MULTICHANNEL SCATTERING THEORY FOR TOEPLITZ OPERATORS WITH PIECEWISE CONTINUOUS SYMBOLS

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ABSTRACT. Self-adjoint Toeplitz operators have purely absolutely continuous spectrum. For Toeplitz operators T with piecewise continuous symbols, we suggest a further spectral classification determined by propagation properties of the operator T, that is, by the behavior of $\exp(-iTt)f$ for $t \to \pm\infty$. It turns out that the spectrum is naturally partitioned into three disjoint subsets: *thick*, *thin* and *mixed* spectra. On the thick spectrum, the propagation properties are modelled by the continuous part of the symbol, whereas on the thin spectrum, the model operator is determined by the jumps of the symbol. On the mixed spectrum, these two types of the asymptotic evolution of $\exp(-iTt)f$ coexist. This classification is justified in the framework of scattering theory. We prove the existence of wave operators that relate the model operators with the Toeplitz operator T. The ranges of these wave operators are pairwise orthogonal, and their orthogonal sum exhausts the whole space, i.e., the set of these wave operators is asymptotically complete.

1. INTRODUCTION

1.1. **Basic notions.** The Toeplitz operator $T = T(\omega)$ with symbol $\omega(\zeta)$ is defined on the Hardy space $\mathbb{H}^2 = \mathbb{H}^2_+$ on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ by the formula

$$(Tu)(\zeta) = (\mathbb{P}\omega u)(\zeta), \quad u \in \mathbb{H}^2, \ \zeta \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{P} = \mathbb{P}_+ : L^2(\mathbb{T}) \to \mathbb{H}^2$ is the orthogonal projection onto $\mathbb{H}^2$. The normalized Lebesgue measure on $\mathbb{T}$ is denoted by $d\mathbf{m}(\zeta) = (2\pi i\zeta)^{-1}d\zeta$. Throughout the paper we assume that $\omega \in L^\infty(\mathbb{T})$, so that the operator T is bounded. If ω is real-valued, then the operator T is self-adjoint.

In the early 60's M. Rosenblum (see [11, 12, 13]) established a number of fundamental spectral properties of self-adjoint Toeplitz operators T. Namely if ω is a

non-constant function, then T is absolutely continuous and its spectrum $\sigma(T)$ fills the interval $[\gamma_1, \gamma_2]$ where

$$\gamma_1 = \operatorname{ess\,inf} \omega, \quad \gamma_2 = \operatorname{ess\,sup} \omega. \tag{1.2}$$

Furthermore, M. Rosenblum (see also the paper [6] by R. S. Ismagilov) has constructed a spectral representation of T and described its spectral multiplicity.

In this paper we consider Toeplitz operators with piecewise continuous symbols ω. It is natural to compare T with the operator $\mathbf{\Omega}$ of multiplication by a piecewise continuous function ω on $L^2(\mathbb{T})$. The spectrum $\sigma(\mathbf{\Omega})$ is the closure $\operatorname{clos} \omega(\mathbb{T})$ of the set $\omega(\mathbb{T})$. A point $\lambda \in \sigma(\mathbf{\Omega})$ is an eigenvalue of $\mathbf{\Omega}$ if and only if the measure $\mathbf{m}\big(\{\zeta \in \mathbb{T} : \omega(\zeta) = \lambda\}\big)$ is positive. If ω is piecewise C^1, then, apart from eigenvalues, the spectrum of $\mathbf{\Omega}$ is absolutely continuous, see Lemma 2.2. Also, the multiplicity of the spectrum is found as the number of solutions of the equation $\omega(\zeta) = \lambda$. In particular, the eigenvalues of $\mathbf{\Omega}$ can have only infinite multiplicity.

Our main aim is to construct scattering theory for Toeplitz operators with piecewise continuous symbols. We start, however, with the case of smooth symbols and prove the existence, isometry and completeness of the wave operators[1] $W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P})$ for the pair $\mathbf{\Omega}, T(\omega)$ and the "identification" $\mathbb{P} : L^2(\mathbb{T}) \to \mathbb{H}^2$. For smooth symbols, we have $\sigma(T) = \sigma(\mathbf{\Omega})$.

For general piecewise continuous symbols, $\sigma(\mathbf{\Omega})$ may have gaps, so only the inclusion $\sigma(\mathbf{\Omega}) \subset [\gamma_1, \gamma_2] = \sigma(T)$ holds. At first glance, this looks counter-intuitive since T is a compression of $\mathbf{\Omega}$. The set $\sigma(T)$ splits into two disjoint subsets $\sigma(\mathbf{\Omega})$ and $\sigma(T) \setminus \sigma(\mathbf{\Omega})$. It turns out that the spectral nature of these two components is qualitatively different. In some sense, the spectrum of the Toeplitz operator T is *thin* on $\sigma(T) \setminus \sigma(\mathbf{\Omega})$, and it is *thick* (or *mixed*) on the set $\sigma(\mathbf{\Omega})$. We give precise definitions of these notions and prove corresponding results in the scattering theory framework, studying the asymptotic behaviour of the time evolution $e^{-iTt} f$ as $t \to \pm\infty$ for various $f \in \mathbb{H}^2$.

1.2. **Truncated Toeplitz matrices.** Previously, Toeplitz operators with discontinuous symbols were extensively studied in a different context: instead of the operator T one considered *truncated* Toeplitz matrices $T_n = T_n(\omega)$ as $n \to \infty$. Here T_n are defined as follows. Let $\mathcal{P}_n \subset \mathbb{H}^2$ be the subspace of all polynomials of degree $\leq n$, and let P_n be the orthogonal projection onto $\mathcal{P}_n$. The operator T_n acting on the space $\mathcal{P}_n$ is defined by the formula $T_n f = \mathsf{P}_n T f$. The difference between the thick and thin components of $\sigma(T)$ becomes clearly visible when one studies the asymptotics of the eigenvalue counting function $N_n(a, b)$ for the matrix T_n on an interval (a, b) as $n \to \infty$. It is straightforward to deduce from the classical result in [4, Sect 5.2] the following formula. If $\mathbf{m}\big(\{\zeta : \omega(\zeta) = a\}\big) = \mathbf{m}\big(\{\zeta : \omega(\zeta) = b\}\big) = 0$,

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[1]See Sect. 3.1, for the precise definition

then
$$\lim_{n\to\infty} n^{-1} N_n(a,b) = \mathbf{m}\big(\{\zeta : a < \omega(\zeta) < b\}\big).$$

In particular, if $(a, b) \subset \sigma(T) \setminus \sigma(\mathbf{\Omega})$, then the right-hand side is zero. This is consistent with our choice of the term "thin spectrum" for the set $\sigma(T) \setminus \sigma(\mathbf{\Omega})$.

Starting with the pioneering Szegő's paper [15], a substantial body of research was focused on the asymptotic behaviour of the determinants of the truncated Toeplitz matrices $T_n = T_n(\omega)$, as $n \to \infty$, with complex-valued ω. The case of discontinuous symbols ω attracted a special attention in connection with the so-called Fischer-Hartwig formula, see [3] and [2, 7].

1.3. **Scattering theory framework.** For smooth symbols ω, the asymptotic behaviour of the evolution $e^{-iTt} f$ as $t \to \pm\infty$ is described by the same formula for all $f \in \mathbb{H}^2$. Precisely, for every f there exists an $f_{\pm} \in L^2(\mathbb{T})$ such that

$$e^{-iTt} f \sim \mathbb{P} e^{-i\mathbf{\Omega}t} f_{\pm}, \quad t \to \pm\infty, \tag{1.3}$$

where the symbol "$\sim$" means that the difference of the left- and right-hand sides tends to zero.

To describe the evolution $e^{-iTt} f$ for discontinuous symbols ω, we distinguish three components of the spectrum $\sigma(T)$. Assume that ω is continuous on $\mathbb{T}$ apart from some finite subset $\mathsf{S} \subset \mathbb{T}$. For each $\eta \in \mathsf{S}$, we set

$$a_- = \min\{\omega(\eta - 0), \omega(\eta + 0)\}, \quad a_+ = \max\{\omega(\eta - 0), \omega(\eta + 0)\},$$

and define

$$\Lambda_\eta = [a_-, a_+], \quad \Upsilon = \bigcup_{\eta \in \mathsf{S}} \Lambda_\eta. \tag{1.4}$$

Clearly,

$$\sigma(T) = \sigma(\mathbf{\Omega}) \cup \Upsilon \quad \text{and} \quad \sigma(T) \setminus \sigma(\mathbf{\Omega}) \subset \Upsilon \subset \sigma(T).$$

Denoting by $E_T(X)$, $X \subset \mathbb{R}$, the spectral projection of the operator T, introduce the pair-wise orthogonal subspaces

$$\left. \begin{aligned} \mathcal{H}_{\text{thin}} &= \operatorname{Ran} E_T\big(\sigma(T) \setminus \sigma(\mathbf{\Omega})\big), \\ \mathcal{H}_{\text{thick}} &= \operatorname{Ran} E_T\big(\sigma(\mathbf{\Omega}) \setminus \Upsilon\big), \\ \mathcal{H}_{\text{mix}} &= \operatorname{Ran} E_T\big(\sigma(\mathbf{\Omega}) \cap \Upsilon\big), \end{aligned} \right\} \tag{1.5}$$

so that

$$\mathcal{H}_{\text{thin}} \oplus \mathcal{H}_{\text{thick}} \oplus \mathcal{H}_{\text{mix}} = \mathbb{H}^2.$$

Definition 1.1. The spectrum of the operator T on the subspaces $\mathcal{H}_{\text{thin}}$, $\mathcal{H}_{\text{thick}}$ and $\mathcal{H}_{\text{mix}}$ is said to be thin, thick and mixed, respectively.
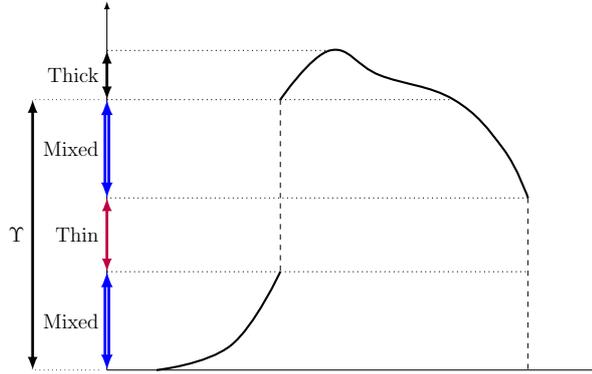
FIGURE 1. Partition into thick, thin and mixed spectra

Up to sets of measure zero, these three components of the spectrum are disjoint. This yields a fine classification of the absolutely continuous spectra for Toeplitz operators with piecewise continuous symbols.

If the symbol ω is continuous, then $\Upsilon = \varnothing$ so that $\sigma(T) = \sigma(\boldsymbol{\Omega})$. In this case $\mathcal{H}_{\mathrm{thick}} = \mathbb{H}^2$ and the whole spectrum is thick. On the other hand, if ω is piecewise constant (but not a constant function), then the Lebesgue measure $|\sigma(\boldsymbol{\Omega})| = 0$ and $\mathcal{H}_{\mathrm{thin}} = \operatorname{Ran} E_T\big(\sigma(T)\big) = \mathbb{H}^2$, i.e., the whole spectrum is thin.

Let us illustrate this partition of the spectrum using the example of symbol ω in Fig 1. According to Theorem 2.5, the spectrum of $T(\omega)$ is simple. In this case the multiplicity of the spectrum of $\boldsymbol{\Omega}$ does not exceed 2, see Theorem 2.4. If for a spectral interval $\Lambda \subset \sigma(\boldsymbol{\Omega})$ it equals 2, then the spectrum of T on Λ is thick. If it equals 1, then the spectrum of T on Λ is mixed. The gap in the spectrum of $\boldsymbol{\Omega}$ produces the thin spectrum of T.

These definitions are justified in terms of the asymptotic behaviour of $e^{-iTt}f$ as $t \to \pm\infty$ for f from each of these subspaces. If $f \in \mathcal{H}_{\mathrm{thick}}$, then the asymptotic formula (1.3) holds. On the subspace $\mathcal{H}_{\mathrm{thin}}$, the evolution is modelled by the Toeplitz operators $T_{\mathrm{s}} = \omega_{\mathrm{s}}$ with symbols ω_{s} that are step functions on $\mathbb{T}$. Here we use that for symbols that are indicators of arcs, the operator $e^{-iT_{\mathrm{s}}t}$ admits an explicit representation, see Sect. 4. On the subspace $\mathcal{H}_{\mathrm{mix}}$, the evolution $e^{-iTt}f$ is, asymptotically, a linear combination of evolutions for the thin and thick cases.

A surprising fact is that, for $f \in \mathcal{H}_{\mathrm{mix}}$, the asymptotic behaviour of $e^{-iTt}f$ as $t \to \pm\infty$ qualitatively depends on the sign of t. For instance, the evolution may be "thick" for one of the signs and "thin" for the other sign. This phenomenon is discussed in Sect. 5.4.

The difference between thick and thin spectra manifests itself also in the structure of the continuous spectrum eigenfunctions $\varphi(\zeta; \lambda)$. As discussed in Sect. 2.3, an arbitrary Toeplitz operator with a simple spectrum can be reduced to the multiplication operator by λ in the space $L^2(\gamma_1, \gamma_2)$ by a unitary operator

$\Phi : \mathbb{H}^2 \to L^2(\gamma_1, \gamma_2)$ defined by a formal relation

$$(\Phi f)(\lambda) = \int_{\mathbb{T}} \overline{\varphi(\zeta; \lambda)} f(\zeta) d\mathbf{m}(\zeta). \tag{1.6}$$

This means that for any $f \in \mathbb{H}^2$,

$$(\Phi T f)(\lambda) = \lambda (\Phi f)(\lambda), \quad \text{a.e.} \quad \lambda \in (\gamma_1, \gamma_2). \tag{1.7}$$

The functions $\varphi(\zeta; \lambda)$ satisfy the equation $T(\omega)\varphi(\lambda) = \lambda\varphi(\lambda)$, in an appropriate generalized sense (see [14, Sect. 5] for details of this construction), and hence can be interpreted as eigenfunctions of the operator $T(\omega)$. We note that
  – eigenfunctions of the thick spectrum have stronger singularities than those of the thin one,
  – on the thick spectrum, the position of singularities depends on the spectral parameter λ while on the thin spectrum, singularities are located at the points of discontinuity irrespectively of the value of λ.

In Section 2.5 we consider two examples where eigenfunctions are given by explicit formulas: one with a smooth symbol, and the other one with a symbol which is the indicator of an arc of $\mathbb{T}$. For brevity we call such symbols *indicator type* symbols. In the first case the entire spectrum is thick, and in the second case the entire spectrum is thin.

1.4. **Analytic background.** We study the evolution $e^{-iTt} f$ by investigating wave operators for the pairs $\mathbf{\Omega}, T$ and T_s, T. Our main results are the existence and completeness of an appropriate set of wave operators.
  From the analytic point of view, our approach comprises three ingredients:

  (i) To investigate the thick spectrum, we use trace class scattering theory which allows us to prove the existence of the wave operators for the pair $\mathbf{\Omega}, T$. In the case of smooth symbols, the trace class framework yields also the completeness of these wave operators.

 (ii) For the thin spectrum, we rely on Cook's criterion which leads to the existence of wave operators for the pairs T_s, T with model Toeplitz operators $T_\mathrm{s} = T(\omega_s)$ whose symbols ω_s are appropriately chosen step functions.

