Counterfactual Worlds*

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January 30, 2021

Abstract

We study an extension of a treatment effect model in which an observed discrete classifier indicates which one of a set of counterfactual processes occurs, each of which may result in the realization of several endogenous outcomes. In addition to the classifier indicating which process was realized, other observed outcomes are delivered by the particular counterfactual process. Models of the counterfactual processes can be incomplete in the sense that even with knowledge of the values of observed exogenous and unobserved variables they may not deliver a unique value of the endogenous outcomes. Thus, relative to the usual treatment effect models, counterfactual outcomes are replaced by counterfactual processes. The determination of endogenous variables in these counterfactual processes may be modeled by the researcher, and impacted by observable exogenous variables restricted to be independent of certain unobservable variables as in instrumental variable models. We study the identifying power of models of this sort that incorporate (i) conditional independence restrictions under which unobserved variables and the classifier variable are stochastically independent conditional on some of the observed exogenous variables and (ii) marginal independence restrictions under which unobservable variables and a subset of the exogenous variables are independently distributed. Building on results in Chesher and Rosen (2017), we characterize the identifying power of these models for fundamental structural relationships and probability distributions of unobservable heterogeneity.

*We gratefully acknowledge financial support from the UK Economic and Social Research Council through grants (RES-589-28-0001, RES-589-28-0002, ES/P008909/1) to the ESRC Centre for Microdata Methods and Practice (CeMMAP) and through the funding of the "Programme Evaluation for Policy Analysis" node of the UK National Centre for Research Methods, and from the European Research Council (ERC) grant ERC-2009-StG-240910-ROMETA. Earlier versions of this paper were presented at the Winter Meeting of the Econometric Society, Philadelphia, January 4th 2014, the CeMMAP/HKUST Conference on Advances in Microeconometrics, Hong Kong University of Science and Technology, May 23rd 2014, the 4th Shanghai Econometrics Workshop, Shanghai University of Finance and Economics, June 29th 2014 and the European Meeting of the Econometric Society, Toulouse, August 27th 2014. Xinyue Bei provided excellent research assistance.

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1 Introduction

In a treatment effect model a discrete classifier indicates which one of a set of counterfactual outcomes is observed. The counterfactual outcomes and the discrete classifier may not be independently distributed because decision makers with beliefs about the counterfactual outcomes may strive to end up in desirable situations. Often little is known about either how the classifier variable is chosen - equivalently how treatment is assigned - or about the relationship between observed and counterfactual outcomes. Functionals of the distribution of treatment effects may then not be point identified, but are typically partially identified. See for instance Manski (1990) as well as Manski (2007) and references therein for several examples.

Many treatment effect models impose a conditional independence restriction, sometimes referred to as unconfoundedness or selection on observables, that counterfactual outcomes and the classifier are independently distributed (unconfounded) conditional on some set of observed exogenous variables.[1] Under some additional restrictions these models can point identify the marginal distributions of the counterfactual outcomes and thus average treatment effects and quantile treatment effects.

In this paper we consider an extension of such models to settings in which the classifier determines which of multiple counterfactual *processes* occur. The econometrician observes each decision maker engaging in one and only one of the counterfactual processes and observes only the realizations of the endogenous outcomes delivered by that process. In contrast to the usual potential outcomes framework, the econometrician may be willing to impose some prior knowledge of the determination of endogenous variables through use of a structural model for the counterfactual processes. Wary of basing inference on highly restrictive models, the econometrician may however come to data with incomplete models of the counterfactual processes that restrict the determination of endogenous outcomes to a possibly non-unique set of values. It is this case that is center stage in this paper.

The models of counterfactual processes specify the role of observed exogenous variables and unobservable variables in the genesis of endogenous outcome variables, as is done in papers that provide partial identification analysis using structural econometric models. Conditional on observed exogenous variables, the unobservables produce stochastic variation in counterfactual processes which deliver the values of outcomes that the econometrician observes.

Given the presence of both a classifier variable and counterfactual processes associated with each realization of the classifier, we consider a combination of restrictions on unobservable variables appearing in the program evaluation and structural econometrics literatures as follows.

1. Conditional independence restrictions. The unobservable variables appearing in the counterfactual processes and the classifier are independently distributed conditional on the observed

[1] For example the models studied in Rubin (1974) and Rosenbaum and Rubin (1983).

exogenous variables. This sort of condition is often used in treatment effect models, commonly referred to as unconfoundedness or selection on observables.

2. Marginal independence restrictions. The unobservable variables appearing in the counterfactual processes and a possibly vector-valued function of the exogenous variables are stochastically independent. In the absence of selection this is a common restriction in instrumental variable models.

The models we study thus contain a combination of conditional and marginal independence restrictions. Our analysis brings together strands from structural econometrics and analysis of causal inference. A contribution of the paper is to provide a characterization of the (sharp) identified sets delivered by models which may be incomplete and embody such a combination of conditional and marginal independence restrictions. Our analysis thus enables the application of a selection on observables restriction with regard to the classifier in conjunction with instrumental variable restrictions with respect to each counterfactual process. Some of the variables that motivate selection on unobservables may also be instruments in the counterfactual processes, but the two sets of variables generally will not coincide because some of the variables considered necessary for selection on unobservables may not be exogenous to the determination of outcomes in each counterfactual process.

Here are examples of cases in which the results of this paper can be applied.

1. Some unemployed workers participate in a training program, others do not. Assignment to the program may not be random. Subsequently the workers engage in one of two counterfactual labor market processes, corresponding to whether or not training was received, and endogenous outcomes such as unemployment duration and wage on re-employment, job tenure and so forth are observed.

2. In a generalization of the Roy model, individuals decide in which of a number of occupations to work whereupon we observe multiple endogenous outcomes that arise in the chosen occupation.

3. Firms decide whether or not to operate in markets distinguished by regulatory regimes and various endogenous outcomes that ensue are observed.

The research reported here is a first step on the way to the study of a broad class of incomplete models that involve a blend of conditional and marginal independence conditions that follow from a combination of selection on observables and instrumental variable restrictions. Possible generalizations in several respects may be fruitful. These are discussed further in the conclusion. The key distinction here, relative to the common use of the potential outcomes framework, is that each possible realization of the treatment may be associated with not just a counterfactual outcome, but a counterfactual process by which a collection of endogenous variables are determined. The determination of endogenous variables in each counterfactual process may happen by way of an incomplete model.

The models studied in this paper impose few restrictions on the determination of the state in which individuals are found. There is just a conditional independence restriction requiring unobservable variables and the classifier variable to be independently distributed conditional on some observed exogenous variables. The way in which the classifier variable is determined is not specified in the models studied in this paper.

2 Structures, Models and Data

This section introduces notation and constructs employed in the rest of the paper.

Notation. We write $\mathcal{R}_A$ to denote the support of random vector A, and $\mathcal{R}_{AB}$ to denote the joint support of random vectors A and B. For any random vectors A, B, $\mathcal{R}_{A|b}$ denotes the support of A conditional on $B = b$. For random variables A and B, $A \perp\!\!\!\perp B$ indicates that A and B are independently distributed. The symbol $\emptyset$ denotes the empty set. Script font ($\mathcal{S}$) is reserved for sets, and sans serif font (S) is reserved for collections of sets. For any set $\mathcal{S}$, $\partial\mathcal{S}$ denotes the boundary of $\mathcal{S}$, $\text{cl}(\mathcal{S})$ denotes the closure of $\mathcal{S}$, and $\mathcal{S}^c$ denotes the complement of $\mathcal{S}$. The sign $\subseteq$ is used to indicate nonstrict inclusion so "$\mathcal{A} \subseteq \mathcal{B}$" includes $\mathcal{A} = \mathcal{B}$. $\mathbb{R}$ denotes the real line. $1[\mathcal{E}]$ denotes the indicator function, taking the value 1 if the event $\mathcal{E}$ occurs and 0 otherwise.

Throughout Y denotes a list of observable endogenous variables, Z denotes a list of observable exogenous variables and U denotes a list of unobservable exogenous variables. Each of these variables may be vector-valued and the observable variables may be discrete or continuous. The variables have support $\mathcal{R}_{YZU}$ on a subset of Euclidean space. Lower case y, z and u denote values of these variables.

With M counterfactual processes there are M components in U, thus: $U = (U_1, \ldots, U_M)$ with only U_m delivering stochastic variation in the m^{th} counterfactual process. Each U_m may be a random variable or vector. Notation U_{-m} denotes U with component U_m omitted.

Some econometric selection models impose the restriction $U_1 = \cdots = U_M$. Examples are given in Heckman and Robb (1985). A number of papers study econometric selection models without this restriction. Such models are described in Heckman, Urzua, and Vytlacil (2008) as models with "essential heterogeneity". Examples can be found in Heckman and Vytlacil (2007) and the references therein. In these econometric selection models it is common to find a discrete choice specification of the determination of the classifier variable and instrumental variable restrictions, see for example Heckman and Vytlacil (2005).

In this paper we study models which have no detailed specification of the determination of the classifier variable. In this respect, like treatment effect models, they are incomplete, and as in those models there is a conditional independence condition. Our models also allow incompleteness in the specification of the processes that deliver counterfactual outcomes, and this specification may include instrumental variable restrictions.

2.1 Structural functions

A model specifies a structural function $h(y, z, u) : \mathcal{R}_{YZU} \to \mathbb{R}$ such that

$$h(Y, Z, U) = 0, \text{ almost surely.} \tag{2.1}$$

This representation of structural functions, used in Chesher and Rosen (2017), will be convenient when models of counterfactual processes are incomplete.

Here the structural function h specifies a composite process composed of a collection of M counterfactual processes. There is a particular discrete component of Y denoted Y_* taking values in $\{1, \ldots, M\}$. This classifier variable is the "treatment", "selection", or "process" indicator. It indicates which of the M counterfactual processes obtains.[2] In many applications it will be correlated with U.

There are additionally M structural functions, $h_m(y, z, u) : \mathcal{R}_{YZU} \to \mathbb{R}$, one for each counterfactual process. The relation between the structural function of the composite process and those of the counterfactual processes is given by

$$h(y, z, u) = \sum_{m=1}^{M} 1[y_* = m] \times h_m(y, z, u). \tag{2.2}$$

Each function h_m is invariant with respect to changes in u_{-m}, holding u_m fixed, and invariant with respect to changes in y_*. If Y_* were exogenously assigned the value m then (2.1) would become

$$h_m(Y, Z, U) = 0, \text{ almost surely}$$

which, in view of (2.2), is equivalent to

$$h(\tilde{Y}, Z, U) = 0, \text{ almost surely,}$$

where $\tilde{Y}$ is the random variable Y with its classifier component replaced by m. Due to the role of the classifier in (2.2), a realization of (Y, Z) delivered by the m^{th} counterfactual process is observed if and only if Y_* has the realized value m. In the language of Heckman and Pinto (2015), setting $Y_* = m$ exogenously is equivalent to "fixing" a variable in a structural model for the purpose of counterfactual analysis as considered by Haavelmo (1943, 1944). In the nomenclature of Pearl

[2] In examples 1-3 below, the classifier variable is the last component of Y.

