Dedicated to the memory of Nicu Boboc

AN ALTERNATIVE PROOF OF WELL-POSEDNESS OF STOCHASTIC EVOLUTION EQUATIONS IN THE VARIATIONAL SETTING

CARLO MARINELLI, LUCA SCARPA, and ULISSE STEFANELLI

Communicated by Lucian Beznea

We present a new proof of well-posedness of stochastic evolution equations in variational form, relying solely on a (nonlinear) infinite-dimensional approximation procedure rather than on classical finite-dimensional projection arguments of Galerkin type.

AMS 2010 Subject Classification: 60H15, 46N30, 35R60.

Key words: stochastic evolution equations, variational approach, monotone operators.

1. INTRODUCTION

Let us consider a stochastic evolution equation of the type

$$dX(t) + A(t, X(t))\,dt = B(t, X(t))\,dW(t), \qquad X(0) = X_0, \tag{1.1}$$

in the so-called variational setting, i.e. where A is a random time-dependent nonlinear maximal monotone operator from a reflexive Banach space V to its dual V', with V densely and continuously embedded in a Hilbert space H. Moreover, W is a cylindrical Wiener process (possibly defined on a further separable Hilbert space), and B is a random time-dependent map with values in a suitable space of Hilbert-Schmidt operators. Precise assumptions on the data of the problem are given in §2 below.

This class of equations was introduced and studied by Pardoux in [11], extending to the stochastic setting the classical well-posedness results by Lions (see, e.g., [6]) for equations without noise. More precisely, in [11] the operator A is time-dependent but non-random and the pair (A, B) needs to satisfy coercivity and boundedness assumptions, complemented by a local Lipschitz continuity condition on B. The general case where A can be random was considered by Krylov and Rozovskiĭ in [5], who also showed that the local Lipschitz continuity on B is not needed (see also [7, 12] for comprehensive treatments and recent developments). Under coercivity and boundedness assumptions on A and a Lipschitz continuity assumption on B (a set of hypotheses to which we shall refer to as *disjoint* assumptions), Pardoux [11, Chapter 3, §1] proved well-posedness of (1.1) by a clever, but at the same time natural extension of the deterministic theory, employing an infinite-dimensional argument based on Picard iterations in suitable spaces of processes. As a second step, assuming that the pair (A, B) satisfies a *joint* coercivity and boundedness assumption and that B is locally Lipschitz continuous, well-posedness for (1.1) is proved by a different method, i.e. by finite-dimensional approximations of Galerkin type (see [11, Chapter 3, §3]). As mentioned above, the joint assumption is shown to imply well-posedness without any local Lipschitz continuity condition in B in [5], again using finite-dimensional approximations.

Our goal is to show that existence of solutions under the general joint assumptions on (A, B), as in [5], can be obtained relying only on infinite-dimensional arguments, i.e. in the same spirit of the approach adopted in the first part of [11]. Moreover, since the existence proof in [5] relies on quite advanced results for finite-dimensional stochastic differential equations, the alternative proof provided here could also be seen as a *simpler* proof. The main idea is, roughly speaking, to regularize the operator B through the resolvent of the operator A. The corresponding regularized problem is then shown to satisfy the stronger *disjoint* hypotheses, so that it can be solved, as in [11, Chapter 3, § 1], using infinite-dimensional techniques only. Uniform estimates on the solutions to the regularized equations are then established, which allow to pass to the limit obtaining a solution to the original problem.

Even though the use of Yosida approximations is a standard tool in the field of nonlinear deterministic and stochastic equations (see, e.g., [1, 2, 3, 8] for just a few examples among an enormous literature), it seems that the type of approximation introduced here is not found elsewhere. Let us also mention that another approach to stochastic equations in variational form, namely by reduction to the deterministic case, is developed in [1, §4.4], where, however, only the case of additive noise is considered. Moreover, well-posedness for certain classes of stochastic equations with multi-valued nonlinear drift term is obtained in [8] using the results in [5, 11] as starting point. The main point in [8] is to regularize the multi-valued drift coefficient by its Yosida approximation, thus obtaining a family of well-posed equations with single-valued drift satisfying the assumptions of the classical variational framework, and to show that the corresponding approximate solutions converge, under appropriate assumptions, among which the Lipschitz continuity of the diffusion coefficient, to a process solving the original equation.

2. SETTING AND MAIN RESULTS

Throughout the paper, $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ stands for a filtered probability space satisfying the so-called usual conditions, where $T > 0$ is a fixed final time, on which all random elements will be defined. Equality of processes is always meant in the sense of indistinguishability, unless otherwise stated. Moreover, U is a separable Hilbert space and W is a cylindrical Wiener process on it; H is a separable Hilbert space identified with its dual, and V is a separable reflexive Banach space continuously and densely embedded in H, so that, denoting the (topological) dual of V by V', $V \hookrightarrow H \hookrightarrow V'$ is a Gelfand triple. The scalar product and norm of H will be denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, while the norms of all other Banach spaces will be indicated by subscripts. Since the duality form between V and V' agrees with the scalar product of H in the usual sense, we shall denote the former by $\langle\cdot,\cdot\rangle$ as well. If E_1 and E_2 are Hilbert spaces, the space of Hilbert-Schmidt operators from E_1 to E_2 will be denoted by $\mathscr{L}^2(E_1; E_2)$.

