_p:X\to Y$ such that, when $x\in \bigcup_{n\leq p}\mathfrak{D}_n^p$ and $p\geq n$, then $d(f(x),g_p(x))\leq 2^{-n+2}$.

The sequence $\{g_p\}$ converges pointwise to f. To see this let $\epsilon>0$ and $x\in X$ be given. Find n such that $2^{-n+2}\leq\epsilon$ and find $p\geq n$ such that $x\in\bigcup\mathfrak{D}_n^p$. Then $d(f(x),g_q(x))\leq\epsilon$ for all $q\geq p$. □

We shall now show that the converse of Theorem 1 is true when X is the unit interval I and Y is complete. This will be done through a series of lemmas.

Lemma 3.

Assume $\mathfrak{B}a_1(I,Y)=\mathfrak{B}o_1(I,Y)$ where Y is a metric space. Then for all $\epsilon>0$ and all $y\in Y$ there exists an open neighbourhood U(y) of y such that, for all nonempty open subsets U_0,U_1 of U(y), there exists $\phi\in\mathfrak{B}a_0(I,Y)$ with $\phi(i)\in U_i$ for $i=0,1$ and $\operatorname{diam}\phi(I)\leq\epsilon$.

Proof.

Assume the statement is not true. Then we can find $\epsilon>0$ and $y\in Y$ such that, in all the open balls $B(y,2^{-n}\epsilon)$, there exist two nonempty open sets, U(n,0) and U(n,1), with the property that all arcs, $\phi\in\mathfrak{B}a_0(I,Y)$, with $\phi(i)\in U(n,i)$ for $i=0,1$, satisfies

$\operatorname{diam}\phi(I)>\epsilon$. For each n and i pick a point $y(n,i)\in U(n,i)$

Let $\alpha:\mathbb{N}\times\{0,1\}\to\mathbb{Q}$ be a 1-1 map such that for any open interval $(t_0,t_1)\subset I$ there exists n with $t_0<\alpha(n,0)<\alpha(n,1)<t_1$. Define the function $f:I\to Y$ by

$$f(t)=\begin{cases} y, \text{ if } t\in I\setminus\alpha(\mathbb{N}\times\{0,1\}), \\ y(n,i), \text{ if } t=\alpha(n,i),\ (n,i)\in\mathbb{N}\times\{0,1\}. \end{cases}$$

Note that f(I) is contained in $B(y,\epsilon/2)$. Since the $y(n,i)$ converge to y, the function f is of the first Borel class, and so, by assumption, there exists a sequence, $\{\phi_k\}\subset\mathcal{B}a_0(I,Y)$, converging pointwise to f.

Define for each m the set $H_m=\bigcup_{k\geq m}\phi_k^{-1}\left(Y\setminus\overline{B(y,\epsilon/2)}\right)$. Then each H_m is open and dense. To see this let $0\leq t_0<t_1\leq 1$ and find n such that $t_0<\alpha(n,0)<\alpha(n,1)<t_1$. The sequence $\{\phi_k\}$ converges pointwise to f, so we can find $k\geq m$ so big that $\phi_k(\alpha(n,i))\in U(n,i)$ for $i=0,1$. By the definition of $U(n,i)$, $i=0,1$, we have $\operatorname{diam}\phi_k(\alpha(n,0),\alpha(n,1))>\epsilon$. So there exists $t_2\in(\alpha(n,0),\alpha(n,1))$ such that $\phi_k(t_2)\in Y\setminus\overline{B(y,\epsilon/2)}$.

Using the Baire category theorem we get $\bigcap_m H_m\neq\emptyset$, which is a contradiction with $\bigcap_m H_m\subseteq f^{-1}(Y\setminus B(y,\epsilon/2))=\emptyset$. □

Lemma 4.

<u>Let</u> Y <u>be complete. Then</u> Y <u>is locally arcwise connected, if and only if, for all</u> $\epsilon>0$ <u>and for all</u> $y\in Y$ <u>there exists an open neighbourhood</u> U <u>of</u> y, <u>such that for all nonempty open sets</u> $U_0,U_1\subset U$ <u>there exists an arc of diameter less than</u> ϵ, <u>starting in</u> U_0 <u>and ending in</u> U_1.

proof.

The first implication is trivial. The second implication of the lemma implies that Y is locally connected. To see this let $\epsilon>0$, $y\in Y$ and let U be an open neighbourhood U of y, such that any two nonempty open sets U_0, U_1 contained in U can be joined with an arc of diameter less than $\epsilon/3$ starting in U_0 and ending in U_1. Let $V\subset Y$ be the set obtained from U by adding to U all arcs intersecting U having diameter less than $\epsilon/3$. We prove that V is connected. Indeed, if $V=G_0\cup G_1$, where G_0 and G_1 are non-empty, disjoint relatively open subsets of V, then each arc in V lies in either G_0 or in G_1. Since each point of $V\setminus U$ lies, by definition, on an arc in V that intersects U, it follows that $U\cap G_0\neq\emptyset$ and that $U\cap G_1\neq\emptyset$. Since $U\cap G_0$ and $U\cap G_1$ are open sets, we can find an arc of diameter less than $\epsilon/3$ joining $U\cap G_0$ and $U\cap G_1$. By definition this arc lies in V, which is impossible since it intersects both G_0 and G_1. Finally we observe that $\mathrm{diam}(U)\leq\epsilon/3$, hence $\mathrm{diam}(V)\leq 2\epsilon/3<\epsilon$, which shows that V is a connected neighbourhood of y of diameter less than ϵ.

Now [7], Theorem 1, p.254 shows that Y is locally arcwise connected. □

Lemma 5.

Let Y be complete and assume that $\mathcal{B}a_1(I,Y)=\mathcal{B}o_1(I,Y)$. Then Y is arcwise connected.

Proof.

Let $y_0,y_1\in Y$. We shall define an arc joining these two points. Use Lemma 3 and Lemma 4 to find open neighbourhoods U_0,U_1 of y_0 and y_1, respectively, such that any pair of points of U_0 can be joined with an arc, and any pair of points of U_1 can be joined with an arc..

The function $f(t)=\begin{cases} y_0, & \text{if } t=0, \\ y_1, & \text{if } t>0, \end{cases}$ from I to Y is of first Borel class. By assumption f is then of first Baire class, so there exists an arc $\phi \in \mathcal{B}a_0(I,Y)$ with $\phi(i) \in U_i$ for $i=0,1$. Join y_0 to $\phi(0)$ and $\phi(1)$ to y_1 with arcs. Then these three arcs together join y_0 and y_1. □

Theorem 2.

