Models of Curves over Local Fields

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I, Omri Faraggi, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work. Chapters 3 and 4 are the result of a joint work with Sarah Nowell.

Abstract

Cluster pictures are a recent innovation which have been developed to study the arithmetic of hyperelliptic curves. The cluster picture of such a curve $C : y^2 = f(x)$ over a local field K is a completely combinatorial object containing the data of the p-adic distances between the roots of f. It determines many invariants associated to C, and most pertinently for us it was used in [19] to calculate the minimal regular model of C when it has semistable reduction.

We extend their results to the case where C has tame reduction, calculating its minimal snc model in terms of its cluster picture. As an application we state a condition in terms of the cluster picture for C to have a K-rational point. In addition we use a generalisation of the cluster picture, the chromatic cluster picture, to work out the minimum regular model of a bihyperelliptic curve with semistable reduction.

Impact Statement

The Birch–Swinnerton-Dyer conjecture is one of the principal unsolved problems in modern mathematics. It unifies two vastly different perspectives on elliptic and hyperelliptic curves, fundamental number theoretic objects which have generated decades of research, and promises to unlock deep secrets of number theory. A proof of the conjecture is not yet within reach, but improving our understanding of (hyper)elliptic curves is an important step towards it. My work aims to be a small step in the right direction, calculating a key local invariant of hyperelliptic curves and bihyperelliptic curves, the minimal model. Techniques used in this thesis will be applicable to greater classes of curves, stimulating future research into minimal models of curves. In addition, it is possible to use the condition given in Chapter 5 for local solubility of hyperelliptic curves to determine what proportion of hyperelliptic curves of a given genus are locally soluble, with additional work to determine the probability of hyperelliptic curves having a given cluster picture.

Outside of academia, number theory plays a crucial role in modern communication and cryptography. Algorithms such as RSA, which use Fermat's Little Theorem, have made the contemporary information boom possible, and more recently elliptic and hyperelliptic curves have been used in cryptographic algorithms (ECC and HCC) to facilitate faster and more secure methods of communication. With more and more of life moving online, from workplaces to currencies, such methods are becoming increasingly necessary. While my work is not directly related to this utilisation of hyperelliptic curves, it exists within the same context.

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Contents

Chapter 1

Introduction

1.1 Models

Frequently in number theory one would like to understand an object locally; it is then possible to stitch together this local information to gain global insight. A classical example of this is the *Hasse-Minkowski Theorem*, which states that two quadratic forms over a number field are equivalent if and only if they are equivalent locally at all places. This fails in general: for example, Selmer showed that $3x^3 + 4y^3 + 5z^3 = 0$ is an elliptic curve with points over $\mathbb{R}$ and $\mathbb{Q}_p$ for all p, but no $\mathbb{Q}$-rational points. However, all hope is not lost. There is still much to learn from studying objects locally.

We will be interested in *curves* over a field K with a discrete valuation $v = v_K$. A natural question is to ask about the reduction of a curve X modulo v. In an ideal world, this would always result in a smooth curve $\widetilde{X}$ over the residue field k. Such a curve is said to have *good reduction* over K. However, we do not live in such a world; in the real world there exist curves of *bad reduction*. Sometimes this problem is assailable — the elliptic curve $y^2 = x^3 + p^6$ has bad reduction but the isomorphic curve $y^2 = x^3 + 1$ has good reduction — but other times the bad reduction is honest and we must overcome it via different means.

In this thesis we calculate an important local invariant of curves which offers a solution to the issue of bad reduction: the *regular model*. Formally for a curve X, a regular model of X is a flat, proper, regular $\mathcal{O}_K$-scheme $\mathscr{X}$ whose generic fibre $\mathscr{X}_K$ is isomorphic to X and whose special fibre $\mathscr{X}_k$ represents the reduction of X modulo v and determines much of the local arithmetic of X. For example, the special fibre can be used to calculate *Tamagawa numbers* and *local root numbers*, invariants which appear in the statements of deep unsolved conjectures such as the *Birch–Swinnerton-Dyer conjecture* and the *Parity conjecture*.

The quest to find regular models of curves has a long and varied history. In the 1960s, Kodaira [24] classified the possible special fibres which can appear in the minimal regular model of an elliptic curve. This classification was also done by Néron [39] in a more arithmetic setting. In [38], Namikawa and Ueno devised a similar (albeit much, much longer) classification for genus 2 curves.

These classifications are useful in their own right, but often we would like to calculate the minimal regular model for a given curve X. In theory, this can be done in the following way: first take any model of X and take its normalisation. This results in a normal model, whose singularities are therefore closed points on the special fibre. A theorem of Lipman from [27] tells us that after a finite number of further normalisations and blow ups we obtain a regular model, and after contracting exceptional components we obtain the minimal regular model. In practice, computing these normalisations and blow ups is a time consuming task. We would prefer a more direct way to calculate models.

The case of genus 1 curves was completed in the 1970s, and can be computed using Tate's algorithm [44] (see also [42, § IV]). In the same spirit, Liu [28] devised an algorithm which can explicitly determine the minimal regular model of a genus 2 curve as a function of the coefficients of the defining equation. Both of these algorithms rely heavily on the classification of possible special fibres, which makes them difficult to generalise to genus $g > 2$ since no such classification exists. Therefore, different approaches are necessary.

More recently, several other cases have been computed. The semistable model of a *hyperelliptic curve* can be deduced from its *cluster picture*, a combinatorial invariant of the curve which we shall describe shortly. This work is due to the Dokchitser brothers, Maistret and Morgan in [19]. Their theorem was extended to the case of hyperelliptic curves with *tame reduction* by Sarah Nowell and the author in [22], using the rich theory of tame quotients of models developed in papers such as [47], [32], [12, § 2] and [23]. T. Dokchitser has devised a way to work out the *minimal snc model* — a model with nonreduced special fibre but manageable singularities — of a curve which is Δ_v*-regular* from its Newton polygon in [15]. Being Δ_v-regular is a rather strict condition, but it has been loosened considerably by Muselli in [37].

Non-hyperelliptic genus 3 curves and their models have also attracted interest in recent years, such as [26], which allows us to calculate whether a genus 3 curve has (potentially) good hyperelliptic or quartic reduction, or bad reduction, using *Dixmier-Ohno* invariants. Further work in this direction has been carried out in [10].

Other than that, there is currently a flurry of activity dedicated to using *MacLane valuations* to study models. This was started in Rüth's thesis [40], and continued in a variety of papers such as [41] and [11]. A large advantage of this method is that it can deal with the wild case as effectively as the tame case. However since this technique is rather tangential to ours, we shall not focus on it too much, and shall concentrate on the cluster picture approach instead.

1.2 Hyperelliptic Curves and Cluster Pictures

Much of the thesis will be devoted to finding models of *hyperelliptic curves*. These are classical objects defined by an equation $y^2 = f(x)$, where f is a polynomial of degree greater than 4. Taking the naïve projective closure of this does *not* result in a smooth curve; the point at infinity is a singularity. Upon normalising, we obtain a smooth curve C which is given by two affine charts:

$$y^2 = f(x) \text{ and } w^2 = v^{2g+2} f(1/v),$$

which are glued together via

$$(x, y) \mapsto (v, w) = (1/x, y/x^{g+1}).$$

Here g is the genus of the curve and it is such that $\deg(f) = 2g + 1$ or $2g + 2$. These curves come with a natural 2-to-1 map to $\mathbb{P}^1$, given by sending a point to its x-coordinate. The ramification points of this cover are precisely the roots of the polynomial f, and we shall denote this set $\mathcal{R}$.

It transpires that the combinatorial data of the p-adic distances between these roots determines much of the local arithmetic of C; we call this configuration the *cluster picture* of C. Cluster pictures were first introduced in [35], and have been used in subsequent papers to calculate various invariants, such as the Galois representation, semistable model, conductor and minimal discriminant of C in [19], the Tamagawa number in [6], the root number in [9] and finally differentials in [25] and [37]. A survey article of many of the key uses is available at [5]. In this thesis, we use cluster pictures to compute the minimal snc model of a C when it has tame reduction, as well as using a generalisation of cluster pictures to compute the semistable model of a curve with maps to two distinct hyperelliptic curves. We also use it to state a condition for a hyperelliptic curve to be locally soluble. More precisely, a cluster picture is defined as follows.

Definition 1.2.1. Let $C : y^2 = f(x)$ be a hyperelliptic curve over K, with $\mathcal{R}$ the set of roots of f. A *cluster* is a non-empty subset $\mathfrak{s} \subseteq \mathcal{R}$ of the form $\mathfrak{s} = D \cap \mathcal{R}$ for some disc $D = z + \pi^n \mathcal{O}_K$, where $z \in \overline{K}$ and $n \in \mathbb{Q}$. If $\mathfrak{s}$ is a cluster and $|\mathfrak{s}| > 1$, $\mathfrak{s}$ is a *proper* cluster and we define its *depth*

$$d_{\mathfrak{s}} = \min_{r, r' \in \mathfrak{s}} v_K(r - r').$$

The *cluster picture* $\Sigma = \Sigma_{C/K}$ is the set of all clusters of the roots of f.

Example 1.2.2. Let $C : y^2 = (x^3 - 13^6)((x - 1)^4 - 13^{16})(x^3 - 8)$ be a hyperelliptic curve over $\mathbb{Q}_{13}$. The cluster picture of C is given in Figure 1.1. The number to the bottom right of a cluster indicates its *relative depth*, the difference between its depth and its parent's (the smallest cluster containing it).

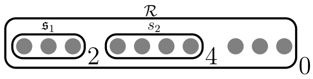


Figure 1.1: Cluster picture of $C : y^2 = (x^3 - 13^6)((x-1)^4 - 13^{16})(x^3 - 8)$.

The set of roots (appearing from left to right in the cluster picture) is

$$\mathcal{R} = \{13^2, 13^2\zeta, 13^2\zeta^2, 1 + 13^4, 1 - 13^4, 1 + 13^4 i, 1 - 13^4 i, 2, 2\zeta, 2\zeta^2\},$$

where ζ is a fixed third root of unity. The first cluster $\mathfrak{s}_1 = \mathcal{R} \cap D_{0,2}$ is the intersection of $\mathcal{R}$ with a p-adic disk centred around 0 of radius 2. The second cluster $\mathfrak{s}_2 = \mathcal{R} \cap D_{1,4}$. Finally, $\mathcal{R}$ itself is a cluster because $\mathcal{R} = \mathcal{R} \cap \mathbb{Z}_{13}$, and since $\mathbb{Z}_{13}$ contains $D_{0,2} = 13^2\mathbb{Z}_{13}$ and $D_{1,4} = 1 + 13^4\mathbb{Z}_{13}$, we draw the clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$ inside the cluster $\mathcal{R}$. Indeed $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are children of $\mathcal{R}$.

There are many advantages to working with cluster pictures. One of the most important is that they allow us to consider hyperelliptic curves in families, parametrised by cluster picture. Since there are only finitely many cluster pictures of a given genus, and the cluster picture determines many invariants of the curve, this allows us to exhaustively classify all the situations which can arise. A possible application would be to classify how many hyperelliptic curves of a given genus are locally soluble (using Theorem 5.1.3); this would require us to calculate the probability of a curve having a given cluster picture. The use of cluster pictures reduces the problem from one about proportions of curves to one about proportions of polynomials. A similar situation for genus 1 curves has been explored in papers such as [13] and [8].

In addition, cluster pictures allow us to give explicit and (relatively) succinct descriptions of invariants in terms of completely combinatorial data. This removes dependence on the exact form of the defining polynomial f, offering a more conceptual understanding of these invariants, as well as allowing us to compute them without the need to follow laborious algorithms. Many results utilising cluster pictures have been implemented computationally in Magma by Best and van Bommel; see [5] for details.

Ultimately, the motivation for calculating these invariants is due to their appearance in the *Birch–Swinnerton-Dyer conjecture* and the *Parity conjecture*. The former conjecture first links the *L-function* of a hyperelliptic curve to the algebraic rank of its Jacobian, and secondly gives a formula for the leading coefficient of the L-function in terms of various invariants of the hyperelliptic curve. Understanding the invariants could lead to a deeper understanding of the conjecture. See for example [45] and [46] where the author verifies the second conjecture numerically up to squares. The latter conjecture links the *global root number* of a hyperelliptic curve to the parity of the algebraic rank of its Jacobian. This is a strictly weaker conjecture, but still has remarkable implications; for example, it implies that any elliptic curve with global root number -1 has infinitely many rational points. This has been proven in various cases given the finiteness of the Tate-Shafarevich group, such as for elliptic curves in [17] and for principally polarised abelian varieties of dimension 2 in [20]. The second of these uses cluster pictures in an integral way.

The other class of curves whose models we will find using cluster pictures are *bihyperelliptic curves*. These are smooth curves with maps to two distinct hyperelliptic curves. They are similar to bielliptic curves, curves with a degree 2 map to an elliptic curve, although we are rather more restrictive in our definition. Our notion of a bihyperelliptic curve insists that it is a $C_2 \times C_2$ cover of $\mathbb{P}^1$, which is not the case for all bielliptic curves. Bihyperelliptic curves arise naturally in the study of the Parity conjecture; indeed let Y be a bihyperelliptic curve and $C : y^2 = f(x)$ and $C' : z^2 = g(x)$ the hyperelliptic curves it maps to, wiht f and g coprime. The Parity conjecture implies a relationship between the Tamagawa numbers of Y, C, C' and the curve C_h : $w^2 = f(x)g(x)$. This is because of the isogeny between the Jabocians of Y and $C \times C' \times C_h$, and the invariance of the BSD formula under isogeny (see [36, Theorem 7.3]). Hence being able to calculate the Tamagawa number of Y is important to comprehend this relationship. This application is currently being researched by Holly Green. We calculate the minimal regular model of Y in the case where Y has semistable reduction. A similar situation is studied in [10], where C and C' are assumed to be elliptic, but not hyperelliptic curves.

It is possible to construct a cluster picture on the set of roots of *both* hyperelliptic curves C and C'. This is not quite sufficient to describe the semistable model of Y; we must also remember the data of which root belongs to which hyperelliptic curve. This motivates our definition of a *chromatic cluster picture*, a cluster picture on the roots of C and C' where the roots coming from C are coloured red and the roots coming from C' are coloured blue. Note that Y is smooth if and only if C and C' have no roots in common, so there is no confusion about which colour to assign a given root. A colouring on the remaining clusters is induced by this condition. We will show that this data is enough to find the minimal regular model of Y, given that Y has semistable reduction.

1.3 Results of Thesis

The main aim of this thesis is to give explicit descriptions via cluster pictures of minimal models of two classes of curves: hyperelliptic curves with tame reduction (i.e., that obtain semistable reduction after a tame extension), and bihyperelliptic curves with semistable reduction. In the first case, we show that the cluster picture, along with the valuation of the leading coefficient of the defining equation, is sufficient to determine the *minimal snc model.*

Theorem 1.3.1 (Theorem 4.1.11)**.** *Let K be a local field with residue field k of characteristic $p > 2$. Let $C : y^2 = f(x)$ be a hyperelliptic curve over K with tame reduction and cluster picture Σ. Let $\mathcal{C}^{\min}$ be the minimal snc model of C over $\mathcal{O}_{K^{\mathrm{ur}}}$. Then the dual graph, with genus and multiplicity, of $\mathcal{C}_k^{\min}$ is completely determined by Σ (with depths) and the valuation of the leading coefficient $v_K(c_f)$ of f.*

Remark 1.3.2. We have defined the cluster picture via the defining equation

f of C. However, we could have defined abstract cluster pictures as a set of subsets of $\{1, \ldots n\}$ with a depth function $d : \Sigma \to \mathbb{Q}$. When we say above that the cluster picture determines the minimal snc model, we really mean the *abstract* cluster picture, i.e., the one that does not depend on the exact nature of the defining equation of C.

The proof is constructive; the structure of $\mathcal{C}_k^{\min}$ is detailed in Theorem 4.1.13 and exact details of the multiplicities of components are Theorem 4.1.19. The action of Frobenius on the components of $\mathcal{C}_k^{\min}$ is also determined by the cluster picture, as well as characters $\epsilon_{\mathfrak{s}}$ which are attached to the clusters of Σ: this is Theorem 4.1.21. As an application, we give a condition (Theorem 5.1.3) for C to have a K-rational point in terms of Σ, assuming that the characteristic of the residue field is large enough for the components of the special fibre to have points. When K does not have algebraically closed residue field, this crucially requires the action of Frobenius on $\mathcal{C}_k^{\min}$.

Corollary 1.3.3. *Let K be a local field with residue field k of characteristic $p > 2$ and let C be a hyperelliptic curve over K such that $p > 2g(C) + 1$. Then the cluster picture of C, along with the action of Frobenius induced on the cluster picture of C and the characters $\epsilon_{\mathfrak{s}}(\mathrm{Frob})$, determines the local solubility of C.*

Our results follow closely from [19], where the authors give an explicit description of the minimal regular model of a hyperelliptic curve with semistable reduction. Indeed, let $C : y^2 = f(x)$ be a hyperelliptic curve with semistable reduction and cluster picture Σ. The minimal regular model $\mathcal{C}$ of C can roughly be described as follows. A cluster $\mathfrak{s} \in \Sigma$ is *principal* if $|\mathfrak{s}| \geq 3$ so long as $\mathfrak{s} \neq \mathcal{R}$, in which case there are a few exceptions. A sufficient condition for $\mathcal{R}$ to be principal is for it to have at least 3 children, none of which have size $2g$. To each principal cluster $\mathfrak{s}$ there is one component $\Gamma_{\mathfrak{s}}$ or two components $\Gamma_{\mathfrak{s}}^+$ and $\Gamma_{\mathfrak{s}}^-$ in $\mathcal{C}_k$. The components $\Gamma_{\mathfrak{s}}$ and $\Gamma_{\mathfrak{s}'}$ of two clusters $\mathfrak{s}$ and $\mathfrak{s}'$ are linked by one or two chains of rational curves if $\mathfrak{s}'$ is a maximal subcluster of $\mathfrak{s}$ (a *child* of $\mathfrak{s}$), or vice versa. *Twins*, clusters of size 2, contribute *loops* to the model: chains of rational curves from the component of their parents to itself.

The action of Frobenius on $\mathcal{C}_k^{\min}$ is also described. The cluster picture inherits a natural action of Frobenius, and the principal components of $\mathcal{C}_k$ are permuted as their corresponding clusters are. There are additional characters $\epsilon_{\mathfrak{s}}$ associated to clusters which determine the action of Frobenius when there are two components associated to a cluster, and the action on the clusters themselves is not sufficient.

Now let $C : y^2 = f(x)$ be a hyperelliptic curve with *tame reduction*; that is, C obtains semistable reduction over a finite extension L/K whose degree is prime to p. We describe the dual graph with genus and multiplicity of the *minimal snc model* $\mathcal{C}^{\min}$ of C in terms of the cluster picture of C. An *snc model* is a model which is regular as a scheme and whose special fibre has smooth components and at worst ordinary double points as singularities.

This is morally similar to the semistable case, but practically somewhat more complicated, as components do not have to be reduced. One particular difference is that we now consider Galois *orbits* of clusters, not just clusters on their own. This makes sense, as inertia acts trivially on the cluster picture of a semistable hyperelliptic curve (see Theorem 2.4.2), but can act non-trivially on the cluster pictures of one with tame reduction.

Orbits of principal clusters give rise to one or two components, and parents are still linked to their children by chains of rational curves. However, there are additional chains of rational curves, *tails*, which intersect the rest of the special fibre in only one place. Orbits of twins can contribute loops or *crossed tails*, whose definition we delay until later. The action of Frobenius is also more involved, but is determined by the cluster picture and the characters $\epsilon_{\mathfrak{s}}$.

For bihyperelliptic curves with semistable reduction, we show that the chromatic cluster picture is sufficient to determine the *minimal regular model.*

Theorem 1.3.4 (Theorem 6.3.1)**.** *Let K be a local field with residue field k of characteristic $p > 2$. Let C_1 and C_2 be two hyperelliptic curves over K.*

Suppose that the bihyperelliptic curve Y arising from C_1 and C_2 has semistable reduction and all the depths in the chromatic cluster picture of Y are integers. Let $\mathscr{Y}^{\min}$ be the minimal regular model of Y. Then the dual graph of $\mathscr{Y}_k^{\min}$ is entirely determined by the chromatic cluster picture of Y.

A description of the dual graph is given in Theorem 6.3.1 and the action of Frobenius in Theorem 6.3.3. These are again similar to above: principal clusters give rise to components of the special fibre, components of parents are linked to parents of children and twins have corresponding loops. Principal components are permuted by Frobenius in the same way as their associated principal clusters, and characters $\epsilon_{\mathfrak{s},1}$ and $\epsilon_{\mathfrak{s},2}$ for $\mathfrak{s}$ a cluster determine the rest of the action of Frobenius.

A possible application of this concerns the Parity Conjecture and in particular the behaviour of Tamagawa numbers in towers of curves. The minimal regular model of Y with Frobenius action allows us to calculate its Tamagawa number c_Y hence and compare c_Y to the Tamagawa numbers of the hyperelliptic curves Y maps to. This application is inspired by work such as [7] and [16]. However, it is beyond the scope of this thesis so we shall dwell on it no further.

1.4 Structure of Thesis

Chapter 2 covers the background material that we will use throughout the thesis, starting with standard definitions and results about models in Section 2.1. We expound on cluster pictures in Section 2.2, illustrating definitions with examples and giving intuition behind some of the most important invariants associated to cluster pictures. The remainder of the background section discusses techniques from the literature for calculating models: using tame quotients in Section 2.3, cluster pictures for semistable hyperelliptic curves in Section 2.4 and finally via Newton polygons in Section 2.5.

The results of [22] concerning minimal snc models of hyperelliptic curves with tame reduction are detailed in Chapters 3 and 4. The first proves the result for the simplest examples: hyperelliptic curves with tame potentially good reduction in Section 3.1 and curves whose cluster pictures have exactly two proper clusters in Section 3.2. The main results, a description of the special fibre of the minimal snc model of a hyperelliptic curve with tame reduction, are Theorems 4.1.13 and 4.1.19. These are proven in Section 4.2. Their application, a condition for a hyperelliptic curve to be locally soluble over K in terms of its cluster picture, is Chapter 5.

The thesis finishes with a chapter on bihyperelliptic curves, Chapter 6. In Theorem 6.3.1 we give the dual graph of the minimal regular model of a bihyperelliptic curve with semistable reduction in terms of its chromatic cluster picture.

1.5 Notation

K	local field	v_K	discrete valuation
$\mathcal{O}_K$	ring of integers	π_K	uniformiser of K
k	residue field of K	p	characteristic of k
$\overline{K}$	algebraic closure of K	K^{ur}	maximal unramified extension of K

Table 1.1: General notation associated to fields, curves, and models

$\Sigma_{C/K}$	(2.2.3)	$\delta(\mathfrak{s},\mathfrak{s}')$	(2.2.14)	$\mathrm{red}_\mathfrak{s}$	(2.4.9)
$\mathfrak{s}$	(2.2.3)	principal	(2.2.8)	$\Delta(C)$	(2.5.3)
$d_\mathfrak{s}$	(2.2.3)	$\mathfrak{s}^*$	(2.2.16)	$\Delta_v(C)$	(2.5.3)
$a_\mathfrak{s}, b_\mathfrak{s}$	(2.2.3)	$g_{\mathrm{ss}}(\mathfrak{s})$	(2.2.17)	v_Δ	(2.5.3)
odd cluster	(2.2.8)	singleton	(2.2.21)	L, F	(2.5.4)
even cluster	(2.2.8)	$\mathfrak{s}_{\mathrm{sing}}$	(2.2.21)	$\Delta(\mathbb{Z}), L(\mathbb{Z}), F(\mathbb{Z})$	(2.5.5)
twin	(2.2.8)	$\nu_\mathfrak{s}$	(2.4.6)	$\overline{\Delta}(\mathbb{Z})\overline{L}(\mathbb{Z}), \overline{F}(\mathbb{Z})$	(2.5.5)
$\mathfrak{s}' < \mathfrak{s}$	(2.2.13)	χ	(2.4.6)	δ_λ	(2.5.6)
$P(\mathfrak{s})$	(2.2.13)	$\lambda_\mathfrak{s}$	(2.4.6)	s_1^L, s_2^L	(2.5.10)
$\widehat{\mathfrak{s}}, \widetilde{\mathfrak{s}}$	(2.2.13)	$\alpha_\mathfrak{s}$	(2.4.6)	$g(\mathfrak{s})$	(3.1.23)

cotwin	(2.2.13)	$\beta_{\mathfrak{s}}$	(2.4.6)	principal orbit	(4.1.1)
übereven	(2.2.13)	$\gamma_{\mathfrak{s}}$	(2.4.6)	λ_X	(4.1.4)
$z_{\mathfrak{s}}$	(2.2.3)	$\theta_{\mathfrak{s}}$	(2.4.6)	K_X	(4.1.2)
$\mathfrak{s} \wedge \mathfrak{s}'$	(2.2.13)	$\epsilon_{\mathfrak{s}}$	(2.4.6)	e_X	(4.1.9)
$\delta_{\mathfrak{s}}$	(2.2.14)	$c_{\mathfrak{s}}$	(2.4.9)	$g(X)$	(4.1.9)

Table 1.2: Notation associated to cluster pictures and Newton polytopes

Chapter 2

Preliminaries

In this chapter we detail some of the background that is necessary for our results. We begin with section 2.1 on models and different types of "nice" models. This is all very standard. After that is a summary of cluster pictures in section 2.2. This is the most important part of the background, as it is a rather novel approach and may be unfamiliar; we shall use cluster pictures extensively. We continue with various strategies to calculate models: via quotients, cluster pictures and Newton polytopes in sections 2.3, 2.4 and 2.5 respectively. Nothing in this chapter is new, but theorems and definitions have been illustrated with examples to demonstrate their importance. Recall that K is a local field with residue field k of characteristic $p > 2$.

2.1 Models

In Section 2.1.1 there is an initial discussion of models and how they relate to the naïve approach of reduction mod ν. We follow this by defining certain "nice" models which are particularly useful: (minimal) regular models in 2.1.2 and semistable models in 2.1.3. A canonical reference for this section is [29], especially Sections 9 and 10. Throughout this section we assume that K has algebraically closed residue field, unless explicitly stated otherwise.

2.1.1 First Definitions

Any good notion of a model of X should remember the original curve X, and allow us to reduce points of X modulo $\mathfrak{m}$. In addition, there should be some overarching structure which allows us to utilise the full power of algebraic geometry. Both of these goals are achieved using $\mathcal{O}_K$-schemes. Since $\mathrm{Spec}(\mathcal{O}_K)$ has two points, the zero ideal and the maximal ideal, any $\mathcal{O}_K$-scheme $\mathscr{X}$ is composed of two fibres — the *generic fibre* $\mathscr{X}_K$, which is a curve over K, and the *special fibre* $\mathscr{X}_k$, which is a curve over k, and will be our reduction modulo $\mathfrak{m}$. We will demand that the former is isomorphic to X, but the latter can, within reason, be whatever we want it to be. Without further ado:

Definition 2.1.1 (Models)**.** Let X/K be a curve. A model $\mathscr{X}$ of X is flat, proper $\mathcal{O}_K$-scheme with generic fibre $\mathscr{X}_K \cong X$. Morphisms of models are morphisms of the underlying schemes which induce an isomorphism on the generic fibres.

Example 2.1.2. The most straightforward way to form a model is to consider a curve over $\mathcal{O}_K$ defined by a single equation. Let $E : y^2 = x^3 + ax + b$ be an elliptic curve over $K = \mathbb{Q}_p^{\mathrm{ur}}$ in Weierstrass form and let $\mathcal{E} = E \times_K \mathcal{O}_K$. Then $\mathcal{E}$ is a model of E. The special fibre of $\mathcal{E}$ is the reduction of E modulo p. Such a model is called a *Weierstrass model.* Now suppose in fact that $E : y^2 = x^3 + p$. We can repeatedly blow up the singular point on the special fibre, starting with $(0, 0)$ on the special fibre of the Weierstrass model, obtaining a sequence of models of E whose special fibres are shown below. This example shows that models are not required to be reduced or irreducible. It also illustrates one of the most effective tools we have to construct new models: blowing up points on the special fibre.

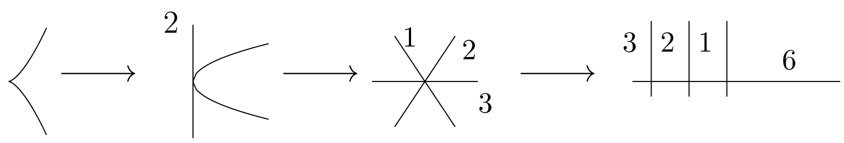


Figure 2.1: Models of the elliptic curve $y^2 = x^3 + p$.

Elliptic curves can have good or bad reduction: there are analogues for models of general curves.

Definition 2.1.3 (Good Reduction)**.** Let X/K be a curve. Then X has *good reduction* over K if there exists a smooth model $\mathscr{X}$ of X (i.e., a model $\mathscr{X}$ such that the underlying scheme is smooth). If no such model exists, X has *bad reduction.* If there exists a finite extension L/K such that $X \times_K L$ has good reduction over L, then X has *potentially good reduction.* If the extension L/K can be chosen such that $[L : K]$ is coprime to p, then X has *tame potentially good reduction*, and it has *wild potentially good reduction otherwise.*

Remark 2.1.4. Throughout this thesis, we will often consider a model $\mathscr{X}$ of a curve X over a field K which does *not* have algebraically closed residue field. In this case, what we mean is the model of X over K^{ur} along with the action of Frobenius on the special fibre $\mathscr{X}_k$; i.e. the action of $\mathrm{Gal}(K^{\mathrm{ur}}/K)$. This is analogous to the difference between split and non-split multiplicative reduction for elliptic curves. For both types of multiplicative reduction, the special fibre of the stable model is a rational curve with a node. However, an elliptic curve E has *split* multiplicative reduction if the tangent lines are defined over K, and *non-split* reduction otherwise. In the latter situation, $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ acts non-trivially on the special fibre, and when we talk about the stable model of E we are also referring to this action.

2.1.2 Regular Models

Weierstrass models, while intuitive, do not provide any information beyond simply reducing modulo $\mathfrak{m}$. We must impose some additional structure on our models in order to utilise their full power. In particular, we frequently demand a model $\mathscr{X}$ be *regular*, i.e. that the underlying scheme be regular. Such a model determines, for example, the Tamagawa number of an elliptic curve — see [42, p 365]. Indeed, if K has algebraically closed residue field, the Tamagawa number is simply the number of multiplicity 1 components in the special fibre of the *minimal regular model*, a canonical choice of regular model.

Definition 2.1.5 (Regular Models)**.** Let X/K be a curve and $\mathscr{X}$ a model of X. Then $\mathscr{X}$ is said to be a regular model (resp. a normal model resp.

a smooth model) if the underlying scheme $\mathscr{X}$ is regular (resp. normal resp. smooth). A regular model $\mathscr{X}$ is *minimal* if any map $\mathscr{X} \to \mathscr{X}'$ to another regular model $\mathscr{X}'$ is an isomorphism.

One of the main strengths of regular models is that we can define an *intersection theory* on $\mathscr{X}$, an incredibly effective tool whose usefulness we shall already see in Theorem 2.1.7. In particular, let $\mathrm{Div}(\mathscr{X})$ be the (Weil) divisor group of $\mathscr{X}$, the free group on codimension one closed integral subschemes of $\mathscr{X}$, and let $\mathrm{Div}_k(\mathscr{X})$ be the subgroup of *vertical divisors*: divisors which are contained in the special fibre $\mathscr{X}_k$. Then there is a well defined local intersection product for any $x \in \mathscr{X}_k$ given by:

$$\begin{aligned} \mathrm{Div}_k(\mathscr{X}) \times \mathrm{Div}(\mathscr{X}) &\longrightarrow \mathbb{Z} \\ (E \cdot E') &= \dim_k \mathcal{O}_{\mathscr{X},x}/(g, g'), \end{aligned}$$

where g and g' are uniformisers for E and E' respectively (informally, functions which vanish to order 1 along E and E' respectively). Roughly, if $E \neq E'$, then $(E \cdot E')$ counts the number of intersections of E and E' with multiplicity. If E and E' intersect everywhere transversally (e.g. in an snc or semistable model), this simply counts the number of intersection points of E and E'.

We have discussed how blowing up allows us to create new models from old models, and blowing up a point on the special fibre of a regular model results in another regular model. Intersection theory allows us to classify when we can go in the other direction.

Definition 2.1.6 (Exceptional Components)**.** Let X be a curve over K and $\mathscr{X}$ a regular model of X. Let $E \in \mathscr{X}_k$ be a component of the special fibre. Suppose there exists another regular model $\mathscr{X}'$ and a morphism $\phi : \mathscr{X} \to \mathscr{X}'$ such that $\phi(E)$ is a point and ϕ is an isomorphism away from E and $f(E)$. Then E is an *exceptional divisor* and ϕ is a *contraction* or *blow down* of E.

Theorem 2.1.7 (Castelnuovo's Criterion)**.** *Let X be a curve over K and $\mathscr{X}$ a regular model of X. Let $E \in \mathscr{X}_k$ be a component of the special fibre. Then E is exceptional if and only if it is isomorphic to $\mathbb{P}^1_k$ and $(E \cdot E) = -1$.*

Proof. [29, 9.3.8] □

This criterion implies the following powerful theorem concerning minimal regular models: namely that they exist, and are unique.

Theorem 2.1.8 (Minimal Regular Models)**.** *Let X/K be a smooth curve of genus $g \geq 1$. Then X admits a minimal regular model over $\mathcal{O}_K$, which is unique up to unique isomorphism.*

Proof. [29, Theorem 9.3.21]. Roughly, a proof proceeds as follow. A regular model exists, since we can take any model of X, and repeatedly normalise and blow up the singularities until the resulting scheme is regular. Such a process terminates by [27]. Then, blowing down any exceptional components, we recover a model with no exceptional components. This is the minimal regular model. □

Remark 2.1.9. In theory, the proof of Theorem 2.1.8 provides an explicit way of calculating the minimal regular model of a given curve. In practice, it is an incredibly time consuming process (as anyone who has ever had to do a blow up by hand can attest), even for a computer. As a result, mathematicians have strived to find more practical ways of finding the minimal regular model of a given curve. This is the motivation for much of this thesis.

2.1.3 Semistable Models

Even on a regular model, the singularities on the special fibre can be rather unpleasant. For example, the minimal regular model of the elliptic curve $y^2 = x^3 + p$ has a cusp on its special fibre. In some settings this is less than ideal. As a result we often demand that the singularities on the special fibre are, in a sense, "as pleasant as possible". Unfortunately, we cannot always demand a smooth model, as not all curves have even potentially good reduction. The mildest possible singularities are *ordinary double points* or *normal crossings*, points which (étale) locally look like the intersection of two coordinate axes. We will define two classes of models whose special fibres only have these as singularities: *snc models*, where the special fibre is allowed to be non-reduced, and *semistable models*, where the special fibre is reduced. Such special fibres have a combinatorial description, the *dual graph* (Definition 2.1.17), which is a very effective tool for studying the underlying models.

Definition 2.1.10 (Snc Models)**.** Let X/K be a curve and $\mathscr{X}$ a model of X. We say $\mathscr{X}$ is an *snc model* if $\mathscr{X}_k$ is an snc divisor: a curve over k whose components are smooth and whose worst singularities are normal crossings. The model $\mathscr{X}$ is a *minimal* snc model if any map $\mathscr{X} \to \mathscr{X}'$ to another snc model $\mathscr{X}'$ is an isomorphism.

Remark 2.1.11. There does not seem to be a widespread consensus in the literature regarding the difference between *normal crossing* (nc) and *strict normal crossing* (snc) models. However, a common convention is to allow the former to have non-smooth components, but not the latter, which is the convention we adopt in this thesis.

Theorem 2.1.12 (Minimal snc Models)**.** *Let X/K be a smooth curve of genus $g \geq 1$. Then X admits a minimal regular snc model over $\mathcal{O}_K$, which is unique up to unique isomorphism.*

Proof. [29, Theorem 9.3.36]. □

Example 2.1.13. In figure 2.2 are the special fibres of the minimal regular and minimal snc models of an elliptic curve of Kodaira Type IV, given for example by Weierstrass equation $y^2 = x^3 - p^2$. Note that they are not the same! The multiplicity 3 component of the minimal snc model is isomorphic to $\mathbb{P}^1$ and has self intersection -1, and so can be blown down by Castelnuovo's Criterion 2.1.7. The result of this blow down is the minimal regular model, which is not snc as its singularity is the intersection point of three curves, not an ordinary double point.

The most important type of snc model is the *semistable model*. Curves with a semistable model are analogous to elliptic curves with multiplicative

(a) Special fibre of the minimal snc model.

(b) Special fibre of the minimal regular model.

Figure 2.2: Elliptic curve of Kodaira Type IV.

reduction, and indeed, an elliptic curve has semistable reduction if and only if it has multiplicative or good reduction. Curves with semistable reduction give rise to particularly nice Galois representations; hence they are ubiquitous in number theory.

Definition 2.1.14 (Semistable Model)**.** An snc model $\mathscr{X}$ of a curve X/K is a *semistable* model if $\mathscr{X}_k$ is reduced and each component of $\mathscr{X}_k$ isomorphic to $\mathbb{P}^1$ intersects the rest of the special fibre in at least 2 points. A curve which admits a semistable model has *semistable reduction.*

A famous theorem of Deligne and Mumford states that after a finite extension, every curve has semistable reduction.

Theorem 2.1.15 (Semistable Reduction Theorem)**.** *Let X/K be a curve. Then there exists a finite, separable extension L/K such that $X_L = X \times_K L$ has semistable reduction.*

Proof. Originally [14]. See also [29, Section 10.4], or [2] for a proof conceptually more similar to the techniques used in this thesis. □

Of particular interest are curves where the extension required for semistability is a tame extension of K.

Definition 2.1.16 (Tame Reduction)**.** Let X/K be a curve and let us suppose that L/K is minimal such that X achieves semistable reduction over L. Then X has *tame reduction* if $[L : K]$ is coprime to p, and *wild reduction* otherwise.

2.1.4 Structure of Semistable Models

Since the singularities of semistable models are all ordinary double points, such models have a nice combinatorial description: the *dual graph*.

Definition 2.1.17 (Dual Graph)**.** Let X/K be a curve with semistable reduction, and let $\mathscr{X}$ be a semistable model of X. We can associate a genus graph Υ to $\mathscr{X}_k$ as follows: to each component Γ there is a vertex $v \in \Upsilon$ of genus $g(\Gamma)$ and two vertices v_i, v_j are linked by $(\Gamma_i \cdot \Gamma_j)$ edges.

Definition 2.1.18 (Thickness)**.** Let X/K be a curve with semistable model $\mathscr{X}$, and let $x \in \mathscr{X}_k$ be a singularity of $\mathscr{X}$. Then locally x is of the form $\mathcal{O}_K[u,v]/(uv-c)$ with $v_K(c) = e_x$. We say that the point x has *thickness* e_x. See [29, Corollary 10.3.22, Definition 10.3.23].

Remark 2.1.19.
(i) There is the equality of genera: $g(X) = g(\Upsilon) + \sum_v g(v)$.

(ii) We can also make Υ into a metric graph where the length of an edge is the thickness of the intersection it represents.

(iii) The graph Υ with this metric is an augmented $\mathbb{Z}$-graph in the sense of [1]. More on this in Section 6.

Lemma 2.1.20. *Let X be a curve over K with semistable reduction and let $\mathscr{X}$ be a semistable model of C. Let L/K be a totally ramified extension of degree e. Then $\mathscr{X} \times_{\mathcal{O}_K} \mathcal{O}_L$ is a semistable model of $X \times_K L$ with the same dual graph as $\mathscr{X}$, but with the lengths of all edges multiplied by e.*

Proof. This follows from [29, Corollaries 10.3.36,10.3.25], noting that the thickness of all double points multiplies by e after extending the field. □

A semistable model has the following structure: there are *principal components* which are linked by chains of rational curves. An snc but not semistable model can in addition have *tails*, which are chains of rational curves intersecting the rest of the special fibre in precisely one point.

Definition 2.1.21 (Principal Components)**.** Let X be a curve over K. Let $\mathscr{X}$ be an snc model of X. Then a component $E \in \mathscr{X}_k$ is *principal* if it is of positive genus, or if it contains three or more singular points of $\mathscr{X}_k$.

Definition 2.1.22 (Linking Chain)**.** Let $\mathscr{X}$ be an snc model of a curve over K. Suppose $E_1, \ldots, E_\lambda$ are smooth irreducible rational components of $\mathscr{X}_k$. A divisor $\mathcal{D} = \bigcup_{i=1}^{\lambda} E_i$ is a *chain of rational curves* if

(i) $(E_i \cdot E_{i+1}) = 1$ for all $1 \leq i < \lambda$ and $(E_i \cdot E_j) = 0$ for $j \neq i \pm 1$,

(ii) $(E_1 \cdot \overline{\mathscr{X}_k \setminus \mathcal{D}}) = 1$,

(iii) $(E_i \cdot \overline{\mathscr{X}_k \setminus \mathcal{D}}) = 0$ for $i \neq 1, \lambda$,

where $(E \cdot F)$ is the usual intersection pairing defined on regular models. If $(E_\lambda \cdot \overline{\mathscr{X}_k \setminus \mathcal{D}}) = 0$ then $\mathcal{D}$ is a *tail*. If $(E_\lambda \cdot \overline{\mathscr{X}_k \setminus \mathcal{D}}) = 1$ then $\mathcal{D}$ is a *linking chain*.

We say a chain of rational curves $\mathcal{D} = \bigcup_{i=1}^{\lambda} E_i$ is a *loop* if $\mathcal{D}$ is a linking chain such that E_1 and E_λ both intersect the same component of $\overline{\mathscr{X}_k \setminus \mathcal{D}}$.

Furthermore, if $(E_\lambda \cdot \overline{\mathscr{X}_k \setminus \mathcal{D}}) = 2$ then $\mathcal{D}$ is a *crossed tail* if E_λ intersects two rational components of $\mathscr{X}_k \setminus \mathcal{D}$, say $E_{\lambda+1}^{+}$ and $E_{\lambda+1}^{-}$, such that $(E_{\lambda+1}^{\pm} \cdot E_\lambda) = 1$ and $(E_{\lambda+1}^{\pm} \cdot \overline{\mathscr{X}_k \setminus E_\lambda}) = 0$. We call the components $E_{\lambda+1}^{\pm}$ the *crosses*.

2.2 Cluster Pictures

This section is an exposition on the key combinatorial object which will be used to describe the various models, the *cluster picture*. We begin in section 2.2.1 by giving initial definitions and examples of cluster pictures, and in section 2.2.2 discuss in brief why cluster pictures are so effective in determining the local arithmetic of C by describing a canonical model of $\mathbb{P}^1$ attached to a cluster picture. Section 2.2.3 defines many of the properties of cluster pictures, and illustrates their utility with examples. We finish in 2.2.4 by relating cluster pictures to Berkovich theory, placing them in a more conceptual framework.

2.2.1 Disks and Clusters

Let $\mathcal{R}$ denote the set of roots of f. The cluster picture of f is a set of subsets (*clusters*) of $\mathcal{R}$ such that a cluster contains elements of $\mathcal{R}$ which are p-adically close together. Roots are p-adically close together if they are contained in a p-adic disk of small radius.

Definition 2.2.1 (p-adic Disk)**.** A *p-adic disk* is a subset of the form

$$D = D_{z,d} = \{x \in \overline{K} \mid v_K(x-z) \geq d\},$$

for some $z \in \overline{K}$ a *centre* and $d \in \mathbb{R}$ the *depth*. Such a disk is *integral* if it has a centre in K^{ur} and $d \in \mathbb{Z}$.

Note that we use valuations and depths rather than absolute values and radii, so in our terminology a p-adic disk of small radius is a p-adic disk of large depth. To each p-adic disk there is an associated valuation (see e.g. [4, Section 1.4.4]).