(iii) In contrast to the smooth case, where there is only one scattering channel described by formula (1.3), for symbols with discontinuities each jump creates a new scattering channel. Thus our final task is to show *asymptotic completeness*, that is to prove that the constructed scattering channels exhaust the whole space. Our proof of the asymptotic completeness relies on the results on spectral multiplicity of Toeplitz operators.

1.5. **Related problems.** We mention two such problems. The first one is the three-particle (and more generally, a multi-particle) quantum-mechanical problem. In this problem, it also makes sense to classify bands of the spectrum as being "thick" or "thin", although the precise meaning of these terms is, of course,

different from the one introduced above for Toeplitz operators. The "thick" spectrum describes scattering states where all particles are asymptotically free, while the "thin" spectrum corresponds to scattering of one of the particles on a bound state of the other two. The "thin" spectrum has essentially two-particle nature.

The second problem is scattering theory for Hankel operators with discontinuous symbols. While Hankel operators with smooth symbols are compact, they acquire an absolutely continuous spectrum if their symbols have jumps. This phenomenon has been known in concrete examples, such as the Carleman operator, see [8, Chapter 10]. A general spectral theory of Hankel operators with discontinuous symbols was built in [9]. Interestingly, continuous spectrum eigenfunctions of Hankel (see [9, Sect. 4]) and Toeplitz operators with discontinuous symbols have singularities of the same type.

1.6. **Structure of the paper.** The paper is organized as follows. In Sect. 2, we collect known facts about Toeplitz operators necessary for the rest of the paper. In particular, we describe in Theorem 2.5 the spectral multiplicity of operators with piecewise continuous symbols ω in terms adapted to our applications to scattering theory. In Sect. 3, we provide some general information about the trace class scattering theory. Then we investigate wave operators for the pair $\mathbf{\Omega}, T$. This is already sufficient for construction of scattering theory for Toeplitz operators with smooth symbols (see Theorem 3.17). Sect. 4 focuses on wave operators for symbols with jump discontinuities. This requires an analysis of the propagator of the model Toeplitz operator, i.e., operator with an indicator type symbol. In Sect. 5, we prove the asymptotic completeness of the set of wave operators constructed in the previous sections. The main result is stated as Theorem 5.9.

1.7. **Notation.** The symbol $\mathbb{D}$ is used to denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. On $\mathbb{T}$ we define the standard measure $d\mathbf{m}(\zeta) = (2\pi i \zeta)^{-1} d\zeta$, $\zeta \in \mathbb{T}$. For any $\zeta_1, \zeta_2 \in \mathbb{T}$, we denote by (ζ_1, ζ_2) the open arc joining ζ_1 and ζ_2 counterclockwise. For a function ω defined on $\mathbb{T}$, the notation $\omega(\zeta \pm 0)$ stands for the limit $\lim_{\varepsilon \downarrow 0} \omega(\zeta e^{\pm i\varepsilon})$, if it exists. Let $\mathbb{1}_\Delta$ be the indicator of a set $\Delta \subset \mathbb{T}$; the operator in $L^2(\mathbb{T})$ of multiplication by this function is also denoted by $\mathbb{1}_\Delta$.

Denote by $F : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ the discrete Fourier transform:

$$(Fu)_n = \int_{\mathbb{T}} \zeta^{-n} u(\zeta) d\mathbf{m}(\zeta), \quad n \in \mathbb{Z}, \ u \in L^2(\mathbb{T}). \tag{1.8}$$

Considered as a mapping of $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$, the operator F is unitary. The notation $\mathbb{H}^2_\pm = \mathbb{H}^2_\pm(\mathbb{T})$ stands for the Hardy classes

$$\mathbb{H}^2_+ = \{f \in L^2(\mathbb{T}) : (Ff)_n = 0, \quad n < 0\},$$
$$\mathbb{H}^2_- = \{f \in L^2(\mathbb{T}) : (Ff)_n = 0, \quad n \geq 0\}.$$

As a rule, we set $\mathbb{H}^2 = \mathbb{H}^2_+$. By $\mathbb{P}_\pm$ we denote the orthogonal projection from $L^2 = \mathbb{H}^2_+ \oplus \mathbb{H}^2_-$ onto $\mathbb{H}^2_\pm$. Usually we use the standard realization of $\mathbb{H}^2$ as the

space of functions u that are analytic on the disk $\mathbb{D}$ and such that the $L^2(\mathbb{T})$-norms $\|u(r \cdot)\|$ are bounded uniformly in $r < 1$; see, for example, the book [5].

It is often convenient to identify $\mathbb{T}$ with $\mathbb{R}/(2\pi\mathbb{Z})$ by writing $\zeta = e^{ix}$ for each $\zeta \in \mathbb{T}$. Thus with every arc $\Delta \subset \mathbb{T}$ we associate an interval $X \subset \mathbb{R}/(2\pi\mathbb{Z})$, and write $\Delta = e^{iX}$. Under this change of variables we have $d\mathbf{m}(\zeta) = (2\pi)^{-1}dx$. Throughout the paper we use the notation $w(x) = \omega(\zeta)$. It is also convenient to set $\omega'(\zeta) := w'(x)$.

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. By $E_A(\Lambda)$ we denote its spectral projection associated with a Borel set $\Lambda \subset \mathbb{R}$; $\mathcal{H}_A^{(\mathrm{ac})}$ is the absolutely continuous subspace of A; $P_A^{(\mathrm{ac})}$ is the orthogonal projection on $\mathcal{H}_A^{(\mathrm{ac})}$ and $A^{(\mathrm{ac})}$ is the restriction of the operator A on $\mathcal{H}_A^{(\mathrm{ac})}$.

Let an operator A and an interval $\Lambda \subset \mathbb{R}$ be such that the operator $E_A(\Lambda)A$ is absolutely continuous. Suppose that there exists a Hilbert space $\mathfrak{N}$ and a unitary mapping $U : E_A(\Lambda)\mathcal{H} \to L^2(\Lambda; \mathfrak{N})$ such that the operator $UE_A(\Lambda)AU^*$ acts as multiplication by independent variable λ on the space $L^2(\Lambda; \mathfrak{N})$. Then we say that the spectral representation of $AE_A(\Lambda)$ is realized on $L^2(\Lambda; \mathfrak{N})$. We denote by $m_A(\Lambda) := \dim \mathfrak{N}$ the multiplicity of the spectrum of A on Λ.

By C, c (with or without indices) we denote various positive constants whose precise values are of no importance.

Unless stated otherwise, throughout the whole paper we assume that ω is a real-valued non-constant function and that $\omega \in L^\infty(\mathbb{T})$.

2. Spectral analysis of Toeplitz and multiplication operators

In this section we study spectra of Toeplitz T and multiplication $\boldsymbol{\Omega}$ operators for piecewise continuous symbols ω.

2.1. **Multiplication operators.** Spectral analysis of the multiplication operator is elementary. It is based on the simple relation $E_{\boldsymbol{\Omega}}(\Lambda) = \mathbb{1}_{\omega^{-1}(\Lambda)}$ valid for any Borel set $\Lambda \subset \mathbb{R}$. The next two lemmas are straightforward.

Lemma 2.1. *Let $\Delta \subset \mathbb{T}$ be an open set. Suppose that $\omega \in C^1(\Delta)$ and $\omega'(\zeta) \neq 0$ for $\zeta \in \Delta$. Then $L^2(\Delta) \subset \mathcal{H}_{\boldsymbol{\Omega}}^{(\mathrm{ac})}$ so that the spectrum of the operator $\boldsymbol{\Omega}$ restricted to the subspace $L^2(\Delta)$ is absolutely continuous. Moreover, if Δ is an arc, then this spectrum is simple.*

Proof. Let Δ be an arc, and let $\Lambda = \omega(\Delta)$. Define the unitary operator $U : L^2(\Delta) \to L^2(\Lambda)$ by the formula

$$(Uf)(\lambda) = \frac{1}{\sqrt{|\omega'(\omega^{-1}(\lambda))|}} f((\omega^{-1}(\lambda)), \quad \lambda \in \Lambda.$$

Since $(U\boldsymbol{\Omega}f)(\lambda) = \lambda(Uf)(\lambda)$ for $f \in L^2(\Delta)$, the restriction of the operator $\boldsymbol{\Omega}$ to $L^2(\Delta)$ is unitarily equivalent to the operator of multiplication by λ on $L^2(\Lambda)$. This

concludes the proof for arcs Δ. In the general case, we only have to apply the result obtained to every constituent arc of Δ. $\square$

Lemma 2.2. *Let $\Delta \subset \mathbb{T}$ be an open set of full measure. Suppose that $\omega \in C^1(\Delta)$. Let*

$$\Delta^{(\pm)} = \{\zeta \in \Delta : \pm\omega'(\zeta) < 0\}, \quad \Delta_0 = \{\zeta \in \Delta : \omega'(\zeta) = 0\}. \tag{2.1}$$

Then

$$P_{\mathbf{\Omega}}^{(\mathrm{ac})} = \mathbb{1}_{\Delta\setminus\Delta_0} = \mathbb{1}_{\Delta^{(+)}} + \mathbb{1}_{\Delta^{(-)}}. \tag{2.2}$$

Proof. Since $\mathbf{m}(\mathbb{T} \setminus \Delta) = 0$ and $\Delta = \Delta_0 \cup \Delta^{(+)} \cup \Delta^{(-)}$ is a union of disjoint sets, we have

$$\mathbb{1} = \mathbb{1}_{\Delta} = \mathbb{1}_{\Delta_0} + \mathbb{1}_{\Delta^{(+)}} + \mathbb{1}_{\Delta^{(-)}}. \tag{2.3}$$

It follows from Lemma 2.1 that

$$\mathbb{1}_{\Delta^{(\pm)}} P_{\mathbf{\Omega}}^{(\mathrm{ac})} = \mathbb{1}_{\Delta^{(\pm)}}. \tag{2.4}$$

By Sard's theorem, $|\omega(\Delta_0)| = 0$ whence $E_{\mathbf{\Omega}}(\omega(\Delta_0))P_{\mathbf{\Omega}}^{(\mathrm{ac})} = 0$. Since

$$\mathbb{1}_{\Delta_0} \le \mathbb{1}_{\omega^{-1}(\omega(\Delta_0))} = E_{\mathbf{\Omega}}(\omega(\Delta_0)),$$

we infer that

$$\mathbb{1}_{\Delta_0} P_{\mathbf{\Omega}}^{(\mathrm{ac})} = 0. \tag{2.5}$$

Putting together (2.3) and (2.4), (2.5), we conclude the proof of (2.2). $\square$

Further results can be obtained under more specific assumptions on the symbol ω.

2.2. **Piecewise continuous symbols.** From now on we assume that the symbol ω is piecewise continuous in the following sense.

Condition 2.3. (i) $\omega = \overline{\omega} \in L^\infty(\mathbb{T})$ *and ω is not a constant function,*
  (ii) *There exists a finite set $\mathsf{S} = \{\eta_k\} \subset \mathbb{T}$ such that $\omega \in C^1(\mathbb{T} \setminus \mathsf{S})$,*
  (iii) *The limits $\omega(\eta_k + 0)$, $\omega(\eta_k - 0)$ exist for all $\eta_k \in \mathsf{S}$. For every $\eta_k \in \mathsf{S}$, either $\omega(\eta_k + 0) \ne \omega(\eta_k - 0)$ or $\omega(\eta_k + 0) = \omega(\eta_k - 0)$ but the derivative ω' is not continuous at η_k.*

Let the sets $\Delta^{(\pm)}$ and the *critical set* Δ_0 be as defined in (2.1) with $\Delta = \mathbb{T}\setminus\mathsf{S}$. The image $\Lambda_{\mathrm{cr}} = \omega(\Delta_0)$ consists of critical values of ω. As already observed earlier, by Sard's theorem, the Lebesgue measure $|\Lambda_{\mathrm{cr}}| = 0$. We introduce also the "threshold" set Λ_{thr} which consists of all values $\omega(\eta_k \pm 0)$, $\eta_k \in \mathsf{S}$, and define the exceptional set

$$\Lambda_{\mathrm{exc}} = \Lambda_{\mathrm{cr}} \cup \Lambda_{\mathrm{thr}}.$$

Since the set Λ_{thr} is finite, $|\Lambda_{\mathrm{exc}}| = 0$. Moreover, the set Λ_{exc} is closed (see [14, Lemma 6.2]).

Let us fix an interval $\Lambda = (\lambda_1, \lambda_2)$ such that

$$\Lambda \subset (\gamma_1, \gamma_2) \setminus \Lambda_{\mathrm{exc}}, \tag{2.6}$$

where γ_1, γ_2 are defined in (1.2). Note that $\omega'(\zeta) \neq 0$ on $\omega^{-1}(\Lambda) \subset \mathbb{T} \setminus \mathsf{S}$. Consider the open set

$$\delta^{(\pm)} = \delta^{(\pm)}(\Lambda) = \Delta^{(\pm)} \cap \omega^{-1}(\Lambda) \quad \text{where} \quad \omega : \mathbb{T} \setminus \mathsf{S} \to \mathbb{R}. \qquad (2.7)$$

The open set $\delta^{(\pm)} \subset \mathbb{T} \setminus \mathsf{S}$ is a union of some number, denoted $n^{(\pm)}$, of disjoint open arcs $\delta_k^{(\pm)} = (\alpha_k^{(\pm)}, \beta_k^{(\pm)})$ such that $\pm \omega'(\zeta) < 0$ for $\zeta \in \delta_k^{(\pm)}$. Observe that $\omega(\alpha_k^{(+)}) = \lambda_2$, $\omega(\beta_k^{(+)}) = \lambda_1$ and $\omega(\alpha_k^{(-)}) = \lambda_1$, $\omega(\beta_k^{(-)}) = \lambda_2$. In particular, we see that $\omega(\delta_k^{(\pm)}) = \Lambda$; see Fig. 2 for illustration. It follows that

$$\omega^{-1}(\Lambda) = \delta^{(+)} \cup \delta^{(-)} = \Big(\bigcup_{k=1}^{n^{(+)}} \delta_k^{(+)} \Big) \cup \Big(\bigcup_{k=1}^{n^{(-)}} \delta_k^{(-)} \Big) \qquad (2.8)$$

where

$$n^{(\pm)} = \#\{\delta_k^{(\pm)}\}. \qquad (2.9)$$

Of course, the arcs $\delta_k^{(\pm)}$ and the numbers $n^{(\pm)}$ depend on the chosen interval Λ. It is easy to show (see [14, Lemma 6.4]) that under assumption (2.6) the numbers $n^{(\pm)}$ are finite.

Let us now calculate the spectral multiplicity of the multiplication operator $\boldsymbol{\Omega}$ on the interval Λ.