(2009), each of the counterfactual processes given by $h_m(y, z, u) = 0$, $m \in \{1, \ldots, M\}$, corresponds to a particular submodel of (2.1).

Associated with the structural function are the zero-level sets

$$\mathcal{Y}(u, z; h) \equiv \{y : h(y, z, u) = 0\},$$

$$\mathcal{U}(y, z; h) \equiv \{u : h(y, z, u) = 0\},$$

which are those values of y and u that satisfy the structural relation $h(y, z, u) = 0$ for given values of (z, u) and (y, z), respectively.

Likewise, associated with each of the M structural functions are the zero-level sets

$$\left.\begin{array}{r} \mathcal{Y}_m(u, z; h) \equiv \{y : h_m(y, z, u) = 0 \wedge y_* = m\} \\ \\ \mathcal{U}_m(y, z; h) \equiv \{u : h_m(y, z, u) = 0\} \end{array}\right\}, \quad m \in \{1, \ldots, M\}.$$

The level set $\mathcal{Y}_m(u, z; h)$ contains the values of y that may arise in the m^{th} counterfactual process when $Z = z$ and $U = u$. In other words, the set $\mathcal{Y}_m(u, z; h)$ is the set of feasible counterfactual outcomes obtained by exogenously shifting the classifier variable y_* to m while holding (z, u) fixed. We allow counterfactual processes to be incomplete, so these sets need not be singleton. Every element $y \in \mathcal{Y}_m(u, z; h)$ has $y_* = m$ and the set $\mathcal{Y}_m(u, z; h)$ is invariant with respect to changes in u_{-m}.

The level set $\mathcal{U}_m(y, z; h)$ gives the values of u that can give rise to the value y of Y in the m^{th} counterfactual process when $Z = z$. This set comprises all vectors $u \in \mathcal{R}_U$ with m^{th} component u_m such that $h_m(y, z, u) = 0$, each such value coupled with every possible value of u_{-m}.

With no restrictions placed on the determination of the classifier Y_*, the zero-level set $\mathcal{Y}(u, z; h)$ for the composite structural function may be written

$$\mathcal{Y}(u, z; h) \equiv \{y : h(y, z, u) = 0\} = \bigcup_{m=1}^{M} \mathcal{Y}_m(u, z; h),$$

since any one of the level sets $\mathcal{Y}_m(u, z; h)$ may be realized. Given a value (y, z) just one of the sets $\mathcal{U}_m(y, z; h)$ is realized, which one being determined by the value y_* of the treatment indicator variable (an element of y), so there is the representation

$$\mathcal{U}(y, z; h) \equiv \{u : h(y, z, u) = 0\} = \mathcal{U}_{y_*}(y, z; h).$$

In this paper we do not consider restrictions placed on the selection of the M counterfactual processes, but suitable restrictions could be added. Models that place restrictions on selection among the counterfactual processes incorporate further information from the particular value of y_* observed. For example, in the Roy Model, the observed value of y_* corresponds to that value of m that achieves the maximum payoff or utility among the M available alternatives.

Example 1. **Treatment effects**. The binary treatment effect model studied in Rosenbaum and Rubin (1983) has counterfactual outcomes U_1 and U_2 and a binary indicator Y_2 equal to 1 if U_1 is observed and equal to 2 if U_2 is observed so that

$$Y_1 = 1[Y_2 = 1] \times U_1 + 1[Y_2 = 2] \times U_2$$

is the observed outcome. This treatment effect model has classifier variable $Y_* = Y_2$ and

$$h_m(y, z, u) = y_1 - u_m, \quad m \in \{1, 2\}$$

with singleton y-level sets:

$$\begin{aligned} \mathcal{Y}_1(u, z; h) &= \{(u_1, 1)\}, \\ \mathcal{Y}_2(u, z; h) &= \{(u_2, 2)\}, \end{aligned}$$

and non-singleton u-level sets:

$$\begin{aligned} \mathcal{U}_1(y, z; h) &= \{(y_1, u_2) : u_2 \in \mathcal{R}_{U_2}\}, \\ \mathcal{U}_2(y, z; h) &= \{(u_1, y_1) : u_1 \in \mathcal{R}_{U_1}\}. \end{aligned}$$

Exogenous variables are excluded from the counterfactual structural functions which involve neither unknown parameters nor unknown functions. There is the following composite structural function:

$$h(y, z, u) = 1[y_2 = 1] \times (y_1 - u_1) + 1[y_2 = 2] \times (y_1 - u_2).$$

□

Example 2. Supermarket choice and demand. A household is observed to shop in one of M supermarkets. In a household's supermarket of choice the endogenous variables: share of total expenditure on food, Y_1, and log total expenditure, Y_2, are observed. For each supermarket, indexed by $Y_3 \in \{1, \ldots, M\}$, there is an incomplete linear model with structural functions as follows.

$$h_m(y, z, u) = y_1 - \alpha_m - \beta_m y_2 - \gamma_m z_1 - u_m, \quad m \in \{1, \ldots, M\}$$

Define $U \equiv (U_1, \ldots, U_M)$ and $Y \equiv (Y_1, Y_2, Y_3)$. There may be exogenous variables Z_2 and a restriction $U \perp\!\!\!\perp (Z_1, Z_2)$ and a conditional independence restriction $U \perp\!\!\!\perp Y_3 | Z$ where $Z \equiv (Z_1, Z_2, Z_3)$.

For example Z_1 may denote various household characteristics, Z_2 earnings of the head of household as used in Blundell, Chen, and Kristensen (2007) for estimation of Engel curves, and Z_3 may denote the location of the household. Location may be an essential determinant in the household's choice of supermarket, important for invoking the conditional independence restriction, but may be considered potentially correlated with household expenditure on food.

There are level sets as follows for each $m \in \{1, \ldots, M\}$:

$$\mathcal{Y}_m(u, z; h) = \{(\alpha_m + \beta_m y_2 + \gamma_m z_1 + u_m, y_2, m) : y_2 \in \mathcal{R}_{Y_2}\},$$

$$\mathcal{U}_m(y, z; h) = \{u \in \mathcal{R}_U : u_m = y_1 - \alpha_m - \beta_m y_2 - \gamma_m z_1\}.$$

The classifier variable $Y_* = Y_3$ and there is the following composite structural function:

$$h(y, z, u) = \sum_{m \in \{1,\ldots,M\}} 1[y_3 = m] \times (y_1 - \alpha_m - \beta_m y_2 - \gamma_m z_1 - u_m).$$

□

Example 3. **Training and labor market processes**. An unemployed worker either does ($Y_3 = 1$), or does not ($Y_3 = 2$), take part in a training program. A binary outcome Y_1 is observed, equal to one if employment is found within one year and zero otherwise. For each state there are incomplete threshold crossing-type models for this binary outcome with structural functions.

$$h_m(y, z, u) = y_1 \times \max(g_m(y_2, z_1) - u_m, 0) + (1 - y_1) \times \max(u_m - g_m(y_2, z_1), 0), \quad m \in \{1, 2\},$$

such that in each process $m \in \{1, 2\}$ employment is found if $U_m > g_m(Y_2, Z_1)$.[3] Here Y_2 is a possibly endogenous, binary variable, for example an indicator of receipt of unemployment benefit, and z_1 is a component of a vector z whose elements are values of observed exogenous variables.[4] There are y-level sets:

$$\mathcal{Y}_m(u, z; h) = \{y \in \mathcal{R}_Y : (2y_1 - 1)(u_m - g_m(y_2, z_1)) \geq 0 \wedge y_3 = m\}.$$

There are u-level sets for each $m \in \{1, 2\}$:

$$\mathcal{U}_m(y, z; h) = \begin{cases} \{u \in \mathbb{R}^2 : u_m \in (-\infty, g_m(y_2, z_1)]\}, & y_1 = 0. \\ \{u \in \mathbb{R}^2 : u_m \in [g_m(y_2, z_1), \infty)\}, & y_1 = 1. \end{cases}$$

[3]If $U_m = g_m(Y_2, Z_1)$ either value of Y_1 is allowed.

[4]State-specific threshold crossing models such as this can arise using mixed proportionate hazard models of unemployment duration (see Example 1 in Chesher (2009)) with state-specific heterogeneity and baseline hazards.

The classifier variable is $Y_* = Y_3$ and the structural function for the composite process is

$$h(y,z,u) = 1[y_3 = 1] \times (y_1 \times \max(g_1(y_2,z_1) - u_1, 0) + (1-y_1) \times \max(u_1 - g_1(y_2,z_1), 0)) \\ + 1[y_3 = 2] \times (y_1 \times \max(g_2(y_2,z_1) - u_2, 0) + (1-y_1) \times \max(u_2 - g_2(y_2,z_1), 0)) \, .$$

An assertion of selection on observables with regard to participation in the training program, yields $U \perp\!\!\!\perp Y_* | Z$. It may not however be plausible to assume that all observable exogenous variables $Z = (Z_1, Z_2, Z_3)$ are independent of unobservable heterogeneity in employment prospects U_m in each counterfactual process. For example Z_3 could denote years of education prior to the training program, while Z_1 denotes local job market conditions and Z_2 denotes other demographic variables that do not affect employment prospects but which may affect eligibility for employment benefits.

□

2.2 Distributions of unobservables

Conditional on $Z = z$ the unobserved random variables $U \equiv (U_1, \ldots, U_M)$ have joint probability distribution $G_{U|Z}(\cdot|z)$ and marginal distributions $G_{U_m|Z}(\cdot|z)$, $m \in \{1, \ldots, M\}$. There are collections of conditional probability distributions as follows:

$$\mathcal{G}_{U|Z} \equiv \{G_{U|Z}(\cdot|z) : z \in \mathcal{R}_Z\},$$

and

$$\mathcal{G}_{U_m|Z} \equiv \{G_{U_m|Z}(\cdot|z) : z \in \mathcal{R}_Z\}, \quad m \in \{1, \ldots, M\}.$$

Here $\mathcal{R}_Z$ denotes the support of the observed exogenous variables and for any set $\mathcal{S} \subseteq \mathcal{R}_{U|z}$, $G_{U|Z}(\mathcal{S}|z)$ denotes the probability mass placed on the set $\mathcal{S}$ by the conditional probability distribution $G_{U|Z}(\cdot|z)$.

Each counterfactual process is associated with a counterfactual structure $\left(h_m, \mathcal{G}_{U_m|Z}\right)$ and a composite process is associated with a composite structure $(h, \mathcal{G}_{U|Z})$.

Models comprise restrictions which limit the set of admissible structures. In the models studied here there are restrictions on structural functions and two types of restrictions on the probability distribution of unobservable variables. Recall Y_* is the element of Y which has the role of selection or classifier variable. This is Y_2 in Example 1 and Y_3 in Examples 2 and 3.

1. **Conditional independence restrictions**. $U \perp\!\!\!\perp Y_* | Z$.