Definition 2.2.2 (Valuation of a Disk)**.** Let $D_{z,d}$ be a p-adic disk. There is an associated p-adic valuation of $\overline{K}(x)$ extending v_K defined by

$$v_D(g) = \inf_{t \in D} v_K(g(t)),$$

or equivalently, writing $g(x) = \sum_i c_i (x-z)^i$,

$$v_D(g) = \min_i (c_i + di).$$

This allows us to define cluster pictures.

Definition 2.2.3 (Clusters)**.** Let $C : y^2 = f(x)$ be a hyperelliptic curve over K and let $\mathcal{R}$ be the set of roots of f. A *cluster* is a non-empty subset $\mathfrak{s} \subseteq \mathcal{R}$ of the form $\mathfrak{s} = D \cap \mathcal{R}$ for some p-adic disc $D = D_{z,d}$. Any such $z = z_{\mathfrak{s}}$ is a *centre* of $\mathfrak{s}$. If $\mathfrak{s}$ is a cluster and $|\mathfrak{s}| > 1$, we say that $\mathfrak{s}$ is a *proper cluster*. For a proper cluster $\mathfrak{s}$ we define its *depth* $d_{\mathfrak{s}}$ to be

$$d_{\mathfrak{s}} = \min_{r,r' \in \mathfrak{s}} v_K(r - r').$$

It is the minimal d for which $\mathfrak{s}$ is cut out by such a disk. We write $D(\mathfrak{s})$ for the disk of depth $d_{\mathfrak{s}}$ cutting out $\mathfrak{s}$ and $d_{\mathfrak{s}} = \frac{a_{\mathfrak{s}}}{b_{\mathfrak{s}}}$ with $a_{\mathfrak{s}}, b_{\mathfrak{s}}$ coprime. The *cluster picture* $\Sigma = \Sigma_{C/K}$ of C is the collection of all clusters of the roots of f. The cluster picture Σ inherits a natural action of $\mathrm{Gal}(\overline{K}/K)$ in the obvious way.

Remark 2.2.4. It is possible to define cluster pictures completely combinatorially, without reference to a hyperelliptic curve: an (abstract) cluster picture on n elements is a subset Σ of the power set $\mathcal{P}(\mathcal{R})$ of $\mathcal{R} = \{1, \dots, n\}$. It is then possible to attach a depth function $d : \Sigma \to \mathbb{Q}$. Isomorphisms of cluster pictures are then defined in the obvious way. This point of view can be very illuminating, but we won't require it in this thesis.

The most important relationship is that between children and their parents.

Definition 2.2.5 (Children)**.** Let Σ be a cluster picture and $\mathfrak{s} \in \Sigma$ is a cluster. Suppose $\mathfrak{s}' \subsetneq \mathfrak{s}$ is a maximal subcluster of $\mathfrak{s}$. Then we say that $\mathfrak{s}'$ is a *child* of $\mathfrak{s}$ and $\mathfrak{s}$ is a *parent* of $\mathfrak{s}'$, written $\mathfrak{s}' < \mathfrak{s}$ and $\mathfrak{s} = P(\mathfrak{s}')$ respectively.

Example 2.2.6. Let $C : y^2 = ((x-i)^3 - 7^6)((x+i)^3 - 7^6)$ be a hyperelliptic curve over $\mathbb{Q}_7$. Its cluster pictures is given below.

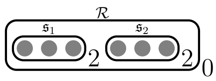


Figure 2.3: Cluster picture of $C : y^2 = ((x-i)^3 - 7^6)((x+i)^3 - 7^6)$.

The set of roots (appearing from left to right in the cluster picture) is

$$\mathcal{R} = \{i + 7^2, i + 7^2\zeta, i + 7^2\zeta^2, i - 7^2, i - 7^2\zeta, i - 7^2\zeta^2\},$$

where ζ is a fixed third root of unity. The important thing to note about this example is that the proper clusters $\mathfrak{s}_1 = \mathcal{R} \cap D_{i,2}$ and $\mathfrak{s}_2 = \mathcal{R} \cap D_{-i,2}$ are permuted by Frobenius since their centres are not in $\mathbb{Q}_7$.

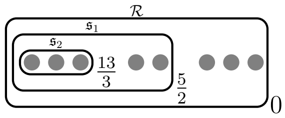


Figure 2.4: Cluster picture of $C : y^2 = (x^3 - p^{13})(x^2 - p^5)(x^3 - 1)$.

Example 2.2.7. Let $C : y^2 = (x^3 - p^{13})(x^2 - p^5)(x^3 - 1)$ be a hyperelliptic curve over $\mathbb{Q}_p$ (where p is such that $\mathbb{Q}_p$ has cube roots of unity). The cluster picture of C is given below.

The set of roots (appearing from left to right in the cluster picture) is

$$\mathcal{R} = \{p^{\frac{13}{3}}, \zeta p^{\frac{13}{3}}, \zeta^2 p^{\frac{13}{3}}, p^{\frac{5}{2}}, -p^{\frac{5}{2}}, 1, \zeta, \zeta^2\},$$

where ζ is a fixed third root of unity. We have that $\mathfrak{s}_1 = \mathcal{R} \cap D_{0,5/2}$, $\mathfrak{s}_2 = \mathcal{R} \cap D_{0,13/3}$ and $\mathcal{R} = \mathcal{R} \cap \mathbb{Z}_p$. The cluster $\mathfrak{s}_2$ is a child of $\mathfrak{s}_1$ but not of $\mathcal{R}$, and $\mathfrak{s}_1$ is a child of $\mathcal{R}$. Inertia permutes the roots in $\mathfrak{s}_2$, and the roots in $\mathfrak{s}_1$, but fixes all proper clusters.

2.2.2 Models of $\mathbb{P}^1$

If our cluster picture Σ is sufficiently nice (see Theorem 2.4.2), we can use it to construct a regular, semistable model $\mathscr{X}^\Sigma$ of $\mathbb{P}^1$ which *separates the points* of $\mathcal{R}$ — in other words, any $r \in \mathcal{R}$ reduces to a unique point of $\mathscr{X}^\Sigma_k$ under the specialisation map $\mathscr{X}^\Sigma_K \to \mathscr{X}^\Sigma_k$. Herein lies the power of cluster pictures; such models behave particularly nicely. For example if we normalise $\mathscr{X}^\Sigma$ in the function field of C, by [43, Lemma 2.1] we obtain a regular model $\mathcal{C}$ of C, and possible after an extension of degree 2, we can guarantee that $\mathcal{C}$ is a regular, semistable of C, and the cover $C \to \mathbb{P}^1$ extends to a map of models $\mathcal{C} \to \mathscr{X}^\Sigma$. This is [31, Theorem 2.3]. This normalisation approach is what is used in [19], and we shall use similar ideas in Section 6 to find the minimal regular model of a bihyperelliptic curve with semistable reduction.

The construction in full can be found at [19, Section 3]. The idea is as follows: we use Σ to construct an *admissible set of integral disks*, a finite collection $\mathcal{D}$ of integral p-adic disks such that $\mathcal{D}$ has a maximal element $D_{\max}$

with respect to inclusion, and if $D \subseteq D' \in \mathcal{D}$, then any integral disk D'' such that $D \subseteq D'' \subseteq D'$ is also in $\mathcal{D}$. Usually, we take $D_{\max} = D_{0,0} = \mathcal{O}_{\overline{K}}$, the unit ball in $\overline{K}$. In particular, we take the minimal admissible collection of disks which contains all of the disks of Σ.

We then construct a model of $\mathbb{P}^1$ inductively. Starting with $\mathbb{P}^1_{\mathcal{O}_K}$, we blow up on the special fibre at any point $\overline{z_D} \in \mathbb{P}^1_k$ such that $z_D \in K$ is a centre of a maximal subdisk $D \subseteq D_{\max}$. After this first step, there is a component Γ_D in the special fibre for any maximal subdisk $D \subseteq D_{\max}$. Now for any such disk D, we blow up on Γ_D at any point corresponding to the centres of its maximal subdisks. Continuing in this way, we obtain a model of $\mathbb{P}^1$ whose components are parametrised by the disks of $\mathcal{D}$.

2.2.3 Properties of Clusters

Clusters possess many properties which are used to determine the local arithmetic of curves. For example, when trying to determine the minimal regular model $\mathcal{C}$ of a hyperelliptic curve C with semistable reduction, these properties tell us which clusters give rise to components in $\mathcal{C}_k$, how many components, their genera and between which components there are linking chains.

Definition 2.2.8. A cluster $\mathfrak{s}$ is *even* (resp. *odd*) if $|\mathfrak{s}|$ is even (resp. odd). Furthermore $\mathfrak{s}$ is a *twin* if $|\mathfrak{s}| = 2$. A cluster $\mathfrak{s}$ is *principal* if $|\mathfrak{s}| \geq 3$ except if either $\mathfrak{s} = \mathcal{R}$ is even and has exactly two children, or if $\mathfrak{s}$ has a child of size $2g$.

Remark 2.2.9. Principal clusters are the most important class of clusters. If C is a hyperelliptic curve with semistable reduction, these components lift to give us the principal components of the minimal regular model $\mathcal{C}$ of C. The proper clusters which are not principal – twins, and in some cases $\mathcal{R}$ – lift to rational curves with two intersection points in $\mathcal{C}$ (and in the case of twins of half integer depth — don't have a corresponding component at all in $\mathcal{C}$).

Example 2.2.10. Let $C : y^2 = (x^2 - p^6)(x^4 - 1)$ be a hyperelliptic curve over $\mathbb{Q}_p$. This is a genus 2 curve with reduction type $\mathrm{I}_{6,0,0}$ in the terminology of [38]. The figure below shows, from left to right, the cluster picture Σ of C, the minimal model of $\mathbb{P}^1$ which separates the points of $\mathcal{R}$ and the minimal regular model $\mathcal{C}$ of C. The cluster $\mathfrak{s}$ is even and not principal (it is a twin), whereas the cluster $\mathcal{R}$ is even and principal. We observe that the component arising from $\mathfrak{s}$ is not principal, it is isomorphic to $\mathbb{P}^1$ and only has two intersection points with the rest of the special fibre.

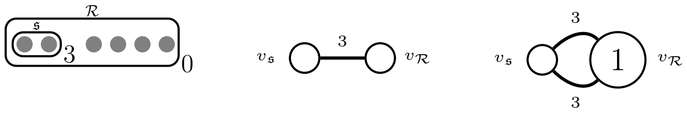


Figure 2.5: From left to right: the cluster picture of $C : y^2 = (x^2 - p^6)(x^4 - 1)$, the model of $\mathbb{P}^1$ separating the points of $\mathcal{R}$ and the minimal regular model of C.

Example 2.2.11. Let $C : y^2 = (x^3 - p^{18})(x^3 - 1)$ be a hyperelliptic curve over $\mathbb{Q}_p$. This is a genus 2 curve with reduction type $\mathrm{I}_0 - \mathrm{I}_0 - 3$ in the terminology of [38]. The figure below shows, from left to right, the cluster picture Σ of C, the minimal model of $\mathbb{P}^1$ which separates the points of $\mathcal{R}$ and the minimal regular model $\mathcal{C}$ of C. We observe that $\mathfrak{s}$ is odd and $\mathcal{R}$ is even. Both $\mathfrak{s}$ and $\mathcal{R}$ are principal — and indeed, they give components of positive genus, $v_{\mathfrak{s}}$ and $v_{\mathcal{R}}$, in $\mathcal{C}$.

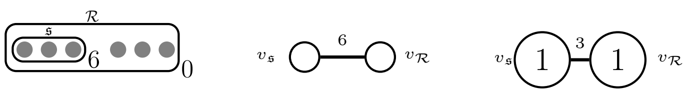


Figure 2.6: From left to right: the cluster picture of $C : y^2 = (x^3 - p^{18})(x^3 - 1)$, the model of $\mathbb{P}^1$ separating the points of $\mathcal{R}$ and the minimal regular model of C.

Example 2.2.12. Let $C : y^2 = (x^3 - p^{12})((x-1)^3 - p^6)$ be a hyperelliptic curve over $\mathbb{Q}_p$. This is also a genus 2 curve with reduction type $\mathrm{I}_0 - \mathrm{I}_0 - 3$ in the terminology of [38]. We observe that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are odd and $\mathcal{R}$ is even. The clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are principal and contribute components of positive genus to $\mathcal{C}$, but $\mathcal{R}$ is not, and while we can assign to it a component in $\mathcal{C}$, it is a rational curve with two intersections, and hence not a principal component.

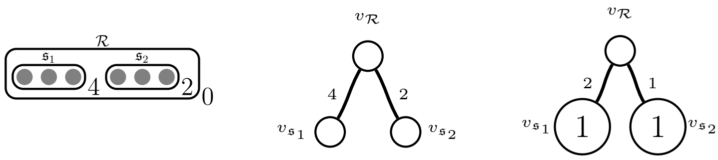


Figure 2.7: From left to right: the cluster picture of $C : y^2 = (x^3 - p^{12})((x - 1)^3 - p^6)$, the model of $\mathbb{P}^1$ separating the points of $\mathcal{R}$ and the minimal regular model of C.

Definition 2.2.13 (Sets of Children)**.** Let $\mathfrak{s}$ be a cluster. If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is a maximal subcluster of $\mathfrak{s}$ then $\mathfrak{s}'$ is a *child* of $\mathfrak{s}$ and $\mathfrak{s}$ is a *parent* of $\mathfrak{s}'$. We write $\mathfrak{s}' < \mathfrak{s}$, and $P(\mathfrak{s}') = \mathfrak{s}$. Denote by $\widehat{\mathfrak{s}}$ the set of all children of $\mathfrak{s}$, and by $\widetilde{\mathfrak{s}}$ the set of all odd children. A cluster is *übereven* if it has only even children. A cluster $\mathfrak{s}$ is a *cotwin* if it has a child of size $2g$ whose complement is not a twin. For clusters $\mathfrak{s}$ and $\mathfrak{s}'$, write $\mathfrak{s} \wedge \mathfrak{s}'$ for the smallest cluster containing $\mathfrak{s}$ and $\mathfrak{s}'$.

Definition 2.2.14. If $\mathfrak{s}$ and $\mathfrak{s}'$ are two clusters then the *distance* between them is $\delta_{\mathfrak{s},\mathfrak{s}'} = d_{\mathfrak{s}} + d_{\mathfrak{s}'} - 2d_{\mathfrak{s}\wedge\mathfrak{s}'}$. For a proper cluster $\mathfrak{s} \neq \mathcal{R}$ define the *relative depth* to be $\delta_{\mathfrak{s}} = \delta_{\mathfrak{s},P(\mathfrak{s})} = d_{\mathfrak{s}} - d_{P(\mathfrak{s})}$.

Remark 2.2.15. The distance between two clusters is precisely the shortest distance between the respective components in the dual graph of $\mathscr{X}_k^{\Sigma}$, the model of $\mathbb{P}^1$ associated to a cluster picture Σ described in Section 2.2.2. The relative distance is the shortest distance from a child to its parent.

Definition 2.2.16. For a cluster $\mathfrak{s}$ that is not a cotwin we write $\mathfrak{s}^*$ for the smallest cluster containing $\mathfrak{s}$, whose parent is not übereven. If no such cluster exists we write $\mathfrak{s}^* = \mathcal{R}$. If $\mathfrak{s}$ is a cotwin, we write $\mathfrak{s}^*$ for its child of size $2g$.

Definition 2.2.17. For a proper cluster $\mathfrak{s}$ we write $g_{\text{ss}}(\mathfrak{s})$ for the *semistable genus of* $\mathfrak{s}$. If $\mathfrak{s}$ is übereven, we set $g_{\text{ss}}(\mathfrak{s}) = 0$. Otherwise the genus is determined by

$$|\tilde{\mathfrak{s}}| = 2g_{\text{ss}}(\mathfrak{s}) + 1, \text{ or } 2g_{\text{ss}}(\mathfrak{s}) + 2.$$

Lemma 2.2.18. *Let Σ be such that $K(\mathcal{R})/K$ is a tame extension, and let $\mathfrak{s} \in \Sigma$ be a proper cluster fixed by Gal$(\overline{K}/K)$.*

(i) There exists a centre $z_{\mathfrak{s}}$ of $\mathfrak{s}$ such that $z_{\mathfrak{s}} \in K$.

(ii) Any child $\mathfrak{s}' < \mathfrak{s}$ is in an orbit of size $b_{\mathfrak{s}}$, except possibly for one child $\mathfrak{s}_f$, where we can choose $z_{\mathfrak{s}_f}$ such that $v_K(z_{\mathfrak{s}_f} - z_{\mathfrak{s}}) > d_{\mathfrak{s}}$, which is fixed by Gal$(\overline{K}/K)$.

Proof. (i) See [19, Lemma B.1]. (ii) See [9, Theorem 1.3]. □

Definition 2.2.19. Let $\mathfrak{s}' < \mathfrak{s}$ be clusters in Σ. Then $\mathfrak{s}'$ is a *stable child* of $\mathfrak{s}$ if the stabiliser of $\mathfrak{s}$ also stabilises $\mathfrak{s}'$. Otherwise $\mathfrak{s}'$ is an *unstable child* of $\mathfrak{s}$.

Remark 2.2.20. Let $\mathfrak{s} \in \Sigma$ be fixed by Gal$(\overline{K}/K)$. If $\mathfrak{s}$ has depth $d_{\mathfrak{s}}$ with denominator > 1 then, by Lemma 2.2.18 (ii), $\mathfrak{s}$ has at most one stable child.

Definition 2.2.21. If $r \in \mathfrak{s}$ is a root which is not contained in a proper child of $\mathfrak{s}$ then we call r a *singleton* of $\mathfrak{s}$. Define $\mathfrak{s}_{\text{sing}}$ to be the set of all singletons of $\mathfrak{s}$. In other words $\mathfrak{s}_{\text{sing}}$ is the set of all children of size 1 of $\mathfrak{s}$.

2.2.4 The Berkovich Projective Line and Clusters

Valuations associated to disks and models of $\mathbb{P}^1$ interact remarkably in Berkovich's theory of analytic spaces [4], which lends a powerful, alternative perspective on cluster pictures. This theory was developed by Berkovich as a non-archimedean analogue to complex analytification, since naïve attempts to develop a theory of non-archimedean analytic spaces fail due to the pathological nature of the topology of $\overline{K}$.

Definition 2.2.22 (Berkovich Analytification)**.** Let $X/\overline{K}$ be a curve. As a set, the *Berkovich analytification* X^{an} of X consists of points of the form $x = (\xi_x, v_x)$, where ξ_x is a point of X and v_x is a valuation on the residue field $\kappa(\xi_x)$ extending the valuation v_K. The space X^{an} is given the weakest topology such that

(i) $\iota : X^{\text{an}} \to X$ given by $x \mapsto \xi_x$ is continuous and,

(ii) for any $U \subseteq X$ open and any $f \in \mathcal{O}_X(U)$, the function $\iota^{-1}(U) \to \mathbb{R}$ given by $x \mapsto v_x(f)$ is continuous.

We say that X^{an} is a *Berkovich curve*.

Remark 2.2.23. The usual definition of the Berkovich analytification uses *absolute values*, as opposed to valuations, and this convention is used throughout the literature on Berkovich curves. However in order to align ourselves with [19], we have given this definition in terms of valuations.

Example 2.2.24 (The Berkovich Projective Line)**.** As a set, $X = \mathbb{P}^1_{\overline{K}}$ consists of closed points: points of $\overline{K}$ and infinity, which have residue field $\overline{K}$; and the generic point ξ_η which has residue field $\overline{K}(t)$. If ξ_x is a closed point of X, then there is a unique valuation on $\overline{K}$ extending v_K — v_K itself. Therefore, there is a unique point $x \in X^{\mathrm{an}}$ arising from ξ_x. These are *type I* points, and in this way $X(\overline{K})$ embeds into X^{an}.

The other points which arise are all of the form (ξ_η, v) where v is a valuation on $\overline{K}(t)$ extending v_K. For every p-adic disk $D = D_{z,r}$, there is a point (ξ_η, v_D) where v_D is the valuation defined in Definition 2.2.2. These are called *type II* points if $r \in v_K(\overline{K})$ and *type III* points otherwise.

In addition there are *type IV* points. These arise due to a curious property of $\overline{K}$ called *spherical incompleteness* — there exist sequences of disks $\mathcal{D} = D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$ such that $\bigcap D_i = \emptyset$. To such a sequence, we can define a valuation $v_{\mathcal{D}}(g) = \sup v_{D_i}(g)$. These are now all of the points of X^{an}.

As a topological space, X^{an} has the structure of a tree, where type I and type IV points are leaves, and at every type II point (ξ_η, v_D) the branches are in 1-to-1 correspondence with the points of $\overline{k}$, as these parametrise the centres of disjoint disks contained in D.

Suppose $\mathscr{X}$ is a model of $X = \mathbb{P}^1$. Then the dual graph of $\mathscr{X}$ embeds canonically into X^{an}: any such model of $\mathbb{P}^1$ arises from a collection of disks, as in Section 2.2.2. Consider the points of X^{an} corresponding to the disks. Since X^{an} has the structure of a tree, there is a unique smallest subspace Υ

of X^{an} containing these points. This is a topological graph, and is in fact the dual graph of $\mathscr{X}$. The complement of Υ is very well behaved — it is a disjoint union of (Berkovich) open disks, which are essentially isomorphic to the Berkovich projective line minus the point at infinity. Such a subspace Υ is called a *skeleton* of X^{an}, and skeletons are in one to one correspondence with semistable models of $\mathbb{P}^1$. Since such models arise from cluster pictures, we have a canonical way of finding cluster pictures inside X^{an}.

This is in fact true much more generally: given a curve X, there exists a *skeleton* Υ of X^{an}, which is a topological graph such that $X^{\mathrm{an}} \setminus \Upsilon$ is a disjoint union of (Berkovich) open disks. Furthermore, such skeletons are in bijection with semistable models of X, where a semistable model is sent to its dual graph. See [3] for more details. Tame morphisms between Berkovich curves are more or less determined by restriction to the skeleton, and hence there is an intimate relationship between tame morphisms of Berkovich curves and tame morphisms of semistable models of algebraic curves.

Many theorems about simultaneous semistable reduction and lifting semistable models, such as can be found in [31] and [30], have been proven in this setting (see for example [1, Section 5]). The link between skeleta of $\mathbb{P}^{1,\mathrm{an}}_K$ and cluster pictures also lends a conceptual justification to cluster pictures. Theorems such as 2.4.11 can be proved in an analytic setting (perhaps even more succinctly), and we shall use this point of view in Section 6.

2.3 Tame Quotients

Let X/K be a curve whose minimal regular model we would like to find, where K has algebraically closed residue field. Finding such a model is not necessarily straightforward, as has been discussed. Suppose we extend the field L/K such that X has semistable reduction over L, and that we know the minimal regular model $\mathscr{Y}$ of X over L. What can we say about the minimal regular model of X over K?

When X has tame reduction (i.e. the extension L/K required is tame), the situation is very well understood, and has been studied thoroughly in papers such as [32], [12], [23] and [47]. We take the quotient $\mathscr{X}$ of $\mathscr{Y}$ by $\mathrm{Gal}(L/K)$, and then resolve the singularities on $\mathscr{X}$. The focus of this section is how exactly to do that. The quotient $\mathscr{X}$ is a normal model of X over K and the singularities of $\mathscr{X}$ are all *tame cyclic quotient singularities*. Their resolution is well understood using Hirzebruch-Jung continued fractions.

The wild case is much more complicated, and much less is known. Some work has been done in that direction (e.g. [33] and [34]), but we shall focus on the tame case.

2.3.1 Taking the Quotient

Let X/K be a curve and let L/K be a tame extension of degree e over which $X_L = X \times_K L$ has semistable reduction. Write $G = \mathrm{Gal}(L/K)$. Since K has an algebraically closed residue field, the extension L/K is totally ramified and G is a cyclic group of order e. Let $\mathscr{Y}$ be a semistable model of X_L. Any $\sigma \in G$ induces an automorphism of $\mathscr{Y}$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\sigma} & \mathscr{Y} \\ \downarrow & & \downarrow \\ \mathcal{O}_L & \xrightarrow{\sigma} & \mathcal{O}_L \end{array}$$

By a gentle abuse of notation, we shall also call this automorphism σ, and think of G as a group of automorphisms of $\mathscr{Y}$. Since $\mathscr{Y}$ is projective, the quotient $\mathscr{X} = \mathscr{Y}/G$ is constructed in the usual way by glueing together the rings of invariants of G-invariant open sets of $\mathscr{Y}$. The scheme $\mathscr{X}$ is a normal model of X over K and as such its singularities are isolated points of the special fibre. We denote the quotient map $q : \mathscr{Y} \to \mathscr{X}$, and this has degree e.

What can we say about the model $\mathscr{X}$? The genera of the components of $\mathscr{X}$ can be calculated via Riemann-Hurwitz: indeed, if $E \in \mathscr{Y}_k$ is a component with pointwise stabiliser I, then $x \in E$ has ramification degree $|I_x|/|I|$, where I_x is the stabiliser of x. The multiplicity of $q(E)$ is also straightforward to calculate:

Proposition 2.3.1. *Let $E \in \mathscr{Y}_k$ be a component with pointwise stabiliser I. Then $q(E) \in \mathscr{Z}_k$ has multiplicity $e/|I|$.*

Proof. [32, Fact IV] □

Resolving a singularity $z \in \mathscr{Z}$ results in a chain of rational curves (Definition 2.1.22): a linking chain if z lies on two components of $\mathscr{Z}_k$, and a tail if z lies on a unique component. The model $\mathscr{X}$, obtained by resolving all of the singularities of $\mathscr{Z}$, is a regular snc model of X over K. To explicitly describe $\mathscr{X}$, we must better understand the singularities of $\mathscr{Z}$.

2.3.2 Tame Cyclic Quotient Singularities

The singularities of $\mathscr{Z}$ are *tame cyclic quotient singularities.* As the name suggests, these are the singularities that arise when taking the quotient of a surface by a cyclic action whose degree is prime to the size of the residue field. The following definition is [12, Definition 2.3.6]:

Definition 2.3.2 (Tame Cyclic Quotient Singularities)**.** Let $\mathscr{X}$ be a curve over $\mathcal{O}_K$ and let $s \in \mathscr{X}$ be a closed point. The point s is a *tame cyclic quotient singularity* if there exists

(i) a positive integer $m > 1$ which is invertible in k,

(ii) a unit $r \in (\mathbb{Z}/m\mathbb{Z})^{\times}$,

(iii) integers $m_1 > 0$ and $m_2 \geq 0$ satisfying $m_1 \equiv -rm_2 \mod m$,

such that $\widehat{\mathcal{O}_{\mathcal{C},s}}$ is isomorphic to the subalgebra of μ_m-invariants in $\mathcal{O}_K[\![t_1, t_2]\!]/(t_1^{m_1}t_2^{m_2} - \pi_K)$ under the action $t_1 \mapsto \zeta_m t_1, t_2 \mapsto \zeta_m^r t_2$. We call the pair (m, r) the *tame cyclic quotient invariants* of s.

Fortunately the resolution of such singularities is very well understood. The result is a chain of rational curves — a linking chain if $m_2 \neq 0$ and a tail otherwise. The self intersections and hence the multiplicities of the components are given by the *Hirzebruch-Jung continued fraction* of $\frac{m}{r}$.

Theorem 2.3.3 (Resolving Tame Cyclic Quotient Singularities)**.** *Let $\mathscr{X}$ be a flat, proper, normal curve over $\mathcal{O}_K$ with smooth generic fibre. Suppose $s \in \mathscr{X}_k$ is a tame cyclic quotient singularity with tame cyclic quotient invariants (m, r), as in Definition 2.3.2 above.*

Consider the Hirzebruch-Jung continued fraction expansion of $\frac{m}{r}$ given by

$$\frac{m}{r} = b_\lambda - \frac{1}{b_{\lambda-1} - \frac{1}{\cdots - \frac{1}{b_1}}},$$

where $b_i \geq 2$ for all $1 \leq i \leq \lambda$. Then the minimal regular resolution of s is a chain of rational curves $\bigcup_{i=1}^{\lambda} E_i$ such that E_i has self intersection $-b_i$.

Proof. [12, Theorem 2.4.1]. □

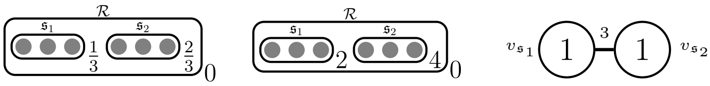
The multiplicities can then be calculated via intersection theory. Write E_0 and $E_{\lambda+1}$ for the components of $\mathscr{X}_k$ which the minimal resolution of s intersects. By the discussion above, these have multiplicities m_1 and m_2 (with $E_{\lambda+1}$ empty if $m_2 = 0$). Suppose E_i has multiplicity μ_i for $0 \leq i \leq \lambda + 1$. Since $E \cdot \mathscr{X}_k = 0$ by [42, Proposition IV.7.3] and each E_i only intersects E_{i-1} and E_{i+1}, we obtain the system of linear equations:

$$E_i \cdot (\mu_{i-1}E_{i-1} + \mu_i E_i + \mu_{i+1}E_{i+1}) = 0,$$
$$\Rightarrow \mu_{i-1} - b_i\mu_i + \mu_{i+1} = 0,$$

for $1 \leq i \leq \mu$. Using the multiplicities of E_0 and $E_{\lambda+1}$, we can solve this system.

Example 2.3.4. Let $C : y^2 = p(x^3 + p)((x-1)^3 + p^2)$ be a hyperelliptic curve over $\mathbb{Q}_p^{\mathrm{ur}}$. This is a type $\mathrm{II}^* - \mathrm{IV}^* - \alpha$ in Namikawa and Ueno's terminology. An extension L of degree 6 is required for semistability, and below we show from left to right: the cluster picture of C over $\mathbb{Q}_p^{\mathrm{ur}}$, the cluster picture of C over L, and the minimal regular model $\mathscr{Y}$ of C over L. The model $\mathscr{Y}$ consists of two genus 1 components linked by a linking chain of length 3 (so with 2 rational curves).

We take the quotient of $\mathscr{Y}$ by $\mathrm{Gal}(L/K)$ and we obtain a normal model $\mathscr{X}$ of C over K with some tame cyclic quotient singularities. In particular,



there are four singularities which lie on a unique component: two on each of the images of the genus 1 components. There is another singularity at an intersection point of two components. The other intersection points are regular points of $\mathscr{X}$. The action of Gal(L/K) on the components of $\mathscr{Y}$ and hence the invariants for all the tame cyclic quotient singularities comes from Theorem 2.4.11 in the next section, but for now we will take it all on faith and try to resolve them. Below we show $\mathscr{X}$ and the result of resolving all the singularities, which is the minimal regular model of C — no additional blow downs are needed.

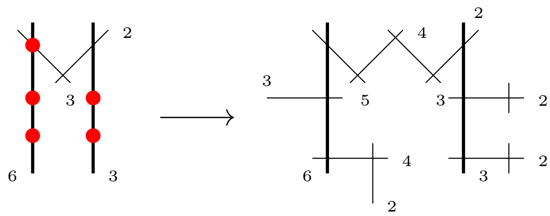


Figure 2.8: Resolving the tame cyclic quotient singularities on the special fibre of $\mathscr{X}$.

First let us examine the singularity which is the intersection point of the two components. This is a tame cyclic quotient singularity with $m_1 = 6, m_2 = 3, m = 3, r = 2$. The Hirzebruch-Jung continued fraction is:

$$\frac{3}{2} = 2 - \frac{1}{2},$$

and so the resolution consists of two rational curves, each with self intersection -2. Solving the system of linear equations $6 - 2\mu_1 + \mu_2 = 0, \mu_1 - 2\mu_2 + 3 = 0$ we obtain $\mu_1 = 5, \mu_2 = 4$. The tame cyclic quotient singularities on the component of multiplicity 6 have invariants $2/1$ and $3/2$, and both of those on the component of multiplicity 3 have invariants $3/2$. After putting all this together, the result is the picture on the right.

2.4 Semistable Hyperelliptic Curves

A description of the semistable model is needed before its quotient can be taken. When C/K is a hyperelliptic curve, this [19, Theorem 8.5], and here we recreate their main results. In Section 2.4.1 we state a criterion in terms of cluster pictures for a hyperelliptic curve to have semistable reduction. Using this, we also find the degree of minimal extension required for semistability, assuming that C has tame reduction. Following on, in Section 2.4.2 we describe the dual graph of the minimal regular model of C over an extension L where C has semistable reduction.

2.4.1 The Semistability Criterion

In order to apply the results of the previous section, we must first find an extension L over which C has semistable reduction. In [19], there is a criterion in terms of the cluster picture of C, which will be most useful for our purposes. We attach an invariant to each cluster $\mathfrak{s}$ as follows, using the valuation attached to the disk $D(\mathfrak{s})$ cutting out $\mathfrak{s}$ (see Definition 2.2.2):

Definition 2.4.1. ($\nu_{\mathfrak{s}}$) Let $C : y^2 = f(x)$ be a hyperelliptic curve with cluster picture Σ, and let $\mathfrak{s} \in \Sigma$ be a cluster. We define

$$\nu_{\mathfrak{s}} = v_{D(\mathfrak{s})}(f) = v_K(c_f) + \sum_{r \in \mathcal{R}} d_{r \wedge \mathfrak{s}},$$

where $D(\mathfrak{s})$ is the smallest p-adic disk cutting out $\mathfrak{s}$ and $v_{D(\mathfrak{s})}$ is the valuation associated to $D(\mathfrak{s})$, as defined in Section 2.2.1.

This allows us to state the semistability criterion. Roughly, these are precisely the conditions on the cluster picture which allow us to construct a model of $\mathbb{P}^1$ which separates the points of $\mathcal{R}$, as discussed in Section 2.2.2.

Theorem 2.4.2 (The Semistability Criterion)**.** *Let $C : y^2 = f(x)$ be a hyperelliptic curve, and let $\mathcal{R}$ be the set of roots of $f(x)$ in $\overline{K}$. Then C has semistable reduction over L if and only if*

(i) the extension $L(\mathcal{R})/L$ has ramification degree at most 2,

(ii) every proper cluster of $\Sigma_{C/L}$ is I_L invariant,

(iii) every principal cluster $\mathfrak{s} \in \Sigma_{C/L}$ has $d_{\mathfrak{s}} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.

Proof. [19, Theorem 1.8] □

Example 2.4.3. Let $C : y^2 = (x^3 - p^{18})(x^3 - 1)$ be a hyperelliptic curve over $\mathbb{Q}_p$. Its cluster picture is shown below, left. It consists of two proper clusters, $\mathcal{R}$ and $\mathfrak{s}$, both of which are principal. The extension $K(\mathcal{R})/K$ has degree 1 and every proper cluster is clearly I_K invariant. Furthermore, $d_{\mathfrak{s}} = 6, d_{\mathcal{R}} = 0$ which are both integers, and $\nu_{\mathfrak{s}} = 18$, $\nu_{\mathcal{R}} = 0$, both of which are even integers. Therefore C satisfies the semistability criterion, and indeed its minimal regular model (shown below, right) is semistable.

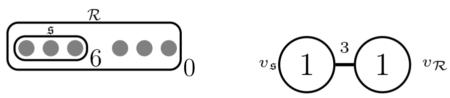


Figure 2.9: Cluster picture and minimal regular model of $C : y^2 = (x^3 - p^{18})(x^3 - 1)$.

Example 2.4.4. Let $C : y^2 = p(x^6 - p^{12})$ be a hyperelliptic curve over $\mathbb{Q}_p$. This is type $\mathrm{I}^*_{0,0,,0}$ in Namikawa and Ueno's terminology. Its cluster picture is shown below, left. There is a unique proper cluster which is therefore I_K invariant, and again $K(\mathcal{R})/K$ has degree 1. Furthermore $d_{\mathcal{R}} = 2$ is an integer, but $\nu_{\mathcal{R}} = 13$, which is odd. Therefore C does not have semistable reduction, and indeed its minimal regular model has a component of multiplicity 2, as shown below. However, C does have semistable reduction over an extension of degree 2.

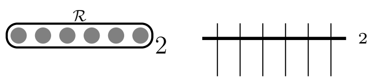


Figure 2.10: Cluster picture and minimal regular model of $C : y^2 = p(x^6 - p^{12})$.

Example 2.4.5. Let $C : y^2 = (x^2 - p)^3 - p^9$ by a hyperelliptic curve. This is a type $2\mathrm{I}_{0,1}$ hyperelliptic curve. The two clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are permuted by inertia. Since they are proper clusters, this implies that C does not have semistable reduction. Its cluster picture and minimal regular model are shown below. The curve C does obtain semistable reduction over an extension of degree 2.

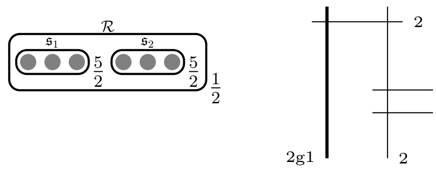


Figure 2.11: Cluster picture and minimal regular model of $C : y^2 = (x^2 - p)^3 - p^9$.

2.4.2 The Special Fibre

Once we have a hyperelliptic curve C with semistable reduction over some extension L/K, we would like to describe its minimal regular model. This is always possible by taking a normal model of C and resolving singularities, but such a method is inefficient. We present an explicit description of the special fibre in terms of the cluster picture of C, along with the equations of the components and the action of Galois.

The idea is: to each principal cluster, there is one or two associated principal components in the special fibre. Two principal clusters $\mathfrak{s}'$ and $\mathfrak{s}$ are linked by a chain of rational curves if $\mathfrak{s}' < \mathfrak{s}$ (or vice versa), and in a few other exceptional cases. A twin $\mathfrak{t}$ contributes a loop from the component of $P(\mathfrak{t})$ to itself. Any $\sigma \in \mathrm{Gal}(\overline{K}/K)$ permutes the principal components as it does the corresponding clusters.

Before we can state the theorem in full, we require several invariants attached to cluster pictures.

Definition 2.4.6 (Invariants and Characters)**.** For $\sigma \in \mathrm{Gal}(\overline{K}/K)$ let

$$\chi(\sigma) = \frac{\sigma(\pi_L)}{\pi_L} \mod \mathfrak{m}.$$

For a proper cluster $\mathfrak{s} \in \Sigma_C$ define

$$\lambda_{\mathfrak{s}} = \frac{\nu_{\mathfrak{s}}}{2} - d_{\mathfrak{s}} \sum_{\mathfrak{s}' < \mathfrak{s}} \left\lfloor \frac{|\mathfrak{s}'|}{2} \right\rfloor.$$

Define $\theta_{\mathfrak{s}} = \sqrt{c_f \prod_{r \notin \mathfrak{s}} (z_{\mathfrak{s}} - r)}$. If $\mathfrak{s}$ is either even or a cotwin, we define $\epsilon_{\mathfrak{s}}$: $\mathrm{Gal}(\overline{K}/K) \to \{\pm 1\}$ by

$$\epsilon_{\mathfrak{s}}(\sigma) \equiv \frac{\sigma(\theta_{\mathfrak{s}^*})}{\theta_{(\sigma\mathfrak{s})^*}} \mod \mathfrak{m}.$$

For all other clusters $\mathfrak{s}$ set $\epsilon_{\mathfrak{s}}(\sigma) = 0$.

Remark 2.4.7. The quantity $\epsilon_{\mathfrak{s}}(\sigma) = -1$ if and only if σ swaps the two points at infinity of $\Gamma_{\mathfrak{s},L}$. When $k = \overline{k}$, $\epsilon_{\mathfrak{s}}(\sigma) = (-1)^{\nu_{\mathfrak{s}^*} - |\mathfrak{s}^*| d_{\mathfrak{s}^*}}$ for σ a generator of inertia since

$$\nu_{\mathfrak{s}^*} = v_K \left(c_f \prod_{r \notin \mathfrak{s}^*} (z_{\mathfrak{s}^*} - r) \right) + \sum_{r \in \mathfrak{s}^*} d_{\mathfrak{s}^*}.$$

Remark 2.4.8. Our $\lambda_{\mathfrak{s}}$ is slightly different from the one defined in [19, Definition 6.1] (and is in fact equal to $\widetilde{\lambda}_{\mathfrak{s}}$, defined in Definition 6.4 of *loc. cit.*). This is because the authors of *loc. cit.* allow components of the special fibre to be singular, whereas we forbid this.

Next we would like to state the equations of our principal components.

Definition 2.4.9 (Reduction)**.** Let $\mathfrak{s} \in \Sigma_{C/K}$ be a principal cluster. Define $c_{\mathfrak{s}} \in k^{\times}$ by

$$c_{\mathfrak{s}} = \frac{c_f}{\pi_L^{v_L(c_f)}} \prod_{r \notin \mathfrak{s}} \frac{z_{\mathfrak{s}} - r}{\pi_L^{v_L(z_{\mathfrak{s}} - r)}} \mod \mathfrak{m},$$

and for $t \in \overline{K}$ define

$$\mathrm{red}_{\mathfrak{s}}(t) = \frac{t - z_{\mathfrak{s}}}{\pi_K^{d_{\mathfrak{s}}}} \mod \mathfrak{m}.$$

For $\mathfrak{s}' < \mathfrak{s}$ define $\mathrm{red}_{\mathfrak{s}}(\mathfrak{s}') = \mathrm{red}_{\mathfrak{s}}(r)$ for any $r \in \mathfrak{s}'$.

Definition 2.4.10 (Component Equations)**.** Let $\mathfrak{s}$ be a principal cluster. Define the hyperelliptic curve $\Gamma_{\mathfrak{s}}/\overline{k}$ by

$$\Gamma_{\mathfrak{s}} : Y^2 = c_{\mathfrak{s}} \prod_{\substack{\mathfrak{o} < \mathfrak{s} \\ \text{odd}}} (X - \mathrm{red}_{\mathfrak{s}}(\mathfrak{o})).$$

Note that if $\mathfrak{s}$ is übereven then this is two disjoint $\mathbb{P}^1$s.

Finally we are ready to state the description of the dual graph of the semistable model.

