Investment Decision Making under Uncertainty: The Impact of Risk Aversion, Operational Flexibility, and Competition

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Statement of Originality

I, Michail Chronopoulos, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

Traditional real options analysis addresses investment under uncertainty assuming a risk-neutral decision maker and complete markets. In reality, however, decision makers are often risk averse and markets are incomplete. Additionally, capital projects are seldom now-or-never investments and can be abandoned, suspended, and resumed at any time.

In this thesis, we develop a utility-based framework in order to examine the impact of operational flexibility, via suspension and resumption options, on optimal investment policies and option values. Assuming a risk-averse decision maker with perpetual options to suspend and resume a project costlessly, we confirm that risk aversion lowers the probability of investment and demonstrate how this effect can be mitigated by incorporating operational flexibility. Also, we illustrate how increased risk aversion may facilitate the abandonment of a project while delaying its temporary suspension prior to permanent resumption.

Besides timing, a firm may have the freedom to scale the investment's installed capacity. We extend the traditional real options approach to investment under uncertainty with discretion over capacity by allowing for a constant relative risk aversion utility function and operational flexibility in the form of suspension and resumption options. We find that, with the option to delay investment, increased risk aversion facilitates investment and decreases the required investment threshold price by reducing the amount of installed capacity.

We explore strategic aspects of decision making under uncertainty by examining how duopolistic competition affects the entry decisions

of risk-averse investors. Depending on the discrepancy between the market share of the leader and the follower, greater uncertainty may increase or decrease the discrepancy in the non-pre-emptive leader's relative value. Furthermore, risk aversion does not affect the loss in the value of the leader for the pre-emptive duopoly setting, but it makes the loss in value relatively less for the leader in a non-pre-emptive duopoly setting.

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Contents

List of Acronyms

CARA	Constant Absolute Risk Aversion
CRRA	Constant Relative Risk Aversion
FONC	First-Order Necessary Condition
GBM	Geometric Brownian Motion
HARA	Hyperbolic Absolute Risk Aversion
MB	Marginal Benefit of Delaying Investment
MC	Marginal Cost of Delaying Investment
$\widetilde{MB}$	Marginal Benefit of Increasing Capacity
$\widetilde{MC}$	Marginal Cost of Increasing Capacity
NPV	Net Present Value
R&D	Research and Development
SOSC	Second-Order Sufficiency Condition

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Chapter 1

Introduction

Managing the investment and operational risk of capital projects is crucial to the economic viability of many industries. Given the deregulation of many sectors of the economy, including infrastructure, and the recent volatility in financial markets, decision rules for managing capital projects should consider discretion over timing and uncertainty in underlying variables. Indeed, accounting for these features often yields significantly different expected project values and optimal investment and operational policies than those from the traditional net present value (NPV) approach, which has been a mainstay for industry and policymakers alike.

Further complicating the use of the expected NPV approach is that capital projects are more complex than simple now-or-never investments since they entail embedded options to make additional decisions at arbitrary points in time in response to the values of realised state variables. Such managerial discretion implies the right, but not the obligation, to undertake decisions in the future; thus, capital projects may be considered as packages of compound financial options. Ever since simple financial call options were valued analytically (Black and Scholes, 1973 and Merton, 1973), they have been amenable to supporting decision making in non-financial settings. Consequently, the field of real options has sought to exploit this application potential in decision making under uncertainty by analysing capital projects as a series of options. The theory of real options indicates that uncertainty and irreversibility create a value of waiting with undertaking capital investments. This happens because the information that becomes available over time allows for uncertainty to be resolved, thereby enabling the decision maker to take better investment decisions at a later time.

Examples of analytical real options theory include McDonald and Siegel (1985, 1986) who explore the optimal timing of investment in an irreversible project with revenues that follow a continuous-time stochastic process. Brennan and Schwartz (1985) show how to value assets whose cash flows depend on highly variable output prices and exploit the property of replicating self-financing portfolios in order to derive optimal policies from managing these assets. Majd and Pindyck (1987) use the contingent claims approach to analyse the flexibility that lies within the time it takes to build an investment project. Bar-Ilan and Strange (1996) present an analytical solution to the investment problem with lags and show that conventional results regarding the effect of price uncertainty on investment are weakened or reversed when there are lags. Deng *et al.* (2001) present a methodology for valuing electricity derivatives by constructing replicating portfolios from electricity futures and the risk-free asset. Gollier *et al.* (2005) examine how a producer faced with a change in the competitive price of electricity will be able to compare a sequence of investments in medium-capacity nuclear power plants with an investment in a high-capacity unit. In other words, they examine how to choose between the flexibility of the modular investment and the efficiency of the high-capacity unit due to increase in economy of scale. Lund (2005) analyses the relationship between investment and uncertainty in real options models and, in addition to the positive effect of uncertainty on the trigger level for revenue relative to cost, identifies an opposing effect on the probability of investment that yields a total effect with ambiguous sign. Malchow-Møller and Thorsen (2005) extend the traditional investment under uncertainty setup with a single investment option to the case of infinite repeated options.

One gap in the real options framework is that it assumes that financial markets are complete, and, therefore, that the investor is risk neutral. This is at odds with how operational research has traditionally tackled decision making under uncertainty, viz., under the premise of a risk-averse agent whose preferences for wealth are summarised by a utility function. Especially for projects that involve undiversifiable risks, such as research and development (R&D) of new products, risk aversion on part of investors should be considered. A recent working paper uses a constant relative risk aversion (CRRA) utility function to illustrate that investment under uncertainty with risk aversion results not only in a further delay in investment but also reduces the probability of investment (Hugonnier and Morellec, 2007). This working paper applies the analytical framework of Karatzas and Shreve (1999), in which the strong Markov property of the geometric Brownian motion (GBM) describing the project's value is exploited to cast the investment timing problem as an optimal stopping one. In order to facilitate the analysis, the expected project value is expressed as the product of the stochastic discount factor and the expected present value of the cash flows from an active project. Subsequently, the value of the investment opportunity may be maximised by selecting the first passage time of the project value to a certain threshold. However, it remains an open question how strong this interaction between risk aversion and uncertainty would be if the project included embedded options, such as operational flexibility, the right to scale the capacity size, or the impact of competition. Through this thesis, we aim to bridge the gaps in real options theory so that it may be more suitable not only for decision making but also for risk assessment.

1.1 Optimal Investment under Operational Flexibility, Risk Aversion, and Uncertainty

Dixit and Pindyck (1994) and McDonald and Siegel (1985, 1986) address the problem of optimal entry to and exit from a project assuming a risk-neutral decision maker with a perpetual option to invest. This canonical real options problem can be solved via either the contingent claims approach, assuming that either markets are complete or the project's unique risk can be diversified, or via dynamic programming, using a subjective discount rate. Contingent claims analysis, however, cannot be used in cases where the project's risk is not diversifiable. This occurs, for example, in R&D projects with technical risk that is idiosyncratic, or in nascent markets that may not have sufficiently developed financial instruments. Furthermore, the decision maker may be inherently risk averse due to the firm's ownership structure, e.g., in the case of a municipal authority or due to costs of financial distress. Dynamic programming can then be used to maximise the expected discounted utility of the lifetime profits of a risk-averse decision maker.

In the real options literature, different types of operational flexibility have been studied mainly under the assumption of a risk-neutral decision maker. For example, Majd and Pindyck (1987) analyse the flexibility that lies within the time it takes to build an investment project. Their analysis is based on the fact that the rate at which the construction of an investment project proceeds is flexible and can, therefore, be adjusted as new information becomes available. Applying contingent claims analysis, they show how traditional discounted cash flow methods understate the value of the project by ignoring this flexibility. However, their analysis is restricted by the assumptions of market completeness and a risk-neutral decision maker. Malchow-Møller and Thorsen (2005) consider the case of repeated investment options, thereby extending the single-option model that was first developed by McDonald and Siegel (1986) in which the option is killed when investment is undertaken. Their analysis shows that when investment options are repeated, the value of waiting is reduced significantly compared to the single-option case and that the simple NPV rule is a better indicator of optimal investment. Also, sensitivity analysis reveals that the effects of parameter changes are very different under the repeated-options approach than in the single-option model. Nevertheless, each of these papers assumes a risk-neutral decision maker.

Since the assumptions of risk neutrality and market completeness are not particularly relevant to most real-life situations, it is important to examine the implications that arise when these assumptions are relaxed. A utility-based framework has been adopted, for example, by Henderson and Hobson (2002), who extend the real options approach to pricing and hedging assets by taking the perspective of a risk-averse decision maker facing incomplete markets. Their analysis is based on Merton (1969) who studies a decision maker facing complete markets seeking to maximise the expected utility of terminal wealth over a fixed and continuous time horizon using a CRRA utility function. Henderson and Hobson (2002) extend Merton's analysis by introducing a second risky asset on which no trading is allowed. In that case, the decision maker has a claim on units of the non-traded asset, and the question is how to price and hedge this random payoff. Furthermore, Henderson (2007) investigates the impact of risk aversion and incompleteness on investment timing and option value by a risk-averse decision maker with an exponential utility function who can choose at any time to undertake an irreversible investment project and receive a risky payoff. To offset some of the risk associated with the unknown investment payoff, the decision maker also trades in a risk-free bond and a risky asset that is correlated with the investment payoff. Results indicate that the higher the decision maker's risk aversion or the lower the correlation between the project value and hedging asset, the lower will the investment threshold and option value be. In particular, there is a parameter region within which the assumptions of complete and incomplete markets yield different results. In this region, and under the assumption of complete markets, the option is never exercised (and investment never occurs), whereas the decision maker exercises the option in the incomplete setting.

More pertinent to our analysis is the working paper by Hugonnier and Morellec (2007), who extend the work of Dixit and Pindyck (1994) and McDonald and Siegel (1986) by illustrating how risk aversion affects investment under uncertainty when the decision maker faces incomplete markets. Instead of using contingent claims, they use an optimal stopping time approach to allow for the decision maker's risk aversion to be incorporated via a CRRA utility function. Their framework is based on a closed-form expression for the expected discounted utility of stochastic cash flows derived by Karatzas and Shreve (1999). The results indicate that risk aversion lowers the likelihood of investment and erodes the value of investment projects. In Chapter 2, we extend Hugonnier and Morellec (2007) by incorporating operational flexibility in the form of suspension and resumption options that can be exercised at any time at no cost. We will show how this flexibility can mitigate the effect of risk aversion and offer insights on how to exercise optimally such suspension and resumption options.

1.2 The Value of Capacity Sizing under Risk Aversion and Operational Flexibility

Although investment in a project with discretion over capacity is a real options problem, the majority of real options models, including the ones that account for competition, either consider only the problem of optimal investment timing (Dixit and Pindyck, 1994 and McDonald and Siegel, 1985, 1986) or address the problem under risk neutrality. In the area of capacity sizing, Dangl (1999) addresses the problem of a risk-neutral firm that invests in a project with continuously scalable capacity using the dynamic programming approach. He finds that uncertainty in future demand leads to an increase in optimal installed capacity and causes investment to be delayed to an extent that even low uncertainty makes waiting and accumulation of further information the optimal decision for large ranges of demand. Following the same approach, Huisman and Kort (2009) examine the same problem in monopoly and duopoly settings and compare their results with the standard model where the firm has no discretion over capacity. They confirm that, compared to the model without capacity choice, increased uncertainty delays investment and leads to higher installed capacity both for the monopolist and the follower, thereby illustrating how the leader invests later in lower capacity for low uncertainty and earlier at higher capacity for higher uncertainty. They also find that when uncertainty is low, the leader invests in lower capacity than the follower in order to deter temporarily the follower's entry and in higher capacity when uncertainty increases. The flexibility to choose between two alternative investment projects of different scales under output price uncertainty has been studied by Décamps, Mariotti, and Villeneuve (2006). Their analysis extends the results of Dixit (1993) where the irreversible choice among mutually exclusive projects under output price uncertainty is considered. Hagspiel *et al.* (2010) also account for the production decision, apart from the optimal investment timing and sizing decisions, by comparing the flexible scenario, where a firm can costlessly adjust production over time with the capacity level as the upper bound, to the inflexible scenario, where a firm fixes production at capacity level from the moment of investment onward. Among other results, they find that the flexible firm invests in higher capacity than the inflexible firm and that the capacity difference increases with uncertainty. Although the flexible firm has an incentive to invest earlier (because flexibility raises the project value), it also has an incentive to invest later (because costs are higher due to the larger capacity level). The latter effect dominates in highly uncertain economic environments.

We contribute to this line of work by analysing the impact of risk aversion on the optimal investment and sizing decisions of a firm. Using an optimal stopping time approach, we allow for risk aversion to be incorporated via a CRRA utility function. We assume that the risk-averse firm is a price taker, that it can suspend and resume operations costlessly depending on the fluctuations of the output price, and that its option to adjust the capacity of the project expires upon investment. This framework is based on a closed-form expression for the expected discounted utility of stochastic cash flows derived by Karatzas and Shreve (1999).

We begin by analysing the case of now-or-never investment and find that increased risk aversion reduces the expected utility of the investment's payoff, thereby creating an incentive to reduce the amount of installed capacity. Furthermore, we find that, without operational flexibility, increased uncertainty leaves the optimal capacity of the project unaffected under risk neutrality and decreases it under risk aversion, while the presence of embedded options to suspend and resume operations increases the value of the now-or-never investment opportunity, thereby motivating the installation of greater capacity.