The following assumptions will be in force throughout the paper.

(I) The operator $A : \Omega \times [0,T] \times V \to V'$ is progressively measurable and hemicontinuous, i.e., for every $x \in V$ the V'-valued process $A(\cdot,\cdot,x)$ is progressively measurable and the map

$$\mathbb{R} \ni r \longmapsto \langle A(\omega, t, x + ry), z\rangle$$

is continuous for every $\omega \in \Omega$, $t \in [0,T]$, and $x, y, z \in V$.

(II) The operator $B \colon \Omega \times [0,T] \times V \to \mathscr{L}^2(U;H)$ is progressively measurable, i.e., for every $x \in V$ the $\mathscr{L}^2(U;H)$-valued process $B(\cdot,\cdot,x)$ is progressively measurable.

(III) There exist constants $c_1 > 0$, $c_2 \geqslant 0$, $p \in\,]1, +\infty[$ and an adapted process $f \in L^1(\Omega \times (0,T))$ such that

$$\begin{aligned}\big\langle A(\omega,t,x) - A(\omega,t,y), x-y\big\rangle - \frac{1}{2}\big\|B(\omega,t,x) - B(\omega,t,y)\big\|^2_{\mathscr{L}^2(U;H)}&\\ \geqslant -c_2\|x-y\|^2&,\end{aligned}$$

$$\big\langle A(\omega,t,x), x\big\rangle - \frac{1}{2}\big\|B(\omega,t,x)\big\|^2_{\mathscr{L}^2(U;H)} \geqslant c_1\|x\|_V^p - c_2\|x\|^2 - f(\omega,t)$$

for every $x, y \in V$, $t \in [0,T]$ and $\omega \in \Omega$.

(IV) There exist a constant $C > 0$ and an adapted process $g \in L^1(\Omega \times (0,T))$ such that, setting $q := p/(p-1)$,

$$\big\|A(\omega,t,x)\big\|^q_{V'} \leqslant C\|x\|_V^p + g(\omega,t)$$

for every $x \in V$, $t \in [0,T]$ and $\omega \in \Omega$.

(V) $X_0 \in L^2(\Omega, \mathscr{F}_0; H)$.

We can give now the definition of strong solution for the equation.

Definition 2.1. A strong solution to (1.1) is a V-valued progressively measurable process X such that

$$X \in L^0(\Omega; C([0,T]; H)) \cap L^0(\Omega; L^p(0,T;V)),$$
$$B(\cdot,\cdot,X) \in L^0(\Omega; L^2(0,T; \mathscr{L}^2(U;H))),$$

and

$$X + \int_0^\cdot A(s, X(s))\, ds = X_0 + \int_0^\cdot B(s, X(s))\, dW(s),$$

as an identity in the sense of indistinguishable V'-valued processes.

The classical well-posedness result [5, 11] for equation (1.1) is as follows.

Theorem 2.2. *There exists a unique strong solution X to* (1.1). *Moreover,*

$$X \in L^2(\Omega; C([0,T]; H)) \cap L^p(\Omega; L^p(0,T;V))$$

and the solution map

$$\begin{aligned} L^2(\Omega, \mathscr{F}_0; H) &\longrightarrow C([0,T]; L^2(\Omega; H)) \\ X_0 &\longmapsto X \end{aligned}$$

is Lipschitz continuous.

As discussed above, in the next section we show that Theorem 2.2 can be proved relying only on infinite-dimensional arguments.

Remark 2.3. The proof of uniqueness of strong solution crucially relies on an Itô formula for the square of the H-norm. In [11] such formula is obtained under the assumption that an operator $C\colon V \to V'$ exists satisfying monotonicity, coercivity and boundedness conditions (see [11, p. 57]). These conditions coincide with those assumed here on $A(\omega,t)$ for all $(\omega,t) \in \Omega \times [0,T]$, hence are automatically verified. In general, the formula remains valid, without any connection to a specific stochastic equation, if the duality map $J\colon V \to V'$ is single-valued, which is the case if, e.g., V' is a strictly convex Banach space. Such an assumption is always satisfied in all applications to SPDEs we know of. On the other hand, the Itô formula in [5] does not require any "geometric" assumption on the Banach space V, but its proof is rather involved and relies on finite-dimensional projections. A simple proof of Itô's formula for the square of the H-norm in the variational setting, which relies just on infinite-dimensional arguments, is available, to the best of our knowledge, only in the case where V is a Hilbert space (see [4], as well as [9]).