Theorem 2.4.11 (Dual Graph of Minimal Regular Model). *Let $C : y^2 = f(x)$ be a hyperelliptic curve over K. Suppose C obtains semistable reduction over L. Let Υ be the dual graph of the special fibre of the minimal regular model of C over $\mathcal{O}_{L^{\mathrm{ur}}}$. Then Υ has a vertex $v_{\mathfrak{s}}$ for every non-übereven cluster $\mathfrak{s}$ and two vertices, $v_{\mathfrak{s}}^{+}$ and $v_{\mathfrak{s}}^{-}$ for every übereven cluster, corresponding to $\Gamma_{\mathfrak{s}}$. Furthermore, these are linked by chains of edges (writing $v_{\mathfrak{s}} = v_{\mathfrak{s}}^{+} = v_{\mathfrak{s}}^{-}$ if $\mathfrak{s}$ is not übereven):*

Name	From	To	Length	Condition
$L_{\mathfrak{s}'}$	$v_{\mathfrak{s}}$	$v_{\mathfrak{s}'}$	$\frac{1}{2}\delta_{\mathfrak{s}'}$	$\mathfrak{s}' < \mathfrak{s}$ both principal, $\mathfrak{s}'$ odd
$L_{\mathfrak{s}'}^{+}$	$v_{\mathfrak{s}}^{+}$	$v_{\mathfrak{s}'}^{+}$	$\delta_{\mathfrak{s}'}$	$\mathfrak{s}' < \mathfrak{s}$ both principal, $\mathfrak{s}'$ even
$L_{\mathfrak{s}'}^{-}$	$v_{\mathfrak{s}}^{-}$	$v_{\mathfrak{s}'}^{-}$		
$L_{\mathfrak{t}}$	$v_{\mathfrak{s}}^{+}$	$v_{\mathfrak{s}}^{-}$	$\delta_{\mathfrak{s}'}$	$\mathfrak{s}$ principal, $\mathfrak{t} < \mathfrak{s}$ twin
$L_{\mathfrak{t}}$	$v_{\mathfrak{s}}^{+}$	$v_{\mathfrak{s}}^{-}$	$\delta_{\mathfrak{s}'}$	$\mathfrak{s}$ principal, $\mathfrak{s} < \mathfrak{t}$ cotwin

Moreover, if $\mathcal{R}$ is not principal:

$L_{\mathfrak{s},\mathfrak{s}'}$	$v_{\mathfrak{s}}$	$v_{\mathfrak{s}'}$	$\frac{1}{2}(\delta_{\mathfrak{s}} + \delta_{\mathfrak{s}'})$	$\mathcal{R} = \mathfrak{s}' \sqcup \mathfrak{s}$, $\mathfrak{s}, \mathfrak{s}'$ both principal, odd
$L_{\mathfrak{s},\mathfrak{s}'}^{+}$	$v_{\mathfrak{s}}^{+}$	$v_{\mathfrak{s}'}^{+}$	$\delta_{\mathfrak{s}} + \delta_{\mathfrak{s}'}$	$\mathcal{R} = \mathfrak{s}' \sqcup \mathfrak{s}$, $\mathfrak{s}, \mathfrak{s}'$ both principal, even
$L_{\mathfrak{s},\mathfrak{s}'}^{-}$	$v_{\mathfrak{s}}^{-}$	$v_{\mathfrak{s}'}^{-}$		
$L_{\mathfrak{s},\mathfrak{t}}$	$v_{\mathfrak{s}}^{+}$	$v_{\mathfrak{s}}^{-}$	$2(\delta_{\mathfrak{s}} + \delta_{\mathfrak{t}})$	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ principal, even, $\mathfrak{t}$ twin

Furthermore, $\sigma \in Gal(\overline{K}/K)$ acts on Υ as:

(i) $\sigma(v_{\mathfrak{s}}^{\pm}) = v_{\sigma(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s}}(\sigma)}$,

(ii) $\sigma(L_{\mathfrak{s}}^{\pm}) = L_{\sigma(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s}}(\sigma)}$,

(iii) for $\mathfrak{t}$ a twin or cotwin $\sigma(L_{\mathfrak{t}}) = \epsilon_{\mathfrak{t}}(\sigma)L_{\sigma(\mathfrak{t})}$, where $-L$ denotes L with the reversed orientation,

and the induced permutation on the remaining edges and vertices. On components, σ acts as

$$\sigma|_{\Gamma_{\mathfrak{s}}}(x,y) = (\chi(\sigma)^{ed_{\mathfrak{s}}}\overline{\sigma}(x), \chi(\sigma)^{e\lambda_{\mathfrak{s}}}\overline{\sigma}(y)) \in \Gamma_{\sigma(\mathfrak{s})}.$$

Proof. [19, Theorem 8.5]. Idea: let $\mathscr{X}^{\Sigma}$ be the model of $\mathbb{P}^1$ associated to Σ, as described in Section 2.2.2. Normalise this in the function field $K(C)$ of C. This is a proper, regular model by [43, Lemma 2.1]. After blowing down components of multiplicity 2, we obtain the description given above. □

Remark 2.4.12. It is possible to give a different but morally similar proof using Berkovich curves. Suppose K has algebraically closed residue field. After an extension L/K, we can assume that the cover $C \to \mathbb{P}^1$ extends to a map of semistable models $\mathcal{C} \to \mathscr{X}^{\Sigma}$ by [31, Theorem 2.3]. Furthermore, this induces a harmonic morphism of augmented $\mathbb{Z}$-graphs on the dual graphs of $\mathscr{X}^{\Sigma}$ and $\mathcal{C}$, in the sense of [1, Section 2]. Comparing the cluster picture Σ and the model $\mathscr{X}^{\Sigma}$, and using this fact, we obtain the same description as above. We can then use Lemma 2.1.20 to move between different fields where C has semistable reduction. We will use this strategy in Section 6 to find the minimal regular model of a bihyperelliptic curve with semistable reduction.

One drawback of this method is it doesn't immediately provide explicit equations for the components, which in turn makes it tricky to deduce the Galois action. However, this is potentially possible with some additional work.

In many of the previous examples we have already drawn the special fibre of a hyperelliptic curve with semistable reduction. Here we present a couple of more involved examples for fun.

Example 2.4.13. Let $C : y^2 = (x^3-p^{24})((x-p^2)^3-p^{18})((x-1)^2-p^{10})(x^3-2)$ be a hyperelliptic curve over $\mathbb{Q}_p^{\mathrm{ur}}$. This curve has a cluster picture (shown below) with five proper clusters: four of them principal and non-übereven, and one twin. The twin contributes a loop of length 10 to the special fibre. The cluster $\mathcal{R}$ has three odd children and so contributes a component $v_{\mathcal{R}}$ of genus 1 to the special fibre. Its unique principal child $\mathfrak{s}_3$ is even and has two odd children, so contributes one component of genus 0 to the special fibre, with two linking chains of length 2 to the component of its parent. The two children of $\mathfrak{s}_3$, $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are both odd with three odd children, and so each contribute a component of genus 1 with a linking chain to $v_{\mathfrak{s}_3}$ of length half the relative depth of the cluster.

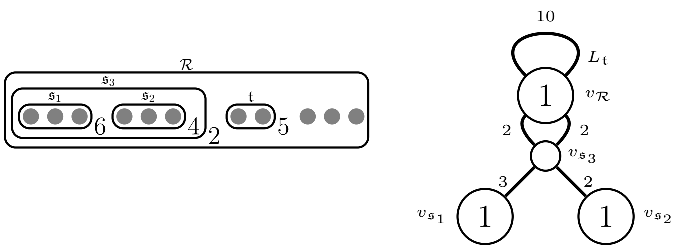


Figure 2.12: Cluster picture and minimal regular model of $C : y^2 = (x^3 - p^{24})((x-p^2)^3 - p^{18})((x-1)^2 - p^{10})(x^3 - 2)$.

Example 2.4.14. Let $C : y^2 = (x^2 - p^6)((x-1)^2 - p^6)((x-2)^2 - p^6)$ be a hyperelliptic curve over $\mathbb{Q}_p^{\mathrm{ur}}$. The key thing to note here is that the top cluster $\mathcal{R}$ is übereven, since all of its children are twins, and hence there are *two* components associated to $\mathcal{R}$ in the special fibre: $v_{\mathfrak{s}}^+$ and $v_{\mathfrak{s}^-}$. Each of its children twins contributes a linking chain of length 6 between these two components.

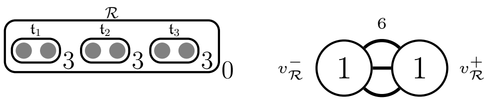


Figure 2.13: Cluster picture and minimal regular model of $C : y^2 = (x^2 - p^6)((x-1)^2 - p^6)((x-2)^2 - p^6)$.

Example 2.4.15. Let $C : y^2 : ((x-i)^3 - p^{12})((x+i)^3 - p^{12})$ be a hyperelliptic curve over $\mathbb{Q}_p$, p such that -1 does not have a square root mod p. The cluster picture and minimal regular model are shown below ($\mathcal{R}$ is not principal so does not contribute a component). The crucial thing to spot is that Frobenius swaps the two clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$, and therefore also swaps the two components $v_{\mathfrak{s}_1}$ and $v_{\mathfrak{s}_2}$.

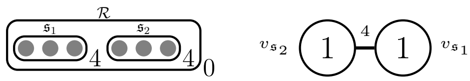


Figure 2.14: Cluster picture and minimal regular model of $C : y^2 : ((x-i)^3 - p^{12})((x+i)^3 - p^{12})$.

2.5 Models via Newton Polygons

The final piece of the puzzle concerns a different approach to calculating models. Following [15], we describe how to use the Newton polygon of a curve to calculate its minimal snc model in the case where it has tame reduction. Another condition required is that the curve be Δ_v*-regular*, which simplifies to a relatively nice condition in the case of hyperelliptic curves. Some work has been done by Muselli in [37] to loosen this condition and apply the construction of Theorem 2.5.11 to a larger class of curves. We use Theorem 2.5.11 to do the heavy lifting of calculating linking chains when we prove our description of the snc model of a hyperelliptic curve with tame reduction.

We briefly recall some definitions of Newton polygons in Section 2.5.1, before describing the main result of [15] in Section 2.5.2. We finish the section off by defining *sloped chains of rational curves*, a definition which will be crucial when stating our main theorems in Section 2.5.3.

2.5.1 Newton Polygons

The results of this section only apply to Δ_v-regular curves. For hyperelliptic curves this condition is that its cluster picture is *nested*; roughly, any cluster can contain at most one proper child.

Definition 2.5.1 (Nested Cluster Picture)**.** A cluster picture Σ is *nested* if for all proper clusters $\mathfrak{s}, \mathfrak{s}' \in \Sigma$ either $\mathfrak{s} \subseteq \mathfrak{s}'$, or $\mathfrak{s}' \subseteq \mathfrak{s}$. If C is a hyperelliptic curve, we say C is *nested* if Σ_C is nested.

Example 2.5.2. Below are three cluster pictures. The first and second are nested, but the rightmost is not, because $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are disjoint.

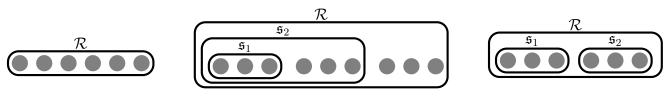


Figure 2.15: Nested and non-nested cluster pictures.

To such a hyperelliptic curve we attach *Newton polytopes*. This can be defined for any curve given by a bivariate polynomial, but we restrict our focus to nested hyperelliptic curves.

Definition 2.5.3 (Newton Polytopes)**.** Let $G(x, y) = y^2 - f(x) = \sum a_{ij}x^i y^j$ be the defining equation of a hyperelliptic curve C over K. The *Newton polytopes* of C over K and $\mathcal{O}_K$ respectively are:

$$\Delta(C) = \text{convex hull } \{(i, j) \mid a_{ij} \neq 0\} \subseteq \mathbb{R}^2,$$
$$\Delta_v(C) = \text{lower convex hull } \{(i, j, v_K(a_{ij})) \mid a_{ij} \neq 0\} \subseteq \mathbb{R}^2 \times \mathbb{R}.$$

Above every point $P \in \Delta$ there is exactly one point $(P, v_K(P)) \in \Delta_v$. This defines a piecewise affine function $v_{\Delta(C)} : \Delta(C) \to \mathbb{R}$. When there is no risk of confusion, we may sometimes write $\Delta = \Delta(C)$, and $\Delta_v = \Delta_v(C)$ and the pair (Δ, v_Δ) determines Δ_v.

Definition 2.5.4 (v-edges and faces)**.** Under the homeomorphic projection $\Delta_v \to \Delta$, the images of the 1 and 2 dimensional open faces of Δ_v are called *v-edges*, and *v-faces* respectively. A v-edge is homeomorphic to an open interval, and a v-face is homeomorphic to an open disc.

Notation 2.5.5. For a v-edge L and a v-face F we write

$$L(\mathbb{Z}) = L \cap \mathbb{Z}^2, \quad F(\mathbb{Z}) = F \cap \mathbb{Z}^2, \quad \Delta(\mathbb{Z}) = (\Delta^{\mathrm{o}}) \cap \mathbb{Z}^2,$$

and $\overline{L}(\mathbb{Z})$, $\overline{F}(\mathbb{Z})$, $\overline{\Delta}(\mathbb{Z})$ to include points on the boundary. We use subscripts to restrict to the set of points P with $v_K(P)$ in a given set, for instance $F(\mathbb{Z})_{\mathbb{Z}} = \{P \in F(\mathbb{Z}) \mid v_\Delta(P) \in \mathbb{Z}\}$.

Definition 2.5.6 (Denominator)**.** The *denominator* δ_λ, for every v-face or v-edge λ is defined to be the common denominator of $v_\Delta(P)$ for $P \in \overline{\lambda}(\mathbb{Z})$. For two alternate, but equivalent, definitions see [15, Notation 3.2].

Remark 2.5.7. We shall see that the denominator of a v-face or v-edge λ, in some sense, tells us the multiplicity of the component or chain of the snc model arising from λ. Roughly, for a v-face F, δ_F is the multiplicity of the component Γ_F, and for a v-edge L, δ_L is the minimum multiplicity appearing in the chain of rational curves arising from L.

We distinguish between v-edges which lie on precisely one or two v-faces of the Newton polytope, the former giving rise to tails and the latter to linking chains (see Definition 2.1.22).

Definition 2.5.8 (Inner and outer edges)**.** A v-edge L is *inner* if it is on the boundary of two v-faces. Otherwise, if L is only on the boundary of one v-face, L is *outer*.

2.5.2 Calculating a Model

Before we state how to calculate an snc model given the Newton polygon, we define a few constants related to v-faces and v-edges which will be necessary for the description.

Definition 2.5.9 (Function on edges)**.** Let L be a v-edge on the boundary of a v-face F. Write

$$L^* = L^*_{(F)} = \text{ the unique affine function } \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z} \text{ with } L^*|_{\bar{L}} = 0, \text{ and } L^*|_F \geq 0.$$

Definition 2.5.10 (Slopes)**.** Let L be a v-edge. If L is inner it bounds two v-faces, say F_1 and F_2. If L is outer it bounds one v-face, say F_1. Choose $P_0, P_1 \in \mathbb{Z}^2$ with $L^*_{(F_1)}(P_0) = 0$, and $L^*_{(F_1)}(P_1) = 1$. The *slopes* $[s_1^L, s_2^L]$ at L are

$$s_1^L = \delta_L(v_1(P_1) - v_1(P_0)), \quad \text{and} \quad s_2^L = \begin{cases} \delta_L(v_2(P_1) - v_2(P_0)) & \text{for } L \text{ inner,} \\ \lfloor s_1^L - 1 \rfloor & \text{for } L \text{ outer,} \end{cases}$$

where v_i is the unique affine function $\mathbb{Z}^2 \to \mathbb{Q}$ that agrees with v_Δ on F_i.

The following is the main theorem of the section:

Theorem 2.5.11 (Minimal snc model from Newton Polytope)**.** *Suppose $C : y^2 = f(x)$ is a nested hyperelliptic curve over K. Then there exists a regular snc model $C_\Delta/\mathcal{O}_K$ of C/K. Its special fibre is as follows:*

(i) Every v-face F of Δ gives a complete smooth curve Γ_F/k of multiplicity δ_F and genus $|F(\mathbb{Z})_{\mathbb{Z}}|$.

(ii) For every v-edge L with slopes $[s_1^L, s_2^L]$ pick $\frac{m_i}{d_i} \in \mathbb{Q}$ such that

$$s_1^L = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \cdots > \frac{m_\lambda}{d_\lambda} > \frac{m_{\lambda+1}}{d_{\lambda+1}} = s_2^L, \text{ with } \begin{vmatrix} m_i & m_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1. \tag{2.1}$$

Then L gives $|\overline{L}(\mathbb{Z})_{\mathbb{Z}}| - 1$ chains of rational curves of length λ from Γ_{F_1} to Γ_{F_2} (if L is outer these chains are tails of Γ_{F_1}) with multiplicities $\delta_L d_1, \ldots, \delta_L d_\lambda$.

Proof. [15, Theorem 3.13]. □

Remark 2.5.12. The model C_Δ is not necessarily the *minimal* snc model of C. However, there is an exact description of which components must be blown down in order to obtain the minimal snc model, which can be found at [15, Section 5]. For nested hyperelliptic curves, the only such components are those arising from twins in certain circumstances. We discuss this in further detail when we apply this theorem in Sections 3.2 and 4.

Remark 2.5.13. In (2.1), $\lambda = 0$ indicates that Γ_{F_1} and Γ_{F_2} intersect $|\overline{L}(\mathbb{Z})_{\mathbb{Z}}| - 1$ times in the inner case, and that L contributes no tails in the outer case.

Remark 2.5.14. An explicit equation for Γ_F is given in [15, Definition 3.7], where it is denoted by $\overline{X}_F$.

Remark 2.5.15. To see that the sequences in Theorem 2.5.11 exist, take all numbers in $[s_2^L, s_1^L] \cap \mathbb{Q}$ of denominator $\leq \max\{\text{denom}(s_1^L), \text{denom}(s_2^L)\}$ in decreasing order. This is essentially a Farey series, so satisfies the determinant condition in (2.1). One can then repeatedly remove, in any order, terms of the form

$$\cdots > \frac{a}{b} > \frac{a+c}{b+d} > \frac{c}{d} > \cdots \mapsto \cdots > \frac{a}{b} > \frac{c}{d} > \ldots,$$

where $(a+c)$ and $(b+d)$ are coprime, until no longer possible. This corresponds to blowing down $\mathbb{P}^1$s of self intersection -1. The resulting minimal sequence is unique (else this would contradict uniqueness of minimal snc model), and still satisfies the determinant condition. If $(s_2^L, s_1^L) \cap \mathbb{Z} = \{N, \ldots, N+a\} \neq \emptyset$ this minimal sequence has the form

$$s_1^L = \frac{m_0}{d_0} > \cdots > \frac{m_h}{d_h} > N+a > \cdots > N > \frac{m_l}{d_l} > \cdots > \frac{m_{\lambda+1}}{d_{\lambda+1}} = s_2^L, \quad (2.2)$$

with $d_0, \ldots, d_h$ strictly decreasing and $d_l, \ldots, d_{\lambda+1}$ strictly increasing. If $(s_2^L, s_1^L) \cap \mathbb{Z} = \emptyset$ this minimal sequence has the form

$$s_1^L = \frac{m_0}{d_0} > \cdots > \frac{m_l}{d_l} > \frac{m_{l+1}}{d_{l+1}} > \cdots > \frac{m_{\lambda+1}}{d_{\lambda+1}} = s_2^L, \quad (2.3)$$

with $d_0, \ldots, d_l$ strictly decreasing, $d_{l+1}, \ldots, d_{\lambda+1}$ strictly increasing, and $d_i > 1$ for all $1 \leq i \leq \lambda$.

Notice that shifting either s_1^L or s_2^L by an integer does not change the denominators d_i, that appear in this sequence. If $s_2 > 0$, which is always the case after shifting by an integer, the numbers $N > \frac{m_l}{d_l} > \cdots > \frac{m_{\lambda+1}}{d_{\lambda+1}}$ are the approximants of the Hirzebruch-Jung continued fraction expansion of s_2^L. Similarly for $\frac{m_0}{d_0} > \cdots > \frac{m_h}{d_h} > N+a$, consider the expansion of $1 - s_1^L$. This makes sense, as we could have also obtained this model via the techniques of Section 2.3, and the resolution of tame cyclic quotient singularities uses Hirzebruch-Jung continued fractions.

Example 2.5.16. Let $C : y^2 = (x^3 - p^5)(x^2 - 1)$ be a hyperelliptic curve over $\mathbb{Q}_p$. This has Namikawa–Ueno type $\mathrm{I}_0 - \mathrm{II}^* - 0$. Below is the cluster picture, Newton polytope Δ and minimal snc model of C.

There are two faces F_1 and F_2 which give two principal components, the former of genus 1 since $(3, 1)$ is a point in the interior of F_1 with integer valuation. There is one inner edge (between the faces F_1 and F_2), which

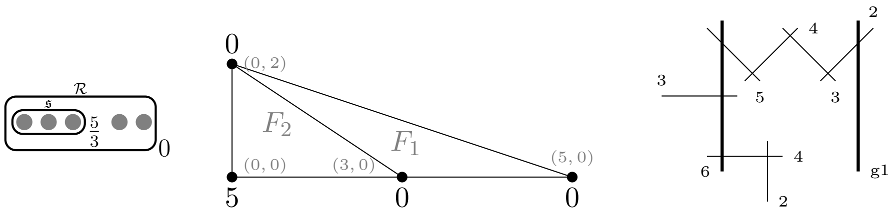


Figure 2.16: Cluster picture, Newton polygon and minimal snc model of C.

we shall label L_I, and four outer edges — the intersection of Δ with the y-axis, which we shall call L_y, the intersection with the x-axis, split into $L_{x,1}$ and $L_{x,2}$ (with $L_{x,i}$ an edge of F_i), and finally the sloped edge L_O. We shall calculate the slopes for the inner edge, and leave the rest as an exercise. The edge L_I intersects $\mathbb{Z}^2$ in two points, both of which have valuation 0, so its denominator is $\delta = 1$. The affine functions v_i are given by $v_1(x,y) = 0$ and $v_2(x,y) = 5 - \frac{5}{2}y - \frac{5}{3}x$. Furthermore, the function $L^*_{I,1}(x,y) = 3y - 2x - 6$ and $L^*_{I,2} = -L^*_{I,1}$. Putting this together, we find $s^L_1 = 0$ and $s^L_2 = -5/6$. This results in the sequence

$$\frac{0}{1} > -\frac{1}{2} > -\frac{2}{3} > -\frac{3}{4} > -\frac{4}{5} > -\frac{5}{6},$$

whose denominators are the multiplicities of the rational curves in the linking chain between the two principal components.

2.5.3 Sloped Chains of Rational Curves

The following definition allows us to disambiguate parts of chains of rational curves arising from v-edges in the Newton polytope of C.

Definition 2.5.17 (Sloped Chain)**.** Let $t_1, t_2 \in \mathbb{Q}$ and $\mu \in \mathbb{N}$. Pick m_i, d_i as in Theorem 2.5.11; that is, such that

$$\mu t_1 = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \cdots > \frac{m_\lambda}{d_\lambda} > \frac{m_{\lambda+1}}{d_{\lambda+1}} = \mu t_2, \text{ and } \begin{vmatrix} m_i & m_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1,$$

with $d_0 \geq \cdots \geq d_l$ and $d_l \leq \cdots \leq d_{\lambda+1}$, for some $0 \leq l \leq \lambda + 1$.

Let $A = \{i \mid 1 \leq i \leq \lambda \text{ and } d_i = 1\}$. If A is non-empty, let a_0 be the minimal element of A and let a_1 the maximal element of A. Suppose

$\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$ is a chain of rational curves where E_i has multiplicity μd_i. Then $\mathcal{C}$ is a *sloped chain of rational curves* with parameters (t_2, t_1, μ) and we split $\mathcal{C}$ into three sections. If $A \neq \emptyset$ we define the following:

(i) $E_1 \cup \cdots \cup E_{a_0-1}$, the *downhill section*,

(ii) $E_{a_0} \cup \cdots \cup E_{a_1}$, the *level section*,

(iii) $E_{a_1+1} \cup \cdots \cup E_{\lambda}$, the *uphill section*.

If instead $A = \emptyset$ we define:

(i) $E_1 \cup \cdots \cup E_l$, the *downhill section*,

(ii) $E_{l+1} \cup \cdots \cup E_{\lambda}$, the *uphill section*,

and there is no *level section*.

We define the length of each section to be the number of E_i contained in it, and each section is allowed to have length 0. For instance, the level section has length 0 if and only if $A = \emptyset$, and the downhill section has length 0 if and only if $1 \in A$.

Remark 2.5.18. A tail is a sloped chain with level section of length 1 and no uphill section. Therefore any tail can be given by just two parameters, namely t_1 and μ (since $t_2 = \frac{1}{\mu}\lfloor \mu t_1 - 1 \rfloor$). We will often refer to a tail as a tail with parameters (t_1, μ). It follows from Remark 2.5.15 that a tail with parameters (t_1, μ) has the same multiplicities as the tail obtained by resolving a tame cyclic quotient singularity with tame cyclic quotient invariants $\frac{1}{\mu t_1}$.

Remark 2.5.19. All of our chains of rational curves, be they tails, linking chains or crossed tails, are sloped chains. For example, a linking chain in a semistable model will consist of only a level section. Both tails and crossed tails in a minimal snc model will have no uphill section.

Chapter 3

Toy Hyperelliptic Curves

This chapter and the next is the product of joint work with Sarah Nowell. In them we give an explicit description of the minimal snc model of a hyperelliptic curve with tame reduction. The published article can be found at [22]. This begins with some base cases of our inductive proof, namely hyperelliptic curves with tame potentially good reduction in Section 3.1 and curves whose cluster pictures have precisely two proper clusters in Section 3.2. These two cases are the building blocks of a hyperelliptic curve with a more complicated cluster picture. Throughout this section we will assume that the residue field k is algebraically closed.

3.1 Potentially Good Reduction

In this section we calculate the minimal snc model of a hyperelliptic curve C/K with genus $g \geq 1$ which has tame potentially good reduction. Recall that C has tame potentially good reduction if there exists a field extension L/K of degree e such that e and p are coprime, and C has a smooth model over $\mathcal{O}_L$. In order to calculate this model, we assume that L is the minimal such extension. The minimal snc model of such a hyperelliptic curve has a rather straightforward description: it consists of a principal component with some tails (in the sense of Definition 2.1.22) whose multiplicities can be explicitly described using the results of Section 2.3. The size and depth of the unique proper cluster $\mathfrak{s}$, as well as the valuation of the leading coefficient c_f will be sufficient to calculate the (dual graph with multiplicity of the) minimal snc model of C over K:

Theorem 3.1.1 (Minimal snc model, tame potentially good reduction)**.** *Let C be hyperelliptic curve over K with tame potentially good reduction. Then the special fibre $\mathscr{X}_k$ of the minimal snc model $\mathscr{X}$ of C/K consists of a principal component Γ of multiplicity e. Furthermore, if $e > 1$ then the following tails intersect the principal component Γ:*

Name	*Number of tails*	*Condition for tail to arise*
T_∞	1	$\mathfrak{s}$ *odd*
$T_\infty^\pm$	2	$\mathfrak{s}$ *even and* $v_K(c_f)$ *even*
T_∞	1	$\mathfrak{s}$ *even,* $e > 2$ *and* $v_K(c_f)$ *odd*
$T_{y_\mathfrak{s}=0}$	$\frac{\lvert\mathfrak{s}_{\text{sing}}\rvert}{b_\mathfrak{s}}$	$e = 2b_\mathfrak{s}$
$T_{x_\mathfrak{s}=0}$	1	$b_\mathfrak{s} \mid \lvert\mathfrak{s}\rvert$, $\lambda_\mathfrak{s} \notin \mathbb{Z}$ *and* $e > 2$
$T_{x_\mathfrak{s}=0}^\pm$	2	$b_\mathfrak{s} \mid \lvert\mathfrak{s}\rvert$ *and* $\lambda_\mathfrak{s} \in \mathbb{Z}$
$T_{(0,0)}$	1	$b_\mathfrak{s} \nmid \lvert\mathfrak{s}\rvert$

Remark 3.1.2. The genus of the central component can be calculated using Riemann Hurwitz, and we prove an explicit formula for it in Proposition 3.1.25.

Example 3.1.3. Let $C : y^2 = x^7 - p^{11}$. The cluster picture and minimal snc model of C are shown below.

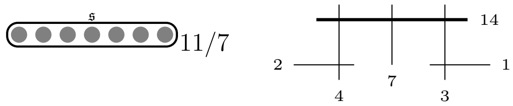


Figure 3.1: Cluster picture and minimal snc model of curve with tame potentially good reduction.

The curve requires an extension of degree 14 for semistability by Theorem 2.4.2, so the central component of its minimal snc model has multiplicity 14. The unique cluster $\mathfrak{s}$ is odd, so there is a tail T_∞ (the rightmost tail). We have $14 = e = 2b_\mathfrak{s} = 7$ so there is a $T_{y_\mathfrak{s}=0}$ tail (the middle tail). Finally, $b_\mathfrak{s} \mid \lvert\mathfrak{s}\rvert$, $\lambda_\mathfrak{s} = 11/2$ and $e_\mathfrak{s} > 2$ so there is a $T_{x_\mathfrak{s}=0}$ tail (the leftmost tail). The multiplicities of the components of the tails are calculated using Proposition

3.1.12. The minimal snc model is *not* the minimal regular model, but this can be straightforwardly obtained by blowing down the exceptional components:

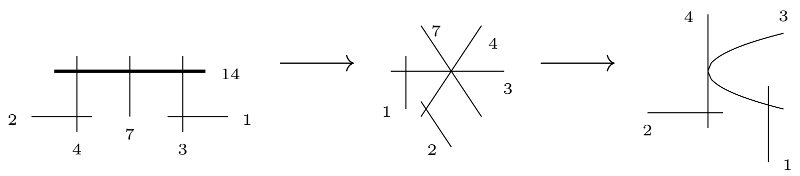


Figure 3.2: Blowing down exceptional components on the special fibre to obtain the minimal regular model.

3.1.1 The Strategy

Since C has tame potentially good reduction, by [19, Theorem 1.8(3)] we can assume (possibly after a Möbius transform) that the cluster picture of C over K consists of a single proper cluster $\mathfrak{s}$. After an appropriate shift of the affine line we can assume that $\mathfrak{s}$ has centre 0 and that C is given by one of the following two equations:

$$y^2 = c_f \prod_{0 \neq r \in \mathcal{R}} (x - u_r \pi^{d_\mathfrak{s}}), \quad \text{or} \quad y^2 = c_f x \prod_{0 \neq r \in \mathcal{R}} (x - u_r \pi^{d_\mathfrak{s}}),$$

if $b \mid |\mathfrak{s}|$ or $b \nmid |\mathfrak{s}|$ respectively, where the $u_r \in K$ are units.

We will proceed in the manner of Section 2.3. Let $\mathscr{Y}$ be the smooth Weierstrass model of C over L. This is in general obtained by a substitution $x_L = \pi^{-d_\mathfrak{s}} x$, $y_L = \pi^{-\lambda_\mathfrak{s}} y$ and will be given by the equation

$$y_L^2 = c_{f,L} f \prod_{0 \neq r \in \mathcal{R}} (x_L - u_r), \quad \text{or} \quad y_L^2 = c_{f,L} x_L \prod_{0 \neq r \in \mathcal{R}} (x_L - u_r),$$

if $b \mid |\mathfrak{s}|$ or $b \nmid |\mathfrak{s}|$ respectively, and where $c_{f,L} = c_f / \pi_K^{v_K(c_f)}$. Let $q : \mathscr{Y} \to \mathscr{X}$ be the quotient map induced by the action of $\mathrm{Gal}(L/K)$. We will explicitly describe the singular points of $\mathscr{X}$, show that they are tame cyclic quotient singularities in the sense of Definition 2.3.2, and give their tame cyclic quotient invariants in Proposition 3.1.12. Theorem 2.3.3 then tells us the self intersection numbers of the rational curves in the tails obtained by resolving the tame cyclic quotient singularities. Using intersection theory, this allows us to describe the special fibre of the minimal snc model $\mathscr{X}$ of C/K in full.

3.1.2 The Automorphism and its Orbits

To describe the singularities on $\mathscr{Z}_k$, we must first explicitly describe the Galois automorphism on the unique component $\Gamma_{\mathfrak{s},L} = \Gamma \subseteq \mathscr{Y}_k$ of the special fibre of the smooth Weierstrass model of C over L. The following fact from [32, Fact IV] describes the singularities of $\mathscr{Z}_k$ in terms of the quotient $q : \mathscr{Y} \to \mathscr{Z}$.

Proposition 3.1.4. *Let $z_1, \ldots, z_d$ be the ramification points of the morphism $q : \Gamma \to \mathscr{Z}_k$. Then $\{z_1, \ldots, z_d\}$ is precisely the set of singular points of $\mathscr{Z}_k$.*

Furthermore, the ramification points of q correspond to points whose preimage is an orbit of size strictly less than e.

Definition 3.1.5 (Small Orbits)**.** Let X be an orbit of points of $\mathscr{Y}_k$. If $|X| < e$, we say that X is a *small orbit*.

So, describing the singular points of $\mathscr{Z}_k$ is equivalent to describing the small orbits of $\mathrm{Gal}(L/K)$. In order to list these orbits, we simplify some cluster invariants from 2.4.6.

Lemma 3.1.6. *Let C/K be a hyperelliptic curve with tame potentially good reduction and unique proper cluster $\mathfrak{s}$. Then:*

$$\nu_{\mathfrak{s}} = |\mathfrak{s}| d_{\mathfrak{s}} + v_K(c_f), \quad \lambda_{\mathfrak{s}} = \frac{\nu_{\mathfrak{s}}}{2} = \frac{|\mathfrak{s}| d_{\mathfrak{s}} + v_K(c_f)}{2}, \quad \epsilon_{\mathfrak{s}} = (-1)^{v_K(c_f)},$$

and any $\sigma \in Gal(\overline{K}/K)$ induces on the special fibre

$$\sigma|_{\Gamma} : (x_{\mathfrak{s}}, y_{\mathfrak{s}}) \longmapsto (\chi(\sigma)^{e d_{\mathfrak{s}}} x_{\mathfrak{s}}, \chi(\sigma)^{e \lambda_{\mathfrak{s}}} y_{\mathfrak{s}}),$$

where $x_{\mathfrak{s}}, y_{\mathfrak{s}}$ are coordinates on the special fibre.

Proof. This follows directly from the definitions in Definition 2.4.6, and Theorem 2.4.11. □

Since $\chi(\sigma)^{ed_{\mathfrak{s}}}$ and $\chi(\sigma)^{e\lambda_{\mathfrak{s}}}$ are non-zero and k is algebraically closed, the only points which can lie in orbits of size strictly less than e are points at infinity, or points where $x_{\mathfrak{s}} = 0$ or $y_{\mathfrak{s}} = 0$. This gives four cases which we will take care to distinguish between, as it will make it easier to describe the minimal snc model for a general cluster picture. With this in mind we make the following definitions:

Definition 3.1.7 (Types of Small Orbits)**.** We split the small orbits that can occur into the following types.

— ∞-*orbits*: orbits on the point(s) at infinity,

— $(y = 0)$-*orbits*: orbits on non-zero roots,

— $(x = 0)$-*orbits*: orbits on the points $(0, \pm\sqrt{c_f})$,

— $(0, 0)$-*orbits*: the orbit on the point $(0, 0)$.

The following lemmas describe in which situations we see these small orbits. We will assume $e > 1$ since no small orbits occur when $e = 1$.

Lemma 3.1.8. *If* $\deg(f)$ *is odd then there is a single* ∞*-orbit consisting of a single point. If* $\deg(f)$ *is even and* $v_K(c_f) \in 2\mathbb{Z}$ *then there are two* ∞*-orbits each of size 1. If* $\deg(f)$ *is even,* $v_K(c_f) \notin 2\mathbb{Z}$ *and* $e > 2$ *then there is a single* ∞*-orbit of size 2.*

Proof. Let $u = 1/x, v = y/x^{g+1}$ denote the coordinates at infinity. The curve C has a single point at infinity $(u, v) = (0, 0)$ if $\deg(f)$ is odd, and two points at infinity $(u, v) = (0, \pm\sqrt{c_f})$ if $\deg(f)$ is even. In the latter case, the action at infinity is given by $\sigma : (0, \sqrt{c_f}) \mapsto (0, \chi(\sigma)^{e\lambda_{\mathfrak{s}}}\sqrt{c_f})$. Therefore, when $\deg(f)$ is even, the points at infinity are swapped if and only if $\chi(\sigma)^{e\lambda_{\mathfrak{s}}} = -1$ for some $\sigma \in \mathrm{Gal}(L/K)$. This is the case if and only if $v_K(c_f)$ is odd. In this case, the orbit at infinity has size 2 and is only a small orbit if $e > 2$. □

Lemma 3.1.9. *If $f(0) = 0$ then there is a single $(0,0)$-orbit consisting of a single point. Otherwise $f(0) \neq 0$, and if $\lambda_{\mathfrak{s}} \in \mathbb{Z}$ then there are two $(x = 0)$-orbits of size 1, else $\lambda_{\mathfrak{s}} \notin \mathbb{Z}$ and $e > 2$ there is a single $(x = 0)$-orbit of size 2.*

Proof. If $f(0) = 0$ then $\{(0,0)\} \in \Gamma$ is the unique $(0,0)$-orbit. If $f(0) \neq 0$ then $(0, \pm\sqrt{c_f}) \in \Gamma$, and these points are swapped by some element of inertia if and only if $\lambda_{\mathfrak{s}} \notin \mathbb{Z}$. If $\lambda_{\mathfrak{s}} \notin \mathbb{Z}$ then the orbit has size 2 hence it is only a small orbit if $e > 2$. □

Lemma 3.1.10. *Either $e = b_{\mathfrak{s}}$ or $e = 2b_{\mathfrak{s}}$, where $b_{\mathfrak{s}}$ is the denominator of $d_{\mathfrak{s}}$. In particular $e = 2b_{\mathfrak{s}}$ if and only if $b_{\mathfrak{s}}\nu_{\mathfrak{s}} \notin 2\mathbb{Z}$.*

Proof. By Theorem 2.4.2, e is the minimal integer such that $ed_{\mathfrak{s}} \in \mathbb{Z}$ and $e\nu_{\mathfrak{s}} \in 2\mathbb{Z}$. Since $ed_{\mathfrak{s}} \in \mathbb{Z}$, we can deduce that $b_{\mathfrak{s}} \mid e$. Furthermore, since $2b_{\mathfrak{s}}\nu_{\mathfrak{s}} \in 2\mathbb{Z}$, $e = b_{\mathfrak{s}}$ or $e = 2b_{\mathfrak{s}}$. It is straightforward to check that the other conditions of Theorem 2.4.2 are satisfied over a field extension of degree e. □

Lemma 3.1.11. *If $e > b_{\mathfrak{s}}$ then there are $\frac{|\mathfrak{s}|}{b_{\mathfrak{s}}}$ $(y = 0)$-orbits if $b_{\mathfrak{s}} \mid |\mathfrak{s}|$, or $\frac{|\mathfrak{s}|-1}{b_{\mathfrak{s}}}$ $(y = 0)$-orbits if $b_{\mathfrak{s}} \nmid |\mathfrak{s}|$.*

Proof. This follows since the non-zero points with $y = 0$ are of the form $(\zeta_{b_{\mathfrak{s}}}^i, 0)$ for $\zeta_{b_{\mathfrak{s}}}$ a primitive $b_{\mathfrak{s}}^{\text{th}}$ root of unity. Note that $(y = 0)$-orbits always have size $b_{\mathfrak{s}}$ so if $e = b_{\mathfrak{s}}$ then the $(y = 0)$-orbits are not small orbits. □

These lemmas allow us to fully describe how many singularities $\mathscr{X}_k$ has. The following proposition tells us that they are tame cyclic quotient singularities in the sense of Definition 2.3.2. Theorem 2.3.3 then allows us to resolve these singularities.

Proposition 3.1.12. *Let $z \in \mathscr{X}_k$ be a singularity which is the image of a Galois orbit $Y \subseteq \mathscr{Y}_k$. Then z is a tame cyclic quotient singularity. In addition, with notation as in Definition 2.3.2, $\frac{m}{r} = \frac{e}{r}$ where $1 \leq r < e$ and $r \mod e$ is given in the following table:*

Orbit Type	$r \mod e$	*Condition*
∞	$e\lambda_{\mathfrak{s}} - e(g(C)+1)d_{\mathfrak{s}}$	$\mathfrak{s}$ *odd*
∞	$-ed_{\mathfrak{s}}\lvert Y\rvert$	$\mathfrak{s}$ *even*
$y_{\mathfrak{s}} = 0$	$e\lambda_{\mathfrak{s}}\lvert Y\rvert$	*None*
$x_{\mathfrak{s}} = 0$	$ed_{\mathfrak{s}}\lvert Y\rvert$	*None*
$(0,0)$	$e\lambda_{\mathfrak{s}}$	*None*

Proof. Recall that for z to be a tame cyclic quotient singularity, there must exist $m > 1$ invertible in k, a unit $r \in (\mathbb{Z}/m\mathbb{Z})^\times$ and integers $m_1 > 0$ and $m_2 \geq 0$ such that $m_1 \equiv -rm_2 \mod m$, and such that $\mathcal{O}_{\mathscr{X},z}$ is equal to the subalgebra of μ_m-invariants in $k[\![t_1,t_2]\!]/(t_1^{m_1}t_2^{m_2} - \pi_K)$ under the action $t_1 \mapsto \zeta_m t_1, t_2 \mapsto \zeta_m^r t_2$. We will show that $m = \frac{e}{|Y|} = |\mathrm{Stab}(Y)|$, $m_1 = e$, $m_2 = 0$ and will explicitly calculate r.

Let $Y \subseteq \mathscr{Y}_k$ be a small orbit and let $Q \in Y$. Then $\mathcal{O}_{\mathscr{X},z}$ is the subalgebra of μ_m-invariants of $\mathcal{O}_{\mathscr{Y},Q}$ under the action of $\mathrm{Stab}(Y)$, where $m = |\mathrm{Stab(Y)}|$. This follows from the definition of $\mathscr{X}$ as the quotient of $\mathscr{Y}$ under the action of $\mathrm{Gal}(L/K)$, which for a generator $\sigma \in \mathrm{Gal}(L/K)$ sends

$$\sigma : \pi_L \longmapsto \chi(\sigma)\pi_L, \quad \sigma : x_{\mathfrak{s}} \longmapsto \chi(\sigma)^{ed_{\mathfrak{s}}} x_{\mathfrak{s}}, \quad \sigma : y_{\mathfrak{s}} \longmapsto \chi(\sigma)^{e\lambda_{\mathfrak{s}}} y_{\mathfrak{s}}.$$

To prove that z is a tame cyclic quotient singularity we must calculate $\mathcal{O}_{\mathscr{Y},Q}$.

First, suppose Y is a $(y_{\mathfrak{s}} = 0)$ or a $(0,0)$-orbit, and write $Q = (x_Q, 0)$. Then $\mathcal{O}_{\mathscr{Y},Q}$ is generated by $\pi_L, x_{\mathfrak{s}} - x_Q$ and $y_{\mathfrak{s}}$. However, since $x_{\mathfrak{s}} - x_Q = uy^2$ for a unit $u \in \mathcal{O}_{\mathscr{Y},Q}$, $\mathcal{O}_{\mathscr{Y},Q}$ is generated by π_L and $y_{\mathfrak{s}}$. Therefore, $\mathcal{O}_{\mathscr{Y},Q} \cong k[\![\pi_L, y_{\mathfrak{s}}]\!]/(\pi_L^e - \pi_K)$, and $\mathcal{O}_{\mathscr{X},z}$ is the subalgebra of μ_m-invariants of this under the action $\pi_L \mapsto \zeta_m \pi_L, y_{\mathfrak{s}} \mapsto \zeta_m^{e\lambda_{\mathfrak{s}}} y_{\mathfrak{s}}$ where $\zeta_m = \chi(\sigma)^{|Y|}$ generates $\mathrm{Stab}(Y)$ (as $\mathrm{Gal}(L/K)$ is cyclic). Let r be such that $0 < r < m$ and $r \equiv e\lambda_{\mathfrak{s}}|Y| \mod m$. Then to prove z is a tame cyclic quotient singularity all that is left to show is that r is a unit in $(\mathbb{Z}/m\mathbb{Z})^\times$ and that $e \equiv 0 \mod m$. The second is clear, and for the first note that since ζ_m^r also generates $\mathrm{Stab}(Y)$, it must be a primitive m^{th} root of unity hence r must be a unit.

If Y is an $(x_{\mathfrak{s}} = 0)$-orbit, then $Q = (0, \pm\sqrt{c_{f,L}})$. By a similar argument to above, $\mathcal{O}_{\mathscr{Y},Q} \cong k[\![\pi_L, x_{\mathfrak{s}}]\!]/(\pi_L^e - \pi_K)$ and $\mathcal{O}_{\mathscr{X},z}$ is the subalgebra of μ_m invariants under the action $\pi_L \mapsto \zeta_m \pi_L$, $x_\mathfrak{s} \mapsto \zeta_m^r x_\mathfrak{s}$, where $m = \frac{e}{|Y|}$ and r is such that $0 < r < m$ and $r \equiv ed_\mathfrak{s}|Y| \mod m$.

If Y is an ∞-orbit, then we can calculate m, r, m_1 and m_2 explicitly by going to the chart at infinity. □

Example 3.1.13. Recall example 3.1.3. Let $C : y^2 = x^7 - p^{11}$. The cluster picture and minimal snc model of C are recreated below.

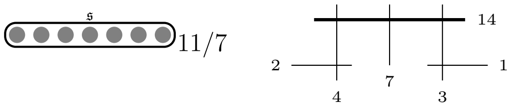


Figure 3.3: Cluster picture and minimal snc model of $C : y^2 = x^7 - p^{14}$.

