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Digital mapping of school mathematics: Three projects to enhance teaching and learning

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Abstract

Most educational jurisdictions prescribe the content of mathematics curricula in print documents, increasingly available as downloadable files. Digital mapping offers another view of curriculum that can depart from the linear structure of these documents and more effectively convey the connections within school mathematics. In this chapter we introduce three digital mapping projects: *Math Mapper 6-8*, the *Dynamic Mathematics Curriculum Network*, and the *Cambridge Mathematics Framework*. Each project draws on distinct theoretical and methodological approaches. Moreover, the connections within school mathematics shown in each map are based on different sources. Despite these differences, each project seeks to enhance mathematics teaching and learning by visually representing connections, making the basis for those connections explicit, ensuring the map can be used in flexible ways, and providing on-demand access to related instructional materials. Initial feedback from the intended audiences for each map reveals the unique ways each project can contribute to mathematics education. Some future directions for digital mapping of school mathematics that emerged from our discussion of shared challenges across projects are also offered.

Keywords: curriculum mapping, learning trajectories, digital mathematics resources

In his discussion of maps as cultural technologies, Siegert argues for a view of maps not as representations of space but as "spaces of representation" (2011, p.14). In this chapter, we explore spaces of representation that can be created through the digital mapping of school mathematics. We define digital mapping as the process by which information is compiled and represented as a digital image. Ifenthaler and Hanewald (2014) note that many educational settings have invested considerable resources in "digital knowledge maps" which they describe as "visual representations that enable enriching, imaginative and transformative ways for teaching and learning" (p.v). We are interested in the potential of digital maps as spaces of representation that offer a transformative view of mathematics curricula by departing from the linear structure of print documents. Digital maps can visually foreground the many connections within school mathematics and offer immediate access to related resources such as instructional materials and current research. Given that learning outcomes for students in K-12 mathematics depend partly on how curriculum policy, design, and enactment are aligned (Schmidt, Wang, & McKnight, 2005) and in light of the ways digital technology enhances the ability of communities to construct, organize and share knowledge (Stahl, 2000), we suggest that digital maps, when placed in the public domain, can facilitate the alignment identified by Schmidt et al. (2005). In doing so, digital maps can significantly enhance mathematics teaching and learning.

In this chapter, we describe three digital mapping projects: *Math-Mapper 6-8,* a learning map developed in the US; the *Dynamic Mathematics Curriculum Network*, a digital network that resulted from a Canadian research project; and the *Cambridge Mathematics Framework*, a knowledge map being developed in England. Each author describes features of their project including: **purpose**; intended **audience**;

unit size of elements in the map; **structure and relationships; positionality and navigation**; processes enabling **extensibility and evolution** of the map; the **language** or mathematical register used; and **connections to other resources**. These descriptions introduce readers to the unique spaces of representation created in each project. Greater appreciation of the features of each map can be achieved by visiting each project website.

Math Mapper 6-8 and the Epistemology of Learning Maps (Jere Confrey)

The learning maps I have been developing are meant to provide insight into the paths a learner's developing knowledge is likely to follow as the learner moves from less to more sophisticated reasoning. The primary goal of these maps is to focus instruction around nine big ideas and to support learner-centered instruction in mathematics. I began building learning maps in 2008 when I undertook a project to visually describe the New York City mathematics curriculum standards using hexagons. In this early work, individual curriculum standards constituted the nodes for the map, and the goal was to assemble the hexagons to show users how the curriculum standards evolved across grades. This work concluded in a map of the Common Core State Standards in Mathematics (CCSS-M) for grades K-8 offering two views. One view showed a set of 18 learning trajectories (LTs). The other highlighted the grades of each set of curriculum standards (Confrey & Maloney, 2014; Confrey et al., 2011). However, the use of curriculum standards as nodes restricted the placement of any standard to a single location and because the United States curriculum standards vary in size, the map also lost consistency of scale. I created a second map.