3. **PROOF OF THEOREM 2.2**

With $\omega \in \Omega$ and $t \in [0,T]$ arbitrary but fixed, assumptions **(I)** and **(III)** imply that $\tilde{A} := A(\omega,t,\cdot) + c_2 I \colon V \to V'$ is maximal monotone. Let us show that the part of $\tilde{A}$ in H, denoted by $\tilde{A}_H$, is maximal monotone (as an operator in H) with domain $\mathsf{D}(\tilde{A}) := \{x \in V : A(\omega,t,x) \in H\}$. It suffices to show that, for any $y \in H$, the equation

$$x + \tilde{A}x = y$$

admits a solution $x \in \mathsf{D}(\tilde{A})$. Since $\tilde{A}$ is coercive on V by assumption **(III)**, it follows by maximal monotonicity that the equation admits a (unique) solution $x \in V$. This obviously implies $\tilde{A}x \in H$, i.e. $x \in \mathsf{D}(\tilde{A})$. We have hence shown that $\tilde{A}_H$ is a maximal monotone operator on H.

For every $\lambda > 0$ we define the resolvent operator

$$J_\lambda \colon \Omega \times [0,T] \times H \longrightarrow V$$

of $\tilde{A}$ setting, for every $x \in H$, $t \in [0,T]$ and $\omega \in \Omega$,

$$J_\lambda(\omega,t,x) + \lambda \tilde{A}(\omega,t,J_\lambda(\omega,t,x)) = x.$$

The maximal monotonicity of $\tilde{A}_H$ implies that, for every $(\omega,t) \in \Omega \times [0,T]$, $J_\lambda(\omega,t,\cdot)$ is a contraction and converges pointwise to the identity map of H as $\lambda \to 0$.

Moreover, we define the Yosida approximation of $\tilde{A}$ as the map

$$\begin{aligned}\tilde{A}_\lambda \colon \Omega \times [0,T] \times H &\longrightarrow H,\\ (\omega,t,x) &\longmapsto \tilde{A}(\omega,t,J_\lambda(\omega,t,x)).\end{aligned}$$

It follows by the contraction property of J_λ that, for every $(\omega,t) \in \Omega \times [0,T]$, $\tilde{A}_\lambda(\omega,t,\cdot)$ is Lipschitz continuous with Lipschitz constant bounded by $1/\lambda$.

3.1. **Regularized equation**

Let us introduce the family of operators indexed by $\lambda > 0$

$$\begin{aligned}B_\lambda \colon \Omega \times [0,T] \times H &\longrightarrow \mathscr{L}^2(U;H)\\ (\omega,t,x) &\longmapsto B(\omega,t,J_\lambda(\omega,t,x)).\end{aligned}$$

Since A, hence also J_λ, and B are progressively measurable, B_λ is progressively measurable as well for every $\lambda > 0$. Moreover, B_λ is Lipschitz continuous in its third argument, uniformly with respect to the other ones: in fact, thanks to assumption **(III)**, one has

$$
\begin{aligned}
\frac{1}{2}\big\|B_\lambda(\omega,t,x)-B_\lambda(\omega,t,y)\big\|^2_{\mathscr{L}^2(U;H)}&\\
&= \frac{1}{2}\big\|B(\omega,t,J_\lambda(\omega,t,x))-B(\omega,t,J_\lambda(\omega,t,y))\big\|^2_{\mathscr{L}^2(U;H)}\\
&\leqslant \big\langle A(\omega,t,J_\lambda(\omega,t,x))-A(\omega,t,J_\lambda(\omega,t,y)),J_\lambda(\omega,t,x)-J_\lambda(\omega,t,y)\big\rangle\\
&\qquad + c_2\|J_\lambda(\omega,t,x)-J_\lambda(\omega,t,y)\|^2\\
&= \big\langle \tilde{A}_\lambda(\omega,t,x)-\tilde{A}_\lambda(\omega,t,y),x-y\big\rangle \leqslant \frac{1}{\lambda}\|x-y\|^2
\end{aligned}
$$

for every $\omega\in\Omega$, $t\in[0,T]$, and $x,y\in H$.

Therefore, since $\tilde{A}_\lambda$ is also a Lipschitz continuous operator on H, well-posedness results for stochastic differential equations on Hilbert spaces (see, e.g., [10, §34]) yields the existence (and uniqueness) of a predictable process $X_\lambda\in L^2(\Omega;C([0,T];H))$ such that

$$
X_\lambda+\int_0^\cdot \tilde{A}_\lambda(s,X_\lambda(s))\,ds = X_0+c_2\int_0^\cdot X_\lambda(s)\,ds+\int_0^\cdot B_\lambda(s,X_\lambda(s))\,dW(s). \tag{3.1}
$$

3.2. **A priori estimates**

We are going to prove estimates on the family of solutions (X_λ) to the regularized equations obtained above that are uniform with respect to λ.