The tail arising from the ∞-orbit is the rightmost. Proposition 3.1.12 states that the singularity from which it arises has tame cyclic quotient invariants$\frac{m}{r}$ where $m = \frac{e}{1} = 14$ since there is a single point at infinity and $r = e\lambda_\mathfrak{s} - e(g(C)+1)d_\mathfrak{s} = 14\left(\frac{11}{2} - 4 \times \frac{11}{7}\right) = 3$. Then we have the Hirzebruch-Jung continued fraction $\frac{14}{3} = 5 - \frac{1}{3}$ and hence the tail from the ∞-orbit consists of two components with self intersection -5 and -3 respectively. Some quick and dirty linear algebra then tells us that their multiplicities must be 3 and 1. Similarly the $(y_\mathfrak{s} = 0)$-orbit has size 7 and gives a singularity with tame cyclic quotient invariants$\frac{2}{1}$ and the $(x_\mathfrak{s} = 0)$-orbit has size 2 and gives a singularity with tame cyclic quotient invariants $\frac{7}{2}$, resolving to give the centre and leftmost tail respectively.

Corollary 3.1.14. *If Y is a $(y_\mathfrak{s} = 0)$-orbit which gives rise to a tame cyclic quotient singularity $z \in \mathscr{X}_k$, then the tame cyclic quotient invariants (m, r) of z are such that $\frac{m}{r} = 2$.*

Proof. The orbit Y is a $(y_\mathfrak{s} = 0)$-orbit hence has size $b_\mathfrak{s}$. Lemma 3.1.10 tells us that, $|Y| < e$ if and only if $e = 2b_\mathfrak{s}$. In this case $e\lambda_\mathfrak{s}|Y| = 2b_\mathfrak{s} \cdot \frac{\nu_\mathfrak{s}}{2} \cdot b_\mathfrak{s} = b_\mathfrak{s}^2 \nu_\mathfrak{s}$. Since $b_\mathfrak{s} = \frac{e}{2}$ and $b_\mathfrak{s}\nu_\mathfrak{s}$ is an odd integer, this gives $e\lambda_\mathfrak{s}|Y| \equiv \frac{e}{2} \mod e$, hence $\frac{m}{r} = 2$. □

3.1.3 Tails

Resolving singularities as in Section 3.1.2 results in tails. These are chains of rational curves intersecting the central component once and intersecting the rest of the special fibre nowhere else. It is useful to distinguish between tails based on the type of orbit they arise from.

Definition 3.1.15 (Tails)**.** Define the following tails based on the type of singularity of $\mathscr{X}_k$ they arise from:

— ∞-*tail*: arising from the blow up of a singularity of $\mathscr{X}_k$ which arose from an ∞-orbit,

— $(y_{\mathfrak{s}} = 0)$-*tail*: arising from the blow up of a singularity of $\mathscr{X}_k$ which arose from an orbit of non-zero roots,

— $(x_{\mathfrak{s}} = 0)$-*tail*: arising from the blow up of a singularity of $\mathscr{X}_k$ which arose from an orbit on the points $(0, \pm\sqrt{c_{f,L}})$,

— $(0,0)$-*tail*: arising from the blow up of a singularity of $\mathscr{X}_k$ which arose from the point $(0,0)$.

Proof of Theorem 3.1.1. The central component Γ is the image of the unique component of $\mathscr{Y}_k$ under q. Since blowing up points on Γ does not affect its multiplicity, this has multiplicity e, by Proposition 2.3.1. The description of the tails follows from Lemmas 3.1.8, 3.1.9, and 3.1.11, since the tails are in a bijective correspondence with the orbits of points of $\mathscr{Y}_k$ of size strictly less than e. We must check that Γ really appears in the minimal snc model. Suppose Γ is exceptional. Then $g(\Gamma) = 0$ and Riemann-Hurwitz tells us

$$\sum_{z \in \mathscr{X}_k} \left(\frac{e}{|q^{-1}(z)|} - 1 \right) \geq e.$$

Therefore there must be at least three ramification points, so Γ intersects at least three tails. $\square$

Remark 3.1.16. The method for calculating the multiplicities of the rational curves in these tails is described in Theorem 2.3.3 using the tame cyclic quotient invariants given in Proposition 3.1.12.

Remark 3.1.17. The central component Γ is the only component of $\mathscr{X}_k$ which may have non-zero genus. Its genus, $g(\Gamma)$, can be calculated via the Riemann-Hurwitz formula. Explicitly,

$$2g_{\mathrm{ss}}(\mathfrak{s}) - 2 = e(2g(\Gamma) - 2) + \sum_{z \in \mathscr{Z}} \left(\frac{e}{|q^{-1}(z)|} - 1 \right).$$

An even more explicit calculation of $g(\Gamma)$ in terms of the Newton polytope is given in Proposition 3.1.25.

3.1.4 Relation to Newton polytopes

Up to this point, this section has described the minimal snc model of a hyperelliptic curve C/K with tame potentially good reduction using the methods from Section 2.3. However, such a hyperelliptic curve has a nested cluster picture so we can also calculate the minimal snc model using Newton polytopes and the techniques described in Section 2.5.1. By the uniqueness of the minimal snc model, these two methods will give the same result: for the reader's sanity, in this section we will show that this is indeed the case. Recall that without loss of generality we can assume that C/K with tame potentially good reduction is given by one of the following two equations:

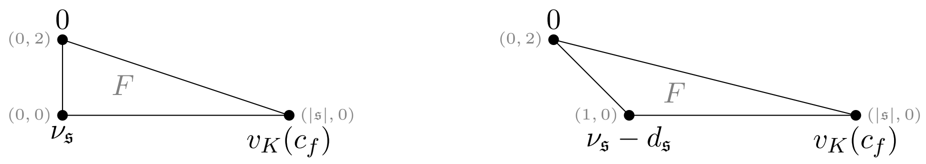
$$y^2 = c_f \prod_{0 \neq r \in \mathcal{R}} (x - u_r \pi_K^{d_\mathfrak{s}}), \qquad \text{if } b_\mathfrak{s} \mid |\mathfrak{s}|,$$

$$y^2 = c_f x \prod_{0 \neq r \in \mathcal{R}} (x - u_r \pi_K^{d_\mathfrak{s}}), \qquad \text{if } b_\mathfrak{s} \nmid |\mathfrak{s}|.$$

The Newton polytope of C is shown in Figure 3.4a if $b_\mathfrak{s} \mid |\mathfrak{s}|$, and in Figure 3.4b if $b_\mathfrak{s} \nmid |\mathfrak{s}|$. In each case there is exactly one v-face of $\Delta_v(C)$, which we shall label F. Therefore, by Theorem 2.5.11, the minimal snc model consists of a central component $\Gamma_\mathfrak{s} = \Gamma_F$, and possibly tails arising from the three outer v-edges of F.

Lemma 3.1.18. *The multiplicity of* $\Gamma_\mathfrak{s} = \Gamma_F$ *is* δ_F*; that is* $\delta_F = e$.

Proof. We will first show that $e \mid \delta_F$, and then that $\delta_F \mid e$. Note that, in both Newton polytopes in Figure 3.4, the valuation map is given by the affine



(a) If $0 \notin \mathcal{R}$. **(b)** If $0 \in \mathcal{R}$.

Figure 3.4: $\Delta_v(C)$ of a hyperelliptic curve C with tame potential good reduction.

function

$$v_\Delta(x, y) = \nu_{\mathfrak{s}} - d_{\mathfrak{s}} x - \frac{\nu_{\mathfrak{s}}}{2} y.$$

Since e is such that $ed_{\mathfrak{s}} \in \mathbb{Z}$ and $e\nu_{\mathfrak{s}} \in 2\mathbb{Z}$, we have that

$$ev_\Delta(x, y) = e\nu_{\mathfrak{s}} - ed_{\mathfrak{s}} x - e\frac{\nu_{\mathfrak{s}}}{2} y \in \mathbb{Z}.$$

As δ_F is the common denominator of all $v_\Delta(x, y)$ for $x, y \in \Delta$, this gives that $\delta_F \mid e$. For the other direction, note that

$$\delta_F \left(v_\Delta(n-1, 0) - v_\Delta(n, 0) \right) = \delta_F d_{\mathfrak{s}} \in \mathbb{Z},$$

and

$$\delta_F \left(v_\Delta(1, 0) - v_\Delta(1, 1) \right) = \delta_F \frac{\nu_{\mathfrak{s}}}{2} \in \mathbb{Z}.$$

By minimality of e, this implies $e \mid \delta_F$. □

Lemma 3.1.19. *The ∞-tails arise from the outer v-edge of $\Delta_v(C)$ between $(0, 2)$ and $(|\mathfrak{s}|, 0)$.*

Proof. We will first check that this v-edge gives the correct number of ∞-tails, and then calculate the slope to check that the multiplicities of the components are the same.

Let us call this v-edge L. By Theorem 2.5.11 then L contributes $|\overline{L}(\mathbb{Z})_{\mathbb{Z}}| - 1$ tails to the snc model. Since the points $(0, 2), (|\mathfrak{s}|, 0) \in \bar{L}(\mathbb{Z})_{\mathbb{Z}}$, it contributes two tails if and only if $P = (\frac{|\mathfrak{s}|}{2}, 1) \in \bar{L}(\mathbb{Z})_{\mathbb{Z}}$. If $\mathfrak{s}$ is odd then $P \notin \bar{L} \cap \mathbb{Z}^2$, hence L contributes one tail. If $\mathfrak{s}$ is even then $v_\Delta(P) = \frac{v_K(c_f)}{2}$, hence $P \in \bar{L}(\mathbb{Z})_{\mathbb{Z}}$ if and only if $v_K(c_f) \in 2\mathbb{Z}$. Therefore L contributes one tail if $\mathfrak{s}$ is even and $v_K(c_f)$ is odd, and two tails if $\mathfrak{s}$ and $v_K(c_f)$ are even. This agrees with Theorem 3.1.1.

A quick calculation tells us that $\delta_L = 2$ if and only if $\mathfrak{s}$ is even and $v_K(c_f) \notin 2\mathbb{Z}$, and that $\delta_L = 1$ otherwise. Therefore, $\delta_L = |Y|$, where Y is the orbit at infinity. The unique surjective affine function which is zero on L and non-negative on F is $L_F^*(x,y) = 2|\mathfrak{s}| - 2x - |\mathfrak{s}|y$ if $\mathfrak{s}$ is odd, and $L_F^*(x,y) = |\mathfrak{s}| - x - \frac{1}{2}|\mathfrak{s}|y$ if $\mathfrak{s}$ is even. Therefore, $s_1^L = (g+1)d_{\mathfrak{s}} - \lambda_{\mathfrak{s}}$ if $\mathfrak{s}$ is odd, and $s_1^L = -d_{\mathfrak{s}}|Y|$ if $\mathfrak{s}$ is even. Since the multiplicities of the components of a tail are the Hirzebruch-Jung approximants of the slopes, we are done after comparing the slopes to the table in Proposition 3.1.12.

If $e = 2$ (when $\mathfrak{s}$ is even and $v_K(c_f)$ is odd) then $s_L^1 \in \mathbb{Z}$, so the associated tail is empty, which agrees with the table in Theorem 3.1.1. □

Lemma 3.1.20. *In both cases, when* $0 \in \mathcal{R}$ *and when* $0 \notin \mathcal{R}$*, the* $(y_{\mathfrak{s}} = 0)$*-tails arise from the outer* v*-edge of* $\Delta_v(C)$ *on the* x*-axis. Also, if* $b_{\mathfrak{s}} \mid |\mathfrak{s}|$ *then the* $(x_{\mathfrak{s}} = 0)$*-tails arise from the* v*-edge between* $(0,0)$ *and* $(0,2)$*. Else the* $(0,0)$*-tail arises from the* v*-edge between* $(1,0)$ *and* $(0,2)$*.*

Proof. This follows after a similar calcuation to Lemma 3.1.19. □

3.1.5 The Curve $C_{\tilde{\mathfrak{s}}}$

To conclude this section, we drop the requirement for C/K to have tame potentially good reduction. We will describe a hyperelliptic curve with potentially good reduction which we associate to a principal cluster $\mathfrak{s} \in \Sigma_C$ with $g_{\mathrm{ss}}(\mathfrak{s}) > 0$. This new curve, which we will denote by $C_{\tilde{\mathfrak{s}}}$, will be invaluable in describing the components of the minimal snc model of C/K which are associated to $\mathfrak{s} \in \Sigma_C$. For $\mathfrak{s} \in \Sigma_{C/K}$ with $g_{\mathrm{ss}}(\mathfrak{s}) > 0$, the cluster picture $\Sigma_{\tilde{\mathfrak{s}}}$ of $C_{\tilde{\mathfrak{s}}}/K$ will be such that the singletons in $\Sigma_{\tilde{\mathfrak{s}}}$ correspond to odd children of $\mathfrak{s}$ and the even children of $\mathfrak{s}$ are in effect discarded. The leading coefficient of $C_{\tilde{\mathfrak{s}}}/K$ is chosen so that everything behaves well, and allows us to make the comparisons we wish between the minimal snc model of C/K and the minimal snc model of $C_{\tilde{\mathfrak{s}}}/K$.

Definition 3.1.21 ($C_{\tilde{\mathfrak{s}}}$)**.** Let C/K be a hyperelliptic curve, not necessarily with tame potentially good reduction. Let $\mathfrak{s} \in \Sigma$ be a principal cluster with $g_{\mathrm{ss}}(\mathfrak{s}) > 0$ such that $\mathfrak{s}$ is fixed by $\mathrm{Gal}(\overline{K}/K)$. Suppose furthermore that centres are chosen such that $\sigma(z_{\mathfrak{s}'}) = z_{\sigma(\mathfrak{s}')}$ for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$, $\mathfrak{s}' \in \Sigma_{C/K}$. We define another hyperelliptic curve $C_{\tilde{\mathfrak{s}}}/K$ by

$$C_{\tilde{\mathfrak{s}}} : y^2 = c_{f_{\mathfrak{s}}} \prod_{\mathfrak{o} \in \tilde{\mathfrak{s}}} (x - z_{\mathfrak{o}}), \text{ where } c_{f_{\mathfrak{s}}} = c_f \prod_{r \notin \mathfrak{s}} (z_{\mathfrak{s}} - r).$$

Write $\Sigma_{\tilde{\mathfrak{s}}/K} = \Sigma_{\tilde{\mathfrak{s}}} = \Sigma(C_{\tilde{\mathfrak{s}}}/K)$ for the cluster picture of $C_{\tilde{\mathfrak{s}}}/K$, and $\mathscr{X}_{\tilde{\mathfrak{s}}}$ for the minimal snc model of $C_{\tilde{\mathfrak{s}}}/K$. The special fibre of the minimal snc model of $C_{\tilde{\mathfrak{s}}}$ is denoted $\mathscr{X}_{\tilde{\mathfrak{s}},k}$, and the central component is denoted $\Gamma_{\tilde{\mathfrak{s}}}$. We also write $\mathcal{R}_{\tilde{\mathfrak{s}}}$ for the set of all roots of $c_{f_{\mathfrak{s}}} \prod_{\mathfrak{o} \in \tilde{\mathfrak{s}}} (x - z_{\mathfrak{o}})$, and define $d_{\tilde{\mathfrak{s}}} = d_{\mathcal{R}_{\tilde{\mathfrak{s}}}}$, $\nu_{\tilde{\mathfrak{s}}} = \nu_{\mathcal{R}_{\tilde{\mathfrak{s}}}}$, and $\lambda_{\tilde{\mathfrak{s}}} = \lambda_{\mathcal{R}_{\tilde{\mathfrak{s}}}}$.

Remark 3.1.22. Let $\mathscr{Y}$ be the minimal semistable model of C over $\mathcal{O}_L$, for some L/K such that C/L is semistable. Let $\mathfrak{s}$ be a principal cluster with $g_{\mathrm{ss}}(\mathfrak{s}) > 0$. If we reduce $C_{\tilde{\mathfrak{s}}} \mod \mathfrak{m}$, we obtain $\Gamma_{\mathfrak{s},L}$, the component of $\mathscr{Y}_k$ corresponding to $\mathfrak{s}$ (see Definition 2.4.10 for the equation of $\Gamma_{\mathfrak{s},L}$). In addition, $c_{f_{\mathfrak{s}}}$ has been carefully chosen so that $d_{\mathfrak{s}} = d_{\tilde{\mathfrak{s}}}$, $\nu_{\mathfrak{s}} = \nu_{\tilde{\mathfrak{s}}}$ and $\lambda_{\mathfrak{s}} = \lambda_{\tilde{\mathfrak{s}}}$. In particular, the automorphisms induced by Galois on $\Gamma_{\mathfrak{s},L}$ and $\Gamma_{\tilde{\mathfrak{s}},L}$ are the same.

Definition 3.1.23 ($e_{\mathfrak{s}}, g(\mathfrak{s})$)**.** For a principal, Galois-invariant cluster $\mathfrak{s}$, define $e_{\mathfrak{s}}$ to be the minimum integer such that $e_{\mathfrak{s}} d_{\mathfrak{s}} \in \mathbb{Z}$ and $e_{\mathfrak{s}} \nu_{\mathfrak{s}} \in 2\mathbb{Z}$. Furthermore, if $g_{\mathrm{ss}}(\mathfrak{s}) > 0$ define $g(\mathfrak{s})$ to be the *genus of* $\Gamma_{\tilde{\mathfrak{s}}}$ and if $g_{\mathrm{ss}}(\mathfrak{s}) = 0$ define $g(\mathfrak{s}) = 0$. We call $g(\mathfrak{s})$ the *genus of* $\mathfrak{s}$.

Remark 3.1.24. By the Semistability Criterion 2.4.2, if $\mathfrak{s}$ is not übereven then $e_{\mathfrak{s}}$ is the minimum integer such that $C_{\tilde{\mathfrak{s}}}$ has semistable reduction over a field extension L/K of degree $e_{\mathfrak{s}}$. In particular, the central component $\Gamma_{\tilde{\mathfrak{s}}}$ of $\mathscr{X}_{\tilde{\mathfrak{s}},k}$ has multiplicity $e_{\mathfrak{s}}$ and genus $g(\mathfrak{s})$. If $e_{\mathfrak{s}} = 1$ then $g_{\mathrm{ss}}(\mathfrak{s}) = g(\mathfrak{s})$, but the converse is not necessarily true.

Proposition 3.1.25. *If $g_{ss}(\mathfrak{s}) > 0$, the genus $g(\mathfrak{s})$ is given by*

$$g(\mathfrak{s}) = \begin{cases} \lfloor \frac{g_{ss}(\mathfrak{s})}{b_\mathfrak{s}} \rfloor & \lambda_\mathfrak{s} \in \mathbb{Z}, \\ \lfloor \frac{g_{ss}(\mathfrak{s})}{b_\mathfrak{s}} + \frac{1}{2} \rfloor & \lambda_\mathfrak{s} \notin \mathbb{Z}, b_\mathfrak{s} \text{ even}, \\ 0 & \lambda_\mathfrak{s} \notin \mathbb{Z}, b_\mathfrak{s} \text{ odd}. \end{cases}$$

Proof. By Theorem 2.5.11, we know $g(\mathfrak{s})$ is given by $|F(\mathbb{Z})_\mathbb{Z}|$. This is the number of interior points with integer valuation of the unique face F of the Newton polytope of $C_{\tilde{\mathfrak{s}}}$. By examining Figure 3.4, we see that all interior points are of the form $(x, 1)$ with $1 \leq x \leq g_{\mathrm{ss}}(\mathfrak{s})$. For such points, $v_\Delta(x, 1) = \lambda_\mathfrak{s} - d_\mathfrak{s} x$. Therefore,

$$g(\mathfrak{s}) = |\{x : 1 \leq x \leq g_{\mathrm{ss}}(\mathfrak{s}), \lambda_\mathfrak{s} - x d_\mathfrak{s} \in \mathbb{Z}\}|.$$

When $\lambda_\mathfrak{s} \in \mathbb{Z}$ this is therefore equal to

$$|\{x : 1 \leq x \leq g_{\mathrm{ss}}(\mathfrak{s}), b_\mathfrak{s} \mid x\}| = \left\lfloor \frac{g_{\mathrm{ss}}(\mathfrak{s})}{b_\mathfrak{s}} \right\rfloor.$$

When $\lambda_\mathfrak{s} \notin \mathbb{Z}$, this is equal to

$$\left| \left\{ x : 1 \leq x \leq g_{\mathrm{ss}}(\mathfrak{s}), x d_\mathfrak{s} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \right\} \right|.$$

When $\lambda_\mathfrak{s} \notin \mathbb{Z}$ and $b_\mathfrak{s}$ is odd this set is always empty, and when $\lambda_\mathfrak{s} \notin \mathbb{Z}$ and $b_\mathfrak{s}$ is even it has size $\left\lfloor \frac{g_{\mathrm{ss}}(\mathfrak{s})}{b_\mathfrak{s}} + \frac{1}{2} \right\rfloor$. □

Lemma 3.1.26. *Let C be a hyperelliptic curve and let $\mathfrak{s} \in \Sigma_C$ be a principal cluster which is fixed by Galois. Let L be an extension such that C is semistable over L, and let σ generate Gal(L/K). Then $\sigma|_{\Gamma_{\mathfrak{s},L}} : \Gamma_{\mathfrak{s},L} \to \Gamma_{\mathfrak{s},L}$ has degree $e_\mathfrak{s}$.*

Proof. The map $\sigma|_{\Gamma_{\mathfrak{s},L}}$ is given by $(x_\mathfrak{s}, y_\mathfrak{s}) \mapsto (\chi(\sigma)^{e_\mathfrak{s} d_\mathfrak{s}} x_\mathfrak{s}, \chi(\sigma)^{e_\mathfrak{s} \lambda_\mathfrak{s}} y_\mathfrak{s})$. The result follows as $e_\mathfrak{s}$, by definition, is the minimal integer such that $e_\mathfrak{s} d_\mathfrak{s}, e_\mathfrak{s} \lambda_\mathfrak{s} \in \mathbb{Z}$. □

3.2 Curves with Two Clusters

The minimal snc model of a general hyperelliptic curve C/K can roughly be described as follows. Each principal cluster of Σ_C has one or two central components, and some tails associated to it. These principal components are linked by chains of rational curves. Section 3.1 will allow us to describe these central components and tails, while this section will be used to describe these linking chains. This includes describing any loops. We will also see the simplest example of the general philosophy that the components of the special fibre of the minimal snc model of C/K associated to a principal cluster $\mathfrak{s}$ "look like" the special fibre of the minimal snc model of $C_{\tilde{\mathfrak{s}}}/K$.

Throughout the rest of this section we will take C/K to be a hyperelliptic curve such that $\Sigma_{C/K}$ consists of exactly two proper clusters: a proper cluster $\mathfrak{s}$ and a unique proper child $\mathfrak{s}' < \mathfrak{s}$. This is illustrated in Figure 3.5. Note that $d_{\mathfrak{s}'} > d_{\mathfrak{s}}$ and $|\mathfrak{s}| > |\mathfrak{s}'|$. If C is such that $\mathfrak{s}$ is even and $|\mathfrak{s}| = |\mathfrak{s}'| + 1$ then C/K has potentially good reduction, this case is covered in Section 3.1. To avoid this case we will assume that if $\mathfrak{s}$ is even then $|\mathfrak{s}| \geq |\mathfrak{s}'| + 2$. Since hyperelliptic curves of this type are nested we can directly apply the methods from Section 2.5.1.

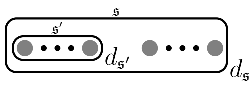


Figure 3.5: Cluster picture with a parent $\mathfrak{s}$ and a unique proper child $\mathfrak{s}'$ with no proper children of its own.

3.2.1 Structure of Special Fibre

For such a hyperelliptic curve C we prove the following structure theorem.

Theorem 3.2.1 (Minimal snc model, unique proper child). *Let C/K be a hyperelliptic curve with cluster picture as in Figure 3.5. If $\mathfrak{s}$ is principal, then the special fibre of the minimal snc model has a component $\Gamma_{\mathfrak{s},K}$ arising from $\mathfrak{s}$ with multiplicity $e_{\mathfrak{s}}$ and genus $g(\mathfrak{s})$. If $\mathfrak{s}'$ is principal then there is a component $\Gamma_{\mathfrak{s}',K}$ arising from $\mathfrak{s}'$ of multiplicity $e_{\mathfrak{s}'}$ and genus $g(\mathfrak{s}')$. These are linked by sloped chain(s) of rational curves with parameters $(t_1 - \delta, t_1, \mu)$, which are described in the following table:*

<table>
<tr><th>Name</th><th>From</th><th>To</th><th>t_1</th><th>δ</th><th>μ</th><th>Conditions</th></tr>
<tr><td>$L_{\mathfrak{s},\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}}$</td><td>$\Gamma_{\mathfrak{s}'}$</td><td>$-\lambda_{\mathfrak{s}}$</td><td>$\delta_{\mathfrak{s}'}/2$</td><td>1</td><td>$\mathfrak{s}$ principal, $\mathfrak{s}'$ odd, principal</td></tr>
<tr><td>$L^{+}_{\mathfrak{s},\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}}$</td><td>$\Gamma_{\mathfrak{s}'}$</td><td rowspan="2">$-d_{\mathfrak{s}}$</td><td rowspan="2">$\delta_{\mathfrak{s}'}$</td><td rowspan="2">1</td><td rowspan="2">$\mathfrak{s}$ principal, $\mathfrak{s}'$ even, principal, $\epsilon_{\mathfrak{s}'} = 1$</td></tr>
<tr><td>$L^{-}_{\mathfrak{s},\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}}$</td><td>$\Gamma_{\mathfrak{s}'}$</td></tr>
<tr><td>$L_{\mathfrak{s},\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}}$</td><td>$\Gamma_{\mathfrak{s}'}$</td><td>$-d_{\mathfrak{s}}$</td><td>$\delta_{\mathfrak{s}'}$</td><td>2</td><td>$\mathfrak{s}$ principal, $\mathfrak{s}'$ even, principal, $\epsilon_{\mathfrak{s}'} = -1$</td></tr>
<tr><td>$L_{\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}}$</td><td>$\Gamma_{\mathfrak{s}}$</td><td>$-d_{\mathfrak{s}}$</td><td>$2\delta_{\mathfrak{s}'}$</td><td>1</td><td>$\mathfrak{s}$ principal, $\mathfrak{s}'$ twin, $\epsilon_{\mathfrak{s}'} = 1$</td></tr>
<tr><td>$T_{\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}}$</td><td>-</td><td>$-d_{\mathfrak{s}}$</td><td>$\delta_{\mathfrak{s}'} + \frac{1}{2}$</td><td>2</td><td>$\mathfrak{s}$ principal, $\mathfrak{s}'$ twin, $\epsilon_{\mathfrak{s}'} = -1$</td></tr>
<tr><td>$L_{\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}'}$</td><td>$-d_{\mathfrak{s}}$</td><td>$2\delta_{\mathfrak{s}'}$</td><td>1</td><td>$\mathfrak{s}$ cotwin, $v_K(c_f) \in 2\mathbb{Z}$</td></tr>
<tr><td>$T_{\mathfrak{s}'}$</td><td>$\Gamma_{\mathfrak{s}'}$</td><td>-</td><td>$-d_{\mathfrak{s}}$</td><td>$\delta_{\mathfrak{s}'} + \frac{1}{2}$</td><td>2</td><td>$\mathfrak{s}$ cotwin, $v_K(c_f) \notin 2\mathbb{Z}$</td></tr>
</table>

The chains where the "To" column has been left empty are crossed tails with crosses of multiplicity 1*. If* $\mathfrak{s}$ *is principal and* $e_{\mathfrak{s}} > 1$ *then* $\Gamma_{\mathfrak{s}}$ *has the following tails with parameters* (t_1, μ)*:*

Name	*Number*	t_1	μ	*Condition*
T_∞	1	$(g(\mathfrak{s})+1)d_{\mathfrak{s}} - \lambda_{\mathfrak{s}}$	1	$\mathfrak{s}$ *odd*
$T^{\pm}_\infty$	2	$-d_{\mathfrak{s}}$	1	$\mathfrak{s}$ *even and* $\epsilon_{\mathfrak{s}} = 1$
T_∞	1	$-d_{\mathfrak{s}}$	2	$\mathfrak{s}$ *even,* $\epsilon_{\mathfrak{s}} = -1$ *and* $e_{\mathfrak{s}} > 2$
$T_{y_{\mathfrak{s}}=0}$	$\lvert\mathfrak{s}_{\text{sing}}\rvert/b_{\mathfrak{s}}$	$-\lambda_{\mathfrak{s}}$	$b_{\mathfrak{s}}$	$e_{\mathfrak{s}} = 2b_{\mathfrak{s}}$

If $\mathfrak{s}'$ *is principal and* $e_{\mathfrak{s}'} > 1$ *then* $\Gamma_{\mathfrak{s}'}$ *has the following tails with parameters* (t_1, μ)*:*

Name	*Number*	t_1	μ	*Condition*
$T_{y_{\mathfrak{s}}=0}$	$\lvert\mathfrak{s}'_{\text{sing}}\rvert/b_{\mathfrak{s}'}$	$-\lambda_{\mathfrak{s}'}$	$b_{\mathfrak{s}'}$	$e_{\mathfrak{s}'} = 2b_{\mathfrak{s}'}$
$T_{x_{\mathfrak{s}}=0}$	1	$-d_{\mathfrak{s}'}$	2	$b_{\mathfrak{s}'} \mid \lvert\mathfrak{s}'\rvert$, $\lambda_{\mathfrak{s}'} \notin \mathbb{Z}$ *and* $e_{\mathfrak{s}'} > 2$
$T^{\pm}_{x_{\mathfrak{s}}=0}$	2	$-d_{\mathfrak{s}'}$	1	$b_{\mathfrak{s}'} \mid \lvert\mathfrak{s}'\rvert$, $\lambda_{\mathfrak{s}'} \in \mathbb{Z}$
$T_{(0,0)}$	1	$-\lambda_{\mathfrak{s}'}$	1	$b_{\mathfrak{s}'} \nmid \lvert\mathfrak{s}'\rvert$

Remark 3.2.2. For this particular type of hyperelliptic curve, $\mathfrak{s}$ will be principal unless it is a cotwin (i.e. if $|\mathfrak{s}'| = 2g(C)$), and $\mathfrak{s}'$ will be principal unless it is a twin. Since we have assumed that $g \geq 2$, these cases cannot coincide. Note that neither $\mathfrak{s}$ nor $\mathfrak{s}'$ can be übereven in this case.

Remark 3.2.3. Suppose $\mathfrak{s}$ is principal. In $\mathscr{X}_k$ we can see most of the components of $\mathscr{X}_{\tilde{\mathfrak{s}},k}$. The central component $\Gamma_{\mathfrak{s}}$ will have the same multiplicity and genus as $\Gamma_{\tilde{\mathfrak{s}}}$, and will have almost the same tails. The only difference being that one or two of the tails (the $(0,0)$-tail in the case $\mathfrak{s}'$ is odd and the $(x_{\mathfrak{s}} = 0)$-tail(s) otherwise) will instead form either part of a linking chain between $\Gamma_{\mathfrak{s}}$ and $\Gamma_{\mathfrak{s}'}$ (in the case $\mathfrak{s}'$ principal); or a loop or a crossed tail associated to $\mathfrak{s}'$ (in the case where $\mathfrak{s}'$ is a twin). We will say that the downhill section of the linking chain *corresponds* to this tail. If the linking chain, loop or crossed tail in $\mathscr{X}_k$ has a non-trivial level section, then all the components of the tails in $\mathscr{X}_{\mathfrak{s},k}$ appear in the linking chain(s) in $\mathscr{X}_k$. If the level section has length zero then some of the lower multiplicity components do not appear - we expand on this in Section 3.2.4.

Similarly, if $\mathfrak{s}'$ is principal, we see most of the components of $\mathscr{X}_{\tilde{\mathfrak{s}}',k}$ in $\mathscr{X}_k$. In this case, $\Gamma_{\mathfrak{s}'}$ has the same tails as $\Gamma_{\tilde{\mathfrak{s}}'}$ except that the infinity tail(s) of the latter are absorbed into the linking chain(s) $L_{\mathfrak{s},\mathfrak{s}'}$ (or the loop or crossed tail arising from $\mathfrak{s}$ if it is a cotwin). In this case, we say that the uphill section of the linking chain *corresponds* to the infinity tail in $\mathscr{X}_{\tilde{\mathfrak{s}}',k}$. We shall see that this is a phenomenon which generalises to the main theorems in Section 4.

Remark 3.2.4. The length of the level section of a linking chain, loop or crossed tail $\mathcal{C} \subseteq \mathscr{X}_k$ (that is, the number of $\mathbb{P}^1$s with multiplicity μ) is equal to $|(\mu(t_1 - \delta), \mu t_1) \cap \mathbb{Z}|$. Let $\mathscr{Y}$ be the minimal regular model of C over L, $q : \mathscr{Y} \to \mathscr{Z}$ be the quotient by $\mathrm{Gal}(L/K)$ and $\phi : \mathscr{X} \to \mathscr{Z}$ the resolution of singularities. Then any irreducible component E in the level section of $\mathcal{C}$ is *not* an exceptional divisor — that is to say, it is the image of μ components of $\mathscr{Y}_k$ which are permuted by $\mathrm{Gal}(L/K)$. This can be seen by looking at the explicit automorphisms on the components of $\mathscr{Y}$ given in [19, Theorem 6.2].

Example 3.2.5. Consider the hyperelliptic curve $C : y^2 = (x^2 - p)(x^3 - p^5)$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$. The special fibre of the minimal snc model of C/K can be seen in Figure 3.6. The central components $\Gamma_{\mathfrak{s}}$, and $\Gamma_{\mathfrak{s}'}$ are labelled and shown in bold.

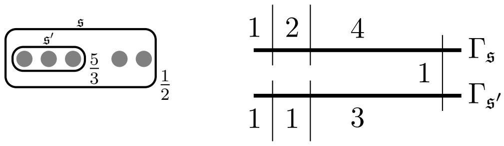


Figure 3.6: Cluster picture and special fibre of the minimal snc model of $C : y^2 = (x^2 - p)(x^3 - p^5)$.

The minimal snc models of the curves $C_{\tilde{\mathfrak{s}}}$ and $C_{\tilde{\mathfrak{s}}'}$ are pictured in Figure 3.7 below.

(a) where $C_{\tilde{\mathfrak{s}}} : y^2 = x(x^2 - p)$.

(b) where $C_{\tilde{\mathfrak{s}}'} : y^2 = p(x^3 - p^5)$.

Figure 3.7: The special fibres of the minimal snc models of $C_{\tilde{\mathfrak{s}}}$ and $C_{\tilde{\mathfrak{s}}'}$

We can see that all the components in both Figures 3.7a and 3.7b also appear in the special fibre of the minimal snc model of C. They are glued together along one of their multiplicity one components which forms the linking chain in Figure 3.6. This illustrates Remark 3.2.3.

3.2.2 The Newton polytope

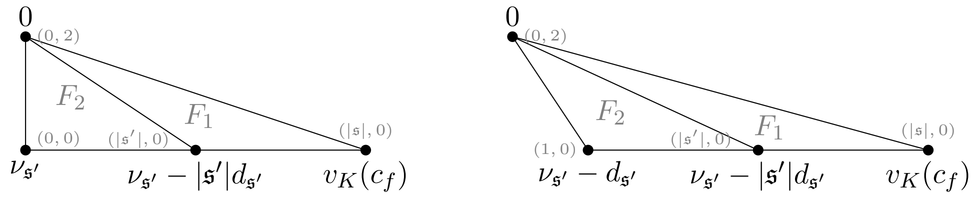
Without loss of generality, we can assume that the defining equation of C/K will be either

$$y^2 = c_f \prod_{r \in \mathcal{R} \setminus \mathfrak{s}'} \left(x - u_r \pi_K^{d_{\mathfrak{s}}}\right) \prod_{r \in \mathfrak{s}'} \left(x - u_r \pi_K^{d_{\mathfrak{s}'}}\right), \tag{3.1}$$

or

$$y^2 = c_f x \prod_{r \in \mathcal{R} \setminus \mathfrak{s}'} \left(x - u_r \pi_K^{d_{\mathfrak{s}}}\right) \prod_{0 \neq r \in \mathfrak{s}'} \left(x - u_r \pi_K^{d_{\mathfrak{s}'}}\right). \tag{3.2}$$

where the u_r are units. If C has defining equation (3.1), then $\nu_{\mathfrak{s}'} = v_K(c_f) + (|\mathfrak{s}| - |\mathfrak{s}'|)d_{\mathfrak{s}} + |\mathfrak{s}'|d_{\mathfrak{s}'}$, and the Newton polytope $\Delta_v(C)$ of C will be as shown in Figure 3.8a. If instead C has defining equation (3.2), the Newton polytope will be as shown in Figure 3.8b.



(a) if C has defining equation (3.1) **(b)** if C has defining equation (3.2)

Figure 3.8: Newton polytope $\Delta_v(C)$ of C.

Lemma 3.2.6. *Let C have Newton polytope as in Figure 3.8a. Then there is an isomorphism $\psi : \overline{F_1} \to \Delta_v(C_{\tilde{\mathfrak{s}}})$, from the closure of the v-face marked F_1 to the Newton polytope of $C_{\tilde{\mathfrak{s}}}$ (whose only v-face we label $F_{\tilde{\mathfrak{s}}}$), shown in Figure 3.9. In particular ψ preserves valuations and $\delta_{F_1} = \delta_{F_{\tilde{\mathfrak{s}}}}$. In this sense we say that F_1* corresponds *to the cluster $\mathfrak{s}$. Similarly the v-face F_2 in Figure 3.8a corresponds to $\mathfrak{s}'$.*

Proof of Lemma 3.2.6. Let us compare the v-face F_1 in Figure 3.8a to the Newton polytope, $\Delta_v(C_{\tilde{\mathfrak{s}}})$, of $C_{\tilde{\mathfrak{s}}}$. This is given in Figure 3.9a if $\mathfrak{s}'$ is even, and given in Figure 3.9b if $\mathfrak{s}'$ is odd.

If $\mathfrak{s}'$ is even we can define

$$\psi : \overline{F_1} \to \Delta_v(C_{\tilde{\mathfrak{s}}}) : (x, y) \mapsto \left(x - \frac{|\mathfrak{s}'|}{2}(2 - y), y \right).$$

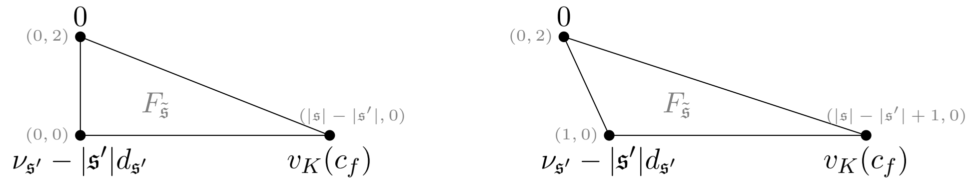
It is easy to show that this is an isomorphism, and that the valuations are preserved. Similarly if $\mathfrak{s}'$ is odd we can define

$$\psi : \overline{F_1} \to \Delta_v(C_{\tilde{\mathfrak{s}}}) : (x, y) \mapsto \left(x - \frac{(|\mathfrak{s}'| + 1)}{2}(2 - y), y \right),$$

which is also an isomorphism that preserves the valuations. In particular, in both cases we have $\delta_{F_1} = \delta_{F_{\tilde{\mathfrak{s}}}}$, and if v_1 is the unique affine function agreeing with $v_{\Delta(C)}$ on F_1, then $v_1(x, y) = v_{\Delta_{\tilde{\mathfrak{s}}}}(\psi(x, y))$, where $v_{\Delta_{\tilde{\mathfrak{s}}}} = v_{\Delta(C_{\tilde{\mathfrak{s}}})}$.

Similarly, we can see that the v-face F_2 in Figure 3.8a corresponds to $\mathfrak{s}'$ by considering the Newton polytope $\Delta_v(C_{\tilde{\mathfrak{s}}'})$ of $C_{\tilde{\mathfrak{s}}'}$. This is shown in Figure 3.10. We see that the map

$$\overline{F_2} \to \Delta_v(C_{\tilde{\mathfrak{s}}'}) : (x, y) \mapsto (x, y)$$



(a) if $\mathfrak{s}'$ is even **(b)** if $\mathfrak{s}'$ is odd

Figure 3.9: Newton polytope $\Delta_v(C_{\tilde{\mathfrak{s}}})$ of $C_{\tilde{\mathfrak{s}}}$,where C is given by either defining equation (3.1), or (3.2).

is an isomorphism that preserves the valuations, that is $v_2(x, y) = v_{\Delta(C_{\tilde{\mathfrak{s}}'})}(x, y)$, and $\delta_{F_2} = \delta_{F_{\tilde{\mathfrak{s}}'}}$, where v_2 is the unique affine function agreeing with $v_{\Delta(C)}$ on F_2.

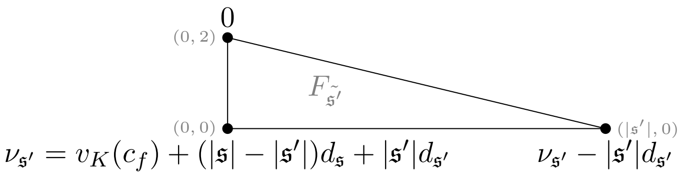


Figure 3.10: Newton polytope $\Delta_v(C_{\tilde{\mathfrak{s}}'})$ of $C_{\tilde{\mathfrak{s}}'}$, where C has defining equation (3.1).

□

Lemma 3.2.7. *Let C have Newton polytope as in Figure 3.8b. Then the v-face marked F_1 in Figure 3.8b corresponds to the cluster $\mathfrak{s}$. That is there is a valuation preserving isomorphism between $\overline{F_1}$ and $\Delta_v(C_{\tilde{\mathfrak{s}}})$, and $\delta_{F_1} = \delta_{F_{\tilde{\mathfrak{s}}}}$, where $F_{\tilde{\mathfrak{s}}}$ is the unique v-face of $\Delta_v(C_{\tilde{\mathfrak{s}}})$. Similarly the v-face marked F_2 on the Newton polytope in Figure 3.8b corresponds to the cluster $\mathfrak{s}'$.*

Proof. Follows by a similar argument to Lemma 3.2.6. □

3.2.3 Proof of Theorem 3.2.1

Without further ado let us prove Theorem 3.2.1. We will begin with some lemmas.

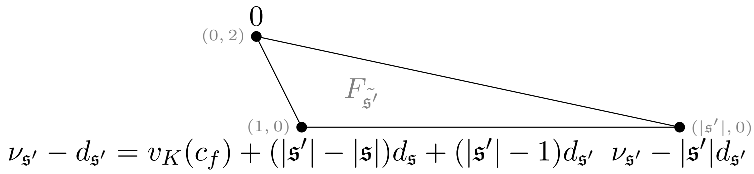


Figure 3.11: Newton polytope $\Delta_v(C_{\tilde{\mathfrak{s}}'})$ of $C_{\tilde{\mathfrak{s}}'}$, where C has defining equation (3.2).

Lemma 3.2.8. *If $\mathfrak{s}$ is principal then the special fibre has an irreducible component $\Gamma_{\mathfrak{s}} = \Gamma_{F_1}$ of multiplicity $e_{\mathfrak{s}}$ and genus $g(\mathfrak{s})$. If $\mathfrak{s}'$ is principal then there is a component $\Gamma_{\mathfrak{s}'} = \Gamma_{F_2}$ of multiplicity $e_{\mathfrak{s}'}$ and genus $g(\mathfrak{s}')$.*

Proof. Follows from Lemmas 3.2.6 and 3.2.7. □

Remark 3.2.9. Lemma 3.2.8 further proves that $\delta_{F_1} = e_{\mathfrak{s}}$ and $\delta_{F_2} = e_{\mathfrak{s}'}$ since, by Theorem 2.5.11, Γ_{F_i} has multiplicity δ_{F_i}.

Lemma 3.2.10. *If $\mathfrak{s}$ is principal and $e_{\mathfrak{s}} > 1$, the following tails of $\Gamma_{\mathfrak{s}}$ arise from outer v-edges of the v-face F_1 in Figure 3.8, with conditions as in Theorem 3.2.1:*

(i) ∞-tail(s) arising from the v-edge connecting $(0,2)$ and $(|\mathfrak{s}|,0)$,

(ii) $(y_{\mathfrak{s}} = 0)$-tail(s) arising from the v-edge connecting $(|\mathfrak{s}'|,0)$ and $(|\mathfrak{s}|,0)$.