<u>Let</u> Y <u>be</u> <u>complete</u> <u>metric.</u> <u>Then</u> <u>the</u> <u>following</u> <u>three</u> <u>statements</u> <u>are</u> <u>equivalent:</u>

(i) Y <u>is</u> <u>connected</u> <u>and</u> <u>locally</u> <u>connected;</u>

(ii) $\mathcal{B}a_1(I,Y)=\mathcal{B}o_1(I,Y)$; <u>and</u>

(iii) $\mathcal{B}a_1(X,Y)=\mathcal{B}o_1(X,Y)\cap\Sigma(X,Y)$ <u>for</u> <u>all</u> <u>metric</u> <u>spaces</u> X.

Remark.

The implication: Y is arcwise connected and locally arcwise connected ⇒ (iii), is the statement of Theorem 1 which does not assume that Y is complete.

Proof.

Theorem 1, p.254 in [7] shows that (i) implies that Y is locally arcwise connected. This again implies that Y is arcwise connected. To see this let $y \in Y$ and let U be the union of all arcs going through y. Since Y is locally arcwise connected U is a non-empty clopen set. Since Y is connected we conclude that U equals Y, and hence that Y is arcwise connected. Now, to obtain the conclusion of the theorem, combine Lemma 3, Lemma 4, Lemma 5 and Theorem 1 and note that every function from I to Y is σ-discrete. □

Note that the proofs of Lemma 3 and Lemma 5 still work when I is replaced by a metric space X containing a homeomorphic copy of I. Therefore we immediately get the following corollary.

Corollary.

Let X be a metric space that contains a homeomorphic copy of I. Let Y be a complete metric space. Then $\mathfrak{B}a_1(X,Y)=\mathfrak{B}o_1(X,Y)\cap\Sigma(X,Y)$, if and only if, Y is locally connected and connected. □

Remark. The remark after Theorem 2 applies here as well.

Next we shall look at some examples. The first will show that Lemma 4 and Lemma 5 fail to be true in general when Y is not complete, and that it is not enough to assume that Y is a K_σ-set in a complete separable metric space, to obtain the conclusions of the lemmas.

We will be working in the Hilbert cube $I^{\mathbb{N}}$, equipped with the complete metric $d((t_i),(s_i))=\sum_i 2^{-i}|t_i-s_i|$, and we will let π_i be the projection on the i'th coordinate.

Example 1.

We shall define a K_σ-set $Y \subset I^{\mathbb{N}}$ such that:

(i) $\mathcal{B}o_1(X,Y)=\mathcal{B}a_1(X,Y)$ for all metric spaces X; and

(ii) Y is not locally arcwise connected nor arcwise connected.

For all p let $Y_p=\{(t_i)\in I^{\mathbb{N}}: t_p=1,\ t_i=0\ \forall i>p\}$. Then $\{Y_p: p\in\mathbb{N}\}$ is a countable and disjoint collection of compact sets. Define $Y=\bigcup_p Y_p$.

(i) Let $f\in\mathcal{B}o_1(X,Y)$. Then $\pi_k\circ f\in\mathcal{B}o_1(X,I)$ for each k, since the projection is continuous. By Theorem 1 there exist sequences of continuous functions $\{g_k^m\}\subseteq\mathcal{B}a_0(X,I)$ converging pointwise to $\pi_k\circ f$ for each k.

Now $g^m=(g_1^m, g_2^m, \dots, g_m^m, 1, 0, 0, \dots)$ is a continous function from X to Y_{m+1} and $\{g^m\}$ converges pointwise to f.

(ii) Let $\phi: I\to Y$ be an arc. Then $\phi^{-1}(Y_p)\neq\emptyset$ for exactly one p for otherwise we could write I as a disjoint union of countably many but at least two nonempty closed sets, which is impossible by the Sierpinski Theorem in [11], p440.

This shows that Y is not arcwise connected. In fact it also shows that Y is not locally arcwise connected, for any open nonempty subset of Y intersects infinitely many of the sets $\{Y_p: p\in\mathbb{N}\}$ and, as we have just seen, a pair of points that do not lie in one of these sets can not be joined with an arc. □

If a metric space Y satisfies $\mathcal{B}a_1(I,Y)=\mathcal{B}o_1(I,Y)$, then Y must have the following two properties which we call P^1 and P^2.

P^1: For all finite sequences of nonempty open sets $U_1,U_2,\dots,U_n\subset Y$ there exists an arc $\phi \in \mathcal{B}a_0(I,Y)$ such that $\phi^{-1}(U_i) \neq \emptyset$ for all $i \leq n$.

P^2: For all $\epsilon > 0$ and all $y \in Y$ there exists an open neighbourhood U of y such that, for all finite sequences of nonempty open sets $U_1, U_2, \ldots, U_n \subset U$, there exists an arc $\phi \in \mathcal{B}a_0(I,Y)$ such that $\phi^{-1}(U_i) \neq \emptyset$ for all $i \leq n$ and such that $\operatorname{diam}\phi(I) \leq \epsilon$.

That P^1 must hold can be shown with an argument similar to that in the proof of Lemma 5, and that P^2 must hold can be shown along the line of the proof of Lemma 3.

Conversely, if a complete space Y satisfied P^1 and P^2 then by Lemma 4 it would be arcwise connected and locally arcwise connected. If Y were also separable, then we would have $\mathcal{B}a_1(X,Y) = \mathcal{B}o_1(X,Y)$ for all metric spaces X. The next example shows that this conclusion fails when Y is not complete. Indeed, it shows that the conclusion fails for a space that satisfies P^1 and P^2 and is a K_σ-set in a complete separable metric space.

Example 2.

<u>There exists a K_σ-subset Y of $I^{\mathbb{N}}$ satisfying P^1 and P^2 but where</u> $\mathcal{B}a_1(I,Y) \neq \mathcal{B}o_1(I,Y)$.

We shall construct the set Y. Consider the subsets of $I^{\mathbb{N}}$

$$Y_p^1 = \{(t_i) \in I^{\mathbb{N}} : t_p = 1,\ t_i = 0\ \ \forall i > p\}$$

$$Y_p^2 = \{(t_i) \in I^{\mathbb{N}} : t_p = 1/2,\ t_i = 0\ \ \forall i > p\}.$$

These sets are disjoint. To see this assume that $Y_{p_1}^{j_1} \cap Y_{p_2}^{j_2}$ is non-empty. We cannot have $p_1 \neq p_2$ because of the restriction on the last non-zero coordinate. For the same reason we cannot have $j_1 \neq j_2$.

The space $I^{\mathbb{N}}$ has a countable base of sets of the form

$$O_1 \times O_2 \times \ldots \times O_k \times I \times I \times \ldots$$

with $O_1, O_2, \ldots, O_k$ non-empty open subsets of I, so the set of all finite sequences of sets from this base is countable. We denote it by

$$\{(U^i_1,\ U^i_2,\ \ldots,\ U^i_{\sigma(i)}\ ) : i \in \mathbb{N}\}.$$

Note that the function $\sigma : \mathbb{N} \to \mathbb{N}$ determines the length of each of the finite sequences.