The integration by parts formula for Hilbert space valued semimartingales yields

$$
\begin{aligned}
\frac{1}{2}\|X_\lambda\|^2+\int_0^\cdot\big\langle\tilde{A}_\lambda(s,X_\lambda(s)),X_\lambda(s)\big\rangle\,ds-\frac{1}{2}\int_0^\cdot\big\|B_\lambda(s,X_\lambda(s))\big\|^2_{\mathscr{L}^2(U;H)}\,ds&\\
= \frac{1}{2}\|X_0\|^2+c_2\int_0^\cdot\|X_\lambda(s)\|^2\,ds+\int_0^\cdot X_\lambda(s)B_\lambda(s,X_\lambda(s))\,dW(s),&
\end{aligned}
$$

where, in the stochastic integral, X_λ is treated as a process taking values in the dual of H.

From now on we shall occasionally suppress the explicit indication of the dependence on ω and t for processes and operators for notational simplicity. Note that, by definition of Yosida approximation,

$$
\begin{aligned}
\big\langle\tilde{A}_\lambda x,x\big\rangle = \big\langle\tilde{A}J_\lambda x,x\big\rangle &= \big\langle\tilde{A}J_\lambda x,J_\lambda x\big\rangle+\big\langle\tilde{A}J_\lambda x,x-J_\lambda x\big\rangle\\
&= \big\langle\tilde{A}J_\lambda x,J_\lambda x\big\rangle+\lambda\big\|\tilde{A}_\lambda x\big\|^2
\end{aligned}
\tag{3.2}
$$

for every $x\in H$. Therefore, denoting the norm of $\mathscr{L}^2(U;H)$ by $\|\cdot\|_2$ for brevity, one has, thanks to assumption **(III)**,

$$\left\langle \tilde{A}_\lambda x, x\right\rangle - \frac{1}{2}\left\|B_\lambda(x)\right\|_2^2 = \left\langle \tilde{A}J_\lambda x, J_\lambda x\right\rangle - \frac{1}{2}\left\|B(J_\lambda x)\right\|_2^2 + \lambda\left\|\tilde{A}_\lambda x\right\|^2$$
$$\geqslant c_1\left\|J_\lambda x\right\|_V^p + \lambda\left\|\tilde{A}_\lambda x\right\|^2 - f,$$

which in turn yields

$$\begin{aligned}\frac{1}{2}\|X_\lambda\|^2 + c_1\int_0^\cdot \left\|J_\lambda X_\lambda(s)\right\|_V^p ds + \lambda\int_0^\cdot \left\|\tilde{A}_\lambda X_\lambda(s)\right\|^2 ds \\ \leqslant \frac{1}{2}\|X_0\|^2 + \int_0^\cdot f(s)\,ds + c_2\int_0^\cdot \|X_\lambda(s)\|^2\,ds \\ + \int_0^\cdot X_\lambda(s)B_\lambda(X_\lambda(s))\,dW(s),\end{aligned}$$

where the stochastic integral on the right-hand side is a martingale because B_λ is Lipschitz continuous. In particular, taking expectation on both sides,

$$\begin{aligned}\frac{1}{2}\,\mathbb{E}\|X_\lambda(t)\|^2 + c_1\,\mathbb{E}\int_0^t \left\|J_\lambda X_\lambda(s)\right\|_V^p ds + \lambda\,\mathbb{E}\int_0^t \left\|\tilde{A}_\lambda X_\lambda(s)\right\|^2 ds \\ \leqslant \frac{1}{2}\,\mathbb{E}\|X_0\|^2 + \mathbb{E}\int_0^t f(s)\,ds + c_2\,\mathbb{E}\int_0^t \|X_\lambda(s)\|^2\,ds\end{aligned} \tag{3.3}$$

for all $t \in [0,T]$. This implies that, for any interval $[0,T_0] \subseteq [0,T]$,

$$\sup_{t\leqslant T_0}\mathbb{E}\|X_\lambda(t)\|^2 \leqslant \mathbb{E}\|X_0\|^2 + 2\,\mathbb{E}\int_0^T f(s)\,ds + 2c_2\,\mathbb{E}\int_0^{T_0}\Big(\sup_{r\leqslant s}\mathbb{E}\|X_\lambda(r)\|^2\Big)\,ds,$$

so that, by Gronwall's inequality, (X_λ) is bounded in $C([0,T];L^2(\Omega;H))$. From this and (3.3) it immediately follows that $(\lambda^{1/2}\tilde{A}_\lambda X_\lambda)$ is bounded in $L^2(\Omega;L^2(0,T;H))$ and that $(J_\lambda X_\lambda)$ is bounded in $L^p(\Omega;L^p(0,T;V))$.
The latter implies, thanks to assumption **(IV)**, that $(\tilde{A}_\lambda X_\lambda)$ is bounded in $L^q(\Omega;L^q(0,T;V'))$. Moreover, since, by assumption **(III)**,

$$\frac{1}{2}\left\|B_\lambda(x)\right\|_2^2 = \frac{1}{2}\left\|B(J_\lambda x)\right\|_2^2 \leqslant \left\langle \tilde{A}J_\lambda x, J_\lambda x\right\rangle + f = \left\langle \tilde{A}_\lambda x, J_\lambda x\right\rangle + f,$$