Proof. This is a consequence of our discussion above, relating F_1 to the Newton polytope of $C_{\tilde{\mathfrak{s}}}$. The conditions in Theorem 3.2.1 for the tails to occur follow since $\epsilon_{\mathfrak{s}}(\sigma) = (-1)^{v_K(c_f)}$ for σ a generator of inertia. □

Lemma 3.2.11. *If $\mathfrak{s}'$ is principal and $e_{\mathfrak{s}'} > 1$, the following tails of $\Gamma_{\mathfrak{s}'}$ arise from outer v-edges of the v-face F_2 in Figure 3.8, with conditions as in Theorem 3.2.1:*

(i) if $b_{\mathfrak{s}'} \mid |\mathfrak{s}'|$, $(x_{\mathfrak{s}} = 0)$-tail(s) arise from the v-edge connecting $(0,0)$ and $(0,2)$,

(ii) if $b_{\mathfrak{s}'} \nmid |\mathfrak{s}'|$, a $(0,0)$-tail arises from the v-edge connecting $(1,0)$ and $(0,2)$,

(iii) in both cases, $(y_{\mathfrak{s}} = 0)$-tail(s) arise from the v-edge intersecting the x-axis.

Proof. This is a consequence of our discussion above, relating F_2 to the Newton polytope of $C_{\tilde{\mathfrak{s}}'}$. The conditions in Theorem 3.2.1 for these tails to occur follow since $\epsilon_{\mathfrak{s}'}(\sigma) = (-1)^{\nu_{\mathfrak{s}'} - |\mathfrak{s}'| d_{\mathfrak{s}'}}$ for σ a generator of inertia. □

In order to find the lengths of the level sections of the linking chains, we must calculate the slopes of the unique inner v-edge L, adjacent to both v-faces F_1 and F_2 in Figure 3.8.

Lemma 3.2.12. *If $\mathfrak{s}'$ is odd $s_1^L = -\lambda_{\mathfrak{s}}$, and $s_2^L = -\lambda_{\mathfrak{s}} - \frac{\delta_{\mathfrak{s}'}}{2}$. Else $s_1^L = -\delta_L d_{\mathfrak{s}}$, and $s_2^L = -\delta_L d_{\mathfrak{s}'}$.*

Proof. Suppose s' is odd. Then the only points in $\overline{L}(\mathbb{Z})$ are the endpoints $(0,2)$ and $(|\mathfrak{s}'|, 0)$, so $\delta_L = 1$. The unique function $L^*_{F_1} : \mathbb{Z}^2 \to \mathbb{Z}$ such that $L^*_{F_1}\big|_L = 0$ and $L^*_{F_1}\big|_{F_1} \geq 0$ is given by

$$L^*_{F_1}(x,y) = 2x + |\mathfrak{s}'|y - 2|\mathfrak{s}'|.$$

To calculate s_1^L and s_2^L we need P_0 and P_1 such that $L^*_{F_1}(P_1) = 1$ and $L^*_{F_1}(P_0) = 0$. We will take $P_0 = (|\mathfrak{s}'|, 0)$ and $P_1 = (\frac{|\mathfrak{s}'|+1}{2}, 1)$. The unique affine function which agrees with v_Δ on F_1 is defined by $v_1(x,y) = \nu_{\mathfrak{s}} - d_{\mathfrak{s}}x - \frac{\nu_{\mathfrak{s}}}{2}y$. Therefore,

$$\begin{aligned} s_1^L = \delta_L(v_1(P_1) - v_1(P_0)) &= \nu_{\mathfrak{s}} - d_{\mathfrak{s}}\frac{|\mathfrak{s}'|+1}{2} - \frac{\nu_{\mathfrak{s}}}{2} - \nu_{\mathfrak{s}} + d_{\mathfrak{s}}|\mathfrak{s}'| \\ &= -\left(\frac{\nu_{\mathfrak{s}}}{2} - d_{\mathfrak{s}}\frac{|\mathfrak{s}'|-1}{2}\right) = -\lambda_{\mathfrak{s}}. \end{aligned}$$

The calculations for s_2^L and $\mathfrak{s}'$ even are similar. □

Proof of Theorem 3.2.1. Recall that $e_{\mathfrak{s}}$ is the minimum integer such that $e_{\mathfrak{s}}d_{\mathfrak{s}} \in \mathbb{Z}$, and $e_{\mathfrak{s}}\nu_{\mathfrak{s}} \in 2\mathbb{Z}$. If $e_{\mathfrak{s}} = 1$ then $d_{\mathfrak{s}}, \lambda_{\mathfrak{s}} \in \mathbb{Z}$, hence the slopes of the outer v-edges of F_1 are integers and $\Gamma_{\mathfrak{s}}$ has no tails. If $e_{\mathfrak{s}} > 1$ then Lemma 3.2.10 describes the tails of $\Gamma_{\mathfrak{s}}$. Similarly if $e_{\mathfrak{s}'} = 1$ then $\Gamma_{\mathfrak{s}'}$ has no tails and if $e_{\mathfrak{s}'} > 1$ then Lemma 3.2.11 describes the tails of $\Gamma_{\mathfrak{s}'}$. The statement on the parameters of the tails and the linking chain follows from Remark 2.5.15 and the calculation of the slopes in Lemma 3.2.12. The multiplicity of the level section is δ_L where L is the inner v-edge between F_1 and F_2.

The two cases left to consider are when $\mathfrak{s}'$ is a twin or when $\mathfrak{s}$ is a cotwin. We will only argue the case where $\mathfrak{s}'$ is a twin, as the case where $\mathfrak{s}$ is a cotwin is proved similarly. Recall from Remark 2.4.7 that for σ a generator of inertia $\epsilon_{\mathfrak{s}'}(\sigma) = (-1)^{\nu_{\mathfrak{s}'}-|\mathfrak{s}'|d_{\mathfrak{s}'}}$. So, $\epsilon_{\mathfrak{s}'}(\sigma) = 1$ if and only if $v_\Delta(|\mathfrak{s}'|, 0) = \nu_{\mathfrak{s}'} - |\mathfrak{s}'|d_{\mathfrak{s}'} \in 2\mathbb{Z}$.

Suppose that $\epsilon_{\mathfrak{s}'} = 1$. Since $v_\Delta(0, 2) = 0 \in 2\mathbb{Z}$ and $|\mathfrak{s}'| = 2$ we have that $(\frac{|\mathfrak{s}'|}{2}, 1) = (1, 1) \in \mathbb{Z}^2$, and $v_\Delta(1, 1) \in \mathbb{Z}$. So, $|\overline{L}(\mathbb{Z})_\mathbb{Z}| = 3$ and by Theorem 2.5.11 there are two linking chains from $\Gamma_\mathfrak{s}$ to the component Γ_{F_2} arising from the v-face F_2 of $\Delta_v(C)$ in Figure 3.8. The component Γ_{F_2} is exceptional by [15, Proposition 5.2] and the linking chains between $\Gamma_\mathfrak{s}$ and Γ_{F_2} are minimal. After blowing down Γ_{F_2}, this results in a loop from $\Gamma_\mathfrak{s}$ to itself.

Suppose instead that, $\epsilon_{\mathfrak{s}'}(\sigma) = -1$. Then there is a single chain of rational curves from $\Gamma_\mathfrak{s}$ to Γ_{F_2}, and Γ_{F_2} has two other rational curves intersecting it transversely (which arise from the v-edge connecting $(0, 0)$ and $(0, 2)$). Therefore, Γ_{F_2} is not exceptional and must appear in the minimal snc model. This means, if we consider Γ_{F_2} as a component of the level section, that this chain of rational curves is a crossed tail. □

3.2.4 Small Distances

Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be the principal clusters such that there is a linking chain $\mathcal{C} \subseteq \mathscr{X}_k$ from $\Gamma_{\mathfrak{s}_1}$ to $\Gamma_{\mathfrak{s}_2}$. If $\mathcal{C}$ has level section of length greater than 0, it is straightforward to compare the multiplicities of $\mathcal{C}$ to those of the corresponding tails (see Remark 3.2.3). All of the multiplicities of the corresponding tails appear in the uphill and downhill sections of $\mathcal{C}$. However, if the level section is empty and the downhill section of $\mathcal{C}$ corresponds to a tail, say $\mathcal{T}_1$, then not all of the multiplicities of $\mathcal{T}_1 \subseteq \mathscr{X}_{\tilde{\mathfrak{s}}_1,k}$ appear in the downhill section of $\mathcal{C}$. Similarly if the uphill section corresponds to a tail, say $\mathcal{T}_2 \subseteq \mathscr{X}_{\tilde{\mathfrak{s}}_2,k}$. We shall show that in this case, $\mathcal{T}_1$ and $\mathcal{T}_2$ "meet" at a component of second least common multiplicity. In other words, if we consider a chain of rational curves $\mathcal{C}'$ such that $\mathcal{C}'$ has level section of length 1, and whose downhill and uphill sections correspond to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, then we "cut out" a section of $\mathcal{C}'$ to obtain $\mathcal{C}$.

Example 3.2.13. Consider the hyperelliptic curves given by $y^2 = (x^4 - p)(x^5 - p^{2+10m})$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$ for $m \in \mathbb{Z}_{\geq 0}$, with cluster pictures shown in Figure 3.12.

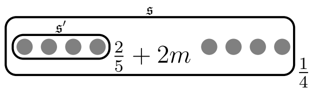
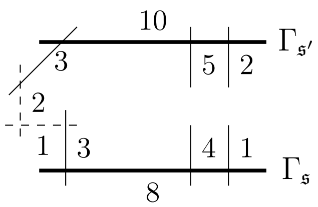
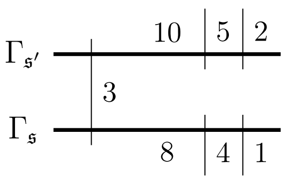


Figure 3.12: Cluster picture Σ_C of $C : y^2 = (x^4 - p)(x^5 - p^{2+10m})$.

The level section of the linking chain between $\Gamma_{\mathfrak{s}}$ and $\Gamma_{\mathfrak{s}'}$ has length m. Figure 3.13 shows the special fibres of the minimal snc models for both when $m = 1$, and the small distance case (when $m = 0$). Here we can see that when $m = 1$ the uphill and downhill sections of the linking chain have a common multiplicity greater than 1, namely 3, and that to obtain the $m = 0$ case we remove the dashed section of the linking chain and glue back along the multiplicity 3 components.



(a) $m = 1$ case.



(b) $m = 0$ case.

Figure 3.13: Example of "cutting out" a section of linking chain to obtain the small distance case.

Theorem 3.2.14 (Small Distances)**.** *Let* $\mathcal{C} = \bigcup_{i=1}^{\lambda} E_i$ *be a sloped chain of rational curves with parameters* (t_2, t_1, μ)*, as in Definition 2.5.17. Suppose that* $\mathcal{C}$ *has level section length* 0 *and* $[\mu t_2, \mu t_1] \subset (0, 1)$*. Suppose* E_i *has multiplicity* μ_i*; the downhill section comprises of* E_i *for* $1 \leq i \leq l$*, for some* $l \in \mathbb{Z}$ *with* $1 \leq l \leq \lambda$*; and all remaining components form the uphill section. Write* $\mu_0 =$

denom(μt_1) *and* $\mu_{\lambda+1} = \text{denom}(\mu t_2)$*. Let* $\mathcal{T}_j = \bigcup_{i=1}^{\lambda_j} F_i^{(j)}$ *for* $j = 1, 2$ *be tails (with* $\mathcal{T}_j$ *possibly empty, in which case* $\lambda_j = 0$*), where* $\mathcal{T}_1$ *has parameters* (t_1, μ) *and* $\mathcal{T}_2$ *has parameters* $(\frac{1}{\mu} - t_2, \mu)$*. Let* $F_i^{(j)}$ *have multiplicity* $\mu_i^{(j)}$ *(and write* $\mu_0^{(j)} = \text{denom}(\mu t_j)$*), and let* $l_j < \max(1, \lambda_j)$ *be maximal such that* $\mu_{l_1}^{(1)} = \mu_{l_2}^{(2)}$*. Then* $l = l_1 = \lambda - l_2$*,* $\mu_i = \mu_i^{(1)}$ *for* $0 \le i \le l_1$ *and* $\mu_{\lambda+1-i} = \mu_i^{(2)}$ *for* $0 \le i \le l_2$*.*

Remark 3.2.15. Let $\mathcal{C}$ be as in Theorem 3.2.14. Since the level section of $\mathcal{C}$ is empty, it must be the case that $(\mu t_2, \mu t_1) \cap \mathbb{Z} = \emptyset$. Therefore, after shifting μt_2 and μt_1 by an integer if necessary, we may insist that $[\mu t_2, \mu t_1] \subseteq [0, 1]$. If $\mu t_2 \in \mathbb{Z}$ (hence $\mathcal{T}_2$ is empty) then it is immediate from Remark 2.5.15 that $\lambda = \lambda_1 - 1$ and $\mu_i = \mu_i^{(1)}$ for $1 \le i \le \lambda$, since the multiplicities come from the same sequence of fractions. A similar conclusion applies if $\mu t_1 \in \mathbb{Z}$. So we are able to assume without loss of generality that $\mu t_2, \mu t_1 \notin \mathbb{Z}$, hence our assumption in Theorem 3.2.14 that $[\mu t_2, \mu t_1] \subset (0, 1)$.

Roughly, Theorem 3.2.14, states that when there is no level section, rather than seeing all of the multiplicities of the tails which the uphill and downhill sections correspond to, the two tails "meet" at the component of minimal shared multiplicity greater than μ. Before we prove this theorem, let us prove a couple of lemmas.

Lemma 3.2.16. *Let* $q_1, q_2 \in \mathbb{Q}$ *with* $[q_1, q_2] \cap \mathbb{Z} = \emptyset$*. Then there is a unique fraction with minimal denominator in the set* $[q_1, q_2] \cap \mathbb{Q}$*, when written with coprime numerator and denominator.*

Proof. Suppose not, and suppose $r_1, r_2 \in [q_1, q_2] \cap \mathbb{Q}$ can be written $r_i = \frac{m_i}{d}$ with m_i, d coprime and d the minimal denominator of elements in the set $[q_1, q_2] \cap \mathbb{Q}$. We will show that there exists a rational number r lying between r_1 and r_2 of denominator $< d$.

Write $r_i = \frac{m_i(d-1)}{d(d-1)}$, and consider the set $S = [m_1(d-1), m_2(d-1)] \cap \mathbb{Z}$. Since $m_2 > m_1$ and $m_1, m_2 \in \mathbb{Z}$, $|S| \ge d$ and there must exist a multiple of d in S. That is, there exists $m \in \mathbb{Z}$ such that $md \in S$. Since m_i and d are coprime, we have $m_1 < md < m_2$. Therefore,

$$\frac{m_1(d-1)}{d(d-1)} < \frac{md}{d(d-1)} < \frac{m_2(d-1)}{d(d-1)} \Longrightarrow r_1 < \frac{m}{d-1} < r_2,$$

which contradicts the minimality of d. $\square$

Lemma 3.2.17. *With notation as in Theorem 3.2.14, there exists some $l_j < \lambda_j$, for $j = 1, 2$, such that $\mu^{(1)}_{l_1} = \mu^{(2)}_{l_2}$.*

Proof. Write $s_i = \mu t_i$. Recall that we assumed that, $[s_2, s_1] \subset (0, 1)$, so $[s_2, s_1] \cap \mathbb{Z} = \emptyset$. Let $\frac{m}{d}$ be the unique fraction of minimal denominator in $[s_2, s_1]$, which exists by Lemma 3.2.16. Then if

$$s_1 = \mu t_1 = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \cdots > \frac{m_\lambda}{d_\lambda} > \frac{m_{\lambda+1}}{d_{\lambda+1}} = \mu t_2 = s_2,$$

is the reduced sequence giving rise to the linking chain $\mathcal{C}$, as in Remark 2.5.15, where $(m_i, d_i) = 1$, $d_0 > \cdots > d_l$ and $d_l < \cdots < d_{\lambda+1}$ for some $1 \le l \le \lambda$, we must have that $d_l = d$.

Consider the following two reduced sequences:

$$\mu t_1 = \frac{m^{(1)}_0}{d^{(1)}_0} > \frac{m^{(1)}_1}{d^{(1)}_1} > \cdots > \frac{m^{(1)}_{\lambda_1}}{d^{(1)}_{\lambda_1}} > \frac{m^{(1)}_{\lambda_1+1}}{d^{(1)}_{\lambda_1+1}} = -1,$$
$$1 - \mu t_2 = \frac{m^{(2)}_0}{d^{(2)}_0} > \frac{m^{(2)}_1}{d^{(2)}_1} > \cdots > \frac{m^{(2)}_{\lambda_2}}{d^{(2)}_{\lambda_2}} > \frac{m^{(2)}_{\lambda_2+1}}{d^{(2)}_{\lambda_2+1}} = -1.$$

These give rise to the multiplicities $\mu^{(j)}_i = \mu \cdot d^{(j)}_i$ for $1 \le i \le \lambda_j$, $j = 1, 2$ of the tails $\mathcal{T}_j$. We will show that there exist $0 \le l_1 < \lambda_1 + 1$ and $0 \le l_2 < \lambda_2 + 1$ with $d^{(1)}_{l_1} = d = d^{(2)}_{l_2}$.

We will first prove that $d^{(1)}_{l_1} = d$ for some $l_1 \in \mathbb{Z}$. Since $[s_2, s_1] \subset (0, 1)$, we have that $s_2 > \lfloor s_1 \rfloor = 0$. So, some fraction of denominator d, say $\frac{m}{d}$, appears in the full sequence of fractions in $[\lfloor s_1 \rfloor, s_1] \cap \mathbb{Q}$ of denominator less than or equal to $\max\{d_0, d_{\lambda+1}\}$. To obtain a reduced sequence, we remove all terms of the form

$$\cdots > \frac{a}{b} > \frac{a+c}{b+d} > \frac{c}{d} > \cdots \mapsto \cdots > \frac{a}{b} > \frac{c}{d} > \ldots,$$

as in Remark 2.5.15. We can only remove $\frac{m}{d}$ if there exists some $q \in \mathbb{Q}$ with $\text{denom}(q) < d$ and $s_1 > q > \frac{m}{d}$. No such q can exist since d is the minimal denominator of any element of $[s_2, s_1] \cap \mathbb{Q}$. Therefore, $\frac{m}{d}$ cannot be removed in the reduction process and so must appear in the reduced sequence. Therefore there exists $0 \le l_1 < \lambda_1 + 1$ such that $d^{(1)}_{l_1} = d$. Proving that there exists $0 \le l_2 < \lambda_2 + 1$ such that $d^{(1)}_{l_1} = d = d^{(2)}_{l_2}$ is done similarly. □

We can now prove Theorem 3.2.14.

Proof of Theorem 3.2.14. The fractions $\frac{m_0}{d_0}, \frac{m_1}{d_1}, \ldots, \frac{m_l}{d_l}$ in the reduced sequence depend only on the elements of $[s_1, \frac{m_l}{d_l}]$ of denominator less than or equal to $\max(d_0, d_{\lambda+1})$, as do the fractions $\frac{m^{(1)}_0}{d^{(1)}_0}, \ldots, \frac{m^{(1)}_{l_1}}{d^{(1)}_{l_1}} = \frac{m_l}{d_l}$. This proves that $d^{(1)}_i = d_i$ hence $\mu_i = \mu^{(1)}_i$ for $0 \le i \le l_1$. Similarly $d^{(2)}_i = d_{\lambda+1-i}$ hence $\mu_{\lambda+1-i} = \mu^{(2)}_i$ for $0 \le i \le l_2$. It remains to show maximality of l_1 and l_2.

Suppose there is some r_1, r_2 such that $\lambda_i > r_i > l_i$ and $\mu^{(1)}_{r_1} = \mu^{(2)}_{r_2} < \mu^{(1)}_{l_1}$. In addition to this, $d^{(1)}_{r_1} = d^{(2)}_{r_2} < d$ (recall $\frac{m}{d}$ is the unique fraction with least denominator in $[s_2, s_1] \cap \mathbb{Q}$). Therefore $q_2 = 1 - \frac{m^{(2)}_{r_2}}{d^{(2)}_{r_2}} \in (s_1, 1]$ and $q_1 = \frac{m^{(1)}_{r_1}}{d^{(1)}_{r_1}} \in [0, s_2)$. Let q' be the unique rational with least denominator d' in $[q_1, q_2]$. By uniqueness, $d' < d^{(1)}_{r_1} < d$. Therefore, $q' \in (s_1, q_2)$ or (q_1, s_2). Suppose for now that $q' \in (s_1, q_2)$, and consider again the reduced sequence

$$1 - \mu t_2 = \frac{m^{(2)}_0}{d^{(2)}_0} > \frac{m^{(2)}_1}{d^{(2)}_1} > \cdots > \frac{m^{(2)}_{\lambda_2}}{d^{(2)}_{\lambda_2}} > \frac{m^{(2)}_{\lambda_2+1}}{d^{(2)}_{\lambda_2+1}} = -1.$$

However $1 - q_2$ cannot appear in this reduced sequence since a fraction with smaller denominator, $1 - q'$, appears to the left of it in the non-reduced sequence. So, at some step in the reduction process $1 - q_2$ would have been removed. Therefore, $q' \notin (s_1, q_2)$. Similarly, one can show that $q' \notin (q_1, s_2)$. This is a contradiction. So no such r_1 and r_2 exist. □

Chapter 4

Hyperelliptic Curves with Tame Reduction

In Chapter 3 we described the minimal snc model of a hyperelliptic curve with tame reduction and a particularly nice cluster picture: with precisely one or two proper clusters. In this section we use those two cases as the base step in an induction which will allow us to describe the minimal snc model of a general hyperelliptic curve C/K with tame reduction.

This description is presented as several theorems. The first, Theorem 4.1.11 states that the cluster picture of C determines the minimal snc model $\mathcal{C}$ of C. Theorem 4.1.13 allows us to construct the dual graph of $\mathcal{C}_k$ by describing the principal components and chains of the special fibre, albeit without multiplicities, genera and lengths of chains. These are in later theorems: Theorem 4.1.18 for the genera and multiplicity of principal components, and Theorem 4.1.19 for the multiplicities and lengths of the chains linking them. Finally in Theorem 4.1.21 we state the action of Frobenius of $\mathcal{C}_k$ in the case where K does not have algebraically closed residue field. These theorems are proven in Section 4.2.

4.1 Structure of Special Fibre

Here we state our main theorems describing the special fibre of the minimal snc model of a hyperelliptic curve with tame reduction, beginning with some key definitions in Section 4.1.1, and continuing to the dual graph in 4.1.2 and the multiplicity and genera of components and lengths of chains in 4.1.3.

4.1.1 Orbits

In the description from [19] of the minimal regular model of a hyperelliptic curve C with *semistable* reduction (Theorem 2.4.11), conditions are given in terms of clusters. In this case inertia acts trivially on the proper clusters of Σ_C. However when C has tame but not necessarily semistable reduction, there can be a non-trivial action of inertia on the proper clusters. It transpires that *inertia orbits* of proper clusters will play the role of proper clusters in the description of the minimal snc model of C. To this end, we extend many of the definitions of 2.2 to orbits of clusters. Roughly, if we apply any adjective to a cluster $\mathfrak{s}$ (such as even or odd), then we will also apply it to the inertia orbit it belongs to.

Definition 4.1.1. Let X be a inertia orbit of clusters. Then X is *übereven* if for all $\mathfrak{s} \in X$, $\mathfrak{s}$ is übereven. Define an orbit X to be *odd*, *even*, and *principal* similarly.

Definition 4.1.2. Let X be an inertia orbit of clusters. Define K_X/K to be the field extension of K of degree $|X|$.

Remark 4.1.3. By Lemma 2.2.18, K_X/K is the minimal field extension over which for any $\mathfrak{s} \in X$, $\sigma \in \mathrm{Gal}(\overline{K}/K_X)$ we have $\sigma(\mathfrak{s}) = \mathfrak{s}$.

Definition 4.1.4. Let X be a Galois orbit of clusters, and choose some $\mathfrak{s} \in X$. Then we define

$$d_X = \frac{a_X}{b_X} = d_{\mathfrak{s}}, \quad \nu_X = \nu_{\mathfrak{s}}, \quad \lambda_X = \lambda_{\mathfrak{s}}, \quad g_{\mathrm{ss}}(X) = g_{\mathrm{ss}}(\mathfrak{s}).$$

Furthermore, we define $\epsilon_X = \epsilon_{\mathfrak{s}}^{|X|}$. This is the restriction of the character $\epsilon_{\mathfrak{s}}$ to the stabiliser of X, i.e. to the group $\mathrm{Gal}(\overline{K}/K_X)$.

Remark 4.1.5. Note that the invariants defined in Definition 4.1.4 are well defined, i.e they do not depend on the choice of $\mathfrak{s} \in X$.

Definition 4.1.6. An orbit X' is a *child* of X, written $X' < X$, if for every $\mathfrak{s}' \in X'$ there exists some $\mathfrak{s} \in X$ such that $\mathfrak{s}' < \mathfrak{s}$. Define $\delta_{X'} = \delta_{\mathfrak{s}'}$ for some $\mathfrak{s}' \in X'$.

Definition 4.1.7. Let X be a principal orbit of clusters with $g_{\mathrm{ss}}(X) > 0$ and choose some $\mathfrak{s} \in X$. Then $C_{\widetilde{X}}$ is defined to be the curve $C_{\tilde{\mathfrak{s}}}$ over K_X. We denote the minimal snc model of $C_{\widetilde{X}}/K_X$ by $\mathscr{X}_{\widetilde{X}}/\mathcal{O}_{K_X}$, and the principal component by $\Gamma_{\widetilde{X}}/k$.

Remark 4.1.8. The exact curve $C_{\widetilde{X}}$ depends on a choice of $\mathfrak{s} \in X$, but the combinatorial description of the special fibre of the minimal snc model will not. Since this is what we need $C_{\widetilde{X}}$ for, we do not need to worry about this.

Definition 4.1.9. Let X be a principal orbit of clusters. Define e_X to be the minimal integer such that $e_X|X|d_{\mathfrak{s}} \in \mathbb{Z}$ and $e_X|X|\nu_{\mathfrak{s}} \in 2\mathbb{Z}$ for all $\mathfrak{s} \in X$. Define $g(X) = g(\mathfrak{s})$ for $\mathfrak{s} \in X$ over K_X, where $g(\mathfrak{s})$ is as defined in Definition 3.1.23.

Remark 4.1.10. Analogously to Section 3.1.5, the curve $C_{\widetilde{X}}/K_X$ is semistable over an extension of K_X of degree e_X and the quotient map $\Gamma_{\mathfrak{s},L} \to \Gamma_{\mathfrak{s},K_X}$ has degree e_X for $\mathfrak{s} \in X$.

4.1.2 The Minimal snc Model

We state here the first of our main theorems. Roughly this tells us that the cluster picture, the leading coefficient of f, and the action of $\mathrm{Gal}(\overline{K}/K)$ on the cluster picture is enough to calculate the structure of the minimal snc model, along with the multiplicities and genera of the components.

Theorem 4.1.11. *Let K be a local field with residue field k of characteristic $p > 2$. Let $C : y^2 = f(x)$ be a hyperelliptic curve over K with tame reduction and cluster picture Σ. Let $\mathcal{C}$ be the minimal snc model of C over $\mathcal{O}_{K^{\mathrm{ur}}}$. Then the dual graph, with genus and multiplicity, of $\mathcal{C}_k$ is completely determined by Σ (with depths) and the valuation of the leading coefficient $v_K(c_f)$ of f.*

Remark 4.1.12. If K does not have algebraically closed residue field, then the Frobenius action on the dual graph is determined by this data, as well as the values of $\epsilon_X(\mathrm{Frob})$ for each orbit of clusters X. See Theorem 4.1.21.

The proof of this will follow from the theorems proved in the rest of this section, and we make this more precise later. First we split Theorem 4.1.11 into several smaller theorems. The first tells us which components appear in the special fibre of the minimal snc model. Roughly, there is a central component for every orbit of principal, non übereven clusters, one or two central components for every orbit of principal übereven clusters, and a chain of rational curves associated to each orbit of twins. These central components are linked by chains of rational curves, and certain central components will also have tails intersecting them. The following theorem gives us the structure of the special fibre but is missing important details such as multiplicities, genera and lengths of these chains. These remaining details will be discussed in a later theorem.

Theorem 4.1.13 (Structure of the snc model)**.** *Let K be a local field with residue field of characteristic $p > 2$. Let C/K be a hyperelliptic curve with tame reduction. Then the special fibre of its minimal snc model is structured as follows. Every principal Galois orbit of clusters X contributes one principal component Γ_X, unless X is übereven with $\epsilon_X(\sigma) = 1$ for σ a generator of inertia, in which case X contributes two central components Γ_X^+ and Γ_X^-.*

These principal components are linked by chains of rational curves, or are intersected transversely by a crossed tail in the following ways (where, for any orbit Y, we write $\Gamma_Y^+ = \Gamma_Y^- = \Gamma_Y$ if Y is not übereven):

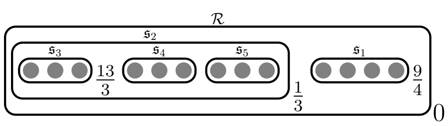
Name	*From*	*To*	*Condition*
$L_{X,X'}$	Γ_X	$\Gamma_{X'}$	$X' < X$ *both principal,* X' *odd*
$L^+_{X,X'}$	Γ^+_X	$\Gamma^+_{X'}$	$X' < X$ *both principal,* X' *even with* $\epsilon_{X'} = 1$
$L^-_{X,X'}$	Γ^-_X	$\Gamma^-_{X'}$	
$L_{X,X'}$	Γ_X	$\Gamma_{X'}$	$X' < X$ *both principal,* X' *even with* $\epsilon_{X'} = -1$
$L_{X'}$	Γ^-_X	Γ^+_X	X *principal,* $X' < X$ *orbit of twins,* $\epsilon_{X'} = 1$
$T_{X'}$	Γ_X	-	X *principal,* $X' < X$ *orbit of twins,* $\epsilon_{X'} = -1$

Note that any chain where the "To" column has been left blank is a crossed tail. If $\mathcal{R}$ is not principal then we also get the following chains of rational curves:

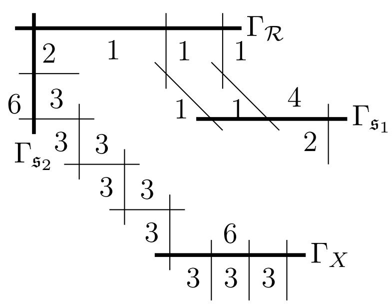
Name	*From*	*To*	*Condition*
$L_{\mathcal{R}}$	$\Gamma_{\mathfrak{s}}^-$	$\Gamma_{\mathfrak{s}}^+$	$\mathcal{R}$ *a cotwin,* $\mathfrak{s} < \mathcal{R}$ *principal of size* $2g$, $\epsilon_{\mathfrak{s}} = 1$
$T_{\mathcal{R}}$	$\Gamma_{\mathfrak{s}}$	-	$\mathcal{R}$ *a cotwin,* $\mathfrak{s} < \mathcal{R}$ *principal of size* $2g$, $\epsilon_{\mathfrak{s}} = -1$
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$\Gamma_{\mathfrak{s}_1}$	$\Gamma_{\mathfrak{s}_2}$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, *with* $\mathfrak{s}_i$ *both principal and odd,* $e_{\mathcal{R}} = 1$
T_X	Γ_X	-	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, *with* $X = \{s_1, s_2\}$ *a principal, odd orbit*
$L^+_{\mathfrak{s}_1,\mathfrak{s}_2}$	$\Gamma^+_{\mathfrak{s}_1}$	$\Gamma^+_{\mathfrak{s}_2}$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *both principal and even,* $e_{\mathcal{R}} = 1$, $\epsilon_{\mathfrak{s}_i} = 1$
$L^-_{\mathfrak{s}_1,\mathfrak{s}_2}$	$\Gamma^-_{\mathfrak{s}_1}$	$\Gamma^-_{\mathfrak{s}_2}$	
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$\Gamma_{\mathfrak{s}_1}$	$\Gamma_{\mathfrak{s}_2}$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *both principal and even,* $e_{\mathcal{R}} = 1$, $\epsilon_{\mathfrak{s}_i} = -1$
T^+_X	Γ^+_X	-	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, *with* $X = \{s_1, s_2\}$ *a principal, even orbit,* $\epsilon_{\mathfrak{s}_i} = 1$
T^-_X	Γ^-_X	-	
T_X	Γ_X	-	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, *with* $X = \{s_1, s_2\}$ *a principal, even orbit,* $\epsilon_{\mathfrak{s}_i} = -1$
$L_{\mathfrak{t}}$	$\Gamma_{\mathfrak{s}}^-$	$\Gamma_{\mathfrak{s}}^+$	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal and even,* $\mathfrak{t}$ *a twin,* $\epsilon_{\mathfrak{t}} = 1$
$T_{\mathfrak{t}}$	$\Gamma_{\mathfrak{s}}$	-	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal and even,* $\mathfrak{t}$ *a twin,* $\epsilon_{\mathfrak{t}} = -1$

Finally, a principal component Γ_X *is intersected transversally by some tails if and only if* $e_X > 1$*. These are explicitly described in Theorem 4.1.19.*

Example 4.1.14. Let $K = \mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 5$, and C/K be the hyperelliptic curve given by $C : y^2 = ((x^3 - p)^3 - p^{15})((x - 1)^4 - p^9)$. The cluster picture and minimal snc model of this hyperelliptic curve are shown below.



(a) Cluster picture $\Sigma_{C/K}$.



(b) Special fibre of the minimal snc model of C/K.

Figure 4.1: $C : y^2 = ((x^3 - p)^3 - p^{15})((x - 1)^4 - p^9)$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$.

The principal orbits are: $X = \{\mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5\}$, $\{\mathfrak{s}_2\}$, $\{\mathfrak{s}_1\}$ and $\{\mathcal{R}\}$, none of which are übereven, and so there are four principal components corresponding to each of these. There are linking chains arising from the parent-child relations $\mathfrak{s}_1 < \mathcal{R}$, $\mathfrak{s}_2 < \mathcal{R}$ and $X < \mathfrak{s}_2$. Finally, $e_{\mathcal{R}} = 1$ so $\Gamma_{\mathcal{R}}$ has no tails, and $e_{\mathfrak{s}_1}, e_{\mathfrak{s}_2}, e_X > 1$ so their corresponding components have tails.

Remark 4.1.15. At no point do we give explicit equations for the principal components $\Gamma_X^{\pm}$. However, these can be calculated using the method laid out in this thesis. In particular, one can take the explicit equations given in Theorem 2.4.11 for the components $\Gamma_{\mathfrak{s},L}^{\pm}$ in the semistable model of C/L and the Galois action on these components, and apply [18, Theorem 1.1].

Before we prove this, let us prove a couple of lemmas. Recall that L is a field over which C has semistable reduction and that $\Gamma_{\mathfrak{s},L}$ is the component associated to a cluster $\mathfrak{s}$ in the special fibre of the minimal semistable model $\mathscr{Y}$ of C over L.

Lemma 4.1.16. *Let $\mathfrak{s}$ be a principal cluster with $g_{ss}(\mathfrak{s}) = 0$.*

(i) If $\mathfrak{s} = \mathcal{R}$ and $\mathfrak{s}$ is not übereven (resp. übereven) then $\Gamma_{\mathfrak{s},L}$ (resp. each of $\Gamma_{\mathfrak{s},L}^{+}$ and $\Gamma_{\mathfrak{s},L}^{-}$) intersects at least two other components.

(ii) If $\mathfrak{s} \neq \mathcal{R}$ and $\mathfrak{s}$ is not übereven (resp. übereven) then $\Gamma_{\mathfrak{s},L}$ (resp. each of $\Gamma_{\mathfrak{s},L}^{+}$ and $\Gamma_{\mathfrak{s},L}^{-}$) intersects at least three other components.

Proof. (i) Let $\mathfrak{s} = \mathcal{R}$ and suppose $\mathfrak{s}$ is not übereven. Since $g_{\mathrm{ss}}(\mathfrak{s}) = 0$, $\mathfrak{s}$ can have at most two odd children and in particular at most two singletons. Since, $g(C) \geq 2$, we have $|\mathfrak{s}| \geq 5$. If $|\mathfrak{s}|$ is odd then $\mathfrak{s}$ must have an even child $\mathfrak{s}'$ and, by Theorem 2.4.11, $\Gamma_{\mathfrak{s},L}$ is intersected by the two linking chains to $\Gamma_{\mathfrak{s}',L}$. Note that, since $\mathfrak{s}$ is principal, $\mathfrak{s}$ cannot be the union of two odd clusters. So, if $|\mathfrak{s}|$ is even then $\mathfrak{s}$ has an even child and we are done by Theorem 2.4.11.

If $\mathfrak{s} = \mathcal{R}$ is übereven then every child of $\mathfrak{s}$ is even. In particular, there are at least two even children $\mathfrak{s}_1$ and $\mathfrak{s}_2$. So, each of $\Gamma_{\mathfrak{s},L}^{\pm}$ intersects $L_{\mathfrak{s}_1}^{\pm}$ and $L_{\mathfrak{s}_2}^{\pm}$ (the linking chains to the children).

(ii) Let $\mathfrak{s} \neq \mathcal{R}$ and suppose $\mathfrak{s}$ is not übereven. Since $\mathfrak{s}$ is principal, we know $|\mathfrak{s}| \geq 3$. Therefore, $\mathfrak{s}$ must have at least one proper child $\mathfrak{s}'$. Suppose that $P(\mathfrak{s})$ is principal. If $\mathfrak{s}' < \mathfrak{s}$ is even then $\Gamma_{\mathfrak{s},L}$ intersects the linking chain to $\Gamma_{P(\mathfrak{s}),L}$ and the two linking chains to $\Gamma_{\mathfrak{s}',L}$. Otherwise $\mathfrak{s}$ must be the union of two odd clusters, hence $\mathfrak{s}$ is even. In this case there are two linking chains to $\Gamma_{P(\mathfrak{s}),L}$ and one to $\Gamma_{\mathfrak{s}',L}$. A similar argument works if $\mathfrak{s}$ is übereven. If $P(\mathfrak{s}) = \mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}_2$ is not principal, the argument is similar, but linking chains to $\Gamma_{P(\mathfrak{s}),L}$ are replaced by linking chains to $\Gamma_{\mathfrak{s}_2,L}$. □

Proposition 4.1.17. *Let $\mathscr{Y}$ be the semistable model of C/L and $\mathscr{Z}$ the image under the quotient map. Let $\mathscr{X}$ be the snc model obtained by resolving the singularities of $\mathscr{Z}$ such that all rational chains are minimal. Let X be a principal orbit of clusters. Let $\Gamma_{X,K} \in \mathscr{X}_k$ be the image of $\Gamma_{\mathfrak{s},L}$ for some $\mathfrak{s} \in X$ under the quotient by Gal(L/K). Then if $g(\Gamma_{X,K}) = 0$ and $(\Gamma_{X,K} \cdot \Gamma_{X,K}) = -1$, $\Gamma_{X,K}$ intersects at least three other components of the special fibre (i.e. blowing down $\Gamma_{X,K}$ would not result in an snc model).*

Proof. If $|X| > 1$, there is a non trivial field extension of K to K_X. Over K_X, each $\mathfrak{s}' \in X$ is fixed by $\mathrm{Gal}(\overline{K}/K_X)$. The Galois group $\mathrm{Gal}(K_X/K)$ then induces an étale morphism $\bigsqcup_{\mathfrak{s}' \in X} \Gamma_{\mathfrak{s}',K_X} \to \Gamma_{X,K}$. Therefore, $g(\Gamma_{X,K}) = g(\Gamma_{\mathfrak{s}',K_X})$, $(\Gamma_{X,K} \cdot \Gamma_{X,K}) = (\Gamma_{\mathfrak{s}',K_X} \cdot \Gamma_{\mathfrak{s}',K_X})$, and $\Gamma_{X,K}$ and $\Gamma_{\mathfrak{s}',K}$ intersect the same number of other components. So, it is enough to prove this proposition when $|X| = 1$, and from now on let $X = \{\mathfrak{s}\}$. When $g(\Gamma_{\mathfrak{s},L}) > 0$, Riemann-Hurwitz implies that

$$\sum_{P \in \Gamma_{\mathfrak{s},K}} \left(\frac{e_{\mathfrak{s}}}{|q^{-1}(P)|} - 1 \right) \geq 2e_{\mathfrak{s}},$$

where $q : \Gamma_{\mathfrak{s},L} \to \Gamma_{\mathfrak{s},K}$ is the quotient by Gal(L/K). So, if $g(\Gamma_{\mathfrak{s},L}) > 0$, there must be at least three points $P \in \Gamma_{\mathfrak{s},K}$ with $|q^{-1}(P)| < e_{\mathfrak{s}}$. These ramification points are singular points by Proposition 3.1.4. After blowing up these singular points, we see that $\Gamma_{\mathfrak{s},K}$ intersects at least three other components of $\mathscr{X}_k$.

It remains to deal with the case when $g(\Gamma_{\mathfrak{s},L}) = 0$. If $e_{\mathfrak{s}} = 1$, Lemma 4.1.16 implies that $\Gamma_{\mathfrak{s},K}$ intersects two or more other components. In this case

$\Gamma_{\mathfrak{s},K}$ will have multiplicity $e_{\mathfrak{s}} = 1$. This tells us that $(\Gamma_{\mathfrak{s},K} \cdot \Gamma_{\mathfrak{s},K}) < -1$, so $\Gamma_{\mathfrak{s},K}$ is not exceptional.

Suppose instead that $e_{\mathfrak{s}} > 1$. We will show that the component $\Gamma_{\mathfrak{s},K}$ intersects at least three components. There are two branch points P_0 and P_∞ of the morphism $q : \Gamma_{\mathfrak{s},L} \to \Gamma_{\mathfrak{s},K}$, the images of $x_{\mathfrak{s}} = 0$ and $x_{\mathfrak{s}} = \infty$ respectively. Both P_0 and P_∞ are singularities. If $q^{-1}(P_0)$ is an intersection point of $\Gamma_{\mathfrak{s},L}$ with another component Γ then P_0 will be the intersection point of $\Gamma_{\mathfrak{s},K}$ and $q(\Gamma)$[1]. Otherwise, blowing up P_0 introduces a component intersecting $\Gamma_{\mathfrak{s},K}$. Similarly for P_∞. If $\mathfrak{s} = \mathcal{R}$ then $q^{-1}(P_\infty)$ will never be an intersection point by [19, Propositions 5.5, 5.20]. Since $\Gamma_{\mathfrak{s},L}$ has two intersection points with other components Q_1 and Q_2, either $q(Q_1) \neq q(Q_2)$, or $q(Q_1) = q(Q_2) \neq P_0$ (since $|q^{-1}(P_0)| = 1$). If $q(Q_1) \neq q(Q_2)$ then these are both intersection points with other components, hence $\Gamma_{\mathfrak{s},K}$ intersects at least 3 components at $P_\infty, q(Q_1)$ and $q(Q_2)$ which are all distinct. If $q(Q_1) = q(Q_2) \neq P_0$ then $P_\infty, q(Q_1)$ and P_0 are distinct intersection points with other components. A similar argument works if $\mathfrak{s} \neq \mathcal{R}$. □

We are now able to prove our structure theorem (Theorem 4.1.13).

Proof of Theorem 4.1.13. First let us find which central components appear. By Theorem 2.4.11, we know there is a component over L for every principal, non-übereven cluster, and we know the action of $\mathrm{Gal}(L/K)$ on these principal components is the same as the action on the clusters. After taking the quotient by $\mathrm{Gal}(L/K)$, there is a component for every orbit of principal, non übereven clusters. Similarly over L, by Theorem 2.4.11 there are two components for every übereven cluster $\mathfrak{s}$. These are swapped by inertia if and only if $\epsilon_{\mathfrak{s}}(\sigma) = -1$ for σ a generator of inertia. After taking the quotient this gives us two components for an übereven orbit X if $\epsilon_X = 1$ and a single component if $\epsilon_X(\sigma) = -1$. Showing which linking chains appear is done similarly, using the

[1] We may have to blow down $q(\Gamma)$ but even then P_0 will remain an intersection point, since the eventual linking chain will be minimal. This follows from Lemmas 4.2.3 and 4.2.4 below.

information given in Theorem 2.4.11.