Theorem 2.4. *Suppose that ω satisfies Condition 2.3, and that an interval Λ satisfies condition (2.6). Let the numbers $n^{(\pm)}$ be defined by formula (2.9). Then the spectral representation of the operator $\boldsymbol{\Omega} E_{\boldsymbol{\Omega}}(\Lambda)$ restricted to the subspace $\operatorname{Ran} \mathbb{1}_{\Delta^{(\pm)}}$ is realized on the space $L^2(\Lambda; \mathbb{C}^{n^{(\pm)}})$. In other words, the spectral multiplicity of the operator $\boldsymbol{\Omega} \mathbb{1}_{\Delta^{(\pm)}}$ on Λ equals $n^{(\pm)}$.*

Proof. It follows from equality (2.8) that

$$E_{\boldsymbol{\Omega}}(\Lambda) \mathbb{1}_{\Delta^{(\pm)}} = \mathbb{1}_{\omega^{-1}(\Lambda)} \mathbb{1}_{\Delta^{(\pm)}} = \sum_{k=1}^{n^{(\pm)}} \mathbb{1}_{\delta_k^{(\pm)}}.$$

According to Lemma 2.1 the spectrum of the operator $\boldsymbol{\Omega} \mathbb{1}_{\Delta^{(\pm)}}$ on each of $n^{(\pm)}$ disjoint arcs $\delta_k^{(\pm)}$ is simple. Since $\omega(\delta_k^{(\pm)}) = \Lambda$ for each $k = 1, 2, \ldots, n^{(\pm)}$, the spectral multiplicity of $\boldsymbol{\Omega} \mathbb{1}_{\Delta^{(\pm)}}$ on Λ equals $n^{(\pm)}$, as required. $\qquad \square$

2.3. **Toeplitz operators: diagonalization.** As explained in [14], the Toeplitz operator $T(\omega)$ can be diagonalized by a unitary operator $\Phi = \Phi(\omega)$ whose integral kernel is given by the generalized eigenfunctions of $T(\omega)$. In the simple spectrum case, the operator $T(\omega)$ has one generalized eigenfunction $\varphi(z; \lambda)$, $z \in \mathbb{D}$, a.e. $\lambda \in (\gamma_1, \gamma_2)$, and the unitary operator $\Phi : \mathbb{H}^2 \to L^2(\gamma_1, \gamma_2)$ is defined by the formula

$$(\Phi f)(\lambda) = \lim_{r \to 1-0} \int_{\mathbb{T}} \overline{\varphi(r\zeta; \lambda)} f(\zeta) d\mathbf{m}(\zeta), \qquad (2.10)$$
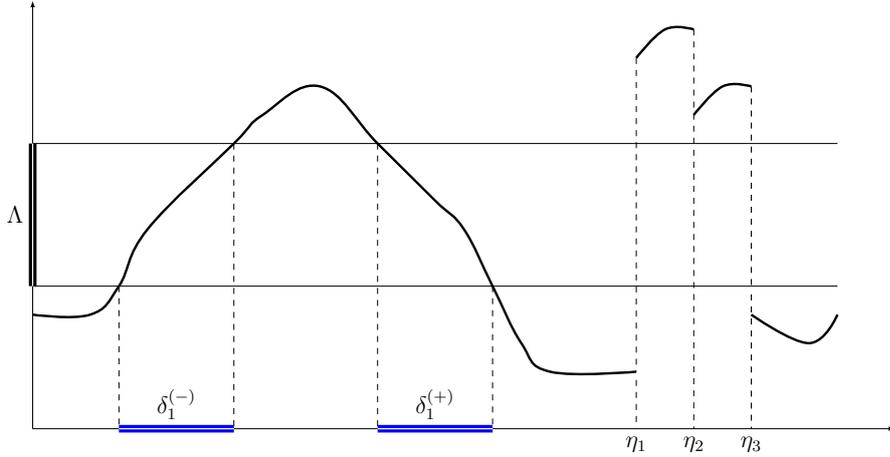
FIGURE 2. Example: $\omega^{-1}(\Lambda) = \delta_1^{(+)} \cup \delta_1^{(-)}$, $\mathsf{S}^{(-)} = \mathsf{S}^{(-)}(\Lambda) = \{\eta_1\}$, $\mathsf{S}^{(+)} = \{\eta_2, \eta_3\}$, $\mathsf{S}^{(+)}(\Lambda) = \{\eta_3\}$.

which is a more precise version of (1.6). The adjoint operator is given by the formula

$$(\Phi^* g)(z) = \int_{\gamma_1}^{\gamma_2} \varphi(z; \lambda) g(\lambda) d\lambda, \quad z \in \mathbb{D},$$

for all $g \in L^2(\gamma_1, \gamma_2)$. Thus, in view of (1.7), the representation

$$(e^{-iTt} f)(z) = \int_{\gamma_1}^{\gamma_2} \varphi(z; \lambda) e^{-i\lambda t} \tilde{f}(\lambda) d\lambda, \quad \tilde{f} = \Phi f, \tag{2.11}$$

holds. Here we do not discuss the eigenfunctions in greater detail since for our purposes we need the representation (2.11) only for the singular symbol (2.17), for which φ is found explicitly, see (2.18).

2.4. **Toeplitz operators: piecewise continuous symbols.** Now we turn our attention to Toeplitz operators with piecewise continuous symbols. Let $\mathsf{S}^{(\pm)}$ consist of $\eta_k \in \mathsf{S}$ such that

$$\pm(\omega(\eta_k - 0) - \omega(\eta_k + 0)) > 0, \quad \eta_k \in \mathsf{S}^{(\pm)},$$

and let $\mathsf{S}_0 \subset \mathsf{S}$ be the set of those η_k where $\omega(\eta_k - 0) = \omega(\eta_k + 0)$ but the derivative $\omega'(\zeta)$ is not continuous at the point η_k. Thus S is the disjoint union

$$\mathsf{S} = \mathsf{S}^{(+)} \cup \mathsf{S}^{(-)} \cup \mathsf{S}_0. \tag{2.12}$$

We associate with every discontinuity $\eta_k \in \mathsf{S}^{(\pm)}$ the interval (the "jump"):

$$\Lambda_k := \Lambda_{\eta_k} = [\omega(\eta_k \pm 0), \omega(\eta_k \mp 0)], \quad \eta_k \in \mathsf{S}^{(\pm)}, \tag{2.13}$$

cf. (1.4). The condition (2.6) guarantees that $\omega(\eta_k \pm 0) \notin \Lambda$ for all $\eta_k \in \mathsf{S}$, and hence each interval Λ_k either contains Λ, or is disjoint from Λ. We are interested only in those singular points η_k, for which $\Lambda \subset \Lambda_k$, and we denote this subset by $\mathsf{S}(\Lambda)$. Introduce also the notation $\mathsf{S}^{(\pm)}(\Lambda) = \mathsf{S}(\Lambda) \cap \mathsf{S}^{(\pm)}$ and put

$$s^{(\pm)} = s^{(\pm)}(\Lambda) = \#\{\mathsf{S}^{(\pm)}(\Lambda)\}. \tag{2.14}$$

Note that $\mathsf{S}(\Lambda) = \mathsf{S}^{(+)}(\Lambda) \cup \mathsf{S}^{(-)}(\Lambda)$. All these objects are illustrated in Fig. 2.

Now we are in a position to quote [14, Theorems 6.5, 6.6].

Theorem 2.5. *Suppose that ω satisfies Condition 2.3, and that an interval Λ satisfies condition (2.6). Let the numbers $n^{(\pm)} = n^{(\pm)}(\Lambda)$ and $s^{(\pm)} = s^{(\pm)}(\Lambda)$ be defined by (2.9) and (2.14), respectively. Then $n^{(+)} + s^{(+)} = n^{(-)} + s^{(-)}$ and the spectral representation of the operator $T(\omega)$ restricted to the subspace $E_{T(\omega)}(\Lambda)\mathbb{H}^2$ is realized on the space $L^2(\Lambda; \mathbb{C}^m)$ where*

$$m = n^{(\pm)} + s^{(\pm)} \tag{2.15}$$

(for both signs " $\pm$ "). In other words, the spectral multiplicity $m_{T(\omega)}(\Lambda)$ of $T(\omega)$ on Λ is finite and it coincides with the number (2.15).

By this theorem, the spectrum of the Toeplitz operator with the symbol in Fig. 2 has multiplicity two on Λ.

2.5. **Examples.** Let us give two examples illustrating Theorem 2.5. Both examples were mentioned in [12] and discussed in more detail in [14].

Example 2.6. First we consider the regular symbol

$$\omega_{\mathrm{r}}(\zeta) = (\zeta + \zeta^{-1})/2,$$

for which $\sigma(T(\omega_{\mathrm{r}})) = [-1, 1]$. Now we have $\mathsf{S} = \varnothing$, $\Delta_0 = \{-1, 1\} \subset \mathbb{T}$, $\Lambda_{\mathrm{exc}} = \Lambda_{\mathrm{cr}} = \{-1, 1\} \subset \mathbb{R}$, $\Delta^{(+)} = (1, -1) \subset \mathbb{T}$, $\Delta^{(-)} = (-1, 1) \subset \mathbb{T}$, $n^{(\pm)} = 1$, $s^{(\pm)} = 0$. By Theorem 2.5, the spectrum of the operator $T(\omega_{\mathrm{r}})$ is simple, and, according to Definition 1.1, it is thick.

The eigenfunctions of the Toeplitz operator $T(\omega_{\mathrm{r}})$ equal

$$\varphi_{\mathrm{r}}(z; \lambda) = \sqrt{\frac{2}{\pi}} \frac{(1 - \lambda^2)^{\frac{1}{4}}}{1 - 2\lambda z + z^2}, \quad \lambda \in (-1, 1). \tag{2.16}$$

They are singular at the points

$$z_{\pm} = \lambda \pm i\sqrt{1 - \lambda^2}.$$

Note that function (2.16) belongs to the Hardy class $\mathbb{H}^p$ for any $p < 1$ but $\varphi_{\mathrm{r}}(\cdot; \lambda) \notin \mathbb{H}^1$.

Example 2.7. The simplest singular symbol is given by the indicator

$$\omega_{\mathrm{s}}(\zeta) = \mathbb{1}_{(\zeta_1, \zeta_2)}(\zeta) \tag{2.17}$$

of an arc $(\zeta_1, \zeta_2) \subset \mathbb{T}$; then $\sigma(T(\omega_s)) = [0,1]$. Now we have $\mathsf{S} = \{\zeta_1, \zeta_2\}$, $\mathsf{S}^{(+)} = \{\zeta_2\}$, $\mathsf{S}^{(-)} = \{\zeta_1\}$, $s^{(\pm)} = 1$ and $\Lambda_{\mathrm{exc}} = \Lambda_{\mathrm{thr}} = \{0,1\}$, $\Delta^{(\pm)} = \varnothing$, $n^{(\pm)} = 0$. By Theorem 2.5, the spectrum of the operator $T(\omega_s)$ is again simple, and, according to Definition 1.1, it is thin.

The eigenfunctions of the Toeplitz operator $T(\omega_s)$ equal

$$\varphi_s(z; \lambda) = \varkappa(\lambda)(1 - z/\zeta_1)^{-1/2 - i\sigma(\lambda)}(1 - z/\zeta_2)^{-1/2 + i\sigma(\lambda)}, \tag{2.18}$$

where

$$\sigma(\lambda) = \frac{1}{2\pi} \ln(\lambda^{-1} - 1) \tag{2.19}$$

and

$$\varkappa(\lambda) = \sqrt{\frac{|\zeta_2 - \zeta_1|}{2\pi\lambda}} e^{-\pi\sigma(\lambda)\mathbf{m}(\zeta_1, \zeta_2)}. \tag{2.20}$$

As mentioned in the Introduction, formulas (2.16) and (2.18) show that the eigenfunctions of the Toeplitz operators have stronger singularities in the smooth case, and that in the singular case the location of these singularities is independent of λ.

3. Toeplitz versus multiplication operators. Smooth symbols

3.1. **Scattering theory. Basic notions.** We refer to the books [10] or [16] for detailed information.

Let A and B be self-adjoint operators in, possibly, different Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, respectively, and let $J : \mathcal{H} \to \mathcal{G}$ be a bounded operator.

Definition 3.1. Under the assumption of their existence, the strong limits

$$\operatorname*{s-lim}_{t \to \pm\infty} e^{iBt} J e^{-iAt} P_A^{(\mathrm{ac})} =: W_\pm(B, A; J) \tag{3.1}$$

are called the wave operators for the pair of self-adjoint operators A and B and the "identification" J.

Here and below all statements about the wave operators $W_\pm(B, A; J)$ (as well as about other objects labelled by "$\pm$") are understood as two independent statements.

Obviously, the wave operator $W_\pm(B, A; J)$ exists if the limit $e^{iBt} J e^{-iAt} f$ as $t \to \pm\infty$ exists on a set of vectors f dense in the subspace $\mathcal{H}_A^{(\mathrm{ac})}$.

Let us list some elementary properties of wave operators following from the mere fact of their existence:

(a) The wave operators are bounded and

$$\|W_\pm(B, A; J)\| \leq \|J\|.$$

(b) The wave operators enjoy the intertwining property

$$E_B(X) W_\pm(B, A; J) = W_\pm(B, A; J) E_A(X), \tag{3.2}$$

where $X \subset \mathbb{R}$ is an arbitrary Borel set.

(c) The ranges $\operatorname{Ran} W_{\pm}(B, A; J)$ of the wave operators satisfy the inclusions

$$\operatorname{Ran} W_{\pm}(B, A; J) \subset \mathcal{G}_B^{(\mathrm{ac})}, \tag{3.3}$$

and their closures are invariant subspaces of the operator B.

(d) If

$$\lim_{t \to \pm\infty} \|J e^{-iAt} P_A^{(\mathrm{ac})} f\| = \|P_A^{(\mathrm{ac})} f\|, \tag{3.4}$$

then $\|W_{\pm}(B, A; J)f\| = \|P_A^{(\mathrm{ac})} f\|$. Thus, the wave operator $W_{\pm}(B, A; J)$ is isometric on $\mathcal{H}_A^{(\mathrm{ac})}$ if condition (3.4) is satisfied for a set of vectors f dense in $\mathcal{H}_A^{(\mathrm{ac})}$. In particular, for $\mathcal{H} = \mathcal{G}$ the wave operators $W_{\pm}(B, A; I)$ are isometric on $\mathcal{H}_A^{(\mathrm{ac})}$.

(e) If the operator $W_{\pm}(B, A; J)$ is isometric on $\mathcal{H}_A^{(\mathrm{ac})}$, then it follows from (b) that the restriction of the operator B to the subspace $\operatorname{Ran} W_{\pm}(B, A; J)$ is unitarily equivalent to the absolutely continuous part $A^{(\mathrm{ac})}$ of the operator A.

(f) If $f = W_{\pm}(B, A; J)f_0$, then

$$\lim_{t \to \pm\infty} \|e^{-iBt} f - J e^{-iAt} P_A^{(\mathrm{ac})} f_0\| = 0. \tag{3.5}$$

For a given $f \in \operatorname{Ran} W_{\pm}(B, A; J)$, the choice of f_0 satisfying (3.5) is in general not unique. One can set $f_0 = W_{\pm}(B, A; J)^* f$ if the operator $W_{\pm}(B, A; J)$ is isometric on $\mathcal{H}_A^{(\mathrm{ac})}$.

(g) If J is compact, then $J e^{-iAt} f \to 0$ as $t \to \pm\infty$ for all $f \in \mathcal{H}_A^{(\mathrm{ac})}$, so that $W_{\pm}(B, A; J) = 0$.

Definition 3.2. A wave operator $W_{\pm}(B, A; J)$ is called complete if the equality holds in (3.3):

$$\operatorname{Ran} W_{\pm}(B, A; J) = \mathcal{G}_B^{(\mathrm{ac})}.$$

If the operator $W_{\pm}(B, A; J)$ is isometric on $\mathcal{H}_A^{(\mathrm{ac})}$ and complete, then the absolutely continuous parts $A^{(\mathrm{ac})}$ and $B^{(\mathrm{ac})}$ of the operators A and B are unitarily equivalent.

