MODIFIED GRAVITY AND COSMOLOGY: An Update by the CANTATA Network

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Abstract

General Relativity and the ΛCDM framework are currently the standard lore and constitute the concordance paradigm. Nevertheless, long-standing open theoretical issues, as well as possible new observational ones arising from the explosive development of cosmology the last two decades, offer the motivation and lead a large amount of research to be devoted in constructing various extensions and modifications.

All extended theories and scenarios are first examined under the light of theoretical consistency, and then are applied to various geometrical backgrounds, such as the cosmological and the spherical symmetric ones. Their predictions at both the background and perturbation levels, and concerning cosmology at early, intermediate and late times, are then confronted with the huge amount of observational data that astrophysics and cosmology are able to offer recently. Theories, scenarios and models that successfully and efficiently pass the above steps are classified as viable and are candidates for the description of Nature.

We list the recent developments in the fields of gravity and cosmology, presenting the state of the art, high-lighting the open problems, and outlining the directions of future research. Its realization is performed in the framework of the COST European Action "Cosmology and Astrophysics Network for Theoretical Advances and Training Actions".

Introduction

The dawn of the 21st century came with very positive prospects for gravity, cosmology and astrophysics. Technological progress made it possible for cosmology to enter to its adulthood and become a precision science, both for its own shake as well as for being the laboratory of gravity, which can now be accurately tested and investigated in scales different than the earth ones. As a result, the opinion that cosmology is one of the main directions that will lead to progress in physics in the near future, is now well established.

"Cosmology and Astrophysics Network for Theoretical Advances and Training Actions" (CANTATA) is a COST European Action established in order to contribute to the front of research in the fields of gravity, cosmology and astrophysics. It involves Institutions from 26 European countries, as well as from 5 countries abroad. CANTATA Collaboration has a variety of interests, which include: i) the classification and definition of theoretical and phenomenological aspects of gravitational interaction that cannot be enclosed in the standard lore scheme but might be considered as signs of alternative theories of gravity, ii) the confrontation of the theoretical predictions with observations at both the background and the perturbation levels, iii) the production of numerical codes to simulate astrophysical and cosmological phenomena, iv) the construction of self-consistent models at various scales and the investigation of the features capable of confirming or ruling out an effective theory of gravity, v) the study of how extended and modified theories of gravity emerge from quantum field theory and how mechanisms produced by the latter may explain cosmological dynamics.

One of the biggest achievements of Physics has been the establishment of the standard cosmological paradigm, which is able to describe the Universe quite successfully through a vast multitude of observations. In this architectural miracle the role of building bricks and joists is played by exquisite observational data, informing us about different epochs, regions and regimes. Evidence ranges from cosmic microwave background fluctuations to supernovae luminosity distances, along with baryon acoustic oscillations, cluster mass measurements and several other probes. However, our concordance model still exhibits open questions. We can mention two of them that have gathered some attention for different reasons.

H_0 **tension** – Firstly, we witness the discrepancy between reputed teams of cosmologists concerning the speed at which astronomical bodies are hurtling away from us, unprivileged observers. In particular, the current value of the Hubble parameter H_0 estimated from CMB data assuming ΛCDM cosmology differs from the direct local distance ladder measurements made by the SH0ES collaboration. This results in the so-called H_0 tension, a concern that has been around for some years now and that does not cease to be a lively source of controversy, quite the opposite [1].

σ_8 **tension** – A second actor in the stage play of cosmological tensions is the apparent discrepancy in the amplitude of the matter power spectrum as set by σ_8, the root-mean-square fluctuations in the matter mass density in a comoving sphere of diameter 8 Mpc. Large compilations of redhift-space distorsions and other dynamical probes show a statistically significant discrepancy with Planck data. If not related to systematics, σ_8 tension is a subject that might lead to changes in the cosmological consensus [2].

We may be tempted to regard these two quandaries as astrophysical rather gravitational, nevertheless gravity is the dominant interaction on the scales of astrophysical interest. Hence, it is clear that these are times when a fluid dialogue between theory and experiment is very much needed. As Albert Einstein said:

> *"A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it."*

Microscopic experiments on gravity – It may seem from the above that the routes to test the gravitational interaction are restricted to macroscopic realms where concentrations of mass and/or energy are very significant. However, even though full characterizations of effects in some modified gravity scenarios are still lacking, our improved understanding implies that microscopic experiments might bring surprises.

The current wide spectrum of enigmas is probably a manifestation of our need to improve and extend the standard body of our knowledge in gravity. Fortunately, even though history indicates that early motivations to proceed beyond General Relativity (GR) were more intellectually motivated by Mathematics, it is Physics itself where such explorations find their roots at present.

1 Theories of Gravity

General Relativity has not ceased its outstanding performance to explain gravitational phenomena with exquisite accuracy in an ever increasing range of scales. Cosmology, on the other hand, has traditionally been a driving force for speculations beyond General Relativity, whose only limitation was the imagination of the theoretical cosmologists. In the last two decades, however, the accumulation of precise cosmological measurements has substantially constrained the permitted theories and we have now the means to robustly rule out wide classes of theories. At the same time, these cosmological observations have triggered investigations seeking for theories beyond General Relativity, mainly motivated by the three fundamental missing ingredients of the standard cosmological model: dark matter, dark energy and inflation.

A very distinctive feature of gravity that actually guided Einstein to its original formulation is its intimate relation with inertia, to the point that it is possible to interpret gravity (at least locally) as a purely inertial effect. This is rooted in the equivalence principle that dictates the universal character of gravity, which in turn lies at the very heart of the possibility to interpret gravity in geometrical terms. We thus arrive at the properties that could be used to *define* gravity from its geometrical side.

From a field theory perspective, General Relativity is a theory that describes the interactions of a massless spin-2 particle. It is profoundly remarkable that by starting with a massless spin-2 particle and imposing some reasonable additional assumptions like Lorentz-invariance, it naturally follows that this particle *must* couple universally (at low energies) to matter fields, and its interactions are precisely those of General Relativity. Thus, from the field theory side, the fundamental defining property of gravity could be identified with its massless spin-2 nature. The structure of General Relativity then results as a particular consequence of the strict rules that govern the interactions of massless particles.

The approaches to modifications of General Relativity come in several fashions, which can be broadly divided into those essentially based on adding new fields, and those that fully embrace its geometrical description and hence the modifications are based on modified geometrical scenarios. Definitely, this separation may be regarded as purely conventional and, as a matter of fact, it is not difficult to go from one to the other in some scenarios. This is clearly illustrated by, for example, gravity theories in a Weyl geometry that can equivalently be regarded as a theory with an extra vector field provided by the Weyl non-metricity trace.

As it occurs many times, however, the starting point or interpretation of the same theory can serve as motivation and inspiration to explore different modified gravity scenarios. It is nevertheless important to keep in mind the basic properties that make General Relativity special among all gravity theories, so that the modifications can be clearly ascribed to the breaking of one of the fundamental assumptions for General Relativity. This is particularly important in helping us to discern truly modified theories from those that are simply General Relativity in disguise.

Modifying General Relativity is an arduous task, not only for its aforementioned exquisite performance to explain observations, but because its internal structure is tightly constrained by consistency conditions that are ultimately imposed by the massless spin-2 nature of the graviton. This delicate structure causes many (infrared) modifications of General Relativity to be doomed from their very conception, and this has fuelled an intense activity in recent years to find theoretically consistent modifications of General Relativity that in turn could play a role in describing the Universe's dark sector or inflation.