2. **Marginal independence restrictions**. There is a function $e(\cdot)$ such that $U \perp\!\!\!\perp e(Z)$.

The function $e(Z)$ is brought into play because conditional independence is required to hold conditional on one set of exogenous variables and marginal independence is required to hold for a different set of exogenous variables. One reason this is desirable is that restricting $U \perp\!\!\!\perp Y_*|Z$ *and* $U \perp\!\!\!\perp Z$ (that is setting $e(Z) = Z$) implies $Y_* \perp\!\!\!\perp U$ which, in many cases, does not capture essential features of a problem. Specifying $e(Z) = Z_1$, a selection of the elements of Z, may be a common choice.[5] In Example 1 it is common to impose $U \perp\!\!\!\perp Y_2|Z$. In Examples 2 and 3 one might have reason to impose the conditional independence restriction $U \perp\!\!\!\perp Y_3|Z$ and the marginal independence restriction $U \perp\!\!\!\perp (Z_1, Z_2)$.

2.3 Data

We consider cases in which realizations of (Y, Z) are obtained *via* an observation process such that the joint distribution of these variables, F_{YZ}, is identified. Of particular importance will be the conditional distributions of Y given Z and Y given (Y_*, Z). For any set $\mathcal{T} \subseteq \mathcal{R}_{Y|z}$, $F_{Y|Z}(\mathcal{T}|z)$ denotes the probability mass placed on the set $\mathcal{T}$ by the conditional probability distribution $F_{Y|Z}(\cdot|z)$ and $F_{Y|Y_*Z}(\mathcal{T}|y_*, z)$ denotes the probability mass placed on the set $\mathcal{T}$ by the conditional probability distribution $F_{Y|Y_*Z}(\cdot|y_*, z)$. The cumulative distribution function of Y given $Z = z$ evaluated at a point t is

$$\mathbb{P}[Y \leq t|Z = z] = F_{Y|Z}(\{y : y \leq t\}\,|z).$$

Likewise

$$\mathbb{P}[Y \leq t|Y_* = y_* \wedge Z = z] = F_{Y|Y_*Z}(\{y : y \leq t\}\,|y_*, z).$$

3 Identification

We ask: what characterizes the set of structures $(h, \mathcal{G}_{U|Z})$ admitted by a model, $\mathcal{M}$, that can deliver the joint distribution of F_{YZ}? This set, denoted $\mathcal{M}^*(F_{YZ})$, is the *identified set* of structures delivered by the model when presented with F_{YZ}. Identified sets of structural features may be obtained as projections of the identified set of structures. We obtain characterizations of identified sets of structures under conditional and marginal independence restrictions building on the results in Chesher and Rosen (2017), henceforth CR17.[6] Our analysis employs random set theory, also used for partial identification analysis in Beresteanu, Molchanov, and Molinari (2011, 2012), Chesher, Rosen, and Smolinski (2013), and Chesher and Rosen (2012a, 2012b, 2013b). This is the first paper explicitly applying these tools in models with conditional independence restrictions. Moreover, we

[5]There is the possibility that conditional independence could be conditional on some function of Z, $d(Z)$, but that is not considered here.

[6]We use the term *identified set* to refer to the collection of **all** structures $\left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M}$ that can generate the joint distribution F_{YZ}. This set is sharp in that there is no structure $\left(h, \mathcal{G}_{U|Z}\right)$ belonging to the identified set that can be distinguished from one generating F_{YZ} on the basis of modeling restrictions and observed data.

are unaware of previous papers featuring the combination of conditional and marginal independence restrictions with regard to the joint distribution of unobserved heterogeneity and observed variables in the class of models considered.

3.1 Restrictions

We impose Restrictions A1 - A3 throughout. These are as in CR17 where they are presented and discussed in Section 2 of that paper.[7] Restriction A4 below extends Restriction A4 of CR17 to the particular cases considered in this paper, while also ensuring that relevant random sets are closed.

Restriction A1: (Y, Z, U) are random vectors defined on a probability space $(\Omega, \mathsf{L}, \mathbb{P})$, endowed with the Borel sets on Ω. The support of (Y, Z, U) is a subset of a finite-dimensional Euclidean space. $\square$

Restriction A2: The joint distribution of (Y, Z), F_{YZ}, is identified by the sampling process. $\square$

Restriction A3: There is an L-measurable function $h(\cdot, \cdot, \cdot) : \mathcal{R}_{YZU} \rightarrow \mathbb{R}$ such that

$$\mathbb{P}\left[h\left(Y, Z, U\right) = 0\right] = 1$$

and there is a collection of conditional distributions

$$\mathcal{G}_{U|Z} \equiv \left\{G_{U|Z}\left(\cdot|z\right) : z \in \mathcal{R}_Z\right\}$$

where for all $\mathcal{S} \subseteq \mathcal{R}_{U|z}$, $G_{U|Z}\left(\mathcal{S}|z\right) \equiv \mathbb{P}\left[U \in \mathcal{S}|z\right]$. $\square$

Restriction A4: The pair $\left(h, \mathcal{G}_{U|Z}\right)$ belongs to a known set of admissible structures $\mathcal{M}$. The model $\mathcal{M}$ contains restrictions as follows. One element of Y, denoted Y_*, only takes values in $\{1, \ldots, M\}$ and U has M components, $U = (U_1, \ldots, U_M)$, each of which may be vectors. The structural function has the form

$$h(y, z, u) = \sum_{m=1}^{M} 1[y_* = m] \times h_m(y, z, u),$$

such that the zero-level sets $\mathcal{Y}(U, Z; h)$ and $\mathcal{U}(Y, Z; h)$ are closed almost surely. $\square$

With regard to Restriction A3, the collection of admissible distributions specified may include

[7] Restriction A2 in CR17 requires that a collection of conditional distributions

$$\mathcal{F}_{Y|Z} \equiv \left\{F_{Y|Z}\left(\cdot|z\right) : z \in \mathcal{R}_Z\right\}$$

is identified by the sampling process. The identification of conditional distributions $F_{Y|Z}\left(\cdot|z\right)$ for all $z \in \mathcal{R}_Z$ and identification of $F_Z(\cdot)$ is equivalent to identification of the joint distribution of Y and Z.

In this paper conditional independence restrictions will require conditioning on components of Y together with Z in places, rather than conditioning on Z alone. This makes the statement of Restriction A2 involving the joint distribution F_{YZ} more natural in the present context.

restrictions on conditional distributions $G_{U|Y_*Z}(\cdot|y_*,z)$, each $(y_*,z) \in \mathcal{R}_{Y_*Z}$, where for all $\mathcal{S} \subseteq \mathcal{R}_{U|y_*z}$, $G_{U|Y_*Z}(\mathcal{S}|y_*,z) \equiv \mathbb{P}[U \in \mathcal{S}|y_*,z]$. In this case the components of $\mathcal{G}_{U|Z}$ are restricted to be such that there exists for each $z \in \mathcal{R}_Z$ conditional distributions $G_{U|Y_*Z}(\cdot|y_*,z)$ satisfying

$$G_{U|Z}(\cdot|z) = \int_{y_* \in \mathcal{R}_{Y_*}} G_{U|Y_*Z}(\cdot|y_*,z)\, dF_{Y_*|Z}(y_*|z).$$

Notation

$$\mathcal{G}_{U|Y_*Z} \equiv \left\{G_{U|Y_*Z}(\cdot|y_*,z) : (y_*,z) \in \mathcal{R}_{Y_*Z}\right\}$$

is used to denote a collection of such conditional distributions where required.

Restriction A4 places restrictions on structural functions $h_m(\cdot,\cdot,\cdot)$ through the specification of admissible pairs $\left(h, \mathcal{G}_{U|Z}\right)$, which may include parametric or shape restrictions. There will in general also be restrictions on the covariation of observable and unobservable exogenous variables embodied in admissible $\mathcal{G}_{U|Z}$. The requirement that the sets $\mathcal{Y}(u,z;h)$ and $\mathcal{U}(y,z;h)$ are closed is a mild restriction that is easily satisfied and is not restrictive in most econometric applications.

It should be noted that Restriction A4 places no restriction on the determination of y_* from the M counterfactual processes. For now we leave this selection process completely unspecified, noting that restrictions on the selection process may be added.

3.2 Identification: foundational results from CR17

This Section extends results given in CR17 in order to provide the basis for the identification analysis to follow. The distinguishing features of these results stems from the need to work with conditional independence restrictions of the sort $U \perp\!\!\!\perp Y_*|Z$. This requires results to be stated conditional on realizations of exogenous variables Z as well as the classifier variable Y_*, rather than conditional on Z alone as in CR17. All of these results apply to the class of models considered in this paper when Restrictions A1 - A3 hold.

Our first result, Theorem 1, proven in the Appendix, builds on Theorem 2 of CR17. This Theorem gives a characterization of identified sets in terms of a selectionability property of the distributions of unobservable variables admitted by a model.[8] Recall that the random set $\mathcal{U}(Y,Z;h)$ which appears in the theorem is defined as

$$\mathcal{U}(Y,Z;h) \equiv \{u \in \mathcal{R}_U : h(Y,Z,u) = 0\}.$$

Theorem 1. *Let Restrictions A1-A3 hold. Then the identified set of structures $\mathcal{M}^*(F_{YZ})$ are those $\left(h, \mathcal{G}_{U|Z}\right)$ admitted by the model $\mathcal{M}$ such that for almost every $z \in \mathcal{R}_Z$ and each $y_* \in \{1,...,M\}$*

[8] The probability distribution, F_A, of a point valued random variable is selectionable with respect to the probability distribution of a random set, $\mathcal{A}$, if there exists a random variable, A, distributed F_A and there exists a random set $\mathcal{A}^*$ with the same probability distribution as $\mathcal{A}$, such that $\mathbb{P}[A \in \mathcal{A}^*] = 1$. See Molchanov (2005).

there exist conditional distributions $G_{U|Y_*Z}(\cdot|y_*, z)$ *defined on measurable subsets of* $\mathcal{R}_U$ *such that*

1. $G_{U|Z}(\cdot|z) = \int\limits_{y_* \in \mathcal{R}_{Y_*}} G_{U|Y_*Z}(\cdot|y_*, z)\, dF_{Y_*|Z}(y_*|z)$.

2. $G_{U|Y_*Z}(\cdot|y_*, z)$ *is selectionable with respect to the conditional distribution of random set* $\mathcal{U}(Y, Z; h)$ *given* $(Y_* = y_* \wedge Z = z)$ *induced by the distribution of* Y *conditional on* $(Y_* = y_* \wedge Z = z)$ *as given by* F_{YZ}.

The following Corollary gives an alternative characterization of the identified set in terms of moment inequalities. This result follows from using Artstein's (1983) Inequality which gives necessary and sufficient conditions for selectionability in terms of containment functionals of random sets. This result is the analog of Corollary 1 in CR17, which uses Artstein's Inequality to produce moment inequalities conditional on realizations of Z rather than on realizations of both Y_* and Z. The proof is a straightforward consequence of the selectionability statement in Theorem 1 and Corollary 1 of CR17 and is omitted.

