Consensus Control for a Class of Linear Multiagent Systems using a Distributed Integral Sliding Mode Strategy

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Abstract

In this paper, a consensus framework is proposed for a class of linear multiagent systems subject to matched and unmatched disturbances in an undirected topology. A linear coordinate transformation is derived so that the consensus protocol design can be conveniently performed. The distributed consensus protocol is developed by using an integral sliding mode strategy. Consensus is achieved asymptotically and all subsystems are globally input-to-state-stable. By using an integral sliding mode control, the subsystems lie on the sliding surface from the initial time, which avoids any sensitivity to disturbances during the reaching phase. By use of an appropriate projection matrix, the size of the equivalent control required to maintain sliding is reduced which reduces the conservatism of the design. MATLAB simulations validate the effectiveness and superiority of the proposed method.

Key Words: Multiagent system, Linear system, Consensus control, Integral sliding mode control, Regular form

I. Introduction

Cooperative control of multiagent systems has received considerable attention in recent years due to its relevance in fields including microgrids, spacecraft formation and industrial cooperative robotics [1]. The behaviour is characterised by cooperation between subsystems via a communication network whereby each subsystem shares information with its neighbours to ensure that all agents reach an agreed goal. Consensus control is a typical and fundamental collective behavior of cooperative control. In a distributed system, consensus control generally focuses on how the agents come to agreement on certain quantities by using their own information together with information received from their neighbours [2]. Consensus control can be widely applied in practice. For instance, in order to increase production, multiple reactors are used to simultaneously perform a chemical reaction where controllers communicate with each other and maintain the temperature, pressure and flow across the reactors in order to maintain consistency of the product.

In process control, external disturbances can seriously affect the behaviour of subsystems. Within a multiagent network this behaviour can spread across the systems because of the interactions between the agents. The presence of such uncertainties can greatly decrease the performance in terms of control accuracy. Robust control is an effective approach to cope with such uncertainty. H_∞ control is a typical robust control strategy which has been widely applied in consensus theory [3][4]. The adaptive control paradigm is also commonly used to deal with disturbances in multiagent systems [5][6]. However, in much of this research, a high control gain is required to suppress disturbances which may be undesirable in practice. In some cases, a disturbance observer can be systematically designed to observe and then compensate for disturbances [7][8]. However, typically well parameterised models are required to define the disturbance observer. Sliding mode control possesses useful characteristics such as total invariance to matched disturbances, straightforward implementation and fast global convergence [9][10]. There are several contributions which consider distributed control using sliding mode approaches. Consensus is achieved using a decoupled distributed sliding mode control for second-order multiagent systems in [2]. Leader-following containment control is investigated for linear systems in [11]. Scaled consensus is studied for linear systems by means of an H_∞ sliding mode control in [12]. It should be noted that during the reaching phase in classical sliding mode control, the system behaviour is still affected by matched disturbances [13]. Integral sliding mode control serves as a solution to this problem as it eliminates the reaching phase. Finite-time consensus is achieved for second-order multiagent systems with disturbances using an integral sliding mode approach in [14]. Fixed-time consensus tracking is studied for second-order nonlinear systems in [15]. The consensus protocols in [14] and [15] are not applicable for more general classes of linear system. A nearly optimal integral sliding-mode consensus protocol is designed for multiagent systems in the presence of matched disturbances in [16]. Note that the unmatched disturbances have not been considered in this work. Consequently, it is valuable to develop a method to cope with matched and unmatched disturbances for linear multiagent systems.

Much of the existing research in distributed control considers consensus for multiagent systems, but does not consider the stability of the subsystems. For example, in [2] [14] [17], second order systems are usually considered as position-velocity systems, in which position increases over time, i.e., the subsystems are unstable after achieving consensus. Theoretically, this is due to the existence of zero eigenvalues in the system matrix, which causes the system to be critically stable, and when subject to disturbances, the states will diverge. However, in physics, the second order system can also act as a mathematical model of a sensor system [18] or a motor system [19]. In these application scenarios, divergence of the states to infinity over time is undesirable. For other known research, though the states reach the equilibrium point ultimately, there is no direct proof of the stability of the subsystems. In [1] [5] and [6], a robust adaptive strategy is utilized to achieve consensus, but it is difficult to synthesize this method to demonstrate stability of the subsystems. As a consequence, it is challenging to develop a consensus protocol which will stabilize the subsystems and where proof of stability can be demonstrated constructively.

Motivated by the above discussion, in this paper a consensus framework is proposed for linear multiagent systems which are subjected to disturbances and uncertainties by using an integral sliding mode strategy. Firstly, the distributed linear system is transformed into a novel regular form by a linear coordinate transformation, which facilitates designing the distributed consensus protocol. In comparison with the traditional regular form [20], the novel regular form inherits the property that matched and unmatched disturbances can be separated. Further the transformed representation facilitates analysis of the consensus error. Secondly, despite the presence of external disturbances, an integral sliding mode strategy is employed so that the states start on the sliding surface. Thirdly, in light of the novel regular form and integral sliding mode strategy, a consensus control protocol is proposed for a distributed linear system, which renders all the subsystems globally input-to-state-stable (ISS). The proposed protocol is fully distributed without requiring global information when compared to [4][21] and [22]. In this paper, the main contributions are twofold. On the one hand, an integral sliding mode based consensus protocol is proposed so that matched disturbances are eliminated while the unmatched disturbances are minimized by the projection theorem. On the other hand, in light of the consensus control framework, consensus for the multiagent system can be achieved asymptotically, while the subsystems are rendered globally ISS.

The rest of this paper is organized as follows. In Section II, some basic concepts are stated, a linear coordinate transformation is given and the problem to be solved is formulated. In Section III, the integral sliding mode control is designed and sliding motion stability is analyzed. In Section IV, consensus and subsystems' stability are analyzed. In Section V, simulation results are analysed and finally in Section VI, conclusions are drawn.

II. Preliminaries and problem formulation

Graph theory is used to illustrate the communication among subsystems [23]. Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ denote an N order undirected graph consisting of a set of nodes $\mathcal{V}=\{v_1,v_2,\ldots,v_N\}$, a set of undirected edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$, and a weighted adjacency matrix $\mathcal{A}=(a_{ij})_{N\times N}$. An undirected edge $\mathcal{E}_{ij}$ in the undirected graph $\mathcal{G}$ is denoted by a pair of unordered nodes (v_i,v_j), which indicates v_i and v_j are neighbours and can communicate with each other. The set of neighbours for node v_i is denoted by $\mathcal{N}_{v_i}=\{v_j\in\mathcal{V}:(v_i,v_j)\in\mathcal{E},i\neq j\}$. The weights $a_{ij}=a_{ji}=1$ in the weighted adjacency matrix $\mathcal{A}$ if and only if the edge (v_i,v_j) exists, and $a_{ij}=a_{ji}=0$ otherwise. Define $a_{ij}=0$ when $i=j$. A path is a sequence of connected edges in a graph, and a graph is connected if there is a path between every pair of vertices.

$0_{n\times m}$ denotes an $n-$row and $m-$column matrix with all the entries being 0. 0_n denotes an $n-$row vector with all the entries being 0. I_m denotes an $m\times m$ identity matrix. Let $\|M\|_F=\sqrt{\sum_{i=1}^{p}\sum_{j=1}^{q}|m_{ij}|^2}$ be the Frobenius norm of $M=(m_{ij})_{p\times q}$. $\|\varpi\|_\infty=\max_{1\leq i\leq n}|\varpi_i|$ denotes an infinite norm of $\varpi\in R^n$. $\|.\|$ denotes the Euclidean norm and is consistently assumed in this paper unless additionally stated. $\lambda_i(P)$ denotes an eigenvalue of $P \in R^{n\times n}$, where $i = 1, 2, \ldots, n$, $\lambda_{\max}(P)$ denotes the maximum eigenvalue of P.

Consider a distributed multiagent system with N subsystems where the communication among subsystems is denoted by an undirected topology graph $\mathcal{G}$. Each subsystem has the following identical nominal linear dynamics which is subject to external disturbances

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + \phi_i(t, x), i = 1, 2, \ldots, N \tag{1}$$

where $x_i(t) \in R^n$, $u_i(t) \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times m}$ are the state, control protocol, system matrix and input matrix of the ith subsystem respectively. The disturbances and uncertainties are lumped together and denoted as $\phi_i(t, x) \in R^n$, and $x \triangleq \left[x_1^T, \ldots, x_N^T\right]^T \in R^{Nn}$.

The following assumptions will be imposed on system (1).

Assumption 1: The pair (A, B) is controllable.

Assumption 2: B has full column rank, i.e., $rank(B) = m$.

Assumption 3: $\phi_i(t, x) \in R^n$ is unknown but bounded, i.e., $\|\phi_i(t, x)\| \le \beta$, where $\beta \in R$ is known.

Assumption 4: The undirected graph $\mathcal{G}$ is connected.