General Relativity and Foundations of Gravity – A deep understanding of the foundations of General Relativity is a necessary kick-off for the investigation of modified gravity. GR is formulated using the language of differential geometry. The geometrical setting used by Einstein consists of a four-dimensional Lorentzian manifold, equipped with a metric structure g and a covariant derivative ∇, or equivalently, a connection $\hat{\Gamma}$. This derivative is assumed to be metric compatible and torsion free, which then uniquely determines the connection coefficients to be the Christoffel symbol components Γ.

Let us briefly dissect these assumptions to get an immediate idea of how one could modify GR. To begin with one does not have to restrict the geometry to four dimensions. Kaluza and Klein are credited with suggestions along those lines [3]. The use of four dimensions relies on our experience of three spatial dimensions and a sense of time, which acts as the fourth dimension, commonly denoted as the zeroth coordinate. One could now assume that there exist other spatial dimensions that have not yet been observed. It is probably fair to say that String Theory has followed that path, point particles (points are zero dimensional objects) being replaced by strings (strings or curves are one-dimensional objects). Bosonic string theory is formulated in a 26-dimensional Lorentzian manifold, while superstring theory is formulated in 10 dimensions. These extra dimensions are dealt with by compactification, which means "rolling up" those dimensions in such a way that they are very small, hence, effectively leading to a four-dimensional space in which Special Relativity and General Relativity are formulated.

The next generalisation concerns the connection $\hat{\Gamma}$ which neither has to be metric compatible nor torsion free. Both, non-metricity and torsion have neat geometrical interpretations [4]; one speaks of an affine connection. Torsion represents the failure of this infinitesimal parallelogram to close. In order to understand the effect of non-metricity on the manifold, let us consider a null vector u^μ, which means it satisfies $g_{\mu\nu}u^\mu u^\nu = 0$. If the covariant derivative of the metric tensor does not vanish, then this vector may no longer be null when parallelly transported. In particular, the light cone structure would no longer be invariant under parallel transport. However, neither the lack of closed infinitesimal parallelograms nor the non-invariance of the light cone structure under parallel transport are reason enough to discard these geometrical concepts from a physical point of view. In the end, any theoretical model of the gravitational field will make certain predictions that an experiment can either verify or falsify.

The entire discussion up to now was independent of the Einstein field equations; it merely assumed that there exists a gravitational theory that can be formulated using differential geometry. Let us now start making some connections between the mathematical formulation and the physical content of our theories. It is a well-established everyday fact that light travels along straight lines, and so do massive particles in the absence of external forces. In classical physics one would refer to these as Fermat's principle and Newton's first law, respectively. In the context of differential geometry things start to get interesting now, as a manifold equipped with a metric structure and an affine connection gives rise to two distinct curves geodesics and autoparallels. Geodesics are the shortest possible curves between two fixed end points, autoparallels are the straightest possible curves between two points. Geodesics are generally introduced by studying curves $\mathcal{C}$ with tangent vectors $T^\mu = dX^\mu/d\lambda$ such that the quantity

$$s = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu} T^\mu T^\nu} d\lambda \,, \tag{1}$$

is extremised. Here, $X^\mu(\lambda)$ are the local coordinates of the curve and λ is the (affine) parameter of the curve. This yields the familiar geodesic equations

$$\frac{dT^\mu}{d\lambda} + \Gamma^\mu_{\sigma\tau} T^\sigma T^\tau = 0 \,. \tag{2}$$

It needs to be emphasised that the geodesic equation, defined via this variational approach, depends on the Christoffel symbol components $\Gamma^\mu_{\sigma\tau}$ only. This follows from the fact that (1) is independent of the affine connection - that is, it depends on the metric tensor and the curve.

On the other hand, we can introduce the straightest possible curves or autoparallels. Let us again consider a curve $\mathcal{C}$ with tangent vector T^μ, then the vector V^σ is parallelly transported along this curve if $T^\mu \nabla_\mu V^\sigma = 0$. The notion of parallel transport allows us to consider curves (defined indirectly) whose tangent vectors are parallelly transported along themselves, the tangent vector is kept as parallel as possible along the curve, hence autoparallel. Using the chain rule and the definition of covariant differentiation, the autoparallel equations are given by $T^\mu \nabla_\mu T^\sigma = 0$, i.e. by

$$\frac{dT^\mu}{d\lambda} + \hat{\Gamma}^\mu_{\sigma\tau} T^\sigma T^\tau = 0 \,. \tag{3}$$

The key difference between (2) and (3) is that two different connections appear in these equations, while their form is identical. It is clear that (3) depends on the symmetric part of the connection, since one can exchange T^σ and T^τ; however, it is important to state that

$$\hat{\Gamma}^\mu_{(\sigma\tau)} \neq \Gamma^\mu_{\sigma\tau} \,, \tag{4}$$

which means that the symmetric part of the affine connection is not the Christoffel symbol. This symmetric part contains the Christoffel symbol, but it also depends on torsion and non-metricity, should these be present.

General Relativity is special in the sense that the shortest possible lines coincide with the straightest possible lines.[1] These considerations have practical implications. By studying the geometric properties of trajectories of test particles one can, in principle, determine whether the connection contains contributions other than those from the Christoffel symbol.

In its standard formulation, the dynamical variables of General Relativity are the 10 metric functions $g_{\mu\nu}$, which are the solutions of the ten Einstein field equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \kappa^2 T_{\mu\nu} \,. \tag{5}$$

Here, $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ stands for the metric energy-momentum-stress tensor, while the gravitational coupling constant κ^2 is given by $\kappa^2 = 8\pi G/c^4$. This is a true tensor equation in the sense that it is valid for all coordinate systems and hence diffeomorphism invariant. In four spacetime dimensions one has four coordinates, which can be arbitrarily changed, which implies that the Einstein field equations can be viewed as six independent equations. When a Hamiltonian analysis is performed on these

[1] If the affine connection differs from the Christoffel symbol components by a totally skew-symmetric piece, geodesics and autoparallels would also coincide.

equations, one finds four primary constraints, thereby reducing the number of propagating degrees of freedom of this theory to two (10 metric components minus four coordinate transformations minus four primary constraints) [5].

A more elegant approach, which somewhat lacks physical motivation from first principles, is the variational approach. The field equations can also be derived from the so-called Einstein-Hilbert action

$$S_{\rm EH} = \frac{1}{2\kappa^2}\int g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\,d^4x = \frac{1}{2\kappa^2}\int R\sqrt{-g}\,d^4x\,, \tag{6}$$

$$S_{\rm matter} = \int \mathcal{L}_{\rm matter}(g,\phi,\nabla\psi)\,d^4x = \int L_{\rm matter}(g,\phi,\nabla\psi)\sqrt{-g}\,d^4x\,, \tag{7}$$

$$S_{\rm total} = S_{\rm EH} + S_{\rm matter}\,, \tag{8}$$

where one varies with respect to the dynamical variable $g_{\mu\nu}$. Here, g is the determinant of the metric tensor $g_{\mu\nu}$, so that $\sqrt{-g}\,d^4x$ is the appropriate volume element when integrating over the manifold. The matter fields are denoted by ψ and the matter Lagrangian can depend on derivatives of the matter fields. It is through this variational approach that one can introduce and motivate various gravitational theories, which can be seen as extensions or modifications of the original theory.