Corollary 1. *Under Restrictions A1-A3 the identified set can be written*

$$\mathcal{M}^*(F_{YZ}) \equiv \left\{ \begin{array}{c} \left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M} : \exists \mathcal{G}_{U|Y_*Z} \text{ s.t. } \forall \mathcal{S} \in \mathsf{F}(\mathcal{R}_U), \\ C(\mathcal{S}, h|y_*, z) \leq G_{U|Y_*,Z}(\mathcal{S}|y_*, z) \text{ a.e. } (y_*, z) \in \mathcal{R}_{Y_*Z}, \\ \text{and } G_{U|Z}(\mathcal{S}|z) = \int\limits_{y_* \in \mathcal{R}_{Y_*}} G_{U|Y_*Z}(\mathcal{S}|y_*, z)\, dF_{Y_*|Z}(y_*|z) \text{ a.e. } z \in \mathcal{R}_Z \end{array} \right\}, \quad (3.1)$$

where $\mathsf{F}(\mathcal{R}_U)$ *denotes the collection of all closed subsets of* $\mathcal{R}_U$ *and*

$$C(\mathcal{S}, h|y_*, z) \equiv \mathbb{P}\left[\mathcal{U}(Y, Z; h) \subseteq \mathcal{S}|y_*, z\right]$$

is the conditional containment functional of the random set $\mathcal{U}(Y, Z; h)$ *applied to set* $\mathcal{S}$.

The collection of sets $\mathsf{F}(\mathcal{R}_U)$ is too large to inspect in practice. Theorem 2 below provides a smaller collection of *core-determining sets*, a concept introduced in Galichon and Henry (2011). Again where CR17 provided results conditional on exogenous variables Z, we provide results conditional on Z and the discrete classifier Y_*, as required for consideration of core-determining sets under conditional independence restrictions involving Y_* and Z. This turns out to be a simple generalization of Theorem 3 of CR17, with a formal statement given in Theorem 2. Subsequent results in Section 3.3 will provide further refinement of collections of core-determining sets that apply under the restrictions imposed in this paper. The formal statement of Theorem 2 and its Corollary are thus provided as a foundation for this refinement, and in order to make its development transparent.[9]

[9] The proof of this Theorem and its Corollary are identical to those of CR17 Theorem 3 and its Corollary upon substituting "y_*, z" for "z" in that paper and are therefore omitted.

First to state the results it is necessary to define two collections of sets, $\mathsf{U}(h, y_*, z)$: the conditional support of the random set $\mathcal{U}(Y, Z; h)$ given $(Y_* = y_* \wedge Z = z)$ and $\mathsf{U}^*(h, y_*, z)$: the collection of the unions of these sets.

Definition 1. *Under Restrictions A1-A3, the* ***conditional support of random set*** $\mathcal{U}(Y, Z; h)$ *given* $(Y_* = y_* \wedge Z = z)$ *is*

$$\mathsf{U}(h, y_*, z) \equiv \left\{\mathcal{U} \subseteq \mathcal{R}_U : \exists y \in \mathcal{R}_{Y|y_* z} \text{ such that } \mathcal{U} = \mathcal{U}(y, z; h)\right\}.$$

The collections of all sets that are unions of elements of $\mathsf{U}(h, y_*, z)$ *is denoted*

$$\mathsf{U}^*(h, y_*, z) \equiv \left\{\mathcal{U} \subseteq \mathcal{R}_U : \exists \mathcal{Y} \subseteq \mathcal{R}_{Y|y_* z} \text{ such that } \mathcal{U} = \mathcal{U}(\mathcal{Y}, z; h)\right\}.$$

In the definition of $\mathsf{U}^*(h, y_*, z)$ we employ the following notation.

$$\forall \mathcal{Y} \subseteq \mathcal{R}_Y, \qquad \mathcal{U}(\mathcal{Y}, z; h) \equiv \bigcup_{y \in \mathcal{Y}} \mathcal{U}(y, z; h)$$

In the statement of Theorem 2 we use the notation

$$\mathcal{H}(\mathcal{M}) \equiv \left\{h : \left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M} \text{ for some } \mathcal{G}_{U|Z}\right\}.$$

We also define for any set $\mathcal{S} \subseteq \mathcal{R}_U$ and any $(h, y_*, z) \in \mathcal{H}(\mathcal{M}) \times \mathcal{R}_{Y_*} \times \mathcal{R}_Z$,

$$\mathsf{U}^{\mathcal{S}}(h, y_*, z) \equiv \{\mathcal{U} \in \mathsf{U}(h, y_*, z) : \mathcal{U} \subseteq \mathcal{S}\},$$

which are those sets on the support of $\mathcal{U}(\mathcal{Y}, Z; h)$ given $(Y_* = y_* \wedge Z = z)$ that are contained in $\mathcal{S}$.

Theorem 2. *Let Restrictions A1-A3 hold. Fix* $(h, y_*, z) \in \mathcal{H}(\mathcal{M}) \times \mathcal{R}_{Y_*} \times \mathcal{R}_Z$ *and a distribution* $G_{U|Y_* Z}(\cdot|y_*, z)$*. Let* $\mathsf{Q}(h, y_*, z) \subseteq \mathsf{U}^*(h, y_*, z)$ *be such that for any* $\mathcal{S} \in \mathsf{U}^*(h, y_*, z)$ *with* $\mathcal{S} \notin \mathsf{Q}(h, y_*, z)$*, there exist nonempty collections* $\mathsf{S}_1, \mathsf{S}_2$ *that are both subsets of* $\mathsf{U}^{\mathcal{S}}(h, y_*, z)$ *with* $\mathsf{S}_1 \cup \mathsf{S}_2 = \mathsf{U}^{\mathcal{S}}(h, y_*, z)$ *such that*

$$\mathcal{S}_1 \equiv \bigcup_{\mathcal{T} \in \mathsf{S}_1} \mathcal{T},\ \mathcal{S}_2 \equiv \bigcup_{\mathcal{T} \in \mathsf{S}_2} \mathcal{T},\ \text{and } G_{U|Y_* Z}(\mathcal{S}_1 \cap \mathcal{S}_2 | y_*, z) = 0, \tag{3.2}$$

with $\mathcal{S}_1, \mathcal{S}_2 \in \mathsf{Q}(h, y_*, z)$*. Then* $C(\mathcal{S}, h|y_*, z) \leq G_{U|Y_* Z}(\mathcal{S}|y_*, z)$ *for all* $\mathcal{S} \in \mathsf{Q}(h, y_*, z)$ *implies that* $C(\mathcal{S}, h|y_*, z) \leq G_{U|Y_* Z}(\mathcal{S}|y_*, z)$ *holds for all* $\mathcal{S} \subseteq \mathcal{R}_U$*, and in particular for* $\mathcal{S} \in \mathsf{F}(\mathcal{R}_U)$*, so that the collection of sets* $\mathsf{Q}(h, y_*, z)$ *is core-determining.*

Finally, Corollary 2 gives conditions under which a core determining set delivers a moment equality rather than a moment inequality.

Corollary 2. *Define*

$$\mathsf{Q}^{E}\left(h,y_{*},z\right)\equiv\{\mathcal{S}\in\mathsf{Q}\left(h,y_{*},z\right):G_{U|Y_{*}Z}(\partial\mathcal{S}|y_{*}z)=0\ \textit{and}$$
$$\forall y\in\mathcal{R}_{Y|y_{*}z}\ \textit{either}\ \mathcal{U}\left(y,z;h\right)\subseteq\mathcal{S}\ \textit{or}\ \mathcal{U}\left(y,z;h\right)\subseteq\mathrm{cl}(\mathcal{S}^{c})\}.$$

Then, under the conditions of Theorem 2, the collection of equalities and inequalities

$$\begin{aligned} C\left(\mathcal{S},h|y_{*},z\right) &= G_{U|Y_{*}Z}\left(\mathcal{S}|y_{*},z\right),\ \textit{all}\ \mathcal{S}\in\mathsf{Q}^{E}\left(h,y_{*},z\right), \\ C\left(\mathcal{S},h|y_{*},z\right) &\leq G_{U|Y_{*}Z}\left(\mathcal{S}|y_{*},z\right),\ \textit{all}\ \mathcal{S}\in\mathsf{Q}^{I}\left(h,y_{*},z\right)\equiv\mathsf{Q}\left(h,y_{*},z\right)\backslash\mathsf{Q}^{E}\left(h,y_{*},z\right). \end{aligned}$$

holds if and only if $C\left(\mathcal{S},h|y_{*},z\right)\leq G_{U|Y_{*}Z}\left(\mathcal{S}|y_{*},z\right)$ *for all* $\mathcal{S}\in\mathsf{Q}\left(h,y_{*},z\right)$.

A consequence of Corollary 2 is that *all* members of a collection $\mathsf{Q}(h,y_{*},z)$ deliver equalities when the structural function h is such that either (i) every set on the conditional support of $\mathcal{Y}(U,Z;h)$ is singleton and/or (ii) every set on the conditional support of $\mathcal{U}(Y,Z;h)$ is singleton.

3.3 Moment inequalities absent restrictions on selection of Y_{*}

A further simplification of the core determining sets obtains when, in addition to Restrictions A1-A3, Restriction A4 is also imposed, absent further restrictions on the determination of Y_{*}. Without such restrictions, all sets $\mathcal{U}$ of the form $\mathcal{U}\left(y,z;h\right)$ for some $(y,z)\in\mathcal{R}_{YZ}$ are such that for all components $m\in\{1,...,M\}$ with $m\neq y_{*}$, $\mathcal{U}_{m}=\mathcal{R}_{U_{m}}$. To state this formally, we define

$$\mathcal{U}_{m}\left(y,z;h\right)\equiv\left\{u_{m}^{*}\in\mathcal{R}_{U_{m}|z}:\exists u\ \text{s.t.}\ u_{m}=u_{m}^{*}\wedge h\left(y,z,u\right)=0\right\}$$

as the projection of $\mathcal{U}\left(y,z;h\right)$ onto $\mathcal{R}_{U_{m}|Z}$. Then we have the simplification that

$$\forall m\neq y_{*},\quad \mathcal{U}_{m}\left(y,z;h\right)=\mathcal{R}_{U_{m}|Z}. \tag{3.3}$$

The conditional support of the random set $\mathcal{U}_{m}\left(Y,Z;h\right)$ conditional on $(Y_{*}=m\wedge Z=z)$ is

$$\mathsf{U}_{m}\left(h,z\right)\equiv\left\{\mathcal{U}_{m}\left(y,z;h\right):y_{*}=m\wedge y\in\mathcal{R}_{Y|y_{*}z}\right\}.$$

The projection of any set $\mathcal{S}\subseteq\mathcal{R}_{U}$ onto $\mathcal{R}_{U_{m}|z}$ is

$$\mathcal{S}_{m}\equiv\left\{u_{m}^{*}\in\mathcal{R}_{U_{m}|z}:\exists u\in\mathcal{S}\ \text{s.t.}\ u_{m}=u_{m}^{*}\right\}.$$