This new map is one component of a digital learning system called *Math Mapper 6-8* (MM6-8) which covers the content of middle grades mathematics in the

United States prior to a full course in algebra (Siemens & Confrey, 2015). The map can be accessed by registering an account at sudds.co. The **purpose** of the MM6-8 map is two-fold. Firstly, the map creates a visual representation of the relationships among the big ideas and sub-constructs within middle school math. We see big ideas as concepts that anchor and coherently connect the content, processes, and forms of argumentation in a discipline; they help avoid viewing mathematics as a set of fragmented topics and skills. The non-linear, hierarchical design of the map is intended to support the use of the map with a variety of curricular materials. Secondly, the map is intended to provide teachers with direct access to empirically-based learning trajectories (LTs) (Clements & Sarama, 2004; Confrey, Toutkoushian, & Shah, 2019; Simon, 1995) which can guide learner-centered instruction and ground the map's related diagnostic assessments. A LT is:

> a researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009, p. 347).

On the map, LTs are specified for each sub-construct and connected to the curriculum standards to help assure the teacher's adequate coverage of standards while focusing their attention on student learning.

The map's **audience** is both students and teachers[1]. The map replaces the linearity of a book's table of contents in favor of multiple levels of visual illustration. The map's hierarchical structure helps to parsimoniously represent the primary relationships among big ideas supplemented with magnification of additional detail. Students who use the map can see how what they are learning connects to a small but powerful set of big ideas. Teachers can use MM6-8 to reexamine instructional materials and curriculum documents, access internet-based resources, and diagnostically assess student progress along LTs (Confrey, Gianopulos, McGowan, Shah, & Belcher, 2017). The map gives teachers access to empirically established ideas about learning using LTs and every level of the trajectories has a related set of assessment tasks. Assessment tests and practice are available to students and teachers accompanied by intuitive student and class reports to guide diagnostically-valid instructional moves.

Structure and relationships in the map are critical to its purpose. The learning map includes: nine big ideas, 25 relational learning clusters (RLCs), and 62 constructs, each of which is associated with a LT. An RLC is a set of 1-4 constructs that create a constellation of ideas to be learned together because their meaning derives from their relationships. For instance, in the RLC "key ratio relations" (Figure 1), the constructs of ratio equivalence (at the bottom), base ratio, and unit ratio co-support and inform each other. Base ratio and unit ratio are parallel constructs on the map. When students move to the next RLC, "building, comparing, and solving

[1] MM6-8 has been developed in ongoing partnership with three districts using the map and assessments and benefitted substantially from their suggestions and observations of their practices.

proportions," the first cluster's elements serve as a foundation.

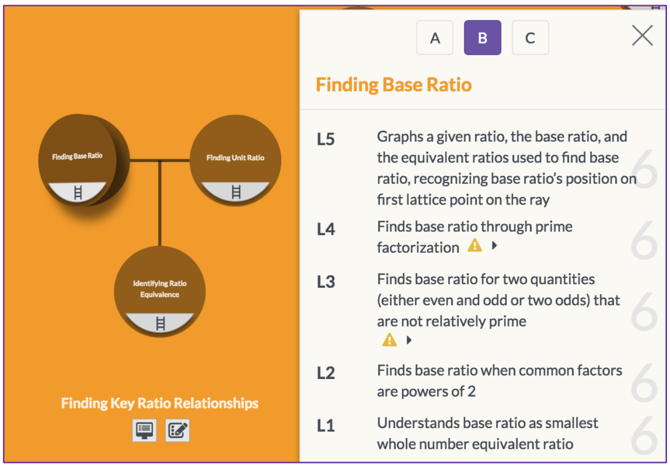

Figure 1. Learning trajectory for base ratio next to the RLC "Key Ratio Relationships"

Critical to the learning map's meaning are the underlying LTs. A LT appears when any construct in a cluster within a big idea is tapped. The LT levels are displayed from bottom to top to parallel a movement upwards in sophistication. Each LT level is labeled with a grade level to help teachers interpret what to expect from students across the grades. For example, the base ratio LT in Figure 1 has five levels. The small yellow triangles signal access to common misconceptions. Tapping one reveals its description on a flip card, with the correct conception on the back.

While developing MM6-8, the **unit sizes** at each level of the map have been a constant source of challenge. Four levels (big idea → RLC → construct → LT) proved necessary to reach the level of specificity needed to capture meaningful distinctions in student thinking. Nine big ideas seemed parsimonious. Only one big idea contains more than four RLCs. To support relatively efficient, valid, and reliable assessment, we limited the number of constructs in a cluster to four. We subsequently added, at the request of teachers, diagnostic assessments and practice at the construct level. Finally, the number of levels in an LT averages 5, depending on the extent of detail in the research, with no LT in the map exceeding 10 levels.