Before discussing other gravitational theories, let us briefly mention the issue of matter couplings. This is, of course, of crucial importance as gravity is universal and is the dominant interaction in the macroscopic world. Lagrangians, which describe scalars (spin 0 particles) or spinors (spin 1/2 particles), typically depend on the fields and their first derivatives, thereby giving rise to equations of motion of at most second order. This also holds for Yang-Mills theories; however, we will focus our discussion on scalars and spinors for now. When the scalar or spinor field actions, formulated in Minkowski space, are formulated on an arbitrary manifold, one replaces the Minkowski metric η with an arbitrary metric g. The partial derivatives are replaced with covariant derivatives. In the scalar field case one simply has $\nabla_\mu\phi = \partial_\mu\phi$, while for spinorial fields the covariant derivative also depends on the connection and we have $\nabla_\mu\psi \neq \partial_\mu\psi$, with ψ standing for a spinor field. The immediate consequence of this is that theories in which variations with respect to the connection are considered will contain source terms when spinor fields are taken into account [6]. Since protons, neutrons and electrons are all spin 1/2 particles, this is an important issue to keep in mind. Finally, when considering Yang-Mills theories we recall that the currents which act as the source terms are conserved and couple to the gauge fields. General Relativity can also be formulated as gauge theories; however, it is not in the form of a typical Yang-Mills theory [7]. The above mentioned approach is often referred to as the principle of minimal coupling, however, many other coupling terms are in principle possible. There are Pauli-type terms and Jordan-Brans-Dicke-type terms where geometrical quantities like the Riemann tensor or the Ricci tensor couple to the matter fields.

Linear Extensions – On manifolds where the connection is metric compatible and torsion free, the Einstein-Hilbert action is the unique action that is linear in a curvature scalar, and the Ricci scalar is the unique linear curvature scalar. In more general spaces with torsion and non-metricity, one can also construct the scalar $\varepsilon^{\mu\nu\kappa\lambda}\hat{R}_{\mu\nu\kappa\lambda}$, which does not vanish in general. This term appears in the so-called Palatini action of General Relativity or the Holst action. It becomes important in the context of Loop Quantum Gravity, where it appears in Ashtekar's choice of variables, which allows the formulation of GR as a Yang-Mills type theory [8].

Let us return to the Einstein-Hilbert action (6) for now. The Riemann curvature tensor and the Ricci tensor can be defined using a general affine connection $\hat{\Gamma}$ alone, without requiring the metric tensor. To make this explicit, it is often written as $\hat{R}_{\mu\nu}$ [9]. Hence, in affine spacetimes the Einstein-Hilbert action can be generalised simply by writing

$$S = \frac{1}{2\kappa^2} \int g^{\mu\nu} \hat{R}_{\mu\nu} \sqrt{-g}\, d^4x \,, \tag{9}$$

$$S_{\mathrm{matter}} = \int L_{\mathrm{matter}}(g, \phi, \nabla\psi) \sqrt{-g}\, d^4x \,, \tag{10}$$

$$S_{\mathrm{total}} = S_{\mathrm{EH}} + S_{\mathrm{matter}} \,. \tag{11}$$

One now considers the metric tensor g and the connection $\hat{\Gamma}$ as *a priori* independent dynamical variables. The matter action also depends on the connection through the covariant derivative; this is completely consistent with the principle of minimal coupling used in General Relativity. This principle states that first one writes all equations covariantly in a four-dimensional Lorentzian manifold, flat Minkowski space, then one replaces all partial derivatives with covariant derivatives and all Minkowski metric tensors with arbitrary metric tensors.

If we assume that the matter part of the action does not depend on the connection and we make independent variations with respect to the metric and connection, we arrive at Einstein's theory of General Relativity. This is often referred to as the Palatini variation; however, things become more subtle when geometries are more general.

In many ways the most natural generalisation of General Relativity is constructed when beginning with (11) and allowing the matter part of the action to depend on the matter fields, the metric and an independent connection. When we now compute the variations with respect to the metric and the independent connection, we arrive at two sets of field equations. Variations with respect to the metric yield equations that resemble the Einstein field equations, while variations with respect to the connection give a new set of field equations which determine the connection. The source term that appears in the latter is often referred to as the hyper-momentum $\Delta^{\lambda}{}_{\mu\nu}$, following a commonly used notation [10]. As the affine connection has no symmetries, the hyper-momentum tensor has, in general, 64 independent components in four dimensions.

Let us now discuss how we can connect these different theories back to General Relativity, using a mathematically consistent approach. The perhaps most elegant way to do it is through the introduction of Lagrange multipliers in the total action (11), so that this action is subsequently extremised subject to constraints. These constraints are introduced so that the geometrical properties of the manifold are controlled. More explicitly, let us, for the time being, extract General Relativity within the framework of metric affine theories. Recall that the two key geometrical assumptions are a metric compatible and torsion-free covariant derivative. In the language of constraints we would write

$$S_{\mathrm{GR}} = \frac{1}{2\kappa^2} \int \Big\{ g^{\mu\nu} \hat{R}_{\mu\nu} + \lambda_{(1)}^{\mu\nu\lambda} T_{\mu\nu\lambda} + \lambda_{(2)}^{\mu\nu\lambda} Q_{\mu\nu\lambda} \Big\} \sqrt{-g}\, d^4x \,, \tag{12}$$

$$S_{\mathrm{total}} = S_{\mathrm{GR}} + S_{\mathrm{matter}} \,, \tag{13}$$

where $T_{\mu\nu\lambda}$ is the torsion tensor and $Q_{\mu\nu\lambda}$ is the non-metricity tensor. Here, $\lambda_{(1)}$ and $\lambda_{(2)}$ are two Lagrange multipliers, which ensure that the affine connection will become the usual Christoffel symbol. Clearly, variations with respect to $\lambda_{(1)}$ give $T_{\mu\nu\lambda} = 0$, while variation with respect to $\lambda_{(2)}$ yields $Q_{\mu\nu\lambda} = 0$. The Minkowski space is the unique space that has vanishing torsion, vanishing non-metricity and is globally flat. However, what makes this approach, using constraints, particularly useful is the ability to systematically study a variety of theories in a uniform setting [11].

Nonlinear Extensions – From a theoretical point of view it is well motivated to consider more general theories, which depend on other scalars constructed out of the Riemann curvature tensor or the Ricci tensor. There is no reason to exclude terms like $c_1 R_{\mu\nu} R^{\mu\nu}$, for example, in a gravitational action. Alternatively, one can consider theories where an arbitrary function of the Ricci scalar is considered. The basic idea underlying this approach is to view General Relativity as the lowest order theory. Other models contain nonlinear functions of total derivative terms, like the Gauss-Bonnet term, for example. The Gauss-Bonnet term is related to a topological number, the Euler characteristic of the manifold. However, when any nonlinear function of any topological quantity is added to the action, it will yield some non-trivial field equations. Definitely, one can also introduce new couplings between the geometry and the matter, different from the minimal coupling. Finally, let us mention that a function f contains uncountably many degrees of freedom, thus it is perhaps not too surprising that various models are able to fit a variety of observational data.

Let us go back to the need of celebrating the ability of General Relativity to describe most of the physical behaviours we have access too, and acknowledge at the same time that many puzzles are still standing. Therefore, all modifications and extensions attempting to alleviate them must have a sensible GR limit.

Horndeski/Galileon theories – A successful GR limit is one of the key aspects of one of the most popular ways to modify gravity: the scalar-tensor theories. In this framework the Lagrangian typically becomes dependent on a new quantity, namely the scalar field ϕ. In the pioneering presentation of this seductive idea, Brans and Dicke carefully tailored the total Lagrangian so that the field equations do not display derivatives of order higher than second. This is important towards guaranteeing that the theory remains ghost free and no Ostrogradski instabilities appear. It is possible to engineer the family of all models with a Lagrangian containing second-order derivatives of the field but still leading to second-order equations of motion. The Lagrangian density is given by [12]

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \tag{14}$$

with

$$\begin{aligned}
\mathcal{L}_2 &= K(\phi,\, X) && (15)\\
\mathcal{L}_3 &= -G_3(\phi,\, X)\Box\phi && (16)\\
\mathcal{L}_4 &= G_4(\phi,\, X)R + G_{4,X}(\phi,\, X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right] && (17)\\
\mathcal{L}_5 &= G_5(\phi,\, X)G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}\left[(\Box\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi\right], && (18)
\end{aligned}$$

with $X = -\phi_{;\nu}\,\phi^{;\nu}/2$, and where $K(\phi,\, X)$, $G_3(\phi,\, X)$, $G_4(\phi,\, X)$, $G_5(\phi,\, X)$ are arbitrary functions, while a comma represents covariant derivative. This is the so called Horndeski theory, which was rediscovered some decades after its first appearance in the framework of Galileon models [13], which can be extended to multiple fields [14].