From Theorem 2 we have that all core determining sets, $\mathcal{S}\in\mathsf{Q}\left(h,y_{*},z\right)$ are unions of sets on the support of $\mathcal{U}\left(y,z;h\right)$. Thus from (3.3) a collection of core-determining sets $\mathsf{Q}\left(h,y_{*},z\right)$ can be supplied such that each $\mathcal{S} \in \mathsf{Q}(h, y_*, z)$ satisfies

$$\forall m \neq y_*, \quad \mathcal{S}_m = \mathcal{R}_{U_m}. \tag{3.4}$$

Consideration of the conditional containment functional applied to such sets then gives

$$C(\mathcal{S}, h|m, z) \equiv \mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{S}|Y_* = m, z] = \mathbb{P}[\mathcal{U}_m(Y, Z; h) \subseteq \mathcal{S}_m|Y_* = m, z], \tag{3.5}$$

which is the probability, conditional on $(Y_* = m \wedge Z = z)$, that the projection of $\mathcal{U}(Y, Z; h)$ onto its m^{th} component is contained in the projection of $\mathcal{S}$ onto its m^{th} component. Consequently, the identified set $\mathcal{M}^*(F_{YZ})$ can be succinctly characterized through inequalities involving only containment functionals for projection level sets $\mathcal{U}_m(Y, Z; h)$ applied to projections of test sets $\mathcal{S}$. We thus define containment functionals for projections of level sets for any test set $\mathcal{S}_m \subseteq \mathcal{R}_{U_m}$ as

$$C_m(\mathcal{S}_m, h|y_*, z) \equiv \mathbb{P}[\mathcal{U}_m(Y, Z; h) \subseteq \mathcal{S}_m|y_*, z]. \tag{3.6}$$

Likewise we have from (3.4) that

$$\forall \mathcal{S} \in \mathsf{Q}(h, y_*, z), \; G_{U|Y_*Z}(\mathcal{S}|m, z) = G_{U_m|Y_*Z}(\mathcal{S}_m|m, z). \tag{3.7}$$

Implications (3.5) and (3.7) together enable us to work in a lower dimensional space, namely that of $\mathcal{R}_{U_m}$ in the construction of core-determining sets, rather than $\mathcal{R}_U$. Specifically, we have that for any $(y_*, z) \in \mathcal{R}_{Y_*Z}$ and any test set $\mathcal{S} \in \mathsf{Q}(h, y_*, z)$, the containment functional inequality

$$C(\mathcal{S}, h|m, z) \leq G_{U|Y_*Z}(\mathcal{S}|m, z), \tag{3.8}$$

appearing in Corollary 1 holds if and only if

$$C_m(\mathcal{S}_m, h|m, z) \leq G_{U_m|Y_*Z}(\mathcal{S}_m|m, z). \tag{3.9}$$

From $\mathcal{S}_m = \mathcal{R}_{U_m}$ for all $m \neq y_*$, (3.8) $\Rightarrow$(3.9) is immediate. The reverse implication is formally proven in the proof of Theorem 3.

Lemma 1 characterizes a collection of core-determining sets on the lower dimensional space $\mathcal{R}_{U_m}$ sufficient to guarantee (3.9) holds for all closed $\mathcal{S}_m \subseteq \mathcal{R}_{U_m}$. Before stating the lemma we require the following definitions for any $(h, m, z) \in \mathcal{H}(\mathcal{M}) \times \mathcal{R}_{Y_*} \times \mathcal{R}_Z$.

$$\mathsf{U}_m^*(h, z) \equiv \{\mathcal{U}_m \subseteq \mathcal{R}_{U_m} : \mathcal{U}_m \text{ is a union of elements of } \mathsf{U}_m(h, z)\},$$

and for any set $\mathcal{S}_m \subseteq \mathcal{R}_{U_m}$,

$$\mathsf{U}^{\mathcal{S}_m}(h,z) \equiv \{\mathcal{U} \in \mathsf{U}_m(h,z) : \mathcal{U} \subseteq \mathcal{S}_m\},$$

which are those sets on the conditional support of $\mathcal{U}_m(Y,Z;h)$ conditional on $(Y_* = m \wedge Z = z)$ that are contained in $\mathcal{S}_m$. With this notation in hand, the proof of the following lemma is a straightforward extension of Theorem 2 and is omitted.

Lemma 1. *Let Restrictions A1-A4 hold. Fix $(h,m,z) \in \mathcal{H}(\mathcal{M}) \times \mathcal{R}_{Y_*} \times \mathcal{R}_Z$ and a distribution $G_{U|Y_*Z}(\cdot|y_*,z)$. Let $\mathsf{Q}_m(h,z) \subseteq \mathsf{U}_m^*(h,z)$, such that for any $\mathcal{S}_m \in \mathsf{U}_m^*(h,z)$ with $\mathcal{S}_m \notin \mathsf{Q}_m(h,z)$, there exist nonempty collections $\mathsf{S}_{m1}, \mathsf{S}_{m2}$ that are both subsets of $\mathsf{U}^{\mathcal{S}_m}(h,z)$ with $\mathsf{S}_{m1} \cup \mathsf{S}_{m2} = \mathsf{U}^{\mathcal{S}_m}(h,z)$ such that*

$$\mathcal{S}_{m1} \equiv \bigcup_{\mathcal{T} \in \mathsf{S}_{m1}} \mathcal{T},\ \mathcal{S}_{m2} \equiv \bigcup_{\mathcal{T} \in \mathsf{S}_{m2}} \mathcal{T},\ \text{and } G_{U|Y_*Z}(\mathcal{S}_{m1} \cap \mathcal{S}_{m2}|y_*,z) = 0, \tag{3.10}$$

with $\mathcal{S}_{m1}, \mathcal{S}_{m2} \in \mathsf{Q}_m(h,z)$. Then $C_m(\mathcal{S}_m, h|m,z) \leq G_{U_m|Y_*Z}(\mathcal{S}_m|m,z)$ for all $\mathcal{S}_m \in \mathsf{Q}_m(h,z)$ implies that $C_m(\mathcal{S}_m, h|m,z) \leq G_{U_m|Y_*Z}(\mathcal{S}_m|m,z)$ holds for all $\mathcal{S}_m \subseteq \mathcal{R}_{U_m}$, and in particular for $\mathcal{S}_m \in \mathsf{F}(\mathcal{R}_{U_m})$, so that the collection of sets $\mathsf{Q}_m(h,z)$ is core-determining.

The following Theorem, proven in the Appendix, uses this lemma *en route* to characterizing the identified set $\mathcal{M}^*(F_{YZ})$ under Restrictions A1-A4 through conditional containment functional inequalities defined on $\mathcal{R}_{U_m}$, $m \in \{1,...,M\}$. The inequalities constrain the structural function h and the marginal distributions of each U_m conditional on (m,z) and z, but provide no further constraint on the joint distribution of any U_m and U_n, $n \neq m$.

Theorem 3. *Let Restrictions A1-A4 hold, with no further restrictions imposed on the determination of the classifier Y_*. Given collection of conditional distributions $\mathcal{G}_{U|Y_*Z}$ we have that*

$$\forall \mathcal{S} \in \mathsf{F}(\mathcal{R}_U),\ C(\mathcal{S},h|m,z) \leq G_{U|Y_*Z}(\mathcal{S}|m,z) \ \ a.e.\ \ (y_*,z) \in \mathcal{R}_{Y_*Z}$$

if and only if

$$\forall m \in \{1,...,M\},\ \forall \mathcal{S} \in \mathsf{Q}_m(h,z),\ C_m(\mathcal{S},h|m,z) \leq G_{U_m|Y_*Z}(\mathcal{S}|m,z) \ \ a.e.\ \ (y_*,z) \in \mathcal{R}_{Y_*Z}.$$

Hence

$$\mathcal{M}^*(F_{YZ}) = \left\{ \begin{array}{c} \left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M} : \exists \mathcal{G}_{U|Y_*Z} \ s.t.\ \forall m \in \{1,...,M\},\ \forall \mathcal{S} \in \mathsf{Q}_m(h,z), \\ C_m(\mathcal{S},h|m,z) \leq G_{U_m|Y_*Z}(\mathcal{S}|m,z) \ \ a.e.\ \ z \in \mathcal{R}_Z,\ and \\ G_{U_m|Z}(\mathcal{S}|z) = \displaystyle\int_{y_* \in \mathcal{R}_{Y_*}} G_{U_m|Y_*Z}(\mathcal{S}|y_*,z)\, dF_{Y_*|Z}(y_*|z) \ \ a.e.\ \ z \in \mathcal{R}_Z \end{array} \right\}.$$

3.4 The identifying power of a conditional independence restriction

The models studied in this paper include a *conditional* independence Restriction CI.

Restriction CI. Let Y_* be the classifier element of Y. Random variables U and Y_* are independently distributed conditional on $Z = z$ for every $z \in \mathcal{R}_Z$.

Restriction CI places restrictions on the collection of distributions $\mathcal{G}_{U|Z}$, namely that for all sets $\mathcal{S} \subseteq \mathcal{R}_{U|Z}$, the conditional distribution of U given (Y_*, Z), $G_{U|Y_*Z}(\cdot|y_*, z)$, satisfies $G_{U|Y_*Z}(\mathcal{S}|y_*, z) = G_{U|Z}(\mathcal{S}|z)$ a.e. $(y_*, z) \in \mathcal{R}_{Y_*Z}$. A consequence is equality of the conditional support of unobserved heterogeneity and its components, that is that $\mathcal{R}_{U|y_*z} = \mathcal{R}_{U|z}$ and $\mathcal{R}_{U_m|y_*z} = \mathcal{R}_{U_m|z}$, for all $m \in \{1, \ldots, M\}$.

In Theorem 4, proven in the Appendix, we build on Theorem 3 to develop a characterization of the identified set when there is a conditional independence condition.

Theorem 4. *Let Restrictions A1-A3 hold. A model $\mathcal{M}$ which embodies Restriction A4 and the conditional independence restriction CI has an identified set $\mathcal{M}^*(\mathcal{F}_{YZ})$ which can be written as*

$$\mathcal{M}^*(F_{YZ}) \equiv \big\{ \big(h, \mathcal{G}_{U|Z}\big) \in \mathcal{M} : \forall m \in \{1, \ldots, M\}, \forall \mathcal{S} \in \mathsf{Q}_m(h, z), \\ C_m(\mathcal{S}, h|m, z) \leq G_{U_m|Z}(\mathcal{S}|z), \ a.e. \ z \in \mathcal{R}_Z \big\}.$$

Here $\mathcal{S} \subseteq \mathcal{R}_{U_m|z}$, and $\mathsf{Q}_m(h, z)$ is a collection of closed subsets of $\mathcal{R}_{U_m|z}$ comprising unions of sets on the conditional support of $\mathcal{U}_m(Y, Z; h)$ given $Z = z$ and $Y_* = m$ defined in Lemma 1.