Positioning nodes and supporting navigation involves other design principles. The four main quadrants (number, statistics and probability, measurement, and algebra) are positioned to support an overall sense of movement from bottom left to upper right. Number is an entry point and movement is towards algebra. Likewise, RLCs within big ideas proceed left to right and bottom to top, again to signal the idea of growth. Within RLCs, lower constructs are likely to be learned before higher ones while those on the same level can be taught in any order. Navigation within the map allows easy and consistent movement with access to resources such as curriculum standards, misconceptions, and assessments handled through tabs and menus.

Coordinating topics across locations in the map was a major challenge in this project, involving significant decisions about where to locate ideas with strong connections to two big ideas. For example, linear regression is a topic with primary connections to statistics (a means to describe covarying relationships), and as an application in algebra (linear functions). For parsimony, I located topics in only one place, though I envisioned developing the idea of “wormholes” to connect disparate topics in future development.

MM6-8 is inherently **extensible and evolving.** Certain clusters have already been revised in light of subsequent research and validation of the diagnostic assessments. For instance, we added a construct, “building up/down with ratios,” based on data showing students floundering in comparing ratio and finding missing values in proportions.

Mapping requires careful attention to **language**, with the challenge being to communicate to users in a precise fashion. Space is limited on a map. For instance, LT levels need to be short but convey a depth of meaning. For example, L5 of “comparing ratios” reads ”compares ratios by examining relative steepness (later slope) of graph”. The term, “steepness”, was needed to avoid teachers prematurely introducing a formal definition of slope. Working for consistency of language use across the map has required frequent discussions and revisions based on practitioner feedback.

We built in **connections to other resources** to help users understand how the map relates to the larger educational enterprise. One such resource is access to the CCSS-M curriculum standards. Also, for each cluster, an icon allows access to illustrative curricular resources from the “Resource library” which is extensible and can be filtered by construct. Another icon accesses the diagnostic assessments and practice resources.

I see the creation of learning maps as a visually expressive activity that is ongoing and dialogic. Many characteristics and design principles of MM6-8 represent the first generation of a learning map. Because our map is linked to practice most vigorously through the use of the diagnostic assessments (n> 75,000 tests), my research team studies the use of the tool focusing on the validation of the LTs on which the maps are based (Confrey et al., 2019); and on how the data from the assessments are used to improve instruction (Confrey et al., 2018). Because our purpose is to promote learner-centered instruction, the map is subject to ongoing revision. In my mind, the admonition that “the map is not the territory” (Korzybski, 1933) reminds one that a map is fundamentally a cultural tool (Siegert, 2011) designed to help one navigate within a space. Its effectiveness is ultimately tied to how well it proves able to do that.

A participatory dynamic curriculum network (Martha Koch & Christine Suurtamm)

The Dynamic Mathematics Curriculum (DMC) Network (www.dynamicmathcurriculum.ca) emerged from our research on making connections within school mathematics more visible through the use of digital technology (Koch, Suurtamm, Lazarus & Masterson, 2018). The concept of the Network originated at a meeting of the Canadian Mathematics Education Study Group as we participated in a working group on reconceptualizing curricula (Davis & Kubota-Zarivnij, 2014). We began to envision a curriculum with elements of school mathematics represented as layers of connected nodes. Since then, with support from the Social Sciences and Humanities Research Council of Canada, we developed a method for engaging mathematics educators in co-creating such a curriculum network and designed a prototype based on their contributions.

We view mathematics as emergent and deeply connected and value non-linear views of mathematics teaching and learning. We also see curriculum as inherently dynamic; a curriculum emerges as each teacher transforms their written curriculum standards and available resources into classroom activity. We see participatory approaches to curriculum design as ways to enhance teaching (Clandinin & Connelly, 1988; Cochran-Smith & Lytle, 2009) and recognize that mathematics teachers, working together, can create a curriculum that responds to their students' needs (Breyfogle, McDuffie & Wohlhuter, 2010; Drake & Sherin, 2006). Researchers have also shown that mathematics learning is enhanced as teachers emphasize connections between concepts, processes, and representations

(Johanning, 2010). Although we find many curricula align with this research by encouraging teachers to make connections as they teach, we also notice curriculum documents and resources typically present mathematics as discrete strands and provide little support for teachers to make connections.