Under Assumption 2, it follows from Lemma 5.3 in [20] that there exists a linear coordinate transformation $\begin{bmatrix}\tilde{z}_{i1}^T & \tilde{z}_{i2}^T\end{bmatrix}^T = T_1 x$ such that (1) can be described as

$$\begin{aligned}\dot{\tilde{z}}_{i1}(t) &= \tilde{A}_{11}\tilde{z}_{i1}(t) + \tilde{A}_{12}\tilde{z}_{i2}(t) + \tilde{\phi}_{i1}(t, \tilde{z}_1, \tilde{z}_2)\\ \dot{\tilde{z}}_{i2}(t) &= \tilde{A}_{21}\tilde{z}_{i1}(t) + \tilde{A}_{22}\tilde{z}_{i2}(t) + B_2 u_i(t) + \tilde{\phi}_{i2}(t, \tilde{z}_1, \tilde{z}_2)\end{aligned} \tag{2}$$

where T_1 is an invertible matrix, $\tilde{z}_{i1}(t) \in R^{n-m}$, $\tilde{z}_{i2}(t) \in R^m$, $\tilde{A}_{11} \in R^{(n-m)\times(n-m)}$, $\tilde{A}_{22} \in R^{m\times m}$, $rank(B_2) = m$, $\tilde{\phi}_{i1}(t, \tilde{z}_1, \tilde{z}_2) \in R^{n-m}$ and $\tilde{\phi}_{i2}(t, \tilde{z}_1, \tilde{z}_2) \in R^m$ are unmatched and matched disturbances respectively, $\tilde{z}_1 \triangleq \left[\tilde{z}_{11}^T, \ldots, \tilde{z}_{N1}^T\right]^T \in R^{N(n-m)}$ and $\tilde{z}_2 \triangleq \left[\tilde{z}_{12}^T, \ldots, \tilde{z}_{N2}^T\right]^T \in R^{Nm}$.

Perform a coordinate transformation $\begin{bmatrix}z_{i1}^T & z_{i2}^T\end{bmatrix}^T = T_2\begin{bmatrix}\tilde{z}_{i1}^T & \tilde{z}_{i2}^T\end{bmatrix}^T = \begin{bmatrix}K_1 & 0_{(n-m)\times m}\\ K_2 & I_m\end{bmatrix}\begin{bmatrix}\tilde{z}_{i1}^T & \tilde{z}_{i2}^T\end{bmatrix}^T$ such that A_{11} in (3) is negative symmetric definite

$$\begin{aligned}\dot{z}_{i1}(t) &= A_{11}z_{i1}(t) + A_{12}z_{i2}(t) + \phi_{i1}(t, z_1, z_2)\\ \dot{z}_{i2}(t) &= A_{21}z_{i1}(t) + A_{22}z_{i2}(t) + B_2 u_i(t) + \phi_{i2}(t, z_1, z_2)\end{aligned} \tag{3}$$

where T_2 is an invertible matrix, $z_{i1}(t) \in R^{n-m}$, $z_{i2}(t) \in R^m$, $A_{11} = K_1\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right)K_1^{-1} \in R^{(n-m)\times(n-m)}$, $A_{22} \in R^{m\times m}$, $\phi_{i1}(t, z_1, z_2) = K_1\tilde{\phi}_{i1}(t, \tilde{z}_1, \tilde{z}_2)$, $\phi_{i2}(t, z_1, z_2) = K_2\tilde{\phi}_{i1}(t, \tilde{z}_1, \tilde{z}_2) + \tilde{\phi}_{i2}(t, \tilde{z}_1, \tilde{z}_2)$ are unmatched and matched disturbances respectively, $z_1 \triangleq \left[z_{11}^T, \ldots, z_{N1}^T\right]^T \in R^{N(n-m)}$ and $z_2 \triangleq \left[z_{12}^T, \ldots, z_{N2}^T\right]^T \in R^{Nm}$.

The steps required to render A_{11} negative symmetric definite are presented as follows:

(a) Apply pole assignment to $\tilde{A}_{11} - \tilde{A}_{12}K_2$. Under Assumption 1, the pair $\left(\tilde{A}_{11}, \tilde{A}_{12}\right)$ is controllable according to Proposition 3.3 in [20], so there exists $K_2 \in R^{m\times(n-m)}$ such that $\tilde{A}_{11} - \tilde{A}_{12}K_2$ has $n-m$ distinct negative eigenvalues $\lambda_i\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right)$, $i = 1, \ldots, n-m$. In this case, $\tilde{A}_{11} - \tilde{A}_{12}K_2$ is Hurwitz stable and $rank\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right) = n - m$.

(b) Since $\tilde{A}_{11} - \tilde{A}_{12}K_2$ has $n-m$ distinct negative eigenvalues, it follows from Theorem 1.3.9 in [24] that $\tilde{A}_{11} - \tilde{A}_{12}K_2$ can be transformed into the corresponding diagonal matrix $\Lambda \triangleq diag(\lambda_1, \ldots, \lambda_{n-m})$ by using the nonsingular matrix $K_1 \in R^{(n-m)\times(n-m)}$, i.e., $A_{11} = K_1\left(\tilde{A}_{11} - \tilde{A}_{12}K_2\right)K_1^{-1}$, then $A_{11} = \Lambda$.

Remark 1: A_{11} being negative symmetric definite in (3) is helpful for consensus protocol design and synthesis. This will play an important role in achieving consensus and ensuring the subsystems' stability.

From the results in [25], $\phi_i(t, x)$, $\phi_{i1}(t, z_1, z_2)$ and $\phi_{i2}(t, z_1, z_2)$ may be expressed in the following form:

$$\left[0_{n-m}^T, \phi_{i2}^T(t, z_1, z_2)\right]^T = T_2 T_1 B B^+ \phi_i(t, x) \tag{4}$$

$$\left[\phi_{i1}^T(t, z_1, z_2), 0_m^T\right]^T = T_2 T_1 B^\perp B^{\perp +} \phi_i(t, x) \tag{5}$$

where $B^{+} \triangleq \left(B^{T}B\right)^{-1}B^{T} \in R^{m\times n}$ is the left inverse of B, and the columns of $B^{\perp} \in R^{n\times(n-m)}$ span the null space of B^{T}, i.e., $B^{T}B^{\perp} = 0_{m\times(n-m)}$. Moreover, the following identity holds

$$BB^{+} + B^{\perp}B^{\perp +} = I_n \tag{6}$$

Definition 1: Consensus is said to be achieved for the distributed multiagent system (1) if for any initial conditions, $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, $\forall i, j = 1, 2, \cdots, N$.

This paper concentrates on utilizing local information to develop a control protocol such that consensus can be achieved when each subsystem (3) is affected by bounded external disturbances. In this case, the consensus problem for (1) can also be solved correspondingly.

Before presenting the main results, some lemmas and definitions are given as follows.

Lemma 1: [26] (Global Invariant Set Theorem) Consider the autonomous system $\dot{x} = f(x)$ with f continuous, and let $V(x)$ be a scalar function with continuous first partial derivatives. Assume that $V(x) \to \infty$ as $\|x\| \to \infty$, and $\dot{V}(x) \le 0$ over the whole state space. Let $\mathcal{R}$ be the set of all points where $\dot{V}(x) = 0$, and $\mathcal{M}$ be the largest invariant set in $\mathcal{R}$. Then all solutions globally asymptotically converge to $\mathcal{M}$ as $t \to \infty$.

Definition 2: [27] Consider the system

$$\dot{x} = f(x, u) \tag{7}$$

Assume that $\dot{x} = f(x, 0)$ has a uniformly asymptotically stable equilibrium point at the origin. The system (7) is said to be globally ISS if there exist a $\mathcal{KL}$ function η, a class $\mathcal{K}$ function ϑ such that

$$\|x\| \le \eta(\|x_0\|, t) + \vartheta(\|u\|_\infty), \forall t \ge 0 \tag{8}$$

for any initial state $x_0 \in R^n$ and any bounded input $u \in R^m$.

Definition 3: [27] A continuously differentiable function $V : R^n \to R$ is said to be an ISS global Lyapunov function on R^n for the system (7) if there exist class $\mathcal{K}_\infty$ functions $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\mathcal{X}$ such that:

$$\varepsilon_1(\|x\|) \le V(x(t)) \le \varepsilon_2(\|x\|), \forall x \in R^n, t > 0 \tag{9}$$

$$\frac{\partial V(x)}{\partial x} f(x, u) \le -\varepsilon_3(\|x\|), \forall u \in R^m : \|x\| \ge \mathcal{X}(\|x\|) \tag{10}$$

Lemma 2: [27] (Globally ISS Theorem) Consider the system (7) and let $V : R^n \to R$ be an ISS global Lyapunov function for this system. Then (7) is globally ISS according to Definition 2 with

$$\vartheta = \varepsilon_1^{-1} \cdot \varepsilon_2 \cdot \chi \tag{11}$$

Remark 2: According to Definition 2, the response of $\dot{x} = f(x, 0)$ with initial state x_0 satisfies

$$\|x\| \le \eta(\|x_0\|, t), \forall t \ge 0 \tag{12}$$

As t increases, $\eta(\|x_0\|, t) \to 0$, then

$$\|x\| \le \vartheta(\|u\|_\infty) \tag{13}$$

Lemma 3: [28] If $\mu_1, \mu_2, \ldots, \mu_n \ge 0$ and $0 < p < q$, then

$$\left(\sum_{i=1}^{n} \mu_i^q\right)^{1/q} \le \left(\sum_{i=1}^{n} \mu_i^p\right)^{1/p} \tag{14}$$

III. Integral sliding mode control protocol design and stability analysis of the sliding motion

This section aims to design an integral sliding mode control protocol and analyze the stability of the sliding motion for the multiagent system (3). To simplify notation, some of the function arguments will be omitted.