Hölder's inequality yields

$$\begin{aligned}\mathbb{E}\int_0^T \left\|B_\lambda(X_\lambda(s))\right\|_2^2 ds &\leqslant \mathbb{E}\int_0^T \left\|\tilde{A}_\lambda X_\lambda\right\|_{V'}\left\|J_\lambda X_\lambda\right\|_V + \left\|f\right\|_{L^1(\Omega\times[0,T])} \\ &\leqslant \left\|\tilde{A}_\lambda X_\lambda\right\|_{L^q(\Omega;L^q(0,T;V'))}\left\|J_\lambda X_\lambda\right\|_{L^p(\Omega;L^p(0,T;V))} \\ &\qquad + \left\|f\right\|_{L^1(\Omega\times[0,T])},\end{aligned}$$

thus proving that $(B_\lambda(X_\lambda))$ is bounded in $L^2(\Omega;L^2(0,T;\mathscr{L}^2(U;H)))$.

3.3. Construction of a solution and its uniqueness

The boundedness of various families of processes indexed by λ obtained above imply, by well-known compactness properties in weak and weak* topologies, that there exist measurable and adapted processes

$$\begin{aligned}
&X \in L^\infty(0,T;L^2(\Omega;H)),\\
&\bar X \in L^p(\Omega;L^p(0,T;V)),\\
&Y \in L^q(\Omega;L^q(0,T;V')),\\
&G \in L^2(\Omega;L^2(0,T;\mathscr{L}^2(U;H)))
\end{aligned}$$

such that

$$\begin{aligned}
X_\lambda &\longrightarrow X && \text{weakly* in } L^\infty(0,T;L^2(\Omega;H)),\\
J_\lambda X_\lambda &\longrightarrow \bar X && \text{weakly in } L^p(\Omega;L^p(0,T;V)),\\
\lambda \tilde A_\lambda X_\lambda &\longrightarrow 0 && \text{in } L^2(\Omega;L^2(0,T;H)),\\
\tilde A_\lambda X_\lambda &\longrightarrow Y && \text{weakly in } L^q(\Omega;L^q(0,T;V')),\\
B_\lambda(\cdot,X_\lambda) &\longrightarrow G && \text{weakly in } L^2(\Omega;L^2(0,T;\mathscr{L}^2(U;H))).
\end{aligned}$$

Since, by definition of Yosida approximation, $X_\lambda - J_\lambda X_\lambda = \lambda \tilde A_\lambda X_\lambda$, the boundedness of $(\lambda^{1/2}\tilde A_\lambda X_\lambda)$ in $L^2(\Omega;L^2(0,T;H))$ implies that

$$X_\lambda - J_\lambda X_\lambda \longrightarrow 0 \qquad \text{in } L^2(\Omega;L^2(0,T;H)),$$

hence X and $\bar X$ are equal $\mathbb{P}\otimes$ Leb-a.e. in $\Omega\times[0,T]$ and belong to

$$L^\infty(0,T;L^2(\Omega;H)) \cap L^p(\Omega;L^p(0,T;V)).$$

Recalling that the linear operator

$$\begin{aligned}
L^r(\Omega;L^1(0,T;E)) &\longrightarrow L^r(\Omega;C([0,T];E))\\
w &\longmapsto \int_0^\cdot w(s)\,ds
\end{aligned}$$

is continuous for any $r\in[1,\infty[$ and any Banach space E, hence also weakly continuous, it follows from the above convergence results that

$$\begin{aligned}
\int_0^\cdot X_\lambda(s)\,ds &\longrightarrow \int_0^\cdot X(s)\,ds && \text{weakly in } L^p(\Omega;C([0,T];V)),\\
\int_0^\cdot \tilde A_\lambda X_\lambda(s)\,ds &\longrightarrow \int_0^\cdot Y(s)\,ds && \text{weakly in } L^q(\Omega;C([0,T];V')).
\end{aligned}$$

Similarly, since the stochastic integral operator

$$\begin{aligned} L^2(\Omega;L^2(0,T;\mathscr{L}^2(U;H)) &\longrightarrow L^2(\Omega;C([0,T];H)) \\ C &\longmapsto \int_0^\cdot C(s)\,dW(s) \end{aligned}$$

is linear and continuous, hence weakly continuous, one gets that

$$\int_0^\cdot B_\lambda(s,X_\lambda(s))\,dW(s) \longrightarrow \int_0^\cdot G(s)\,dW(s) \qquad \text{weakly in } L^2(\Omega;C([0,T];H)).$$

This implies, taking the weak limit in the space $L^q(\Omega;C([0,T];V'))$ as $\lambda\to 0$ in equation (3.1),

$$X + \int_0^\cdot Y(s)\,ds = X_0 + c_2\int_0^\cdot X(s)\,ds + \int_0^\cdot G(s)\,dW(s) \qquad \text{in } V'. \tag{3.4}$$