Massive Gravity and Bigravity – Another extension of the Einstein-Hilbert action through the addition of extra degrees of freedom, but of tensor nature this time, are massive gravity and bigravity theories. Although the first attempts towards the direction of adding a mass term for the graviton were performed in the late 1930's, the consistent formulation was possible only recently. The corresponding action is written as [15]

$$S_{\mathrm{mG}} = \int \mathrm{d}^4x \left[\frac{1}{2\kappa^2}\sqrt{-g}R_g - \frac{m^2}{\kappa^2}\sqrt{-g}\sum_{n=0}^{4}\alpha_n e_n\left(\sqrt{g^{-1}f}\right)\right]. \tag{19}$$

One can generalize the action for ghost-free massive gravity into bigravity by including an Einstein-Hilbert kinetic term $\frac{M_f^2}{2}\sqrt{-f}R_f$ for the reference metric [16]. The two metrics, g and f, are only allowed to interact through potential interactions encoded in the elementary symmetric polynomials $e_n(S)$ of the matrix square root $S^\mu_\nu = (\sqrt{g^{-1}f})^\mu_\nu$, which satisfies $S^\mu_\alpha S^\alpha_\nu \equiv g^{\mu\alpha} f_{\alpha\nu}$. These polynomials take the form $e_0(S) = 1$, $e_1(S) = [S]$, $e_2(S) = \frac{1}{2}\left([S]^2 - [S^2]\right)$, $e_3(S) = \frac{1}{6}\left([S]^3 - 3[S][S^2] + 2[S^3]\right)$ and $e_4(S) = \det(S)$. The implications of such a theory could be interesting in the effective dark sectors [17].

In Fig. 1 we present a schematic categorisation of the Tensor-Vector-Scalar class of theories, arising by adding new fields to General Relativity. Nevertheless, unstoppable as curiosity and imagination are, the seeking for generalisations of GR opens new possibilities.

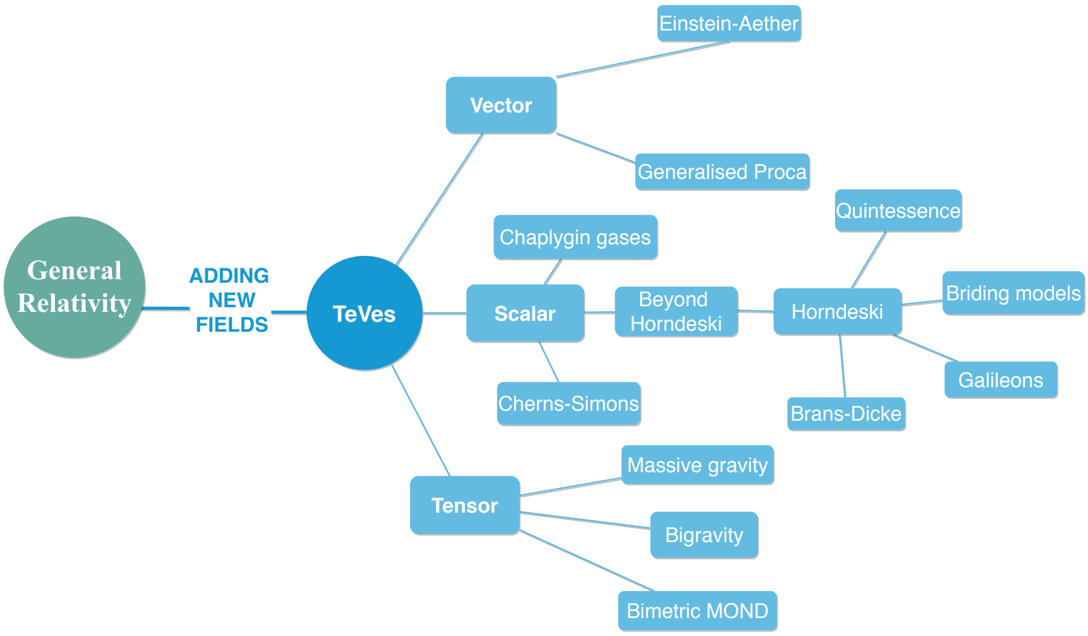

Figure 1: *Schematic categorisation of the Tensor-Vector-Scalar (TeVeS) class of theories, arising by adding new fields to General Relativity.*

$f(R)$ **gravity** – One gravitational modification that has met significant popularity is the $f(R)$ proposal, in which the Ricci scalar R in the original Einstein-Hilbert action is extended. The corresponding action is

$$S = \frac{1}{2\kappa^2} \int f(R)\sqrt{-g}\, d^4x. \tag{20}$$

With their generality, tractability and flexibility, these scenarios offer chances to reproduce a wide assortment of cosmological kinematics, and they are able to provide mechanisms to explain either early or late-time acceleration [18].

In Fig. 2 we present a schematic categorisation of the theories arising by adding higher-order invariants in the Lagrangian of General Relativity.

Extra dimensions – One of the most tempting routes of modification of General Relativity has been the possibility that our spacetime has more than four dimensions, with the extra ones inaccessible to low-energy exploration methods. The necessary compactification procedure typically results in specific modifications of the gravitational Lagrangian [19], which can be found among the many covered in the huge assortment of possibilities [20].

Metric-Affine Gravity – As we mentioned above, there are two main building blocks in a gravitational theory: differential geometry as its mathematical formulation on one hand, and the Lagrangian as the encoder of its physical content on the other. The necessary association among them emerges when we eventually understand how a manifold, furnished with a metric and a connection, dictates how all the particles that live in it move, and conversely how all the

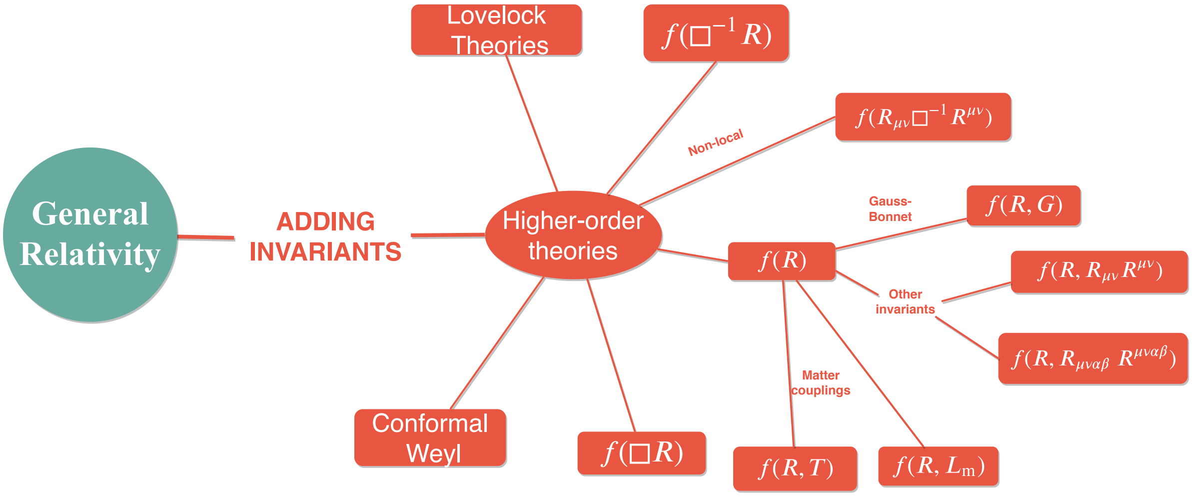

Figure 2: *Schematic categorisation of the higher-order class of theories, arising by adding higher-order invariants in the Lagrangian of General Relativity.*

physical manifestations of those particles affect the metric [21]. This geometric route brings us formally to the geodesics equation, but with the caveat that the involved connection should not necessarily be the Levi-Civita one, which is metric compatible and torsion-free (and therefore symmetric). Hence, the metric-affine structure opens the way to the construction of a very big class of modifications and extensions of gravity [22].