For each i there exists a q(i) such that $\pi_q(U^i_j) = I$ for all $q \geq q(i)$ and all $j \leq \sigma(i)$. Let $p(1) = \max\{q(1), 3\}$ and define inductively $p(i+1) = \max\{q(i+1), (p(i)+1)\}$. Then $\{p(i)\}$ is a strictly increasing sequence of integers, all bigger than 3, and such that $U^i_j \cap Y^1_{p(i)} \neq \emptyset$ and $U^i_j \cap Y^2_{p(i)} \neq \emptyset$ for all i and $j \leq \sigma(i)$. To see this let us fix i and j and write $U^i_j = V \times I^{\mathbb{N}}$ where V is an open subset of $I^{p(i)-1}$. Let $(t_1, t_2, \ldots, t_{p(i)-1}) \in V$. Then $(t_1, t_2, \ldots, t_{p(i)-1}, 1, 0, 0, \ldots) \in U^i_j \cap Y^1_{p(i)}$ and $(t_1, t_2, \ldots, t_{p(i)-1}, \frac{1}{2}, 0, 0, \ldots) \in U^i_j \cap Y^2_{p(i)}$.

For each i and $j \leq \sigma(i)$ pick a point $u^i_j \in U^i_j \cap Y^1_{p(i)}$. For all i let $y^i = (0,1,0,0,\ldots,0,1,0,0,\ldots)$, where the second 1 is on the p(i)'-th coordinate. Note that the distance between y^i and $y = (0,1,0,0,\ldots)$ is $2^{-p(i)}$, and that $2^{-p(i)} \leq 2^{-3}$ for all i.

For each i we will define a continuous piecewise linear function $\phi^1_i : I \to Y^1_{p(i)}$ by

$$\phi^1_i(t) = \begin{cases} y^i, \text{ if } t \leq \frac{1}{2\sigma(i)}, & \\ y^i + (t - \frac{j}{\sigma(i)} + \frac{1}{2\sigma(i)}) 2\sigma(i)(u^i_j - y^i), \text{ if } \frac{j}{\sigma(i)} - \frac{1}{2\sigma(i)} < t \leq \frac{j}{\sigma(i)} & , j \leq \sigma(i)-1. \\ u^i_j + (t - \frac{j}{\sigma(i)}) 2\sigma(i)(y^i - u^i_j), \text{ if } \frac{j}{\sigma(i)} < t \leq \frac{j}{\sigma(i)} + \frac{1}{2\sigma(i)}. & \end{cases}$$

Note that $Y^1_{p(i)}$ is convex and that $\phi^1_i(I)$ is the union of $\sigma(i)$ straight-line segments joining y^i to the points u^i_j.

Let $N = \{i \in \mathbb{N} : \operatorname{diam}\left(\bigcup_{j \leq \sigma(i)} U^i_j\right) \leq 1/2\}$. For each $i \in N$ let $\phi^2_i : I \to Y^2_{p(i)}$ be a piecewise linear continuous function satisfying:

$$\phi_i^2(I)\cap U_j^i\neq\varnothing \quad \forall j\leq\sigma(i); \text{ and}$$

$$\operatorname{diam}\phi_i^2(I)\leq\operatorname{diam}\Big(\bigcup_{j\leq\sigma(i)} U_j^i\Big).$$

We can construct these functions by the method used to construct the functions ϕ_i^1.

The space providing the example is defined by

$$Y=Y_2^1 \cup Y_2^2 \cup \bigcup_{i\in\mathbb{N}} \phi_i^1(I) \cup \bigcup_{i\in N} \phi_i^2(I).$$

Note the following points:

Y is a K_σ-set;

any arc $\psi(I)\subset Y$ must be contained in one of the disjoint sets of which Y is the union;

the subsets $\{\phi_i^1(I)\}$ of Y ensures that Y satisfies P^1; and

the subsets $\{\phi_i^2(I): i\in N\}$ of Y ensures that Y satisfies P^2.

Let us assume that $\mathcal{B}a_1(I,Y)=\mathcal{B}o_1(I,Y)$ and seek a contradiction to prove that Y is indeed the example. Define the first Borel class function $f:I\to Y$ by

$$f(t)=\begin{cases}(t,\frac{1}{2},0,0,\ldots), & \text{if } t<1,\\ (1,1,0,0,\ldots), & \text{if } t=1.\end{cases}$$

Then, by assumption, we can find a sequence of functions, $\{\psi_n\}\subset\mathcal{B}a_0(I,Y)$, converging pointwise to f. When n is sufficiently large the points $\psi_n(0)$ and $\psi_n(1)$ are near $(t,\frac{1}{2},0,0,\ldots)$ and $(1,1,0,0,\ldots)$. We may assume that $\psi_n(I)\subseteq \bigcup_{i\in\mathbb{N}}\phi_i^1(I)$ for all n, for all arcs contained in one of the sets $\phi_i^2(I)$, $i\in\mathbb{N}$, would have too small a diameter. In fact $\psi_n(I)\subseteq\phi_i^n(I)$ for some $i=i(n)$.

Let $y=(0,1,0,0,\ldots)$, which is in Y_2^1, and for each m define the set $H_m=\bigcup_{n\geq m}\psi_n^{-1}(B(y,\frac{1}{4}))$. Then clearly H_m is open and $y^i\in B(y,\frac{1}{4})$ for all i. To see that

H_m is also dense in $[\frac{1}{2},1]$ let a and b be such that $\frac{1}{2}<a<b<1$. When ϵ is sufficiently small there is no line that intersects $B(f(a),\epsilon)$, $B(f(b),\epsilon)$ and $\{y\}\cup\bigcup_i\{y^i\}$. However, for large n we have $\psi_n(a)\in B(f(a),\epsilon)$ and $\psi_n(b)\in B(f(b),\epsilon)$. But $\psi_n(I)\subseteq\phi_i^1(I)$ for some i, so, since $\psi_n((a,b))$ cannot be contained in one of the linear segments of $\phi_i^1(I)$, we see that $\psi_n((a,b))$ cannot lie on just one of the line segments making up $\phi_i^1(I)$. Therefore $y^i\in\psi_n((a,b))$ and so $\psi_n((a,b))\cap B(y,\frac{1}{4}))\neq\emptyset$.

Hence each H_m is dense and open in $[\frac{1}{2},1]$ and so $\bigcap_m H_m\cap[\frac{1}{2},1]\neq\emptyset$ by the Baire category theorem, which is a contradiction with $\bigcap_m H_m\cap[\frac{1}{2},1]\subseteq f^{-1}(\overline{B(y,\frac{1}{4})})=\emptyset$. We conclude that $\mathcal{B}a_1(I,Y)\neq\mathcal{B}o_1(I,Y)$ as required. □

Example 3.