Regarding the collections of distributions $\mathcal{G}_{U|Z}$, the identified set in Theorem 4 only places restrictions on the distributions, $G_{U_m|Z}(\cdot|z)$, $m \in \{1, \ldots, M\}$. Data is never informative about the covariation of U_m and $U_{m'}$, for any $m \neq m'$, although the joint distribution of U_m and $U_{m'}$ can be bounded using information on their marginal distributions using for instance the Fréchet-Hoeffding bounds as in Heckman, Smith, and Clements (1997).

Example 1 continued. In this classical treatment effect model the projected u-level sets $\mathcal{U}_m(Y, Z; h)$ are singleton sets and a small modification to the argument that leads to Corollary 2 leads to the conclusion that the inequalities in the definition of $\mathcal{M}^*(\mathcal{F}_{YZ})$ in Theorem 4 reduce to equalities. For any set $\mathcal{S} \subseteq \mathsf{Q}_m(h, z)$,

$$G_{U_m|Z}(\mathcal{S}|z) = F_{Y_1|Y_2,Z}(\mathcal{S}|m, z)$$

and it follows that for $m \in \{1, \ldots, M\}$ each conditional distribution function of U_m given $Z = z$ is point identified by the conditional distribution function of Y_1 given $Y_2 = m$ and $Z = z$. Consequently, each marginal distribution function of U_m is point identified by the expected value with respect to Z of the conditional distribution function of Y_1 given $Y_2 = m$ and $Z = z$, leading directly to familiar results on point identification of the Average and Quantile Treatment Effects. Without further restrictions the joint distribution of potential outcomes and various functionals thereof are generally not point identified but are partially identified, see for example results of Heckman, Smith, and Clements (1997) as well as Fan, Guerre, and Zhu (2017) using covariates to tighten the identified set. □

3.5 The additional identifying power of marginal independence conditions

Theorem 4 provides a characterization of the identified set of structures delivered by a model of counterfactual processes embodying Restriction A4 and the conditional independence restriction CI. Models featuring counterfactual processes more complex than found in the treatment effects case may additionally feature marginal independence restrictions. We consider Restriction MI.

Restriction MI. Let $e(Z)$ be a vector-valued function of Z. Random variables U_m and $e(Z)$ are independently distributed for each $m \in \mathcal{R}_{Y_*}$.

Restriction MI restricts the set of admissible structures $\left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M}$ to be those with U_m and $e(Z)$ independently distributed for all $m \in \{1, \ldots, M\}$. The function $e(\cdot)$ could for example be a function that selects certain elements from Z, for example, with $Z = (Z_1, Z_2)$, $e(Z) = Z_1$.[10] In the classical treatment effect model in which there is no conditional independence restriction and an instrumental variable is independently distributed of each potential outcome Kitagawa (2020) characterizes sharp bounds on the marginal distribution of each potential outcome.[11]

Theorem 5 provides a characterization of the identified set delivered by a model embodying the conditional *and* marginal independence restrictions CI and MI.

Theorem 5. *Let Restrictions A1-A3 hold. A model $\mathcal{M}$ which embodies Restriction A4 and the independence restrictions CI and MI has an identified set $\mathcal{M}^*(\mathcal{F}_{Y|Z})$ which can be written as follows.*

$$\mathcal{M}^*\left(F_{YZ}\right) \equiv \left\{ \begin{array}{c} \left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M} : \forall m \in \{1, \ldots, M\},\ \forall \mathcal{S} \in \mathsf{Q}_m\left(h, z\right), \\ C_m\left(\mathcal{S}, h|m, z\right) \leq G_{U_m|Z}\left(\mathcal{S}|z\right),\ a.e.\ z \in \mathcal{R}_Z \end{array} \right\},$$

where $\mathsf{Q}_m\left(h, z\right)$ is the collection of core determining sets defined in Lemma 1.

This characterization appears the same as that of Theorem 4, but it differs because now admissible structures $\left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M}$ are required to be such that $\mathcal{G}_{U|Z}$ satisfies Restriction MI in addition to Restriction CI. Thus the identified set of Theorem 5 is subset of that of Theorem 4 because the conditional containment inequality must hold for some $\left(h, \mathcal{G}_{U|Z}\right)$ in this more restrictive collection of admissible structures.

Sharpness is immediate because for any $\mathcal{S} \in \mathcal{R}_{U_m}$, under Restriction CI

$$C_m\left(\mathcal{S}, h|m, z\right) \leq G_{U_m|Z}\left(\mathcal{S}|z\right) \Rightarrow C_m\left(\mathcal{S}, h|m, z\right) \leq G_{U_m|Y_*Z}\left(\mathcal{S}|m, z\right).$$

[10]It would be easy to relax the marginal independence restriction to $U_m \perp\!\!\!\perp e_m(Z)$, $m \in \{1, \ldots, M\}$.

[11]Kitagawa (2020) additionally considers the identifying power of the pair of potential outcomes being jointly independent of the instrument, as well as a monotonicity restriction.

This is required to hold for all (m,z) and for all core-determining sets, so the selectionability statement of Theorem 1 is satisfied. Again, the difference with Theorem 4 is that the distributions $G_{U_m|Z}$ are now required to belong to more restrictive collections of conditional distributions, namely we have as a requirement of admissible structures that for each $e \in \mathcal{R}_{e(Z)}$,

$$G_{U_m|Z}(\mathcal{S}|Z \in \mathcal{Z}_e) = G_{U_m}(\mathcal{S}), \text{ where } \mathcal{Z}_e \equiv \{z : e(Z) = e\}. \tag{3.11}$$

The characterization of $\mathcal{M}^*(\mathcal{F}_{YZ})$ in Theorem 5 produces interesting observable implications that may not appear immediate, but which provide bounds on $\left(h, \mathcal{G}_{U|Z}\right)$, potentially non-sharp in isolation. These implications may prove beneficial in developing sufficient conditions for point identification of $\left(h, \mathcal{G}_{U|Z}\right)$ or features of $\left(h, \mathcal{G}_{U|Z}\right)$ in particular models. Two such implications follow.

1. For any $m \in \mathcal{R}_{Y_*}$, $e \in \mathcal{R}_{e(Z)}$, and any $\mathcal{S} \subseteq \mathcal{R}_{U_m}$,

$$E\left[C_m(\mathcal{S}, h|m, Z)\,|e(Z) = e\right] \leq G_{U_m}(\mathcal{S}). \tag{3.12}$$

   This follows from integrating both sides of the inequality

$$C_m(\mathcal{S}, h|m, z) \leq G_{U_m|Z}(\mathcal{S}|z) \tag{3.13}$$

   as follows. First, starting with the left hand side of (3.13),

$$\begin{aligned}\frac{1}{F_Z(\mathcal{Z}_e)} \int_{z \in \mathcal{Z}_e} C_m(\mathcal{S}, h|m, z)\, dF_Z(z) &= E\left[C_m(\mathcal{S}, h|m, Z)\,|Z \in \mathcal{Z}_e\right] \\ &= E\left[C_m(\mathcal{S}, h|m, Z)\,|e(Z) = e\right].\end{aligned}$$

   Then likewise multiplying the right hand side of (3.13) by $\frac{1}{F_Z(\mathcal{Z}_e)}$ and integrating we obtain

$$\frac{1}{F_Z(\mathcal{Z}_e)} \int_{z \in \mathcal{Z}_e} G_{U_m|Z}(\mathcal{S}|z)\, dF_Z(z) = G_{U_m|Z}(\mathcal{S}|Z \in \mathcal{Z}_e) = G_{U_m}(\mathcal{S}),$$

   where the final equality follows from Restriction MI.

   It is interesting to note that the expression $E\left[C_m(\mathcal{S}, h|m, Z)\,|e(Z) = e\right]$ is a conditional expectation of the containment functional $C_m(\mathcal{S}, h|m, Z)$ holding m fixed, which may in general differ from $C_m(\mathcal{S}, h|Y_* = m, e(Z) = e)$.

2. For any $m \in \mathcal{R}_{Y_*}$, $e \in \mathcal{R}_{e(Z)}$, and any $\mathcal{S} \subseteq \mathcal{R}_{U_m}$,

$$C_m(\mathcal{S}, h|Z \in \mathcal{Z}_e) \leq G_{U_m|Z}(\mathcal{S}|Z \in \mathcal{Z}_e) = G_{U_m}(\mathcal{S}),$$

by Restriction MI.

Remarks

1. Since the bounded probabilities $G_{U_m}(\mathcal{S}) = G_{U_m|Z}(\mathcal{S}|Z \in \mathcal{Z}_e)$ do not depend on the value e of $e(Z)$, for each value m and $\mathcal{S}$ only the supremum of the lower bounding expression over values $e \in \mathcal{R}_{e(Z)}$ is instrumental in (3.12).

2. In the common case in which $Z = (Z_1, Z_2)$ and $e(Z) = Z_1$ is a selection of the elements in Z,

$$E_Z\left[\cdot|e(Z) = e\right] = E_{Z_2}\left[\cdot|Z_1 = e\right].$$

3. Applying the inequalities appearing in the characterization of $\mathcal{M}(F^*_{YZ})$, to both $\mathcal{S}$ and its complement $\mathcal{S}^c$ a two-sided inequality is obtained:

$$E_Z\left[C_m\left(\mathcal{S}, h|m, z\right)|e(Z) = e_L\right] \leq G_{U_m}\left(\mathcal{S}\right) \leq 1 - E_Z\left[C_m\left(\mathcal{S}^c, h|m, z\right)|e(Z) = e_U\right],$$

which must hold for all $(e_L, e_U) \in \mathcal{R}_{e(Z)}$.

Example 3 continued. In this example the structural function $h(y, z, u)$ is characterized by the collection of parameters $\theta \equiv \{(g_1(0, z_1), g_1(1, z_1), g_2(0, z_1), g_2(1, z_1)) : z_1 \in \mathcal{R}_{Z_1}\}$, which is the collection of threshold values for determination of the binary employment outcomes across both values of $m \in \{1, 2\}$, both values of the endogenous variable Y_2, and all values of z_1 on the support of Z_1. Recall $m = 1$ for people who attend a training program and $m = 2$ for people who do not. Thus, $g_1(0, z_1)$ is the threshold parameter for a person with $Z_1 = z_1$ who does attend a training program and is not in receipt of benefit payment ($Y_2 = 0$). We can normalize the threshold functions so that each U_m is marginally uniformly distributed on the unit interval and then there is the following representation.[12]

$$\text{In state } m\text{:} \quad Y_1 = \begin{cases} 0 \quad , & 0 \leq U_m \leq g_m(Y_2, Z_1) \\ \\ 1 \quad , & g_m(Y_2, Z_1) \leq U_m \leq 1 \end{cases} .$$

The setup here is similar to that in Chesher and Rosen (2013) and Section 8.2 of Chesher and Rosen (2020), but in those analyses there was only one state, such that $U_1 = U_2$ (denoted U) and $g_1(y_2, Z_1) = g_2(y_2, Z_1)$. Additionally, in both earlier studies there was no conditional independence restriction but there was a marginal independence restriction $U \perp\!\!\!\perp Z$ in an IV model for a binary outcome.