Although modifying gravity through the modification of the Einstein-Hilbert action is both interesting and theoretically and phenomenologically successful, the above metric-affine considerations lead to a novel way to construct gravitational modifications, namely to change the underlying geometrical foundations. In Fig. 3 we present a schematic categorisation of the theories arising by modifying the geometry of General Relativity.

Palatini and Hybrid Metric-Palatini Gravity – Palatini formalism allows to relax the "standard" convention between the metric and the connection, leaving the latter to be specified by first principles. The metric-affine consideration progresses initially from the construction of a curvature scalar using the connection and the metric tensor as independent quantities, but then it can be enlarged considerably by constructing Lagrangians with other scalar terms built from the symmetric part of the Ricci tensor and its contractions with the metric tensor [23]. These settings offer many new attractive possibilities, such as the removal or smoothing out of cosmological singularities which are generic in GR scenarios. A related approach is hybrid-metric Palatini gravity, where the standard R term coming from the metric connection is added to extra terms depending on an alternative curvature scalar, $\mathcal{R}$, derived from an independent connection [24]. Such terms offer a richer phenomenology in the evolution of matter inhomogeneities.

Teleparallel Gravity, Foundations, Modifications and Cosmology – A completely separate route to describe gravity as a manifestation of geometry can be pursued too: teleparallel gravity and its extensions. In these theories gravity is still geometrized, nevertheless geometry is characterized by torsion instead of curvature, which in turn is achieved if one uses the teleparallel, curvature-free connection. In particular, using that a general connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$ can be decomposed as [25]:

$$\hat{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}, \tag{21}$$

where $K^{\alpha}{}_{\mu\nu} = \frac{1}{2}T^{\alpha}{}_{\mu\nu} + T_{(\mu}{}^{\alpha}{}_{\nu)}$ is the contorsion tensor and $L^{\alpha}{}_{\mu\nu} = \frac{1}{2}Q^{\alpha}{}_{\mu\nu} - Q_{(\mu}{}^{\alpha}{}_{\nu)}$ the disformation

[26], we can have an alternative formulation of the Einstein-Hilbert action, namely

$$S_{\mathrm{TEGR}} = \frac{1}{2\kappa^2} \int \mathbb{T} e \, d^4x \,, \tag{22}$$

with $\mathbb{T} = K_{\nu\rho}{}^{\nu} K^{\lambda}{}_{\lambda}{}^{\rho} - K_{\nu\rho}{}^{\nu} K^{\lambda}{}_{\lambda}{}^{\rho}$, where e denotes the determinant of the tetrad field e^A_μ, which satisfies $g_{\mu\nu} = e^A_\mu e^B_\nu \eta_{AB}$. This is the standard formulation of the Teleparallel Equivalent of General Relativity where the tetrad is the independent dynamical variable.

While basic teleparallel gravity is completely equivalent with GR at the level of equations, their modifications correspond to different classes of modified gravity. In particular, considering nonlinear extensions we can obtain $f(\mathbb{T})$ gravity as [27]

$$S_{f(\mathbb{T})} = \frac{1}{2\kappa^2} \int f(\mathbb{T}) e \, d^4x \,, \tag{23}$$

$$S_{\mathrm{matter}} = \int L_{\mathrm{matter}}(g, \phi, \nabla\psi) e \, d^4x \,, \tag{24}$$

$$S_{\mathrm{total}} = S_{f(\mathbb{T})} + S_{\mathrm{matter}} \,. \tag{25}$$

Apart from the interesting cosmological phenomenology [28], it is certainly very appealing that these theories allow to waive the equivalence principle, thus re-framing gravitation as an interaction more similar to the other fundamental ones, as well as being a gauge theory. The torsional framework opens the way to many novel theories of gravity, by using higher-order torsion invariants [29], boundary terms [30], scalar-torsion constructions, teleparallel Horndeski, symmetric teleparallel theories [31], etc.

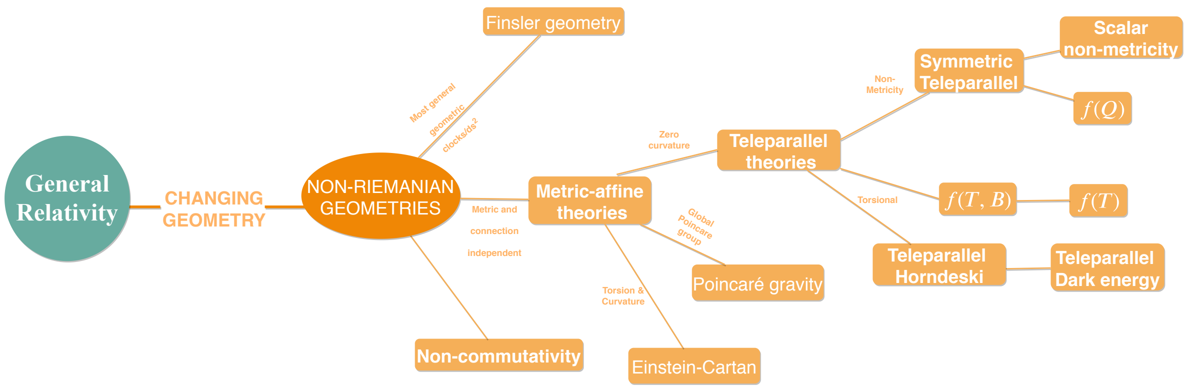

Figure 3: *Schematic categorisation of the theories arising by modifying the geometry of General Relativity.*

Finsler Gravity – Even within the radical procedure of modifying gravity through modifying geometry, there are more radical paths, such as in Finsler geometry and gravity. These geometries extend in a natural way the Riemannian one, by allowing the physical quantities to have a dependence on the observer 4-velocity [32], which in turn reflects the Lorentz-violating character of the kinematics [33]. Furthermore, they may play an important role in quantum gravity considerations and therefore to quantum gravity phenomenology.

As we see, modifying gravity opens the road towards quantum considerations, which was expected since it was known for a long time that modifications of the Einstein-Hilbert action can improve the renormalizability of GR and thus potentially open the way towards a quantum description of gravity. In Fig. 4 we present a schematic categorisation of the theories arising by the use of quantum arguments.