Accordingly, in Phase One of our research we devised a method of using collaborative problem solving to prompt educators to articulate the connections they make as they engage in mathematical thinking. In a series of video-recorded sessions, we asked participants to note aspects of mathematics that came to mind as they worked on a mathematical task in a small group. Each group then created a physical model connecting the mathematical content and processes they had identified. Figure 2 shows one model where participants represented the mathematical processes of problem solving, collaboration, and visualization using coloured pipe cleaners entwined around each of the connections the group had made between the concepts they had identified.

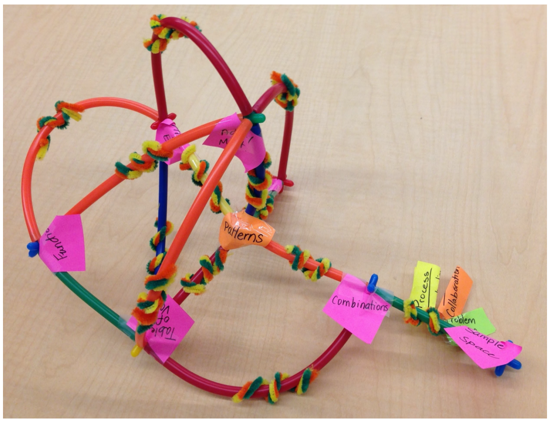

Figure 2. Research participants’ model of mathematics concepts and processes

Analysis of these models became the basis for the first iteration of the DMC Network posted on the project website. Participants sometimes referred to their province’s curriculum as they created their models but the DMC Network itself is not linked to one set of curriculum standards. In Phase Two, we invited other mathematics educators to visit the website and provide feedback on the first iteration of the Network using an “Add to the Network” tab. After analyzing their proposed contributions, we added several new nodes and connections and renamed some existing nodes. Educators from both phases were also invited to comment on features of the website and propose relevant instructional resources. Mathematicians, teacher educators, graduate students, teachers, and university-based researchers participated and the most recent iteration of the Network shown on the website is based on their collective views and experiences. The screenshot in Figure 3 provides a view of the website and the dynamic network showing the “Relations/functions” node.

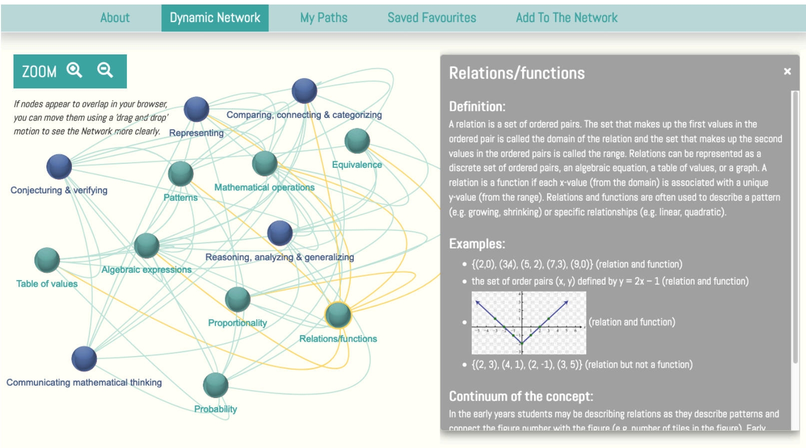

Figure 3. Screenshot from DMC Network focused on Relations/functions node

Our goal in this project has been to foreground connections in school mathematics by gathering and representing the connections educators make as they engage in problem solving. The **purpose** of the Network is to provide a prototype of what such a curriculum might look like. We anticipate the main **audience** for the Network would be teachers as they enact their curriculum and make instructional plans. Teacher educators may also use the Network to help pre-service and in-service teachers become more aware of these connections.

As with the other maps in this chapter, we wrestled with **unit size** as we developed the Network. Participants were comfortable including different grain sizes of mathematics ideas as they built their models. We were committed to ensuring the Network reflected their contributions and this has resulted in a lack of consistency of scale (noted in relation to *Math Mapper 6-8*). For instance, some nodes in the Network encompass more complex concepts (e.g. “Proportionality”) than others (e.g. “Table of values”). We also found unit size to be challenging because participants represented mathematics concepts and processes as interwoven in their models. To reflect this view, yet show some distinction between concepts and processes, we opted to use green nodes for concepts (e.g. “Algebraic expressions”) and blue nodes for processes (e.g. “Conjecturing & verifying”). We view this as an interim approach to resolving this aspect of unit size.