Next, we account for the option to invest and find that in contrast to Hugonnier and Morellec (2007) who, using the same framework, show that increased risk aversion erodes option value and increases the required investment threshold, increased risk aversion facilitates investment by reducing the amount of installed capacity. We show that with higher risk aversion, the incentive to avoid exposure to unfavourable market conditions by decreasing the amount of installed capacity is more profound than the incentive to delay investment due to the decrease of the project's expected utility.

1.3 Duopolistic Competition under Risk Aversion and Uncertainty

The majority of real options models account for the problem of optimal investment timing without considering competition (McDonald and Siegel 1985, 1986), while the ones that do, assume risk neutrality (Dixit and Pindyck, 1994). In the area of competition, Smets (1993) first combined real options valuation techniques with game theory concepts, thus developing a continuous-time model of strategic real option exercise under product market competition, assuming that entry is irreversible, demand is stochastic and simultaneous investment occurs only when the role of the leader is defined exogenously. Williams (1993) provides the first rigorous derivation of a Nash-equilibrium in a real options framework.

Following Smets (1993), Grenadier (1996) develops an equilibrium framework for option exercise strategies and particularly focuses on the behaviour of real-estate markets in order to emphasise on the applicability of the model. Through this framework, he proposes a potential explanation for certain market phenomena such as why some real estate markets have been prone to pronounced bursts of development activity, while others have been characterised by smooth patterns of development over time.

Weeds (2002) considers irreversible investment in competing research projects with uncertain returns under a winner-takes-all patent system. The technological success of the project is probabilistic, while the economic value of the patent to be won evolves stochastically over time. Her results indicate that in a pre-emptive leader-follower equilibrium firms invest sequentially and option values are reduced by competition. However, a symmetric outcome may also occur in which investment is more delayed than the single-firm counterpart. Comparing this with the optimal cooperative investment pattern, investment is found to be more delayed when firms act non-cooperatively as each holds back from investing in the fear of starting a patent race.

Grenadier (2002) provides a general and tractable solution approach for deriving the equilibrium investment strategies of firms in a Cournot-Nash framework with more than two competitors. Each firm faces a sequence of investment opportunities and must determine an exercise strategy for its path of investment. The resulting equilibrium has potentially wide applications. For example, a real estate developer contemplating the construction of an office building will choose his optimal time at which to begin construction, contingent upon his beliefs about the decisions of other developers in the market. The model permits a rather simple solution approach to the derivation of equilibrium exercise strategies in a continuous-time stochastic setting.

Thijssen *et al.* (2002) consider the problem of investment timing under uncertainty in a duopoly framework and propose a method in order to deal with the problem of coordination, that arises when both firms want to enter the market first, involving the use of symmetric mixed strategies. They study first-mover advantage and war of attrition (second-mover advantage) games, thereby extending the strategy spaces and equilibrium concepts introduced in Fudenberg and Tirole (1985).

Shackleton *et al.* (2004) analyse the entry decisions of competing firms in a two-player stochastic real option game, when rivals earn different but correlated uncertain profitabilities from operating. They determine an explicit measure for the expected time of each firm being active in the market and the probability of both rivals entering within a finite time. In the presence of entry costs, decision thresholds exhibit hysteresis, the range of which is decreasing in the correlation between competing firms. A measure of the expected time of each firm being active in the market and the probability of both rivals entering within a finite time are explicitly calculated. The former (latter) is found to decrease (increase) with the volatility of relative firm profitabilities implying that market leadership is shorter-lived the more uncertain the industry environment.

Smit and Trigeorgis (2006) illustrate the use of real options valuation and game theory principles to analyse prototypical investment opportunities involving important competitive and strategic decisions under uncertainty. They use examples from innovation cases, alliances and acquisitions to discuss strategic and competitive aspects, relevant in a range of industries like consumer electronics and telecoms. Particularly, they focus on whether it is optimal to compete independently or coordinate via strategic alliances.

Grzegorz and Kort (2006) analyse the situation where two firms have an opportunity to invest in a profit-enhancing investment project and face different (effective) investment costs. They relax the assumption that the duopolistic rivals are identical, is motivated by the existence of many sources of potential cost asymmetry. Investment cost asymmetries may be due to either liquidity constraints, that increase the investment cost for a firm that faces capital market imperfections, or due to different degree of organisational flexibility at implementing a new production technology. Their results indicate that within a certain range of asymmetry level, a marginal increase in the investment cost of the firm with the cost disadvantage can enhance its own value and reduce its rival's value.

Huisman and Kort (1999) examine how the deterministic duopoly framework of Fudenberg and Tirole (1985) is affected when uncertainty is introduced. According to Fudenberg and Tirole (1985), under large first-mover advantages, a pre-emption equilibrium occurs with dispersed adoption timings since it is essential for each firm to move quickly and pre-empt investment by its rivals. The introduction of uncertainty creates an opposing force since now there is a positive option value of waiting that becomes larger with higher uncertainty, thereby delaying investment. In the simultaneous investment and pre-emptive equilibrium cases, the results of Huisman and Kort (1999) agree with those of Fudenberg and Tirole (1985); however, in the stochastic case, uncertainty raises the required entry threshold for both firms as it increases the value of waiting. Finally, if first-mover advantages are lower but sufficiently large for the pre-emptive equilibrium to result in the deterministic model, then Huisman and Kort (1999) show that sufficiently high uncertainty results in simultaneous investment equilibrium, thereby reducing the number of scenarios where the pre-emptive equilibrium is optimal.

Paxson and Pinto (2005) extend the traditional real options approach that treats the number of units sold and the price per unit as an aggregate variable by presenting a rivalry model in which the profits per unit and the number of units sold are both stochastic variables. They examine a pre-emptive setting (where both firms fight for the leader's position) and a non-pre-emptive setting (where the role of the leader is defined exogenously). Their results indicate that the triggers of both the leader and the follower increase in both settings as the correlation between the profits per unit and the quantity of units increases since then the aggregate volatility involving the number of units and the profits per unit also increases. Furthermore, they illustrate how the value of the active leader increases by more than the value of her investment opportunity when the number of units sold while being alone in the market increases. This, in turn, increases the non-pre-emptive leader's incentive to invest, thereby reducing the discrepancy between the pre-emptive leader's and non-pre-emptive leader's entry thresholds. Finally, they illustrate how increasing first-mover advantages create an incentive for the pre-emptive leader to enter the market sooner since then the entry of the follower is less damaging.

Unlike earlier studies concerning investment strategies in the electricity market, Takashima *et al.* (2008) show the effect of competition on market entry and the strategies of firms with different types of power plants. They analyse the entry strategies into the electricity market of two firms that have power plants under price uncertainty and competition and consider firms with either a thermal power plant or a nuclear power plant. Among other results, they show that for a nuclear power plant the entry threshold of the leader is higher compared to a liquified natural gas thermal power plant, since the latter has mothballing options that facilitate investment. Also, compared to the firm with a coal power plant or an oil thermal power plant, a firm with a nuclear power plant tends to be the leader because variable and construction costs for a nuclear power plant are lower compared to those of a coal power plant, while the oil thermal power plant may have lower construction cost but has variable cost that is twice as much as that of the nuclear power plant.

Huisman and Kort (2009) model not only the timing but also the size of the investment. They consider a monopoly setting as well as a duopoly setting and compare the results with the standard models in which the firms do not have capacity choice. They identify the region of demand where the leader can choose either to deter temporarily or to accommodate the entry of the follower and find that the leader can choose the deterrence strategy only up to a certain high level of demand. If the demand is higher than that level, then it is optimal for the follower to enter at the same time as the leader. Similarly, if the demand is low, then it is not optimal for the leader to choose the deterrence strategy as this would result in negative profits. Also, at high levels of demand, the leader's optimal strategy is either to deter or to accommodate the entry of the follower. However, the region in which the leader can choose either one of the two strategies decreases with uncertainty, thereby increasing the range of demand where the leader chooses the deterrence strategy.

Extending the traditional approach that considers only two competing firms, Bouis *et al.* (2009) analyse investments in new markets where more than two identical competitors are present. In the setting including three firms, they find that if entry of the third firm is delayed, then the second firm has an incentive to invest earlier because this firm can enjoy the duopoly market structure for a longer time. This reduces the investment incentive for the first firm, which now faces a shorter period in which it can enjoy monopoly profits, and, thus, it invests later. This effect is denoted as the accordion effect and is also observed when the number of competing firms is greater. Indeed, with more than three firms competing, exogenous demand changes affect the timing of entry of the first, third, fifth, etc., investor in the same qualitative way, while the entry of the second, fourth, sixth, etc., investor is affected in exactly the opposite qualitative way. In other words, if a delay is observed for the "odd" investors, then the "even" investors will invest sooner.

Each of these papers assumes a risk-neutral decision maker, and, as a result, the implications of risk aversion are not addressed. We contribute to this line of work by developing a utility-based framework in order to examine how optimal investment decisions under uncertainty are affected by competition and risk aversion. This is relevant to a knowledge-based sector in which firms compete to launch a new product while simultaneously facing costs of financial distress or shareholder pressure. In order to describe the preferences of the two firms, we apply a CRRA utility function and determine the optimal strategies that maximise the expected utility of their future profits in both pre-emptive and non-pre-emptive settings.

We confirm the results of Hugonnier and Morellec (2007) by showing that risk aversion lowers the expected utility of the project, thereby delaying the entry of the leader and the follower in both pre-emtpive and non-pre-emptive settings. We also find that, relative to the monopolist, the non-pre-emptive leader is hurt less from the follower's entry than the pre-emptive leader since the former has the flexibility to delay entry into the market. Interestingly, risk aversion does not impact the relative loss in the pre-emptive leader's value due to the follower's entry, but makes the non-pre-emptive leader relatively better off. Furthermore, we show that higher uncertainty reduces the loss in value of the pre-emptive leader relative to the monopolist by delaying the entry of the follower, thereby allowing the pre-emptive leader to enjoy monopoly profits for longer time. In the non-pre-emptive duopoly setting, we show that if the discrepancy between the market share of the leader and the follower is small, then the impact of uncertainty on the leader's option value is more profound and offsets the loss in value due to the follower's entry. By contrast, a large discrepancy in market share makes the increase in option value less profound as it increases the first-mover advantage and, at the same time, increases the impact of the follower's entry, thereby making the non-pre-emptive leader worse off.

1.4 Structure of the Thesis

The remainder of the thesis is structured as follows. In Chapter 2, we incorporate the concept of risk in the real options framework by constructing a utility-based framework in order to examine the impact of risk aversion and uncertainty on the optimal investment timing decisions of a risk-averse decision maker. We formulate the problem using a nested optimal stopping time approach and a CRRA utility function to determine the optimal time of investment that maximises the decision maker's expected utility of future profits and examine the impact of operational flexibility in terms of having the ability to suspend and resume operations at any time. Also, numerical examples that illustrate the interaction among risk aversion, uncertainty, and operational flexibility are provided for each case. In Chapter 3, we assess how the flexibility to adjust capacity impacts the value of an option to invest. We extend the traditional real options approach to investment under uncertainty with discretion over capacity by allowing for risk aversion, through a CRRA utility function, and operational flexibility in the form of suspension and resumption options. In Chapter 4, we examine how duopolistic competition affects the entry decisions of risk-averse investors. We also explore how the impact of competition on the value of a firm under two different oligopolistic frameworks varies with risk aversion and uncertainty. Chapter

5 concludes with a discussion about the findings and limitations of the current approaches. Future research recommendations in these areas are also provided.

Chapter 2

Optimal Investment under Operational Flexibility, Risk Aversion, and Uncertainty

Fluctuating global economic conditions require responsive strategies in order to ensure the effectiveness of investment decisions. The withdrawal of Honda in 2008 from Formula One (Financial Times, 2008), for instance, was made in light of the rapidly deteriorating conditions facing the global auto industry and reflects the impact of the global financial and economic crisis. Indeed, when market uncertainty increases and decision makers are risk averse, the discretion to abandon, modify, or suspend existing projects becomes of greater importance. Here, we examine the impact of such operational flexibility, in terms of being able to suspend and resume the project at any time, on optimal investment policies and option values. We analyse the case where the decision maker exhibits risk aversion and has perpetual options to suspend and resume a project at no cost. Under these conditions, we address the question of how investment decisions are affected by risk aversion, operational flexibility, and uncertainty. First, we develop a theoretical framework for investment under uncertainty with risk aversion and operational flexibility in order to derive optimal investment and operational thresholds. Second, we show how risk aversion interacts with operational flexibility to affect optimal investment policy. Third, we provide managerial insights for operational decisions based on analytical and numerical results.

We proceed in Section 2.1 by formulating the problem using the nested optimal stopping time approach and a CRRA utility function to determine the optimal time of investment that maximises the decision maker's expected utility of future profits. The impact of operational flexibility, in terms of having the ability to suspend and resume operations, is examined in Section 2.2. We first analyse the case where the investment is irreversible (2.2.1) and then introduce operational flexibility in the form of a single abandonment option (2.2.2), a combined suspension-resumption option (2.2.3), and finally complete flexibility where the decision maker has an infinite number of perpetual options to suspend and resume operations (2.2.4). Section 2.3 provides numerical examples for each case and examines the effects of volatility and risk aversion on the optimal investment, suspension, and resumption thresholds. We illustrate the interaction among risk aversion, uncertainty, and operational flexibility and present managerial insights to enable more informed investment and operational decisions. Section 2.4 concludes and offers directions for future research.