To ensure these principal components do in fact appear in the minimal snc model, we must check that they cannot be blown down. Any central component $\Gamma_{X,K} \in \mathscr{X}_k$ is the image of $\Gamma_{\mathfrak{s},L} \in \mathscr{Y}_k$ for some $\mathfrak{s} \in X$. A central component $\Gamma_{X,K}$ can only be blown down if $g(\Gamma_{X,K}) = 0$, and $(\Gamma_{X,K} \cdot \Gamma_{X,K}) = -1$. However, by Proposition 4.1.17, any central component $\Gamma_{X,K}$ with $g(\Gamma_{X,K}) = 0$ and $(\Gamma_{X,K} \cdot \Gamma_{X,K}) = -1$ intersects at least three other components of the special fibre. Therefore, if $\Gamma_{X,K}$ were to be blown down, $\mathscr{X}_k$ would no longer be an snc divisor. So $\Gamma_{X,K}$ must appear in the special fibre of the minimal snc model. □

4.1.3 A More Explicit Description

Theorem 4.1.13 describes the structure of the special fibre, but says nothing about the multiplicity or genera of the components, or the action of Frobenius. The following theorems fill in these details. The first focuses on the principal components, the second describes the chains of rational curves present in the special fibre, and the last gives the Frobenius action.

Theorem 4.1.18 (Principal Components)**.** *Let K and C/K be as in Theorem 4.1.13. Let X be a principal orbit of clusters in Σ. If X is not übereven then Γ_X has multiplicity $|X|e_X$ and genus $g(X)$. If X is übereven with $\epsilon_X(\sigma) = 1$ for σ a generator of inertia then Γ_X^+ and Γ_X^- have multiplicity $|X|e_X$ and genus 0, and if $\epsilon_X(\sigma) = -1$ then Γ_X has multiplicity $2|X|e_X$ and genus 0.*

Proof. Let X be a principal, non-übereven orbit, and choose some $\mathfrak{s} \in X$. Recall that K_X is the minimal field extension of K such that the clusters of X are fixed by $\mathrm{Gal}(\overline{K}/K_X)$, and L is the minimal field extension of K such that C is semistable over L. The image $\Gamma_{\mathfrak{s},K_X}$ of $\Gamma_{\mathfrak{s},L}$ after taking the quotient by $\mathrm{Gal}(L/K_X)$ has multiplicity e_X, since the action on $\Gamma_{\mathfrak{s},L}$ has multiplicity e_X (by Lemma 3.1.26). There are $|X|$ such components, which are permuted by $\mathrm{Gal}(K_X/K)$ in the minimal snc model of C/K_X. So, Γ_X has multiplicity $|X|e_X$ by [32, Fact IV]. The multiplicities of components corresponding to übereven clusters follows similarly, being careful to account for whether $\Gamma_{\mathfrak{s},L}^+$ and $\Gamma_{\mathfrak{s},L}^-$

are swapped by $\mathrm{Gal}(L/K)$ in the semistable model (which happens precisely when $\epsilon_{\mathfrak{s}}(\sigma) = -1$).

To find the genus of the central components, note that if $g(\Gamma_{\mathfrak{s},L}) = 0$ then $g(\Gamma_{X,K}) = 0$. So let us assume that $g(\Gamma_{\mathfrak{s},L}) > 0$. In this case, as mentioned in Remark 3.1.22, $\Gamma_{\mathfrak{s},L}$ is isomorphic to the special fibre of the smooth model of $C_{\tilde{\mathfrak{s}}}$ over L. Furthermore, the action on $\Gamma_{\mathfrak{s},L}$ is the same as the action on $\Gamma_{\tilde{\mathfrak{s}},L}$. Hence, the genus of $\Gamma_{\mathfrak{s},K_X}$ is $g(X)$, and also the genus of $\Gamma_{X,K}$. □

Theorem 4.1.19 (Description of Chains)**.** *Let K and C/K be as in Theorem 4.1.13. Let X be a principal orbit of clusters with $e_X > 1$. Choose some $\mathfrak{s} \in X$ of depth $d_{\mathfrak{s}}$ with denominator $b_{\mathfrak{s}}$. Then the principal component(s) associated to X are intersected transversely by the following sloped tails with parameters (t_1, μ) (writing $\Gamma_X = \Gamma_X^+ = \Gamma_X^-$ if X is not übereven):*

Name	*From*	*Number*	t_1	μ	*Condition*
T_∞	Γ_X	1	$(g+1)d_{\mathcal{R}} - \lambda_{\mathcal{R}}$	1	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *odd*
$T_\infty^\pm$	$\Gamma_X^\pm$	2	$-d_{\mathcal{R}}$	1	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *even*, $\epsilon_{\mathcal{R}} = 1$
T_∞	Γ_X	1	$-d_{\mathcal{R}}$	2	$X = \{\mathcal{R}\}$, $\mathcal{R}$ *even*, $e_{\mathcal{R}} > 2$, $\epsilon_{\mathcal{R}} = -1$
$T_{y_{\mathfrak{s}}=0}$	Γ_X	$\frac{\lvert\mathfrak{s}_{\mathrm{sing}}\rvert\lvert X\rvert}{b_X}$	$-\lambda_X$	b_X	$\lvert\mathfrak{s}_{\mathrm{sing}}\rvert \geq 2$, *and* $e_X > b_X/\lvert X\rvert$
$T_{x_{\mathfrak{s}}=0}$	Γ_X	1	$-d_X$	$2\lvert X\rvert$	*X has no stable child,* $\lambda_X \notin \mathbb{Z}$, $e_X > 2$ *and either* $g_{\mathrm{ss}}(X) > 0$ *or X is übereven*
$T_{x_{\mathfrak{s}}=0}^\pm$	$\Gamma_X^\pm$	2	$-d_X$	$\lvert X\rvert$	*X has no stable child,* $\lambda_X \in \mathbb{Z}$, *and either* $g_{\mathrm{ss}}(X) > 0$ *or X is übereven*
$T_{(0,0)}$	Γ_X	1	$-\lambda_X$	$\lvert X\rvert$	*X has a stable singleton or* $g_{\mathrm{ss}}(X) = 0$, *X is not übereven and X has no proper stable odd child*

The central components are intersected by the following sloped chains of ratio-

nal curves with parameters $(t_1 - \delta, t_1, \mu)$*:*

Name	t_1	δ	μ	*Condition*
$L_{X,X'}$	$-\lambda_X$	$\delta_{X'}/2$	$\lvert X'\rvert$	$X' < X$ *both principal,* X' *odd*
$L^+_{X,X'}$	$-d_X$	$\delta_{X'}$	$\lvert X'\rvert$	$X' < X$ *both principal,* X' *even with* $\epsilon_{X'} = 1$
$L^-_{X,X'}$				
$L_{X,X'}$	$-d_X$	$\delta_{X'}$	$2\lvert X'\rvert$	$X' < X$ *both principal,* X' *even with* $\epsilon_{X'} = -1$
$L_{X'}$	$-d_X$	$2\delta_{X'}$	$\lvert X'\rvert$	X *principal,* $X' < X$ *orbit of twins,* $\epsilon_{X'} = 1$
$T_{X'}$	$-d_X$	$\delta_{X'} + \frac{1}{\mu}$	$2\lvert X'\rvert$	X *principal,* $X' < X$ *orbit of twins,* $\epsilon_{X'} = -1$

If $\mathcal{R}$ *is not principal we get additional sloped chains with parameters* $(t_1 - \delta, t_1, \mu)$ *as follows:*

Name	t_2	δ	μ	*Condition*
$L_{\mathcal{R}}$	$-d_{\mathcal{R}}$	$2\delta_{\mathfrak{s}}$	1	$\mathcal{R}$ *a cotwin,* $\mathfrak{s} < \mathcal{R}$ *child of size* $2g$, $v_K(c_f) \in 2\mathbb{Z}$
$T_{\mathcal{R}}$	$-d_{\mathcal{R}}$	$\delta_{\mathfrak{s}} + \frac{1}{\mu}$	2	$\mathcal{R}$ *a cotwin,* $\mathfrak{s} < \mathcal{R}$ *child of size* $2g$, $v_K(c_f) \notin 2\mathbb{Z}$
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$(g(\mathfrak{s}_1)+1)d_{\mathfrak{s}_1} - \lambda_{\mathfrak{s}_1}$	$\frac{1}{2}\delta_{\mathfrak{s}_1,\mathfrak{s}_2}$	1	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *principal, odd,* $e_{\mathcal{R}} = 1$
L_X	$(g(\mathfrak{s}_1)+1)d_{\mathfrak{s}_1} - \lambda_{\mathfrak{s}_1}$	$\frac{1}{2}\delta_{\mathfrak{s}_1,\mathfrak{s}_2}$	2	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{\mathfrak{s}_1, \mathfrak{s}_2\}$ *principal, odd orbit*
$L^+_{\mathfrak{s}_1,\mathfrak{s}_2}$	$d_{\mathfrak{s}_1}$	$\delta_{\mathfrak{s}_1,\mathfrak{s}_2}$	1	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *principal, even,* $e_{\mathcal{R}} = 1$, $\epsilon_{\mathfrak{s}_i} = 1$
$L^-_{\mathfrak{s}_1,\mathfrak{s}_2}$				
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$d_{\mathfrak{s}_1}$	$\delta_{\mathfrak{s}_1,\mathfrak{s}_2}$	2	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $\mathfrak{s}_i$ *principal, even,* $e_{\mathcal{R}} = 1$, $\epsilon_{\mathfrak{s}_i} = -1$
L^+_X	$d_{\mathfrak{s}_1}$	$\delta_{\mathfrak{s}_1,\mathfrak{s}_2}$	2	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{\mathfrak{s}_1, \mathfrak{s}_2\}$ *principal, even orbit, and* $\epsilon_{\mathfrak{s}_i} = 1$
L^-_X				
T_X	$d_{\mathfrak{s}_1}$	$\delta_{\mathfrak{s}_1,\mathfrak{s}_2} + \frac{1}{\mu}$	4	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, $X = \{\mathfrak{s}_1, \mathfrak{s}_2\}$ *principal, even orbit, and* $\epsilon_{\mathfrak{s}_i} = -1$

$L_{\mathfrak{t}}$	$d_{\mathfrak{s}}$	$2\delta_{\mathfrak{s},\mathfrak{t}}$	1	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal even,* $\mathfrak{t}$ *twin,* $\epsilon_{\mathfrak{t}} = 1$
$T_{\mathfrak{t}}$	$d_{\mathfrak{s}}$	$\delta_{\mathfrak{s},\mathfrak{t}} + \frac{1}{\mu}$	2	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal even,* $\mathfrak{t}$ *twin,* $\epsilon_{\mathfrak{t}} = -1$

Finally, the crosses of any crossed tail have multiplicity $\frac{\mu}{2}$*.*

Proof. Postponed to Section 4.2. □

Remark 4.1.20. Let X be a principal orbit of clusters in Σ_C. As in Remark 3.2.3, we make a comparison between the rational chains intersecting a central component, $\Gamma_X \in \mathscr{X}_k$ to the tails in the special fibre of the minimal snc model $\mathscr{X}_{\widetilde{X}}$. This comparison makes sense when $g(\Gamma_{\mathfrak{s},L}) > 0$ for some $\mathfrak{s} \in X$. The central component $\Gamma_X \in \mathscr{X}_k$ will have the same genus as the central component $\Gamma_{\widetilde{X}} \in \mathscr{X}_{\widetilde{X},k}$ and multiplicity multiplied by $|X|$. It will have the same tails (with all multiplicities multiplied by $|X|$) except these tails will make up part of the linking chains intersecting Γ_X in the following cases:

(i) If $X \neq \mathcal{R}$ and $P(X)$ is principal, an ∞-tail in $\mathscr{X}_{\widetilde{X},k}$ will form the uphill section of one of the linking chains $L^{\pm}_{P(X),X}$,

(ii) If $X < \mathcal{R}$ and $\mathcal{R}$ is not principal, then any ∞-tail in $\mathscr{X}_{\widetilde{X},k}$ will form the uphill section of a chain: the linking chain between $\Gamma_{\mathfrak{s}_1}$ and $\Gamma_{\mathfrak{s}_2}$ if $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ and $X = \{\mathfrak{s}_1\}$; the crossed tail if $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ and $X = \{\mathfrak{s}_1, \mathfrak{s}_2\}$; and the loop or crossed tail arising from $\mathcal{R}$ if $\mathcal{R}$ is a cotwin,

(iii) a $(y_{\mathfrak{s}} = 0)$-tail will form the downhill section of a linking chain $L_{X,X'}$ if there exists some $X' < X$, a non-trivial orbit of odd, principal children,

(iv) a $(x_{\mathfrak{s}} = 0)$-tail will form the downhill section of a linking chain $L^{\pm}_{X,X'}$ if there exists some $\{\mathfrak{s}'\} = X' < X$, a stable even child,

(v) a $(0,0)$-tail will form the downhill section of a linking chain $L_{X,X'}$ if there exists some $\{\mathfrak{s}'\} = X' < X$, a stable odd child.

where again, all multiplcities are multiplied by $|X|$.

We finish with a description of the Frobenius action on the components of the minimal snc model (or equivalently, on the dual graph).

Theorem 4.1.21 (Frobenius Action)**.** *Let K be a local field and let C/K be a curve with tame reduction and minimal snc model $\mathcal{C}$ over $\mathcal{O}_{K^{\mathrm{ur}}}$. Then the Frobenius automorphism,* Frob*, acts on the components of $\mathcal{C}$ as:*

(i) $\mathrm{Frob}(\Gamma_X^{\pm}) = \Gamma_{\mathrm{Frob}(X)}^{\pm\epsilon_X(\mathrm{Frob})}$,

(ii) $\mathrm{Frob}(L_{X,\,X'}^{\pm}) = L_{\mathrm{Frob}(X),\,\mathrm{Frob}(X')}^{\pm\epsilon_{X'}(\mathrm{Frob})}$,

(iii) a loop L_X is sent to $\epsilon_X(\mathrm{Frob})L_{\mathrm{Frob}(X)}$, a crossed tail T_X to $\epsilon_X(\mathrm{Frob})T_{\mathrm{Frob}(X)}$,[2]

(iv) tails are permuted as $\mathrm{Frob}(T_\infty^{\pm}) = T_\infty^{\pm\epsilon_X(\mathrm{Frob})}$, $\mathrm{Frob}(T_{x_\mathfrak{s}=0}^{\pm}) = T_{x_\mathfrak{s}=0}^{\pm 1^{v(c_X)}}$, and $(y_\mathfrak{s}=0)$-tails are permuted as the corresponding roots of the cluster pictures are.

Proof. Let C have semistable reduction over a Galois extension L of K, and let $\mathscr{Y}$ be the minimal semistable model of C over L^{ur}. Then Frob acts on the components of $\mathscr{Y}_k$ as required by Theorem 2.4.11. Let $\mathscr{X}$ be the quotient of $\mathscr{Y}$ be $\mathrm{Gal}(L^{\mathrm{ur}}/K^{\mathrm{ur}})$. By considering G-invariant open affines, we see that the following square commutes:

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\mathrm{Frob}} & \mathscr{Y} \\ \downarrow q & & \downarrow q \\ \mathscr{X} & \xrightarrow{\mathrm{Frob}} & \mathscr{X} \end{array}$$

So Frob permutes the components of $\mathscr{X}$ as required. Since all central components are components of $\mathscr{X}$, this proves (i).

It remains to show that, after resolving the singularities on $\mathscr{X}$, Frobenius acts on the components as desired. Consider a single blow up of an ideal sheaf

[2] $-L_X$ is same loop but with reversed orientation. $-T_X$ is the same crossed tail but with crosses swapped.

$\mathfrak{J}$ corresponding to an orbit of points under Frobenius. Denote the resulting scheme $\mathscr{Z}'$. The Frobenius automorphism on $\mathscr{Z}$ extends to an automorphism on $\mathscr{Z}'$, which must also be induced by Frobenius. Note that the exceptional components of $\mathscr{Z}'$ are permuted by Frobenius in the same way as the corresponding singularities of $\mathscr{Z}$ are. So it is sufficient to show that Frobenius acts on the singularities of $\mathscr{Z}$ as expected.

The action on singularities on linking chains is determined by the action on the rest of the linking chain. The action on the linking chain is entirely determined by the action on the central components they link, except in the case that there are two linking chains between central components. In this case, they are swapped if and only if $\epsilon_X(\mathrm{Frob}) = -1$. This follows from [19, Theorem 8.15] and the commutative square above. This proves (ii). Loops and crossed tails can be dealt with similarly to prove (iii).

If there are two infinity tails, the singularities they arise from are the images of two points at infinity of a component of $\mathscr{Y}_k$ (see the proof of Theorem 4.1.13). Points at infinity of a component $\Gamma_{\mathfrak{s}}$ of $\mathscr{Y}_k$, arising from a cluster $\mathfrak{s}$, are swapped by Frobenius if and only if $\epsilon_{\mathfrak{s}}(\mathrm{Frob}) = -1$. This proves the first condition of (iv). The singularities giving rise to $(y_{\mathfrak{s}} = 0)$-tails are images of roots of $f(x)$, and those giving rise to $(x_{\mathfrak{s}} = 0)$-tails are images of the points $(0, \pm\sqrt{c_X})$, hence (iv). □

4.1.4 Examples

We present some examples to illustrate the theorem. So far all examples we have given have had principal top cluster; here we also provide examples with $\mathcal{R}$ not principal.

Example 4.1.22. This table shows $\mathscr{X}_k$, of the minimal snc model $\mathscr{X}$ for the different Kodaira-Néron types of elliptic curves with tame potentially semistable reduction (for which it is sufficient to take $p \geq 5$). Our table differs from the table found in [42, p 365], where instead the special fibers of the *minimal regular models* for the different types of elliptic curves are shown. This makes a difference for type II, III or IV elliptic curves, whereas for all the other types the minimal regular model is snc. These special fibres can be read off straight from Theorems 4.1.13 and 4.1.19: one does not need to follow any laborious algorithm.

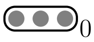
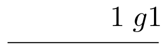
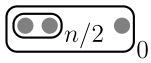
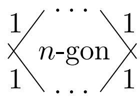
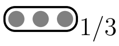
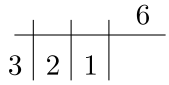
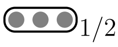
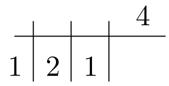
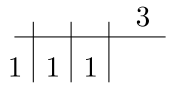
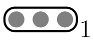
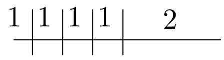
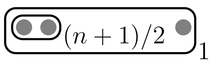
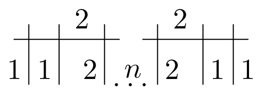
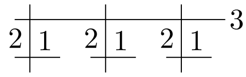
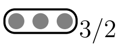
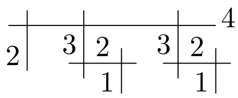
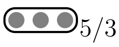
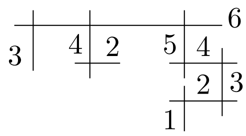
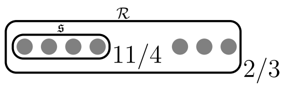
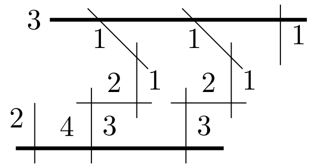
Type	Cluster Picture	$\mathscr{X}_k$
I_0		
I_n		
II		
III		
IV		
I_0^*		
I_n^*		
IV^*		
III^*		
II^*		

Table 4.3: Kodaira-Néron types of elliptic curves with $p \geq 5$.

Example 4.1.23. Let C over $K = \mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 5$ be the hyperelliptic curve given by $C : y^2 = f(x) = (x^3 - p^2)(x^4 - p^{11})$. The cluster picture of C/K consists of two proper clusters $\mathcal{R}$ and $\mathfrak{s}$, shown in Figure 4.2a. The special fibre $\mathscr{X}_k$ of the minimal snc model $\mathscr{X}$ of C/K is shown in Figure 4.2b.



(a) Cluster picture $\Sigma_{C/K}$.



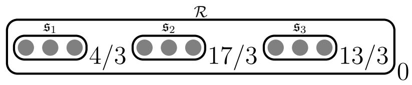
(b) Special fibre of the minimal snc model of C/K.

Figure 4.2: $C : y^2 = (x^3 - p^2)(x^4 - p^{11})$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$.

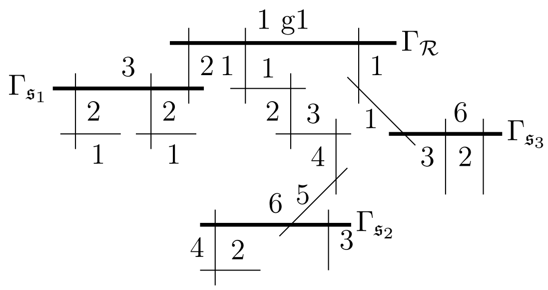
Define elliptic curves C_1 and C_2 over K by $C_1 : y^2 = f_1(x) = x^3 - p^2$ and $C_2 : y^2 = p^2 f_2(x) = p^2(x^4 - p^{11})$ respectively. Note that $f(x) = f_1(x) \cdot f_2(x)$. The roots of $f_1(x)$ contribute the roots in $\mathcal{R} \backslash \mathfrak{s}$, and the roots of $f_2(x)$ contribute the roots in $\mathfrak{s}$. The coefficient in the defining equation of C_2 is chosen to somehow "see" the roots of f_1. It is interesting to compare the minimal snc models of C_i to that of C for $i = 1, 2$. Note that C_1 and C_2 are type IV and type III* elliptic curves respectively, as shown in Table 4.3. It appears that the roots of f_1 and f_2 are making their own contributions to $\mathscr{X}_k$, as both the special fibres of the minimal snc models of C_i can be seen as "submodels" of $\mathscr{X}_k$ for $i = 1, 2$. This shows how $\mathcal{R}$ and $\mathfrak{s}$ each make their own contribution to $\mathscr{X}_k$. Since $\mathfrak{s}$ is an even child of $\mathcal{R}$, and $\epsilon_{\mathfrak{s}} = 1$, there are two linking chains between their contributions in $\mathscr{X}_k$.

Example 4.1.24. Let $K = \mathbb{Q}_p^{\mathrm{ur}}$ for $p \geq 5$, and C/K be the hyperelliptic curve given by $C : y^2 = (x^3 - p^4)((x - 1)^3 - p^{17})((x - 2)^3 - p^{13})$.

The central components of the minimal snc model of C (Figure 4.3b), which arise from clusters in $\Sigma_{C/K}$ (4.3a), are labeled. Note that $\mathcal{R}$ contributes components to the model which look like those appearing in the minimal snc model of a type I_0 elliptic curve; $\mathfrak{s}_1$ those of a type IV* elliptic curve; $\mathfrak{s}_2$ those of a type II* elliptic curve; and $\mathfrak{s}_3$ those of a type II elliptic curve. The special



(a) Cluster picture $\Sigma_{C/K}$.



(b) Special fibre of the minimal snc model of C/K.

Figure 4.3: $C : y^2 = (x^3 - p^4)((x-1)^3 - p^{17})((x-2)^3 - p^{13})$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$.

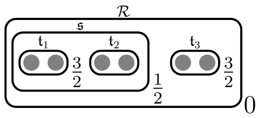
fibers of the minimal snc models of these Kodaira types are all shown in Table 4.3 in Example 4.1.22. This reflects the general phenomenon discussed in remark 4.1.20 that the chains intersecting a central component arising from a cluster $\mathfrak{s}$ "correspond" to the tails of a hyperelliptic curve constructed from $\mathfrak{s}$.

Example 4.1.25. Let $C : y^2 = ((x^2-p)^2 + p^4)\,((x-1)^2 - p^3)$ be a hyperelliptic curve over K. Note that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are swapped by $\mathrm{Gal}(\overline{K}/K)$ and denote their orbit by X. This is a hyperelliptic curve of Namikawa-Ueno type II_{2-4} as in [38, p. 183]. Note $\mathfrak{s}$ is übereven and $\epsilon_{\mathfrak{s}} = 1$, hence $\mathfrak{s}$ gives rise to two components; X is an orbit of twins with $\epsilon_X = 1$, so gives rise to a linking chain, and $\mathcal{R}$ is a cotwin (Definition 2.2.13) so gives rise to a linking chain. Also $e_{\mathfrak{s}} = 2$ so $\Gamma_{\mathfrak{s}}^{\pm}$ are both intersected by tails.

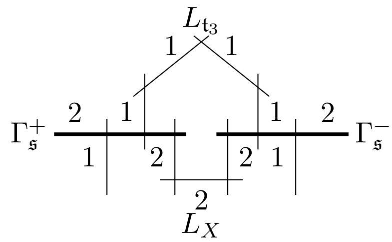
Example 4.1.26. Let C/K be the hyperelliptic curve given by $C : y^2 = x(x^2-p)\,((x-1)^3 - p^2)$. This is a curve of Namikawa-Ueno type $\mathrm{IV} - \mathrm{III} - 0$ as in [38, p. 167]. Observe that $\mathcal{R}$ is not principal so gives rise to a linking chain between $\Gamma_{\mathfrak{s}_1}$ and $\Gamma_{\mathfrak{s}_2}$.

4.2 The Proof

To prove Theorem 4.1.19, we will proceed by induction on two things: the number of proper clusters in $\Sigma_{C/K}$, and the degree $e = [L : K]$ of the minimal extension L/K such that C/L is semistable. The base cases for these are when

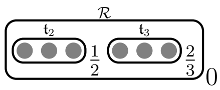


(a) Cluster picture $\Sigma_{C/K}$.

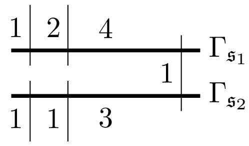


(b) The special fibre of the minimal snc model of C/K.

Figure 4.4: $C : y^2 = ((x^2 - p)^2 + p^4)((x-1)^2 - p^3)$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$, whose minimal snc model is reminiscent of a hot tub, according to a friend of the author's.



(a) Cluster picture $\Sigma_{C/K}$.



(b) Special fibre of the minimal snc model of C/K.

Figure 4.5: $C : y^2 = x(x^2 - p)((x-1)^3 - p^2)$ over $K = \mathbb{Q}_p^{\mathrm{ur}}$.

$\Sigma_{C/K}$ consists of a single proper cluster (which is covered in Section 3.1, in particular Theorem 3.1.1 and Proposition 3.1.12), and when C has semistable reduction over K i.e. $e = 1$ (which is covered in Section 2.4).

The proof itself is split into two sections: the first when the top cluster $\mathcal{R}$ is principal, and the second when it isn't. The second is technically not needed, as there is always a Möbius transform to take a hyperelliptic curve to an isomorphic one whose cluster picture has a principal top cluster. However, many of the hyperelliptic curves encountered in the wild (such as many genus 2 curves) have a non-principal top cluster. Proving the theorem in this way allows us to explicitly describe the minimal snc model of such curves without having to make a Möbius transform.

4.2.1 Principal Top Cluster

We start by assuming that the top cluster $\mathcal{R}$ is principal, and that it has a inertia invariant proper child $\mathfrak{s}$. We will calculate the tails of $\Gamma^{\pm}_{\mathcal{R},K}$ and, if $\mathfrak{s}$ is principal, $\Gamma^{\pm}_{\mathfrak{s},K}$. We will also calculate the linking chain(s) (or the chain arising from $\mathfrak{s}$ if $\mathfrak{s}$ is a twin) between them. This will be done by comparing the linking chain(s) to those in the special fibre of the minimal snc model of another hyperelliptic curve over K, which we will call C^{new}. We will write $C^{\mathrm{new}} : y^2 = f^{\mathrm{new}}(x)$, and denote the set of roots of f^{new} over $\overline{K}$ by $\mathcal{R}^{\mathrm{new}}$. The curve C^{new}/K is chosen so that $\Sigma_{C^{\mathrm{new}}/K}$ has a unique proper cluster $\mathfrak{s}^{\mathrm{new}} \neq \mathcal{R}^{\mathrm{new}}$, enabling us to apply the results of Section 3.2. We will then use induction to deduce the components of the model arising from the subclusters of $\mathfrak{s}$. Finally, we will remove the assumption that $\mathfrak{s}$ is inertia invariant.

Lemma 4.2.1. *Let $\mathcal{R}$ be principal and suppose that $e_{\mathcal{R}} > 1$. The tails of the central component(s) associated to $\mathcal{R}$ are as described in Theorem 4.1.19.*

Proof. First suppose that $\mathcal{R}$ is not übereven. Let $\mathscr{Y}$ be the semistable model of C/L and consider $\Gamma_{\mathcal{R},L} \subseteq \mathscr{Y}$. The stabiliser of $\mathcal{R}$ has order $e_{\mathcal{R}}$. Under the quotient map, a Galois orbit T of points of $\Gamma_{\mathcal{R},L}$ gives rise to a singularity on $\Gamma_{\mathcal{R},K}$ lying on precisely one component of $\mathscr{X}_K$ if and only if $|T| < e_{\mathcal{R}}$ and the points of T lie on $\Gamma_{\mathcal{R},L}$ and no other components of $\mathscr{Y}_k$.

Suppose that $g(\Gamma_{\mathcal{R},L}) = 0$. There are only two orbits with size less than $e_{\mathcal{R}}$, which after an appropriate shift we can assume are at $x_{\mathcal{R}} = 0$ and $x_{\mathcal{R}} = \infty$. The point at ∞ certainly lies on no other component of $\mathscr{Y}_k$ by [19, Propositions 5.5,5.20], so $\Gamma_{\mathcal{R},K}$ will always have ∞-tails. By Theorem 2.4.11, the point $x_{\mathcal{R}} = 0$ lies on no other component of $\mathscr{Y}_k$ if and only if $\mathcal{R}$ has no stable proper odd child. This is because if $\mathfrak{s} < \mathcal{R}$ is a stable odd child then $L_{\mathcal{R},\mathfrak{s}}$ intersects $\Gamma_{\mathcal{R},L}$ at $x_{\mathcal{R}} = 0$, however no other linking chain to a child will ever intersect $\Gamma_{\mathcal{R},L}$ at $x_{\mathcal{R}} = 0$. Therefore $\Gamma_{\mathcal{R},K}$ will have a $(0,0)$-tail if and only if it has no stable proper odd child. The description of the tails follows.

Suppose instead that $g(\Gamma_{\mathcal{R},L}) > 0$. The orbits of points on $\Gamma_{\mathcal{R},L}$ of size less than $e_{\mathcal{R}}$ are the same as the small orbits of points on $\Gamma_{\widetilde{\mathcal{R}},L}$, which are described in Lemmas 3.1.8 - 3.1.11. To complete the description, we must calculate when these small orbits are intersection points with other components. We do this using the explicit description of the components of $\mathscr{Y}_k$ given in Theorem 2.4.11 and how they glue in [19, Proposition 5.5]. From this, we can deduce that the points at ∞ never lie on a component other than $\Gamma_{\mathcal{R},L}$, $(y_{\mathfrak{s}} = 0)$-orbits are intersection points if and only if $\mathfrak{s}$ has a non-trivial orbit of proper odd children, $(x_{\mathfrak{s}} = 0)$-orbits are intersection points if and only if $\mathfrak{s}$ has a stable even child, and the $(0,0)$-orbit is an intersection point if and only if $\mathcal{R}$ has a proper stable odd child.

Now suppose $\mathcal{R}$ is übereven. Then each $\Gamma^{\pm}_{\mathcal{R},L}$ has two orbits of size less than $e_{\mathcal{R}}$, $\{x_{\mathcal{R}} = 0\}$ and $\{x_{\mathcal{R}} = \infty\}$. The points at ∞ do not lie on any other components of $\mathscr{Y}_k$. The points at 0 lie on no other component of $\mathscr{Y}_k$ if and only if $\mathcal{R}$ has no stable child. So, $\Gamma^{\pm}_{\mathcal{R},K}$ has a $(x_{\mathfrak{s}} = 0)$-tail if and only if $\mathcal{R}$ does not have a stable child. The description of the tails follows. □

Lemma 4.2.2. *Let $\mathfrak{s} < \mathcal{R}$ be a principal, Galois invariant cluster with $e_{\mathfrak{s}} > 1$. Then the tails intersecting the central component(s) assosciated to $\mathfrak{s}$ are as described in Theorem 4.1.19.*

Proof. The proof is similar to that of the previous lemma, noting that all of the orbits at infinity are the intersection points of $\Gamma^{\pm}_{\mathfrak{s},L}$ and the linking chain between $\Gamma^{\pm}_{\mathcal{R},L}$ and $\Gamma^{\pm}_{\mathfrak{s},L}$. □

Following is a technical lemma allowing us to compare the chain(s) appearing between $\Gamma_{\mathcal{R},K}$ and $\Gamma_{\mathfrak{s},K}$ to those of a simpler curve C^{new}.

Lemma 4.2.3. *Let $\mathfrak{s}_1, \mathfrak{s}_2$ be two inertia invariant principal clusters (resp. a principal cluster and a twin) such that either $\mathfrak{s}_2 < \mathfrak{s}_1$, or $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ is not principal. Then any linking chain between $\Gamma^{\pm}_{\mathfrak{s}_1,K}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,K}$ (resp. the chain of rational curves arising from $\mathfrak{s}_2$ intersecting $\Gamma^{\pm}_{\mathfrak{s}_1,K}$) is determined entirely by $\lambda_{\mathfrak{s}_i}$ $\bmod \mathbb{Z}$, the parity of $|\mathfrak{s}_2|$, $d_{\mathfrak{s}_i}$, and when $\mathcal{R}$ is not principal $d_{\mathcal{R}}$.*

Proof. Assume that both $\mathfrak{s}_i$ are principal, inertia invariant clusters. From Section 2.3, a linking chain between $\Gamma^{\pm}_{\mathfrak{s}_1,K}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,K}$ is completely determined by the length and number of linking chains between $\Gamma^{\pm}_{\mathfrak{s}_1,L}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,L}$, the order of the action of $\mathrm{Gal}(L/K)$ on any individual component of a linking chain between $\Gamma^{\pm}_{\mathfrak{s}_1,L}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,L}$, and the nature of the singularities at the intersection points of components after taking the quotient. Recall from Theorem 2.4.11 that there is one linking chain, say $\mathcal{C}$, between $\Gamma^{\pm}_{\mathfrak{s}_1,L}$ and $\Gamma^{\pm}_{\mathfrak{s}_2,L}$ if $\mathfrak{s}_2$ is odd and two linking chains, say $\mathcal{C}^+$ and $\mathcal{C}^-$, if $\mathfrak{s}_2$ is even. We will write $\mathcal{C} = \mathcal{C}^+ = \mathcal{C}^-$ if $\mathfrak{s}_2$ is odd. Theorem 2.4.11 tells us that the length of $\mathcal{C}^{\pm}$ is determined by $\delta(\mathfrak{s}_1, \mathfrak{s}_2)$, which is given in terms of $d_{\mathfrak{s}_1}$ and $d_{\mathfrak{s}_2}$ (and $d_{\mathcal{R}}$ in the case where $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ is not principal).

Let P be an intersection point of components $E_1, E_2 \in \{\Gamma_{\mathfrak{s}_1,L}, \Gamma_{\mathfrak{s}_2,L}, \mathcal{C}^{\pm}\}$, and σ_{E_i} the induced $\mathrm{Gal}(\overline{K}/K)$ action on E_i for a generator $\sigma \in \mathrm{Gal}(L/K)$. Suppose $\sigma^a_{E_1}$, and $\sigma^b_{E_2}$, generate the stabilisers of P in E_1 and E_2 respectively. Then $q(P)$ is a tame cyclic quotient singularity with parameters

$$n = \gcd(o(\sigma^a_{E_1}), o(\sigma^b_{E_2})), \quad r = \begin{cases} \frac{d^{-a}_{E_1} d^b_{E_2}}{n^2} & \mathfrak{s}_2 \text{ even}, \\ \frac{\lambda^{-a}_{E_1} \lambda^b_{E_2}}{n^2} & \mathfrak{s}_2 \text{ odd}, \end{cases}$$
$$m_1 = o(\sigma^a_{E_1})/n, \text{ and } m_2 = o(\sigma^b_{E_2})/n,$$

where $o(\tau)$ is the order of $\tau \in \mathrm{Gal}(L/K)$. In other words, the tame cyclic quotient singularity is determined entirely by the automorphisms on the E_i and the parity of $\mathfrak{s}_2$. Therefore, since the automorphisms on E_i are determined entirely by the invariants in the statement of the theorem (by [19, Theorem 6.2]), we are done. The case where $\mathfrak{s}_2$ is a twin follows similarly. □

For the following lemma we first need some notation. Recall that a child of $\mathfrak{s} \in \Sigma_{C/K}$ is *stable* if it has the same stabiliser as $\mathfrak{s}$. Let $\widehat{\mathfrak{s}}^{\mathrm{f}}$ denote the set of stable children of $\mathfrak{s}$, and $\widehat{\mathfrak{s}}^{\mathrm{nf}}$ denote the set of unstable children of $\mathfrak{s}$.

Lemma 4.2.4. *Let C/K be a hyperelliptic curve with $\mathcal{R}$ principal, and let $\mathfrak{s} < \mathcal{R}$ be a Galois invariant proper child. We can construct a hyperelliptic curve, C^{new}, such that the cluster picture $\Sigma_{C^{\mathrm{new}}}$ of C^{new} consists of two proper clusters $\mathfrak{s}^{\mathrm{new}} < \mathcal{R}^{\mathrm{new}}$, where $|\mathfrak{s}| \equiv |\mathfrak{s}^{\mathrm{new}}| \mod 2, d_{\mathcal{R}} = d_{\mathcal{R}^{\mathrm{new}}}, d_{\mathfrak{s}} = d_{\mathfrak{s}^{\mathrm{new}}}$ and $\lambda_{\mathcal{R}} - \lambda_{\mathcal{R}^{\mathrm{new}}}, \lambda_{\mathfrak{s}} - \lambda_{\mathfrak{s}^{\mathrm{new}}} \in \mathbb{Z}$.*

Proof. Let C^{new} be the hyperelliptic curve over K defined by $C^{\text{new}} : y^2 = c_f f_{\mathcal{R}} f_{\mathfrak{s}}$, where

$$f_{\mathcal{R}} = \begin{cases} \prod_{\mathfrak{s} \neq \mathfrak{o} \in \widetilde{\mathcal{R}}} (x - z_{\mathfrak{o}}) & |\widetilde{\mathcal{R}} \setminus \mathfrak{s}| \geq 2, \\ \pi_K^{|\widehat{\mathcal{R}} \setminus \widetilde{\mathcal{R}}| d_{\mathcal{R}}} \prod_{\mathfrak{s} \neq \mathfrak{s}' < \mathcal{R}} (x - z_{\mathfrak{s}'}) & \text{otherwise,} \end{cases}$$

$$f_{\mathfrak{s}} = \begin{cases} \prod_{\mathfrak{o} \in \widetilde{\mathcal{R}}} (x - z_{\mathfrak{o}}) & |\widetilde{\mathfrak{s}}| \geq 2, \\ \prod_{\mathfrak{o} \in \widetilde{\mathfrak{s}}^{\text{f}}} (x - z_{\mathfrak{o}}) \prod_{\mathfrak{s}' \in \widehat{\mathfrak{s}}^{\text{nf}}} (x - z_{\mathfrak{s}'}) & |\widetilde{\mathfrak{s}}| \leq 1 \text{ and } |\widehat{\mathfrak{s}}^{\text{nf}}| \text{ even,} \\ \prod_{\mathfrak{o} \in \widetilde{\mathfrak{s}}^{\text{f}}} (x - z_{\mathfrak{o}}) \prod_{\mathfrak{s}' \in \widehat{\mathfrak{s}}^{\text{nf}}} (x - z_{\mathfrak{s}'})(x + z_{\mathfrak{s}'}) & |\widetilde{\mathfrak{s}}| \leq 1 \text{ and } |\widehat{\mathfrak{s}}^{\text{nf}}| \text{ odd.} \end{cases}$$

It is clear that $\Sigma_{C_{\text{new}}/K}$ consists of proper two clusters which we will call $\mathcal{R}^{\text{new}}$ and $\mathfrak{s}^{\text{new}}$, where $\mathcal{R}_{\text{new}}$ consists of the roots of $f_{\mathcal{R}} \cdot f_{\mathfrak{s}}$, and $\mathfrak{s}^{\text{new}}$ consists of the roots of $f_{\mathfrak{s}}$. It follows that $\mathfrak{s}^{\text{new}} < \mathcal{R}^{\text{new}}$. It remains to check how the cluster invariants of $\mathcal{R}^{\text{new}}$ and $\mathfrak{s}^{\text{new}}$ compare to those of $\mathcal{R}$ and $\mathfrak{s}$. Since any root in a cluster can be taken as its center, it is immediate that $d_{\mathcal{R}} = d_{\mathcal{R}^{\text{new}}}$ and $d_{\mathfrak{s}} = d_{\mathfrak{s}^{\text{new}}}$. By comparing $\deg(f_{\mathfrak{s}})$ to $|\mathfrak{s}|$ we see that $|\mathfrak{s}| \equiv |\mathfrak{s}^{\text{new}}| \mod 2$.

It remains to check that $\lambda_{\mathcal{R}} - \lambda_{\mathcal{R}^{\text{new}}}, \lambda_{\mathfrak{s}} - \lambda_{\mathfrak{s}^{\text{new}}} \in \mathbb{Z}$. Let us begin with the first. By construction, $\mathfrak{s}^{\text{new}}$ is odd if and only if $\mathfrak{s}$ is. Therefore, if $|\widetilde{\mathcal{R}} \setminus \mathfrak{s}| \geq 2$ it follows that $\lambda_{\mathcal{R}^{\text{new}}} = \lambda_{\mathcal{R}}$. Else,

$$2(\lambda_{\mathcal{R}^{\text{new}}} - \lambda_{\mathcal{R}}) = v_K(c_f) + |\widetilde{\mathcal{R}}| d_{\mathcal{R}} + |\widehat{\mathcal{R}} \setminus \widetilde{\mathcal{R}}| d_{\mathcal{R}} - v_K(c_f) - |\widetilde{\mathcal{R}}| d_{\mathcal{R}} = 2|\widehat{\mathcal{R}} \setminus \widetilde{\mathcal{R}}| d_{\mathcal{R}}.$$

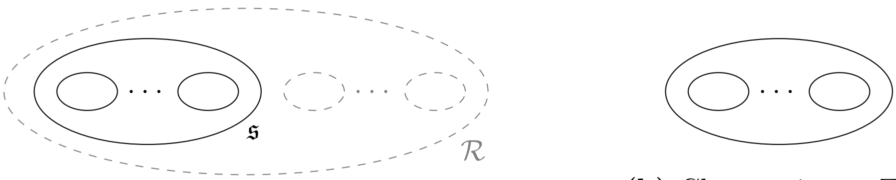
If $d_{\mathcal{R}} \in \mathbb{Z}$, then clearly $\lambda_{\mathcal{R}^{\text{new}}} - \lambda_{\mathcal{R}} \in \mathbb{Z}$. Otherwise, $d_{\mathcal{R}} \notin \mathbb{Z}$. By Lemma 2.2.18, the children of $\mathcal{R}$ must lie in orbits of size $b_{\mathcal{R}} > 1$. Therefore, any such orbit must be an orbit of even children of $\mathcal{R}$, since $\mathfrak{s}$ is fixed and there is at most one child not equal to $\mathfrak{s}$. Hence, $|\widehat{\mathcal{R}} \setminus \widetilde{\mathcal{R}}| d_{\mathcal{R}} \in \mathbb{Z}$, and so $\lambda_{\mathcal{R}^{\text{new}}} - \lambda_{\mathcal{R}} \in \mathbb{Z}$. It can be checked similarly that $\lambda_{\mathfrak{s}^{\text{new}}} - \lambda_{\mathfrak{s}} \in \mathbb{Z}$. □

By the above lemmas and Theorem 3.2.1, we have proved the statements in Theorem 4.1.13 about the linking chain(s) between $\Gamma^{\pm}_{\mathfrak{s},K}$ and $\Gamma^{\pm}_{\mathcal{R},K}$ where $\mathfrak{s} < \mathcal{R}$ is a inertia invariant proper child.