Let us note an important special case $\mathcal{H} = \mathcal{G}$, $J = I$ (the identity operator). For short, we write $W_{\pm}(B, A; I) = W_{\pm}(B, A)$. Of course, the operators $W_{\pm}(B, A)$ are isometric on the subspace $\mathcal{H}_A^{(\mathrm{ac})}$. The completeness of $W_{\pm}(B, A)$ is equivalent to the existence of $W_{\pm}(A, B)$, and in this case

$$W_{\pm}(A, B) = W_{\pm}^*(B, A).$$

3.2. **Existence of wave operators.** The following condition (known as Cook's criterion) of the existence of wave operators (3.1) is quite elementary, but it requires the knowledge of the "free" evolution e^{-iAt}. In view of our applications, we assume that A and B are bounded operators.

Theorem 3.3. *Suppose that the operator A is absolutely continuous and*

$$\int_0^{\pm\infty} \|(BJ - JA)e^{-iAt}f\|dt < \infty$$

for a set of elements f dense in $\mathcal{H}$. Then the corresponding wave operator $W_{\pm}(B, A; J)$ exists.

On the contrary, the trace class method treats the operators A and B on an equal footing. The ideal of trace class operators is denoted by $\mathfrak{S}_1$. First, we recall the classical Kato - Rosenblum theorem.

Theorem 3.4. *Suppose that $\mathcal{H} = \mathcal{G}$, $J = I$ and $B - A \in \mathfrak{S}_1$. Then the wave operators $W_{\pm}(B, A)$ exist, are isometric on $\mathcal{H}_A^{(ac)}$ and are complete, that is,*

$$\operatorname{Ran} W_{\pm}(B, A) = \mathcal{H}_B^{(ac)}.$$

An extension of this result to arbitrary J is due to D. Pearson.

Theorem 3.5. *Suppose that*

$$BJ - JA \in \mathfrak{S}_1.$$

Then the wave operators $W_{\pm}(B, A; J)$ exist.

3.3. **Duplex Toeplitz operators.** Let us return to Toeplitz operators. Here we compare the Toeplitz operator $T = T(\omega)$ defined by (1.1) with the multiplication operator $\mathbf{\Omega}$ by a bounded function ω,

$$(\mathbf{\Omega}f)(\zeta) = \omega(\zeta)f(\zeta), \quad \zeta \in \mathbb{T},$$

acting on the space $\mathcal{H} = L^2(\mathbb{T})$. In this subsection ω may be complex-valued. Recall that $\mathbb{P}_{\pm}$ are the orthogonal projections in $L^2(\mathbb{T})$ onto the Hardy spaces $\mathbb{H}_{\pm}^2(\mathbb{T})$.

It is convenient to introduce on $L^2(\mathbb{T})$ the *duplex* Toeplitz operator

$$\mathbf{T} = \mathbf{T}(\omega) = \mathbb{P}_+\mathbf{\Omega}\mathbb{P}_+ + \mathbb{P}_-\mathbf{\Omega}\mathbb{P}_- \tag{3.6}$$

and the *symmetrized* Hankel operator

$$\mathbf{H} = \mathbf{H}(\omega) = \mathbb{P}_+\mathbf{\Omega}\mathbb{P}_- + \mathbb{P}_-\mathbf{\Omega}\mathbb{P}_+ \tag{3.7}$$

(these operators are symmetric if $\omega = \overline{\omega}$). Then

$$\mathbf{\Omega} = \mathbf{T} + \mathbf{H}. \tag{3.8}$$

Under very general assumptions on its symbol ω the operator $\mathbf{H}$ is trace class, so that $\mathbf{T}$ can be viewed as a perturbation of the multiplication operator $\mathbf{\Omega}$.

Let $T_{\pm} = T_{\pm}(\omega)$ be the restriction of the operator $\mathbf{T}$ onto the subspace $\mathbb{H}_{\pm}^2(\mathbb{T})$. Clearly, $T = T_+$. Define the unitary operator V on $L^2(\mathbb{T})$ by the formula

$$(Vf)(\zeta) = \bar{\zeta}f(\bar{\zeta}), \quad \zeta \in \mathbb{T}.$$

Evidently, $V : \mathbb{H}_{\pm}^2(\mathbb{T}) \to \mathbb{H}_{\mp}^2(\mathbb{T})$ and $V\mathbb{P}_{\pm} = \mathbb{P}_{\mp}V$. The following fact is almost obvious.

Lemma 3.6. *Let $\tilde{\omega}(\zeta) = \omega(\overline{\zeta})$. Then*

$$VT_-(\omega)f = T_+(\tilde{\omega})Vf, \quad \forall f \in \mathbb{H}^2_-(\mathbb{T}), \tag{3.9}$$

so that the operators $T_-(\omega)$ and $T_+(\tilde{\omega})$ are unitarily equivalent.

Proof. Since $V(\omega g) = \tilde{\omega}Vg$ for all $g \in L^2(\mathbb{T})$, we see that

$$V\mathbb{P}_-(\omega\mathbb{P}_-f) = \mathbb{P}_+V(\omega\mathbb{P}_-f) = \mathbb{P}_+(\tilde{\omega}V\mathbb{P}_-f) = \mathbb{P}_+(\tilde{\omega}\mathbb{P}_+Vf), \quad \forall f \in L^2(\mathbb{T}),$$

whence (3.9) follows. □

Thus, for a real-valued bounded ω, the operators $T_+(\omega)$ and $T_-(\omega) = V^*T_+(\tilde{\omega})V$ are absolutely continuous on $\mathbb{H}^2_+$ and $\mathbb{H}^2_-$ respectively, and $\mathbf{T}$ is absolutely continuous on $L^2(\mathbb{T})$.

3.4. **Smooth symbols.** Recall one of the equivalent definitions of the Besov class $\mathbb{B}^1_{11} = \mathbb{B}^1_{11}(\mathbb{T})$: a function $\omega \in \mathbb{B}^1_{11}$ if

$$\int_{\mathbb{T}}\int_{\mathbb{T}} |\eta - 1|^{-2}|\omega(\zeta\eta) + \omega(\zeta\eta^{-1}) - 2\omega(\zeta)|d\mathbf{m}(\zeta)d\mathbf{m}(\eta) < \infty.$$

A list of basic properties of the Besov spaces can be found in the book [8, Appendix 2.6]. For us, it is useful to remember that

$$\mathbb{W}^{2,1}(\mathbb{T}) \subset \mathbb{B}^1_{11} \subset \mathbb{W}^{1,1}(\mathbb{T}) \subset C(\mathbb{T})$$

and that $\mathbb{P}_\pm\mathbb{B}^1_{11} \subset \mathbb{B}^1_{11}$.

We need the well known criterion of V. V. Peller for a Hankel operator to be trace class. The function $\omega(\zeta)$ is not required to be real-valued here.

Theorem 3.7. [8, Theorem 6.1.1] *The inclusion $\mathbb{P}_\pm\mathbf{\Omega}\mathbb{P}_\mp \in \mathfrak{S}_1$ holds if and only if $\mathbb{P}_\pm\omega \in \mathbb{B}^1_{11}$.*

Corollary 3.8. *The symmetrized Hankel operator* (3.7) *satisfies*

$$\mathbb{P}_-\mathbf{\Omega}\mathbb{P}_+ + \mathbb{P}_+\mathbf{\Omega}\mathbb{P}_- \in \mathfrak{S}_1 \tag{3.10}$$

if and only if $\omega \in \mathbb{B}^1_{11}$.

Proof. If $\omega \in \mathbb{B}^1_{11}$, then $\mathbb{P}_-\omega \in \mathbb{B}^1_{11}$ and $\mathbb{P}_+\omega \in \mathbb{B}^1_{11}$ so that

$$\mathbb{P}_-\mathbf{\Omega}\mathbb{P}_+ \in \mathfrak{S}_1 \quad \text{and} \quad \mathbb{P}_+\mathbf{\Omega}\mathbb{P}_- \in \mathfrak{S}_1 \tag{3.11}$$

by Theorem 3.7. Conversely, inclusion (3.10) implies both inclusions (3.11), and hence $\mathbb{P}_-\omega \in \mathbb{B}^1_{11}$ and $\mathbb{P}_+\omega \in \mathbb{B}^1_{11}$ again by Theorem 3.7. □

Putting together Theorems 3.4 and 3.7, we obtain the following result for the self-adjoint duplex Toeplitz operator $\mathbf{T}(\omega)$. Recall that this operator is absolutely continuous.

Theorem 3.9. *Suppose that $\omega = \bar{\omega} \in \mathbb{B}^1_{11}$. Then the wave operators $W_\pm(\mathbf{T}(\omega), \mathbf{\Omega})$ exist, are isometric on the absolutely continuous subspace of $\mathbf{\Omega}$ and are complete, that is,*

$$\operatorname{Ran} W_\pm(\mathbf{T}(\omega), \mathbf{\Omega}) = L^2(\mathbb{T}).$$

In view of the general property (3.2), we also have

Corollary 3.10. *The intertwining property*

$$\mathbf{T}(\omega)W_{\pm}(\mathbf{T}(\omega),\mathbf{\Omega}) = W_{\pm}(\mathbf{T}(\omega),\mathbf{\Omega})\,\mathbf{\Omega}$$

holds, and the operator $\mathbf{T}(\omega)$ *is unitarily equivalent to the absolutely continuous part* $\mathbf{\Omega}^{(\mathrm{ac})}$ *of the operator* $\mathbf{\Omega}$.

Theorem 3.9 leads to the following result for the Toeplitz operators $T_{\pm}(\omega) = \mathbb{P}_{\pm}\mathbf{T}(\omega)\mathbb{P}_{\pm}$ on the subspaces $\mathbb{H}^2_{\pm}(\mathbb{T})$.

Corollary 3.11. *Let* $\omega = \bar{\omega} \in \mathbb{B}^1_{11}$. *Then, for both signs "$\pm$", the wave operators*

$$W_{\pm}(T_{+}(\omega),\mathbf{\Omega};\mathbb{P}_{+}) \quad \text{and} \quad W_{\pm}(T_{-}(\omega),\mathbf{\Omega};\mathbb{P}_{-}) \tag{3.12}$$

exist and

$$W_{\pm}(\mathbf{T}(\omega),\mathbf{\Omega}) = W_{\pm}(T_{+}(\omega),\mathbf{\Omega};\mathbb{P}_{+}) + W_{\pm}(T_{-}(\omega),\mathbf{\Omega};\mathbb{P}_{-}). \tag{3.13}$$

Furthermore, the relations

$$\operatorname{Ran} W_{\pm}(T_{+}(\omega),\mathbf{\Omega};\mathbb{P}_{+}) = \mathbb{H}^2_{+} \quad \text{and} \quad \operatorname{Ran} W_{\pm}(T_{-}(\omega),\mathbf{\Omega};\mathbb{P}_{-}) = \mathbb{H}^2_{-} \tag{3.14}$$

hold true.

Proof. In view of definition (3.6), the existence of wave operators on the right-hand side of (3.13) and the formula (3.13) itself follow from Theorem 3.9 due to the identities $e^{i\mathbf{T}t}\mathbb{P}_{\pm} = e^{iT_{\pm}t}\mathbb{P}_{\pm}$ and

$$\mathbb{P}_{+}W_{\pm}(\mathbf{T},\mathbf{\Omega}) = W_{\pm}(T_{+}(\omega),\mathbf{\Omega};\mathbb{P}_{+}), \quad \mathbb{P}_{-}W_{\pm}(\mathbf{T},\mathbf{\Omega}) = W_{\pm}(T_{-}(\omega),\mathbf{\Omega};\mathbb{P}_{-}).$$

These relations, together with the completeness of $W_{\pm}(\mathbf{T},\mathbf{\Omega})$, imply (3.14). $\quad\square$

According to (3.2), all wave operators constructed above possess the intertwining property, for example,

$$T_{+}(\omega)W_{\pm}(T_{+}(\omega),\mathbf{\Omega};\mathbb{P}_{+}) = W_{\pm}(T_{+}(\omega),\mathbf{\Omega};\mathbb{P}_{+})\,\mathbf{\Omega}.$$

The isometry of the wave operators (3.12) is discussed in Section 3.6 in a more general setting.

3.5. **Local smoothness.** In the following assertion we do not assume that the symbol ω is smooth on the whole unit circle $\mathbb{T}$. We first consider the duplex Toeplitz operator $\mathbf{T}$ defined by (3.6).

Theorem 3.12. *Let* $\omega = \bar{\omega} \in L^{\infty}(\mathbb{T})$ *and* $\varphi\omega \in \mathbb{B}^1_{11}(\mathbb{T})$ *for some function* $\varphi \in C^{\infty}(\mathbb{T})$. *Denote by* J_{φ} *the operator of multiplication by* φ *in the space* $L^2(\mathbb{T})$. *Then the wave operators* $W_{\pm}(\mathbf{T}(\omega),\mathbf{\Omega};J_{\varphi})$ *exist.*

Proof. According to Theorem 3.5 it suffices to check that

$$\mathbf{T}J_\varphi - J_\varphi\mathbf{\Omega} \in \mathfrak{S}_1, \quad \mathbf{T} = \mathbf{T}(\omega). \tag{3.15}$$

By (3.8), we have

$$\mathbf{T}J_\varphi - J_\varphi\mathbf{\Omega} = (\mathbf{\Omega} - \mathbf{H})J_\varphi - J_\varphi\mathbf{\Omega} = -\mathbf{H}J_\varphi.$$

Using definition (3.7) of the operator $\mathbf{H}$, we now see that (3.15) is equivalent to the inclusion

$$\mathbf{H}J_\varphi = \mathbb{P}_+\mathbf{\Omega}\mathbb{P}_-J_\varphi + \mathbb{P}_-\mathbf{\Omega}\mathbb{P}_+J_\varphi \in \mathfrak{S}_1. \tag{3.16}$$

Two terms here are quite similar. Consider, for example, the first one:

$$\mathbb{P}_+\mathbf{\Omega}\mathbb{P}_-J_\varphi = \mathbb{P}_+\mathbf{\Omega}J_\varphi\mathbb{P}_- + \mathbb{P}_+\mathbf{\Omega}(\mathbb{P}_-J_\varphi - J_\varphi\mathbb{P}_-). \tag{3.17}$$

According to Theorem 3.7 the operator $\mathbb{P}_+(\mathbf{\Omega}J_\varphi)\mathbb{P}_- \in \mathfrak{S}_1$ because $\omega\varphi \in \mathbb{B}^1_{11}$. Since $\varphi \in C^\infty$, Theorem 3.7 also implies that

$$\mathbb{P}_-J_\varphi - J_\varphi\mathbb{P}_- = \mathbb{P}_-J_\varphi\mathbb{P}_+ - \mathbb{P}_+J_\varphi\mathbb{P}_- \in \mathfrak{S}_1. \tag{3.18}$$

It follows that the operator (3.17) is trace class which yields inclusions (3.16) and hence (3.15). □

Similarly to Corollary 3.11, we have

Corollary 3.13. *Under the assumptions of Theorem 3.12 the wave operators*

$$W_\pm(T_+(\omega), \mathbf{\Omega}; \mathbb{P}_+J_\varphi) \quad and \quad W_\pm(T_-(\omega), \mathbf{\Omega}; \mathbb{P}_-J_\varphi)$$

exist for both signs "$\pm$" and

$$W_\pm(\mathbf{T}(\omega), \mathbf{\Omega}; J_\varphi) = W_\pm(T_+(\omega), \mathbf{\Omega}; \mathbb{P}_+J_\varphi) + W_\pm(T_-(\omega), \mathbf{\Omega}; \mathbb{P}_-J_\varphi). \tag{3.19}$$

Let us, finally, replace $\varphi \in C^\infty(\mathbb{T})$ by the characteristic function of an arc Δ.