2.1 Problem Formulation and Assumptions

We assume that a risk-averse decision maker holds the perpetual option to invest in a project that yields stochastic revenues and produces a single unit of output per annum over an infinite lifetime. Prior to investment, the decision maker's initial wealth is invested in a risk-free asset with rate of return $r > 0$. Let K be the amount of wealth the decision maker gives up in order to cover the fixed and irrecoverable cost of investment and c be the deterministic variable operating cost of the project. As the operating cost, c, is incurred in perpetuity, the present value of these costs at the time of investment equals $K + \frac{c}{r}$, which we assume is the decision maker's initial wealth. Also, time is continuous and denoted by $t \geq 0$, and the value of the project's exogenous output price, P_t, follows a geometric Brownian motion (GBM) :

$$dP_t = \mu P_t dt + \sigma P_t dZ_t, \ P_0 > 0 \tag{2.1}$$

Here, μ is the growth rate, σ is the proportional variance, Z_t is the standard Brownian motion, and P_0 is the initial value of the project's output price. All values and rates are expressed in real terms. The decision maker's preferences are described by an increasing and concave utility function, $U(\cdot)$. Hence, our analysis can accommodate hyperbolic absolute risk aversion (HARA), constant absolute risk aversion (CARA), and CRRA utility functions. To enable comparisons with Hugonnier and Morellec (2007), we apply the same utility function, i.e., a CRRA utility function:

$$U(P_t) = \begin{cases} \frac{P_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \geq 0 \ \& \ \gamma \neq 1 \\ \ln(P_t) & \text{if } \gamma = 1 \end{cases} \tag{2.2}$$

We follow the framework of Hugonnier and Morellec (2007) for decomposing cash flows into disjoint time intervals. We denote by $P_{\tau_j}^{(i)}$ the output price at time τ_j, $j = 1, 2, 3...$, at which we exercise an investment ($j = 1$), suspension ($j = 2, 4, 6, ...$), or resumption option ($j = 3, 5, 7, ...$) when $i = 0, 1, 2, 3...$ subsequent embedded options exist. For example, $P_{\tau_1}^{(0)}$ is the price at which we exercise an investment option without operational flexibility, $P_{\tau_2}^{(0)}$ is the price at which we exercise an abandonment option, and $P_{\tau_2}^{(1)}$ is the price at which we exercise a suspension option with a resumption option still available, etc. Suppose now that we have a perpetually operating project that we start at random time τ_1. Thus, up to time τ_1, we earn an instantaneous cash flow of $c + rK$ per time unit with utility $U(c+rK)$ discounted at our subjective rate of time preference, $\rho > \mu$. Once we invest in the project, we swap this certain cash flow for a risky one, P_t per time unit, with utility $U(P_t)$, as shown in Figure 2.1.

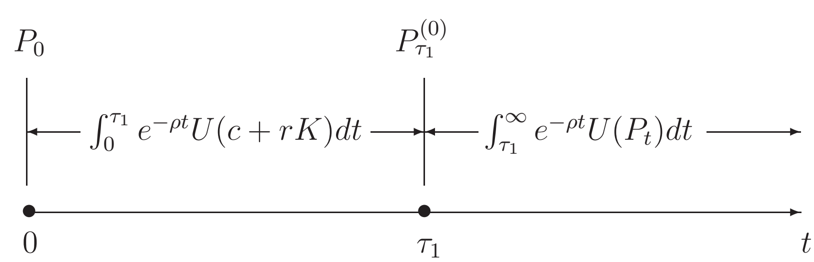


Figure 2.1: Irreversible investment under risk aversion

Using the law of iterated expectations and the strong Markov property of the

GBM, which states that price values after time τ_1 are independent of the values before τ_1 and depend only on the value of the process at τ_1, the time-zero discounted expected utility of the cash flows is:

$$\mathbb{E}_{P_0}\left[\int_0^{\tau_1} e^{-\rho t}U(c+rK)dt + \int_{\tau_1}^{\infty} e^{-\rho t}U(P_t)dt\right] = \int_0^{\infty} e^{-\rho t}U(c+rK)dt + \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right] V_1\left(P_{\tau_1}^{(0)}\right) \quad (2.3)$$

where

$$V_1\left(P_{\tau_1}^{(0)}\right) = \mathbb{E}_{P_{\tau_1}^{(0)}}\left[\int_0^{\infty} e^{-\rho t}\left[U\left(P_t\right) - U\left(c+rK\right)\right] dt\right] \quad (2.4)$$

is the expected utility of the project's cash flows, discounted to τ_1. Here, $\mathbb{E}_{P_0}$ denotes the expectation operator, which is conditional on the initial value, P_0, of the price process and reflects the randomness of both τ_1 and P_t in 2.3.

Now, we extend this framework by allowing for an abandonment option at random time $\tau_2 > \tau_1$. The value of the output price at which the option to abandon the project is exercised is denoted by $P_{\tau_2}^{(0)}$, as shown in Figure 2.2.

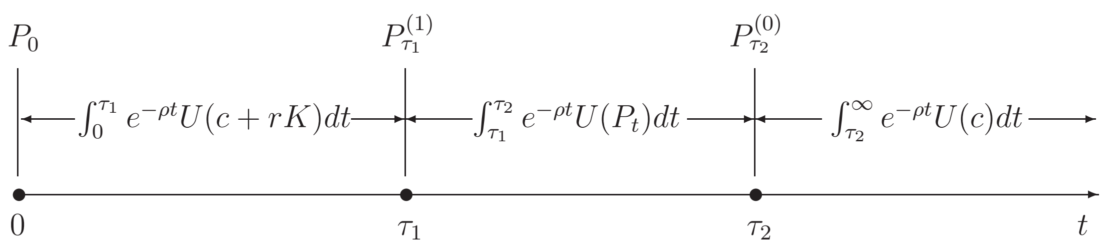


Figure 2.2: Investment under risk aversion with a single abandonment option

In this case, the expected discounted utility of all future cash flows equals:

$$\int_0^{\infty} e^{-\rho t}U(c+rK)dt + \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right]\left[V_1\left(P_{\tau_1}^{(1)}\right) + \mathbb{E}_{P_{\tau_1}^{(1)}}\left[e^{-\rho(\tau_2-\tau_1)}\right] V_2\left(P_{\tau_2}^{(0)}\right)\right] \quad (2.5)$$

where

$$V_2\left(P_{\tau_2}^{(0)}\right) = \mathbb{E}_{P_{\tau_2}^{(0)}}\left[\int_0^{\infty} e^{-\rho t}\left[U\left(c\right) - U\left(P_t\right)\right] dt\right] \quad (2.6)$$

is the expected utility of the cash flows from abandonment, discounted to τ_2.

Finally, we allow for a subsequent resumption option at random time $\tau_3 > \tau_2$. The output price at which the resumption option is exercised is denoted by $P_{\tau_3}^{(0)}$ as shown in Figure 2.3.

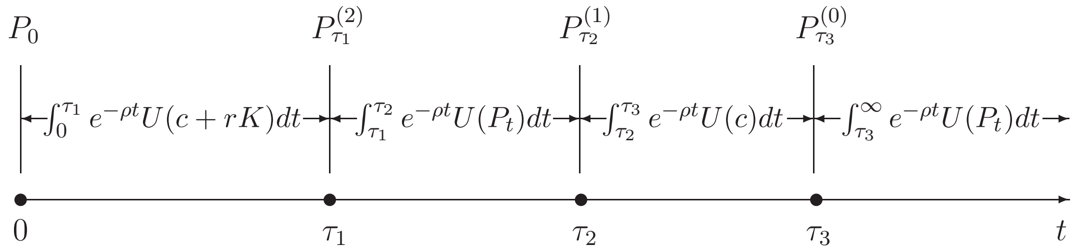


Figure 2.3: Investment under risk aversion with one suspension and one resumption option

Here, the expected discounted utility of all future cash flows is:

$$\int_0^\infty e^{-\rho t} U(c + rK)dt + \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right]\left[V_1\left(P_{\tau_1}^{(2)}\right) + \mathbb{E}_{P_{\tau_1}^{(2)}}\left[e^{-\rho(\tau_2-\tau_1)}\right]\left[V_2\left(P_{\tau_2}^{(1)}\right)\right.\right.$$
$$\left.\left. + \ \mathbb{E}_{P_{\tau_2}^{(1)}}\left[e^{-\rho(\tau_3-\tau_2)}\right] V_3\left(P_{\tau_3}^{(0)}\right)\right]\right] \tag{2.7}$$

where

$$V_3\left(P_{\tau_3}^{(0)}\right) = \mathbb{E}_{P_{\tau_3}^{(0)}}\left[\int_0^\infty e^{-\rho t}\left[U\left(P_t\right) - U\left(c\right)\right] dt\right] \tag{2.8}$$

is the expected utility of the cash flows from resumption, discounted to τ_3. Following the same reasoning, we can extend the model to include complete operational flexibility, i.e., infinitely many suspension and resumption options.

2.2 Analytical Results

2.2.1 Investment without Operational Flexibility

Since this problem has already been examined by Hugonnier and Morellec (2007), we summarise the results for ease of reference, to allow for comparisons, and to provide further insights. Let $F_{\tau_j}^{(i)}(\cdot)$ denote the value of an option that is exercised at time τ_j, $j = 1, 2, 3, ...$, with $i = 0, 1, 2, 3, ...$ subsequent embedded options remaining. $F_{\tau_1}^{(0)}(\cdot)$ refers to an investment option without operational flexibility, $F_{\tau_2}^{(0)}(\cdot)$ refers to an abandonment option, while $F_{\tau_2}^{(1)}(\cdot)$ refers to a suspension with one resumption option, and so on. We define the value of the incremental investment opportunity, $F_{\tau_1}^{(0)}(P_0)$, as follows:

$$F_{\tau_1}^{(0)}(P_0) \quad = \quad \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right] V_1\left(P_{\tau_1}^{(0)}\right) \tag{2.9}$$

By $\mathcal{S}$, we denote the set of stopping times of the filtration generated by the price process.

Using Theorem 9.18 of Karatzas and Shreve (1999) for the CRRA utility function in (2.2), we obtain Proposition 2.2.1. All proofs can be found in the appendix.

Proposition 2.2.1 *Under a CRRA utility-of-wealth function, the expected utility of a perpetual GBM discounted to P_0 is given by 2.10:*

$$\mathbb{E}_{P_0}\int_0^\infty e^{-\rho t}U(P_t)dt = \mathcal{A}U(P_0) \tag{2.10}$$

where $\mathcal{A} = \frac{\beta_1\beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} > 0$ and $\beta_1 > 1$, $\beta_2 < 0$ are the solutions to the following quadratic equation:

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - \rho = 0 \tag{2.11}$$

Since the expected discount factor is $\mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right] = \left(\frac{P_0}{P_{\tau_1}^{(0)}}\right)^{\beta_1}$ (Dixit and Pindyck, 1994, p.315), (2.9) can be written as follows:

$$F_{\tau_1}^{(0)}(P_0) \quad = \quad \max_{P_{\tau_1}^{(0)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(0)}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_1}^{(0)}\right) - \frac{U(c+rK)}{\rho}\right] \tag{2.12}$$

Applying first-order necessary condition (FONC) for this unconstrained maximisation problem we obtain Proposition 2.2.2.

Proposition 2.2.2 *Under a CRRA utility-of-wealth function, the optimal investment threshold is:*

$$P_{\tau_1}^{(0)^*} = (c + rK)\left[\frac{\beta_2 + \gamma - 1}{\beta_2}\right]^{\frac{1}{1-\gamma}} \tag{2.13}$$

The second-order sufficiency condition (SOSC) requires the objective function to be concave at $P_{\tau_1}^{(0)^*}$, which we show in Proposition 2.2.3.

Proposition 2.2.3 *The objective function (2.9) is strictly concave at $P_{\tau_1}^{(0)^*}$ iff $\gamma < 1$.*

Clearly, as $\left[\frac{\beta_2+\gamma-1}{\beta_2}\right]^{\frac{1}{1-\gamma}} > 1$, this implies that $P_{\tau_1}^{(0)^*} > c + rK$. Thus, (2.13) implies that the option to invest should be exercised only when the critical value, $P_{\tau_1}^{(0)^*}$, exceeds the amortised investment cost, $c + rK$, by a positive quantity. This, in turn, implies that uncertainty and risk aversion drive a wedge between the optimal investment threshold and the amortised investment cost. The size of this wedge, as we will show later, depends on the levels of uncertainty, risk aversion, and operational flexibility.