Language selection and use was also a concern in our research. As we compiled the original models and reviewed Phase Two participant input, we found a range of terms used to express ideas. We created a chart of similar and related terms and referred to that chart, sometimes making additions to it, as we decided which terms to use in the Network. Our goal has been to use terms that will make sense to as many users and contributors as possible.

The **structure and relationships** in the Network reflect complexity thinking (Davis & Simmt, 2003; Doll, 2008; St. Julien, 2005). A complex system, represented as a network, is more flexible than a linear or hierarchical structure since one can move from node to node in non-linear ways. It is also generative and adaptive as a result of interactions between its elements. These characteristics, within the digital interface we designed, mean an unlimited number of nodes and connections are possible and that some nodes may connect to many parts of the Network (e.g. “Algebraic expressions” currently connects to 12 nodes), while others have fewer connections (e.g. “Proportionality” currently connects to 4 nodes). As it grows, the Network also reflects multiple and emergent approaches to mathematics. Indeed, the first iteration had 10 nodes with 31 connections, growing to 13 nodes with 59 connections after Phase Two.

With respect to **positionality and navigation**, in the DMC Network the position of a node is determined solely by the connections between that node and other nodes. Users can click and drag any node in the Network to a different place on their screen and the connections from the node will be maintained. Clicking on a node also highlights all connections between that node and other nodes. These features, along with a +/- zoom tool, help users focus on a specific part of the Network. Clicking on a node or a connection also reveals definitions, explanations, examples, **connections to other resources** and a description of the continuum of the concept across grades. These resources have also been derived from our analysis of input from participants. Readers can explore other navigational features, including some still in the design phase, by visiting our website.

As with the other maps in this chapter, **extensibility and evolution** are key to our participatory approach. The problem-solving task we gave participants in Phase One of our research tended to prompt algebraic thinking and mathematics ideas often encountered in grades 6 to 10. Engaging participants with a different task would be one way to extend the Network to other topics. In theory, the extensibility and evolution of the Network is without limits as users contribute their ideas through the "Add to the network" feature. That said, moderating participants' suggestions has been challenging. Thus far, the research team reviews each submission and applies guidelines to determine which proposed changes should be made in the Network. One criterion we use for nodes is trying to avoid redundancy. For connections, a key criterion is our ability to follow and verify the mathematical thinking of the proposed connection (the "Add to the network" tool asks participants to explain their thinking for each proposed change). Moderating these contributions is a rich research opportunity but is also time consuming. More feasible strategies would be needed if the Network were to move from research project to widespread use.

Our project has provided insights about the ways collaborative problem solving prompts mathematical thinking and helps teachers articulate the connections they see in school mathematics. We have been encouraged by the enthusiasm and thoughtful contributions of participants and have learned a great deal about digital affordances and constraints. Our original vision for the Network included interconnected layers of nodes, strikingly similar to the layers of the Cambridge Mathematics Framework (see Figure 4). We felt these layers could effectively represent the many dimensions of mathematics teaching and learning. However, some of our participants suggested a virtual reality interface would be preferable as it would allow users to move freely within the connections space. Translating these ideas into reality captured our imagination but exceeded our financial resources. Thus, some of our participants' most innovative suggestions remain future visions.

Mapping knowledge about learning in the Cambridge Mathematics Framework (Ellen Jameson)

In the Cambridge Mathematics Framework (CM) project (www.cambridgemaths.org) a team of designers, teachers, and researchers are developing a framework for expressing mathematics learning experiences and key relationships between those experiences. The CM Framework is a tool for the dynamic generation of maps which highlight connections between ideas and experiences in school mathematics. These connections do not represent the paths that students follow, or that teachers plan. Rather, they highlight some important ways in which mathematical experiences can contribute to or depend on one another as students develop their understanding. The maps, and associated content, are representations of knowledge about mathematics learning, as interpreted from reports of research and practice according to our methodology (Jameson, 2019; Jameson, Gould & McClure, 2018). The maps are derived from a database which stores all connected content, and a front-end interface that allows us to create, edit, search, filter, group and map the content.