The sliding function is presented as follows

$$
\begin{aligned}
s_i(t) = \alpha_i G \Bigg(& \begin{bmatrix} z_{i1}^T(t) & z_{i2}^T(t) \end{bmatrix}^T - \begin{bmatrix} z_{i1}^T(t_0) & z_{i2}^T(t_0) \end{bmatrix}^T \\
& - \int_{t_0}^{t} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_{i1}^T(\tau) & z_{i2}^T(\tau) \end{bmatrix}^T + \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} u_i^{con}(\tau)\, d\tau \Bigg)
\end{aligned} \tag{15}
$$

where $s_i(t)$ is a sliding-mode variable, $G \in R^{m\times n}$ is a projection matrix that will be designed later according to the projection theorem and satisfies $rank\left(G \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} \right) = m$, $\alpha_i \in R$ is a small positive parameter which can be chosen by the designer, $z_{i1}(t_0)$ and $z_{i2}(t_0)$ are the initial values and $u_i^{con}(t)$ is a consensus control protocol that is defined by

$$
u_i^{con}(t) = B_2^{-1} \left(\sum_{j=1}^{N} a_{ij}\left(z_{j2}(t) - z_{i2}(t)\right) + A_{12}^T \sum_{j=1}^{N} a_{ij}\left(z_{j1}(t) - z_{i1}(t)\right) \right) - A_{21} z_{i1}(t) - A_{22} z_{i2}(t) \tag{16}
$$

The corresponding sliding surface is

$$
\left\{ \left(z_{11}^T, \cdots, z_{N1}^T, z_{12}^T, \cdots, z_{N2}^T\right)^T \middle| \, s_i(t) = 0_m, \forall i = 1, 2, \cdots, N \right\} \tag{17}
$$

where $s_i(t)$ is defined in (15).

The control protocol for the multiagent system (3) is given by

$$
u_i(t) = u_i^{dis}(t) + u_i^{con}(t) \tag{18}
$$

where $u_i^{dis}(t)$ is a discontinuous control protocol and selected as

$$
u_i^{dis}(t) = -\rho \frac{\left(G \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} \right)^T s_i(t)}{\left\| \left(G \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} \right)^T s_i(t) \right\|} \tag{19}
$$

where $\rho > \beta \|B^+\|_F$ is a control gain.

Next, the behaviour when each subsystem is subjected to disturbance effects will be analyzed when the system is controlled by the discontinuous control protocol (19). Closing the loop in (3) with (18), the derivative of $s_i(t)$ with respect to time is given by

$$
\begin{aligned}
\dot{s}_i(t) &= \alpha_i G \left(\begin{bmatrix} \dot{z}_{i1}^T & \dot{z}_{i2}^T \end{bmatrix}^T - \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_{i1}^T & z_{i2}^T \end{bmatrix}^T + \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} u_i^{con} \right) \right) \\
&= \alpha_i G \Bigg(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_{i1}^T & z_{i2}^T \end{bmatrix}^T + \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} \left(u_i^{dis} + u_i^{con}\right) + \begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T \\
&\quad - \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_{i1}^T & z_{i2}^T \end{bmatrix}^T + \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} u_i^{con} \right) \Bigg) \\
&= \alpha_i G \left(\begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} u_i^{dis} + \begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T \right)
\end{aligned} \tag{20}
$$

The equivalent discontinuous control u_{ieq}^{dis} is obtained from this as

$$
u_{ieq}^{dis}(t) = -\left(G \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} \right)^{-1} G \begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T \tag{21}
$$

By substituting (21) as $u_i^{dis}(t)$ in (3), the sliding dynamics can be obtained as

$$\begin{aligned}\dot{z}_{i1}(t) &= A_{11}z_{i1}(t) + A_{12}z_{i2}(t) + \phi_{i1}(t, z_1, z_2)\\ \dot{z}_{i2}(t) &= A_{21}z_{i1}(t) + A_{22}z_{i2}(t) + B_2u_i^{con}(t) - B_2\left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^{-1} G\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T\end{aligned} \tag{22}$$

As can be seen, the action of the integral sliding mode control strategy has transformed the original disturbances $\begin{bmatrix}\phi_{i1}^T & \phi_{i2}^T\end{bmatrix}^T$ into the following equivalent disturbances

$$\phi_{ieq}(t, z_1, z_2) \triangleq \begin{bmatrix}\phi_{i1}\\ -B_2\left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^{-1} G\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T\end{bmatrix} = \left(I_n - \begin{bmatrix}0_{(n-m)\times n}\\ B_2\left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^{-1} G\end{bmatrix}\right)\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T \tag{23}$$

Theorem 1: Since $G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}$ has full rank, $B^+(T_2T_1)^{-1}$ is a matrix which minimizes the norm of $\phi_{ieq}(t, z_1, z_2)$, i.e.,

$$G^* = B^+(T_2T_1)^{-1} = \arg\min_{G\in R^{m\times n}} \left\|\left(I_n - \begin{bmatrix}0_{(n-m)\times n}\\ B_2\left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^{-1} G\end{bmatrix}\right)\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T\right\| \tag{24}$$

***Proof*:** Notice that

$$\left\|\left(I_n - \begin{bmatrix}0_{(n-m)\times n}\\ B_2\left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^{-1} G\end{bmatrix}\right)\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T\right\| = \left\|\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T - \begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\varphi_i\right\| \tag{25}$$

where $\varphi_i = \left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^{-1} G\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T$. Thus (24) can be transformed into

$$\varphi_i^* = \arg\min_{\varphi_i\in R^m} \left\|\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T - \begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\varphi_i\right\| \tag{26}$$

which has $\varphi_i^* = B^+(T_2T_1)^{-1}\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T$ as a solution according to the classical projection theorem in page 51 of [29]. Making $G = B^+(T_2T_1)^{-1}$, it can be obtained that $\varphi_i = B^+(T_2T_1)^{-1}\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T = \varphi_i^*$, which implies that (24) is true.

Remark 3: By substituting $\varphi_i^* = B^+(T_2T_1)^{-1}\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T$ into (25) and combining (6), it follows that $\|\phi_{ieq}^*\| = \left\|\begin{bmatrix}\phi_{i1}^T & 0_m^T\end{bmatrix}^T\right\|$, i.e., the norm of the equivalent disturbances is driven by the unmatched disturbances and the effects of the disturbances are minimized by designing the projection matrix G optimally.

Theorem 2: Assume Assumptions 1-3 hold. Then the control from (19) can keep the subsystem (3) on the sliding surface (17) from the initial time with $G = B^+(T_2T_1)^{-1}$.

***Proof*:** Substitute the discontinuous element from (19) with $G = B^+(T_2T_1)^{-1}$ into (20). Then

$$\dot{s}_i(t) = \alpha_i B^+(T_2T_1)^{-1}\left(-\rho T_2T_1B\frac{s_i}{\|s_i\|} + \begin{bmatrix}\phi_{i1}^T & \phi_{i2}^T\end{bmatrix}^T\right) \tag{27}$$

A Lyapunov candidate function is selected as

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N} s_i^T s_i \tag{28}$$

Combining with (4), (5) and (27), the derivative of $V_1(t)$ is given by

$$\begin{aligned}\dot{V}_1(t) &= \sum_{i=1}^{N} s_i^T \dot{s}_i \\ &= \sum_{i=1}^{N} s_i^T \alpha_i B^+ (T_2 T_1)^{-1} \left(-\rho T_2 T_1 B \frac{s_i}{\|s_i\|} + \begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T \right) \\ &= \sum_{i=1}^{N} \alpha_i \left(-\rho \|s_i\| + s_i^T B^+ \phi_i \right) \\ &\leq \sum_{i=1}^{N} \alpha_i \left(-\rho \|s_i\| + \|s_i\| \left\| B^+ \phi_i \right\| \right) \\ &= \sum_{i=1}^{N} -\alpha_i \|s_i\| \left(\rho - \left\| B^+ \phi_i \right\| \right) \\ &\leq \sum_{i=1}^{N} -\alpha_i \|s_i\| \left(\rho - \left\| B^+ \right\|_F \|\phi_i\| \right) \\ &\leq \sum_{i=1}^{N} -\alpha_i \|s_i\| \left(\rho - \beta \left\| B^+ \right\|_F \right)\end{aligned} \tag{29}$$

According to Lemma 3, it follows that

$$\begin{aligned}\dot{V}_1(t) &\leq \sum_{i=1}^{N} -\sigma_i \|s_i\| \\ &\leq -\sigma_{\min} \sum_{i=1}^{N} \|s_i\| \\ &\leq -\sigma_{\min} \sqrt{V_1}\end{aligned} \tag{30}$$

where $\sigma_i = \alpha_i \left(\rho - \beta \|B^+\|_F \right)$, $\sigma_{\min}$ is the minimum among the σ_i.

It follows that the subsystem (3) will slide on the surface (17) despite the presence of the disturbances [20]. Because the subsystem starts on the sliding surface at the initial time, it will remain on the sliding surface thereafter, i.e., $s = \dot{s} = 0$ when $t \geq 0$.

IV. Consensus and Stability Analysis of Subsystems

In this section, consensus will be analyzed for the distributed system in the presence of the control protocol. The stability of each subsystem is then considered.

When the subsystem is restricted on the sliding surface (17), substitute $G = B^+ (T_2 T_1)^{-1}$ and the consensus control protocol (16) into (22). The sliding dynamics can then be described as

$$\begin{aligned}\dot{z}_{i1}(t) &= A_{11} z_{i1}(t) + A_{12} z_{i2}(t) + \phi_{i1}(t, z_1, z_2) \\ \dot{z}_{i2}(t) &= \zeta_i(t, z_1, z_2)\end{aligned} \tag{31}$$

where $\zeta_i(t, z_1, z_2) = \sum_{j=1}^{N} a_{ij} \left(z_{j2}(t) - z_{i2}(t) \right) + A_{12}^T \sum_{j=1}^{N} a_{ij} \left(z_{j1}(t) - z_{i1}(t) \right)$.

Assumption 5: [9] $\|\phi_{i1}(t, z_1, z_2)\| \leq \gamma_i(t, z_1, z_2) \|z_{i1}\|$, where $\gamma_{\max}(t, z_{i1}, z_{j1}) \leq -\lambda_{\max}(A_{11})$, $\gamma_{\max}$ is the maximum among the γ_i, $i = 1, \ldots, N$.