[12]As noted previously, if $U_m = g_m(Y_2, Z_1)$ then either value of Y_1 is permitted.

Here, we similarly think of employment Y_1 as a binary outcome of interest for which there is an endogenous included variable Y_2 such as receipt of unemployment benefits and Z_1 and Z_2 as included and excluded exogenous variables in its determination. There are however additional observable variables Z_3 important for motivating the conditional independence restriction $U \perp\!\!\!\perp Y_3|Z$, or selection on observables for selection into the training program, where $Z = (Z_1, Z_2, Z_3)$. These additional variables Z_3 are not however restricted to be independent of unobservable variables U_m.

To illustrate the roles of the two different types of restrictions, define conditional probabilities consistently estimable from data, as follows.

$$f_{ij}(z, m) \equiv \mathbb{P}[Y_1 = i \wedge Y_2 = j | Y_3 = m, Z = z], \quad (i, j) \in \{0, 1\} \times \{0, 1\}, \ m \in \{1, 2\}, \ z \in \mathcal{R}_Z.$$

Applying Theorem 4, under the conditional independence restriction alone, $(U_1, U_2) \perp\!\!\!\perp Y_3|Z$, the identified set of structures $(\theta, \mathcal{G}_{U|Z})$ is characterized by the following inequalities which hold for $m \in \{1, 2\}$ and almost every $z \in \mathcal{R}_Z$.

If $g_m(0, z_1) \leq g_m(1, z_1)$ then

$$f_{00}(z, m) \leq G_{U_m|Z}\left([0, (g_m(0, z_1)] | z\right) \leq f_{00}(z, m) + f_{01}(z, m), \tag{3.14}$$

$$f_{00}(z, m) + f_{01}(z, m) \leq G_{U_m|Z}\left([0, (g_m(1, z_1)] | z\right) \leq 1 - f_{11}(z, m), \tag{3.15}$$

while for $g_m(0, z_1) \geq g_m(1, z_1)$

$$f_{00}(z, m) + f_{01}(z, m) \leq G_{U_m|Z}([0, g_m(0, z_1)] | z) \leq 1 - f_{10}(z, m), \tag{3.16}$$

$$f_{01}(z, m) \leq G_{U_m|Z}([0, g_m(1, z_1)] | z) \leq f_{00}(z, m) + f_{01}(z, m). \tag{3.17}$$

There are two separate pairs of inequalities for each z according to whether the threshold $g_m(\cdot, z_1)$ is weakly increasing in its first argument. This is because the inequalities implied by the general characterization given in Theorem 4 simplify as such according to the change in the direction of the threshold. Note that the change in the direction of the threshold corresponds to the opposite sign of the conditional on z average treatment effect $g_m(0, z_1) - g_m(1, z_1)$.

The reduction of the characterization of the identified set into two different sets of inequalities depending on the sign of the treatment effect also occurs in the IV binary outcome models studied in Chesher and Rosen (2013) and Section 8.2 of Chesher and Rosen (2020). However, those invoke marginal independence restrictions but not conditional independence restrictions. So while the form of the inequalities appears similar, they are different. To compare to the inequalities obtained in the IV binary outome model in those prior analyses, consider the inequalities obtained for a single fixed m, since those analyses did not feature a classifier taking multiple values. In those models the unobservable U ($= U_m$ here) is independent of Z. With the normalization in place imposing uniformly distributed U there is $G_{U_m|Z}([0, g_m(\cdot, z_1)] | z) = g_m(\cdot, z_1)$. By contrast, the conditional independence restriction that $(U_1, U_2, U_3) \perp\!\!\!\perp Y_3|Z$ used to obtain inequalities (3.14)-(3.17) allows for the conditional distribution of each unobservable $G_{U_m|Z}(\cdot|z)$ to vary with respect to z. So while the uniform normalization implies that $E_Z\left[G_{U_m|Z}((-\infty, t]|Z)\right] = t$ for all $t \in [0,1]$, evaluation of the conditional distributions in the middle of the inequalities in (3.14)-(3.17) do not simplify. Substantively, with only Restriction CI, the conditional distributions of unobservable heterogeneity in each of the m processes can vary with the value of Z conditioned upon.

Now consider imposing Restriction MI in addition to Restriction CI. Restriction MI asserts that within each counterfactual process there are observable variables Z_1 and Z_2 that are independent of unobservable heterogeneity specific to that process. The former variables Z_1 are included exogenous variables in the process and Z_2 are excluded exogenous instruments. Restriction MI is however silent regarding Z_3, which are variables that may be important for asserting selection on observables with regard to the classifier, but which may not be suitable instruments in the counterfactual process, such as perhaps years of education prior to the rollout of the job training program.

Thus Restriction MI additionally asserts further to the conditional independence restriction that for each m, $U_m \perp\!\!\!\perp (Z_1, Z_2)$. The inequalities (3.12) then deliver the following additional inequalities which hold for $m \in \{1,2\}$ and almost every $(z_1, z_2) \in \mathcal{R}_{(Z_1,Z_2)}$.

If $g_m(0, z_1)) \leq g_m(1, z_1)$ then

$$E_{Z_3}\left[f_{00}(Z,m)|z_1,z_2\right] \leq g_m(0,z_1) \leq E_{Z_3}\left[f_{00}(Z,m) + f_{01}(Z,m)|z_1,z_2\right], \quad (3.18)$$

$$E_{Z_3}\left[f_{00}(Z,m) + f_{01}(Z,m)|z_1,z_2\right] \leq g_m(1,z_1) \leq E_{Z_3}\left[1 - f_{11}(Z,m)|z_1,z_2\right], \quad (3.19)$$

while if $g_m(0, z_1) \geq g_m(1, z_1)$

$$E_{Z_3}\left[f_{00}(Z,m) + f_{01}(Z,m)|z_1,z_2\right] \leq g_m(0,z_1) \leq E_{Z_3}\left[1 - f_{10}(Z,m)|z_1,z_2\right], \quad (3.20)$$

$$E_{Z_3}\left[f_{01}(Z,m)|z_1,z_2\right] \leq g_m(1,z_1) \leq E_{Z_3}\left[f_{00}(Z,m) + f_{01}(Z,m)|z_1,z_2\right]. \quad (3.21)$$

Relative to the implications (3.14)-(3.17) under the conditional independence assumption alone, these implications can be seen to hold by integrating the expressions in each inequality with respect to the conditional distribution of Z_3 given $(Z_1, Z_2) = (z_1, z_2)$. Under Restriction MI for each $m \in \{1,2\}$ the conditional distribution of U_m given $(Z_1, Z_2) = (z_1, z_2)$ is the same for all values (z_1, z_2), and under the normalization that U_m is uniformly distributed on $[0,1]$ it follows that $G_{U_m|Z}([0, g_m(t, z_1)]|z_1, z_2)$ simplifies to $g_m(t, z_1)$ for any t with $0 \leq g_m(t, z_1) \leq 1$.

The characterization delivered by the implications (3.14)-(3.17) and (3.18)-(3.21) almost surely for each m differ from those of the binary outcome IV models studied in Chesher and Rosen (2013) and Section 8.2 of Chesher and Rosen (2020), each of which had no classifier variable and no conditional independence restrictions. Nonetheless, while the precise inequalities in this example differ, they share similar structure to those obtained in the prior analyses, and similar methods from the literature on estimation and inference using moment inequalities could be used in an application, such as inference methods from Chernozhukov, Lee, and Rosen (2013) used in Chesher and Rosen (2020).

4 Concluding remarks

We have presented an extension of a treatment effect model in which a discrete classifier variable indicates in which one of a number of counterfactual processes an individual engages. While it is common in treatment effect models for a value of a discrete classifier to give rise to a counterfactual outcome, here the discrete classifier gives rise to a counterfactual process that may deliver values of multiple endogenous variables. The modeling of such processes can be beneficial when the researcher asserts a model for these counterfactual processes incorporating observed realizations of endogenous and exogenous variables.

Regarding selection of the discrete classifier, we have allowed the possibility that the researcher invokes a selection on observables assumption as is widely used in the study of treatment effects. With regard to each counterfactual process, we have allowed for the specification of models for the determination of endogenous variables, as is done in structural econometric modeling, incorporating instrumental variable restrictions to deal with issues of endogeneity.

We have thus provided identification analysis for models that incorporate a blend of conditional independence restrictions and marginal independence restrictions. This combines the use of selection on observables restrictions from the treatment effects literature with instrumental variable restrictions from structural econometrics. Both types of restrictions may be useful, but their implications differ. In applications, the variables used in support of selection on observables will not always be the same as those that are exogenous with respect to the ensuing counterfactual process. Consequently the results in this paper are developed for cases in which conditional independence and marginal independence restrictions are invoked with respect to different collections of observed exogenous variables.

Using tools from random set theory and building in particular on CR17, we have developed characterizations of the (sharp) identified sets delivered by these models. The identification analysis here can serve as a starting point for applications in which there is a model for counterfactual processes subsequent to determination of a classifier, and in which the distinction between those variables required to justify unconfoundedness at the selection stage and those exogenous to the counterfactual processes is important.

We have considered models of counterfactual processes which may be incomplete, but our results also apply when the counterfactual models are complete. Incompleteness can arise when these processes feature endogenous variables whose determination cannot be completely specified, as in incomplete IV models, when the determination of outcomes involves multiple equilibria and no equilibrium selection mechanism is specified, when a process is defined by inequality restrictions as in some auction models, and when only some parts of a simultaneous equations system that determines values of endogenous variables are specified.

The models considered place no structure on the determination of the classifier variable but impose a conditional independence restriction requiring the unobservable variables that deliver stochastic variation in the counterfactual processes and the classifier variable to be independently distributed conditional on some observed exogenous variables.

Several further avenues for analysis are possible. One may for example aim to exploit economic restrictions on the determination of the process in which an individual is engaged, for example a model of choice. There may be settings featuring a continuum of processes rather than the discrete classification considered here, and/or a researcher may wish to consider conditional independence restrictions involving endogenous and exogenous variables as in control function models. Future research building upon the framework herein may additionally entail study of the identifying power of alternative covariation restrictions, such as conditional mean and quantile independence.