Consider the subset of $\mathbb{R}^2$

$$M=\{(x,\sin\tfrac{\pi}{x}):0<x\leq 1\}\cup\{0\}\times[-1,2]\cup[0,1]\times\{2\}\cup\{1\}\times[0,2]$$

The space M is metric, separable, compact, complete, arcwise connected and a continuous image of the real line. But M is not locally arcwise connected and so, by Theorem 2, we have $\mathcal{B}a_1(I,M)\neq\mathcal{B}o_1(I,M)$.

4. Second Baire class functions.

Lemma 6.

If $\bigcup_{\alpha<\omega_1} \mathcal{B}a_\alpha(X,Y) = \bigcup_{\alpha<\omega_1} \mathcal{B}o_\alpha(X,Y)$, then, for all $q\in\mathbb{N}$, for all sequences, $U_1,\ldots,U_q$, of nonempty open subsets of Y and for all sequences, $x_1,\ldots,x_q$, of distinct points of X, there exists a continuous function $\phi:X\to Y$ with $\phi(x_i)\in U_i$ for all $i\le q$.

Proof.

Let $U_1,\ldots,U_q$ be a sequence of nonempty open subsets of Y and $x_1,\ldots,x_q$ be a sequence of distinct points of X. Pick for each $i\le q$ a point $y_i\in U_i$. Define the σ-discrete first Borel class function

$$f(x)=\begin{cases} y_i, \text{ if } x=x_i,\ q\ge i>1, \\ y_1, \text{ otherwise.}\end{cases}$$

Then by assumption f is a Baire class α function, for some $\alpha<\omega_1$, and $f(x_i)\in U_i$ for all $i\le q$.

Observe that, whenever a non-continuous function g of Baire class β satisfies $g(x_i)\in U_i$ for all $i\le q$, then there exists $\eta<\beta$ and a function h of Baire class η, such that $h(x_i)\in U_i$ for all $i\le q$.

Apply this observation to f to yeild a function, f_1, of strictly lower Baire class than that of f, with $f_1(x_i)\in U_i$ for all $i\le q$. If f_1 is continuous, then f_1 satisfies the conclusion of the lemma. If f_1 is not continuous, we apply our observation to f_1 to yeild a function f_2 of strictly lower Baire class than that of f_1. We repeat this process until it halts. The process will halt, since a strictly decreasing sequence of ordinals can only be finite. The function we have when the process halts satisfies the conclusion of the lemma. □

Lemma 7.

If Y is separable, metric and X is metric such that for all $q \in \mathbb{N}$, for all sequences, $F_1,\ldots,F_q$, of disjoint closed subsets of X and for all sequences, $U_1,\ldots,U_q$, of nonempty open subsets of Y, there exists a continuous function $\phi: X \to Y$ with $\phi(F_i) \subseteq U_i$ for all $i \leq q$, then $\mathcal{B}a_2(X,Y) = \mathcal{B}o_2(X,Y)$.

Proof.

We will first show that all first Borel class functions from X to Y with finite range are of first Baire class. So let $f: X \to Y$ by a first Borel class function with finite range. Let $\{y_i : i \leq q\}$ be the finite range, and let, for each m, $\{F_{im} : i \leq q\}$ be a disjoint family of closed sets such that $f^{-1}(y_i) = \bigcup_m F_{im}$ for every $i \leq q$. By hypothesis we can find continuous functions, $\phi_n : I \to Y$, $n \in \mathbb{N}$, so that $\phi_n(\bigcup_{m \leq n} F_{im}) \subseteq B(y_i, 2^{-n})$ for all $i \leq q$. Then the sequence $\{\phi_n\}$ converges pointwise to f.

Now let $h: X \to Y$ be of second Borel class. We shall show that h is of second Baire class. In [1], pages 389-391, it is proved that all second class Borel functions from a metric space to a separable metric space are pointwise limits of first Borel class functions with finite range. By the argument above, all these first Borel class functions are of first Baire class. Thus h is of second Baire class. □

Theorem 3.

Let Y be a separable metric space. Let X be a metric space containing a homeomorphic copy of the unit interval. The following statements are equivalent:

(i) $\mathcal{B}a_2(I,Y)=\mathcal{B}o_2(I,Y)$;

(ii) For all $q\in\mathbb{N}$, for all sequences, $x_1,\dots,x_q$, of distinct points of I and for all sequences, $U_1,\dots,U_q$, of nonempty open subsets of Y, there exists a continuous function $\phi:I\to Y$ with $\phi(x_i)\in U_i$ for all $i\leq q$;

(iii) For all $q\in\mathbb{N}$, for all sequences, $F_1,\dots,F_q$, of disjoint closed subsets of X and for all sequences, $U_1,\dots,U_q$, of nonempty open subsets of Y, there exists a continuous function $\phi:X\to Y$ with $\phi(F_i)\subseteq U_i$ for all $i\leq q$;

(iv) $\mathcal{B}a_2(X,Y)=\mathcal{B}o_2(X,Y)$;

(v) For each finite (countable, infinite) ordinal $\alpha\geq 2$ we have $\mathcal{B}a_\alpha(X,Y)$ equal to $\mathcal{B}o_\alpha(X,Y)$ ($\mathcal{B}o_{\alpha+1}(X,Y)$); and

(vi) $\bigcup_{\alpha<\omega_1}\mathcal{B}a_\alpha(X,Y)=\bigcup_{\alpha<\omega_1}\mathcal{B}o_\alpha(X,Y)$.

Proof.

(i)$\Rightarrow$(ii) follows from the argument of Lemma 6.

(ii)$\Rightarrow$(iii). By Tietze's theorem let $f:X\to I$ be a continuous extension of the function defined by $f(x)=\frac{i}{q}\Leftrightarrow x\in F_i$. By (ii) we can find a continuous function $\phi:I\to Y$ with $\phi(\frac{i}{q})\in U_i$ for all $i\leq q$. The function $\phi$of satisfies the condition in (iii).

(iii)$\Rightarrow$(iv) is Lemma 7.

(iv)$\Rightarrow$(v) follows from the remark in section 1, or from Banach's theorem in [1], page 394.

(v)$\Rightarrow$(vi) is trivial.

(vi)$\Rightarrow$(i). By Lemma 6 we have that I satisfies (ii), since $I \subseteq X$. We have already noted that (ii) implies (iv) so I satisfies (iv), which is the statement of (i). □

Let X be a topological space and let Y be a metric space. Every σ-discrete second Borel class function from X to Y is, by [2], Theorem 5, the pointwise limit of a sequence of σ-discrete first Borel class functions, each of which have discrete range.