We now turn our focus to the components of $\mathscr{X}_k$ which arise from $\mathfrak{s}$ and its subclusters. In order to do this, we construct another new hyperelliptic curve, which we shall call C', given by

$$C' : y^2 = c_f' \prod_{r \in \mathfrak{s}} (x - r), \text{ where } c_f' = c_f \prod_{r \notin \mathfrak{s}} (z_{\mathfrak{s}} - r). \tag{4.1}$$

Note that C' is also semistable over L, and let $\mathscr{Y}'$ be the semistable model of C' over L. Comparing the cluster pictures of C' and C, we see that the cluster picture $\Sigma_{C'}$ appears within the cluster picture Σ_C of C. This is illustrated in Figure 4.6. In particular, $\mathfrak{s}$ and all of its subclusters in Σ_C are drawn in solid black in Figure 4.6a. These are exactly the clusters that make up $\Sigma_{C'}$, also shown in solid black.



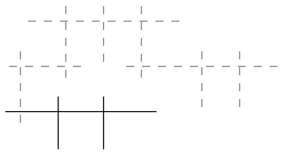
(a) Cluster picture Σ_C.

(b) Cluster picture $\Sigma_{C'}$.

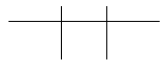
Figure 4.6: Comparison of the cluster pictures of C and C'

The leading coefficient of C' has been chosen so that the corresponding clusters in Σ_C and $\Sigma_{C'}$ have the same cluster invariants. Therefore, there is a closed immersion $\mathscr{Y}_k' \to \mathscr{Y}_k$ which commutes with the action of $\mathrm{Gal}(\overline{K}/K)$. We can see this by calculating the explicit equations of the components of $\mathscr{Y}'$ and using the explicit Galois action on these components given in Theorem 2.4.11. Therefore, this immersion also commutes with the quotient by $\mathrm{Gal}(L/K)$.

After taking this quotient by $\mathrm{Gal}(L/K)$, and performing any appropriate blow ups and blow downs, we obtain a closed immersion $\overline{\mathscr{X}_k' \setminus T_\infty} \to \mathscr{X}_k$, where $\mathscr{X}'$ is the minimal snc model of C'/K and T_∞ is the set of infinity tails of $\mathscr{X}_k'$. We remove the infinity tails since in the small distance case (see Section 3.2.4) the whole tails do not appear in $\mathscr{X}_k$. By our inductive hypothesis we can calculate $\mathscr{X}_k'$. This gives us a full description of the components of $\mathscr{X}_k$ which arise from the subclusters of $\mathfrak{s}$.



(a) Special fibre $\mathscr{Y}_k$, of the minimal snc model of C/L.



(b) Special fibre $\mathscr{Y}'_k$, of the minimal snc model of C'/L.

Figure 4.7: Comparison of the special fibres of the minimal snc models of C and C'

Finally let us remove the assumption that $\mathfrak{s}$ is $\mathrm{Gal}(\overline{K}/K)$ invariant. Let $X < \mathcal{R}$ be a non-trivial orbit of children. Extend K by degree $|X|$ to the field K_X, the minimal extension such that each cluster in X is fixed by $\mathrm{Gal}(\overline{K}/K_X)$. By our inductive hypothesis (since C/K_X needs an extension of degree strictly less than C/K does in order to have semistable reduction), we can calculate the minimal snc model of C over K_X, which we denote $\mathscr{X}_X$. Since each cluster of X is fixed by $\mathrm{Gal}(L/K_X)$, there is a divisor $D_\mathfrak{s}$ corresponding to every cluster $\mathfrak{s} \in X$ and all of the subclusters of $\mathfrak{s}$. Let $D_X = \bigcup_{\mathfrak{s} \in X} D_\mathfrak{s}$ be the union of these divisors. Since $\mathrm{Gal}(K_X/K)$ simply permutes these divisors, the quotient by $\mathrm{Gal}(K_X/K)$ is an étale morphism, and the image of D_X consists of precisely the same components as $D_\mathfrak{s}$ for some $\mathfrak{s} \in X$, but with all the multiplicities multiplied by $|X|$. See Figure 4.8 for an illustration. This concludes the proof when $\mathcal{R}$ is principal.

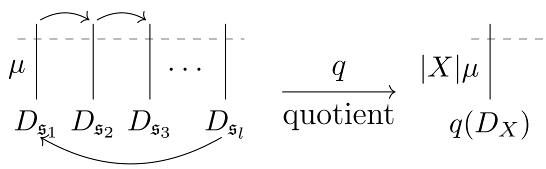


Figure 4.8: Divisors $D_{\mathfrak{s}_i}$, where $X = \{\mathfrak{s}_1 \dots, \mathfrak{s}_l\}$, are permuted by $\mathrm{Gal}(K_X/K)$.

4.2.2 Non-Principal Top Cluster

Now suppose that $\mathcal{R}$ is not principal. If $\mathcal{R}$ is a cotwin, then the contribution to the special fibre of the minimal snc model from $\mathcal{R}$ can be deduced using Remark 3.2.3 and Lemmas 4.2.3 and 4.2.4. The contribution of $\mathfrak{s} < \mathcal{R}$, the child of size $2g$, can be calculated by induction using a curve C' as in (4.1) above.

If $\mathcal{R}$ is not principal and not a cotwin then $\mathcal{R}$ is even and the union of two proper children. In this case, we will write $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$. Here the s_i are either fixed or swapped by inertia. We will deal with the case when s_i are swapped at the end of this section, so for now suppose that both s_i are fixed by inertia. The first of these lemmas shows that there is a Möbius transform taking a certain class of curves with $\mathcal{R}$ not principal to the curves we studied in Section 3.2.

Lemma 4.2.5. *Let C/K be a hyperelliptic curve with cluster picture $\Sigma_{C/K}$, and set of roots $\mathcal{R}$.*

(i) Let $\mathfrak{s} \in \Sigma_{C/K}$ be a cluster with centre $z_{\mathfrak{s}}$. Write every root $r \in \mathfrak{s}$ as $r = z_{\mathfrak{s}} + r_h$, where $v_K(r_h) \geq d_{\mathfrak{s}}$. Then there exists at most one $r \in \mathfrak{s}$ such that $v_K(r_h) > d_{\mathfrak{s}}$.

(ii) If $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $d_{\mathcal{R}} \geq 0$, where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are both fixed by $Gal(L/K)$, have no proper children, and $z_{\mathfrak{s}_1} = 0$. Then the Möbius transform $\psi : r \mapsto \frac{1}{r}$ takes C to a new curve C_M which has cluster picture $\Sigma_M = \{\mathcal{R}_M = \mathfrak{s}_{1,M}, \mathfrak{s}_{2,M}\}$, with $\mathfrak{s}_{1,M} = \{\frac{1}{r} : 0 \neq r \in \mathfrak{s}_1\}$, $\mathfrak{s}_{2,M} = \{\frac{1}{r} : r \in \mathfrak{s}_2\}$, $d_{\mathfrak{s}_{1,M}} = -d_{\mathfrak{s}_1}$ and $d_{\mathfrak{s}_{2,M}} = d_{\mathfrak{s}_2} - 2d_{\mathcal{R}}$.

Proof. (i) Suppose there are two roots r and r' such that $v_K(r_h), v_K(r'_h) > d_{\mathfrak{s}}$. Then $d_{\mathfrak{s}} = v_K(r - r') = v_K(r_h - r'_h) \geq \min(v_K(r_h), v_K(r'_h)) > d_{\mathfrak{s}}$.

(ii) Since $z_{\mathfrak{s}_1} = 0$, we have that $v_K(r) = d_{\mathfrak{s}_1}$ for any $0 \neq r \in \mathfrak{s}_1$. Note also that, $v_K(z_{\mathfrak{s}_2}) = d_{\mathcal{R}}$, hence $v_K(r) = d_{\mathcal{R}}$ for any $r \in \mathfrak{s}_2$. The statement then follows from the fact that $v_K\left(\frac{1}{x} - \frac{1}{y}\right) = v_K(x - y) - v_K(x) - v_K(y)$. □

Remark 4.2.6. Note that $\delta_{\mathfrak{s}_{1,M}} = \delta_{\mathfrak{s}_1} + \delta_{\mathfrak{s}_2}$, $\lambda_{\mathfrak{s}_{1,M}} = \lambda_{\mathfrak{s}_1} - (g(\mathfrak{s})+1)d_{\mathfrak{s}}$ and $\lambda_{\mathfrak{s}_2} - \lambda_{\mathfrak{s}_{2,M}} = (|s_1| - |s_2|)d_{\mathcal{R}} \in 2\mathbb{Z}$.

The next lemma is analogous to Lemma 4.2.4, it constructs a new curve, which we will again call C^{new}, to which we can apply Lemma 4.2.5. This will allow us to calculate the linking chain(s) between $\Gamma^{\pm}_{\mathfrak{s}_1}$ and $\Gamma^{\pm}_{\mathfrak{s}_2}$, by using Lemma 4.2.3.

Lemma 4.2.7. *Let $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $\mathfrak{s}_i$ both fixed by Galois. Then there exists a hyperelliptic curve $C^{\text{new}} : y^2 = f^{\text{new}}(x)$ whose set of roots of f^{new} we denote by $\mathcal{R}^{\text{new}}$, such that $\mathcal{R}^{\text{new}} = \mathfrak{s}_1^{\text{new}} \sqcup \mathfrak{s}_2^{\text{new}}$, where $\mathfrak{s}_i^{\text{new}}$ has no proper children, $|\mathfrak{s}_i| - |\mathfrak{s}_i^{\text{new}}| \in 2\mathbb{Z}$, $d_{\mathfrak{s}_i} = d_{\mathfrak{s}_i^{\text{new}}}$ and $\lambda_{\mathfrak{s}_i} - \lambda_{\mathfrak{s}_i^{\text{new}}} \in \mathbb{Z}$ for $i = 1, 2$.*

Proof. For $i = 1, 2$ define

$$f_{\mathfrak{s}_i} = \begin{cases} \prod_{\mathfrak{o} \in \widetilde{\mathfrak{s}}_i} (x - z_{\mathfrak{o}}) & g(\Gamma_{\mathfrak{s}_i, L}) > 0, \\ \prod_{\mathfrak{o} \in \widehat{\mathfrak{s}}_i^{\text{f}}} (x - z_{\mathfrak{o}}) \prod_{\mathfrak{s}' \in \widehat{\mathfrak{s}}_i^{\text{nf}}} (x - z_{\mathfrak{s}'}) & g(\Gamma_{\mathfrak{s}_i, L}) = 0 \text{ and } |\widehat{\mathfrak{s}}_i^{\text{nf}}| \text{ even}, \\ \prod_{\mathfrak{o} \in \widehat{\mathfrak{s}}_i^{\text{f}}} (x - z_{\mathfrak{o}}) \prod_{\mathfrak{s}' \in \widehat{\mathfrak{s}}_i^{\text{nf}}} (x - z_{\mathfrak{s}'})(x + z_{\mathfrak{s}'}) & g(\Gamma_{\mathfrak{s}_i, L}) = 0 \text{ and } |\widehat{\mathfrak{s}}_i^{\text{nf}}| \text{ odd}. \end{cases}$$

Let $f^{\text{new}} = c_f f_{\mathfrak{s}_1} f_{\mathfrak{s}_2}$, so $C^{\text{new}} : y^2 = c_f f_{\mathfrak{s}_1} f_{\mathfrak{s}_2}$. Proving this satisfies the conditions in the statement of this lemma is similar to the proof of Lemma 4.2.4. □

So, if $\mathcal{R}$ is not principal and a union of two clusters s_i which are fixed by inertia then, by Lemma 4.2.7, Lemma 4.2.3, and Lemma 4.2.5, we know now the linking chain(s) between $\Gamma^{\pm}_{\mathfrak{s}_1}$ and $\Gamma^{\pm}_{\mathfrak{s}_2}$. We can calculate the components associated to $\mathfrak{s}_i$ and its subclusters by induction, constructing a curve as in (4.1). Therefore this gives us the full special fibre of minimal snc model of C/K when $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ is not principal and $\mathfrak{s}_i$ are fixed by inertia.

It remains to consider the case when $\mathcal{R} = s_1 \sqcup \mathfrak{s}_2$ is not principal and $\mathfrak{s}_i$ are swapped by inertia. This is solved by extending the field K to K_X, an extension of degree two. Here, C/K_X has a non principal top cluster $\mathcal{R}' = \mathfrak{s}_1' \sqcup \mathfrak{s}_2'$, where s_i' are both proper clusters, and are fixed by $\text{Gal}(\overline{K}/K_X)$. So we can apply the above lemmas to find the special fibre of the minimal snc model of C/K_X. Taking the quotient by $\mathrm{Gal}(K_X/K)$, which we know how to do by Section 2.3, gives the special fibre of the minimal snc model of C/K. This completes the cases when $\mathcal{R}$ is not principal.

Proof of Theorem 4.1.11. We are done by combining the theorems proved in the rest of the section. □

Chapter 5

Local Solubility

An important application of regular models of curves concerns finding K-rational points, as there is an intimate connection between the k-rational points of the special fibre and the K-rational points of the generic fibre. If K is henselian (which for us it always is), we can lift k points of the special fibre to the generic fibre.

In this section we present a condition for hyperelliptic curve C to have a K-rational point in terms of the cluster picture of C. This allows us to straightforwardly check whether a given curve has a K-rational point without explicitly calculating the model. In addition we can check whether families of hyperelliptic curves (classified by their cluster picture) have K-rational points. With additional work to calculate what proportion of hyperelliptic curves have a given cluster picture (which has been done for elliptic curves in [13] and genus 1 curves in [8]), this would allow us to find the proportion of hyperelliptic curves (of a given genus) which have a K-rational point.

5.1 The Condition

Lemma 5.1.1. *Let $C : y^2 = f(x)$ be a hyperelliptic curve with tame reduction over a local field K and let $\mathcal{C}$ be a regular snc model of C. Suppose that the residue field of K has size $q > 2(g(C)^2 - 1)$. Then any smooth component Γ of the special fibre which is fixed by Frobenius has a smooth k-rational point.*

Proof. Assume f has odd degree, since the even case is dealt with similarly.

Note that any smooth component fixed by Frobenius must be defined over k. The Hasse-Weil bound then states that $|\#\Gamma(k) - (q+1)| \leq 2g(\Gamma)\sqrt{q}$, and hence $\Gamma(k)$ is non-empty if $q > 2(g(\Gamma)^2 - 1)$. Since $g(\Gamma) \leq g(C)$, we are done if $\mathcal{C}$ is a smooth model.

Now assume $\mathcal{C}$ is a semistable model. If Γ is genus 0 then it has k-points, so assume it has positive genus and comes from a principal cluster $\mathfrak{s}$. Let I be the number of intersection points of other components. Then Γ has a smooth k-rational point if $q - I > 2(g(\Gamma)^2 - 1)$, and hence it has a smooth k-point if $2g(C)^2 - I > 2g(\Gamma)^2$. Suppose $\mathfrak{s} = \mathcal{R}$ is odd, and that $\mathfrak{s}$ has s_s children of size 1, s_o proper odd children and s_e proper even children. Then by [19, Theorem 8.5], $I = s_o + 2s_e$ and $2g(\Gamma) + 1 = s_o + s_s$. We also know that $2g(C) + 1 = \deg(f) \geq s_s + 2s_e + 3s_o$. Putting these together the result follows, and similarly if $\mathfrak{s}$ is even or $\mathfrak{s} \neq \mathcal{R}$

Now assume C has tame reduction. A similar argument works using [22, Theorem 7.12,7.18], noting that since Γ is a smooth component it has $e_{\mathfrak{s}} = 1$ and its only intersection points are loops or linking chains to other principal components. □

Proposition 5.1.2. *Let X/K be a curve over a field K with residue field k of size $q > 2(g(C)^2 - 1)$ and let $\mathscr{X}$ be a regular snc model of X. Then X has a K-rational point if and only if $\mathscr{X}_k$ has a component of multiplicity 1 which is fixed by Frobenius and has a k-rational point.*

Proof. By [29, Corollary 9.1.32] there is a reduction map $\mathrm{red} : X(K) \to \mathscr{X}_k(k)$ landing in the smooth locus of $\mathscr{X}_k$, which is onto since K is henselian. Therefore $X(K)$ is empty if and only if smooth locus of $\mathscr{X}_k(k)$ is empty. Since the smooth locus of $\mathscr{X}_k(k)$ consists of points lying on components of multiplicity 1 fixed by Frobenius, the result follows. □

Theorem 5.1.3. *Let C be a hyperelliptic curve with tame reduction over a local field K and let $\mathcal{C}$ be the minimal snc model of C. If $\mathcal{R}$ is principal, then $\mathcal{C}_k$ has a component of multiplicity 1 fixed by Frobenius precisely if at least one of the follow occurs:*

(i) *there is a principal cluster* $\mathfrak{s}$ *fixed by* $Gal(\overline{K}/K)$ *with* $e_{\mathfrak{s}} = 1$*, and if in addition* $\mathfrak{s}$ *is übereven, the character* $\epsilon_{\mathfrak{s}}$ *is trivial on* $Gal(\overline{K}/K)$*;*

(ii) *there is a principal cluster* $\mathfrak{s}$ *fixed by* $Gal(\overline{K}/K)$ *(and* $\epsilon_{\mathfrak{s}}$ *trivial if* $\mathfrak{s}$ *übereven) with* $e_{\mathfrak{s}} > 1$ *and at least one of the following:*

(a) $\mathfrak{s} = \mathcal{R}$ *and either* $\mathcal{R}$ *is odd or* $\mathcal{R}$ *is even and* $\epsilon_{\mathcal{R}}$ *is the trivial character,*

(b) $\mathfrak{s}$ *has a stable child of size* 1 *or* $g(\mathfrak{s}) = 0$*,* $\mathfrak{s}$ *is not übereven and* $\mathfrak{s}$ *has no proper stable odd child,*

(c) $\mathfrak{s}$ *has no stable proper child,* $\lambda_{\mathfrak{s}} \in \mathbb{Z}$*,* $v_K(c_{\mathfrak{s}})$ *is even and either* $g(\mathfrak{s}) > 0$ *or* $\mathfrak{s}$ *is übereven,*

(d) *the children of size* 1 *of* $\mathfrak{s}$ *are fixed by* $Gal(\overline{K}/K)$*;*

(iii) *there is a pair of principal clusters* $\mathfrak{s}' < \mathfrak{s}$*, both fixed by* $Gal(\overline{K}/K)$*, either with* $\mathfrak{s}'$ *odd and* $[-\lambda_{\mathfrak{s}} - \delta_{\mathfrak{s}'}/2, -\lambda_{\mathfrak{s}}] \cap \mathbb{Z} \neq \emptyset$*, or* $\mathfrak{s}'$ *even, the character* $\epsilon_{\mathfrak{s}'}$ *trivial on* $Gal(\overline{K}/K)$ *and* $[-d_{\mathfrak{s}'}, -d_{\mathfrak{s}}] \cap \mathbb{Z} \neq \emptyset$*;*

(iv) *there is a twin* $\mathfrak{t}$ *fixed by* $Gal(\overline{K}/K)$ *and either:*

(a) *the character* $\epsilon_{\mathfrak{t}}$ *is trivial and* $[-d_{\mathfrak{t}}, -d_{P(\mathfrak{t})}] \cap \mathbb{Z} \neq \emptyset$*,*

(b) *the character* $\epsilon_{\mathfrak{t}}$ *is trivial on inertia,* $\epsilon_{\mathfrak{t}}(\mathrm{Frob}) = -1$*,* $d_{\mathfrak{t}} \in \mathbb{Z}$ *and* $\nu_{\mathfrak{t}} \in 2\mathbb{Z}$ *or,*

(c) *the character* $\epsilon_{\mathfrak{t}}$ *is non-trivial on inertia and* $v_K(c_{\mathfrak{t}})$ *even;*

If $\mathcal{R}$ *is not principal, then* $\mathcal{C}_k$ *has a component of multiplicity* 1 *fixed by Frobenius in the additional following cases:*

(v) *there is a cotwin* $\mathfrak{s} < \mathfrak{t}$ *fixed by* $Gal(\overline{K}/K)$ *and either:*

(a) *the character* $\epsilon_{\mathfrak{t}}$ *is trivial and* $[-d_{\mathfrak{s}}, -d_{\mathfrak{t}}] \cap \mathbb{Z} \neq \emptyset$*,*

(b) *the character* $\epsilon_{\mathfrak{t}}$ *is trivial on inertia,* $\epsilon_{\mathfrak{t}}(\mathrm{Frob}) = -1$*,* $d_{\mathfrak{t}} \in \mathbb{Z}$ *and* $\nu_{\mathfrak{t}} \in 2\mathbb{Z}$ *or,*

(c) *the character* $\epsilon_{\mathfrak{t}}$ *is non-trivial on inertia and trivial on Frobenius;*

(vi) the top cluster $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ *is not principal and either:*

(a) $\mathfrak{s}_1$ *is odd, fixed by* $Gal(\overline{K}/K)$ *and* $[-\lambda_{\mathcal{R}} - \delta_{\mathfrak{s}_1}/2, -\lambda_{\mathcal{R}}] \cap \mathbb{Z} \neq \emptyset$*,*

(b) $\mathfrak{s}_1$ *is odd, fixed by inertia but not Frobenius and* $d_{\mathcal{R}} \in \mathbb{Z}$*,*

(c) $\mathfrak{s}_1$ *is odd, inertia swaps* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *and* $\epsilon_{\mathcal{R}}(\mathrm{Frob}) = 1$*,*

(d) $\mathfrak{s}_1$ *is even, fixed by* $Gal(\overline{K}/K)$*,* $\epsilon_{\mathfrak{s}_1}$ *is trivial and* $[-d_{\mathfrak{s}_1}, -d_{\mathcal{R}}] \cap \mathbb{Z} \neq \emptyset$*,*

(e) $\mathfrak{s}_1$ *is even, fixed by inertia but not Frobenius,* $\epsilon_{\mathfrak{s}_1}$ *is trivial and* $d_{\mathcal{R}} \in \mathbb{Z}$*,*

(f) $\mathfrak{s}_1$ *is even, inertia swaps* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$*,* $\epsilon_{\mathfrak{s}_1}$ *is trivial and* $\epsilon_{\mathcal{R}}(\mathrm{Frob}) = 1$*.*

Proof. For brevity, call components of multiplicity 1 which are fixed by Frobenius *good.* If $\mathcal{C}$ has a good component, it must either by a principal component of part of a chain of rational curves. We investigate these cases separately, beginning with principal components. The principal components are parametrised by principal clusters[1], and by Theorem 4.1.18 a good component must come from a principal cluster $\mathfrak{s}$ fixed by Galois. In addition, $\mathfrak{s}$ must have $e_{\mathfrak{s}} = 1$, i.e. $d_{\mathfrak{s}} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$, and if $\mathfrak{s}$ is übereven, the character $\epsilon_{\mathfrak{s}}$ must be trivial. In all these cases we do indeed have a good component. This is case (i).

We are left to find the cases where there is a good component in a chain of rational curves. Chains of rational curves appear in several flavours: linking chains between principal components, loops from a principal component to itself, and tails (including the crosses of crossed tails). A principal cluster $\mathfrak{s}$ fixed by Galois with $e_{\mathfrak{s}} > 1$ contributes tails to the special fibre. If $\mathfrak{s} = \mathcal{R}$ then the ∞-tails will have a good component if $\mathcal{R}$ is odd or if $\mathcal{R}$ is even and the character $\epsilon_{\mathcal{R}}$ is trivial by the first three rows of the first table in Theorem 4.1.19, noting that $\epsilon_{\mathcal{R}}(\mathrm{Frob})$ swaps the ∞-tails if and only if it is -1 by Theorem 4.1.21. If $\mathfrak{s}$ is a general principal cluster then a $(0,0)$-tail will always have a good component. An $(x_{\mathfrak{s}} = 0)$-tail will have a good component

[1]Except in crossed tails but these always have even multiplicity.

so long as $\lambda_{\mathfrak{s}} \in \mathbb{Z}$ and $v_K(c_{\mathfrak{s}})$ — the former ensures that there are two $(x_{\mathfrak{s}} = 0)$-tails by Theorem 4.1.11 and the latter that they are fixed by Frobenius, by Theorem 6.3.3. A $(y_{\mathfrak{s}} = 0)$-tail will have a good component if and only if the singletons of $\mathfrak{s}$ are fixed by Galois. This is case (ii).

Suppose L is a linking chain arising from a pair of orbits $X' < X$. By Theorem 4.1.19, the lowest common multiple of multiplicities of the components of L is $|X'|$, so $X = \mathfrak{s}$, $X' = \mathfrak{s}'$ must be clusters fixed by Galois. If $\mathfrak{s}'$ is odd then L has a good component if and only if $[-\lambda_{\mathfrak{s}} - \delta_{\mathfrak{s}'}/2, -\lambda_{\mathfrak{s}}] \cap \mathbb{Z} \neq \emptyset$. See Remark 2.5.15 for more details. If $\mathfrak{s}'$ is even then L has a good component if and only if $\epsilon_{\mathfrak{s}'}$ is trivial and $[-d_{\mathfrak{s}'}, -d_{\mathfrak{s}}] \cap \mathbb{Z} \neq \emptyset$. This is because $\sigma \in \mathrm{Gal}(\overline{K}/K)$ swaps the two linking chains connecting $\Gamma_{\mathfrak{s},L}$ and $\Gamma_{\mathfrak{s}',L}$ if and only if $\epsilon_{\mathfrak{s}'}(\sigma) = -1$ This is case (iii).

Now suppose L is a loop arising from an orbit of twins. By the same argument as above, this must in fact arise from a twin $\mathfrak{t}$ fixed by Galois with $\epsilon_{\mathfrak{t}}$ trivial on inertia. If $\epsilon_{\mathfrak{t}}$ is further trivial on Frobenius then L will have a good component if and only if it has a component of multiplicity 1, which occurs precisely when $[-d_{\mathfrak{t}}, -d_{P(\mathfrak{t})}] \cap \mathbb{Z} \neq \emptyset$. If $\epsilon_{\mathfrak{t}}(\mathrm{Frob}) = -1$ then Frobenius inverts the loop, but there can still be a component fixed by Frobenius if L is a loop of odd length. This happens precisely when $d_{\mathfrak{t}}$ is an integer.

If T is instead a crossed tail, it must arise from a twin $\mathfrak{t}$ fixed by Galois with $\epsilon_{\mathfrak{t}}$ non-trivial on inertia. In this case, the crosses of T have multiplicity 1, and by Theorem 6.3.3 they are fixed by Frobenius if and only if $v_K(c_{\mathfrak{t}})$ is even. This and the previous paragraph is case (iv).

Cases (v) and (vi), where $\mathcal{R}$ is a non-principal cluster, are dealt with similarly.

□

Corollary 5.1.4. *Let C be a hyperelliptic curve with tame reduction over a local field K with residue field k of characteristic $p > 2$. Suppose that $|k| = q$ is such that $q > 2(g(C)^2 - 1)$. Then C has a K-rational point in precisely the cases described in Theorem 5.1.3.*

Proof. By Lemma 5.1.1, any smooth components of the special fibre fixed by Frobenius have a k-point. By Theorem 5.1.3, a smooth component fixed by Frobenius exists and hence by Proposition 5.1.2 C has a K-rational point. □

5.2 Examples

We give some examples to illustrate our theorem. All curves are over $\mathbb{Q}_p$, with $p > 5$ so that all multiplicity 1 components of the special fibre which are fixed by Frobenius have points.

Example 5.2.1. Let $C : y^2 = (x^4 - p^{17})(x^3 - p^2)$. We immediately observe that $\mathcal{R}$ is principal, odd and $e_{\mathcal{R}} > 1$, so C must have a K-rational point by condition (ii)(a).

Example 5.2.2. Let $C : y^2 = p(x^6 - p^2)$. The cluster picture of C consists of a unique principal cluster $\mathfrak{s}$ of depth 1/2. Therefore, $e_{\mathfrak{s}} > 1$ and condition (i) is not satisfied. Conditions (iii)-(vi) are clearly not satisfied, so we are left to check condition (ii). The character $\epsilon_{\mathcal{R}}(\sigma) = (-1)^{v_K(c_f)}$ for σ a generator of inertia, and in this case $v_K(c_f) = 1$ and so $\epsilon_{\mathcal{R}}$ is non-trivial and (ii)(a) is not satisfied. Since $\mathfrak{s}$ has no stable singleton, it has no $(0,0)$-tail by Theorem 4.1.19 and $\lambda_{\mathfrak{s}} \notin \mathbb{Z}$ so it has a unique $x_{\mathfrak{s}} = 0$-tail. Therefore (ii)(b) and (c) are not satisfied. Galois acts non trivially on the singletons of $\mathfrak{s}$, and so finally (ii)(d) is not satisfied. Therefore C has no K-rational point. Indeed, the minimal snc model of C, shown below, has no component of multiplicity 1 and so certainly C cannot have a K-rational point.

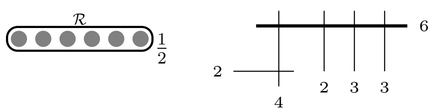


Figure 5.1: Cluster picture and minimal snc model of $C : y^2 = p(x^6 - p^2)$.

Example 5.2.3. Let $C : y^2 = p((x-1)^2 + p^2)((x-\zeta_3)^2 + p^2)((x-\zeta_3^2)^2 + p^2)$, with $p \equiv -1 \bmod 3$ and ζ_3 a fixed cube root of unity. The cluster picture of C is shown below. There is a unique principal cluster $\mathcal{R}$, which is übereven. However, $e_{\mathcal{R}} = 2$ and $\epsilon_{\mathcal{R}}(\sigma) = (-1)^v_K(c_f) = -1$ for σ a generator of $\mathrm{Gal}(\mathbb{Q}_p(\sqrt{2})/\mathbb{Q}_p)$ and so $\epsilon_{\mathcal{R}}$ is non-trivial. Therefore condition (i) and (ii) are not satisfied. Since there is only 1 principal cluster (iii) is not satisfied. Therefore we are left to check (iv). But the three twins are permuted by Frobenius and hence (iv) is not satisfied. Therefore C has no K-rational points. This can also be seen from looking at the minimal snc model — the components of multiplicity 1 are the crosses of the three crossed tails. But the crossed tails are permuted by Frobenius as the corresponding clusters are, so there is no component of multiplicity 1 fixed by Frobenius.

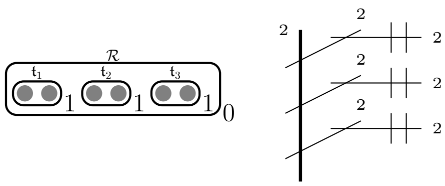


Figure 5.2: Cluster picture and minimal snc model of $C : y^2 = p((x-1)^2 + p^2)((x-\zeta_3)^2 + p^2)((x-\zeta_3^2)^2 + p^2)$.

Example 5.2.4. Let $C : y^2 = p(x^3 - p^2)((x-1)^3 - p^2)$. This is a hyperelliptic curve of Namikawa-Ueno type $\mathrm{II}^* - \mathrm{II}^* - \alpha$. The two principal clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have $e_{\mathfrak{s}_i} = 6$ and so (i) does not apply. Quick inspection reveals that (ii)-(v) don't either. Continuing, we see that the top cluster $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ is not principal and $\mathfrak{s}_1$ is odd and fixed by Galois. However, $[-\lambda_{\mathcal{R}} - \delta_{\mathfrak{s}_1}/2, -\lambda_{\mathcal{R}}] = [-11/6, -3/2]$ which does not contain an integer. Therefore condition (vi) does not give us a K-rational point, and therefore C cannot have any K-rational points. This can also be seen from the minimal snc model as there is no component of multiplicity 1.

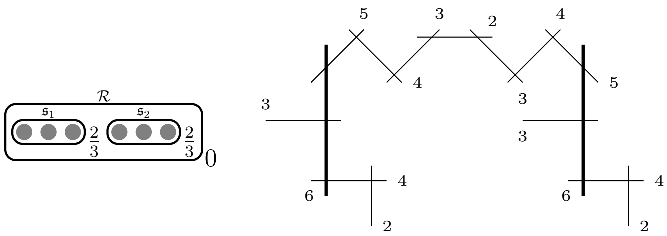


Figure 5.3: Cluster picture and minimal snc model of $C : y^2 = p(x^3 - p^2)((x-1)^3 - p^2)$.

Chapter 6

Models of Bihyperelliptic Curves

We now shift our attention towards *bihyperelliptic curves*, curves with maps to two distinct hyperelliptic curves. Such curves arise naturally when studying the parity conjecture. The results of this section can also be found in [21].

We define a generalisation of cluster pictures, the *chromatic cluster picture*, which we associate to a bihyperelliptic curve Y. We then show that this combinatorial object is sufficient to determine the dual graph with genera of the special fibre of the minimal regular model $\mathscr{Y}^{\min}$ of Y in the case where Y has semistable reduction. We do this by giving an explicit description of the dual graph in terms of the chromatic cluster picture.

Our description of $\mathscr{Y}_k^{\min}$ is very much in the spirit of Theorem 2.4.11 — to a cluster $\mathfrak{s}$ we associate 1, 2 or 4 components of the special fibre, and the components of $\mathfrak{s}$ and $\mathfrak{s}'$ are linked by a chain of $\mathbb{P}^1$s if $\mathfrak{s}'$ is a child of $\mathfrak{s}$ (or vice versa). The length of this chain is determined by $\delta_{\mathfrak{s}'}$. In Theorem 6.3.3, we also give the action of Frobenius on the the dual graph of $\mathscr{Y}_k^{\min}$.

In chromatic cluster pictures, red roots will be represented by spheres ●, blue roots by hexagons ⬢ and purple roots by diamonds ◆. Red clusters will be denoted with dotted lines, blue clusters by dashed lines, purple cluster by dot-dash lines and black clusters by solid lines. Sometimes, we will need an empty cluster, which will look like this: ◉.

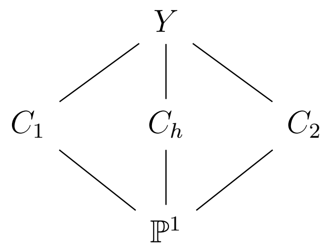
6.1 Bihyperelliptic Curves

Let us begin by defining bihyperelliptic curves: they are smooth projective curves with maps to two distinct hyperelliptic curves, all defined over the same base field.

Definition 6.1.1. Let $C_1 : y_1^2 = f_1(x) = c_1\widetilde{f_1}(x)$ and $C_2 : y_2^2 = f_2(x) = c_2\widetilde{f_2}(x)$ be hyperelliptic curves, given by their affine models and with $\widetilde{f_i}$ monic. Then $C_h : y_h^2 = f_h(x) = c_h\widetilde{f_h}(x)$ is their *composite curve*, where the set of roots of f_h is the roots of $f_1 f_2$ of multiplicity 1 and $c_h = c_1 c_2$. The curve Y, the normalisation of the projective closure of

$$\left\{ \begin{array}{c} y_1^2 = f_1(x) \\ y_2^2 = f_2(x) \end{array} \right\},$$

is a *bihyperelliptic curve*. The curves fit into a tower:



such that $Y/\mathbb{P}^1$ is a Galois cover with Galois group $C_2 \times C_2$.

Remark 6.1.2. A perhaps more suitable notion of a bihyperelliptic curve would be any curve Y which has a degree 2 map to a hyperelliptic curve C, directly mirroring the definition for bielliptic curves. However, the cover $Y/\mathbb{P}^1$ is not necessarily Galois in this case. Since we require this for our purposes, we shall restrict to the case given in Definition 6.1.1.

Example 6.1.3. Suppose $y_1^2 = 2(x-1)(x^2-p^4)$ and $y_2^2 = (x-1)(x^2+p^4)$. Then $y_h^2 = 2(x^2-p^4)(x^2+p^4)$ is their composite curve.

6.2 Chromatic Cluster Pictures

The combinatorial objects we will use to calculate the semistable model will be *chromatic cluster pictures*. A chromatic cluster picture is similar to a cluster picture, but instead of one polynomial we take the roots of two polynomials f_1 and f_2, and we colour the roots red and blue to indicate whether they are roots of f_1 or f_2, or purple if the root belongs to both. In this way, the chromatic cluster picture contains the information of the cluster pictures associated to f_1, f_2 and f_h, their composite curve defined above.

Definition 6.2.1. A *chromatic cluster picture* Σ_χ is a cluster picture Σ on a set $\mathcal{R}$ with a colouring function $c : \mathcal{R} \longrightarrow \{\text{red, blue, purple}\}$, assigning to each root a colour.

This induces a colouring on the remaining clusters as follows:

(i) clusters with an odd number of blue children and an even number of red children (resp. an odd number of red children and an even number of blue children) are coloured blue (resp. red),

(ii) clusters with an odd number of blue children and an odd number of red children are coloured purple,

(iii) all other clusters are coloured black.

where purple children are included in both the set of red *and* blue clusters. Blue, red and purple clusters are called *chromatic clusters.*

Clusters with purple children, or clusters with both blue and red children have *polychromatic children*, whereas clusters whose only chromatic children are red or blue have *monochromatic children.*

We define the *red* (resp. *blue*) cluster picture Σ_1 (resp. Σ_2) associated to Σ_χ to be the subset of Σ where the only clusters of size 1 are the red (resp. blue) ones. We forget the colouring on the rest of the clusters.

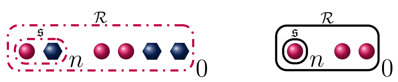
Lemma 6.2.2. *Let $\mathfrak{s} \in \Sigma$ be a cluster with no purple roots. If $\mathfrak{s}$ is odd then $\mathfrak{s}$ is red or blue, and if $\mathfrak{s}$ is even then $\mathfrak{s}$ is purple or black. Furthermore, purple clusters are odd in Σ_1 (i.e., they have odd size when we count only red children) and Σ_2 and even in Σ, red clusters are odd in Σ_1 and Σ and even in Σ_2, blue clusters are odd in Σ_2 and Σ and even in Σ_1, and finally black clusters are even in all of Σ_1, Σ_2 and Σ.*

Proof. This follows by induction on the size of the cluster, noting that it is trivially true for singletons. $\square$

Remark 6.2.3. The red and blue cluster pictures aren't cluster pictures (or chromatic cluster pictures) in the conventional sense — they are cluster pictures with some additional p-adic disks. They can have "empty" clusters which contain no roots, or clusters which only contain another cluster and nothing else. However, they give rise to the same *admissible collection of discs*, in the sense of [19, Definition 3.4], as Σ_h. By admissible collection of disks we mean a non-empty collection $\mathcal{D}$ of disks of integer radius and centre in $\mathcal{O}_K$ (an *integral disk*) such that there is a maximal element with respect to inclusion and if $D_1 \subseteq D_2$ are both in $\mathcal{D}$ then every integral disk $D_1 \subseteq D \subseteq D_2$ is also in $\mathcal{D}$. Therefore we can apply the results of [19] to them.

We will be interested in the chromatic cluster pictures of bihyperelliptic curves. In other words, if $Y : \{y_1^2 = f_1, y_2^2 = f_2\}$ is a bihyperelliptic curve, then its chromatic cluster picture arises from colouring the roots of f_1 red, the roots of f_2 blue, the roots arising from both purple and the rest of the clusters according to the rules (ii) - (iv).

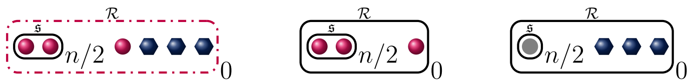
Example 6.2.4. Below (left) is the chromatic cluster picture of $Y : \{y_1^2 = (x-p^n)(x^2-1), y_2^2 = (x+p^n)(x^2-2)\}$. On the right is the red cluster picture, which is equivalent to the blue cluster picture.



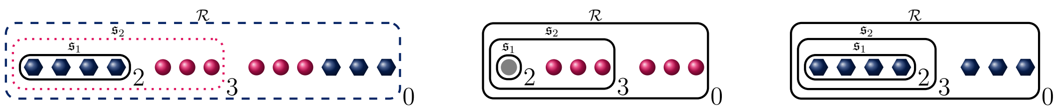
The cluster $\mathfrak{s} \in \Sigma_\chi$ contains both an odd number of red children and an odd number of blue children (one of each), and hence is purple. The cluster $\mathcal{R} \in \Sigma_\chi$ contains three red children (since the purple child counts as a red and a blue child) and three blue children and hence is also purple.

The red and blue clusters are equivalent, and in both the "cluster" $\mathfrak{s}$ is a p-adic disk containing a single root. This wouldn't be a cluster in a conventional cluster picture.

Example 6.2.5. From left to right, we have the chromatic, red and blue cluster pictures of $Y : \{y_1^2 = (x^2 - p^n)(x-1), y_2^2 = x^3 - 2\}$. The cluster $\mathfrak{s} \in \Sigma_\chi$ has an even number of both red and blue children and hence is black. On the other hand, $\mathcal{R}$ has an odd number of both red and blue children and hence is coloured purple. Note that in the blue cluster picture $\mathfrak{s}$ is an empty cluster (i.e. it is a p-adic disk which contains no roots).



Example 6.2.6. From left to right, the chromatic, red, and blue cluster picture of the bihyperelliptic curve $Y : \{y_1^2 = (x^3 - p^9)(x^3 - 2), y_2^2 = (x^4 - p^{20})(x^3 - 1)\}$. In the red cluster picture $\mathfrak{s}_1$ is an empty cluster, and in the blue cluster picture $\mathfrak{s}_2$ has a unique child. Neither of these are traditionally clusters.



The following is an important definition, the clusters which will contribute principal components to the semistable model.

Definition 6.2.7. Let $\mathfrak{s}$ be a cluster. Then $\mathfrak{s}$ is *chromatically principal* if $|\mathfrak{s}| \geq 3$, except in the following cases:

(i) $\mathfrak{s} = \mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, with $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of the same colour, or one of $\mathfrak{s}_1$ or $\mathfrak{s}_2$ a singleton or a twin,

(ii) $\mathfrak{s} = \mathcal{R}$ has a unique proper child $\mathfrak{s}'$ of size $2g(C_h)$ such that either $\mathfrak{s}'$ is purple or $\mathfrak{s}$ and $\mathfrak{s}'$ are both black.

We can characterise this as follows:

Lemma 6.2.8. *A cluster $\mathfrak{s}$ is chromatically principal if and only if it is principal in at least one of $\Sigma_1, \Sigma_2, \Sigma_h$.*

Proof. If $\mathfrak{s} \neq \mathcal{R}$ this is clear as its size in Σ_χ is the same as its size in Σ_h. Therefore, suppose $\mathfrak{s} = \mathcal{R}$. Suppose $\mathfrak{s} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $\mathfrak{s}_1$ and $\mathfrak{s}_2$ the same colour. Then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have the same parity in each of $\Sigma_1, \Sigma_2, \Sigma_h$ by Lemma 6.2.2, noting that purple roots do not have an effect since they contribute to the size of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in each of the cluster pictures. If $\mathfrak{s}$ has a unique proper child of size $2g(C_h)$ then Σ_1 has a child of size $2g(C_1)$, and similarly for Σ_2 and Σ_h. The other direction can be checked similarly (for example, if $\mathfrak{s} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $\mathfrak{s}_1$, $\mathfrak{s}_2$ principal of different colours, then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ will have opposite parities in at least one of Σ_1, Σ_2 or Σ_h). □

We must update our definition of cotwin to exclude a case which cannot happen for classic cluster pictures:

Definition 6.2.9. A cluster $\mathfrak{s}$ is a *cotwin* if it has a child $\mathfrak{s}'$ of size $2g(C_h)$ whose complement isn't a twin, unless $\mathfrak{s}$ is purple and $\mathfrak{s}'$ is black.

We finish the section with a definition which will be needed in stating our main theorem.

Definition 6.2.10. The *chromatic genus* $g_\chi(\mathfrak{s})$ of a principal cluster $\mathfrak{s}$ is defined as follows. If $\mathfrak{s}$ is polychromatic then $g_\chi(\mathfrak{s}) = |\mathfrak{s}_\chi| - 3$ when $\mathfrak{s}$ is the top cluster and f_1 and f_2 both have even degree, or the top cluster $\mathcal{R} = \mathfrak{s} \bigsqcup \mathfrak{s}'$ is not principal, $\mathfrak{s}$ is even and f_1 and f_2 both have even degree; and $g_\chi(\mathfrak{s}) = |\mathfrak{s}_\chi| - 2$ otherwise. If $\mathfrak{s}$ has monochromatic children or is übereven then $g_\chi(\mathfrak{s}) = g(\mathfrak{s})$.

