Note on 'Normalisation for Bilateral Classical Logic with some Philosophical Remarks'

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My claim that the reduction step for case (2) of maximal formulas with reductio and non-contradiction 'does only what it is supposed to do: it removes one maximal formula and introduces no complications' (Kürbis, 2021, 548) was rash, as pointed out to me by Pedro del Valle-Inclan.[1] If there are more assumptions in the assumption class $[+A]^i$ than the one displayed in the original deduction (occurring as top formulas in Π' and Π''), then these remain undischarged in the reduced deduction. In this note I give a solution to the problem. But first, two further corrections:

p.549, line 9f: 'the right premise is not conclusion of an elimination rule' should read 'the right premise is not maximal' (that is, it is not conclusion of $+\vee E$, $-\wedge E$ or reductio).
p.550, line 11: 'reductio' should be 'non-contradiction'.

Reduction step (1) for maximal formulas with reductio and non-contradiction also works in case some of the formulas discharged by reductio are premises of non-contradiction, if it is applied with strategy. I repeat it here for convenience:

$$\cfrac{\begin{array}{c}\Sigma\\ +A\end{array} \qquad \cfrac{\begin{array}{c}[+A]^i\\ \Pi\\ \bot\end{array}}{-A}\,i}{\begin{array}{c}\bot\\ \Xi\end{array}} \quad\rightsquigarrow\quad \begin{array}{c}\Sigma\\ [+A]\\ \Pi\\ \bot\\ \Xi\end{array}$$

and also the situation under consideration:

$$\cfrac{\begin{array}{c}\Sigma\\ +A\end{array} \qquad \cfrac{\begin{array}{c}\cfrac{[+A]^i \qquad \begin{array}{c}\Pi'\\ -A\end{array}}{\bot}\\ \Pi''\\ \bot\end{array}}{-A}\,i}{\begin{array}{c}\bot\\ \Xi\end{array}}$$

[1] To whom also many thanks for discussion.

with further formulas in assumption class i left implicit. The case to be avoided is that applying the reduction step introduces maximal formulas of the same degree as the one removed: in the example, this happens if Π' and Σ conclude $-A$ and $+A$ by introduction rules, or if Π' concludes $-A$ by reductio, $+\vee E$ or $-\wedge E$ and Σ concludes $+A$ by one of these rules, too. If both are concluded by introduction rules, remove them as part of the reduction step by applying the appropriate procedure given under case (d) of the reduction steps for maximal formulas; the latter break up formulas into subformulas and thus any resulting new maximal formulas are of lower degree than the one removed. In the other cases, the conclusion $-A$ of Π' is itself maximal, but of one degree lower than the maximal formula $-A$ to be removed. Recall that if both premises of non-contradiction are maximal, the degree of the right premises is the degree of the formula plus 1. Thus applying the reduction procedure increases the degree of the conclusion $-A$ of Π' by one, as afterwards it stands to the right of another maximal formula. The strategy of the proof of Theorem 1 requires applying the reduction steps to maximal formulas of highest degree such that none others of that degree stand above it. So all maximal formulas above the lower $-A$ have at most its degree *qua* formula (i.e. counting only the number of connectives). Thus one way of dealing with this problem is to remove all maximal formulas of that degree that stand above the lower $-A$ before applying the reduction step that removes it. More economical would be to focus only on the troublesome cases and to remove all and only those maximal formulas that are premises of non-contradiction the other premise of which is in assumption class i before applying the reduction step that removes the lower $-A$. A better solution altogether may, however, be to introduce a special measure taking care of maximal formulas that are conclusions of reductio and premises of non-contradiction. The above strategy effectively requires a subsidiary induction to show that, while the lower maximal formula $-A$ is kept fixed, the other maximal formulas of its degree *qua* formula that stand above it are removable from the deduction. The more economical strategy could use a method similar to that employed by Stålmarck in his normalisation proof for unilateral classical logic (Stålmarck, 1991) and associate the maximal formulas that are the conclusion of reductio and premise of non-contradiction with those assumptions discharged by reductio that are premises of non-contradiction and stand next to formulas that are themselves maximal.

References

Kürbis, N. (2021). Normalisation for bilateral classical logic with some philosophical remarks. *The Journal of Applied Logics 8*(2), 531–556.

Stålmarck, G. (1991). Normalisation theorems for full first order classical natural deduction. *Journal of Symbolic Logic 52*(2), 129–149.