We are going to show, using an adaptation of a classical argument from the theory of maximal monotone operators (cf. [5] as well as [7, § 4.2]), that $Y = A(\cdot,X)+c_2X$ and $G=B(X)$. Itô's formula for the square of the H-norm applied to (3.1) yields

$$\begin{aligned} \frac12\big\|e^{-c_2\cdot}X_\lambda\big\|^2 &+ \int_0^\cdot e^{-2c_2s}\big\langle \tilde A_\lambda X_\lambda(s)), X_\lambda(s)\big\rangle\,ds \\ &\qquad - \frac12\int_0^\cdot e^{-c_2s}\big\|B_\lambda(X_\lambda(s))\big\|^2_{\mathscr{L}^2(U;H)}\,ds \\ &= \frac12\|X_0\|^2 + \int_0^\cdot e^{-c_2s}X_\lambda(s)B_\lambda(X_\lambda(s))\,dW(s). \end{aligned}$$

Since the stochastic integral on the right-hand side is a martingale, as seen above, one has, applying inequality (3.2) and taking expectations on both sides,

$$\begin{aligned} &\frac12\,\mathbb{E}\big\|e^{-c_2\cdot}X_\lambda\big\|^2 - \frac12\,\mathbb{E}\|X_0\|^2 \\ &\qquad \leqslant \mathbb{E}\int_0^\cdot e^{-2c_2s}\Big(-\big\langle \tilde A J_\lambda X_\lambda(s), J_\lambda X_\lambda(s)\big\rangle + \frac12\big\|B(J_\lambda X_\lambda(s))\big\|^2_{\mathscr{L}^2(U;H)}\Big)\,ds. \end{aligned}$$

Let $\varphi\in L^2(\Omega;C([0,T];H))\cap L^p(\Omega;L^p(0,T;V))$.

Thanks to assumptions **(III)–(IV)**, one has that $\tilde A\varphi\in L^q(\Omega;L^q(0,T;V'))$ and $B(\varphi)\in L^2(\Omega;L^2(0,T;\mathscr{L}^2(U;H)))$, so that the term within parentheses on right-hand side can be rewritten as

$$\begin{aligned} &-\big\langle \tilde A J_\lambda X_\lambda - \tilde A\varphi, J_\lambda X_\lambda - \varphi\big\rangle + \frac12\big\|B(J_\lambda X_\lambda) - B(\varphi)\big\|^2_{\mathscr{L}^2(U;H)} \\ &\qquad - \big\langle \tilde A\varphi, J_\lambda X_\lambda\big\rangle - \big\langle \tilde A J_\lambda X_\lambda - \tilde A\varphi, \varphi\big\rangle \\ &\qquad - \frac12\big\|B(\varphi)\big\|^2_{\mathscr{L}^2(U;H)} + \big\langle B(J_\lambda X_\lambda), B(\varphi)\big\rangle_{\mathscr{L}^2(U;H)}, \end{aligned}$$

where the sum of the two terms in the first row is negative by assumption **(III)**. One is thus left with

$$
\begin{aligned}
&\frac{1}{2}\,\mathbb{E}\big\|e^{-c_2\cdot}X_\lambda\big\|^2-\frac{1}{2}\,\mathbb{E}\|X_0\|^2\\
&\qquad\leqslant \mathbb{E}\int_0^\cdot e^{-2c_2s}\Big(-\big\langle\tilde A\varphi(s),J_\lambda X_\lambda(s)\big\rangle-\big\langle\tilde AJ_\lambda X_\lambda(s)-\tilde A\varphi(s),\varphi(s)\big\rangle\\
&\qquad\qquad-\frac{1}{2}\big\|B(\varphi(s))\big\|^2_{\mathscr L^2(U;H)}+\big\langle B(J_\lambda X_\lambda(s)),B(\varphi(s))\big\rangle_{\mathscr L^2(U;H)}\Big)\,ds,
\end{aligned}
$$

from which it follows that, for every nonnegative $\psi\in L^\infty(0,T)$,

$$
\begin{aligned}
&\frac{1}{2}\,\mathbb{E}\int_0^T\psi(t)\big(e^{-2c_2t}\|X_\lambda(t)\|^2-\|X_0\|^2\big)\,dt\\
&\qquad\leqslant \mathbb{E}\int_0^T\psi(t)\Big(\int_0^t e^{-2c_2s}\Big(-\big\langle\tilde A\varphi(s),J_\lambda X_\lambda(s)\big\rangle-\big\langle\tilde AJ_\lambda X_\lambda(s)-\tilde A\varphi(s),\varphi(s)\big\rangle\\
&\qquad\qquad-\frac{1}{2}\big\|B(\varphi(s))\big\|^2_{\mathscr L^2(U;H)}+\big\langle B(J_\lambda X_\lambda(s)),B(\varphi(s))\big\rangle_{\mathscr L^2(U;H)}\Big)\,ds\Big)\,dt.
\end{aligned}
$$