Theorem 3.14. *Let $\omega = \bar{\omega} \in L^\infty(\mathbb{T})$, and let $\Delta \subset \mathbb{T}$ be an open arc. Suppose that $\varphi\omega \in \mathbb{B}^1_{11}(\mathbb{T})$ for all functions $\varphi \in C_0^\infty(\Delta)$. Then the "local" wave operators $W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P}\mathbb{1}_\Delta)$ exist.*

Proof. The wave operators $W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P}J_\varphi)$ exist due to Corollary 3.13. The operator $\mathbf{\Omega}$ commutes with J_φ and $\mathbb{1}_\Delta$, and hence so does $P_\mathbf{\Omega}^{(ac)}$. As a consequence,

$$W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P}J_\varphi) = W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P}\mathbb{1}_\Delta)J_\varphi.$$

Since $C_0^\infty(\Delta)$ is dense in $L^2(\Delta)$, the existence of $W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P}\mathbb{1}_\Delta)$ follows. □

3.6. **Isometry of wave operators.** We start with a relatively standard (cf., for example, [16, Lemma 2.6.4]) analytic result. Recall that $\mathbb{W}^{2,1}_{\mathrm{loc}}(\Delta) \subset C^1(\Delta)$. As usual, we use the notation $w(x) = \omega(e^{ix})$, $w'(x) = \omega'(e^{ix})$.

Lemma 3.15. *Let $\Delta \subset \mathbb{T}$ be an arc. Suppose that $\omega \in \mathbb{W}^{2,1}_{\mathrm{loc}}(\Delta)$ and $\omega'(\zeta) \neq 0$ for $\zeta \in \Delta$. Then*

$$\operatorname*{s-lim}_{t\to\pm\infty} \mathbb{P}_+ e^{-i\boldsymbol{\Omega} t} \mathbb{1}_\Delta = 0 \quad if \quad \pm\omega'(\zeta) > 0 \quad for \quad \zeta \in \Delta, \tag{3.20}$$

and

$$\operatorname*{s-lim}_{t\to\pm\infty} \mathbb{P}_- e^{-i\boldsymbol{\Omega} t} \mathbb{1}_\Delta = 0 \quad if \quad \pm\omega'(\zeta) < 0 \quad for \quad \zeta \in \Delta. \tag{3.21}$$

Proof. Let us check, for example, (3.20). Let $F : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ be the discrete Fourier transform defined in (1.8). For all $u \in L^2(\mathbb{T})$, we have

$$\|\mathbb{P}_+ u\|^2 = \sum_{n=0}^{\infty} |(Fu)_n|^2. \tag{3.22}$$

Let us use the representation

$$(Fe^{-i\boldsymbol{\Omega} t} \mathbb{1}_\Delta f)_n = \int_\Delta \zeta^{-n} e^{-i\omega(\zeta)t} f(\zeta) d\mathbf{m}(\zeta) = (2\pi)^{-1} \int_X e^{-inx-iw(x)t} \mathbf{f}(x) dx,$$

where $f \in L^2(\mathbb{T})$, $\mathbf{f}(x) = f(e^{ix})$ and $X \subset \mathbb{R}/(2\pi\mathbb{Z})$ is an interval such that $e^{iX} = \Delta$. Assuming that $f \in C_0^\infty(\Delta)$, that is, $\mathbf{f} \in C_0^\infty(X)$, we integrate by parts to get

$$(Fe^{-i\boldsymbol{\Omega} t} f)_n = -i(2\pi)^{-1} \int_X e^{-inx-iw(x)t} \left(\frac{\mathbf{f}(x)}{n + w'(x)t}\right)' dx. \tag{3.23}$$

Using that $\pm w'(x) \geq c > 0$ on $\operatorname{supp} \mathbf{f}$ and $\pm t > 0$, we obtain the bound

$$n + w'(x)t \geq n + c|t|.$$

Furthermore, the condition $\omega \in \mathbb{W}^{2,1}_{\mathrm{loc}}(\Delta)$ ensures that $w \in \mathbb{W}^{2,1}_{\mathrm{loc}}(X)$ and hence $w'' \in L^1_{\mathrm{loc}}(X)$. Thus (3.23) yields an estimate

$$|(Fe^{-i\boldsymbol{\Omega} t} f)_n| \leq C(n + |t|)^{-1}.$$

Substituting this estimate into (3.22) where $u = e^{-i\boldsymbol{\Omega} t} f$, we see that

$$\|\mathbb{P}_+ e^{-i\boldsymbol{\Omega} t} f\|^2 \leq C \sum_{n=0}^{\infty} (n + |t|)^{-2} \leq C_1 |t|^{-1}.$$

Since $C_0^\infty(\Delta)$ is dense in $L^2(\Delta)$, this leads to (3.20). Relation (3.21) can be checked quite similarly. $\square$

Let us discuss now the isometry of the "local" wave operators $W_\pm(T, \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_\Delta)$ where, as usual, $\mathbb{P} = \mathbb{P}_+$. Under the assumptions below their existence follows from Theorem 3.14 because $\varphi\omega \in \mathbb{B}^1_{1,1}(\mathbb{T})$ if $\omega \in \mathbb{W}^{2,1}_{\mathrm{loc}}(\Delta)$ and $\varphi \in C_0^\infty(\Delta)$.

Theorem 3.16. *Let $\omega = \bar{\omega} \in \mathbb{W}^{2,1}_{\text{loc}}(\Delta)$ for some arc $\Delta \subset \mathbb{T}$. If $\pm\omega'(\zeta) < 0$ for $\zeta \in \Delta$, then the operator $W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta})$ is isometric on $\operatorname{Ran}\mathbb{1}_{\Delta}$ and*

$$W_{\mp}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta}) = 0. \tag{3.24}$$

Proof. Equality (3.24) follows directly from (3.20). Similarly, relations (3.21) imply that $W_{\pm}(T_{-}(\omega), \boldsymbol{\Omega}; \mathbb{P}_{-}\mathbb{1}_{\Delta}) = 0$. Since (cf. (3.19))

$$W_{\pm}(\mathbf{T}(\omega), \boldsymbol{\Omega}; \mathbb{1}_{\Delta}) = W_{\pm}(T_{+}(\omega), \boldsymbol{\Omega}; \mathbb{P}_{+}\mathbb{1}_{\Delta}) + W_{\pm}(T_{-}(\omega), \boldsymbol{\Omega}; \mathbb{P}_{-}\mathbb{1}_{\Delta}),$$

we see that

$$W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta}) = W_{\pm}(\mathbf{T}(\omega), \boldsymbol{\Omega}; \mathbb{1}_{\Delta}).$$

By property (d) of wave operators (see Section 3.1), the operator on the right is isometric on $\operatorname{Ran}\mathbb{1}_{\Delta}$ because $\mathbb{1}_{\Delta} P^{(\text{ac})}_{\boldsymbol{\Omega}} = \mathbb{1}_{\Delta}$ according to Lemma 2.1. Hence the same is true for the operator on the left. $\square$

3.7. **Main result for smooth symbols.** The results obtained so far can be summarized in the following two theorems. Recall that the sets $\Delta^{(\pm)}$ were defined in (2.1).

Theorem 3.17. *Suppose that $\omega = \bar{\omega} \in L^{\infty}(\mathbb{T})$ and that $\omega \in \mathbb{W}^{2,1}_{\text{loc}}(\Delta)$ for some open set $\Delta \subset \mathbb{T}$ of full measure. Then*

(i) *the wave operators $W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P})$ exist, and the intertwining property*

$$T(\omega)W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}) = W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P})\,\boldsymbol{\Omega}$$

holds.

(ii) *The operators $W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P})$ are isometric on $\operatorname{Ran}\mathbb{1}_{\Delta^{(\pm)}}$ or, equivalently,*

$$W^{*}_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P})W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}) = \mathbb{1}_{\Delta^{(\pm)}}. \tag{3.25}$$

Moreover, the relations

$$W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(\pm)}}) = W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}) \quad and \quad W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(\mp)}}) = 0 \tag{3.26}$$

are satisfied.

(iii) *The restriction of the Toeplitz operator $T(\omega)$ to the subspace*

$$\mathcal{H}^{(\pm)} = \operatorname{Ran}W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P})$$

of $\mathbb{H}^{2}$ is unitarily equivalent to the multiplication operator $\boldsymbol{\Omega}$ on the subspace $\operatorname{Ran}\mathbb{1}_{\Delta^{(\pm)}}$ of $L^{2}(\mathbb{T})$.

Proof. The wave operators $W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta})$ exist by Theorem 3.16. This implies the existence of $W_{\pm}(T(\omega), \boldsymbol{\Omega}; \mathbb{P})$ because the set Δ has full measure. The intertwining property is a direct consequence of the existence of wave operators, see (3.2). By (2.2) we have the equality

$$W_{\pm}(T, \boldsymbol{\Omega}; \mathbb{P}) = W_{\pm}(T, \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(+)}}) + W_{\pm}(T, \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(-)}}).$$

Now relations (3.25) and (3.26) follow from Theorem 3.16 applied separately to $\Delta^{(+)}$ and to $\Delta^{(-)}$. $\square$

The next assertion shows that if $\Delta = \mathbb{T}$, then the wave operators $W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P})$ are complete.

Theorem 3.18. *If $\omega = \bar{\omega} \in \mathbb{W}^{2,1}(\mathbb{T})$, then all conclusions of Theorem 3.17 are true and*

$$\operatorname{Ran} W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P}) = \mathbb{H}^2 \tag{3.27}$$

or, equivalently,

$$W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P}) W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P})^* = I$$

(the identity operator on $\mathbb{H}^2$). In this case the operators $T(\omega)$ and the restriction of $\mathbf{\Omega}$ to $\operatorname{Ran} \mathbb{1}_{\Delta^{(\pm)}}$ are unitarily equivalent.

Proof. The completeness (3.27) is equivalent to the first equality (3.14). $\qquad\square$

Let us state two consequences of this result. The first one concerns the spectral multiplicity of the Toeplitz operator $T(\omega)$.

Corollary 3.19. *Let an interval Λ satisfy condition (2.6), and let the numbers $n^{(\pm)} = n^{(\pm)}(\Lambda)$ be defined by formula (2.9). Then $n^{(+)} = n^{(-)}$ and the spectral representation of the operator $T(\omega) E_{T(\omega)}(\Lambda)$ is realized on the space $L^2(\Lambda; \mathbb{C}^{n^{(\pm)}})$. In other words, the spectral multiplicity of the operator $T(\omega)$ on Λ equals $n^{(+)} = n^{(-)}$.*

Proof. According to Theorem 2.4 the spectral multiplicity of the operator $\mathbf{\Omega} \mathbb{1}_{\Delta^{(\pm)}}$ on Λ equals $n^{(\pm)}$. It follows from Theorem 3.18 that the same statement is true for the operator $T(\omega)$. This automatically implies that $n^{(+)} = n^{(-)}$. $\qquad\square$

The second corollary is a direct consequence of the asymptotic completeness (3.27).

Corollary 3.20. *For every $f \in \mathbb{H}^2$ asymptotic relation (1.3) (with $T = T(\omega)$) is satisfied where necessarily*

$$\mathbb{1}_{\Delta^{(\pm)}} f_\pm = W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P})^* f.$$

For example, we can set $f_\pm = W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P})^* f$.

Theorem 3.18 and its corollaries conclude our construction of scattering theory for Toeplitz operators $T(\omega)$ with smooth symbols ω. Since $\sigma(T(\omega)) = \sigma(\mathbf{\Omega})$, the spectra of such Toeplitz operators are thick (see Definition 1.1). Note also that the scattering theory approach presented here gives (see Corollary 3.19) an independent proof of Theorem 2.5 for smooth symbols.

If the symbol ω has jump discontinuities, then the equality (3.27) is no longer true. To ensure the asymptotic completeness in this case, we need to take into account the wave operators produced by jumps of ω. Such wave operators will be constructed in the next section.

4. Jump discontinuities. A model operator

4.1. **A model singularity.** As a model operator, we choose the Toeplitz operator $T_s = T(\mathbb{1}_{(\zeta_1,\zeta_2)})$ whose symbol $\mathbb{1}_{(\zeta_1,\zeta_2)}$ has two jumps at the points ζ_1 and ζ_2. The unitary operator $\Phi_s = \Phi(\mathbb{1}_{(\zeta_1,\zeta_2)}) : \mathbb{H}^2 \to L^2(0,1)$, diagonalizing T_s, is given by formula (2.10) with $\varphi(z;\lambda) = \varphi_s(z;\lambda), z \in \mathbb{D}$, defined in (2.18). We suppose that $\tilde{f} = \Phi_s f \in C_0^\infty(0,1)$ so that also $\hat{f} := \varkappa \tilde{f} \in C_0^\infty(0,1)$ where $\varkappa$ is given by (2.20). The integral (2.11) takes the form

$$(e^{-iT_s t} f)(z) = (1 - z/\zeta_1)^{-1/2}(1 - z/\zeta_2)^{-1/2} F(z,t) \tag{4.1}$$

where

$$F(z,t) = \int_0^1 (1 - z/\zeta_1)^{-i\sigma(\lambda)}(1 - z/\zeta_2)^{i\sigma(\lambda)} \hat{f}(\lambda) e^{-i\lambda t} d\lambda \tag{4.2}$$

with the real-valued $\sigma(\lambda)$ defined by (2.19).

We are interested in the behaviour of the function (4.1) as $|t| \to \infty$. It is natural to expect that neighborhoods of the points ζ_1 and ζ_2 give the main contributions to the asymptotics of (4.1). To see this, we have to estimate the integral in (4.2). Note that $|\arg(1 - z/\zeta_j)| \leq \pi/2$ and hence

$$\left| (1 - z/\zeta_j)^{i(\pm 1)^j \sigma(\lambda)} \right| = \exp\left((\mp 1)^j \sigma(\lambda) \arg(1 - z/\zeta_j) \right) \leq e^{\pi|\sigma(\lambda)|/2}. \tag{4.3}$$

Lemma 4.1. *Suppose that $\tilde{f} \in C_0^\infty(0,1)$. Then for all $p \geq 0$, we have the estimate*

$$|(e^{-iT_s t} f)(z)| \leq C_p |1 - z/\zeta_1|^{-1/2}|1 - z/\zeta_2|^{-1/2}$$
$$\times \left(1 + \left| \ln |1 - z/\zeta_1| \right| + \left| \ln |1 - z/\zeta_2| \right| \right)^p |t|^{-p}, \tag{4.4}$$

with a constant C_p independent of $t \neq 0$ and $z \in \mathbb{D}$. In particular, if $|z - \zeta_1| \geq c > 0$ and $|z - \zeta_2| \geq c > 0$, then

$$|(e^{-iT_s t} f)(z)| \leq C_p |t|^{-p}, \quad \forall p \geq 0.$$

Proof. Using that $\hat{f} = \varkappa \tilde{f} \in C_0^\infty(0,1)$ and integrating in (4.2) by parts p times, we get

$$F(z,t) = (it)^{-p} \int_0^1 e^{-i\lambda t} \frac{d^p}{d\lambda^p}\left((1 - z/\zeta_1)^{-i\sigma(\lambda)}(1 - z/\zeta_2)^{i\sigma(\lambda)} \hat{f}(\lambda) \right) d\lambda,$$

Taking into account (4.3), we see that the derivative under the integral sign does not exceed

$$\tilde{C}_p \left(1 + \left| \ln |1 - z/\zeta_1| \right| + \left| \ln |1 - z/\zeta_2| \right| \right)^p.$$

In view of (4.1), this leads to the required bound. $\qquad \square$

This lemma shows that $(e^{-iT_s t} f)(z)$ "lives" in neighborhoods of the points ζ_1 and ζ_2 as $|t| \to \infty$. This result can be made more precise if one takes into account the dependence on the sign of t as $t \to \pm\infty$.