Our goal in developing the ability to generate multiple maps from a larger set of connections is to provide flexibility and additional perspectives on the relationships between mathematical ideas and experiences. This can be useful when developing curriculum objectives, planning curriculum scope and sequence, designing resources, and developing assessments. The ability to generate multiple maps can also contribute to teacher education and professional development. Research suggests that when people such as teachers, designers, researchers and policymakers, interpret the curriculum they are working with using a shared frame of reference, it may be possible to better explain and support coherent implementation of that curriculum (Remillard & Heck, 2014; Schmidt et al., 2005; Cunningham, 2017). Greater coherence, in turn, can have a positive impact on students' opportunity to learn mathematics by providing a shared logic for what is being taught and lowering barriers to learning that some students may otherwise not have the support to overcome (Schmidt et al., 2005).

Coherence in a mathematics curriculum depends on what can be known about mathematics and the learning process (Cobb, 1988; Tall, 2013; Thurston, 1990) as well as on what allows expert perspectives to converge (Pring, 2012; Schmidt et al., 2005) or to coordinate (Hall, Morley, & Chen, 2005; Robutti et al., 2016; Thurston, 1990) in line with some shared understanding. In our view, coherence in students' mathematical experiences depends in part on coherence in how professionals in different areas of mathematics education understand and support these experiences in their curricular environment (McClure, 2015). To address this need, the CM Framework is a network of mathematical ideas and experiences, as understood and documented by professionals in mathematics education research and practice - that is, of professional knowledge about student learning.

The **purpose** of this network is to support coherence in mathematics education. The CM Framework relates mathematical ideas to one another, in a way that is not specific to any particular curriculum. At the same time, the Framework relates mathematical ideas to things that may be applicable to a particular curriculum context such as curriculum standards, tasks, assessment components and professional development activities. The Framework can generate maps to represent these relationships in ways which are useful for different audiences.

Our **audiences** include anyone who holds and develops professional knowledge in mathematics education. However, the audiences for whom we feel we need to provide the most support are those who work most directly with large amounts of curriculum content: curriculum designers, resource designers, teachers, and teacher educators. An important part of professional knowledge in many of these roles is knowledge about someone else's knowledge. For example, maps of professional mathematicians' knowledge could be expected to be quite different from maps of teachers' or educational researchers' knowledge of student learning, which in turn would be different from maps of students' learning trajectories. We have made decisions about how to represent researchers' and teachers' knowledge of student learning in the CM Framework based on our purpose and audiences.

After some exploratory writing, we formalised the **structure and relationships** of the CM Framework in an *ontology* which defines what features appear in the network, what they mean to us and our audiences, and how they can be related to each other (Jameson et al., 2019). This ontology structures our database, determining how we create, store, search, filter and display content. The ontology helps us maintain consistency in the way mathematical ideas and experiences are represented. Similarly, it provides a guide to interpreting maps from the CM Framework for reviewers, and a version of it may help our audiences do the same.

We express mathematical content in our maps as *waypoints*, which we define as "'places where learners acquire knowledge, familiarity or expertise". Our definition is based on characterisations of learning sequences by Michener (1978) and Swan (2014). Each waypoint contains a summary of the mathematical idea (the 'what') and its part in the wider narrative (the 'why'), and lists examples of 'student actions' that would provide opportunities to experience the mathematics in meaningful ways. Some waypoints have particular roles; *exploratory waypoints* "indicate a place where ideas can be played with in a less formal or more playful way, as part of building mathematical intuition" and *landmark waypoints* are places where "ideas are brought together such that the whole experience may seem greater than the sum of its parts" (Jameson et al., 2019, p. 4). Waypoints are related to one another by *themes*. A theme might be a concept, skill or procedure, since a student might use or develop a mathematical idea in various ways. Themes representing development have a direction, leading from one waypoint to another. Waypoints are arranged from left to right roughly according to the 'order' created by these directed connections.

Research Summaries are documents in the CM Framework which tell the story of a group of waypoints and themes. They include a literature review, an interactive map of the waypoints and themes, and a section which describes how research has influenced the structure and contents of the map. An example of a Research Summary is available on our website along with a description of our ontology (Jameson et al., 2019).