Another way of expressing (2.13) is to relate the marginal benefit (MB) of waiting to invest with its marginal cost (MC):

$$\left(\frac{P_0}{P_{\tau_1}^{(0)^*}}\right)^{\beta_1}\left[\mathcal{A}P_{\tau_1}^{(0)^*-\gamma} + \frac{\beta_1}{P_{\tau_1}^{(0)^*}}\frac{U(c+rK)}{\rho}\right] = \left(\frac{P_0}{P_{\tau_1}^{(0)^*}}\right)^{\beta_1}\frac{\beta_1\mathcal{A}}{P_{\tau_1}^{(0)^*}}U\left(P_{\tau_1}^{(0)^*}\right) \tag{2.14}$$

The first term on the left-hand side of (2.14) is positive and represents the incremental project value created by waiting until the price is higher. Multiplied by the discount factor, it is a positive, decreasing function of the output price, as waiting longer enables the project to start at a higher initial price; however, the rate at which this benefit accrues diminishes due to the effect of discounting. The second term is positive and represents the reduction in the MC of waiting to invest due to saved investment and operating costs. Together, these two terms constitute the MB of delaying investment. The MC of waiting to invest on the right-hand side of (2.14) is positive and reflects the opportunity cost of forgone cash flows discounted appropriately. For low price values, it is worthwhile to postpone investment since the MB is greater than the MC according to Corollary 2.2.1.

Corollary 2.2.1 *The MB curve is steeper than the MC curve at* $P_{\tau_1}^{(0)^*}$.

As risk aversion increases, the MC of waiting to invest decreases relatively more than the MB. This happens because the MC consists entirely of risky cash flows and, therefore, is affected more by risk aversion. As a result, the marginal utility of the investment's payoff increases, thereby increasing the incentive to postpone investment. This leads to Proposition 2.2.4.

Proposition 2.2.4 *The optimal investment threshold is increasing with risk aversion.*

Finally, for a fixed level of risk aversion, the optimal investment threshold increases as the economic environment becomes more uncertain. This happens because greater uncertainty causes the value of waiting to increase, which in turn increases the opportunity cost of investing. Proposition 2.2.5 verifies this intuition.

Proposition 2.2.5 *The optimal investment threshold is increasing with volatility.*

2.2.2 Investment with a Single Abandonment Option

Here, the value of the investment opportunity is:

$$\begin{aligned}
F_{\tau_1}^{(1)}(P_0) &\equiv \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[\int_{\tau_1}^{\infty} e^{-\rho t}\left[U(P_t) - U(c + rK)\right]dt\right. \\
&\qquad\qquad \left. + \sup_{\tau_2 \geq \tau_1} \mathbb{E}_{P_{\tau_1}^{(1)}}\left[\int_{\tau_2}^{\infty} e^{-\rho t}\left[U(c) - U(P_t)\right]dt\right]\right] \\
&= \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\left[V_1\left(P_{\tau_1}^{(1)}\right) + \sup_{\tau_2 \geq \tau_1} \mathbb{E}_{P_{\tau_1}^{(1)}}\left[e^{-\rho(\tau_2 - \tau_1)}V_2\left(P_{\tau_2}^{(0)}\right)\right]\right]\right] \\
&= \max_{P_{\tau_1}^{(1)} \geq P_0}\left(\frac{P_0}{P_{\tau_1}^{(1)}}\right)^{\beta_1}\left[V_1\left(P_{\tau_1}^{(1)}\right) + F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right)\right] \qquad (2.15)
\end{aligned}$$

The value of the output price at which we exercise the abandonment option is $P_{\tau_2}^{(0)}$, and the maximised value of the option to abandon a just-activated project is denoted by $F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right)$, i.e:

$$\begin{aligned} F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right) &= \max_{P_{\tau_2}^{(0)} \leq P_{\tau_1}^{(1)}} \left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)}}\right)^{\beta_2} V_2\left(P_{\tau_2}^{(0)}\right) \\ \Rightarrow \quad F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right) &= \max_{P_{\tau_2}^{(0)} \leq P_{\tau_1}^{(1)}} \left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)}}\right)^{\beta_2} \left[\frac{U(c)}{\rho} - \mathcal{A}U\left(P_{\tau_2}^{(0)}\right)\right] \end{aligned} \tag{2.16}$$

We solve this compound real options problem backward by first determining the optimal abandonment threshold price, $P_{\tau_2}^{(0)^*}$. The FONC for this unconstrained maximisation problem is expressed as:

$$\frac{\beta_1}{1-\beta_1-\gamma} P_{\tau_2}^{(0)^*\,1-\gamma} + c^{1-\gamma} = 0 \tag{2.17}$$

Solving (2.17) with respect to $P_{\tau_2}^{(0)^*}$, we obtain the following expression for the optimal abandonment threshold:

$$P_{\tau_2}^{(0)^*} = c\left[\frac{\beta_1+\gamma-1}{\beta_1}\right]^{\frac{1}{1-\gamma}} \tag{2.18}$$

To ensure the existence of a local maximum at $P_{\tau_2}^{(0)^*}$, the SOSC has to be verified.

Proposition 2.2.6 *The objective function (2.16) is strictly concave at $P_{\tau_2}^{(0)^*}$ iff $\gamma < 1$.*

Since $\left[\frac{\beta_1+\gamma-1}{\beta_1}\right]^{\frac{1}{1-\gamma}} < 1$, (2.18) implies that $P_{\tau_2}^{(0)^*} < c$, i.e., the option to abandon operations permanently should be exercised when the operating cost, c, exceeds the critical value, $P_{\tau_2}^{(0)^*}$, by a positive quantity. Hence, uncertainty and risk aversion again drive a wedge between the critical value, $P_{\tau_2}^{(0)^*}$, and the operating cost, c. The size of this wedge is affected by volatility, risk aversion, and operational flexibility.

In contrast to the previous section, we now express (2.17) by relating the MB from accelerating abandonment of the project with the MC. Note that unlike in the investment stage, an incremental increase in the threshold value implies that abandonment is accelerated:

$$-\left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2}\frac{\beta_2}{P_{\tau_2}^{(0)^*}}\frac{U(c)}{\rho}=\left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2}\left[\mathcal{A}P_{\tau_2}^{(0)^*-\gamma}-\frac{\beta_2\mathcal{A}}{P_{\tau_2}^{(0)^*}}U\left(P_{\tau_2}^{(0)^*}\right)\right]\quad(2.19)$$

The left-hand side of (2.19) is the MB of accelerating abandonment and represents the recovery of the operating cost from shutting down the project. This term is positive, indicating that abandoning operations at a higher price level (i.e., more quickly) increases the expected utility of the salvageable operating cost. The right-hand side of (2.19) is the MC of accelerating abandonment. The first term corresponds to killing the revenues of the project at a higher price level, while the second term is also positive and corresponds to the increase in the MC from speeding up abandonment. This term represents the increase in the opportunity cost from waiting less, thereby forgoing information. As risk aversion increases, the decision maker appears more willing to terminate operations and, thus, avoid potential losses as Proposition 2.2.7 states.

Proposition 2.2.7 *The optimal abandonment threshold is increasing with risk aversion.*

The behaviour of the optimal abandonment threshold when the level of uncertainty changes can be determined using the FONC with respect to σ^2. This leads to the following proposition.

Proposition 2.2.8 *The optimal abandonment threshold is decreasing with volatility.*

Proposition 2.2.8 implies that the greater the uncertainty, the more reluctant the decision maker is to abandon an active project. Intuitively, this happens because she would not want to abandon the project due to a temporary downturn, which is more likely when volatility is higher.

By moving back to the investment stage, we now solve the decision maker's investment timing problem given the solution to the optimal exercise of the abandonment option:

$$F_{\tau_1}^{(1)}(P_0) = \max_{P_{\tau_1}^{(1)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(1)}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_1}^{(1)}\right) - \frac{U(c+rK)}{\rho} \right.$$
$$\left. + \left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \left[\frac{U(c)}{\rho} - \mathcal{A}U\left(P_{\tau_2}^{(0)^*}\right)\right]\right] \tag{2.20}$$

Substituting in $P_{\tau_2}^{(0)^*}$ and applying the FONC to (2.20) leads to the following non-linear equation that gives the optimal investment threshold:

$$\frac{\beta_2}{1-\beta_2-\gamma} P_{\tau_1}^{(1)^*1-\gamma} + (c+rK)^{1-\gamma} - \frac{\rho(\beta_1-\beta_2)}{\beta_1}(1-\gamma)F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)^*}\right) = 0 \tag{2.21}$$

By comparing (2.21) and (2.13), we can show that the optimal investment threshold decreases due to the embedded abandonment option as follows:

Proposition 2.2.9 *The optimal investment threshold when an abandonment option is available is lower compared to an irreversible investment opportunity, ceteris paribus.*

In order to illustrate Proposition 2.2.9, we express (2.21) by relating the MB of waiting to invest to the MC as shown in (2.22).

$$\left(\frac{P_0}{P_{\tau_1}^{(1)^*}}\right)^{\beta_1} \left[\mathcal{A}P_{\tau_1}^{(1)^*-\gamma} + \frac{\beta_1}{P_{\tau_1}^{(1)^*}}\frac{U(c+rK)}{\rho} \right.$$
$$\left. -(\beta_2-\beta_1)\left(\frac{P_{\tau_1}^{(1)*}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \frac{\mathcal{A}}{P_{\tau_1}^{(1)^*}}U\left(P_{\tau_2}^{(0)^*}\right)\right] =$$
$$\left(\frac{P_0}{P_{\tau_1}^{(1)^*}}\right)^{\beta_1} \frac{1}{P_{\tau_1}^{(1)^*}} \left[\beta_1\mathcal{A}U\left(P_{\tau_1}^{(1)^*}\right) - (\beta_2-\beta_1)\left(\frac{P_{\tau_1}^{(1)^*}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \frac{U(c)}{\rho}\right] \tag{2.22}$$

Compared to the case of investment without operational flexibility (2.14), the MB and MC of delaying investment have now increased due to the additional terms on each side of (2.22). These terms are positive and correspond to the MB and MC from the embedded abandonment option. In fact, the MC increases by more than the MB since, at abandonment, the expected utility of the salvageable operating cost is greater than the expected utility of the forgone cash flows. Thus, the marginal utility of the payoff from delaying investment decreases, thereby increasing the incentive to invest. Intuitively, the abandonment option reduces the decision maker's insecurity since she can now terminate her investment in case the output price drops significantly.

2.2.3 Investment with a Single Suspension and Resumption Option

With a single suspension and resumption option, the value of the investment opportunity is:

$$
\begin{aligned}
F_{\tau_1}^{(2)}(P_0) \equiv{}& \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[\int_{\tau_1}^{\infty} e^{-\rho t}\left[U(P_t) - U(c + rK)\right] dt\right. \\
&+ \sup_{\tau_2 \geq \tau_1} \mathbb{E}_{P_{\tau_1}^{(2)}}\left[\int_{\tau_2}^{\infty} e^{-\rho t}\left[U(c) - U(P_t)\right] dt\right. \\
&\left.\left.+ \sup_{\tau_3 \geq \tau_2} \mathbb{E}_{P_{\tau_2}^{(1)}}\left[\int_{\tau_3}^{\infty} e^{-\rho t}\left[U(P_t) - U(c)\right] dt\right]\right]\right] \\
={}& \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\left[V_1\left(P_{\tau_1}^{(2)}\right) + \sup_{\tau_2 \geq \tau_1} \mathbb{E}_{P_{\tau_1}^{(2)}}\left[e^{-\rho(\tau_2 - \tau_1)}\left[V_2\left(P_{\tau_2}^{(1)}\right)\right.\right.\right.\right. \\
&\left.\left.\left.\left.+ \sup_{\tau_3 \geq \tau_2} \mathbb{E}_{P_{\tau_2}^{(1)}}\left[e^{-\rho(\tau_3 - \tau_2)} V_3\left(P_{\tau_3}^{(0)}\right)\right]\right]\right]\right]\right] \\
={}& \max_{P_{\tau_1}^{(2)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(2)}}\right)^{\beta_1}\left[V_1\left(P_{\tau_1}^{(2)}\right) + F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)}\right)\right]
\end{aligned}
\tag{2.23}
$$

Here, $P_{\tau_2}^{(1)}$ is the threshold price at which we suspend the investment project. The last term, $F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)}\right)$, is the maximised value of the option to suspend a just-activated project with a subsequent resumption option and is defined as:

$$
\begin{aligned}
F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)}\right) &\equiv \max_{P_{\tau_2}^{(1)} \le P_{\tau_1}^{2}} \left(\frac{P_{\tau_1}^{(2)}}{P_{\tau_2}^{(1)}}\right)^{\beta_2} \mathbb{E}_{P_{\tau_2}^{(1)}}\left[\int_0^\infty e^{-\rho t}\left[U(c)-U(P_t)\right]dt\right.\\
&\quad \left. + \max_{P_{\tau_3}^{(0)} \ge P_{\tau_2}^{(1)}} \left(\frac{P_{\tau_2}^{(1)}}{P_{\tau_3}^{(0)}}\right)^{\beta_1} \mathbb{E}_{P_{\tau_3}^{(0)}}\left[\int_0^\infty e^{-\rho t}\left[U(P_t)-U(c)\right]\right]dt\right]\\
&= \max_{P_{\tau_2}^{(1)} \le P_{\tau_1}^{(2)}} \left(\frac{P_{\tau_1}^{(2)}}{P_{\tau_2}^{(1)}}\right)^{\beta_2}\left[\frac{U(c)}{\rho} - \mathcal{A}U\left(P_{\tau_2}^{(1)}\right) + F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)}\right)\right] \qquad (2.24)
\end{aligned}
$$