Lemma 4.2. *Let $j = 1$, $k = 2$ for $t > 0$ and $j = 2$, $k = 1$ for $t < 0$. Assume that $|z - \zeta_j| \leq c_j < 1$ and $|z - \zeta_k| \geq c_k > 0$. Then for all $p \geq 0$, we have the estimates*

$$|(e^{-iT_s t} f)(z)| \leq C_p |1 - z/\zeta_j|^{-1/2} \big(\big| \ln |1 - z/\zeta_j| \big| + |t| \big)^{-p}, \tag{4.5}$$

with a constant C_p independent of $t \neq 0$ and $z \in \mathbb{D}$.

Proof. Consider, for example, the case $j = 1$, $k = 2$, $t > 0$. The integral (4.2) can be rewritten as

$$F(z, t) = \int_0^1 e^{-i\lambda t - i\sigma(\lambda) \ln(1 - z/\zeta_1)} (1 - z/\zeta_2)^{i\sigma(\lambda)} \hat{f}(\lambda) d\lambda.$$

Integrating by parts once, we see that

$$F(z, t) = i \int_0^1 e^{-i\lambda t - i\sigma(\lambda) \ln(1 - z/\zeta_1)} \left(\frac{(1 - z/\zeta_2)^{i\sigma(\lambda)} \hat{f}(\lambda)}{t + \sigma'(\lambda) \ln(1 - z/\zeta_1)} \right)' d\lambda, \tag{4.6}$$

where $\sigma'(\lambda) = -(2\pi\lambda(1 - \lambda))^{-1} \leq -2\pi^{-1}$. Since $|1 - \zeta_1/z| \leq c_1 < 1$, we have

$$\text{Re} \ln(1 - z/\zeta_1) = \ln |1 - z/\zeta_1| \leq \ln c_1 < 0,$$

so that

$$|t + \sigma'(\lambda) \ln(1 - z/\zeta_1)| \geq c(-\ln|1 - z/\zeta_1| + t).$$

We also have the estimate $|\ln(1 - z/\zeta_2)| \geq c_2 > 0$. Using also (4.3) we conclude that the integral (4.6) is bounded by $C(-\ln|1 - z/\zeta_1| + t)^{-1}$. Further integrations by parts show that

$$|F(z, t)| \leq C_p(-\ln|1 - z/\zeta_1| + t)^{-p}$$

for all $p \geq 0$. Substituting this estimate into (4.1), we get (4.5) for $j = 1$. $\square$

Corollary 4.3. *Let $j = 1$ for $t > 0$ and $j = 2$ for $t < 0$. Then*

$$\int_{(\zeta_j e^{-i\varepsilon}, \zeta_j e^{i\varepsilon})} |(e^{-iT_s t} f)(\zeta)|^2 d\mathbf{m}(\zeta) \leq C_p |t|^{-p}$$

for all $p \geq 0$ if ε is sufficiently small.

Thus the function $(e^{-iT_s t} f)(z)$ tends to concentrate near ζ_2 (resp. ζ_1) as $t \to +\infty$ (resp. $t \to -\infty$).

4.2. **Jump discontinuities.** Here we consider symbols $\omega(\zeta)$ with a jump discontinuity at some point $\zeta_0 \in \mathbb{T}$. We suppose that both one-sided limits

$$\omega(\zeta_0 \pm 0) := \lim_{\varepsilon \to \pm 0} \omega(\zeta_0 e^{i\varepsilon}) = \alpha_\pm$$

exist, $\alpha_+ \neq \alpha_-$, and

$$\omega(\zeta) - \alpha_\pm = O(|\ln|\zeta - \zeta_0||^{-\rho}) \quad \text{for some} \quad \rho > 3/2 \tag{4.7}$$

as $\zeta \to \zeta_0 \pm 0$. Pick some $\varepsilon \in (0, 2\pi)$ and define the symbol $\omega_0 = \omega_0^{(\varepsilon)}$ by

$$\left.\begin{array}{ll} \omega_0(\zeta) = \alpha_- & \text{for} \quad \zeta \in (e^{-i\varepsilon}\zeta_0, \zeta_0) \\ \omega_0(\zeta) = \alpha_+ & \text{for} \quad \zeta \notin (e^{-i\varepsilon}\zeta_0, \zeta_0) \end{array}\right\} \quad \text{if} \quad \alpha_+ < \alpha_-, \tag{4.8}$$

and

$$\left.\begin{array}{ll} \omega_0(\zeta) = \alpha_+ & \text{for} \quad \zeta \in (\zeta_0, e^{i\varepsilon}\zeta_0) \\ \omega_0(\zeta) = \alpha_- & \text{for} \quad \zeta \notin (\zeta_0, e^{i\varepsilon}\zeta_0) \end{array}\right\} \quad \text{if} \quad \alpha_+ > \alpha_-. \tag{4.9}$$

Now we apply the results of Section 4.1 to the unitary group $\exp(-iT(\omega_0)t)$. Note that

$$e^{-iT(\omega_0)t} = e^{-i\alpha_+ t} \exp\left(-i(\alpha_- - \alpha_+)T(\mathbb{1}_{(e^{-i\varepsilon}\zeta_0, \zeta_0)})t\right) \quad \text{if} \quad \alpha_+ < \alpha_-, \tag{4.10}$$

and

$$e^{-iT(\omega_0)t} = e^{-i\alpha_- t} \exp\left(-i(\alpha_+ - \alpha_-)T(\mathbb{1}_{(\zeta_0, e^{i\varepsilon}\zeta_0)})t\right) \quad \text{if} \quad \alpha_+ > \alpha_-. \tag{4.11}$$

Theorem 4.4. *Let $\omega = \bar{\omega} \in L^\infty(\mathbb{T})$, and let condition (4.7) be satisfied for some numbers $\alpha_\pm \in \mathbb{R}$. Define the symbol ω_0 by the formula (4.8) or (4.9). If $\alpha_\pm < \alpha_\mp$, then the wave operator $W_\pm(T(\omega), T(\omega_0))$ exists.*

Proof. Consider, for example, the wave operator $W_+(T(\omega), T(\omega_0))$ for the case $\alpha_+ < \alpha_-$ and set $\omega_s = \mathbb{1}_{(e^{-i\varepsilon}\zeta_0, \zeta_0)}$. According to Theorem 3.3 it suffices to check that

$$\int_1^\infty \|(\omega - \omega_0)e^{-iT(\omega_s)t}f\|dt < \infty \tag{4.12}$$

for functions f such that $\tilde{f} = \Phi(\omega_s)f \in C_0^\infty(0, 1)$. Let $\delta(\zeta_0) \subset \mathbb{T}$, $\delta(\zeta_0, \varepsilon) \subset \mathbb{T}$ be disjoint neighbourhoods of the points ζ_0, $e^{-i\varepsilon}\zeta_0$ and

$$\Sigma(\zeta_0, \varepsilon) = \mathbb{T} \setminus \left(\delta(\zeta_0) \cup \delta(\zeta_0, \varepsilon)\right).$$

Lemma 4.1 and Corollary 4.3 show that

$$\|e^{-iT(\omega_s)t}f\|_{L^2(\Sigma(\zeta_0,\varepsilon))} + \|e^{-iT(\omega_s)t}f\|_{L^2(\delta(\zeta_0,\varepsilon))} \leq C_p t^{-p}, \quad \forall p \geq 0. \tag{4.13}$$

In a neighborhood of the point ζ_0, we use condition (4.7) so that

$$\omega(\zeta) - \omega_0(\zeta) = O(|\ln|\zeta - \zeta_0||^{-\rho}), \quad \rho > 3/2,$$

and hence in view of (4.4)

$$|(\omega(\zeta) - \omega_0(\zeta))(e^{-iT(\omega_s)t}f)(\zeta)| \leq C_p|1 - \zeta/\zeta_0|^{-1/2}|\ln|1 - \zeta/\zeta_0||^{-\rho+p}t^{-p}.$$

If $p < \rho - 1/2$, then the right-hand side is in $L^2(\delta(\zeta_0))$, and it follows that

$$\|(\omega - \omega_0)e^{-iT(\omega_s)t}f\|_{L^2(\delta(\zeta_0))} \leq C_p t^{-p}. \tag{4.14}$$

Since $\rho > 3/2$, we can choose $p > 1$. Combining estimates (4.13) and (4.14), we see that the integral (4.12) converges. $\qquad\square$

Remark 4.5. The symbol $\omega_0 = \omega_0^{(\varepsilon)}$ depends on $\varepsilon \in (0, 2\pi)$, and hence so does the wave operator $W_\pm(T(\omega), T(\omega_0^{(\varepsilon)}))$. However this dependence is trivial. Indeed, for two different positive ε and ν, we have

$$e^{iTt}e^{-iT_0^{(\nu)}t} = \left(e^{iTt}e^{-iT_0^{(\varepsilon)}t}\right)\left(e^{iT_0^{(\varepsilon)}t}e^{-iT_0^{(\nu)}t}\right)$$

where we have set $T = T(\omega)$, $T_0^{(\varepsilon)} = T(\omega_0^{(\varepsilon)})$, for short. By Theorem 4.4, the left-hand side and the first factor on the right converge, as $t \to \pm\infty$, to $W_\pm(T, T_0^{(\nu)})$ and $W_\pm(T, T_0^{(\varepsilon)})$, respectively. Again by Theorem 4.4, the second factor on the right converges to the operator

$$W_\pm(T_0^{(\varepsilon)}, T_0^{(\nu)}) =: U_\pm(\varepsilon, \nu),$$

so that

$$W_\pm(T, T_0^{(\nu)}) = W_\pm(T, T_0^{(\varepsilon)})U_\pm(\varepsilon, \nu). \tag{4.15}$$

Since $U_\pm(\varepsilon, \nu)^* = U_\pm(\nu, \varepsilon)$ and $U_\pm(\varepsilon, \nu)U_\pm(\nu, \varepsilon) = I$, the operators $U_\pm(\varepsilon, \nu)$ are unitary. Note that, by definitions (4.10) and (4.11), they do not depend on α_+ or α_-.

Although we omit the dependence of the symbol ω_0 on ε, we always keep in mind the relation (4.15).

4.3. **Orthogonality of the channels.** Let us show that the ranges of the wave operators constructed in Theorems 3.17 and 4.4 are orthogonal to each other. Consider first the wave operators corresponding to jumps of the symbol. Recall that, for an arc $(\zeta_1, \zeta_2) \subset \mathbb{T}$, the unitary operator $\Phi(\mathbb{1}_{(\zeta_1, \zeta_2)}) : \mathbb{H}^2 \to L^2(0, 1)$ is defined by formulas (1.6), (2.18).

Theorem 4.6. *Let $(\zeta_1, \zeta_2) \subset \mathbb{T}$, $(\zeta_1', \zeta_2') \subset \mathbb{T}$ where $\zeta_j \neq \zeta_k'$ for all $j, k = 1, 2$. Suppose that, for some $\alpha, \alpha' \in \mathbb{R}$, the wave operators $W_\pm(T(\omega), T(\alpha\mathbb{1}_{(\zeta_1, \zeta_2)}))$ and $W_\pm(T(\omega), T(\alpha'\mathbb{1}_{(\zeta_1', \zeta_2')}))$ exist. Then the subspaces*

$$\operatorname{Ran} W_\pm(T(\omega), T(\alpha\mathbb{1}_{(\zeta_1, \zeta_2)})) \quad \text{and} \quad \operatorname{Ran} W_\pm(T(\omega), T(\alpha'\mathbb{1}_{(\zeta_1', \zeta_2')}))$$

of $\mathbb{H}^2$ are orthogonal to each other.

Proof. Denote $T_s = T(\mathbb{1}_{(\zeta_1, \zeta_2)})$, $T_s' = T(\mathbb{1}_{(\zeta_1', \zeta_2')})$. It suffices to check that

$$\lim_{|t| \to \infty} (e^{-i\alpha T_s t}f, e^{-i\alpha' T_s' t}f') = 0 \tag{4.16}$$

for all $f, f' \in \mathbb{H}^2$ such that $\Phi(\mathbb{1}_{(\zeta_1, \zeta_2)})f \in C_0^\infty(0, 1)$ and $\Phi(\mathbb{1}_{(\zeta_1', \zeta_2')})f' \in C_0^\infty(0, 1)$. Let $\delta \subset \mathbb{T}$ be a neighborhood of the set $\{\zeta_1, \zeta_2\}$ such that $\operatorname{dist}\{\zeta_j', \delta\} > 0$ for both

$j = 1, 2$. Similarly, let $\delta' \subset \mathbb{T}$ be a neighborhood of the set $\{\zeta_1', \zeta_2'\}$ such that $\text{dist}\{\zeta_j, \delta'\} > 0$ for both $j = 1, 2$. We split the integral in (4.16) over $\mathbb{T}$ in three integrals: over δ, δ' and $\Sigma = \mathbb{T} \setminus (\delta \cup \delta')$, According to Lemma 4.1 $\|e^{-i\alpha T_s t} f\|_{L^2(\delta')} \to 0$, $\|e^{-i\alpha' T_s' t} f'\|_{L^2(\delta)} \to 0$ and both factors $\|e^{-i\alpha T_s t} f\|_{L^2(\Sigma)} \to 0$, $\|e^{-i\alpha' T_s' t} f'\|_{L^2(\Sigma)} \to 0$ as $|t| \to \infty$. This implies (4.16). □

Next, we compare the wave operators for the pair $\mathbf{\Omega}$, $T(\omega)$ with those for the pair $T(\alpha \mathbb{1}_{(\zeta_1, \zeta_2)}), T(\omega)$.

Theorem 4.7. *Let $(\zeta_1, \zeta_2) \subset \mathbb{T}$ and $\alpha \in \mathbb{R}$. Suppose that the wave operators $W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P})$ and $W_\pm(T(\omega), T(\alpha \mathbb{1}_{(\zeta_1, \zeta_2)}))$ exist. Then the subspaces*

$$\text{Ran} \, W_\pm(T(\omega), \mathbf{\Omega}; \mathbb{P}) \quad and \quad \text{Ran} \, W_\pm(T(\omega), T(\alpha \mathbb{1}_{(\zeta_1, \zeta_2)}))$$

of $\mathbb{H}^2$ are orthogonal to each other.

Proof. It suffices to check that

$$\lim_{|t| \to \infty} (\mathbb{P} e^{-i\mathbf{\Omega} t} f, e^{-iT_s t} f_0) = 0, \quad T_s = T(\alpha \mathbb{1}_{(\zeta_1, \zeta_2)}),$$

for all $f \in \mathcal{H}_{\mathbf{\Omega}}^{(\text{ac})}$ and all $f_0 \in \mathbb{H}^2$ such that $\Phi(\mathbb{1}_{(\zeta_1, \zeta_2)}) f_0 \in C_0^\infty(0, 1)$. Let $\psi_\epsilon \in C^\infty(\mathbb{T})$ be a real-valued function such that $\psi_\epsilon(\zeta) = 1$ in ϵ-neighborhoods of the points ζ_1 and ζ_2 and $\psi_\epsilon(\zeta) = 0$ away from 2ϵ-neighborhoods of these points. Put $\phi_\epsilon(\zeta) = 1 - \psi_\epsilon(\zeta)$. Clearly,

$$|(\mathbb{P} e^{-i\mathbf{\Omega} t} \psi_\epsilon f, e^{-iT_s t} f_0)| \le \|\psi_\epsilon f\| \|f_0\|$$

tends to 0 as $\epsilon \to 0$ uniformly in t.