By the weak lower semicontinuity of the norm in $L^2(\Omega;L^2(0,T;H))$, one has

$$
\begin{aligned}
&\mathbb{E}\int_0^T\psi(t)\big(e^{-2c_2t}\|X(t)\|^2-\|X_0\|^2\big)\,dt\\
&\qquad\qquad\leqslant \liminf_{\lambda\to0}\mathbb{E}\int_0^T\psi(t)\big(e^{-2c_2t}\|X_\lambda(t)\|^2-\|X_0\|^2\big)\,dt.
\end{aligned}
$$

Moreover, since $B(\varphi)\in L^2(\Omega;L^2(0,T;\mathscr L^2(U;H)))$, the weak convergence results proved above yield

$$
\begin{aligned}
&\lim_{\lambda\to0}\mathbb{E}\int_0^T\psi(t)\Big(\int_0^t e^{-2c_2s}\Big(-\big\langle\tilde A\varphi(s),J_\lambda X_\lambda(s)\big\rangle-\big\langle\tilde AJ_\lambda X_\lambda(s)-\tilde A\varphi(s),\varphi(s)\big\rangle\\
&\qquad\qquad-\frac{1}{2}\big\|B(\varphi(s))\big\|^2_{\mathscr L^2(U;H)}+\big\langle B(J_\lambda X_\lambda(s)),B(\varphi(s))\big\rangle_{\mathscr L^2(U;H)}\Big)\,ds\Big)\,dt\\
&=\mathbb{E}\int_0^T\psi(t)\Big(\int_0^t e^{-2c_2s}\Big(-\big\langle\tilde A(\varphi(s)),X(s)\big\rangle-\big\langle Y(s)-\tilde A(\varphi(s)),\varphi(s)\big\rangle\\
&\qquad\qquad-\frac{1}{2}\big\|B(\varphi(s))\big\|^2_{\mathscr L^2(U;H)}+\big\langle G(s),B(\varphi(s))\big\rangle_{\mathscr L^2(U;H)}\Big)\,ds\Big)\,dt.
\end{aligned}
$$

We then deduce that

$$
\begin{aligned}
&\frac{1}{2}\,\mathbb{E}\int_0^T\psi(t)\big(e^{-2c_2t}\|X(t)\|^2-\|X_0\|^2\big)\,dt\\
&\qquad\leqslant \mathbb{E}\int_0^T\psi(t)\Big(\int_0^t e^{-2c_2s}\Big(-\big\langle\tilde A\varphi(s),X(s)\big\rangle-\big\langle Y(s)-\tilde A\varphi(s),\varphi(s)\big\rangle
\end{aligned}
$$

$$-\frac{1}{2}\big\|B(\varphi(s))\big\|^2_{\mathscr{L}^2(U;H)} + \big\langle G(s), B(\varphi(s))\big\rangle_{\mathscr{L}^2(U;H)}\Big)\,ds\Big)\,dt.$$

Itô's formula (see [5] or [7, §4]) applied to the limit equation (3.4) implies that there exists a modification of X, denoted by the same symbol for simplicity, such that $X \in L^2(\Omega; C([0,T];H))$ and

$$\frac{1}{2}e^{-2c_2\cdot}\|X\|^2 + \int_0^\cdot e^{-2c_2 s}\big\langle Y(s), X(s)\big\rangle\,ds - \frac{1}{2}\int_0^\cdot e^{-2c_2 s}\big\|G(s)\big\|^2_{\mathscr{L}^2(U;H)}\,ds$$
$$= \frac{1}{2}\|X_0\|^2 + \int_0^\cdot e^{-2c_2 s}X(s)G(s)\,dW(s).$$

Substituting this identity on the left-hand side of the last inequality and rearranging the terms yields

$$\mathbb{E}\int_0^T \psi(t)\int_0^t e^{-2c_2 s}\Big(\big\langle Y(s) - \tilde{A}\varphi(s), \varphi(s) - X(s)\big\rangle$$
$$+ \frac{1}{2}\big\|G(s) - B(\varphi(s))\big\|^2_{\mathscr{L}^2(U;H)}\Big)\,ds\,dt \leqslant 0.$$

Choosing $\varphi = X$ immediately yields $G = B(X)$, hence also, in particular,

$$\mathbb{E}\int_0^T \psi(t)\int_0^t e^{-2c_2 s}\big\langle Y(s) - \tilde{A}\varphi(s), \varphi(s) - X(s)\big\rangle\,ds\,dt \leqslant 0.$$

Let $\delta \in \mathbb{R}_+$, $v \in V$, and $\bar{\varphi} \in L^\infty(\Omega\times[0,T])$. Choosing $\varphi = X + \delta\bar{\varphi}v$ and taking the limit as $\delta \to 0$ yields

$$\mathbb{E}\int_0^T \psi(t)\int_0^t e^{-2c_2 s}\bar{\varphi}(s)\big\langle Y(s) - \tilde{A}\varphi(s), v\big\rangle\,ds\,dt \leqslant 0.$$

By a classical localisation argument, recalling that ψ and $\bar{\varphi}$ have been chosen arbitrarily, one has

$$\big\langle Y - \tilde{A}\varphi, v\big\rangle \leqslant 0 \qquad \forall v \in V$$

a.e. in $\Omega \times [0,T]$. Since v has also been chosen arbitrarily in V, it follows that

$$\big\langle Y - \tilde{A}\varphi, X - \varphi\big\rangle \geqslant 0$$

a.e. in $\Omega \times [0,T]$. Since $\tilde{A}$ is maximal monotone, this implies that $Y = \tilde{A}X$ a.e. in $\Omega \times [0,T]$. We have thus shown that X is a strong solution to equation (1.1).