Analogously, we define $F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)}\right)$ to be the maximised value of the option to resume forever a just-suspended project, and $P_{\tau_3}^{(0)}$ the threshold price at which we exercise the option to resume the investment project:

$$
\begin{aligned}
F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)}\right) &= \max_{P_{\tau_3}^{(0)} \ge P_{\tau_2}^{(1)}} \left(\frac{P_{\tau_2}^{(1)}}{P_{\tau_3}^{(0)}}\right)^{\beta_1} V_3\left(P_{\tau_3}^{(0)}\right)\\
&= \max_{P_{\tau_3}^{(0)} \ge P_{\tau_2}^{(1)}} \left(\frac{P_{\tau_2}^{(1)}}{P_{\tau_3}^{(0)}}\right)^{\beta_1} \mathbb{E}_{P_{\tau_3}^{(0)}}\left[\int_0^\infty e^{-\rho t}\left(U(P_t)-U(c)\right)dt\right]\\
&= \max_{P_{\tau_3}^{(0)} \ge P_{\tau_2}^{(1)}} \left(\frac{P_{\tau_2}^{(1)}}{P_{\tau_3}^{(0)}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_3}^{(0)}\right) - \frac{U(c)}{\rho}\right] \qquad (2.25)
\end{aligned}
$$

We solve this compound real options problem backward by first determining the optimal resumption threshold price. The FONC yields:

$$
P_{\tau_3}^{(0)^*} = c\left[\frac{\beta_2+\gamma-1}{\beta_2}\right]^{\frac{1}{1-\gamma}} \qquad (2.26)
$$

Differentiating (2.26) with respect to γ, we obtain the following proposition:

Proposition 2.2.10 *The optimal resumption threshold is increasing with risk aversion.*

Next, we step back to when the investment project is active in order to decide when to suspend operations, i.e., sub-problem (2.24). Applying the FONC, we obtain the following non-linear equation that gives the optimal suspension threshold:

$$\frac{\beta_1}{1-\beta_1-\gamma}P_{\tau_2}^{(1)^*1-\gamma}+c^{1-\gamma}-\frac{\rho(\beta_1-\beta_2)}{\beta_2}(1-\gamma)F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)^*}\right)=0 \tag{2.27}$$

Proposition 2.2.11 *The optimal suspension threshold is higher than the optimal abandonment one.*

To illustrate Proposition 2.2.11, we will examine the relationship between the MB from accelerating suspension and its MC, which is described in the following equation:

$$-\left(\frac{P_{\tau_1}^{(2)^*}}{P_{\tau_2}^{(1)^*}}\right)^{\beta_2}\frac{1}{P_{\tau_2}^{(1)^*}}\left[\frac{\beta_2 U(c)}{\rho}-(\beta_1-\beta_2)\left(\frac{P_{\tau_2}^{(1)^*}}{P_{\tau_3}^{(0)^*}}\right)^{\beta_1}\mathcal{A}U\left(P_{\tau_3}^{(0)^*}\right)\right]=$$
$$\left(\frac{P_{\tau_1}^{(2)^*}}{P_{\tau_2}^{(1)^*}}\right)^{\beta_2}\left[\mathcal{A}P_{\tau_2}^{(1)^*-\gamma}-\frac{\beta_2}{P_{\tau_2}^{(1)^*}}\mathcal{A}U\left(P_{\tau_2}^{(1)^*}\right)\right.$$
$$\left.+(\beta_1-\beta_2)\left(\frac{P_{\tau_2}^{(1)^*}}{P_{\tau_3}^{(0)^*}}\right)^{\beta_1}\frac{1}{P_{\tau_2}^{(1)^*}}\frac{U(c)}{\rho}\right] \tag{2.28}$$

The left-hand side of (2.28) is the MB of accelerating suspension. The first term is the MB of accelerating abandonment, while the second term represents the MB from the embedded resumption option. Since the latter term is positive, the MB of suspension has increased compared to the case of abandonment in (2.19). The right-hand side of (2.28) is the MC of accelerating suspension. The first two terms correspond to the MC of accelerating abandonment, while the third term represents the MC from the embedded option to resume operations. Since this term is always positive, it causes the MC of abandonment to increase. Although both the MB and MC increase due to the embedded resumption option, the former increases relatively more since at resumption, the expected utility of the risky cash flows is greater than the expected utility of the operating cost. Thus, the marginal utility of the payoff from suspending operations increases, which in turn increases the incentive to suspend operations. As a result, MB and MC curves intersect at a higher level of the output price, thereby indicating that the embedded resumption option facilitates suspension. Intuitively, the decision maker is more willing to suspend operations since now, unlike in the case of permanent abandonment, she can recover the lost cash flows by exercising her resumption option.

Finally, we move to the investment stage to solve the complete problem taking $P_{\tau_3}^{(0)^*}$ and $P_{\tau_2}^{(1)^*}$ as fixed:

$$\begin{aligned} F_{\tau_1}^{(2)}(P_0) &= \max_{P_{\tau_1}^{(2)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(2)}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_1}^{(2)}\right) - \frac{U(c+rK)}{\rho}\right. \\ &\quad \left. + \left(\frac{P_{\tau_1}^{(2)}}{P_{\tau_2}^{(1)^*}}\right)^{\beta_2} \left[\frac{U(c)}{\rho} - \mathcal{A}U\left(P_{\tau_2}^{(1)^*}\right) + F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)^*}\right)\right]\right] \end{aligned} \tag{2.29}$$

The optimal investment threshold is obtained numerically by solving the following non-linear equation resulting from the FONC:

$$\frac{\beta_2}{1-\beta_2-\gamma} P_{\tau_1}^{(2)^* 1-\gamma} + (c+rK)^{1-\gamma} - \frac{\rho(\beta_1-\beta_2)}{\beta_1}(1-\gamma) F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)^*}\right) = 0 \tag{2.30}$$

Proposition 2.2.12 *The optimal investment threshold when a single suspension and a single resumption option are available is lower than for an investment opportunity with a single abandonment option.*

Intuitively, the suspension and resumption options facilitate investment because they provide the decision maker the subsequent option to halt the project in case of a downturn and then to resume it. Propositions 2.2.11 and 2.2.12 lead to the insight that additional flexibility facilitates investment and operational decisions, thereby resulting in an increase of the optimal suspension threshold and a decrease of the optimal investment threshold.

2.2.4 Investment with Complete Operational Flexibility

Following the methodology of McDonald (2006), suppose that we are now operating an investment project with infinitely many perpetual options to suspend and resume operations. The symmetry of the problem suggests that the optimal values of the output prices at which these options are exercised will not be affected by additional flexibility, i.e., each time we suspend or resume operations, we still have infinitely many options left. Therefore, each resumption and suspension threshold will be affected equally by flexibility. We let $P_{\tau_e}^{(\infty)}$, where e stands for *even* (i.e., 2,4,6,...), denote the common threshold at which all suspension options are exercised, and $P_{\tau_o}^{(\infty)}$, where o stands for *odd* (i.e., 3,5,7,...), denote the common threshold at which all resumption options are exercised. Hence, the value of an operating project activated at $P_{\tau_o}^{(\infty)}$ can be written as follows:

$$\begin{aligned} & V_3\left(P_{\tau_o}^{(\infty)}\right) + \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} V_2\left(P_{\tau_e}^{(\infty)}\right) + \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1} V_3\left(P_{\tau_o}^{(\infty)}\right) + ... \\ &= \sum_{i=0}^{\infty} \left\{ \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1} \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} \right\}^i \left\{ V_3\left(P_{\tau_o}^{(\infty)}\right) + \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} V_2\left(P_{\tau_e}^{(\infty)}\right) \right\} \end{aligned} \tag{2.31}$$

Since $\sum_{i=0}^{\infty} \left\{ \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1} \right\}^i$ is a geometric series with $\left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1} < 1$, we know that:

$$\sum_{i=0}^{\infty} \left\{ \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1} \right\}^i = \frac{1}{1 - \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1}} \tag{2.32}$$

Therefore, the decision maker's problem in an active state is:

$$F_{\tau_e}^{(\infty)}(P_{\tau_o}^{\infty}) = \max_{P_{\tau_e}^{\infty} \leq P_{\tau_o}^{(\infty)}} \frac{V_3\left(P_{\tau_o}^{(\infty)}\right) + \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} V_2(P_{\tau_e}^{(\infty)})}{1 - \left(\frac{P_{\tau_o}^{(\infty)}}{P_{\tau_e}^{(\infty)}}\right)^{\beta_2} \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1}} \tag{2.33}$$

It follows that the option to resume a currently suspended project with infinitely many resumption and suspension options, given that the current value of the output price is $P_{\tau_e}^{(\infty)}$, is:

$$F_{\tau_o}^{(\infty)}\left(P_{\tau_e}^{(\infty)}\right) = \max_{P_{\tau_o}^{(\infty)} \geq P_{\tau_e}^{(\infty)}} \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1} F_{\tau_e}^{(\infty)}\left(P_{\tau_o}^{(\infty)}\right) \tag{2.34}$$

In order to solve for $P_{\tau_1}^{(\infty)^*}$, $P_{\tau_o}^{(\infty)^*}$, and $P_{\tau_e}^{(\infty)^*}$, we first substitute (2.33) into (2.34) and use as an initial guess for $P_{\tau_e}^{(\infty)}$ the price at which the option to abandon the investment project is exercised, i.e., $P_{\tau_e}^{(\infty)} = P_{\tau_2}^{(0)^*}$. Thus, we obtain an equation that we then maximise with respect to $P_{\tau_o}^{(\infty)}$. The estimate of $P_{\tau_o}^{(\infty)}$ we obtain this way is then substituted into (2.33), which we maximise with respect to $P_{\tau_e}^{(\infty)}$. This procedure is iterated until each solution converges. As we will demonstrate numerically in Section 2.3.4, the optimal suspension and resumption thresholds converge toward the operating cost, c. Intuitively, each time that additional flexibility becomes available, the optimal suspension threshold increases and the optimal resumption threshold decreases. Assuming that $P_{\tau_{2j}}^{(i)} < c$ and $P_{\tau_{2j+1}}^{(i)} > c$, $\forall\ i < \infty$ and $\forall\ j = 1, 2, 3, ...$, this implies that $\lim_{i\to\infty} P_{\tau_{2j}}^{(i)} = c$ and $\lim_{i\to\infty} P_{\tau_{2j+1}}^{(i)} = c$, $\forall j$.

Finally, we take $P_{\tau_o}^{(\infty)^*}$, $P_{\tau_e}^{(\infty)^*}$, $F_{\tau_o}^{(\infty)}\left(P_{\tau_e}^{(\infty)^*}\right)$, and $F_{\tau_e}^{(\infty)}\left(P_{\tau_o}^{(\infty)^*}\right)$ as given and solve the investment problem for investment threshold, $P_{\tau_1}^{(\infty)}$, assuming investment cost, K:

$$F_{\tau_1}^{(\infty)}(P_0) \equiv \max_{P_{\tau_1}^{(\infty)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(\infty)}}\right)^{\beta_1} \Bigg[V_1\left(P_{\tau_1}^{(\infty)}\right) + \left(\frac{P_{\tau_1}^{(\infty)}}{P_{\tau_e}^{(\infty)^*}}\right)^{\beta_2} \left[V_2\left(P_{\tau_e}^{(\infty)^*}\right) + F_{\tau_o}^{\infty}\left(P_{\tau_e}^{(\infty)^*}\right)\right]\Bigg] \tag{2.35}$$

2.3 Numerical Results

2.3.1 Investment without Operational Flexibility

Suppose we have a project with $K = \$100$, $c = \$10$, $\sigma \in [0, 0.2]$, and $P_0 = \$13.6$. We set $r = \rho = 0.05$ and $\mu = 0.01$. Figure 2.4 shows that the investment threshold, $P_{\tau_1}^{(0)^*}$, increases in risk aversion, γ, for a fixed volatility, σ. This happens because the underlying expected utility of the project decreases with γ, thereby raising the required threshold for investment. Hence, increased risk aversion reduces the incentive to invest. Second, $P_{\tau_1}^{(0)^*}$ increases in σ for fixed γ because greater uncertainty increases the value of waiting and, thus, the opportunity cost of investing.

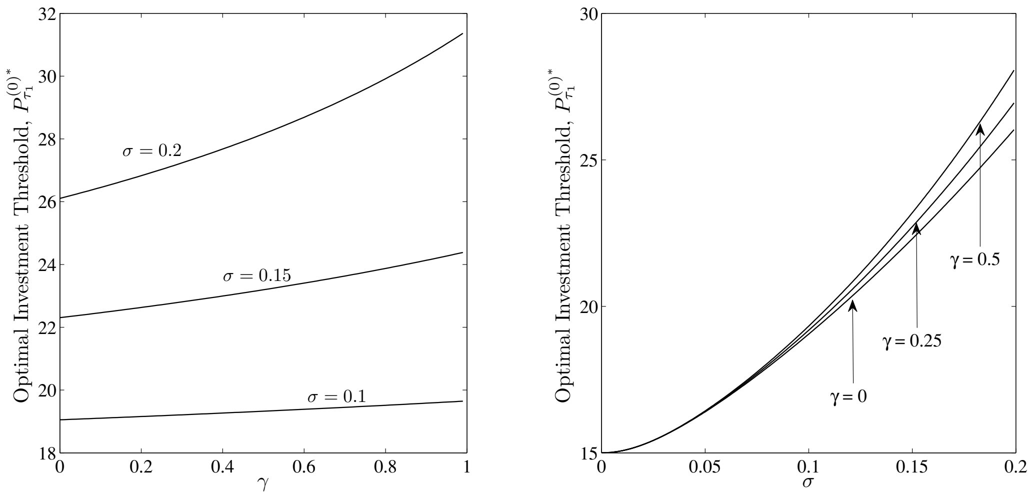


Figure 2.4: Optimal investment threshold versus γ for $\sigma = 0.1, 0.15, 0.2$ (left), and optimal investment threshold versus σ for $\gamma = 0, 0.25, 0.5$ (right).