Thus, we only have to show that

$$\lim_{|t| \to \infty} (\mathbb{P} e^{-i\mathbf{\Omega} t} \phi_\epsilon f, e^{-iT_s t} f_0) = 0, \quad \mathbb{P} = \mathbb{P}_+,$$

for a fixed $\epsilon > 0$. Recall that according to (3.18) the commutator

$$[\mathbb{P}, \phi_\epsilon] = \mathbb{P}_+ \phi_\epsilon - \phi_\epsilon \mathbb{P}_+ = \mathbb{P}_+ \phi_\epsilon \mathbb{P}_- - \mathbb{P}_- \phi_\epsilon \mathbb{P}_+$$

is compact (actually, it belongs to the trace class). Thus property (g), see Sect. 3.1, implies that

$$\lim_{|t| \to \infty} \| [\mathbb{P}, \phi_\epsilon] e^{-i\mathbf{\Omega} t} f \| = 0,$$

and hence it suffices to check that

$$\lim_{|t| \to \infty} (\mathbb{P} e^{-i\mathbf{\Omega} t} f, \phi_\epsilon e^{-iT_s t} f_0) = 0. \tag{4.17}$$

According to Lemma 4.1 we have the estimate

$$\|\phi_\epsilon e^{-iT_s t} f_0\| \le C_p |t|^{-p}, \quad \forall p > 0,$$

whence (4.17) follows. □

5. Putting things together. Main result

Now we are in a position to develop scattering theory for Toeplitz operators $T = T(\omega)$ with piecewise continuous symbols ω. From now on, we always assume that Condition 2.3 is satisfied with some finite set $\mathsf{S} = \{\eta_1, \eta_2, \dots\}$.

5.1. **The existence of wave operators.** Here, we state several immediate consequences of the results established in Sections 3 and 4.

The first theorem is a special case of parts (i) and (ii) of Theorem 3.17 with the set $\Delta = \mathbb{T} \setminus \mathsf{S}$.

Theorem 5.1. *Suppose that Condition 2.3 is satisfied and that $\omega \in \mathbb{W}^{2,1}_{\mathrm{loc}}(\mathbb{T} \setminus \mathsf{S})$. Let the open sets $\Delta^{(\pm)} \subset \mathbb{T} \setminus \mathsf{S}$ be as defined in (2.1). Then the wave operators $W_\pm(T, \mathbf{\Omega}; \mathbb{P})$ exist and are isometric on the subspaces $\mathrm{Ran}\, \mathbb{1}_{\Delta^{(\pm)}}$. Moreover, they satisfy relations (3.26).*

Next, we consider the wave operators produced by the discontinuities of ω. Let $\mathsf{S}^{(\pm)}, \mathsf{S}_0$, be the sets in the union (2.12). Every singular point $\eta_k \in \mathsf{S}^{(\pm)}$ (but not $\eta_k \in \mathsf{S}_0$) produces a new channel of scattering. Following the construction in Section 4.2, we choose an arbitrary $\varepsilon \in (0, 2\pi)$ and introduce auxiliary symbols

$$\left. \begin{aligned} \omega_k(\zeta) &= \omega(\eta_k - 0) \quad \text{for} \quad \zeta \in (e^{-i\varepsilon}\eta_k, \eta_k) \\ \omega_k(\zeta) &= \omega(\eta_k + 0) \quad \text{for} \quad \zeta \notin (e^{-i\varepsilon}\eta_k, \eta_k) \end{aligned} \right\} \quad \text{if} \quad \eta_k \in \mathsf{S}^{(+)} \qquad (5.1+)$$

and

$$\left. \begin{aligned} \omega_k(\zeta) &= \omega(\eta_k + 0) \quad \text{for} \quad \zeta \in (\eta_k, e^{i\varepsilon}\eta_k) \\ \omega_k(\zeta) &= \omega(\eta_k - 0) \quad \text{for} \quad \zeta \notin (\eta_k, e^{i\varepsilon}\eta_k) \end{aligned} \right\} \quad \text{if} \quad \eta_k \in \mathsf{S}^{(-)}. \qquad (5.1-)$$

The Toeplitz operator $T_k = T(\omega_k)$ has simple absolutely continuous spectrum that coincides with the interval Λ_k defined in (2.13).

Theorem 4.4 implies the following result.

Theorem 5.2. *Suppose that the condition (4.7) holds at some point $\eta_k \in \mathsf{S}^{(\pm)}$. Let the symbol ω_k be defined by relations (5.1+) or (5.1–). Then the wave operator $W_\pm(T, T_k)$ exists and is isometric on $\mathbb{H}^2$.*

Corollary 5.3. *The spectral representation of the operator T restricted to the subspace $\mathrm{Ran}\, W_\pm(T, T_k)$ is realized on the space $L^2(\Lambda_k)$.*

Although symbols (5.1+) and (5.1–) depend on ε, Remark 4.5 shows that the dependence of $W_\pm(T, T_k)$ on ε is trivial.

The next result follows from Theorems 4.6 and 4.7. It shows that different scattering channels are orthogonal to each other.

Theorem 5.4. *Let the assumptions of Theorem 5.1 hold. Suppose also that the condition (4.7) is satisfied at all points $\eta_k \in \mathsf{S}^{(\pm)}$. Then the ranges $\mathrm{Ran}\, W_\pm(T, T_k)$ for all $\eta_k \in \mathsf{S}^{(\pm)}$ are orthogonal to $\mathrm{Ran}\, W_\pm(T, \mathbf{\Omega}; \mathbb{P})$ and are pairwise orthogonal to each other.*

It follows from the above results that

$$\operatorname{Ran} W_{\pm}(T, \boldsymbol{\Omega}; \mathbb{P}) \oplus \bigoplus_{\eta_k \in \mathsf{S}^{(\pm)}} \operatorname{Ran} W_{\pm}(T, T_k) \subset \mathbb{H}^2. \tag{5.2}$$

Our final objective is to establish the equality in (5.2). In other words, we intend to prove the asymptotic completeness of the wave operators involved.

5.2. **Counting multiplicities.** Our proof of the asymptotic completeness requires an elementary result of a general nature (cf. [17, Theorem 1.5.7]) concerning self-adjoint operators with finite spectral multiplicity.

Theorem 5.5. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $\tilde{A}$ be its restriction to its invariant subspace $\widetilde{\mathcal{H}}$. Suppose that, for some interval $\Lambda \subset \mathbb{R}$, both operators A and $\tilde{A}$, are unitarily equivalent to the operator of multiplication by independent variable λ on the space $L^2(\Lambda; \mathbb{C}^m)$ where $m < \infty$. Then $\widetilde{\mathcal{H}} = \mathcal{H}$ and $\tilde{A} = A$.*

Proof. We may assume that $\mathcal{H} = L^2(\Lambda; \mathbb{C}^m)$ and $(Af)(\lambda) = \lambda f(\lambda)$, $f \in \mathcal{H}$, a.e. $\lambda \in \Lambda$. Since $\tilde{A}$ is a restriction of A, it acts as multiplication by λ in the direct integral (see, for example, §§7.1, 7.2 of the book [1])

$$\int_{\Lambda}^{\oplus} G(\lambda) d\lambda$$

where $G(\lambda)$ is a measurable family of some subspaces of $\mathbb{C}^m$. At the same time, $\tilde{A}$ is unitarily equivalent to multiplication by λ on the space $L^2(\Lambda; \mathbb{C}^m)$. It follows that $\dim G(\lambda) = m$ for a.e. $\lambda \in \Lambda$. Since $m < \infty$, we conclude that $G(\lambda) = \mathbb{C}^m$ and hence $\widetilde{\mathcal{H}} = \mathcal{H}$ and $\tilde{A} = A$, as required. $\qquad\square$

Corollary 5.6. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let A_k, $k = 0, \ldots, N$, $N < \infty$, be its restrictions to pairwise orthogonal invariant subspaces $\mathcal{H}_k$. Suppose that, for some interval $\Lambda \subset \mathbb{R}$, the operators A and A_k, $k = 0, \ldots, N$, are unitarily equivalent to the operators of multiplication by independent variable λ in the spaces $L^2(\Lambda; \mathbb{C}^m)$, $m < \infty$, and $L^2(\Lambda; \mathbb{C}^{m_k})$, $k = 0, \ldots, N$, respectively. Assume that*

$$\sum_{k=0}^{N} m_k = m.$$

Then

$$\bigoplus_{k=0}^{N} \mathcal{H}_k = \mathcal{H} \quad \text{and} \quad \bigoplus_{k=0}^{N} A_k = A.$$

Proof. It suffices to apply Theorem 5.5 to the subspace $\widetilde{\mathcal{H}} = \oplus_{k=0}^{N} \mathcal{H}_k \subset \mathcal{H}$ and the operator $\tilde{A} = \oplus_{k=0}^{N} A_k$. $\qquad\square$

5.3. **Asymptotic completeness.** In order to use Corollary 5.6 for the proof of the asymptotic completeness, we need to find spectral multiplicities for the operator T restricted to the subspaces on the left-hand side of (5.2). The required result for $\mathrm{Ran}\, W_{\pm}(T, T_k)$ is given by Corollary 5.3.

Let us now consider the first term in (5.2). Below we systematically use the intertwining property (3.2).

Lemma 5.7. *Suppose that the conditions of Theorem 5.1 hold, and that an interval Λ satisfies (2.6). Then the spectral representation of the operator T restricted to the subspace*

$$\mathcal{H}^{(\pm)}(\Lambda) := \mathrm{Ran}\left(E_T(\Lambda)W_{\pm}(T, \mathbf{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(\pm)}})\right) \tag{5.3}$$

is realized on the space $L^2\left(\Lambda; \mathbb{C}^{n^{(\pm)}}\right)$.

Proof. Let $\delta^{(\pm)} = \delta^{(\pm)}(\Lambda)$ be defined by (2.7). By Theorem 2.4, the spectral representation of the operator $\mathbf{\Omega}$ restricted to $L^2(\delta^{(\pm)})$ is realized on the space $L^2(\Lambda; \mathbb{C}^{n^{(\pm)}})$. According to Theorem 5.1 and the intertwining property (3.2), the operator

$$E_T(\Lambda)W_{\pm}(T, \mathbf{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(\pm)}}) = W_{\pm}(T, \mathbf{\Omega}; \mathbb{P}\mathbb{1}_{\delta^{(\pm)}}) : L^2(\delta^{(\pm)}) \to \mathcal{H}^{(\pm)}(\Lambda),$$

is unitary. Therefore the spectral multiplicities of the operators $\mathbf{\Omega}|_{L^2(\delta^{(\pm)})}$ and $T|_{\mathcal{H}^{(\pm)}(\Lambda)}$ on the interval Λ coincide, and both equal $n^{(\pm)}$. $\square$

First, we verify a "local" form of the asymptotic completeness. Recall that the intervals Λ_k are defined by (2.13), the subset $\mathsf{S}(\Lambda) \subset \mathsf{S}$ is distinguished by the condition $\Lambda \subset \Lambda_k$ for $\eta_k \in \mathsf{S}(\Lambda)$, and $\mathsf{S}^{(\pm)}(\Lambda) = \mathsf{S}(\Lambda) \cap \mathsf{S}^{(\pm)}$.

Theorem 5.8. *Suppose that the symbol ω satisfies Condition 2.3 with some finite set S, and that $\omega \in \mathbb{W}^{2,1}_{\mathrm{loc}}(\mathbb{T} \setminus \mathsf{S})$. Let Λ be an interval satisfying (2.6), and assume that condition (4.7) holds at each point $\eta_k \in \mathsf{S}^{(\pm)}(\Lambda)$. Then the equality*

$$\mathrm{Ran}\left(E_T(\Lambda)W_{\pm}(T, \mathbf{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(\pm)}})\right)$$

$$\oplus \bigoplus_{\eta_k \in \mathsf{S}^{(\pm)}(\Lambda)} \mathrm{Ran}\left(E_T(\Lambda)W_{\pm}(T, T_k)\right) = E_T(\Lambda)\mathbb{H}^2 \tag{5.4}$$

holds for both signs "+" and "−".

Proof. Note that the orthogonal sum over η_k on the left-hand side of (5.4) contains $s^{(\pm)} = \#(\mathsf{S}^{(\pm)}(\Lambda))$ terms. Thus, using the notation

$$\mathcal{H}^{(\pm)}_k(\Lambda) := \mathrm{Ran}\left(E_T(\Lambda)W_{\pm}(T, T_k)\right), \quad \eta_k \in \mathsf{S}^{(\pm)}(\Lambda),$$

$\mathcal{H}(\Lambda) := E_T(\Lambda)\mathbb{H}^2$, and definition (5.3), we can rewrite (5.4) as

$$\mathcal{H}^{(\pm)}(\Lambda) \oplus \bigoplus_{k=1}^{s^{(\pm)}} \mathcal{H}^{(\pm)}_k(\Lambda) = \mathcal{H}(\Lambda). \tag{5.5}$$

In order to prove (5.5), we make the following observations. According to Lemma 5.7 the operator T restricted to the subspace $\mathcal{H}^{(\pm)}(\Lambda)$ has spectral multiplicity $n^{(\pm)}$. Similarly, Corollary 5.3 shows that, for every $k = 1, 2, \ldots, s^{(\pm)}$ the operator T on the subspace $\mathcal{H}_k^{(\pm)}(\Lambda)$ has spectral multiplicity 1.

Now we use Corollary 5.6 with the spaces $\mathcal{H}(\Lambda)$,

$$\widetilde{\mathcal{H}}(\Lambda) = \mathcal{H}^{(\pm)}(\Lambda) \oplus \bigoplus_{k=1}^{s^{(\pm)}} \mathcal{H}_k^{(\pm)}(\Lambda) \subset \mathcal{H}(\Lambda),$$

the operators

$$A = T|_{\mathcal{H}(\Lambda)}, \quad A_0 = T|_{\mathcal{H}^{(\pm)}(\Lambda)}, \quad A_k = T|_{\mathcal{H}_k^{(\pm)}(\Lambda)},$$

and multiplicities

$$m_0 = n^{(\pm)}, \quad m_k = 1, \quad k = 1, 2, \ldots, s^{(\pm)}.$$

According to Theorem 2.5, the spectral multiplicity of the operator A equals

$$m = n^{(\pm)} + s^{(\pm)} = \sum_{k=0}^{s^{(\pm)}} m_k.$$

Thus Corollary 5.6 entails (5.5), which completes the proof. $\qquad\square$

The next theorem constitutes the main result of the paper.

Theorem 5.9. *Suppose that the symbol ω satisfies Condition 2.3 with some finite set S, and that $\omega \in \mathbb{W}_{\mathrm{loc}}^{2,1}(\mathbb{T} \setminus \mathsf{S})$. Assume that condition (4.7) holds at each point $\eta_k \in \mathsf{S}^{(\pm)}$. Then*

(i) *The wave operators $W_\pm(T(\omega), \boldsymbol{\Omega}; \mathbb{P})$ exist and satisfy relation (3.26). These operators are isometric on the subspaces $\mathrm{Ran}\, \mathbb{1}_{\Delta^{(\pm)}}$ of $L^2(\mathbb{T})$.*

(ii) *Let $\eta_k \in \mathsf{S}^{(\pm)}$, and let the symbols ω_k be defined by formulas (5.1+), (5.1−). Then the wave operators $W_\pm(T(\omega), T(\omega_k))$ exist and are isometric.*

(iii) *The ranges of all operators $W_\pm(T(\omega), \boldsymbol{\Omega}; \mathbb{P})$ and $W_\pm(T(\omega), T(\omega_k))$ are orthogonal to each other.*

(iv) *The asymptotic completeness holds:*

$$\mathrm{Ran}\, W_\pm(T(\omega), \boldsymbol{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(\pm)}}) \oplus \bigoplus_{\eta_k \in \mathsf{S}^{(\pm)}} \mathrm{Ran}\, W_\pm(T(\omega), T(\omega_k)) = \mathbb{H}^2 \qquad (5.6)$$

for both signs "+" and "−".