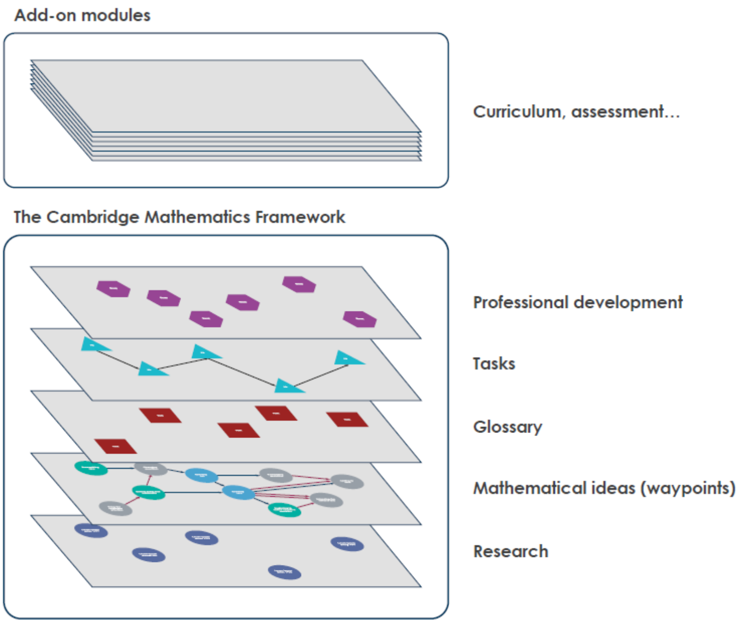

Figure 4: Connected layers within the CM Framework and external add-on modules

In the CM Framework, there is not one overall **unit size.** Rather, the unit size depends on the way the CM Framework is being used. For example, a single node in the network such as a waypoint, can function as a unit, but so can a Research Summary and a group of Research Summaries could outline a subdomain. Features of waypoints, themes, Research Summaries and other layers (examples are shown in Figure 4) are designed to provide different ways of engaging with and interpreting the CM Framework for different audiences. Some people may be looking for a 'way down' to get a perspective on more detailed content, while others may need a 'way up' to position their detailed understanding within a bigger picture. Likewise, some may be working at a time scale of a few weeks, while others may be designing for learning over a few months, a few years, or a decade.

Connections to other resources are managed within this layered structure. These resources might be for designers (such as curriculum statements for curriculum comparison or revision), for teachers and teacher educators (such as professional development activities), or for both (such as glossary definitions of mathematical terms). The structure of the CM Framework allows us to maintain connections within and between layers, so that research sources can be linked to waypoints and to Research Summaries, tasks can be linked to mathematical content which is linked to research, and so on. This structure also opens up other possibilities for connections to outside resources depending on future directions.

In the maps that are generated from the CM Framework, positioning of waypoints is key to **navigation**. We use a left-to-right order of "dependencies," where, roughly speaking, elements toward the right have more waypoints "leading in" to them. The vertical axis does not represent anything in a typical waypoint map. However, for some purposes, we might split a map horizontally into two regions, with the upper region representing some “grouping” category, such as curriculum statements, so that connections to the network of waypoints “underneath” can be examined more easily. At other times, we use the vertical axis to stack waypoints with equivalent “dependencies” in order to demonstrate that designers have leeway when ordering these waypoints in their curriculum or resource. Navigation of the wider network involves building a subset of waypoints which can be added to or subtracted from the rest of the network to focus on a particular topic.

As with the other projects in this chapter, **language** is essential for articulating meaning in the CM Framework. Our research sources, which influence the content and structure of the Framework, are mainly in English. However, we collaborate with researchers familiar with research in other languages (including Spanish, French, German, Japanese, and Chinese) in an effort to identify perspectives and findings of which we might otherwise have been unaware - though this still leaves important gaps that we seek to fill over time. While our familiarity with English-language research strongly influences our work, we hope through collaboration our work will be applicable outside that context as well. We also use a glossary to define key mathematical terms in the Framework. The structure of this glossary allows us to offer multiple definitions, which is important since contexts for learning which draw on a particular term can vary widely. We imagine this being useful in the future as a way to create definitions specific to certain user roles or regions. In fact, we have created a survey app which allows teachers to rate a family of definitions for a term according to usefulness and accuracy for their role and context, and as of this writing we are collecting data on our first set of terms.

The CM Framework is a large undertaking, and **extensiblity and evolution** are key in our design. Not only will new areas of content continue to be added and existing areas fleshed out, but as new research becomes available we will need to revise existing content. Our graph database, network structure and tools for online access work together to make these processes part of the work that we do on a daily basis. As our work progresses, we have been piloting the use of the CM Framework for curriculum and resource design. The feedback we have received thus far has been that the flexibility and perspectives afforded by the Framework are very valuable. However, we are also mindful of what we call the “map of gaps” - the ideas and paths not shown because they are unknown to us, research not consulted because of the constraints of time or language, or perspectives not taken because they are too distant from our own or those of our collaborators. While any interpretation would be susceptible to these issues, maps may be more susceptible as they can “feel” visually complete. We hope the flexibility and possibility for extension and curation in the CM Framework will help audiences probe beyond one view and learn from multiple perspectives, as designing this structure has certainly done for our learning.