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A Proofs

Proof of Theorem 1. Theorem 2 of CR17 states that under Restrictions A1-A3 of that paper, identical to Restrictions A1-A3 here, the identified set of structures $\left(h, \mathcal{G}_{U|Z}\right)$ are those such that

$$G_{U|Z}\left(\cdot|z\right) \precsim \mathcal{U}\left(Y, z; h\right) \text{ when } Y \sim F_{Y|Z}\left(\cdot|z\right), \text{ a.e. } z \in \mathcal{R}_Z, \tag{A.1}$$

where "$\precsim$" means "is selectionable with respect to the distribution of", as in CR17. This statement has the following interpretation. (1) There exists a random vector $\tilde{U}$ such that for almost every $z \in \mathcal{R}_Z$, $\tilde{U} \sim G_{U|Z}\left(\cdot|z\right)$ conditional on $Z = z$. (2) There exists a random variable $\tilde{Y}$ such that for almost every $z \in \mathcal{R}_Z$, $\tilde{Y} \sim F_{Y|Z}\left(\cdot|z\right)$ conditional on $Z = z$. (3) $\tilde{U}$ and $\tilde{Y}$ reside on probability space $(\Omega, \mathsf{L}, \mathbb{P})$ and $\mathbb{P}\left[\tilde{U} \in \mathcal{U}\left(\tilde{Y}, Z; h\right) |Z = z\right] = 1$ a.e. $z \in \mathcal{R}_Z$.

To prove the theorem it is required to show that (A.1) is equivalent to the existence of a collection of conditional distributions $\mathcal{G}_{U|Y_*Z} \equiv \left\{G_{U|Y_*Z}\left(\cdot|y_*, z\right) : (y_*, z) \in \mathcal{R}_{Y_*Z}\right\}$ such that:

A: For almost every $z \in \mathcal{R}_Z$:

$$G_{U|Z}\left(\cdot|z\right) = \int\limits_{y_* \in \mathcal{R}_{Y_*}} G_{U|Y_*Z}\left(\cdot|y_*, z\right) dF_{Y_*|Z}\left(y_*|z\right), \text{ and} \tag{A.2}$$

B: For almost every $(y_*, z) \in \mathcal{R}_{Y_*Z}$:

$$G_{U|Y_*Z}\left(\cdot|y_*, z\right) \precsim \mathcal{U}\left(Y, z; h\right) \text{ when } Y \sim F_{Y|Y_*Z}\left(\cdot|y_*, z\right). \tag{A.3}$$

To show this start with (A.1), from which we have, that there exist $\tilde{U}$ and $\tilde{Y}$ with

$$1 = \mathbb{P}\left[\tilde{U} \in \mathcal{U}\left(\tilde{Y}, Z; h\right) |Z = z\right] = \int\limits_{\mathcal{R}_{Y_*|z}} \mathbb{P}\left[\tilde{U} \in \mathcal{U}\left(\tilde{Y}, Z; h\right) |\tilde{Y}_* = y_*, Z = z\right] dF_{Y_*|Z}\left(y_*|z\right),$$

where $\tilde{Y} \sim F_{Y|Z}\left(\cdot|z\right)$ conditional on $Z = z$. This can hold if and only if

$$\mathbb{P}\left[\tilde{U} \in \mathcal{U}\left(\tilde{Y}, Z; h\right) |\tilde{Y}_* = y_*, Z = z\right] = 1 \text{ a.e. } (y_*, z) \in \mathcal{R}_{Y_*Z},$$

with $\tilde{Y} \sim F_{Y|Z}\left(\cdot|z\right)$.

Now define $G_{U|Y_*Z}\left(\cdot|y_*, z\right)$ such that for any $\mathcal{S} \in \mathcal{R}_U$,

$$G_{U|Y_*Z}\left(\mathcal{S}|y_*, z\right) \equiv \mathbb{P}\left[\tilde{U} \in \mathcal{S}|\tilde{Y}_* = y_*, Z = z\right].$$

Consequently, from Restriction A3 and the first consequence of (A.1) above,

$$G_{U|Z}(\mathcal{S}|z) = \mathbb{P}\left[\tilde{U} \in \mathcal{S}|Z = z\right],$$

and then from the law of total probability (A.2) holds. Then we have that (A.3) holds since

1. There exists a random vector $\tilde{U}$ such that for almost every $z \in \mathcal{R}_Z$, $\tilde{U} \sim G_{U|Z}(\cdot|z)$ conditional on $Z = z$, and such that for almost every $(y_*, z) \in \mathcal{R}_{Y_*Z}$, $\tilde{U} \sim G_{U|Y_*Z}(\cdot|y_*, z)$ conditional on $Z = z$, $Y_* = y_*$.

2. There exists a random variable $\tilde{Y}$ such that for almost every $z \in \mathcal{R}_Z$, $\tilde{Y} \sim F_{Y|Z}(\cdot|z)$ conditional on $Z = z$.

3. $\tilde{U}$ and $\tilde{Y}$ belong to probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}\left[\tilde{U} \in \mathcal{U}\left(\tilde{Y}, Z; h\right) |\tilde{Y}_* = y_*, Z = z\right] = 1$ a.e. $(y_*, z) \in \mathcal{R}_{Y_*Z}$.

That (A.2) and (A.3) imply (A.1) is immediate, and so equivalence is proved. □

Proof of Theorem 3. Fix $(m, z) \in \mathcal{R}_{Y_*Z}$. From Lemma 1 we have that

$$\forall \mathcal{S} \in \mathsf{Q}_m(h, z), \; C_m(\mathcal{S}, h|m, z) \leq G_{U_m|Y_*Z}(\mathcal{S}|m, z)$$

implies that

$$\forall \mathcal{S} \in \mathsf{F}(\mathcal{R}_{U_m}), \; C_m(\mathcal{S}, h|m, z) \leq G_{U_m|Y_*Z}(\mathcal{S}|m, z).$$

We need to show that (3.9),

$$C_m(\mathcal{S}, h|m, z) \leq G_{U_m|Y_*Z}(\mathcal{S}|m, z), \tag{A.4}$$

for all $\mathcal{S} \in \mathsf{F}(\mathcal{R}_{U_m})$ implies that (3.8),

$$C(\mathcal{S}, h|m, z) \leq G_{U|Y_*Z}(\mathcal{S}|m, z).$$

for all $\mathcal{S} \in \mathsf{F}(\mathcal{R}_U)$.

To show this, start with

$$C(\mathcal{S}, h|m, z) \leq G_{U|Y_*Z}(\mathcal{S}|m, z). \tag{A.5}$$

for an arbitrary $\mathcal{S} \in \mathsf{F}(\mathcal{R}_U)$.

First suppose that it does not hold that the projection of $\mathcal{S}$ onto its n_{th} component, $\mathcal{S}_n$, $n \neq m$, is equal to $\mathcal{R}_{U_n}$. All elements of the support of $\mathcal{U}(Y, Z; h)$ conditional on $(Y_*, Z) = (m, z)$ have $\mathcal{U}_n(Y, Z; h) = \mathcal{R}_{U_n}$, implying that $C(\mathcal{S}, h|m, z) = 0$ and (A.5) is trivially satisfied.

We now turn to sets $\mathcal{S}$ with n^{th} projection $\mathcal{S}_n$, $n \neq m$, equal to $\mathcal{R}_{U_n}$. In this case $C(\mathcal{S}, h|m, z) = C_m(\mathcal{S}_m, h|m, z)$ and $G_{U|Y_*Z}(\mathcal{S}|m, z) = G_{U_m|Y_*Z}(\mathcal{S}_m|m, z)$, so that (A.5) is in fact equivalent to (A.4), completing the proof. □

Proof of Theorem 4. We start with the characterization given in Theorem 3:

$$\mathcal{M}^*(F_{YZ}) = \left\{ \begin{array}{c} \left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M} : \exists \mathcal{G}_{U|Y_*Z} \text{ s.t. } \forall m \in \{1, ..., M\}, \forall \mathcal{S} \in \mathsf{Q}_m(h, z), \\ C_m(\mathcal{S}, h|m, z) \leq G_{U_m|Y_*Z}(\mathcal{S}|m, z) \text{ a.e. } z \in \mathcal{R}_Z, \text{ and} \\ G_{U_m|Z}(\mathcal{S}|z) = \int\limits_{y_* \in \mathcal{R}_{Y_*}} G_{U_m|Y_*Z}(\mathcal{S}|y_*, z)\, dF_{Y_*|Z}(y_*|z) \text{ a.e. } z \in \mathcal{R}_Z \end{array} \right\}.$$

Using Restriction CI $G_{U_m|Y_*Z}(\mathcal{S}_m|m, z) = G_{U_m|Z}(\mathcal{S}_m|z)$ so we obtain

$$\mathcal{M}^*(F_{YZ}) = \left\{ \begin{array}{c} \left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M} : \forall m \in \{1, ..., M\}, \forall \mathcal{S} \in \mathsf{Q}_m(h, z), \\ C_m(\mathcal{S}, h|m, z) \leq G_{U_m|Z}(\mathcal{S}|z) \text{ a.e. } z \in \mathcal{R}_Z \end{array} \right\},$$

equivalently

$$\mathcal{M}^*(F_{YZ}) = \left\{ \left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M} : \sup_{(m,z) \in \mathcal{R}_{Y_*Z}} \sup_{\mathcal{S} \in \mathsf{Q}_m(h,z)} C_m(\mathcal{S}, h|m, z) - G_{U_m|Y_*Z}(\mathcal{S}|z) \leq 0 \right\}. \quad \square$$

Proof of Theorem 5. The Theorem is proved using the same argument as in the proof of Theorem 4 but now with structures $\left(h, \mathcal{G}_{U|Z}\right)$ required to belong to a more restrictive set such that Restriction CI and Restriction MI both hold. Thus the set of structures $\left(h, \mathcal{G}_{U|Z}\right)$ satisfying these restrictions (i.e. those such that $\left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M}$) and also satisfying the condition stated in the Theorem, namely

$$\forall m \in \{1, \ldots, M\}, \forall \mathcal{S} \in \mathsf{Q}_m(h, z), C_m(\mathcal{S}, h|m, z) \leq G_{U_m|Z}(\mathcal{S}|z), \text{ a.e. } z \in \mathcal{R}_Z$$

are by Theorem 3 precisely those $\left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M}$ satisfying

$$\forall \mathcal{S} \in \mathsf{F}(\mathcal{R}_U), C(\mathcal{S}, h|m, z) \leq G_{U|Y_*Z}(\mathcal{S}|m, z) \text{ a.e. } (y_*, z) \in \mathcal{R}_{Y_*Z},$$

where the conditional distribution of U given (Y_*, Z) satisfies the conditional independence restriction $G_{U|Y_*Z}(\mathcal{S}|m, z) = G_{U|Z}(\mathcal{S}|z)$. Application of Artstein's Inequality as in Corollary 1 then gives that this collection of $\left(h, \mathcal{G}_{U|Z}\right) \in \mathcal{M}$ satisfies the selectionability criteria of Theorem 1, namely that $G_{U|Y_*Z}(\cdot|m, z)$ is selectionable with respect to the conditional distribution of random set $\mathcal{U}(Y, Z; h)$ given $(Y_* = m \wedge Z = z)$ induced by the distribution of Y conditional on $(Y_* = m \wedge Z = z)$ as given by F_{YZ}, a.e. $(m, z) \in \mathcal{R}_{Y_*Z}$. Thus $\mathcal{M}^*(F_{YZ})$ is the identified set of structures $\left(h, \mathcal{G}_{U|Z}\right)$. □