Proof. Assertions (i), (ii) and (iii) are direct consequences of Theorems 5.1, 5.2 and 5.4, respectively.

Let us check (iv). Since Λ_{exc} is closed, the set $G = (\gamma_1, \gamma_2) \setminus \Lambda_{\mathrm{exc}}$ is open. According to Theorem 5.8 the relation (5.4) is true for every constituent open subinterval of G. For every such subinterval Λ, the orthogonal sum over $\eta_k \in \mathsf{S}^{(\pm)}(\Lambda)$ in (5.4) coincides with the sum over all $\eta_k \in \mathsf{S}^{(\pm)}$. Indeed, if $\eta_k \in \mathsf{S}^{(\pm)} \setminus \mathsf{S}^{(\pm)}(\Lambda)$, then

$E_{T_k}(\Lambda) = 0$ and hence, by the intertwining property (3.2), $E_T(\Lambda)W_\pm(T, T_k) = W_\pm(T, T_k)E_{T_k}(\Lambda) = 0$. Therefore summing relations (5.4) over all constituent subintervals of G, we obtain this relation for the set G itself:

$$\operatorname{Ran}\left(E_T(G)W_\pm(T, \mathbf{\Omega}; \mathbb{P}\mathbb{1}_{\Delta^{(\pm)}})\right)$$
$$\oplus \bigoplus_{\eta_k \in \mathsf{S}^{(\pm)}} \operatorname{Ran}\left(E_T(G)W_\pm(T, T_k)\right) = E_T(G)\mathbb{H}^2, \qquad (5.7)$$

where $T = T(\omega)$, $T_k = T(\omega_k)$. Since the set Λ_{exc} has measure zero and the operator T is absolutely continuous, we have $E_T(G) = I$. Thus (5.7) coincides with (5.6). $\quad\square$

5.4. **Classification of the spectrum.** Let us come back to the classification of the spectrum given by Definition 1.1. In this subsection we relate the subspaces $\mathcal{H}_{\mathrm{thin}}, \mathcal{H}_{\mathrm{thick}}$ defined in (1.5) with the subspaces on the left-hand side of (5.6). We suppose that the conditions of Theorem 5.9 are satisfied, and define the subspaces

$$\begin{cases} \mathcal{H}_{\mathrm{thick}}^{(\pm)} = \operatorname{Ran} W_\pm(T, \mathbf{\Omega}; \mathbb{P}), \quad T = T(\omega), \\ \mathcal{H}_{\mathrm{thin}}^{(\pm)} = \bigoplus_{\eta_k \in \mathsf{S}^{(\pm)}} \operatorname{Ran} W_\pm(T, T_k), \quad T_k = T(\omega_k), \end{cases} \qquad (5.8)$$

of $\mathbb{H}^2$.

The statement below (cf. Corollary 3.20) is a direct consequence of the definition of the wave operators.

Lemma 5.10. *For every $f \in \mathcal{H}_{\mathrm{thick}}^{(\pm)}$, asymptotic relation (1.3) holds with $f^{(\pm)} = W_\pm(T, \mathbf{\Omega}; \mathbb{P})^* f$. For every $f \in \mathcal{H}_{\mathrm{thin}}^{(\pm)}$, asymptotic relation*

$$e^{-iTt}f \sim \sum_{\eta_k \in \mathsf{S}^{(\pm)}} e^{-T_k t} f_k^{(\pm)}, \quad t \to \pm\infty, \qquad (5.9)$$

holds with $f_k^{(\pm)} = W_\pm(T, T_k)^* f$.

Using Theorem 5.9 it is easy to find a relation between the thick and thin subspaces defined by (1.5) and the subspaces (5.8).

Lemma 5.11. *Under the assumptions of Theorem 5.9, we have*

$$\mathcal{H}_{\mathrm{thick}} \subset \mathcal{H}_{\mathrm{thick}}^{(+)} \cap \mathcal{H}_{\mathrm{thick}}^{(-)} \qquad (5.10)$$

and

$$\mathcal{H}_{\mathrm{thin}} \subset \mathcal{H}_{\mathrm{thin}}^{(+)} \cap \mathcal{H}_{\mathrm{thin}}^{(-)}. \qquad (5.11)$$

Proof. First we check (5.10). Let $f \in \mathcal{H}_{\mathrm{thick}}$. By definition (1.5), this means that $f = E_T(\sigma(\mathbf{\Omega}) \setminus \Upsilon)f$. It now follows from formula (5.6) that

$$f = E_T(\sigma(\mathbf{\Omega}) \setminus \Upsilon)W_\pm(T, \mathbf{\Omega}; \mathbb{P})f^{(\pm)} + \sum_{\eta_k \in \mathsf{S}^{(\pm)}} E_T(\sigma(\mathbf{\Omega}) \setminus \Upsilon)W_\pm(T, T_k)f_k^{(\pm)} \quad (5.12)$$

for some $f^{(\pm)} \in L^2(\mathbb{T})$ and $f_k^{(\pm)} \in \mathbb{H}^2$. In view of the intertwining property (3.2) we can rewrite (5.12) as

$$f = W_{\pm}(T, \mathbf{\Omega}; \mathbb{P})E_{\mathbf{\Omega}}(\sigma(\mathbf{\Omega}) \setminus \Upsilon)f^{(\pm)} + \sum_{\eta_k \in \mathsf{S}^{(\pm)}} W_{\pm}(T, T_k)E_{T_k}(\sigma(\mathbf{\Omega}) \setminus \Upsilon)f_k^{(\pm)}. \quad (5.13)$$

All terms in the sum over η_k vanish because $\sigma(T_k) \cap \big(\sigma(\mathbf{\Omega}) \setminus \Upsilon\big) = \varnothing$. Thus it follows from (5.13) that

$$f = W_{\pm}(T, \mathbf{\Omega}; \mathbb{P})E_{\mathbf{\Omega}}(\sigma(\mathbf{\Omega}) \setminus \Upsilon)f^{(\pm)} \in \mathcal{H}_{\text{thick}}^{(\pm)}$$

for both signs "$\pm$".

The inclusion (5.11) is verified in a similar way. Precisely, if $f \in \mathcal{H}_{\text{thin}}$, then $f = E_T(\sigma(T) \setminus \sigma(\mathbf{\Omega}))f$. Therefore using again formula (5.6) and the intertwining property (3.2), we find that

$$f = W_{\pm}(T, \mathbf{\Omega}; \mathbb{P})E_{\mathbf{\Omega}}(\sigma(T) \setminus \sigma(\mathbf{\Omega}))f^{(\pm)}$$
$$+ \sum_{\eta_k \in \mathsf{S}^{(\pm)}} W_{\pm}(T, T_k)E_{T_k}(\sigma(T) \setminus \sigma(\mathbf{\Omega}))f_k^{(\pm)}.$$

Since the first term on the right is zero, we see that

$$f = \sum_{\eta_k \in \mathsf{S}^{(\pm)}} W_{\pm}(T, T_k)E_{T_k}(\sigma(T) \setminus \sigma(\mathbf{\Omega}))f_k^{(\pm)} \in \mathcal{H}_{\text{thin}}^{(\pm)}$$

for both signs "$\pm$". $\qquad \square$

Combining Lemmas 5.10 and 5.11, we find an asymtotic behavior of $e^{-iTt}f$ for $f \in \mathcal{H}_{\text{thick}}$ and $f \in \mathcal{H}_{\text{thin}}$.

Theorem 5.12. *For every $f \in \mathcal{H}_{\text{thick}}$, asymptotic relations (1.3) are satisfied for both signs "$\pm$" with $f^{(\pm)} = W_{\pm}(T, \mathbf{\Omega}; \mathbb{P})^*f$. For every $f \in \mathcal{H}_{\text{thin}}$, asymptotic relations (5.9) are satisfied for both signs "$\pm$" with $f_k^{(\pm)} = W_{\pm}(T, T_k)^*f$.*

For $f \in \mathcal{H}_{\text{mix}}$, the asymtotics of $e^{-iTt}f$ as $t \to \pm\infty$ may contain both terms $\mathbb{P}e^{-i\mathbf{\Omega}t}f^{(\pm)}$ and $e^{-iT_kt}f_k^{(\pm)}$. This is illustrated with the explicit example considered in the next subsection. It exhibits all three types of spectrum.

5.5. **Example.** Consider the symbol shown in Fig. 1. For convenience we copy this figure again with more detailed labelling, see Fig. 3. Below we use notation (1.4) and (2.1).

Assume that $\omega'(\zeta) > 0$ on the arcs $(\varkappa, \eta)$ and (η, ν), and $\omega'(\zeta) < 0$ on the arc $(\nu, \varkappa)$. Thus the spectrum of $T(\omega)$ is simple and it coincides with the interval $[0, d]$. Also,

$$\Delta^{(+)} = (\nu, \varkappa), \ \Delta^{(-)} = (\varkappa, \eta) \cup (\eta, \nu), \quad \mathsf{S}^{(+)} = \{\varkappa\}, \ \mathsf{S}^{(-)} = \{\eta\}$$

and $\Lambda_{\varkappa} = [0, b]$, $\Lambda_{\eta} = [a, c]$ so that $\Upsilon = [0, c]$. By Definition 1.1, the thin, thick and mixed spectra coincide with the sets $[a, b]$, $[c, d]$ and $[0, a] \cup [b, c]$, respectively.
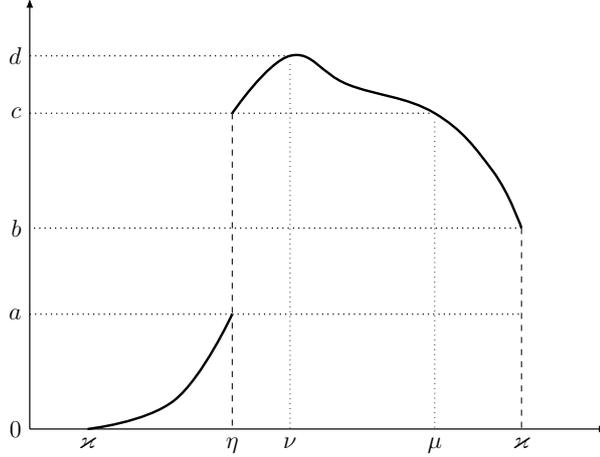
FIGURE 3. Example

The model jump symbols are

$$\omega_+(\zeta) = b\mathbb{1}_{(\varkappa e^{-i\varepsilon}, \varkappa)}(\zeta), \quad \omega_-(\zeta) = a + (c-a)\mathbb{1}_{(\eta, \eta e^{i\varepsilon})}(\zeta),$$

with a fixed $\varepsilon \in (0, 2\pi)$. It is clear that $\Lambda_{\mathrm{exc}} = \{0, a, b, c, d\}$. Thus the set $(0, d) \backslash \Lambda_{\mathrm{exc}}$ is the union of four intervals,

$$(0, d) \setminus \Lambda_{\mathrm{exc}} = (a, b) \cup (c, d) \cup (0, a) \cup (b, c),$$

each of which satisfies (2.6). Consider them one by one. Below we use the notation (2.7): $\delta^{(\pm)} = \Delta^{(\pm)} \cap \omega^{-1}(\Lambda)$.

<u>Thin spectrum</u>. Let $\Lambda = (a, b)$, so that

$$\delta^{(+)} = \delta^{(-)} = \varnothing, \quad n^{(+)} = n^{(-)} = 0, \quad s^{(+)} = s^{(-)} = 1.$$

According to Theorem 5.12, for every $f \in E_T(\Lambda)\mathbb{H}^2 = \mathcal{H}_{\mathrm{thin}}$ we have

$$e^{-iTt}f \sim e^{-iT(\omega_\pm)t}f^{(\pm)}, \quad t \to \pm\infty,$$

with $f^{(\pm)} = W_\pm(T, T(\omega_\pm))^* f$. This is consistent with (5.11).

<u>Thick spectrum.</u> Let $\Lambda = (c, d)$, so that

$$\delta^{(+)} = (\nu, \mu), \ \delta^{(-)} = (\eta, \nu), \quad n^{(+)} = n^{(-)} = 1, \quad s^{(+)} = s^{(-)} = 0.$$

According to Theorem 5.12, for every $f \in E_T(\Lambda)\mathbb{H}^2 = \mathcal{H}_{\mathrm{thick}}$ we have

$$e^{-iTt}f \sim \mathbb{P}e^{-i\boldsymbol{\Omega}t}f^{(\pm)}, \quad t \to \pm\infty,$$

with $f^{(\pm)} = W_\pm(T, \boldsymbol{\Omega}; \mathbb{P})^* f$. This is consistent with (5.10).

<u>Mixed spectrum</u>. Let $\Lambda = (0, a)$, so that

$$\delta^{(+)} = \varnothing, \ \delta^{(-)} = (\varkappa, \eta), \quad n^{(+)} = 0, \ n^{(-)} = 1 \quad s^{(+)} = 1, \ s^{(-)} = 0.$$

The asymptotic completeness (5.4) takes the form

$$\operatorname{Ran} W_+(T, T(\omega_+))E_{T(\omega_+)}(\Lambda) = \operatorname{Ran} W_-\big(T, \mathbf{\Omega}; \mathbb{P}\mathbb{1}_{(\varkappa,\eta)}\big) = E_T(\Lambda)\mathbb{H}^2. \qquad (5.14)$$

According to (3.5), it follows from (5.14) that for every $f \in E_T(0, a)\mathbb{H}^2$ we have

$$e^{-iTt}f \sim e^{-iT(\omega_+)t}f^{(+)},\ t \to \infty \quad \text{and} \quad e^{-iTt}f \sim \mathbb{P}e^{-i\mathbf{\Omega}t}f^{(-)},\ t \to -\infty, \qquad (5.15)$$

with

$$f^{(+)} = W_+(T, T(\omega_+))^*f \in E_{T(\omega_+)}(0, a)\mathbb{H}^2$$
$$\text{and} \quad f^{(-)} = W_-\big(T, \mathbf{\Omega}; \mathbb{P}\big)^*f \in E_{\mathbf{\Omega}}(0, a)L^2(\mathbb{T}).$$

Let $\Lambda = (b, c)$, so that

$$\delta^{(+)} = (\mu, \varkappa),\ \delta^{(-)} = \varnothing, \quad n^{(+)} = 1,\ n^{(-)} = 0, \quad s^{(+)} = 0,\ s^{(-)} = 1.$$

The asymptotic completeness (5.4) takes the form

$$\operatorname{Ran} W_+\big(T, \mathbf{\Omega}; \mathbb{P}\mathbb{1}_{(\mu,\varkappa)}\big) = \operatorname{Ran} W_-(T, T(\omega_-))E_{T(\omega_-)}(\Lambda) = E_T(\Lambda)\mathbb{H}^2. \qquad (5.16)$$

It follows from (5.16) that for $f \in E_T(b, c)\mathbb{H}^2$ the asymptotics of $e^{-iTt}f$ is given by relations similar to (5.15).

Thus, on the mixed spectrum, the operator T has different evolution properties as $t \to \infty$ and $t \to -\infty$.

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