Assumption 6: The closed-loop system (31) does not involve the case that $z_{i2} \to \infty$ when $\zeta_i \to 0_m$.

Remark 4: According to Assumption 6, both the disturbances $\begin{bmatrix} \phi_{i1}^T & \phi_{i2}^T \end{bmatrix}^T$ and control inputs (18) do not drive $z_{i2} \to \infty$ in the system (3), i.e., z_{i2} does not correspond to the unbounded cases from [26] (page 122). Recalling that the disturbances are bounded, the consensus control protocol (16) is linear, and the discontinuous control protocol (19) is bounded, then z_{i2} will be bounded when $\zeta_i \to 0_m$.

Theorem 3: Suppose Assumptions 1-5 hold. The distributed system (31) can achieve consensus asymptotically.

***Proof*:** The consensus problem can be transformed into the following stabilisation problem:

$$\begin{aligned} \dot{e}_i^a(t) &= A_{11} e_i^a(t) + A_{12} e_i^b(t) + e_i^{\phi_1}(t, z_1, z_2) \\ \dot{e}_i^b(t) &= \zeta_i(t, z_1, z_2) - \bar{\zeta}(t, z_1, z_2) \end{aligned} \tag{32}$$

where $e_i^a(t) \triangleq \left(e_{i1}^a, \ldots, e_{i,n-m}^a\right)^T = z_{i1} - \frac{1}{N}\sum_{j=1}^{N} z_{j1}$, $e_i^b(t) \triangleq \left(e_{i1}^b, \ldots, e_{i,m}^b\right)^T = z_{i2} - \frac{1}{N}\sum_{j=1}^{N} z_{j2}$, $e_i^{\phi_1}(t, z_1, z_2) \triangleq \phi_{i1} - \frac{1}{N}\sum_{j=1}^{N}\phi_{j1}$, $\bar{\zeta}(t, z_1, z_2) \triangleq \frac{1}{N}\sum_{j=1}^{N}\zeta_j$.

Based on the errors defined above, $\zeta_i(t, z_1, z_2)$ can be rewritten as

$$\zeta_i(t, z_1, z_2) = \sum_{j=1}^{N} a_{ij}\left(e_j^b - e_i^b\right) + A_{12}^T \sum_{j=1}^{N} a_{ij}\left(e_j^a - e_i^a\right) \tag{33}$$

Because $a_{ij} = a_{ji}$, for $\bar{\zeta}(t, z_1, z_2)$ it can be obtained that

$$\begin{aligned} \bar{\zeta}(t, z_1, z_2) &= \frac{1}{N}\sum_{j=1}^{N}\zeta_j \\ &= \frac{1}{N}\sum_{j=1}^{N}\left(\sum_{k=1}^{N} a_{jk}\left(e_k^b - e_j^b\right) + A_{12}^T \sum_{k=1}^{N} a_{jk}\left(e_k^a - e_j^a\right)\right) \\ &= \frac{1}{2N}\sum_{j=1}^{N}\sum_{k=1}^{N} a_{jk}\left[\left(e_k^b - e_j^b\right) + \left(e_j^b - e_k^b\right) + A_{12}^T\left(e_k^a - e_j^a\right) + A_{12}^T\left(e_j^a - e_k^a\right)\right] \\ &= 0_m \end{aligned} \tag{34}$$

A Lyapunov candidate function is constructed as

$$V_2(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m}\int_0^{e_{ik}^a - e_{jk}^a} a_{ij} y dy + \frac{1}{2}\sum_{i=1}^{N}\left(e_i^b\right)^T e_i^b \tag{35}$$

The derivative of V_2 along the errors e_i^a and e_i^b is given by

$$\begin{aligned} \dot{V}_2(t) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m} a_{ij}\left(e_{ik}^a - e_{jk}^a\right)\left(\dot{e}_{ik}^a - \dot{e}_{jk}^a\right) + \sum_{i=1}^{N}\left(e_i^b\right)^T \dot{e}_i^b \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n-m} a_{ij}\left(e_{ik}^a - e_{jk}^a\right)\dot{e}_{ik}^a + \sum_{i=1}^{N}\left(e_i^b\right)^T \dot{e}_i^b \\ &= \sum_{i=1}^{N}\left(\dot{e}_i^a\right)^T \sum_{j=1}^{N} a_{ij}\left(e_i^a - e_j^a\right) + \sum_{i=1}^{N}\left(e_i^b\right)^T \dot{e}_i^b \end{aligned} \tag{36}$$

Combined with (32), it can be obtained that

$$\begin{aligned} \dot{V}_2(t) &= \sum_{i=1}^{N}\left(A_{11} e_i^a + A_{12} e_i^b + e_i^{\phi_1}\right)^T \sum_{j=1}^{N} a_{ij}\left(e_i^a - e_j^a\right) + \sum_{i=1}^{N}\left(e_i^b\right)^T\left(\sum_{j=1}^{N} a_{ij}\left(e_j^b - e_i^b\right) + A_{12}^T\sum_{j=1}^{N} a_{ij}\left(e_j^a - e_i^a\right)\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(e_i^a)^T A_{11}^T\left(e_i^a - e_j^a\right) + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(e_i^b\right)^T\left(e_j^b - e_i^b\right) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(\left(e_i^{\phi_1}\right)^T - \left(e_j^{\phi_1}\right)^T\right)\left(e_i^a - e_j^a\right) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(e_i^a - e_j^a\right)^T A_{11}^T\left(e_i^a - e_j^a\right) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(e_i^b - e_j^b\right)^T\left(e_i^b - e_j^b\right) + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\phi_{i1}^T\left(e_i^a - e_j^a\right) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_{i1} - z_{j1})^T A_{11}^T\left(z_{i1} - z_{j1}\right) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_{i2} - z_{j2})^T\left(z_{i2} - z_{j2}\right) + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\phi_{i1}^T\left(z_{i1} - z_{j1}\right) \end{aligned} \tag{37}$$

Further, note that A_{11} is negative definite, so A_{11}^T is negative definite. Combined with Assumption 5, the following inequalities can be obtained

$$\begin{aligned}\dot{V}_2(t) &\le \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\lambda_{\max}\left(A_{11}^T\right)\|z_{i1}-z_{j1}\|^2 - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|z_{i2}-z_{j2}\|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|\phi_{i1}\|\|z_{i1}-z_{j1}\| \\ &\le \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\lambda_{\max}\left(A_{11}^T\right)\|z_{i1}-z_{j1}\|^2 - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|z_{i2}-z_{j2}\|^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(\gamma_i\|z_{i1}\|+\gamma_j\|z_{j1}\|\right)\|z_{i1}-z_{j1}\| \\ &\le \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(\lambda_{\max}\left(A_{11}^T\right)+\gamma_{\max}\right)\left(\|z_{i1}\|+\|z_{j1}\|\right)\|z_{i1}-z_{j1}\| - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|z_{i2}-z_{j2}\|^2\end{aligned} \tag{38}$$

The analysis of (38) is presented as follows: $\left(\lambda_{\max}\left(A_{11}^T\right)+\gamma_{\max}\right)\left(\|z_{i1}\|+\|z_{j1}\|\right)\|z_{i1}-z_{j1}\| \le 0$, equality holds if and only if $z_{i1}-z_{j1}=0_{n-m}$ ($z_{i1}=z_{j1}=0_{n-m}$ included); $\|z_{i2}-z_{j2}\|^2 \ge 0$, equality holds if and only if $z_{i2}-z_{j2}=0_m$. Hence, $\dot{V}_2 \le 0$. Referring to Lemma 1, it can be obtained that (a) $V_2(t)$ is radially unbounded over e_i^a and e_i^b; (b) Since the undirected graph is connected, if $\dot{V}_2 \equiv 0$, then $z_{i1}\equiv z_{j1}$, $z_{i2}\equiv z_{j2}$, $\forall i,j=1,2,\ldots,N$. That is, $\lim_{t\to\infty}\|z_{i1}-z_{j1}\|=0$ and $\lim_{t\to\infty}\|z_{i2}-z_{j2}\|=0$, $\forall i,j=1,2,\ldots,N$, i.e., $\lim_{t\to\infty}\|x_i-x_j\|=0$, $\forall i,j=1,2,\cdots,N$. Based on the above analysis, system (31) can be driven to consensus asymptotically.

It can then be obtained from (31) that $\dot{z}_{i2}$ goes to 0_m asymptotically. Due to the presence of the unmatched disturbances $\phi_{i1}(t,z_1,z_2)$, the evolution of $z_{i1}(t)$ and $z_{i2}(t)$ should be discussed.

Theorem 4: Suppose Assumptions 1-6 hold. The subsystem (31) is globally ISS where z_{i1} is the state and both z_{i2} and ϕ_{i1} are considered as inputs.

***Proof*:** According to Assumption 3 $\|\phi_i\|\le\beta$, thus $\|\phi_{i1}\|\le\beta$. From Assumption 6 and Remark 4, z_{i2} is bounded. Referring to Lemma 2, the inputs z_{i2} and ϕ_{i1} are both bounded.