Figure 2.5 illustrates the MB and MC of waiting to invest, for $\sigma = 0.2$ and $\gamma = 0, 0.25$. For low prices, it is worthwhile to postpone investment as the MB is greater than the MC. As risk aversion increases, the MC, which consists entirely of risky cash flows and, hence, gets affected more by risk aversion, decreases by more than the MB. As a result, the marginal utility of the payoff when investment is delayed increases, which, in turn, decreases the incentive to invest and causes the optimal investment threshold to increase with risk aversion.

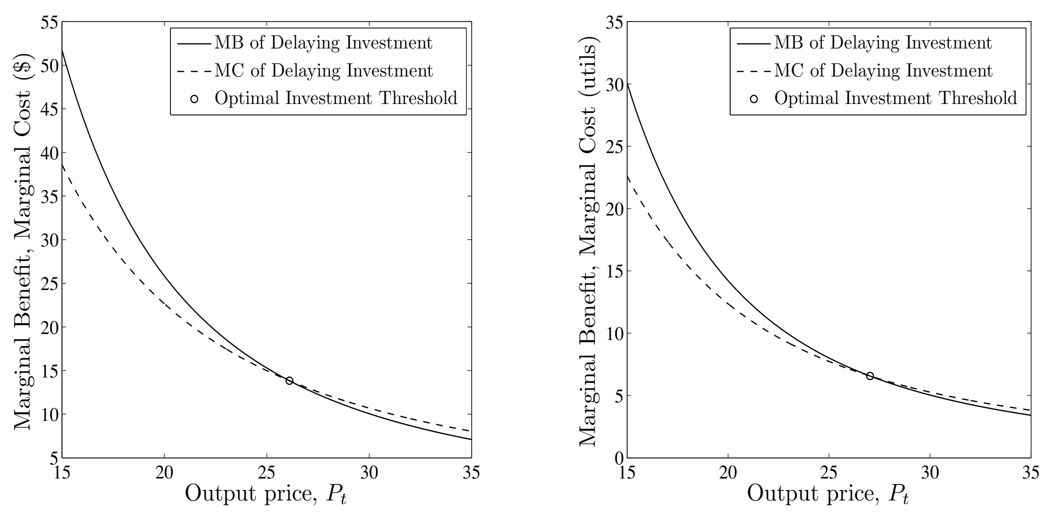


Figure 2.5: Marginal benefit versus marginal cost under risk neutrality (left) and risk aversion, $\gamma = 0.25$, (right) for an irreversible investment opportunity

Figure 2.6 illustrates the impact of volatility, σ, and risk aversion, γ, on the value of the option to invest and the value of the project. In the graph on the left, we plot the value of the project as well as the option value for $\sigma = 0.1, 0.15, 0.2$ holding $\gamma = 0.25$. As uncertainty increases, the project value decreases, but the value of the option to invest, evaluated at the initial level of the output price, increases due to greater waiting value. Consequently, the value of the option to wait also increases, thereby increasing the investment threshold. In the graph on the right, we plot the value of the project and the option value for $\gamma = 0, 0.25, 0.5$ holding $\sigma = 0.2$. The graph indicates that as risk aversion increases, the decision maker requires a higher price before exercising the option to invest. This is due to the decreased expected utility of the project, which decreases the value of the option to invest and increases the investment threshold.

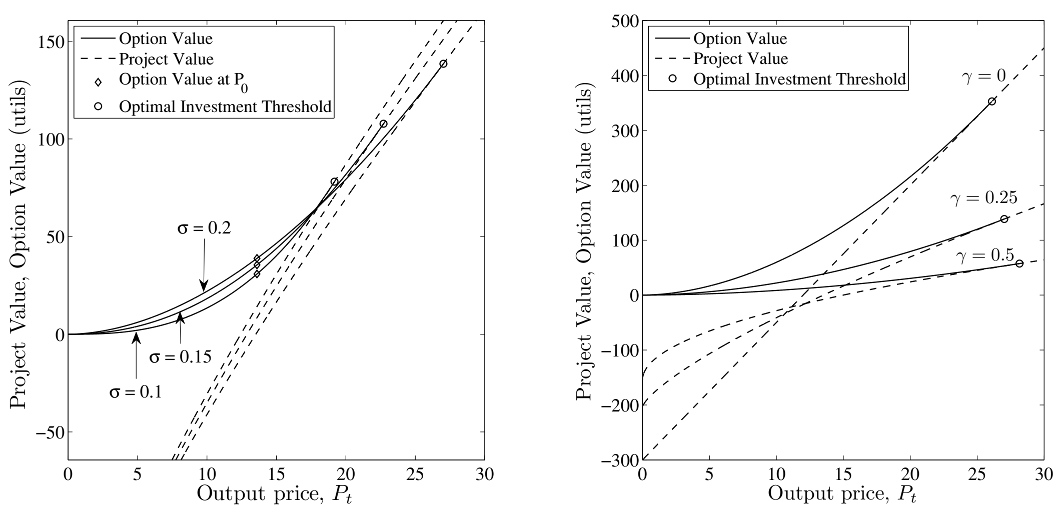


Figure 2.6: Option value and project value versus P_t for $\gamma = 0.25$ and $\sigma = 0.1, 0.15, 0.2$ (left), and option value and project value versus P_t for $\sigma = 0.2$ and $\gamma = 0, 0.25, 0.5$ (right)

2.3.2 Investment with a Single Abandonment Option

Increasing flexibility by adding an abandonment option decreases the optimal investment threshold. The proportional increase in option value due to the subsequent abandonment option is larger for higher levels of uncertainty and risk aversion. Both of these results are illustrated in Figure 2.7. In the graph on the left, we compare the case of investment without operational flexibility to that of investment with a single abandonment option. We plot the value of the project and the value of the investment opportunity for $\gamma = 0.25$ and $\sigma = 0.2$. The graph on the right illustrates how the proportional increase in option value due to the subsequent abandonment option fluctuates with risk aversion for three levels of volatility.

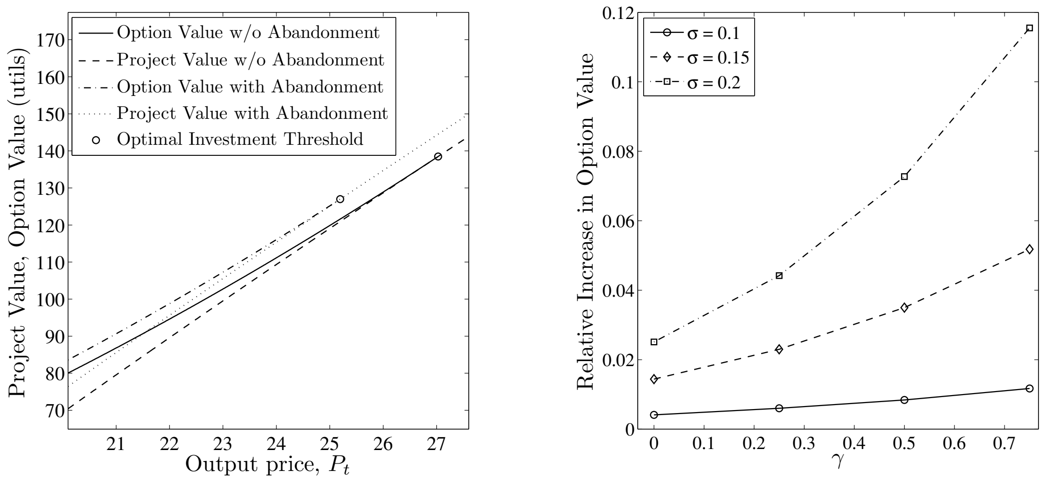


Figure 2.7: Effect of the abandonment option on optimal investment threshold and option value

Although both risk aversion and uncertainty increase the option value of abandonment, the impact of each factor on $P_{\tau_2}^{(0)^*}$ is different. While risk aversion increases the abandonment threshold due to a decrease in project value, uncertainty lowers the abandonment threshold because it increases its opportunity cost. In particular, Figure 2.8 indicates that as risk aversion increases, for a fixed level of volatility, the decision maker becomes more willing to abandon the project in order to avoid potential losses. An increase in uncertainty, however, leads to a decrease in the optimal abandonment threshold.

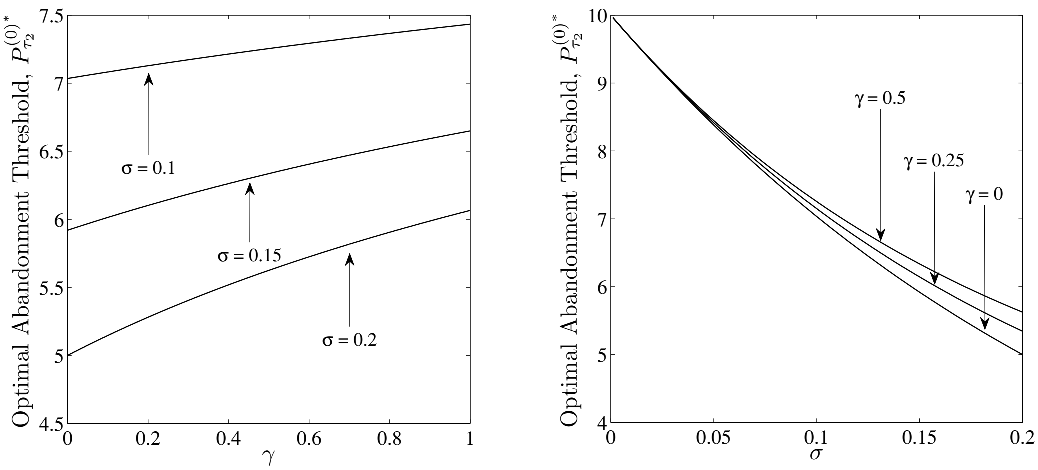


Figure 2.8: Optimal abandonment threshold versus γ for $\sigma = 0.1, 0.15, 0.2$ (left), optimal abandonment threshold versus σ for $\gamma = 0, 0.25, 0.5$ (right)

For large price values, the MB of accelerating abandonment is less than the MC, and, therefore, it is optimal to continue, as Figure 2.9 illustrates. As risk aversion increases, both the MB and MC of accelerating abandonment decrease. However, the MC, which consists entirely of risky cash flows and, therefore, gets affected more by risk aversion, decreases relatively more. As a result, the marginal utility of the payoff from accelerating abandonment increases, which, in turn, increases the incentive to abandon the project and results in an increased optimal abandonment threshold.

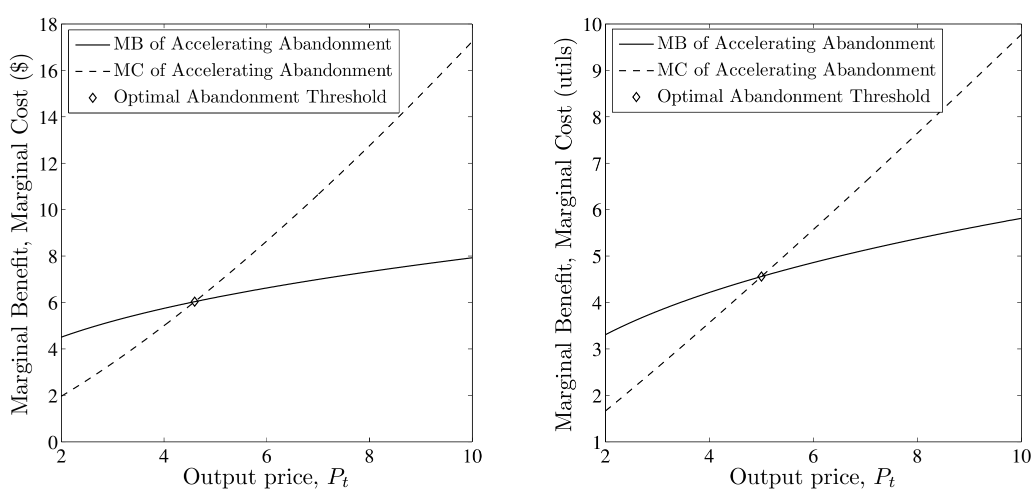


Figure 2.9: Marginal benefit versus marginal cost under risk neutrality (left) and risk aversion, $\gamma = 0.25$, (right) for an irreversible abandonment opportunity

Using the same parameter values as in Section 2.3.1, we plot the MB and MC of waiting to invest versus P_t. The embedded abandonment option causes the marginal utility of the payoff from delaying investment to decrease, which, in turn, increases the incentive to invest. This happens because the MC increases relatively more than the MB, and as a result, the MB and MC curves intersect at a lower level of $P_{\tau_1}^{(1)^*}$, as Figure 2.10 illustrates.