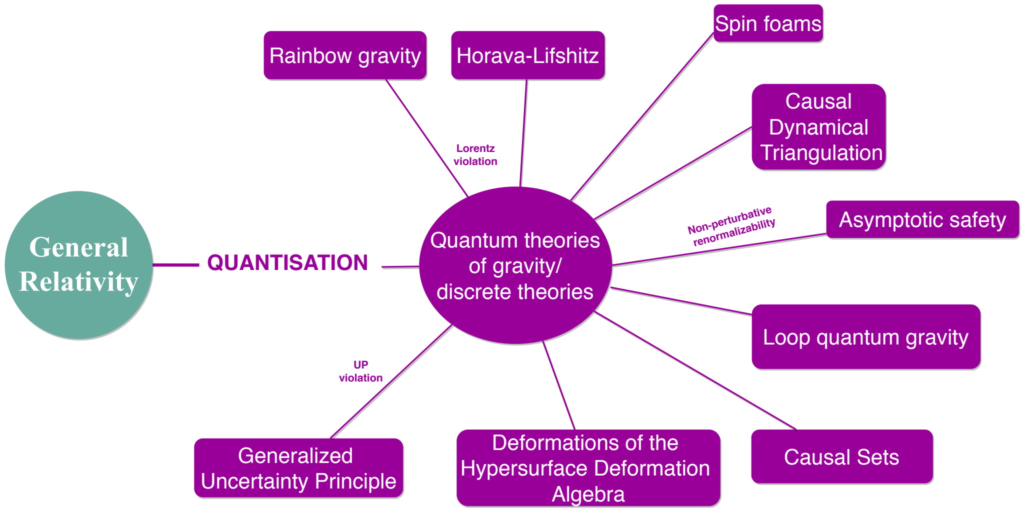

Figure 4: *Schematic categorisation of the theories arising by the use of quantum arguments.*

Non-local Gravity and Quantum Cosmology – Definitely, incorporating quantum considerations leads to a wide spectrum of possibilities, which lead to very interesting cosmological phenomenology, exploring the quantum side of gravity through extensions of Einstein's framework. Examples of such classes are non-local gravity [34], gravity's rainbow [35], classicalizing gravity [36], Eddington-inspired-Born-Infeld gravity [37], as well as other theories based on quantum approaches [38], which prove to lead to interesting cosmological phenomenology [39].

2 Testing Relativistic Effects

The confrontation of gravitational theories with experimental and observational data is a fundamental step in the scientific process. The analysis of relativistic effects is not only necessary but essential for this purpose. Laboratory tests typically search for fifth force effects in the form of short range interactions, which can introduce departures from Newton's law via Yukawa-type corrections mediated by some kind of massive degree of freedom. In some cases this may require going beyond the linearized approximation due to the existence of screening mechanisms that may hide these (chameleon) interactions, which poses severe experimental challenges for current technologies such as atomic interferometry, torsion balance experiments, Casimir force, dipole moment tests, etc. Screening mechanisms can also be constrained via Lunar ranging, by measuring cosmic filaments, and by probing the nonlinear regime of cosmological perturbations.

The effects of modified gravitational dynamics may also arise via nonlinearities induced by the stress-energy densities rather than by new propagating degrees of freedom, thus leading to new phenomena, which do not involve fifth force interactions. This can have nontrivial effects even in scenarios involving elementary particles if one focuses on aspects not related to curvature but to non-metricity and/or torsion. However, from an effective field theory perspective, such new interactions could fit naturally in an extended matter framework; the universality of certain couplings could reveal an underlying geometric structure, thus showing that elementary particle experiments could complement astrophysical tests to unveil modified gravity effects.

The possibility of having new gravitational physics induced by non-linearities in the matter sector may also have an impact on the structural properties of self-gravitating systems. In very low mass stars, where the equation of state of the gas is well understood, modifications in the Newtonian dynamics can change the threshold for sustained hydrogen burning reactions, offering new observables in the search for departures from the predictions of GR. In compact objects such as neutron stars, on the other hand, these interactions could lead to new degeneracies with the matter sector, complicating even more the quest for the properties of the nuclear matter equation of state. In order to break such degeneracies, it is important to identify observables that may lead to universal relations able to tell different gravity theories apart. Some of these relations involve the moment of inertia, asteroseismology, quasi-normal modes, etc, and it has been shown that massive scalar degrees of freedom could manifest themselves clearly in sufficiently separated binary systems and in quasi-normal modes spectra.

Orbital motions and lensing are also key probes for modifications of gravity. The parametrised post-Newtonian formalism developed in the 1970s allows us to confront very different types of theories with observations by just computing certain key coefficients in the appropriate limit and gauge choice. This formalism must be extended in order to accommodate new theories, which do not quite fit within this original framework. Additionally, beyond stellar objects and the slow motion limit, strong gravity effects, such as gravitational waves and strong lensing, typically involving black holes, also offer a glimpse of potentially new gravitational phenomena, including the quantum regime.

CANTATA researchers whose theoretical work makes strong contact with observations, contribute towards testing relativistic effects and finding observational signatures. This allows them to perform masterly forecasts of constraints to be placed on theoretical frameworks by the coming generation of cosmological observations. One should remember in this context the words by Vera Rubin:

"Science progresses best when observations force us to alter our preconceptions."

Laboratory Experiments, Quantum Effects, and Constraints – One important area of this Letter of Interest are tests of gravity at scales well below 1 Mpc, that is, non-cosmological ones, ranging from earth laboratory tests to orbits around compact objects. Within this domain, screening mechanisms in scalar-tensors theories with a scalar field coupled to matter have gathered a lot of interest, since the screening treats differently traditional gravity tests and laboratory tests, in the sense that it allows effects to evade detection by the former while revealing to the latter, as some kind of reward for clever and inspired novel experiments is offered. Several screening mechanisms have been proposed so far, and among the most popular we find those with canonical kinetic terms, such as the chameleon and the symmetron mechanisms [40], as well as the Vainshtein mechanism, which emerges from derivative non-linearities [41]. Thus, one should focus on tests on chameleon scalar fields through different approaches, for example vacuum chambers, atom interferometry, torsion balances, Casimir force searchers and others. Finally, when one allows for non-canonical kinetic terms to enter the picture, he/she can find other mechanisms such as K-mouflage, a screening possibility which has, up to now, not been tightly constrained by gravitational wave observations.

Parametrized Post-Newtonian Formalism – If Earth-based laboratories are at the bottom of the scale ladder of experiments in order to test modifications of gravity, the Solar System is the obvious next: a weak-field limit realm, which is typically treated under the parametrized post-Newtonian (PPN) formalism. PPN is an in-between stratagem to wrap-up conjectures about both observations and theories being considered, which connects the former with the latter through a set of parameters. In fact, a large class of extensions of GR can be accommodated into the customary formulation, however the community does not stopped there, and broader versions are devised too [42]. Stringent bounds have been obtained through refined experiments and can shed light on the viability of modifications of gravity with significant impact on those parameters and the subsequent ones.

Stars as Tests of Modified Gravity – But the prevailing weak-field/strong-field battle in physics explorations demands to also turn our attention to representatives of largest curvature and highest densities regimes: compact stars and black holes. Modeling stellar structure is a demanding task in the default GR setting, and modifications of gravity introduce additional difficulties, both from the mathematical, as well as form and physical perspectives [43]. Hardly any features can be studied from a general formulation, and individual investigations [44] are typically the most efficient way to proceed [45]. New knowledge will, for instance, allow to determine possible changes in the mass-radius relation of neutron stars [46], as it will depend on the new parameters characterizing the gravitational modifications [47]. Additionally, one could even use helioseismology as a precision probe of fifth forces and modified gravity at astrophysical scales [48]. A similar procedure operates on black holes [49], in the sense that the famous no-hair theorem may need a tweak in the presence of additional gravitational degrees of freedom [50].

Compact Objects in General Relativity and Beyond – These are some of the surprises that compact objects (whether extreme or not) may bring. In the case of neutron stars the issue of the equation-of-state dependence is largely eliminated when considering properly scaled quantities [51], which then yield universal relations [52]. Possible deviations from General Gelativity [53] are expected to lead to constraints through observations of, for instance, binary systems [54].