A Lyapunov candidate function is constructed as

$$V_3(t)=\frac{1}{2}z_{i1}^T z_{i1} \tag{39}$$

Let $-1<\theta<0$, then the derivative of $V_3(t)$ is given by

$$\begin{aligned}\dot{V}_3(t) &= z_{i1}^T\dot{z}_{i1} \\ &= z_{i1}^T\left(A_{11}z_{i1}+A_{12}z_{i2}+\phi_{i1}\right) \\ &= (1+\theta)z_{i1}^T A_{11}z_{i1} + z_{i1}^T A_{12}z_{i2} + z_{i1}^T\phi_{i1} - \theta z_{i1}^T A_{11}z_{i1} \\ &\le (1+\theta)\lambda_{\max}(A_{11})\|z_{i1}\|^2\end{aligned} \tag{40}$$

provided that $z_{i1}^T A_{12}z_{i2}+z_{i1}^T\phi_{i1}-\theta z_{i1}^T A_{11}z_{i1}\le 0$.

Assume that $\left\|z_{i1}^T A_{12}z_{i2}+z_{i1}^T\phi_{i1}\right\| \le \left\|\theta z_{i1}^T A_{11}z_{i1}\right\|$, or equivalently

$$\|z_{i1}\| \ge \frac{\|A_{12}\|_F\|z_{i2}\|+\|\phi_{i1}\|}{\lambda_{\max}(A_{11})\theta} \tag{41}$$

According to Lemma 2, it can be shown that $\varepsilon_1(\|z_{i1}\|)=\varepsilon_2(\|z_{i1}\|)=\frac{1}{2}\|z_{i1}\|^2$, $-\varepsilon_3(\|z_{i1}\|)=(1+\theta)\lambda_{\max}(A_{11})\|z_{i1}\|^2$, where z_{i1} is taken as the state and z_{i2} and ϕ_{i1} as the inputs in Lemma 2. It follows that the subsystem is globally ISS with

$$\vartheta(\|u\|)=\frac{\|A_{12}\|_F\|z_{i2}\|+\|\phi_{i1}\|}{\lambda_{\max}(A_{11})\theta} \tag{42}$$

Therefore, appealing to Remark 2, z_{i1} is bounded with

$$\|z_{i1}\| \le \frac{\|A_{12}\|_F\|z_{i2}\|+\|\phi_{i1}\|}{\lambda_{\max}(A_{11})\theta} \tag{43}$$

Remark 5: The above results indicate that the Lyapunov function V_3 is negative definite along the trajectories of z_{i1} whenever the trajectories are outside of the ball defined by $\|z_{i1}^*\|=\frac{\|A_{12}\|_F\|z_{i2}\|+\|\phi_{i1}\|}{\lambda_{\max}(A_{11})\theta}$, and the trajectories will remain ultimately bounded by the ball of radius $\frac{\|A_{12}\|_F\|z_{i2}\|+\|\phi_{i1}\|}{\lambda_{\max}(A_{11})\theta}$.

Remark 6: This section considers stability of the subsystems. Note that the stability of the subsystems is not considered in [1]. When the subsystem dynamics (1) is a class of second-order systems, the states may diverge due to the existence of disturbances. See the appendix for a detailed analysis.

V. Simulations and Analysis

In this section, two simulation examples are presented to demonstrate the validity of the proposed method.

Example 1. This example aims to demonstrate the effectiveness of the theoretical results in the presence of matched and unmatched disturbances. Consider a multiagent system with four subsystems, whose topology connection is shown in Fig.1.

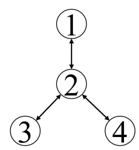


Fig. 1: Undirected graph with 4 subsystems

The dynamics of each subsystem is given by

$$\dot{x}_i = \begin{bmatrix} 1 & 5 & 8 & 3 \\ 4 & 7 & 5 & 9 \\ 11 & 5 & 4 & 3 \\ 9 & 6 & 0 & 9 \end{bmatrix} x_i + \begin{bmatrix} 5 & 7 \\ 4 & 1 \\ 0 & 5 \\ -8 & 6 \end{bmatrix} u_i + \phi_i \tag{44}$$

where the initial states are selected as follows:

$$\begin{aligned} &x_1(0) = \begin{bmatrix} -5 & 7 & 6 & 8 \end{bmatrix}^T, x_2(0) = \begin{bmatrix} 11 & 3 & -10 & -4 \end{bmatrix}^T \\ &x_3(0) = \begin{bmatrix} 8 & -3 & -1 & 0 \end{bmatrix}^T, x_4(0) = \begin{bmatrix} -4 & 6 & 0 & -2 \end{bmatrix}^T \end{aligned} \tag{45}$$

The disturbances are as follows:

$$\phi_i = \begin{bmatrix} \breve{t}_{11}\gamma_i \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) + \breve{t}_{13}\left(0.1\cos\left(x_{i3}\right)\right) + \breve{t}_{14}\left(0.5\sin\left(t\right)\right) \\ \breve{t}_{21}\gamma_i \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) + \breve{t}_{23}\left(0.1\cos\left(x_{i3}\right)\right) + \breve{t}_{24}\left(0.5\sin\left(t\right)\right) \\ \breve{t}_{31}\gamma_i \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) + \breve{t}_{33}\left(0.1\cos\left(x_{i3}\right)\right) + \breve{t}_{34}\left(0.5\sin\left(t\right)\right) \\ \breve{t}_{41}\gamma_i \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) + \breve{t}_{43}\left(0.1\cos\left(x_{i3}\right)\right) + \breve{t}_{44}\left(0.5\sin\left(t\right)\right) \end{bmatrix} \tag{46}$$

where $T_2T_1 = \left[\widehat{t}_{ij}\right]_{4\times4}$, $(T_2T_1)^{-1} = \left[\breve{t}_{ij}\right]_{4\times4}$, $\gamma_i = 0.1$, $i, j = 1, 2, 3, 4$.

The coordinate transformation matrices are

$$T_1 = \begin{bmatrix} -0.4943 & 0.3465 & 0.7856 & -0.1356 \\ -0.1300 & 0.8433 & -0.3951 & 0.3404 \\ -0.4880 & -0.3904 & 0.0000 & 0.7807 \\ 0.7076 & 0.1279 & 0.4762 & 0.5062 \end{bmatrix}, T_2 = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.3561 & 0.9858 & 1.0000 & 0.0000 \\ -0.0737 & 0.6324 & 0.0000 & 1.0000 \end{bmatrix} \tag{47}$$

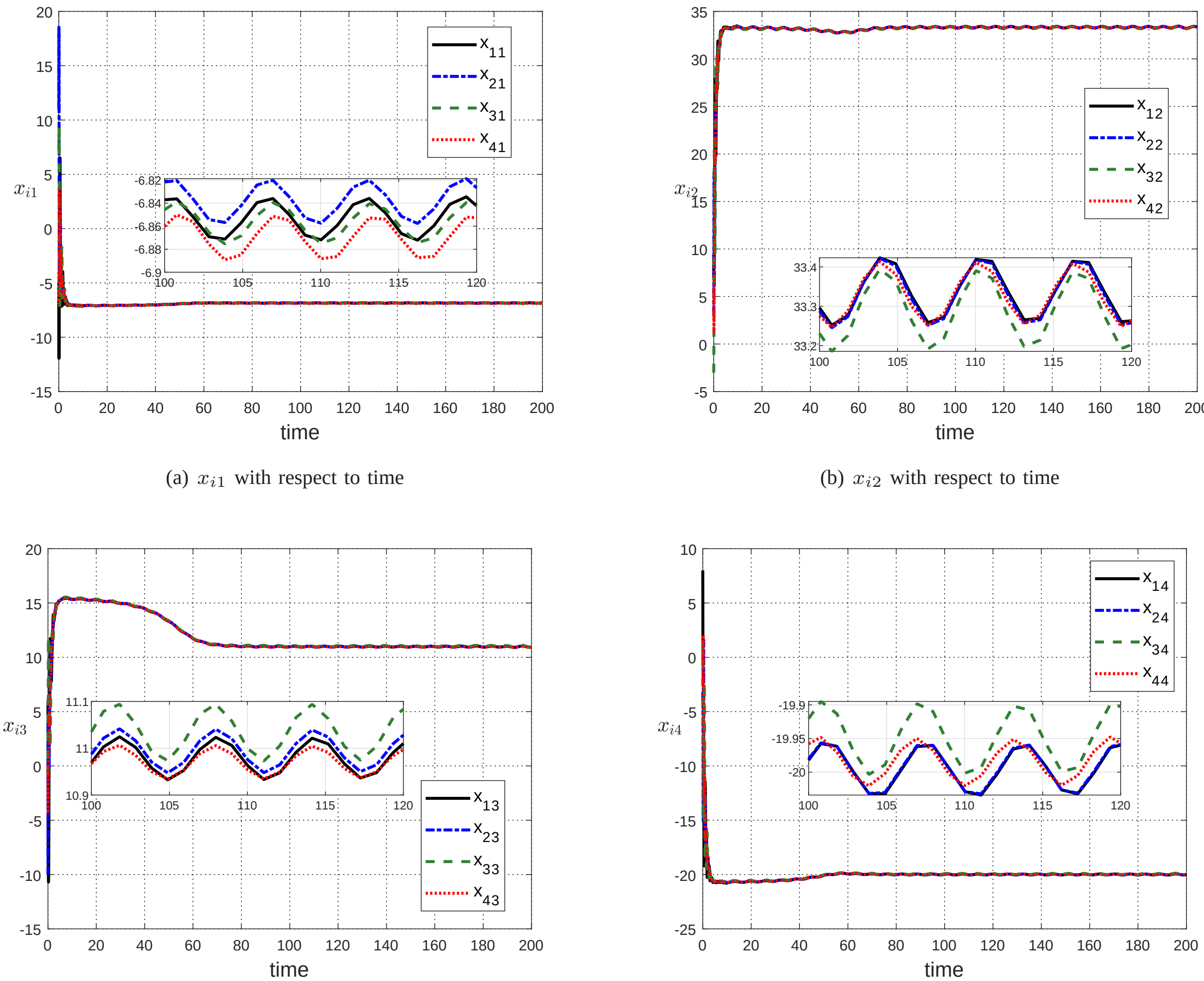
and other parameters are selected as $\beta = 1.00, \alpha_i = 0.0001, \rho = 0.15, G = \begin{bmatrix} 3.6493 & 10.1017 \\ 0.4614 & -7.5056 \end{bmatrix}$.