In order to prove the Lipschitz continuous dependence of the solution with respect to the initial datum (from which uniqueness follows), let X_1, X_2 be strong solutions to (1.1), in the sense of Theorem 2.2, with initial data X_0^1 and X_0^2, respectively. Itô formula (as in [5] or [7]) yields

$$
\begin{aligned}
&\frac{1}{2}\big\|(X_1 - X_2)\big\|_H^2 \\
&\qquad + \int_0^\cdot \Big(\big\langle AX_1 - AX_2, X_1 - X_2\big\rangle - \frac{1}{2}\big\|B(X_1) - B(X_2)\big\|^2_{\mathscr{L}^2(U;H)}\Big)(s)\,ds \\
&\qquad\qquad = \frac{1}{2}\big\|X_0^1 - X_0^2\big\|^2 + \int_0^\cdot \big((X_1 - X_2)(B(X_1) - B(X_2))\big)(s)\,dW(s),
\end{aligned}
$$

where the stochastic integral on the right-hand side is a martingale because X_1, $X_2 \in L^2(\Omega; C([0,T]; H))$ and $B(X_1)$, $B(X_2) \in L^2(\Omega; L^2(0,T; \mathscr{L}^2(U;H)))$. Therefore, taking expectations on both sides and using assumption **(III)**,

$$
\mathbb{E}\big\|(X_1 - X_2)(t)\big\|^2 \leqslant \mathbb{E}\big\|X_0^1 - X_0^2\big\|^2 + 2c_2 \int_0^t \mathbb{E}\big\|(X_1 - X_2)(s)\big\|^2\,ds \qquad \forall t \in [0,T],
$$

from which the conclusion follows thanks to Gronwall's inequality. The proof of Theorem 2.2 is thus complete.

Acknowledgments. CM gratefully acknowledges the hospitality of the Interdisziplinäres Zentrum für Komplexe Systeme, Universität Bonn, Germany. LS and US are partially funded by the FWF-DFG grant I 4354, the FWF grants F 65, P 32788, and M 2876, and the Vienna Science and Technology Fund (WWTF) through Project MA14-009.

REFERENCES

[1] V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces.* Springer, New York, 2010.

[2] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam, 1973.

[3] T. E. Govindan, *Yosida approximations of stochastic differential equations in infinite dimensions and applications.* Springer, Cham, 2016.

[4] N. V. Krylov, *A relatively short proof of Itô's formula for SPDEs and its applications.* Stoch. Partial Differ. Equ. Anal. Comput. **1** (2013), *1*, 152–174.

[5] N. V. Krylov and B. L. Rozovskiĭ, *Stochastic evolution equations.* In: *Current problems in mathematics*, Vol. 14 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979, pp. 71–147, 256.

[6] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Gauthier-Villars, Paris, 1969.

[7] W. Liu and M. Röckner, *Stochastic partial differential equations: an introduction.* Springer, Cham, 2015.

[8] W. Liu and M. Stephan, *Yosida approximations for multivalued stochastic partial differential equations driven by Lévy noise on a Gelfand triple.* J. Math. Anal. Appl. **410** (2014), *1*, 158–178.

[9] C. Marinelli and L. Scarpa, *Well-posedness of monotone semilinear SPDEs with semimartingale noise.* Preprint, arXiv:1805.07562.

[10] M. Métivier, *Semimartingales.* Walter de Gruyter & Co., Berlin, 1982.

[11] E. Pardoux, *Equations aux derivées partielles stochastiques nonlinéaires monotones.* Ph.D. thesis, Université Paris XI, 1975.

[12] B. L. Rozovsky and S. V. Lototsky, *Stochastic evolution systems.* Springer, Cham, 2018.

University College London
Department of Mathematics
Gower Street, London WC1E 6BT, U.K.
URL: `http://goo.gl/4GKJP`

University of Vienna
Faculty of Mathematics
Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
`luca.scarpa@univie.ac.at`

University of Vienna
Faculty of Mathematics
Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
and
Vienna Research Platform on Accelerating
Photoreaction Discovery,
University of Vienna,
Währinger Str. 17, 1090 Vienna, Austria
and
Istituto di Matematica Applicata e Tecnologie
Informatiche "E. Magenes" – CNR
via Ferrata 1, I-27100 Pavia, Italy
`ulisse.stefanelli@univie.ac.at`