Testing Gravity with Standard Sirens – When we move to (much) larger scales we must publicize that efforts of CANTATA researchers, with their expertise on numerics and computation, have served the international effort towards the design of future accurate tests of gravity (modified or standard) and the improvement of current ones. Some of our colleagues are making distinguished contributions to large teams, expected to lead the design and operation of terrestrial and space-based surveys, which will be operating in the near future. A glimpse of the breathtaking future is offered by observations of gravitational waves, which have brought strong implications for modified theories of gravity [55]. At the cosmological background level, an additional friction that adds to the Hubble one appears, and this modifies the amplitude of the waves, whereas an effective (anomalous) mass and consequent atypical speed alter the phase [56].

Gravitational Waves – A special role is expected to be played by a field which, although old in its theoretical background and in its practical design, has reached full maturity only in the most recent years, eventually accomplishing extraordinary results (the Physics Nobel Prize in 2017): *gravitational wave astronomy*. There are many sophisticated and challenging effects that could arise through the physics of gravitational waves in the case where modified gravity scenarios are considered, as not only additional polarisations emerge (up to four extra ones), but also they could get mixed up and frequency mutations might be produced too [57]. Actually, the detection of these additional polarisation modes represents a significant technical challenge, as well as the possibility of refractive behavior [58]. Into the bargain, the absolutely greatest promise of gravitational waves is the cosmological realm. Events that can be regarded as standard sirens (may be able to) probe the redshift evolution of the luminosity distance of the gravitational wave source, which is proportional to the inverse of the amplitude [59]. In principle, it can be different from the electromagnetic luminosity distance of a companion event, and therefore their ratio will offer a test for parameters associated with physics beyond GR as well as with the dark energy features [60].

Gravitational Lensing – The ambitious and broad next step would then be to propose experiments to provide information on the motion of test particles, which could reveal the specific spacetime geometric properties. In this respect, we should search for deeper insights into the boundless question of how geometry affects the motion of particles and in particular of gravitational lensing. Nevertheless, in order to paint a master work of art and not a mere sketch, us physicists have to associate the equations of motion governing the pertinent trajectories with a physical framework, that is, we need to match particles and fields [61].

3 Cosmology and Observational Discriminators

Along the above lines, the next crucial step is to combine measurements with observations of the large scale structure coming from surveys which will probe the Universe in different scale regimes as compared to the size of the horizon. Cosmology is (among others) our gravity laboratory. Hence, detailed investigation of the cosmological applications of various theories of gravity, and the extraction of suitable observational discriminators, can provide valuable knowledge.

In forthcoming years we expect many terrestrial and space-based advanced surveys to be launched and/or become fully operative (among them, *Euclid* and SKA), throwing us directly into a new highly-upgraded era of *precision cosmology.* All of them will provide us with data of unprecedented precision about the large-scale structure, giving us the possibility for very accurate tests of gravity (and modified gravity theories, specifically) on scales spanning many orders of magnitude. A phenomenological summary of all the possible insights is given below. One of the main outputs of the above mentioned surveys will be, among other things, data related to the clustering of galaxies. Nevertheless, there could be possible subtle effects, which might influence those data, i.e., relativistic effects on the number counts, which emerge as powerful complementary tool, to be used in addition to more standard and well-established ones.

Phenomenological Tests of Gravity on Cosmological Scales – The impacts of deviations from GR on cosmological observables could stat by studying the cosmic microwave background (CMB) and cosmic large-scale structure (LSS) [62]. This phenomenological approach can be implemented in numerical codes computing theoretical predictions for cosmological observables [63]. Current constraints on departures from GR from the Planck, KiDS and DES collaborations are expected to be modified by major upcoming cosmological LSS surveys, that will provide data in the present decade [64].

Effective Field Theory of Dark Energy – Related to the previous discussion, one analytic method based on the perturbation theory framework is the Effective Field Theory (EFT) approach. EFT can capture interesting effects on scales smaller than the intermediate ones, and in particular the onset of the transition from GR to a modified gravity regime, which is crucial for the research objectives of our Action [65]. In particular, it depicts physical effects germane to macroscopic scales by integrating out short-distance features which thus appear on long-distance characteristics as extra/perturbative parameters. This formalism supports any dark energy or modified gravity model possessing one additional scalar degree of freedom. Such a theoretical framework, has become very fruitful in recent years by providing interesting and stringent constraints on both (standard) dark energy models and modified gravity theories, assisted additionally by numerical codes (now widely used in the cosmological community), which have been developed and improved by the members of CANTATA, and which have helped to optimise the calculation of the most important quantities that are needed to apply such framework to real data in particular, cosmological perturbations.

Spatial curvature – We wish to add to this Letter of Interest another bold departure from (cosmological) orthodoxy that has been put forward according to reliable evidence coming from *Planck* data: the spatial curvature of the Universe might be non-zero [66]. Such a possibility could leave imprints on the universe evolution [67]. The debate is alive, and probably it might be closed by a thoughtful identification of degeneracy-breaking datasets, namely the cosmic chronometers, which, perhaps, do not rely heavily on our understanding on galactic evolution.

Relativistic Effects – Among the abundance of outputs of future large-scale structure surveys we must mention the sensitivity of the clustering of galaxies to the (specific) theory of gravity under play. Effects less tangible than density perturbations and redshift-space distorsions will place unprecedented constrains, which are obtained by confronting pertinent gauge-invariant quantities (the two Bardeen potentials being among them) with the information provided by the power spectrum [68] and its multipole expansion [69]. Hence, one is able to explore effects which are neglected in current surveys [70]. Note that the importance of these studies is twofold. On one hand they vindicate the role of galaxies as baryonic (and therefore electromagnetically accessible) concentrations of matter for the study of the Universe, and on the other they offer a fundamental channel to explore the vital role of dark matter [71]. In addition, it is relevant that the interplay between baryons and dark matter, which takes place in regions with high matter density [72], allows to test the equivalence principle too [73].

Galaxy Clusters and Modified Gravity – Having stated the paramount importance of galaxies to understand modified theories of gravity, we cannot forget that the Universe offers us even better laboratories, namely the galaxy clusters [74]. These are the largest clearly-observable self-gravitating structures for which we can retrieve multi-messenger astronomical data covering a wide range of wavelengths and complementary information. Their ambivalence as both astrophysical and cosmological objects can help us to discriminate between gravitational effects and cumbersome astrophysical phenomena, through the prospect of kinematic, thermal and lensing explorations, yet again resorting to perturbative quantities considered in other contributions, such as the gravitational slip [75]. Hence, among others, they can provide valuable constraints and information about modifications of gravity.

Probing Modified Gravity with Non-linear Structure Formation – Different modified gravity theories can be degenerate with regard to both the background cosmology and the growth rate of linear perturbations. Hence, it is crucial to identify new probes that can be used to break these degeneracies. For a flavour on the sort of discriminating criteria clusters can offer significant information when time evolution of scalar fields is addressed [76]. We should study the effects of modified gravity in the nonlinear regime of structure formation, using numerical simulations to study possible characteristic features that could be imprinted in galaxies and clusters of galaxies by modified gravity theories, and which could help to eventually discriminate among General Relativity and alternative gravities. The aim is to predict possible smoking guns of modified gravity and of screening mechanisms at cluster-of-galaxies scales.

Testing the Dark Universe with Cosmic Shear – Clearly, all complementary routes offered by tests of dynamical features, for instance weak lensing, are destined to play a most relevant role [77]. The very-hard-to-spot effects are again encoded in the Bardeen potentials, and forecasts have been carried out for surveys such as *Euclid* in the theoretical context of Horndeski theories and other modifications [78]. Such probes will become crucial in the near future, due to some of the surveys we have introduced above, which will be accurate enough, and will observe such a huge amount of galaxies as to make it feasible at unique levels.