Let us assume that X and Y together has the property that, for every closed subset F of X, every continuous function with discrete range $g:F \to Y$ and every $\epsilon > 0$, there exists a continuous function $h:X \to Y$ such that $d(h(x),g(x)) \leq \epsilon$ for all $x \in F$.

Then, using an argument similar to that in Lemma 2, we can show that every σ-discrete first Borel class function with discrete range is of first Baire class. Indeed, let $g:X \to Y$ be a σ-discrete first Borel class function with discrete range. Let $\{y_\gamma : \gamma \in \Gamma\}$ be the discrete range and let, for each n, $\{F_{\gamma n} : \gamma \in \Gamma\}$ be a discrete family of closed sets such that $g^{-1}(y_\gamma) = \bigcup_n F_{\gamma n}$ for each γ. Define continuous functions, g_m by letting $g_m(x) = y_\gamma$, if and only if, $x \in \bigcup_{n \leq m} F_{\gamma n}$, and note that $\bigcup_{\gamma \in \Gamma} \bigcup_{n \leq m} F_{\gamma n}$ is closed for every m. Let $h_m:X \to Y$ be a continuous function such that $d(h_m(x),g_m(x)) \leq 2^{-m}$ for all $x \in \bigcup_{\gamma \in \Gamma} \bigcup_{n \leq m} F_{\gamma n}$. Then the sequence of functions $\{h_m\}$ converges pointwise to g, and hence g is of first Baire class. Thus $\mathcal{B}o_2(X,Y) \cap \Sigma(X,Y) = \mathcal{B}a_2(X,Y)$ and we have proved the following theorem.

Theorem 4.

Let X be a topological space and let Y be a metric space. Assume that for every closed subset F of X, every continuous function with discrete range $g:F\to Y$ and every $\epsilon>0$, there exists a continuous function $h:X\to Y$ such that $d(h(x),g(x))\leq\epsilon$ for all $x\in F$. Then $\mathcal{B}o_2(X,Y)\cap\Sigma(X,Y)=\mathcal{B}a_2(X,Y)$. □

5. References.

[1] Kuratowski, Topology vol.1., Academic Press, New York, 1966.

[2] R.W.Hansell, On Borel mappings and Baire functions, Trans. Amer. Math. Soc. 194 (1974) 195-211.

[3] R.W.Hansell, Extended Bochner Measurable Selectors, Math. Ann. 277 (1987), 79-94.

[4] S.Rolewicz, On inversion of non-linear transformations, Studia Math. 17 (1958) 79-83.

[5] C.A.Rogers, Functions of the first Baire class, J. London Math. Soc. (2) 37 (1988) 535-544.

[6] L.G.Brown, Baire functions and extreme points, Amer. Math. Monthly, 79 (1972), 1016-1018.

[7] Kuratowski, Topology, Vol. 2., Academic Press, New York, 1968.

[8] R.W.Hansell, Borel measurable mappings for nonseparable metric spaces, Trans.Amer.Math.Soc. 161 (1971) 145-169.

[9] W.G.Fleissner, An axiom for nonseparable Borel theory, Trans.Amer.Math.Soc. 251 (1979) 309-328.

[10] S.Banach, Oeuvres, Volume 1, 207-217.

[11] Engelkind, General Topology, Polish Scientific Publishers, 1977.

A Banach space with a Lipschitz Gateaux-smooth bump has w^*-fragmentable dual.

Introduction.

In [1] Namioka, Phelps and Preiss proved a theorem which stated that a Banach-space with a Gateaux-smooth norm is weak Asplund. Later, in [2], Ribarska showed that the idea of the proof in [1] could be used to obtain a stronger theorem, namely that the existence of a Gateaux-smooth norm implies that the dual is w^*-fragmentable. The w^*-fragmentability of the dual implies the weak Asplund property (see [8]). Then in [3] Deville remarked that the Namioka-Phelps-Preiss theorem still holds with the strictly weaker (see [6] and [7]) assumption of existence of a Lipschitz Gateaux-differentiable bump.

The aim of this paper is to use the technique of the Namioka-Phelps-Preiss theorem to prove that the existence of a Lipschitz Gateaux-differentiable bump implies that the dual is w^*-fragmentable. In fact, we shall assume a slightly weaker condition; let us state the theorem.

Theorem.

Let $(E,\|.\|)$ be a Banach-space such that there exists a Lipschitz bump-function, $\phi:E\to\mathbb{R}$, satisfying

$$\liminf_{t\to 0^+} t^{-1}[\ \phi(x+tu) + \phi(x-tu) - 2\phi(x)\ ] \geq 0,$$

for all x,u.

Then (E^*,w^*) is fragmentable. □

Before we proceed with the proof we shall state the necessary definitions and theorems. For further background the reader is referred to [5].

A bump-function is a real-valued function assuming only ~~positive~~ non-negative values and with bounded, non-empty support. One should note that we do not need the positivity assumption, since, if we have a bump which assumes both negative and positive values, we can take the positive part.

If X is a topological space and ρ is a metric on it we say that ρ fragments X if, for every $\epsilon>0$ and for every non-empty subset Y of X, there exists a non-empty relatively open subset U of Y whose ρ-diameter is at most ϵ. A topological space is said to be fragmentable if there exists a fragmenting metric on it.

A Banach space E is called weak Asplund if every convex function on E is Gateaux-differentiable at a dense G_δ-subset of its domain. In [8] we find the following theorem: If (E^*,w^*) is fragmentable then E is weak Asplund.

A well-ordered family $\mathcal{U} = \{\ U_\xi : 0<\xi<\xi_0\ \}$ of subsets of the topological space X is said to be a relatively open partitioning of X if $U_0=\emptyset$, U_ξ is a relatively open subset of $X \setminus (\bigcup_{\eta<\xi} U_\eta)$ for every ξ, $0<\xi<\xi_0$ and $X=\bigcup \mathcal{U}$.

A family $\mathcal{U}$ is said to be a σ-relatively open partitioning of X if $\mathcal{U}=\bigcup_n \mathcal{U}^n$, where $\mathcal{U}^n$, n=1,2,... are relatively open partitionings of X and $\mathcal{U}$ is a separating σ-relatively open partitioning of X if $\mathcal{U}$ separates points. By [4], Theorem 1.9 a topological space is fragmentable if and only if it admits a separating σ-relatively open partitioning.

To ease notation we shall define

$$\Delta f(x,u) = f(x+u) + f(x-u) - 2f(x)$$

for all f,x and u.

With all the artillery in position we can attack the proof. Let us begin with a

Lemma

The following two statements are equivalent.