It can be verified by computations that $\|\phi_i\| \le \beta$. In addition, by coordination transformation, it can be obtained that

$$\phi_{i1} = \gamma_i \begin{bmatrix} \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2\right) \\ 0 \end{bmatrix} \tag{48}$$

$\|\phi_{i1}\| \leq \gamma_i \|z_{i1}\| = \gamma_i \sqrt{\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2} + \widehat{t}_{13}x_{i3} + \widehat{t}_{14}x_{i4}\right)^2 + \left(\widehat{t}_{21}x_{i1} + \widehat{t}_{22}x_{i2} + \widehat{t}_{23}x_{i3} + \widehat{t}_{24}x_{i4}\right)^2}$ can be verified.

To avoid chattering in the implementation, a boundary layer approximation is used such that $\left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^T s_i(t) / \left\|\left(G\begin{bmatrix}0_{(n-m)\times m}\\ B_2\end{bmatrix}\right)^T s_i(t) + \delta\right\|$ is used to replace (19), where δ is a small positive scalar and selected as $\delta = 0.01$. The simulation results are shown as Fig.2−4.



(a) x_{i1} with respect to time

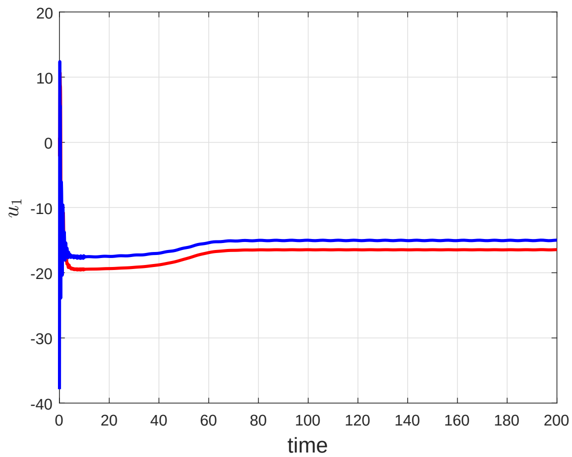
(b) x_{i2} with respect to time

(c) x_{i3} with respect to time

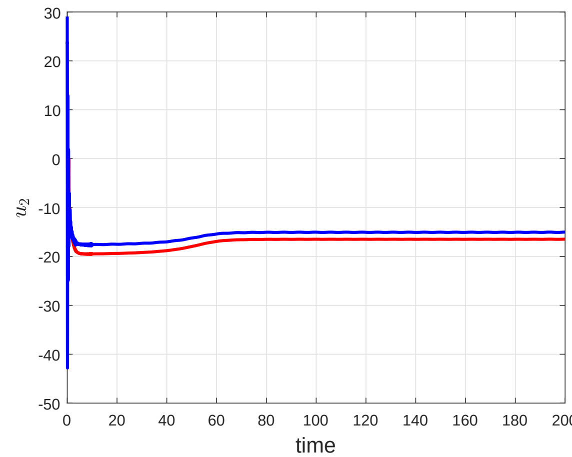
(d) x_{i4} with respect to time

Fig. 2: Subsystems' states with respect to time

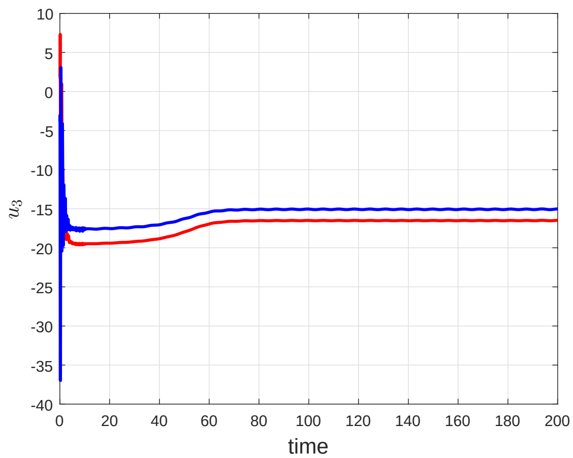
Fig.2 shows the subsystems' states with respect to time. As can be seen, in the presence of matched and unmatched disturbances, the system achieves consensus. Fig.3 shows the subsystems' control inputs with respect to time. It can be seen that the control inputs remain bounded after the subsystems are stabilized, and there is no obvious chattering in the control signal. Fig.4 shows the sliding variable with respect to time. It is seen that every subsystem starts on the sliding surface from the beginning which avoids sensitivity to matched disturbances in the reaching phase.



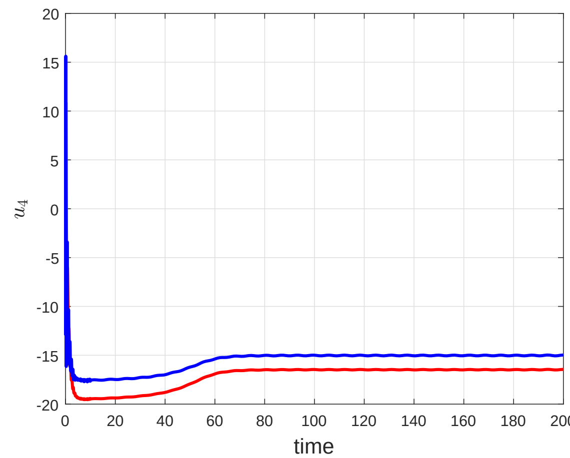
(a) u_1 with respect to time



(b) u_2 with respect to time



(c) u_3 with respect to time



(d) u_4 with respect to time

Fig. 3: Subsystems' control inputs with respect to time

Example 2. Consider the multiagent system whose topology connection is also shown as Fig.1. To further test the proposed distributed protocol, the protocol (3) developed in [1] which uses an adaptive scheme will be compared with the method proposed in this paper. The dynamics of each subsystem is given by

$$\dot{x}_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} u_i + \phi_i \tag{49}$$

where the initial states are selected as follows:

$$x_1(0) = \begin{bmatrix} 1 & 2 \end{bmatrix}^T, x_2(0) = \begin{bmatrix} -1 & -2 \end{bmatrix}^T, x_3(0) = \begin{bmatrix} 3 & 4 \end{bmatrix}^T, x_4(0) = \begin{bmatrix} -3 & -4 \end{bmatrix}^T \tag{50}$$

The disturbances are as follows:

$$\phi_i = \begin{bmatrix} \breve{t}_{11}\gamma_i \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2}\right)^2\right) + 0.01\breve{t}_{12}\left(\cos\left(x_{i1}\right) + \sin\left(t\right)\right) \\ \breve{t}_{21}\gamma_i \sin\left(\left(\widehat{t}_{11}x_{i1} + \widehat{t}_{12}x_{i2}\right)^2\right) + 0.01\breve{t}_{22}\left(\cos\left(x_{i1}\right) + \sin\left(t\right)\right) \end{bmatrix} \tag{51}$$

where $T_2T_1 = \left[\widehat{t}_{ij}\right]_{2\times 2}$, $(T_2T_1)^{-1} = \left[\breve{t}_{ij}\right]_{2\times 2}$, $\gamma_i = 0.08,\ i, j = 1, 2.$

For the protocol proposed in this paper, the coordinate transformation matrices are

$$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{52}$$

and the other parameters are selected as $\beta = 0.15, \alpha_i = 0.0001, \rho = 0.4, G = \begin{bmatrix} 0.4 & -0.4 \end{bmatrix}, \delta = 0.0001$.

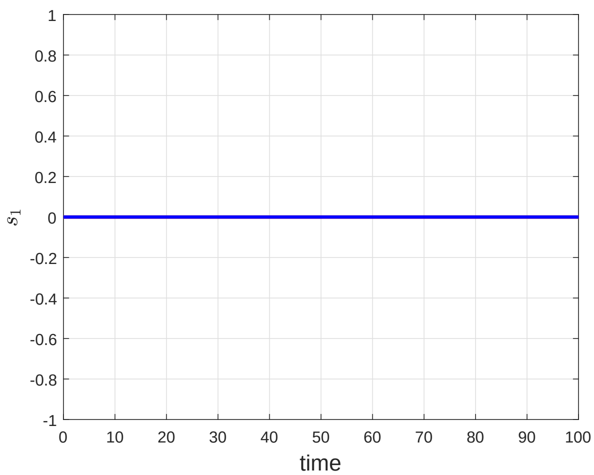
It can be verified by computations that $\|\phi_i\| \leq \beta$. In addition, by coordination transformation, it can be obtained that

$$\phi_{i1} = \gamma_i \sin\left(\left(\widehat{t}_{11} x_{i1} + \widehat{t}_{12} x_{i2}\right)^2\right) \tag{53}$$

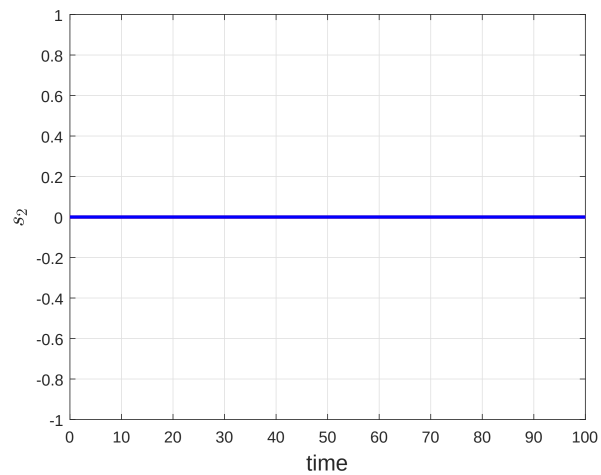
$\|\phi_{i1}\| \leq \gamma_i \|z_{i1}\| = \gamma_i \sqrt{\left(\widehat{t}_{11} x_{i1} + \widehat{t}_{12} x_{i2}\right)^2}$ can be verified.