Definition 6.2.11. If $\mathfrak{s}$ is a cluster then $\mathfrak{s}_\chi$ is the set of chromatic children of $\mathfrak{s}$. If $\mathfrak{s}, \mathfrak{s}'$ are two proper clusters then $\sigma(\mathfrak{s}, \mathfrak{s}') = -1$ in the following cases:

(i) $\mathfrak{s}$ and $\mathfrak{s}'$ each have monochromatic children of opposite colours,

(ii) $\mathfrak{s}$ is black übereven and $\mathfrak{s}'$ is black with monochromatic, blue children;

and 1 otherwise. The chromatic

6.3 Statement of Results

6.3.1 The Theorem

The rough idea of the theorem is that principal clusters give rise to components in the special fibre, and components of $\mathfrak{s}$ are linked to the components of the children of $\mathfrak{s}$. The number of components and linking chains, as well as the lengths of the linking chains, are determined by the properties of the clusters.

Theorem 6.3.1. *Let K be a local field of odd residue characteristic and let $C_1 : y_1^2 = f_1(x)$ and $C_2 : y_2^2 = f_2(x)$ be two distinct hyperelliptic curves over K which are double covers of the same $\mathbb{P}^1 : y = x$. Let Y be the bihyperelliptic curve arising from C_1 and C_2, such that Y has semistable reduction and all the depths in the chromatic cluster picture of Y are integers. Then the dual graph of $\mathscr{Y}_k^{\min}$ is entirely determined by the chromatic cluster picture of Y.*

In particular, each principal cluster $\mathfrak{s}$ contributes vertices of genus $g_\chi(\mathfrak{s})$ to the dual graph of $\mathscr{Y}_k^{min}$. If $\mathfrak{s}$ is not übereven: 1 vertex $v_\mathfrak{s}$ if $\mathfrak{s}$ has polychromatic children and 2 vertices $v_\mathfrak{s}^+, v_\mathfrak{s}^-$ if $\mathfrak{s}$ has monochromatic children; and if übereven: 2 vertices $v_\mathfrak{s}^+, v_\mathfrak{s}^-$ if $\mathfrak{s}$ has chromatic children and 4 vertices $v_\mathfrak{s}^{+,+}, v_\mathfrak{s}^{+,-}, v_\mathfrak{s}^{-,+}, v_\mathfrak{s}^{-,-}$ if $\mathfrak{s}$ has no chromatic children.

These are linked by edges as follows:

Name	From	To	Length	Condition
$L_{\mathfrak{s}'}^+$	$v_\mathfrak{s}^+$	$v_{\mathfrak{s}'}^{\sigma}$	$\frac{1}{2}\delta_{\mathfrak{s}'}$	$\mathfrak{s}' < \mathfrak{s}$ both principal, $\mathfrak{s}'$ chromatic
$L_{\mathfrak{s}'}^-$	$v_\mathfrak{s}^-$	$v_{\mathfrak{s}'}^{-\sigma}$		
$L_{\mathfrak{s}'}^{+,+}$	$v_\mathfrak{s}^{+,+}$	$v_{\mathfrak{s}'}^{+,+}$	$\delta_{\mathfrak{s}'}$	$\mathfrak{s}' < \mathfrak{s}$ both principal, $\mathfrak{s}'$ black
$L_{\mathfrak{s}'}^{+,-}$	$v_\mathfrak{s}^{+,-}$	$v_{\mathfrak{s}'}^{+,-}$		
$L_{\mathfrak{s}'}^{-,+}$	$v_\mathfrak{s}^{-,+}$	$v_{\mathfrak{s}'}^{-,+}$		
$L_{\mathfrak{s}'}^{-,-}$	$v_\mathfrak{s}^{-,-}$	$v_{\mathfrak{s}'}^{-,-}$		
$L_\mathfrak{t}$	$v_\mathfrak{s}^+$	$v_\mathfrak{s}^-$	$\delta_\mathfrak{t}$	$\mathfrak{t} < \mathfrak{s}$, $\mathfrak{t}$ chromatic twin, $\mathfrak{s}$ principal
$L_\mathfrak{t}^+$	$v_\mathfrak{s}^{+,+}$	$v_\mathfrak{s}^{\sigma,-\sigma}$	$2\delta_\mathfrak{t}$	$\mathfrak{t} < \mathfrak{s}$, $\mathfrak{t}$ black twin, $\mathfrak{s}$ principal
$L_\mathfrak{t}^-$	$v_\mathfrak{s}^{-,-}$	$v_\mathfrak{s}^{-\sigma,\sigma}$		

where $\sigma = \sigma(\mathfrak{s}, \mathfrak{s}')$; $v_\mathfrak{s}^{\pm,+} = v_\mathfrak{s}^{\pm,-} = v_\mathfrak{s}^{\pm}$ if $\mathfrak{s}$ is non-übereven with monochro-

matic red children; $v_{\mathfrak{s}}^{+,\pm} = v_{\mathfrak{s}}^{-,\pm} = v_{\mathfrak{s}}^{\pm}$ *if* $\mathfrak{s}$ *is non-übereven with has monochromatic blue children;* $v_{\mathfrak{s}}^{\pm,\pm} = v_{\mathfrak{s}}^{+}$, $v_{\mathfrak{s}}^{\pm,\mp} = v_{\mathfrak{s}}^{-}$ *if* $\mathfrak{s}$ *is übereven with chromatic children; and* $v_{\mathfrak{s}}^{+} = v_{\mathfrak{s}}^{-} = v_{\mathfrak{s}}$ *if* $\mathfrak{s}$ *is non-übereven with polychromatic children.*

Moreover, if $\mathcal{R}$ *is not principal*[1]*, then there are the following additional edges:*

Name	*From*	*To*	*Length*	*Condition*
$L_{\mathfrak{t}}$	$v_{\mathfrak{s}}^{+}$	$v_{\mathfrak{s}}^{-}$	$\delta_{\mathfrak{s}}$	$\mathfrak{s} < \mathfrak{t}$, $\mathfrak{t}$ *cotwin,* $\mathfrak{s}$ *purple*
$L_{\mathfrak{t}}^{+}$	$v_{\mathfrak{s}}^{+,+}$	$v_{\mathfrak{s}}^{\sigma,-\sigma}$	$2\delta_{\mathfrak{s}}$	$\mathfrak{s} < \mathfrak{t}$, $\mathfrak{t}$ *cotwin,* $\mathfrak{s}$ *black*
$L_{\mathfrak{t}}^{-}$	$v_{\mathfrak{s}}^{-,-}$	$v_{\mathfrak{s}}^{-\sigma,\sigma}$		
$L_{\mathfrak{s},\mathfrak{s}'}^{+}$	$v_{\mathfrak{s}}^{+}$	$v_{\mathfrak{s}'}^{+}$	$\frac{1}{2}(\delta_{\mathfrak{s}} + \delta_{\mathfrak{s}'})$	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$, $\mathfrak{s}, \mathfrak{s}'$ *both principal, same chromatic colour*
$L_{\mathfrak{s},\mathfrak{s}'}^{-}$	$v_{\mathfrak{s}}^{-}$	$v_{\mathfrak{s}'}^{-}$		
$L_{\mathfrak{s},\mathfrak{s}'}^{+,+}$	$v_{\mathfrak{s}}^{+,+}$	$v_{\mathfrak{s}'}^{+,+}$	$\delta_{\mathfrak{s}} + \delta_{\mathfrak{s}'}$	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$, $\mathfrak{s}, \mathfrak{s}'$ *both principal, black*
$L_{\mathfrak{s},\mathfrak{s}'}^{+,-}$	$v_{\mathfrak{s}}^{+,-}$	$v_{\mathfrak{s}'}^{+,-}$		
$L_{\mathfrak{s},\mathfrak{s}'}^{-,+}$	$v_{\mathfrak{s}}^{-,+}$	$v_{\mathfrak{s}'}^{-,+}$		
$L_{\mathfrak{s},\mathfrak{s}'}^{-,-}$	$v_{\mathfrak{s}}^{-,-}$	$v_{\mathfrak{s}'}^{-,-}$		
$L_{\mathfrak{t}}$	$v_{\mathfrak{s}}^{+}$	$v_{\mathfrak{s}}^{-}$	$\delta_{\mathfrak{s}} + \delta_{\mathfrak{t}}$	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal,* $\mathfrak{t}$ *twin, both purple*
$L_{\mathfrak{t}}^{+}$	$v_{\mathfrak{s}}^{+,+}$	$v_{\mathfrak{s}}^{\sigma,-\sigma}$	$2(\delta_{\mathfrak{s}} + \delta_{\mathfrak{t}})$	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$, $\mathfrak{s}$ *principal,* $\mathfrak{t}$ *twin, both black*
$L_{\mathfrak{t}}^{-}$	$v_{\mathfrak{s}}^{-,-}$	$v_{\mathfrak{s}}^{-\sigma,\sigma}$		

Remark 6.3.2. Possibly after a totally ramified extension, we can still think of proper, not principal clusters as contributing components to the special fibre. However, these components may not be principal. In other words, they contribute components isomorphic to $\mathbb{P}^1$ which intersects the rest of the special fibre in precisely two places. So for example, a loop $L_{\mathfrak{t}}$ arising from a chromatic twin can be thought of as a component $v_{\mathfrak{t}}$ with two linking chains $L_{\mathfrak{t}}^{+}, L_{\mathfrak{t}}^{-}$ to $v_{P(\mathfrak{t})}$ of length $\frac{1}{2}\delta_{\mathfrak{t}}$ (i.e., the linking chains arising in the first two

[1] Recall that if $\mathcal{R}$ is purple and has a unique proper child $\mathfrak{s}'$ of size $2g(C_h)$ which is black, then $\mathcal{R}$ is principal and doesn't fall in this category.

rows of the first table). We have chosen not to state our theorems in this way since the component $v_\mathfrak{t}$ sometimes only appears in the minimal regular model after a totally ramified extension (and has multiplicity 2 otherwise), but for the purposes of our proof we will usually take this point of view. That is, we will go to a totally ramified extension, treat all proper clusters as if they give us components, and then use Lemma 2.1.20 to move between totally ramified extensions.

Theorem 6.3.3. *Denote the Frobenius automorphism by* Frob*. It acts on* $\mathscr{Y}_k^{\min}$ *in the following way:*

(i) $\mathrm{Frob}(\Gamma_\mathfrak{s}^{\pm}) = \Gamma_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s},i}(\mathrm{Frob})}$ *for* $\mathfrak{s}$ *with chromatic children,* $i \in \{1, 2, h\}$ *with* $\mathfrak{s} \in \Sigma_i$ *übereven,*

(ii) $\mathrm{Frob}(\Gamma_\mathfrak{s}^{\pm,\pm}) = \Gamma_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s},2}(\mathrm{Frob}),\pm\epsilon_{\mathfrak{s},1}(\mathrm{Frob})}$ *for* $\mathfrak{s}$ *übereven with no chromatic children,*

(iii) $\mathrm{Frob}(L_\mathfrak{s}^{\pm}) = \Gamma_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s},i}(\mathrm{Frob})}$ *for* $\mathfrak{s}$ *chromatic,* $i \in \{1, 2, h\}$ *with with* $\mathfrak{s} \in \Sigma_i$ *even,*

(iv) $\mathrm{Frob}(L_\mathfrak{s}^{\pm,\pm}) = L_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s},2}(\mathrm{Frob}),\pm\epsilon_{\mathfrak{s},1}(\mathrm{Frob})}$ *for* $\mathfrak{s}$ *black,*

(v) $\mathrm{Frob}(L_\mathfrak{t}) = \epsilon_{\mathfrak{t},h}(\mathrm{Frob})L_{\mathrm{Frob}(\mathfrak{t})}$ *for* $\mathfrak{t}$ *a chromatic twin, where* $-L$ *denotes* L *with the opposite orientation,*

(vi) $\mathrm{Frob}(L_\mathfrak{t}^{\pm}) = \epsilon_{\mathfrak{t},j}(\mathrm{Frob})L_{\mathrm{Frob}(\mathfrak{t})}^{\pm\epsilon_{\mathfrak{t},i}(\mathrm{Frob})}$ *for* $\mathfrak{t}$ *a black twin,* $i, j \in \{1, 2\}$ *such that* $\mathfrak{t}$ *is empty in* Σ_i *and* $i \neq j$*.*

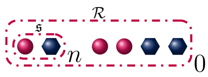
where $\Gamma_\mathfrak{s}^{\pm,\pm}$ *is the component corresponding to the vertex* $v_\mathfrak{s}^{\pm,\pm}$ *from Theorem 6.3.1.*

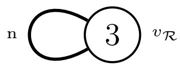
Remark 6.3.4. For simplicity of proof, we have added the technical condition that the depth of all clusters are integers, even though there exist semistable curves with clusters of half integer depth (for example, any bihyperelliptic curve where C_1 is the elliptic curve given by $y_1^2 = (x^2 - p^3)(x - 1)$). However, this is only a very mild restriction, since this condition can always be attained by going to the ramified extension of K of degree 2. The dual graph of $\mathscr{Y}_k$ over K is then the same as the dual graph of $\mathscr{Y}_k$ over $K(\sqrt{\pi})$, except the lengths of all linking chains are halved (see Lemma 2.1.20).

6.3.2 Examples

We present three examples to illustrate our theorem.

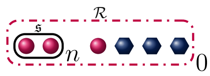
Example 6.3.5. Let $C_1 : y_1^2 = (x-p^n)(x^2-1)$ and $C_2 : y_2^2 = (x+p^n)(x^2-2)$ be elliptic curves, with composite curve $C_h : y_h^2 = (x^2-p^{2n})(x^2-1)(x^2-2)$ and associated bielliptic curve Y. The chromatic cluster picture of Y and the dual graph of $\mathscr{Y}_k^{\min}$ are below.

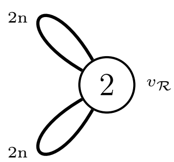




The top cluster $\mathcal{R}$ of size 6 is not übereven and has polychromatic children and so contributes one component of genus 3. The twin $\mathfrak{s}$ is purple and has depth n and so contributes a single loop of length n from the component of genus 3 to itself. In order to understand the Frobenius automorphism ϕ, we note that the twin $\mathfrak{s}$ is only even in Σ_h, not Σ_1 or Σ_2, so the action of Frobenius ϕ on the loop is $\phi(L_{\mathfrak{s}}) = \epsilon_{\mathfrak{s},h}(\phi)L_{\mathfrak{s}}$. We can calculate $\epsilon_{\mathfrak{s},h} = \left(\frac{2}{p}\right)$ and so the loop is inverted if and only if 2 is not a quadratic residue mod p.

Example 6.3.6. Let $C_1 : y_1^2 = (x^2-p^{2n})(x-1)$ and $C_2 : y_2^2 = x^3-2$ be two elliptic curves with composite curve $C_h : y_h^2 = (x^2-p^n)(x-1)(x^3-2)$ and associated bielliptic curve Y. The chromatic cluster picture of Y and the dual graph of $\mathscr{Y}_k^{\min}$ are below.





The top cluster $\mathcal{R}$ is not übereven and has polychromatic children so contributes one component of genus 2. The twin is black and has depth n

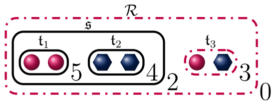
so contributes two loops of length $2n$. Frobenius, ϕ acts on the loops of $\mathfrak{s}$ as $\mathrm{Frob}(L_{\mathfrak{s}}^{\pm}) = \epsilon_{\mathfrak{s},1} L_{\mathfrak{s}}^{\pm\epsilon_{\mathfrak{s},2}(\phi)}$, since $\mathfrak{s}$ is empty in Σ_2. We can calculate $\epsilon_{\mathfrak{s},1}(\mathrm{Frob}) = \left(\frac{-1}{p}\right)$ and $\epsilon_{\mathfrak{s},2} = \left(\frac{-2}{p}\right)$, and so the action of Frobenius swaps the loops if -2 is not a quadratic residue mod p, and the loops are inverted if -1 is not a quadratic residue mod p.

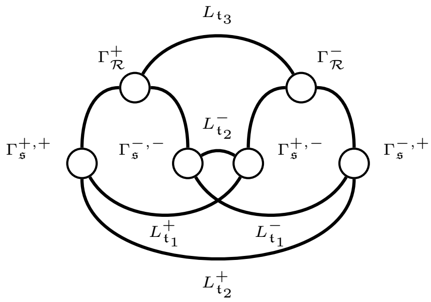
Example 6.3.7. Let $C_1 : y_1^2 = (x^2 - p^{2n})(x-1)$ and $C_2 : y_2^2 = (x-p^n)(x^2-2)$ be two elliptic curves and $C_h : y_h^2 = (x+p^n)(x-1)(x^2-2)$ by their composite curve. Let Y be their associated bihyperelliptic. The chromatic cluster picture of Y and the dual graph of $\mathscr{Y}_k^{\min}$ are below.

The twin $\mathfrak{s}$ is chromatic and so contributes a single loop on the component of its parent. The cluster $\mathcal{R}$ is principal with polychromatic children and so contributes a single cluster of genus 2.

Example 6.3.8. Let $C_1 : y_1^2 = ((x+p^2)^2 - p^{14})(x-2+p^3)$ and $C_2 : y_2^2 = ((x-p^2)^2 - p^{12})(x-2-p^3)$ be hyperelliptic curves over K and let Y be their associated bihyperelliptic curve. The chromatic cluster picture of Y consists of the übereven top cluster $\mathcal{R}$ with chromatic children, the übereven cluster $\mathfrak{s}$ with no chromatic children, and three twins $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$, the first two black with monochromatic children and the latter chromatic with polychromatic children. Note that $\mathcal{R}$ is principal despite being the disjoint union of two clusters as its two children are purple and black. The most subtle part of theorem is illustrated here: that the components arising from $\mathcal{R}$, and the loops arising from $\mathfrak{t}_1$ and $\mathfrak{t}_2$ link to different pairs of $\Gamma_{\mathfrak{s}}^{+,+}, \Gamma_{\mathfrak{s}}^{+,-}, \Gamma_{\mathfrak{s}}^{-,+}$ and $\Gamma_{\mathfrak{s}}^{-,-}$; for example, $\Gamma_{\mathcal{R}}^{+}$ links to $\Gamma_{\mathfrak{s}}^{+,+}$ and $\Gamma_{\mathfrak{s}}^{-,-}$, whereas $L_{\mathfrak{t}_1}^{+}$ links to $\Gamma_{\mathfrak{s}}^{+,+}$ and $\Gamma_{\mathfrak{s}}^{+,-}$. This is because $\mathcal{R}$ is übereven with chromatic children whereas $\mathfrak{t}_1$ is not übereven with monochromatic (red) children.

Remark 6.3.9. The above is an example of a curve whose minimal regular model has a special fibre with a non-planar dual graph (its dual graph is a





$K_{3,3}$). It is in fact an example of minimal genus, since the special fibre is totally degenerate (all of the components have genus 0).

6.4 Proof

The strategy of proof will be as follows: let $\mathscr{X}$ be the minimal model of $\mathbb{P}^1$ which separates the branch points of the map $Y \to \mathbb{P}^1$. We construct such a model from Σ_h following the techniques of [19, Sections 3-4]. Normalising $\mathscr{X}$ in the function field $K(Y)$ gives a regular model $\mathscr{Y}$ of Y, which results in a semistable model $\mathscr{Y}^{\min}$ after blowing down components of multiplicity greater than 1. We also obtain models $\mathcal{C}_1, \mathcal{C}_h, \mathcal{C}_2$ of C_1, C_h and C_2 respectively by normalising in the appropriate function fields. The special fibres of these intermediate models are computed using the result of [19, Sections 5-6].

Notation 6.4.1. There are many cases where we shall wish to refer to some object associated to a cluster $\mathfrak{s}$ for each of the curves $\mathbb{P}^1, C_1, C_2, C_h$ and Y. For example, we may wish to refer to the components arising from $\mathfrak{s}$. In this case, the component(s) in $\mathscr{Y}_k$ will appear without subscript: $\Gamma_{\mathfrak{s}}$, and those in $\mathscr{X}_k$ (resp. $\mathcal{C}_{1,k}, \mathcal{C}_{h,k}, \mathcal{C}_{2,k}$) will be denoted $\Gamma_{\mathfrak{s},\mathbb{P}^1}$ (resp. $\Gamma_{\mathfrak{s},1}, \Gamma_{\mathfrak{s},h}, \Gamma_{\mathfrak{s},2}$).

Remark 6.4.2. The models $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_h$ are not necessarily *minimal* models, as they come from the red (resp. blue resp. chromatic) cluster picture of Y - the admissible collection of disks arising from these is *not* in general the same as that arising from the cluster picture of C_1 (resp. C_2, C_h).

Lemma 6.4.3. *Let Y/K be a bihyperelliptic curve with semistable reduction. Then $\mathscr{Y}$, the normalisation of $\mathscr{X}$ in $K(Y)$, is a proper regular model of Y.*

Proof. The normalisation of $\mathscr{X}$ in $K(Y)$ is isomorphic to the normalisation of $\mathcal{C}_1$ in $K(Y)$, so it is sufficient to prove that the latter is a proper regular model of Y. Let $\varphi_1 : C_1 \to \mathbb{P}^1$ be the canonical double cover and write $\mathcal{D} = (\varphi_1^{-1}(f_2)) = \sum m_i\Gamma_i$, the divisor of (the pullback of) f_2 on $\mathcal{C}_1$. By [43, Lemma 2.1], it is sufficient to prove that

(i) $\varphi_1^{-1}(f_2)$ is not a square as a rational function on $\mathcal{C}_1$,

(ii) any two Γ_i for which m_i is odd do not intersect and,

(iii) any Γ_i for which m_i is odd is regular.

Since C_2 is a hyperelliptic curve, $\varphi_1^{-1}(f_2)$ is not a square as a rational function on $\mathcal{C}_1$. Furthermore, the horizontal components of $\mathcal{D}$ do not intersect since $\mathscr{X}$ was chosen such that the roots of f_2 specialise to distinct point of $\mathscr{X}_k$. We are left to consider the vertical components of $\mathcal{D}$. Note that any vertical component of odd multiplicity must arise as the preimage of some $E \in (f_2)_{\text{vert}}$ which appears with odd multiplicity. The component E has either one or two preimages in $\mathcal{C}_1$. In the first case, the preimage Γ is regular by [19, Theorem 5.2]. Since E does not intersect any other component of (f_2) of odd multiplicity, Γ cannot intersect a component of $\mathcal{D}$. In the second case, the two components are still regular and do not intersect each other, and cannot intersect any other component $\mathcal{D}$ as E does not intersect any other component of (f_2) of odd multiplicity. □

Proposition 6.4.4. *Let Y be a semistable bihyperelliptic curve as in Theorem 6.3.1 and $\mathscr{Y}$ the model obtained by normalising. Let $\mathscr{Y}^{\min}$ be the minimal regular model of $\mathscr{Y}$. Then each principal cluster $\mathfrak{s}$ contributes the following components to $\mathscr{Y}_k^{\min}$: if not übereven,* 1 *component $\Gamma_{\mathfrak{s}}$ if $\mathfrak{s}$ has polychromatic children and* 2 *components $\Gamma_{\mathfrak{s}}^+, \Gamma_{\mathfrak{s}}^-$ if $\mathfrak{s}$ has monochromatic children; and if übereven:* 2 *components $\Gamma_{\mathfrak{s}}^+, \Gamma_{\mathfrak{s}}^-$ if $\mathfrak{s}$ has chromatic children and* 4 *components $\Gamma_{\mathfrak{s}}^{+,+}, \Gamma_{\mathfrak{s}}^{+,-}, \Gamma_{\mathfrak{s}}^{-,+}, \Gamma_{\mathfrak{s}}^{-,-}$ if $\mathfrak{s}$ has no chromatic children.*

Proof. Consider a principal cluster $\mathfrak{s}$ and its corresponding component $\Gamma_{\mathfrak{s},\mathbb{P}^1}$ in the model of $\mathbb{P}^1$. If $\mathfrak{s}$ is not übereven, then $\Gamma_{\mathfrak{s},\mathbb{P}^1}$ lifts to one component

$\Gamma_{\mathfrak{s},h}$ in $\mathcal{C}_h$ and so lifts to either one or two components in $\mathscr{Y}_k$. Suppose $\mathfrak{s}$ has polychromatic children. Then $\mathfrak{s}$ has one corresponding component $\Gamma_{\mathfrak{s},1} \in \mathcal{C}_1$, and $\Gamma_{\mathfrak{s},1}$ contains branch points of the morphism $\mathscr{Y} \to \mathcal{C}_1$ (corresponding to the blue children of $\mathfrak{s}$), and hence must lift to a single component in $\mathscr{Y}_k$. If $\mathfrak{s}$ has monochromatic (e.g. red) children then it has two components in either $\mathcal{C}_1$ or $\mathcal{C}_2$ (in this case $\mathcal{C}_2$), and hence must have two associated components in $\mathscr{Y}$.

If $\mathfrak{s}$ is übereven, then it has two components in $\mathcal{C}_h$ so lifts to either two or four in $\mathscr{Y}_k$. If $\mathfrak{s}$ has chromatic (e.g. red) children then it has a single component in either $\mathcal{C}_1$ or $\mathcal{C}_2$ (in this case $\mathcal{C}_1$), so can only lift to two components in $\mathscr{Y}_k$. If $\mathfrak{s}$ has no chromatic children, then it has two corresponding components in each of $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_h$. Since $C_2 \times C_2$ acts on the components of $\mathscr{Y}_k$ arising from $\mathfrak{s}$, and their images are the $\Gamma^{\pm}_{\mathfrak{s},i}$ under the quotient of the three non trivial subgroups of $C_2 \times C_2$, there must be four components corresponding to $\mathfrak{s}$ in $\mathscr{Y}_k$.

Since $\mathfrak{s}$ is principal, it is principal in one of Σ_1, Σ_2 or Σ_h, say Σ_1. Then by Theorem 2.4.11, any component $\Gamma_{\mathfrak{s},1} \in \mathcal{C}_1$ arising from $\mathfrak{s}$ either has positive genus, or intersects at least three other components. Therefore the same can be said for any $\Gamma \in \mathscr{Y}_k$ arising from $\mathfrak{s}$. We cannot blow such components down, and hence the same components appear in $\mathscr{Y}_k^{\min}$. □

Proposition 6.4.5. *Let $Y, \mathscr{Y}^{\min}$ be as in Proposition 6.4.4, and let $\mathfrak{s}' < \mathfrak{s}$ be principal clusters of Y. Then the components of $\mathfrak{s}$ and $\mathfrak{s}'$ in $\mathscr{Y}_k$ are linked by two chains if $\mathfrak{s}'$ is chromatic and four otherwise, as described in the statement of Theorem 6.3.1. Furthermore, if $\mathfrak{t} < \mathfrak{s}$ is a twin or $\mathfrak{s} < \mathfrak{t}$ a cotwin then there is one loop if the child is chromatic and two loops if the child is black, and if $\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$ is not principal then the components of $\mathfrak{s}$ and $\mathfrak{s}'$ are linked as in the statement of Theorem 6.3.1.*

Proof. Assume that we are in the case where $\mathfrak{s}' < \mathfrak{s}$ are both principal, since the other cases are checked similarly using Remark 6.3.2. It is clear that components have linking chains to the components corresponding to their parents, since this is the case for the model of $\mathbb{P}^1$. Therefore, we have to calculate how many linking chains there are (2 or 4) and precisely which components are linked to which others. The lengths of the linking chains is proved separately in Lemma 6.4.6.

Suppose that $\mathfrak{s}' < \mathfrak{s}$ with $\mathfrak{s}'$ chromatic. Then there are several cases for the different children $\mathfrak{s}'$ can have. If $\mathfrak{s}'$ is red, then $\Gamma_{\mathfrak{s},1}$ and $\Gamma_{\mathfrak{s}',1}$ are linked by one chain and $\Gamma^{\pm}_{\mathfrak{s},2}$ and $\Gamma^{\pm}_{\mathfrak{s}',2}$ are linked by two chains. Similarly if $\mathfrak{s}'$ is blue (swapping 1 and 2). In either case, $\Gamma^{\pm}_{\mathfrak{s}}$ and $\Gamma^{\pm}_{\mathfrak{s}'}$ are linked by two chains. Similarly if $\mathfrak{s}'$ is purple then $\Gamma_{\mathfrak{s},h}$ and $\Gamma_{\mathfrak{s}',h}$ are linked by two chains but $\Gamma_{\mathfrak{s},1}$ and $\Gamma_{\mathfrak{s}',1}$ are linked by one chain so we get two linking chains upstairs.

Now suppose that $\mathfrak{s}'$ is black. Then $\Gamma^{\pm}_{\mathfrak{s}',1}$ has two linking chains up to the components of its parent, as does $\Gamma_{\mathfrak{s}',2}$, so by a similar argument to Proposition 6.4.4 the components of $\mathfrak{s}'$ have four linking chains up to the components of $\mathfrak{s}$. Therefore the number of linking chains is correct. We must check that the correct components are linked.

This is done on a case by case basis. First assume $\mathfrak{s}, \mathfrak{s}'$ are not übereven. If $\mathfrak{s}$ has polychromatic children then it only has one component and everything is correct up to relabelling. Similarly if $\mathfrak{s}'$ has polychromatic children. So assume both $\mathfrak{s}$ and $\mathfrak{s}'$ have monochromatic children. If they have monochromatic children of the same colour (say red), their components are linked as in the tower of models in Figure 6.1

If $\mathfrak{s}$ and $\mathfrak{s}'$ have monochromatic children of different colours (e.g. $\mathfrak{s}$ red and $\mathfrak{s}'$ blue), then their components are linked as in the tower of models in Figure 6.2.

If $\mathfrak{s}$ or $\mathfrak{s}'$ is übereven, the different cases can be checked similarly.

□

Lemma 6.4.6. *Let $Y, \mathscr{Y}^{\min}$ be as in Proposition 6.4.4, and let $\mathfrak{s}' < \mathfrak{s}$ be two principal clusters. Then any linking chain $L_{\mathfrak{s}'}$ arising from this pair has length $\frac{1}{2}\delta_{\mathfrak{s}'}$ if $\mathfrak{s}'$ is chromatic and $\delta_{\mathfrak{s}'}$ otherwise. If $\mathfrak{t} < \mathfrak{s}$ is a twin or $\mathfrak{s} < \mathfrak{t}$ a cotwin then any loop arising from $\mathfrak{t}$ has length $\delta_{\mathfrak{t}}$ if the child is chromatic, and $2\delta_{\mathfrak{t}}$*

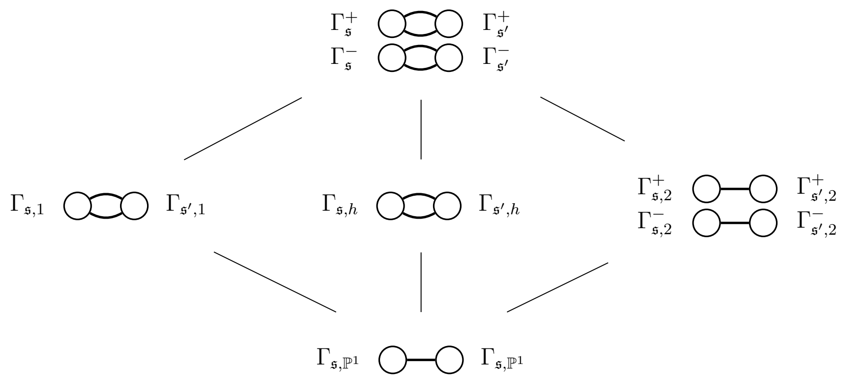


Figure 6.1: Tower of models when $\mathfrak{s}$ and $\mathfrak{s}'$ have monochromatic children of the same colour.

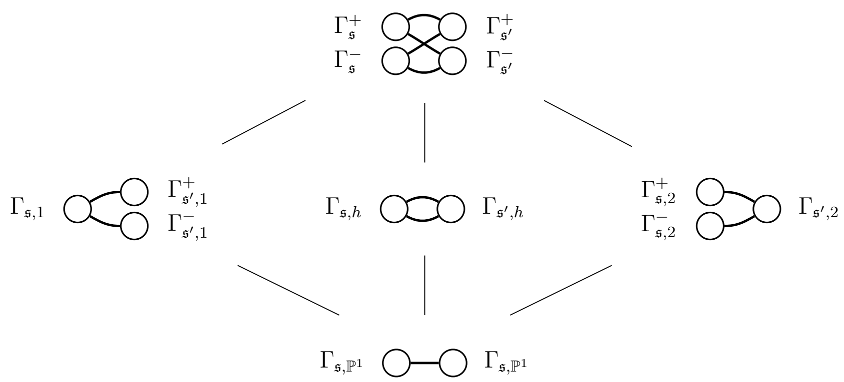


Figure 6.2: Tower of models when $\mathfrak{s}$ and $\mathfrak{s}'$ have monochromatic children of different colours.

otherwise. If $\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$ is not principal, then any length of a linking chain arising from $\mathfrak{s}$ and $\mathfrak{s}'$ has length as described in Theorem 6.3.1.

Proof. Possibly after a finite extension L/K^{ur}, the map $Y \to \mathbb{P}^1$ extends to a map of models $\mathscr{Y} \to \mathscr{X}$ by [31, Theorem 2.3]. Furthermore, this map induces a harmonic morphism of augmented $\mathbb{Z}$-graphs, in the sense of [1, Section 2] (see also Sections 5,8) on the dual graphs of $\mathscr{Y}$ and $\mathscr{X}$. The length of an edge between two vertices in an augmented $\mathbb{Z}$-graph is the thickness of the intersection point of the components the vertices represent. Therefore the distance between two vertices of degree ≥ 3 is exactly the length of the linking chain between them.

If $\mathfrak{s}' < \mathfrak{s}$ are principal, the lemma follows, noting that a linking chain from a chromatic cluster to its parent has two preimages in $\mathscr{Y}'$ (so the length halves), but a linking chain from a black cluster to its parent has four preimages so the length stays the same.

If $\mathfrak{t} < \mathfrak{s}$ is a chromatic twin then (possibly after a field extension), we can think of the loop $L_\mathfrak{t}$ as consisting of a component $\Gamma_\mathfrak{t}$, the unique lift of $\Gamma_{\mathfrak{t},\mathbb{P}^1}$ with two linking chains to $\Gamma_\mathfrak{s}$. Since $\mathfrak{t}$ is chromatic, by the argument above the linking chains will both have length $\frac{1}{2}\delta_\mathfrak{t}$ and hence the total loop will have length $\delta_\mathfrak{t}$. A similar argument is made if $\mathfrak{t}$ is a black twin, or if $\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$ is not principal. □

Proposition 6.4.7. *Let $Y, \mathscr{Y}^{\min}$ be as in Proposition 6.4.4 and let $\mathfrak{s}$ be a principal cluster of Y. Then the components associated to $\mathfrak{s}$ have genus $g_\chi(\mathfrak{s})$.*

Proof. Let $\mathfrak{s}$ be a principal cluster. First suppose $\mathfrak{s}$ is not übereven and has polychromatic children. In this case there is a unique component $\Gamma_\mathfrak{s}$ arising from $\mathfrak{s}$. This is then a direct application of Riemann-Hurwitz. The children of $\mathfrak{s}$ correspond to points on the component $\Gamma_{\mathfrak{s},\mathbb{P}^1}$ as in [19, Definition 3.7], and by Proposition 6.4.5 the points arising from chromatic children are precisely the non-infinity branch points of $\Gamma_\mathfrak{s} \to \Gamma_{\mathfrak{s},\mathbb{P}^1}$. In addition, there is an extra branch point at infinity, unless $\mathfrak{s}$ is the top cluster and f_1 and f_2 both have even degree, or the top cluster $\mathcal{R} = \mathfrak{s} \bigsqcup \mathfrak{s}'$ is not principal, $\mathfrak{s}$ is even and f_1 and f_2 both have even degree.

If $\mathfrak{s}$ is not übereven and has monochromatic children, then the components $\Gamma_\mathfrak{s}^+$ and $\Gamma_\mathfrak{s}^-$ are each isomorphic to $\Gamma_{\mathfrak{s},h}$ and so have genus $g(\mathfrak{s})$. The same is true if $\mathfrak{s}$ is übereven and has chromatic children, except with $\Gamma_{\mathfrak{s},1}$ if $\mathfrak{s}$ has red children and $\Gamma_{\mathfrak{s},2}$ if blue. If $\mathfrak{s}$ is übereven with no chromatic children then its 4 components must have genus $0 = g(\mathfrak{s})$ as well. □

Theorem 6.4.8. *Denote the Frobenius automorphism by* Frob*. It acts on the dual graph of $\mathscr{Y}_k^{\min}$ in the following way:*

(i) $\mathrm{Frob}(\Gamma_{\mathfrak{s}}^{\pm}) = \Gamma_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s},i}(\mathrm{Frob})}$ *for* $\mathfrak{s}$ *with chromatic children,* $i \in \{1,2,h\}$ *with* $\mathfrak{s} \in \Sigma_i$ *übereven,*

(ii) $\mathrm{Frob}(\Gamma_{\mathfrak{s}}^{\pm,\pm}) = \Gamma_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s},2}(\mathrm{Frob}),\pm\epsilon_{\mathfrak{s},1}(\mathrm{Frob})}$ *for* $\mathfrak{s}$ *übereven with no chromatic children,*

(iii) $\mathrm{Frob}(L_{\mathfrak{s}}^{\pm}) = \Gamma_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s},i}(\mathrm{Frob})}$ *for* $\mathfrak{s}$ *chromatic,* $i \in \{1,2,h\}$ *with with* $\mathfrak{s} \in \Sigma_i$ *even,*

(iv) $\mathrm{Frob}(L_{\mathfrak{s}}^{\pm,\pm}) = L_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s},2}(\mathrm{Frob}),\pm\epsilon_{\mathfrak{s},1}(\mathrm{Frob})}$ *for* $\mathfrak{s}$ *black,*

(v) $\mathrm{Frob}(L_{\mathfrak{t}}) = \epsilon_{\mathfrak{t},h}(\mathrm{Frob}) L_{\mathrm{Frob}(\mathfrak{t})}$ *for* $\mathfrak{t}$ *a chromatic twin, where* $-L$ *denotes* L *with the opposite orientation,*

(vi) $\mathrm{Frob}(L_{\mathfrak{t}}^{\pm}) = \epsilon_{\mathfrak{t},j}(\mathrm{Frob}) L_{\mathrm{Frob}(\mathfrak{t})}^{\pm\epsilon_{\mathfrak{t},i}(\mathrm{Frob})}$ *for* $\mathfrak{t}$ *a black twin,* $i,j \in \{1,2\}$ *such that* $\mathfrak{t}$ *is empty in* Σ_i *and* $i \neq j$.

Proof. By Proposition 6.4.4, the components we must blow down to obtain $\mathscr{Y}_k^{\min}$ from $\mathscr{Y}_k$ are all in linking chains, so it is sufficient to calculate the action of Frobenius on $\mathscr{Y}_k$ as the action on the shortened linking chains is the same as the originals. The action of Frobenius commutes with the quotient maps, so we can deduce the action of Frobenius on the components of $\mathscr{Y}_k$ from the corresponding action of Frobenius on $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_h$, which is known by Theorem 2.4.11. First we focus on clusters. For a principal cluster $\mathfrak{s}$, the set $\mathcal{E}_{\mathfrak{s}} = \{\Gamma_{\mathfrak{s}}^{\pm,\pm}\}$ is mapped to $\mathcal{E}_{\mathrm{Frob}(\mathfrak{s})} = \{\Gamma_{\phi(\mathfrak{s})}^{\pm,\pm}\}$ by Frobenius, since Frobenius maps the images of $\mathcal{E}_{\mathfrak{s}}$ to the images of $\mathcal{E}_{\mathrm{Frob}(\mathfrak{s})}$ in $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_h$. It remains to show which component of $\mathcal{E}_{\mathfrak{s}}$ is mapped to which of $\mathcal{E}_{\mathrm{Frob}(\mathfrak{s})}$.

If $\mathfrak{s}$ is a principal cluster with polychromatic children then $\mathcal{E}_{\mathfrak{s}}$ consists of one component and hence there is nothing to verify. If $\mathfrak{s}$ is a principal cluster with monochromatic, red children (so therefore $\mathfrak{s} \in \Sigma_2$ is even) then there are two components $\Gamma_{\mathfrak{s}}^{+}, \Gamma_{\mathfrak{s}}^{-}$ corresponding to $\mathfrak{s}$, which are the lifts of two components $\Gamma_{\mathfrak{s},2}^{+}, \Gamma_{\mathfrak{s},2}^{-} \in \mathcal{C}_{2,k}$. Therefore Frob acts on $\Gamma_{\mathfrak{s}}^{\pm}$ as it does on $\Gamma_{\mathfrak{s},2}^{\pm}$. But by Theorem 2.4.11, $\mathrm{Frob}(\Gamma_{\mathfrak{s},2}^{\pm}) = \Gamma_{\phi(\mathfrak{s}),2}^{\pm\epsilon_2(\mathrm{Frob})}$, and so $\mathrm{Frob}(\Gamma_{\mathfrak{s}}^{\pm}) =$

$\Gamma^{\pm\epsilon_2(\mathrm{Frob})}_{\mathrm{Frob}(\mathfrak{s})}$. Similarly if $\mathfrak{s}$ has monochromatic, blue children or is übereven with polychromatic children.

Now suppose that $\mathfrak{s}$ is an übereven cluster with no chromatic children. In this case there are two components arising from $\mathfrak{s}$ in $\mathcal{C}_{i,k}$ for $i = 1, 2$, $\Gamma^{+}_{\mathfrak{s},i}$ and $\Gamma^{-}_{\mathfrak{s},i}$. Consider $i = 1$. In $\mathcal{C}_1$, the action of Frobenius is $\mathrm{Frob}(\Gamma^{\pm}_{\mathfrak{s},1}) = \Gamma^{\pm\epsilon_1(\mathrm{Frob})}_{\mathrm{Frob}(\mathfrak{s}),1}$. The component $\Gamma^{+}_{\mathfrak{s},1}$ lifts to the components $\Gamma^{+,+}_{\mathfrak{s}}$ and $\Gamma^{-,+}_{\mathfrak{s}}$ and so the set $\{\Gamma^{+,\pm}_{\mathfrak{s}}, \Gamma^{-,\pm}_{\mathfrak{s}}\}$ is mapped to $\{\Gamma^{+,\pm\epsilon_1 \mathrm{Frob}}_{\mathfrak{s}}, \Gamma^{-,\pm\epsilon_1 \mathrm{Frob}}_{\mathfrak{s}}\}$. The same argument for $i = 2$ implies that the set $\{\Gamma^{\pm,+}_{\mathfrak{s}}, \Gamma^{\pm,-}_{\mathfrak{s}}\}$ is mapped to the set $\{\Gamma^{\pm\epsilon_2(\mathrm{Frob}),+}_{\mathfrak{s}}, \Gamma^{\pm\epsilon_2(\mathrm{Frob}),-}_{\mathfrak{s}}\}$. Combining these gives the action of Frobenius.

For linking chains between components of principal clusters $\mathfrak{s}' < \mathfrak{s}$, the action on the whole linking chain is determined by the action on any component in the linking chain. Suppose D is a p-adic disk with $\mathfrak{s}' < D < \mathfrak{s}$. If $\mathfrak{s}'$ is chromatic (i.e., case (iii)) then $\Gamma_{D,\mathbb{P}^1}$ has two preimages in $\mathscr{Y}$ and these are permuted like the principal components in (i). If $\mathfrak{s}'$ is black (case (iv)), then $\Gamma_{D,\mathbb{P}^1}$ has four preimages in $\mathscr{Y}_k$ and these are permuted like the principal components in case (ii).

Loops associated to twins and linking chains between $\mathfrak{s}$ and $\mathfrak{s}'$ when $\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$ are not principal can be dealt with using Remark 6.3.2, as in proofs in the rest of the section. □

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