(1) There exists a Lipschitz bump-function, $\phi:E\to\mathbb{R}$, satisfying, for all x,u,

$$\liminf_{t\to 0^+} t^{-1}[\ \phi(x+tu) + \phi(x-tu) - 2\phi(x)\ ] \geq 0.$$

(2) There exists a function $p_0:E\to\mathbb{R}$ having the following properties:

(i) $p_0(x) \geq 0$;

(ii) $p_0(\lambda x) = |\lambda| p_0(x)$;

(iii) $\exists$ c,C such that $B(0,c) \subset \{\ x : p_0(x)\leq 1\ \} \subset B(0,C)$;

(iv) p_0 is Lipschitz; and

(v) $\forall x,u\in E \quad \limsup_{t\to 0^+} t^{-1}[\![\ p_0(x+tu) + p_0(x-tu) - 2p_0(x)\ ]\!] \leq 0.$

Proof of lemma.

The implication (2)⇒(1) follows immediately upon defining

$$\phi(x) = (\,1 - p_0^2(x)\,)^+.$$

and $0 \in \operatorname{supp}\phi$

So let us assume that ϕ is a bump with $\emptyset \neq \operatorname{supp}\phi \subset B(0,C)$, Lipschitz-constant K, satisfying $\liminf_{t\to 0^+} t^{-1}\Delta\phi(x,tu) \geq 0,$ for all x,u. Following Leduc [9] we define

$$h(x) = \int \phi(tx)dt \text{ and } p_0(x) = h(x)^{-1}.$$

Now conditions (i), (ii) and (iii) are trivially satisfied. We shall prove (iv) and (v).

First note that we have $|x|/c \leq p_0(x) \leq |x|/C$ for all x. Given x and y we can assume that $0 \neq |x| \leq |y|$. If $c|y| \leq C|x|$ then

$$
\begin{aligned}
|\, p_0(x) - p_0(y)\,| &= |x|\,|\,p_0(x/|x|) - p_0(y/|x|)\,| \\
&\leq |x|\, p_0\!\left(\frac{x}{|x|}\right) p_0\!\left(\frac{y}{|x|}\right) 2\int \left|\,\phi\!\left(t\frac{x}{|x|}\right) - \phi\!\left(t\frac{y}{|x|}\right)\right| dt \\
&\leq |x| \,\frac{1}{c^2}\,\frac{|y|}{|x|}\, K^0 \left|\frac{x}{|x|} - \frac{y}{|x|}\right| C^2 \\
&\leq \frac{C^3}{c^3} K\, |\, x - y\, |.
\end{aligned}
$$

In case $c|y| \geq C|x|$ we have

$$
\begin{aligned}
0 \leq p_0(y) - p_0(x) &\leq |y|/c - |x|/C \leq |y-x|/c + |x|/c - |x|/C \\
&\leq |y-x|/c + (|y|-|x|)/C \leq (1/c + 1/C)|y-x|.
\end{aligned}
$$

This proves (iv). To prove (v) note that since ϕ is Lipschitz we get by Fatou's lemma that

$$\limsup\; t^{-1}\big[2h(x) - h(x+tu) - h(x-tu)\big] \leq 0.$$

The differentiability assertion now follows by the chainrule. □

Proof of theorem.

First apply the lemma to yield a function p_0 satisfying (i) to (v). For the sake of easy reference the conditions (i) to (iv) of the lemma will be called condition (α). Note that (v) is not part of condition (α).

Let B^* be the dual unit ball. Since $E^* = \bigcup_m mB^*$ it suffices to show that B^* is a fragmentable space in the weak star topology. To do this it suffices to produce a separating σ-relatively open partitioning of (B^*,w^*).

Given a function p satisfying condition (α) we let $\tilde{p}$ be the norm, equivalent to the original norm, having unit ball $\overline{\text{conv}}\{\ x : p(x)\leq 1\ \}$. Note that $\tilde{p}\leq p$.

We shall use the following basic construction.

Let U be contained in B^* and let p satisfy condition (α). Given ϵ and β we construct a relatively open partitioning

$$\mathfrak{B} = \{\ V_\xi : 0\leq\xi<\xi_0\ \}$$

of (U,w^*) and associate to every element V_ξ of $\mathfrak{B}$ a non-negative real s_ξ, an element $e_\xi \in E$ with $p(e_\xi)=1$ and a new function p_ξ, satisfying condition (α), in the following way.

Let $V_0=\emptyset$, $s_0=0$ and e_0,p_0 be arbitrary. Assume we have constructed V_η, s_η, e_η and p_η for all $\eta<\xi$ and consider $R_\xi = U\setminus(\bigcup_{\eta<\xi} V_\eta)$.

If $R_\xi=\emptyset$ we put $\xi_0=\xi$ and stop the process.

If not, we put

$$s_\xi=\sup\{\ \langle x^*,x\rangle : x^*\in R_\xi,\ p(x)\leq 1\ \}.$$

If $s_\xi=0$ it means that $R_\xi=\{0\}$, by condition (α). Then put $V_\xi=\{0\}$, $\xi_0=\xi+1$, let e_ξ and p_ξ be arbitrary and stop.

If $s_\xi>0$ then there exists $e_\xi\in E$ with $p(e_\xi)=1$ such that

$$V_\xi = \{ x^*\in R_\xi : \langle x^*,e_\xi\rangle > (1-\epsilon)s_\xi \}\neq\emptyset.$$

Moreover, V_ξ is w^*-relatively open in R_ξ. We put

$$q_\xi(x) = \inf\{ p_0(x-\lambda e_\xi) : \lambda\in\mathbb{R} \},$$

for all $x\in E$, and

$$p_\xi^2 = p^2+ \beta^2 q_\xi^2.$$

Note that p_ξ satisfies condition (α) and that the process stops.

Thus the relatively open partitioning $\mathfrak{B}$ of U is constructed.

We shall now construct the σ-relatively open partitioning $\mathcal{U} = \bigcup_n \mathcal{U}^n$ of (B^*,w^*). Fix two sequences of positive real numbers $\frac{1}{4}>\epsilon_1>\epsilon_2>\ldots$ and $\beta_1>\beta_2>\ldots$ such that

$$\sum\beta_k^2<3,\ \sum\frac{\sqrt{\epsilon_k}}{\beta_k}<\infty \text{ and } \sum\epsilon_k<\infty.$$

Applying the basic construction with $U=B^*$, $p=p_0$, $\epsilon=\epsilon_1$ and $\beta=\beta_1$ we obtain a relatively open partitioning $\mathcal{U}^1=\{ U_\xi^1 : 0\leq\xi<\xi_1 \}$ of (B^*,w^*) and for all $\xi<\xi_1$ we get $s_\xi^1\geq0$, $e_\xi^1\in E$ and p_ξ^1 satisfying condition (α) with $p_\xi^1(e_\xi^1)=1$.