For the protocol (3) proposed in [1], the parameters are selected as $\Gamma = \begin{bmatrix} 1.0000 & 2.4495 \\ 2.4495 & 6.0000 \end{bmatrix}, K = \begin{bmatrix} -1.0000 & -2.4495 \end{bmatrix}, \bar{d}_i(0) = 0, \bar{e}_i(0) = 0, \tau_i = 10, \varepsilon_i = 10, \kappa_i = 0.5, \varphi_i = 0.05, \psi_i = 0.05$.

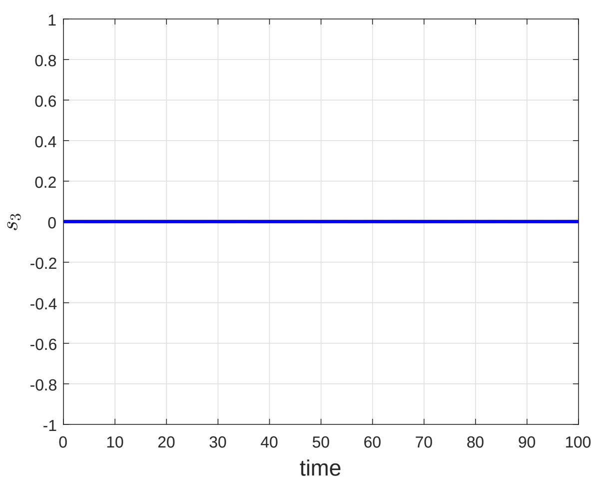
The simulation results are shown as Fig.5−6, where the solid lines denote the method proposed in this paper, labeled as 2020; the dashed lines denote the method proposed in [1], labeled as 2014.



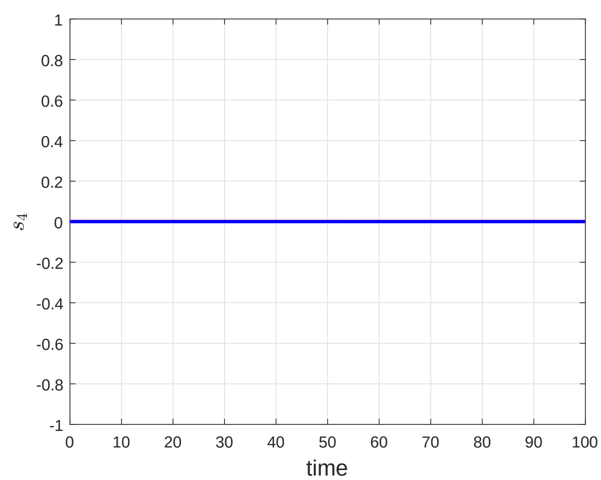
(a) s_1 with respect to time



(b) s_2 with respect to time

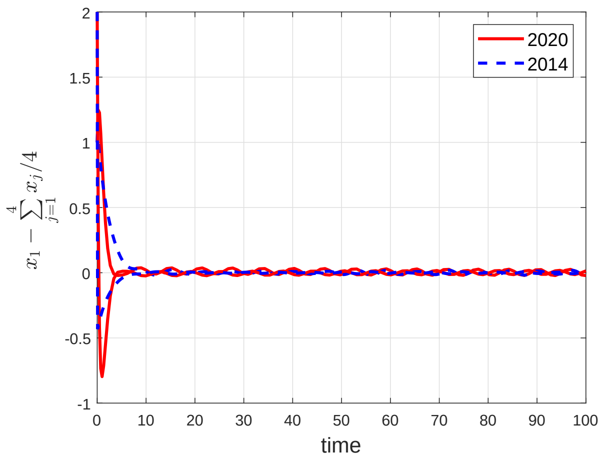


(c) s_3 with respect to time

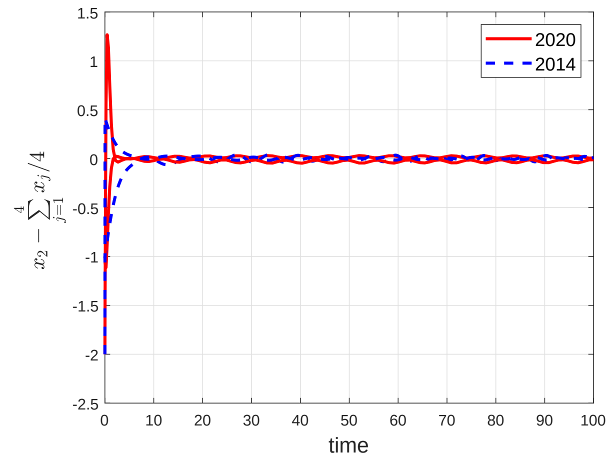


(d) s_4 with respect to time

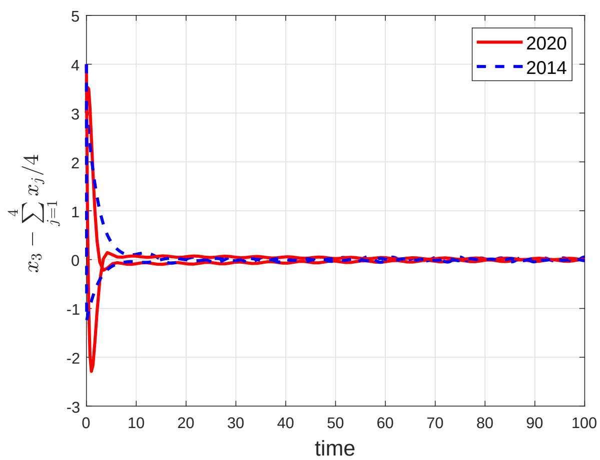
Fig. 4: Subsystems' sliding motion with respect to time



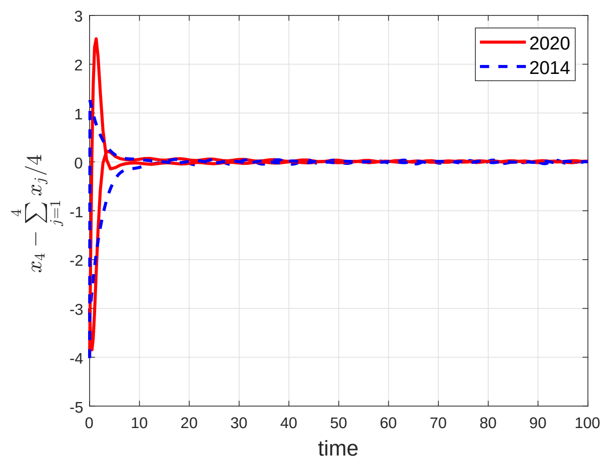
(a) Consensus errors with respect to time for the first subsystem



(b) Consensus errors with respect to time for the second subsystem

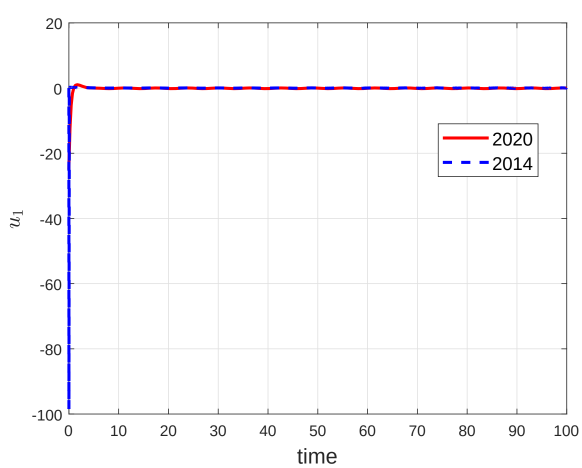


(c) Consensus errors with respect to time for the third subsystem

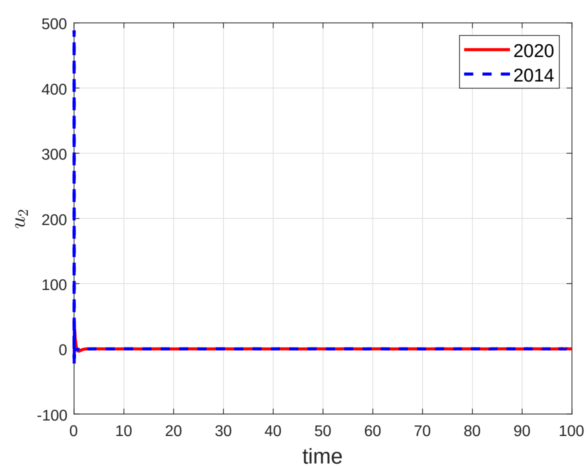


(d) Consensus errors with respect to time for the fourth subsystem

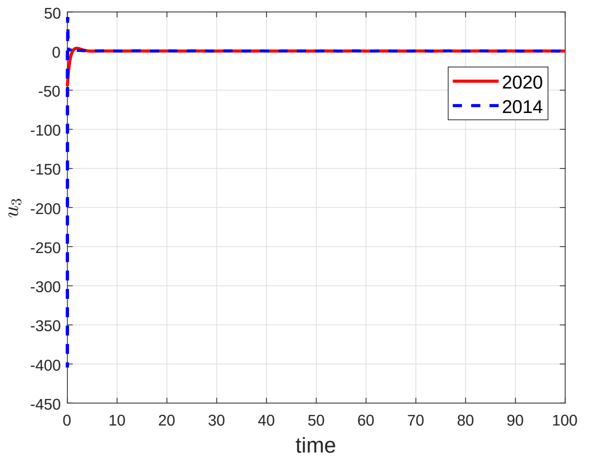
Fig. 5: Consensus errors with respect to time in two protocols