Affordances, Challenges and Future Directions

These projects illustrate many aspects of the design and use of digital maps that can enhance mathematics teaching and learning. In each project, innovative approaches are used to visually represent connections within school mathematics. The connections in the CM Framework are derived from the research literature and interpreted and prioritised by a team of designers with educator backgrounds. In MM6-8, connections are based on empirically-researched learning trajectories. In the DMC Network connections are continually being gathered from mathematics educators. Deeper mathematics understandings can be achieved for students and teachers when greater emphasis is placed on connections among concepts and processes (Johanning, 2010). Yet, showing connections visually and in a non-linear manner is only one way these maps contribute to more effective teaching and learning. Given the capacity of digital technology for storing and accessing information, these maps also provide immediate access to related instructional resources, curriculum standards, assessments, and in some cases, learning trajectories or other research literature. Moreover, these maps are enormously flexible allowing individuals to use them in ways that meet their specific needs. The connections shown in each map are more general than a single curriculum document and each map uses different tools to enable users to focus on particular subsets of the connections. In addition, in all three projects, the maps allow for multiple paths through each space.

Given these characteristics, these maps are spaces of representation that foster shared and emergent understandings of mathematics and mathematics teaching and learning. For instance, in MM6-8 teachers are able to link big ideas in mathematics to empirically-based learning trajectories while students can see how what they are learning connects to those big ideas. Feedback from users of the CM Framework suggest it has enhanced the design of digital mathematics resources being developed by industry. Previously, such resources were often developed with more limited input from subject domain experts. In the DMC Network project, analysis of participants' comments suggests that articulating the connections they percieve and explaining those connections to others led to new ways of thinking about mathematics concepts and processes. In a similar way, we found the maps fostered emergent understandings across the three project teams. Recognizing commonalities in the design and development approaches we used prompted generative conversations about mathematics and led to a deeper appreciation of mathematics education in each of our jurisdictions. In these ways, whether used by policymakers, educational resource designers, researchers, teacher educators, teachers or students, digital maps can make a substantial contribution to more effective mathematics teaching and learning.

The challenges encountered in each project also suggest some future directions for this work. The unit size within digital maps is one area for further consideration. In MM6-8, consistent unit size is maintained within one level of the map while magnification enables more fine-grained information to be displayed. In essence, multiple unit sizes are supported by the affordances of digital technology. Allowing for multiple unit sizes also creates maps that can be used in more flexible ways, as posited by the CM Framework team. Multiple unit sizes might also more closely reflect the ways users see connections in school mathematics, as noted in the DMC Network. At the same time, allowing for multiple unit sizes can compromise consistency of scale. The potential impact on users of this sort of compromise needs to be more fully understood.

Another challenge is that significant resources are required both for the initial development and the continued evolution of these maps. In each project, additional aspects of school mathematics can be included and the connections represented can grow and change in response to new research (for MM6-8 and CM Frameworks) or as members of the mathematics education community contribute their ideas (for DMC Network). At the same time, fully representing the inherently multifaceted nature of mathematics and of mathematics teaching and learning remains somewhat elusive. The connections in school mathematics are more layered and complex than we can effectively represent, even with the affordances of digital technology. Thus, Confrey looks for a wormhole to more fully connect aspects of the MM6-8 learning map, participants in Koch and Suurtamm's project suggest the use of virtual reality or the addition of sound and motion to the DMC Network, and CM Framework designers endeavour to enable switching rapidly between multiple representations and views. Continuing to push the horizons toward these possibilities could mean these digital maps become even more imaginative spaces of representation, more fully capturing the richness of mathematics and the complexity of mathematics teaching and learning.

At the same time, further study of how the current iterations of these first-generation maps are being used to facilitate decision-making, in discussions among and between curriculum and resource designers, teachers, teacher educators and/or policy-makers will provide important insights into how these maps contribute to more effective and equitable mathematics teaching and learning within and across educational contexts.

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