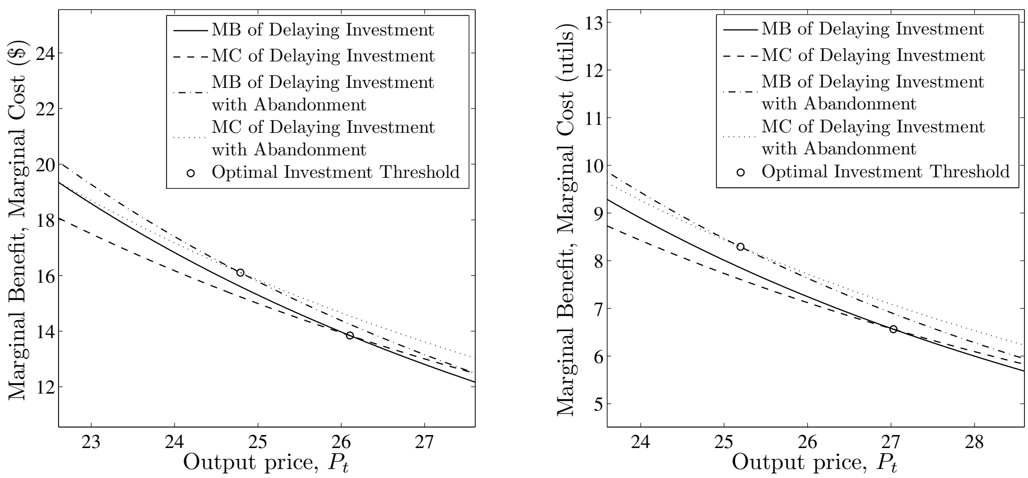


Figure 2.10: Marginal benefit versus marginal cost under risk neutrality (left) and risk aversion, $\gamma = 0.25$ (right) for an investment opportunity with an embedded abandonment option

2.3.3 Investment with a Single Suspension and a Single Resumption Option

Having the option to suspend operations combined with an option to resume them permanently increases the value of the investment opportunity further and decreases the optimal investment threshold as the left panel in Figure 2.11 illustrates. Moreover, the percentage increase in option value due to the subsequent resumption option is greater compared to the case of investment with a single abandonment option, i.e.,

$$\frac{F_{\tau_1}^{(1)}(P_0) - F_{\tau_1}^{(0)}(P_0)}{F_{\tau_1}^{(0)}(P_0)} < \frac{F_{\tau_1}^{(2)}(P_0) - F_{\tau_1}^{(0)}(P_0)}{F_{\tau_1}^{(0)}(P_0)} \tag{2.36}$$

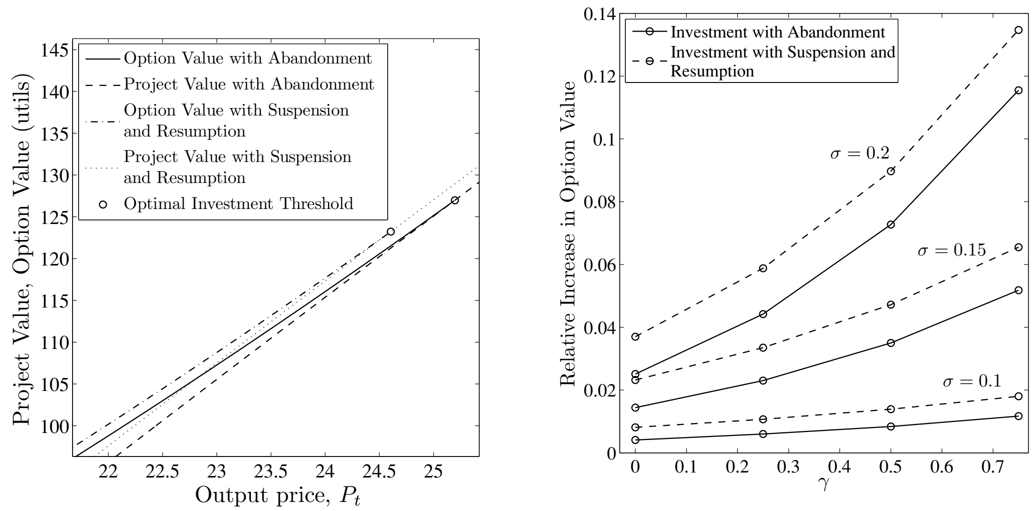


Figure 2.11: Effect of the resumption option on optimal investment threshold and option value

In Figure 2.12, we illustrate the impact of the additional resumption option on the MB and MC of waiting to invest. The embedded resumption option increases the MC relatively more than the MB, and, as a result, the marginal utility of the payoff decreases further, thereby increasing the incentive to invest. Thus, the MB and MC curves intersect at a lower level of the output price compared to the case of investment with abandonment.

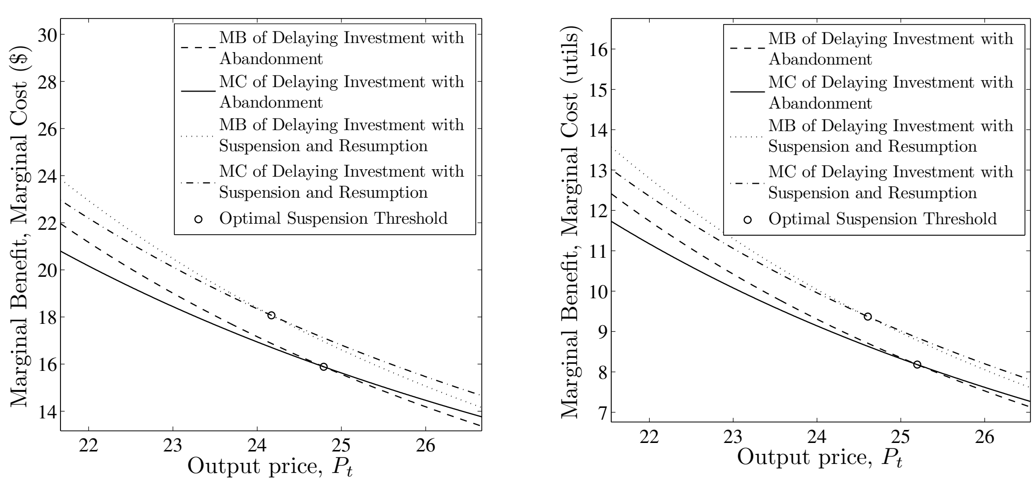


Figure 2.12: Marginal benefit versus marginal cost under risk neutrality (left) and risk aversion, $\gamma = 0.25$, (right) for an investment opportunity with a suspension and resumption option

Interestingly, the results also indicate that the decision maker is less willing to suspend operations as the level of risk aversion increases. This outcome may seem counterintuitive, but it can be explained by the fact that as risk aversion increases, the following two opposing effects take place. First, the marginal utility of accelerating suspension increases with risk aversion, thereby increasing the likelihood of suspension. This happens because the MC of the abandonment option decreases faster with risk aversion than the MB. Second, the marginal utility of delaying resumption from a suspended state increases with risk aversion, thereby decreasing the likelihood of resumption. Here, the MC of the embedded resumption option decreases faster than the MB. Thus, higher risk aversion reduces the marginal value of the payoff from the resumption option, which makes suspension less attractive. Under the assumption of costless suspension and resumption and for the values of the parameters used here, we observe that the impact of risk aversion on the embedded resumption option dominates and postpones the suspension of the project. Figure 2.13 illustrates the impact of risk aversion and uncertainty on the optimal suspension threshold. The graph on the left indicates that as risk aversion increases, the wedge between the MB of suspension and the MB of abandonment decreases, thereby indicating that the impact of risk aversion on the embedded resumption option is more profound and results in the decreased likelihood of suspension. On the other hand, as in the previous section, the suspension threshold decreases with uncertainty since the decision maker is inclined to wait for uncertainty to be resolved before exercising the suspension option.

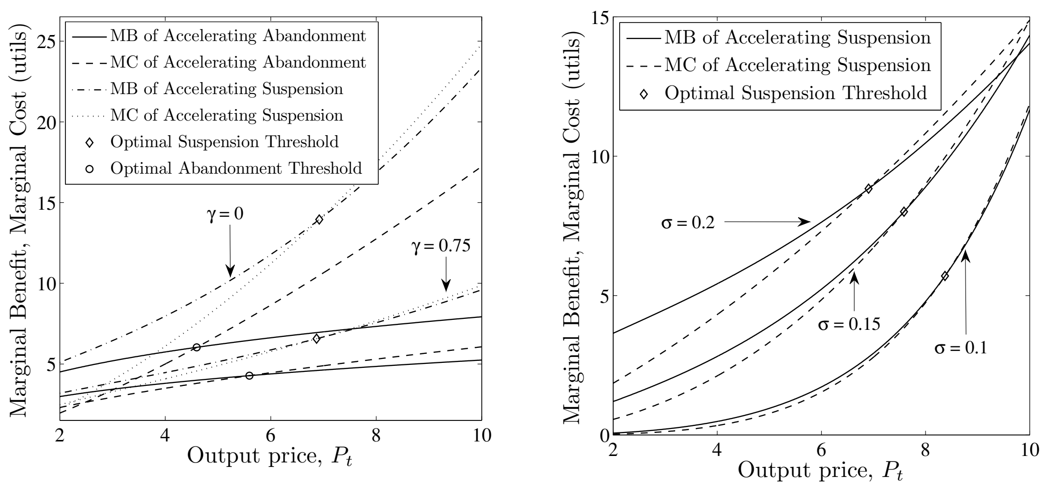


Figure 2.13: MB and MC of abandonment versus MB and MC of suspension for $\sigma = 0.2$ (left), and MB of suspension versus MC of suspension for $\gamma = 0.25$ and $\sigma = 0.1, 0.2, 0.15$ (right)

Also interesting is that by allowing a further abandonment option after resumption, the aforementioned counterintuitive result is no longer observed. Due to the additional abandonment option, the marginal utility of the payoff from the option to suspend operations increases faster with risk aversion than in the case of suspension with a subsequent option of permanent resumption. In particular, the rate of this increase is greater than the rate at which the marginal utility of the payoff from the embedded call option increases. Hence, the impact of risk aversion on the embedded suspension and abandonment options is now greater than that on the single resumption option and, thus, causes the likelihood of suspension to increase with risk aversion. In fact, we observe that the impact of risk aversion on an optimal suspension threshold dominates when the number of subsequent options to suspend operations exceeds the number of the options to resume them.

Figure 2.14 summarises the impact of operational flexibility and risk aversion on the optimal decision thresholds. The direction of the arrows indicates greater operational flexibility. Here, additional flexibility facilitates all operational decisions and causes the optimal investment and resumption thresholds to decrease and the optimal suspension threshold to increase. Meanwhile, the impact of risk aversion on the optimal investment and operational thresholds diminishes as additional flexibility becomes available.

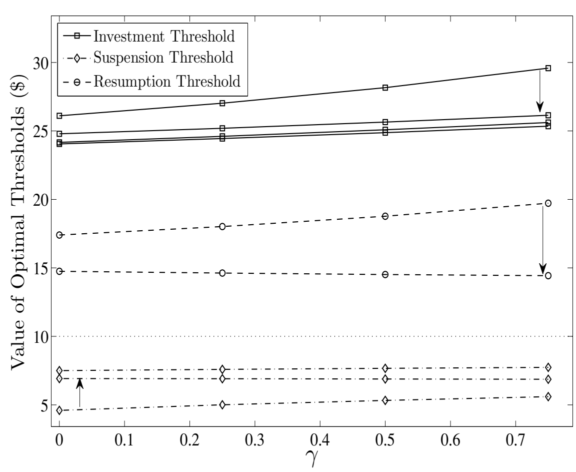


Figure 2.14: Impact of operational flexibility and risk aversion on optimal decision thresholds

2.3.4 Investment with Complete Operational Flexibility

In Figure 2.15, the left figure compares the case of investment with complete flexibility to that of investment with a single suspension and a single resumption option for $\sigma = 0.2$ and $\gamma = 0.25$. Now, the ability to suspend and resume operations at any time increases the value of the investment opportunity, which reduces further the investment threshold price. Also, the proportional increase in option value is greater than that in the case of investment with a single suspension and a single resumption option as the figure on the right illustrates. Finally, according to the numerical results, the optimal suspension and optimal resumption thresholds under complete flexibility are equal to the operating cost, c. Intuitively, additional flexibility facilitates the suspension and resumption of the investment project and, as a result, causes the optimal suspension threshold to increase and the optimal resumption threshold to decrease. Assuming that no rational decision maker would exercise a suspension option at $P^{(i)}_{\tau_{2j}} > c$ and a resumption option at $P^{(i)}_{\tau_{2j+1}} < c$, we can expect both of these thresholds to converge toward the operating cost as additional flexibility becomes available. Thus, as $i \to \infty$, we expect that $P^{(i)}_{\tau_{2j}} \to c$ and $P^{(i)}_{\tau_{2j+1}} \to c$. Hence, the ability to suspend and resume operations costlessly at any time completely mitigates the impact of risk aversion and volatility on the optimal operational thresholds and drives them to the same level as in the risk-neutral case.