If we have constructed relatively open partitionings $\mathcal{U}^1,\mathcal{U}^2,\ldots,\mathcal{U}^n$ we consider $U_\xi^n \in \mathcal{U}^n=\{ U_\xi^n : 0\leq\xi<\xi_n \}$, and apply the basic construction to $U=U_\xi^n$ with $p=p_\xi^n$, $\epsilon=\epsilon_{n+1}$ and $\beta=\beta_{n+1}$.

We then get a relatively open partitioning $\mathcal{U}_\xi^n = \{ U_{\xi\eta}^n : 0\leq\eta<\eta_\xi^n \}$ of U_ξ^n to every element of which a function $p^n_{\xi\eta}$ satisfying condition (α) and a point $e^n_{\xi\eta}$ with $p^n_{\xi\eta}(e^n_{\xi\eta})=1$ are associated.

Now $\mathcal{U}^{n+1} = \cup\{ \mathcal{U}^n_\xi : 0\leq\xi<\xi_n \}$ is a relatively open partitioning of B^* if its elements $U^n_{\xi\eta}$ are ordered alphabetically (see [4], prop. 1.5) and $\mathcal{U}^{n+1}$ is a refinement of $\mathcal{U}^n$.

We shall now see that $\mathcal{U}$ is separating.

Let us assume the contrary, i.e. there exist two points $x_1^*,x_2^*\in B^*$ such that they belong to the same element V_n of $\mathcal{U}_n$ for all n. Let s_n, p_n and e_n be the associated reals, functions and points, respectively, from the construction. By assumption $s_n>0$ for all n.

The construction immediately gives us that the following properties hold:

(i) $B^* = V_0 \supset V_1 \supset V_2 \supset\ldots\supset \cap V_n \supset \{x_1^*,x_2^*\}$;

(ii) $s_n = \sup\{ \langle x^*,x\rangle : x^*\in V_n,\ p_{n-1}(x)\leq 1 \} > 0$;

(iii) $V_n \subset \{ x^*\in V_{n-1} : \langle x^*,e_n\rangle > (1-\epsilon_n)s_n \}$ for all n;

(iv) $p_{n-1}(e_n)=1$ for all n; and

(v) $p_n^2 = p_{n-1}^2 + \beta_n^2 q_n^2$, where $q_n(x) = \inf\{ p_0(x-\lambda e_n) : \lambda\in\mathbb{R} \}$.

We will derive the following facts.

(a) $\{p_n\}$ is an increasing sequence of functions, all satisfying condition (α), converging uniformly on bounded sets to a function p_∞ satisfying condition (α) and $p_0 \le p_\infty \le 2p_0$.

To see this note that

$$p_0^2 \le p_n^2 = p_0^2 + \sum_{k=1}^{n} \beta_k^2 q_k^2 \le (1+ \sum_{k=1}^{n} \beta_k^2) p_0^2 \le 4p_0^2,$$

so the sequence converges and $p_0 \le p_\infty \le 2p_0$. Hence, if $\|x\| \le K$, then

$$p_\infty^2(x) - \sum_1^n \beta_k^2 q_k^2(x) - p_0(x) = \sum_{n+1}^{\infty} \beta_k^2 q_k^2(x) \le \sum_{n+1}^{\infty} \beta_k^2 p_0^2(x) \le \sum_{n+1}^{\infty} \beta_k^2 \frac{K^2}{c^2} \to 0.$$

Thus the sequence p_n^2 converges uniformly on bounded sets, hence so does p_n.

(b) $s_n \ge s_{n+1} \to s_\infty \ge 0$.

Using $V_{n+1} \subset V_n$ and $p_n \ge p_{n-1}$ we have

$$0 < s_{n+1} = \sup\{ <x^*,x> : x^* \in V_{n+1}, p_n(x) \le 1 \}$$
$$\le \sup\{ <x^*,x> : x^* \in V_n, p_{n-1}(x) \le 1 \} = s_n .$$

(c) $e_n \to e_\infty$ with $p_\infty(e_\infty)=1$.

To prove this we will show that

$$\tilde{p}_0(e_{n+1} - e_n) \le 10\frac{\sqrt{\epsilon_n}}{\beta_n} + 4\epsilon_n$$

for n sufficiently large. The conclusion then follows since $\frac{1}{C}\|\cdot\| \le \tilde{p}_0$, $\sum \frac{\sqrt{\epsilon_n}}{\beta_n} < \infty$ and $\sum \epsilon_n < \infty$.

Let n be a positive integer. We have $p_{n-1}(e_n)=1$ and $q_n(e_n)=0$ so $p_n(e_n)=1$. Let us estimate $p_{n-1}(e_{n+1})$.

$$s_n p_{n-1}(e_{n+1}) \ge \sup\{ <x_1^*,x> : p_{n-1}(x) \le 1\} p_{n-1}(e_{n+1}) \ge <x_1^*,e_{n+1}> > (1-\epsilon_{n+1})s_{n+1},$$

by positive homogeneity. On the other hand

$$(1-\epsilon_n)s_n < <x_1^*,e_n> \le \sup\{ <x_1^*,x> : p_n(x) \le 1 \} p_n(e_n) \le s_{n+1}.$$

Therefore

$$p_{n-1}(e_{n+1}) > (1-\epsilon_{n+1})\frac{s_{n+1}}{s_n} > (1-\epsilon_{n+1})(1-\epsilon_n) > (1-2\epsilon_n) > 0.$$

We can now estimate

$$q_n^2(e_{n+1}) = \frac{p_n^2(e_{n+1}) - p_{n-1}^2(e_{n+1})}{\beta_n^2} = \frac{1 - p_{n-1}^2(e_{n+1})}{\beta_n^2}$$

$$< \frac{4\epsilon_n(1-\epsilon_n)}{\beta_n^2} < 4\frac{\epsilon_n}{\beta_n^2}$$

or

$$q_n(e_{n+1}) < 2\frac{\sqrt{\epsilon_n}}{\beta_n} .$$

We can express e_{n+1} as $e_{n+1}=\lambda_n e_n+u_n$, where $\lambda_n \in \mathbb{R}$ and $u_n \in E$ satisfies

$$p_0(u_n) = q_n(e_{n+1}) < 2\frac{\sqrt{\epsilon_n}}{\beta_n} ,$$

so

$$p_n(u_n) \leq 2p_0(u_n) < 4\frac{\sqrt{\epsilon_n}}{\beta_n} .$$

Note that for all e such that $p_n(e) \leq 1$ we have $\langle x_1^*, e\rangle \leq s_{n+1}$, so $\langle x_1^*, e\rangle \leq s_{n+1}$ for all e with $\tilde{p}_n(e) \leq 1$, hence $\tilde{p}_n^*(x_1^*) \leq s_{n+1} \leq s_n$.