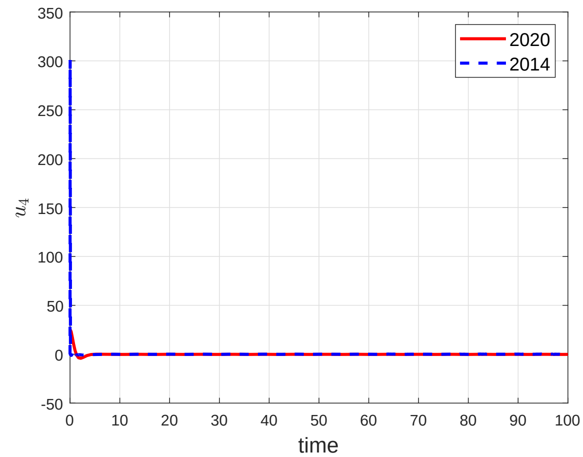
(a) u_1 with respect to time



(b) u_2 with respect to time



(c) u_3 with respect to time

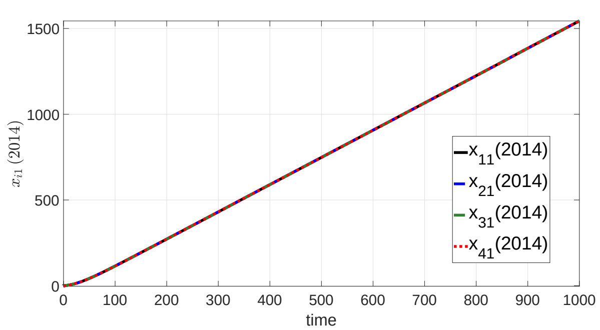


(d) u_4 with respect to time

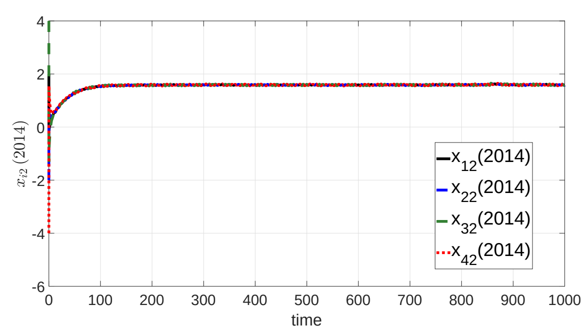
Fig. 6: Subsystems' control inputs with respect to time in two protocols

From Fig.5, the consensus errors with respect to time using the proposed method (2020) differ very little from those of the method in [1] (2014) in terms of the settling time and overshoot. However, it can been seen that the control inputs of the method in [1] are several times higher than for the proposed method for an initial period of time (Fig.6), which is energy-consuming.

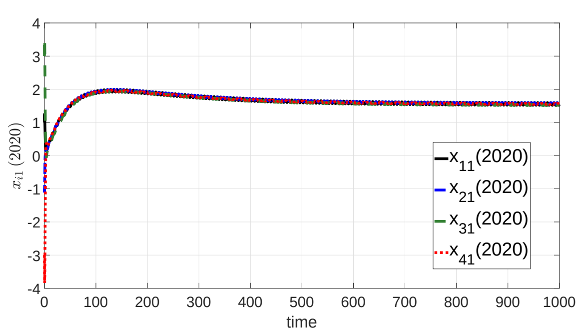
Stability of the subsystems is not considered when designing the protocol (3) in [1], and the states correspondingly diverge. This can be verified by substituting numerical values into the system matrix (54) in the appendix. No matter what values $\bar{d}_i$ take, it can be seen that there are two zero eigenvalues in the system matrix. The simulation results also illustrate this point, as shown in Fig.7 (a) and (b). With the proposed approach, the negative symmetric definitness of A_{11} guarantees the state evolution with the protocol devised in this paper are ultimately bounded as seen in Fig.7 (c) and (d).



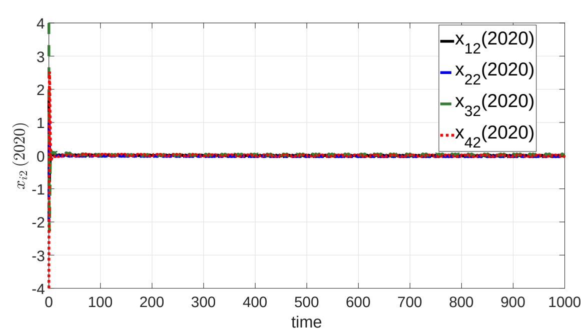
(a) x_{i1} with respect to time in [1]



(b) x_{i2} with respect to time in [1]



(c) x_{i1} with respect to time in this paper



(d) x_{i2} with respect to time in this paper

Fig. 7: Subsystems' states with respect to time

VI. Conclusion

A consensus framework is proposed for a class of linear multiagent systems in the presence of matched and unmatched disturbances. An integral sliding mode strategy is utilized to ensure the subsystems lie on the sliding surface from the initial time. The impact of the disturbances are minimized according to the projection theorem. A consensus protocol is designed and analyzed applying a linear coordinate transformation and the global invariant set theorem. The stability of each subsystem is guaranteed by appealing to results on global ISS. Numerical simulations show the validity and superiority of the proposed method. Future work will focus on experimental testing and practical application of the proposed method. In terms of theory, an interesting direction is to discuss the consensus of nonlinear systems using a sliding mode strategy.

Appendix

The analysis of the case where the subsystem states are diverging in [1] is presented as follows:

(a) Substitute the consensus protocol (3, [1]) into the subsystem dynamics (2, [1]), to obtain a lumped form:

$$\dot{x} = \left[I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK)\right] x + (I_N \otimes B)(R + F) \tag{54}$$

It should be noted that in this appendix, ($*$, [1]) refers to the corresponding equation ($*$) in [1], and the notations also refer to the ones in [1] unless otherwise stated.

(b) Here, an eigenvalue can be acquired by the system matrix $\left[I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK)\right]$ in (53) by which stability of the subsystem can be judged.

(b.1) When the subsystem dynamics (1, [1]) is in a linear second-order form, then $A \triangleq \begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}$ and $B \triangleq \begin{bmatrix} 0 \\ \bar{b}_2 \end{bmatrix}$, where $\bar{a}_{12}, \bar{b}_2 \in R$. To guarantee the controllability of the subsystem, $\bar{a}_{12} \neq 0$ and $\bar{b}_2 \neq 0$. $K \triangleq \begin{bmatrix} \bar{k}_1 & \bar{k}_2 \end{bmatrix}$, where $\bar{k}_1, \bar{k}_2 \in R$.

(b.2) Calculate the elements item by item as follows for the system matrix $\left[I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK)\right]$:

$$I_N \otimes A = diag\left(\underbrace{\begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}}_{N}\right) \tag{55}$$

$$\bar{D}\mathcal{L} = \left[\bar{d}_i \mathcal{L}_{ij}\right]_{N \times N}, i, j = 1, \ldots, N \tag{56}$$

$$BK = \begin{bmatrix} 0 & 0 \\ k_1 b_2 & k_2 b_2 \end{bmatrix} \tag{57}$$

$$\left(\bar{D}\mathcal{L}\right) \otimes (BK) = \left[\bar{d}_i \mathcal{L}_{ij} \begin{bmatrix} 0 & 0 \\ k_1 b_2 & k_2 b_2 \end{bmatrix}\right]_{N \times N} = \left[\begin{bmatrix} 0 & 0 \\ \bar{d}_i \mathcal{L}_{ij} k_1 b_2 & \bar{d}_i \mathcal{L}_{ij} k_2 b_2 \end{bmatrix}\right]_{N \times N} \tag{58}$$

then

$$\begin{aligned} & I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK) \\ & = diag\left(\underbrace{\begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \bar{a}_{12} \\ 0 & 0 \end{bmatrix}}_{N}\right) + \left[\begin{bmatrix} 0 & 0 \\ \bar{d}_i \mathcal{L}_{ij} k_1 b_2 & \bar{d}_i \mathcal{L}_{ij} k_2 b_2 \end{bmatrix}\right]_{N \times N} \\ & \triangleq \left[\bar{\Lambda}_{ij}\right]_{2N \times 2N} \end{aligned} \tag{59}$$

In (58), $\bar{\Lambda}_{i1} + \bar{\Lambda}_{i3} + \bar{\Lambda}_{i5} + \ldots + \bar{\Lambda}_{i(2N-1)} = 0$, then $\bar{\alpha} \begin{bmatrix} \underbrace{1, 0, \ldots, 1, 0}_{2N} \end{bmatrix}^T$ is an eigenvector of (58), where $\bar{\alpha} \in R$ and $\bar{\alpha} \neq 0$, and the corresponding eigenvalue is 0.

(c) Consider now where there is at least one 0 in the eigenvalues of $\left[I_N \otimes A + \left(\bar{D}\mathcal{L}\right) \otimes (BK)\right]$ so that the subsystem is critically stable. For (53), although the nonlinear term R can compensate the disturbance term F, the disturbances still have an effect on the critically stable subsystem and as a consequence the states diverge. This covers the analysis of [1].

In addition to [1], there are other contributions [2][5][6][14] where the states may diverge for a second-order subsystem when subjected to disturbances.

VII. Declaration of conflicting interests

The author(s) declare no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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