Conclusions

The route to erudition in this limitless field is grievous. One may be led to a multifaceted and more thorough understanding of (her/his favourite flavour of) gravity through questions such as whether the correct weak-field limit is attainable, whether the quantum properties are accessible, whether instabilities occur, and whether the initial-value problem is well posed. Nevertheless, and here comes the crux of the matter, no proficient understanding of a modified gravity framework can be reached without an analysis of the formation (and sustenance) of structures. In this context we highlight again the key question of whether the perturbations of the cosmological background are capable of leaving a blueprint agreeable with the currently observed patterns in the cosmic microwave background and the large-scale structure itself. The intimate connection between theory and observations is therefore an unbreakable bond, and the seed sown by our Collaboration's work will surely thrive and feed our knowledge-hungry community. As a team, our feeling is that:

"This is the way" - The Mandalorian

Acknowledgements

This work is supported by COST (European Cooperation in Science and Technology) and is based upon work from CANTATA COST action CA15117, EU Framework Programme Horizon 2020.

Y.A. is supported by LabEx ENS-ICFP: ANR-10-LABX-0010/ANR-10-IDEX-0001-02 PSL*.

J.B.J. would like to thank David Figueruelo, for his help in compiling and unifying parts of the manuscript.

J.L.B-S, B.K. and J.K. gratefully acknowledge support by the DFG Research Training Group 1620 *Models of Gravity* and the COST Action CA16104. JLBS would like to acknowledge support from the DFG project BL 1553, and the FCT projects PTDC/FISOUT/28407/2017 and PTDC/FIS-AST/3041/2020.

M.B.L.'s work is supported by the Basque Foundation of Science Ikerbasque. She also would like to acknowledge the partial support from the Basque government Grant No. IT956-16 (Spain) and from the project FIS2017-85076-P (MINECO/AEI/FEDER, UE).

G.C. thanks M. Maggiore for useful feedback; his work is supported by the I+D grant FIS2017-86497-C2-2-P of the Spanish Ministry of Science, Innovation and Universities.

J.A.R.C. acknowledges financial support from the MINECO (Spain) Projects No. FIS2016-78859-P (AEI/ FEDER) and No. PID2019-107394GB-I00.

A.d.l.C.-D. acknowledges financial support from NRF Grants No.120390, Reference: BSFP190-416431035, No.120396, Reference: CSRP190405427545, and No 101775, Reference: SFH1507-27131568; Project No. FPA2014-53375-C2-1-P from the Spanish Ministry of Economy and Science, MICINN Project No. PID2019-108655GB-I00, Project No. FIS2016-78859-P from the European Regional Development Fund and Spanish Research Agency (AEI).

A.-C. D. is supported by a PhD contract of the program *FPU 2015* (SpanishMinistry of Economy and Competitiveness) with reference FPU15/05406, and acknowledges further support from the Spanish project No. FIS2017-84440-C2-1-P (MINECO/FEDER,EU), i-LINK1215 (CSIC), and the projects SEJI/2017/042 and PROMETEO/2020/079(Generalitat Valenciana).

J.M.E. is grateful to M. Zumalacárregui, J. Beltrán, L. Heisenberg and J. García-Bellido for fruitful discussions and collaborations on topics covered in this contribution; he has been supported by the Spanish FPU Grant No. FPU14/01618, the Research Project FPA2015-68048-03-3P (MINECO-FEDER) and the Centro de Excelencia Severo Ochoa Program SEV- 2016-0597, by NASA through the NASA Hubble Fellowship grant HST-HF2-51435.001-A awarded by the Space Telescope Science Institute, which is operated by the Association of Universities for Research in Astronomy, Inc., for NASA, under contract NAS5-26555, and by the Kavli Institute for Cosmological Physics through an endowment from the Kavli Foun- dation and its founder Fred Kavli.

N.F. and S.P. thank Louis Perenon for useful discussions; the research of NF is supported by Fundação para a Ciência e a Tecnologia (FCT) through national funds (UID/FIS/04434/2013), by FEDER through COMPETE2020 (POCI-01-0145-FEDER-007672) and by FCT project "DarkRipple – Spacetime ripples in the dark gravitational Universe" with ref. number PTDC/FIS-OUT/29048/2017, and S.P. acknowledges support from the NWO and the Dutch Ministry of Education, Culture and Science (OCW), and also from the D-ITP consortium, a program of the NWO that is funded by the OCW.

L.Á.G. is grateful to Cecília Nagy for generating the figure, to Roberto Casadio for discussions o and to David Hobill for collaboration in the subject over many years; he was supported by the Hungarian National Research Development and Innovation Office (NKFIH) in the form of the Grant No. 123996.

L.K. and L.P. thank Levon Pogosian, Alex Zucca, Savvas Nesseris and George Alestas for their help with the MGCAMB and MGCOSMOMC packages; all the runs were performed in the Hydra Cluster of the Institute of Theoretical Physics (IFT) in Madrid, their research is co-financed by Greece and the European Union (European Social Fund- ESF) through the Operational Programme "Human Resources Development, Education and Lifelong Learning" in the context of the project "Strengthening Human Resources Research Potential via Doctorate Research" (MIS-5000432), implemented by the State Scholarships Foundation (IKY).

J.P.M. and F.S.N.L. acknowledge funding from the Fundação para a Ciência e a Tecnologia (FCT, Portugal) research projects No. UID/FIS/04434/2020, No. PTDC/FIS-OUT/29048/2017 and No. CERN/FIS-PAR/0037/2019; F.S.N.L. additionally acknowledges support from the FCT Scientific Employment Stimulus contract with reference CEECIND/04057/2017.

P.M.M. acknowledges financial support from the project FIS2016-78859- P (AEI/FEDER, UE).

M.M. has received the support of a fellowship from "la Caixa" Foundation (ID 100010434), with fellowship code LCF/BQ/PI19/11690015.

D.F.M. thanks M. Gronke, R. Hagala, A. Hammami, C. Llineares, M. Lima, R. Voivodic, and H. Winther whose collaborations resulted in the articles which this review is based on; his work was partially supported by the Research Council of Norway and the simulations where performed in NOTUR.

G.J.O. is supported by the Spanish project FIS2017-84440-C2-1-P (MINECO/FEDER, EU), the project H2020-MSCA-RISE-2017 Grant FunFiCO-777740, the project PROMETEO/2020/079 (Generalitat Valenciana), and the Edital 006/2018 PRONEX (FAPESQ-PB/CNPQ, Brazil, Grant 0015/2019).

V.P. and A.S. thank Jean-Luc Starck for useful comments.

D.R.G. is funded by the *Atracción de Talento Investigador* programme of the Comunidad de Madrid (Spain) No. 2018-T1/TIC-10431, and acknowledges further support from the Ministerior de Ciencia, Innovación y Universidades (Spain) project No. PID2019-108485GB-I00/AEI/10.13039/501100011033, the FCT projects No. PTDC/FIS-PAR/31938/2017 and PTDC/FIS-OUT/29048/2017, the Spanish project No. FIS2017-84440-C2-1-P (MINECO/FEDER, EU), the project PROMETEO/2020/079 (Generalitat Valenciana), and the Edital 006/2018 PRONEX (FAPESQ-PB/CNPQ, Brazil) Grant No. 0015/2019.

E.N.S is supported in part by the USTC Fellowship for international professors.

I.S. is partially supported by a 2019 "Research and Education" grant from Fondazione CRT. The OAVdA is managed by the Fondazione Clément Fillietroz-ONLUS, which is supported by the Regional Government of the Aosta Valley, the Town Municipality of Nus and the "Unité des Communes valdôtaines Mont-Émilius". I.D.S. is funded by European Structural and Investment Funds and the Czech Ministry of Education, Youth and Sports (Project CoGraDS — CZ.02.1.01/0.0/0.0/$15_0$03/0000437).

A. W. is supported by the EU through the European Regional Development Fund CoE program TK133 "The Dark Side of the Universe".

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