We now have

$$|\lambda_n - 1| = \frac{|\langle x_1^*, e_{n+1} - u_n - e_n\rangle|}{\langle x_1^*, e_n\rangle} \leq \frac{|\langle x_1^*, e_{n+1} - e_n\rangle| + |\langle x_1^*, u_n\rangle|}{\langle x_1^*, e_n\rangle}$$

$$\leq \frac{2\epsilon_n s_n + 4\frac{\sqrt{\epsilon_n}}{\beta_n}s_n}{(1-2\epsilon_n)s_n} \leq 4\epsilon_n + 8\frac{\sqrt{\epsilon_n}}{\beta_n}$$

Consequently, using that $\tilde{p}_0(e_n) \leq p_{n-1}(e_n) = 1$, we have

$$\tilde{p}_0(e_{n+1} - e_n) = \tilde{p}_0((\lambda_n - 1)e_n + u_n) \leq |\lambda_n - 1|\tilde{p}_0(e_n) + \tilde{p}_0(u_n)$$

$$\leq 4\epsilon_n + 10\frac{\sqrt{\epsilon_n}}{\beta_n}$$

so $\{e_n\}$ converges to e_∞.

The fact that $p_\infty(e_\infty)=1$ is immediate from $p_n \to p_\infty$ uniformly on bounded sets, $e_n \to e_\infty$ and $p_{n-1}(e_n)=1$. This establishes (c)

For all n we have $(1-\epsilon_n)s_n < \langle x_1^*, e_n \rangle < \tilde{p}_n^*(x_1^*)\tilde{p}_n(e_n) \leq s_{n+1}$. Letting $n \to \infty$ we observe that $s_\infty = \langle x_1^*, e_\infty \rangle = \langle x_2^*, e_\infty \rangle$.

Note that $\tilde{p}_\infty$ is an equivalent norm with $\tilde{p}_\infty \leq p_\infty$.

For all x with $p_\infty(x) \leq 1$ we have $\langle x_1^*, x \rangle \lesssim s_n$ for all n, so $\langle x_1^*, x \rangle \leq s_\infty$ whenever $\tilde{p}_\infty(x) \leq 1$. It follows that $\tilde{p}_\infty^*(x_1^*) \leq s_\infty$.

Now $\tilde{p}_\infty(e_\infty) \leq p_\infty(e_\infty) = 1$, so

$$s_\infty = \langle x_1^*, e_\infty \rangle \leq \tilde{p}_\infty^*(x_1^*) \leq s_\infty,$$

$$\tilde{p}_\infty^*(x_1^*) = s_\infty \quad \text{and} \quad \tilde{p}_\infty(e_\infty) = 1.$$

Hence x_1^* and x_2^* belong to the subdifferential of $\tilde{p}_\infty$ at e_∞. We must now show that $\tilde{p}_\infty$ is Gateaux-differentiable at e_∞, since this implies that $x_1^* = x_2^*$, a contradiction.

Let K be the Lipschitz constant of p_0, and note that we have

$$\begin{aligned} &\Delta p_0^2(x,tu) \\ &= [\, p_0(x+tu) + p_0(x-tu)\,]\Delta p_0(x,tu) + [\, p_0(x+tu) - p_0(x-tu)\,]^2 \\ &\quad - [\, p_0(x) - p_0(x+tu)\,]^2 - [\, p_0(x) - p_0(x-tu)\,]^2 \\ &\leq [\, p_0(x+tu) + p_0(x-tu)\,]\Delta p_0(x,tu) + 4K^2t^2|u|^2. \end{aligned}$$

Let $\{\gamma_k\}$ be reals such that $q_k(e_\infty) = p_0(e_\infty - \gamma_k e_k)$ for all k, and note that

$$q_k^2(e_\infty+tu) + q_k^2(e_\infty-tu) - 2q_k^2(e_\infty) \leq p_k^2(e_\infty-\gamma_k e_k+tu) + p_k^2(e_\infty-\gamma_k e_k-tu) - 2p_k^2(e_\infty)$$

for all t and u.

By convexity we get

$$0 \leq \limsup t^{-1}\Delta\tilde{p}_\infty^2(e_\infty,tu)$$

$$\leq \limsup t^{-1}\Delta p_\infty^2(e_\infty,tu)$$

$$= \limsup t^{-1}[\ \Delta p_0^2(e_\infty,tu) + \sum\beta_k^2\Delta q_k^2(e_\infty,tu)\ ]$$

$$\leq \limsup t^{-1}[\ \Delta p_0^2(e_\infty,tu) + \sum\beta_k^2\Delta p_0^2(e_\infty-\gamma_k e_k,tu)\ ]$$

$$\leq \limsup t^{-1}\Delta p_0^2(e_\infty,tu)$$

$$+ \limsup t^{-1}\Bigg(\sum_{k<N}\beta_k^2 \Big([\ p_0^2(e_\infty-\gamma_k e_k+tu) + p_0^2(e_\infty-\gamma_k e_k-tu)\ ]\Delta p_0(e_\infty-\gamma_k e_k,tu) + 4K^2t^2|u|^2 \Big)$$

$$+ \sum_{k\geq N}\beta_k^2 \Big([\ p_0^2(e_\infty-\gamma_k e_k+tu) + p_0^2(e_\infty-\gamma_k e_k-tu)\ ]2K|u|t + 4K^2t^2|u|^2 \Big) \Bigg)$$

$$\leq M \sum_{k\geq N}\beta_k^2,$$

for some constant M, and this sum converges to 0 as $N\to\infty$.

Thus we see that $\tilde{p}_\infty$ is Gateaux-differentiable at e_∞; this is the contradiction we we sought and the proof is concluded. □

References.

[1] - Namioka, Phelps and Preiss, Smooth Banach spaces, weak Asplund spaces and monotone or usco mappings, to appear in Israel J. of Math.

[2] - Ribarska, The dual of a Gateaux smooth Banach space is weak star fragmentable, to appear.

[3] - Deville, Godefroy, Zizler, A variational principle using bump functions, to appear.

[4] - Ribarska, Internal characterisation of fragmentable spaces, Mathematika 34, 1987, 243-257.

[5] - Phelps, Convex functions, monotone operators and differentiability, Lect. notes in math. nr 1364, Springer Verlag, Berlin-Heidelberg-New York (1989).

[6] - Haydon, A counterexample in several questions about scattered compact spaces, to appear.

[7] - Haydon, Bump functions, to appear.

[8] - Christensen and Kenderov, Dense strong continuity of mappings and the Radon-Nikodym property, Math. Scand., 54 (1984), 70-78.

[9] - Leduc, Densitè de certaines famillès d'hyperplans tangents, C.R.Acad.Sci (Paris) Serie A 270 (1970), 326-328.