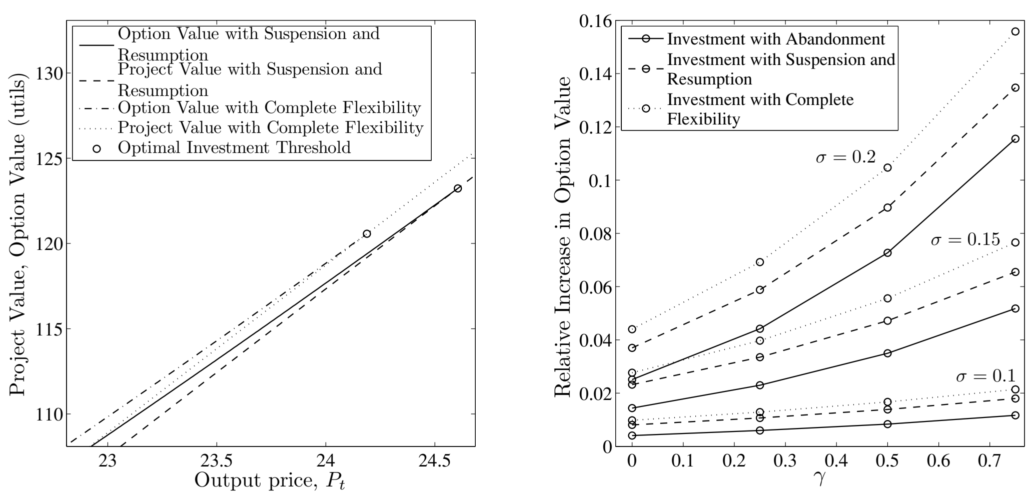


Figure 2.15: Impact of complete flexibility on the optimal investment threshold and option value

2.4 Conclusions

In a world of increasing economic uncertainty, the need to examine the interaction between risk aversion and operational flexibility, so as to provide optimal investment and operational decisions, is of great essence. In this chapter, an effort is made to extend the results of McDonald and Siegel (1985, 1986) and Hugonnier and Morellec (2007) to examine how investment and operational decisions are affected by situations of uncertainty encountered by risk-averse decision makers. Although the impact of risk aversion has already been demonstrated in Hugonnier and Morellec (2007), its implications when combined with operational flexibility have not been thoroughly examined yet. Here, we develop the results regarding the problem of optimal investment under the assumption of risk aversion and operational flexibility assuming that the decision maker faces incomplete markets. We demonstrate how operational flexibility facilitates investment and operational decisions by increasing the likelihood of investment, suspension, and resumption of the investment project. We show that risk aversion provides an incentive for decision makers to delay the investment and resumption of the investment project and speed up their decision to abandon it. Moreover, we describe how an environment of increasing uncertainty may affect the optimal investment policy and lead to hysteresis. Also, we provide insights regarding the behaviour of the optimal suspension threshold when the level of risk aversion changes. Finally, we demonstrate how operational flexibility becomes more valuable as risk aversion increases and the economic environment becomes more volatile.

In order to test the robustness of the model, we can either apply a different stochastic process, such as arithmetic Brownian motion or a mean-reverting process, or an alternative utility function. Other aspects of the real options literature, e.g., dealing with endogenous capacity (Dangl, 1999) and the time-to-build problem, may also be investigated under the framework outlined in this chapter. In Chapter 3, we assume that apart from the option to choose the optimal time of investment, a firm also has the freedom to scale the capacity of the project. Incorporating the same utility-based framework as in Chapter 2, we analyse how investment and sizing decisions are affected by risk aversion and uncertainty.

Chapter 3

The Value of Capacity Sizing under Risk Aversion and Operational Flexibility

Apart from discretion over the timing of investments, a firm typically also has the freedom to determine the scale of the investment in the form of installed capacity. Additionally, capital projects are seldom now-or-never investments and can be abandoned, suspended, and resumed at any time. Examples of such real options problems can be found in the area of infrastructure projects. For instance, in the area of American railways, investment in freight railway capacity will be needed as capacity will have to increase by 90% in order to meet forecast demand of 2035. Among other reasons, this is also due to the change in the pattern of trade as the Panama canal opens a second lane, thereby doubling its capacity and allowing it to carry bigger container vessels and bulk ships. Coming through to Gulf of Mexico and East Coast ports, these vessels will increase the need for better rail links inland (The Economist, 2010).

Although the traditional real options approach addresses the value of flexibility and capacity sizing in capital budgeting decisions, the interaction between an investor's risk tolerance and the optimal capacity to be installed remains an open question. In this chapter, we analyse the impact of uncertainty, risk aversion, and operational flexibility on the optimal investment timing and sizing decisions of a firm in order to assess the degree to which discretion over capacity impacts the value of the option to invest. Thus, the contribution of Chapter 3 is threefold. First, we develop a theoretical framework for investment with capacity sizing under uncertainty with risk aversion and operational flexibility, and we derive closed-form expressions for the optimal investment and operational thresholds as well as the project's optimal capacity. Second, we illustrate how the optimal investment timing and sizing decisions are affected by the interaction among risk aversion, volatility, and operational flexibility. Third, we provide managerial insights for sizing and operational decisions based on analytical and numerical results.

We proceed by setting up the problem in Section 3.1. In Section 3.2, we analyse the impact of uncertainty and risk aversion on the optimal investment timing and sizing decisions when investment is irreversible, and assess how the option value changes due to discretion over capacity. We use a nested optimal stopping-time approach to examine the impact of embedded abandonment, suspension, and resumption options. In Section 3.3, we provide numerical examples for each case in order to quantify the impact of capacity flexibility and to illustrate the effects of volatility and risk aversion on the optimal investment threshold and optimal capacity. Finally, Section 3.4 concludes the chapter, discusses its limitations, and offers directions for future research.

3.1 Setup

3.1.1 Problem Formulation and Notation

Assume that a firm holds a perpetual option to invest in a project with an infinite lifetime that yields stochastic revenues. Time is continuous and denoted by $t \geq 0$, and the output price at time t, P_t (\$/unit), follows a GBM:

$$dP_t = \mu P_t dt + \sigma P_t dZ_t, \ P_0 > 0 \tag{3.1}$$

Here, μ is the annual growth rate, σ is the proportional standard deviation, dZ_t is an increment of the standard Brownian motion process, and P_0 is the initial value of the output price. Initially, the capital required for the realisation of the project is invested in a bond with a risk-free rate $r > 0$.

We denote by $P_{\tau_j}^{(i)}$ the output price at time τ_j, $j = 1, 2, 3, ...$, at which we exercise an investment ($j = 1$), suspension ($j = 2, 4, 6, ...$), or resumption option ($j = 3, 5, 7, ...$) when $i = 0, 1, 2, 3, ...$, subsequent embedded options exist. For example, $P_{\tau_1}^{(0)}$ is the price at which we exercise an investment option without operational flexibility, $P_{\tau_2}^{(0)}$ is the price at which we exercise an abandonment option, and $P_{\tau_2}^{(1)}$ is the price at which we exercise a suspension option with a resumption option still available, etc. Depending on the level of operational flexibility and the output price at investment, we denote by $\tilde{m}^{(i)}(\cdot)$ (units/annum), the annual output corresponding to a now-or-never investment opportunity, and by $m^{(i)}(\cdot)$ the annual output of the project when the option to defer investment is available. The value of an opportunity that is exercised at time τ_j, with i subsequent embedded options remaining, is denoted by $F_{\tau_j}^{(i)}(\cdot)$, while $\tilde{F}^{(i)}(\cdot)$ denotes the maximised value of a now-or-never investment opportunity. Exercising this investment opportunity implies knowledge of the output price at investment and, for this reason, the only variable is the capacity corresponding to the initial output price.

Finally, the deterministic variable operating cost of the project is denoted by c (\$/unit), and the deterministic cost of investment by $K\left(m^{(i)}\right)$ or $K\left(\tilde{m}^{(i)}\right)$ (\$), which we assume behaves as follows:

$$K\left(m^{(i)}\right) = b {m^{(i)}}^{\lambda} \tag{3.2}$$

The parameters b and λ are constants such that $b > 0$ and $\lambda > 1$, i.e., $K(\cdot)$ is a convex function of $m^{(i)}$ or $\tilde{m}^{(i)}$. The particular choice of λ implies an increasing average cost and, as a result, this model is more suitable for describing project that exhibit diseconomies of scale, e.g., renewable-energy power plants. All values and rates are expressed in real terms. The present value of the investment's total cost, i.e., investment and operating cost, at investment equals $K\left(m^{(i)}\right) + \frac{cm^{(i)}}{r}$. Depending on the project's operational flexibility, the firm can determine the optimal investment threshold as well as the corresponding output of the project *ex ante*, and, thus, determine the amount of wealth that should be invested initially in the risk-free asset.

The firm's preferences are described by an increasing and concave utility function, $U(\cdot)$. Thus, we can accommodate a wide range of utility functions such as HARA, CARA, and CRRA utility functions. In our analysis, we apply a CRRA utility function defined as follows:

$$U(P_t) = \begin{cases} \frac{P_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \geq 0 \ \&\ \gamma \neq 1 \\ \ln(P_t) & \text{if } \gamma = 1 \end{cases} \tag{3.3}$$

3.1.2 Irreversible Investment

We begin by decomposing the cash flows into disjoint time intervals. Suppose that we have a perpetually operating project that we start at a random time, τ_1. As the capacity of the project is fixed at investment, it depends on the output price at τ_1, i.e., $m^{(0)} \equiv m^{(0)}\left(P_{\tau_1}^{(0)}\right)$. Up to time τ_1, the firm earns an instantaneous cash flow of $cm^{(0)} + rK\left(m^{(0)}\right)$ per time unit with utility $U\left(cm^{(0)} + rK\left(m^{(0)}\right)\right)$ discounted at its subjective rate of time preference, $\rho > \mu$. At τ_1, when the output price is $P_{\tau_1}^{(0)}$, the firm swaps this risk-free cash flow for a risky one, $m^{(0)}P_t$, with utility $U\left(m^{(0)}P_t\right)$ as illustrated in Figure 3.1.

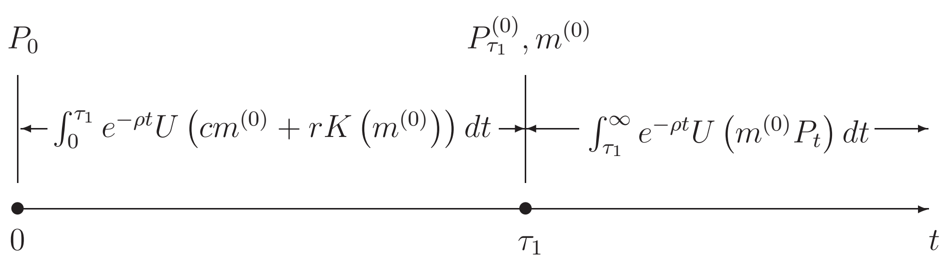


Figure 3.1: Irreversible investment under risk aversion

The time-zero discounted expected utility of the cash flows is:

$$\mathbb{E}_{P_0}\left[\int_0^{\tau_1} e^{-\rho t}U\left(cm^{(0)} + rK\left(m^{(0)}\right)\right) dt + \int_{\tau_1}^{\infty} e^{-\rho t}U\left(m^{(0)}P_t\right) dt\right] =$$

$$\int_0^{\infty} e^{-\rho t}U\left(cm^0 + rK\left(m^{(0)}\right)\right) dt + \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right] V_1\left(P_{\tau_1}^{(0)}, m^{(0)}\right) \tag{3.4}$$

where,

$$V_1\left(P_{\tau_1}^{(0)}, m^{(0)}\right) = \mathbb{E}_{P_{\tau_1}^{(0)}}\left[\int_0^\infty e^{-\rho t}\left[U\left(m^{(0)}P_t\right) - U\left(cm^{(0)} + rK\left(m^{(0)}\right)\right)\right] dt\right] \quad (3.5)$$

is the expected utility of the project's cash flows, discounted to τ_1, given a capacity of $m^{(0)}$. Here, $\mathbb{E}_{P_0}$ denotes the expectation operator, which is conditional on the initial value of the price process.

3.1.3 Investment with a Single Abandonment Option

Now, we extend the previous framework and incorporate operational flexibility by allowing for an abandonment option at a random time, $\tau_2 > \tau_1$. Again, the capacity of the project is fixed at investment and, thus, depends on $P_{\tau_1}^{(1)}$, i.e., $m^{(1)} \equiv m^{(1)}\left(P_{\tau_1}^{(1)}\right)$. The value of the output price at which the option to abandon the project is exercised is denoted by $P_{\tau_2}^{(0)}$ as shown in Figure 3.2. In this case, the time-zero expected discounted utility of all future cash flows equals:

$$\int_0^\infty e^{-\rho t} U\left(cm^{(1)} + rK\left(m^{(1)}\right)\right) dt + \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right]\Big[V_1\left(P_{\tau_1}^{(1)}, m^{(1)}\right) + \mathbb{E}_{P_{\tau_1}^{(1)}}\left[e^{-\rho(\tau_2-\tau_1)}\right] V_2\left(P_{\tau_2}^{(0)}, m^{(1)}\right)\Big] \quad (3.6)$$

where

$$V_2\left(P_{\tau_2}^{(0)}, m^{(1)}\right) = \mathbb{E}_{P_{\tau_2}^{(0)}}\left[\int_0^\infty e^{-\rho t}\left[U\left(cm^{(1)}\right) - U\left(m^{(1)}P_t\right)\right] dt\right] \quad (3.7)$$

is the expected utility of the project's cash flows discounted to τ_2. Notice that the operating cost of the project is recovered upon abandonment.

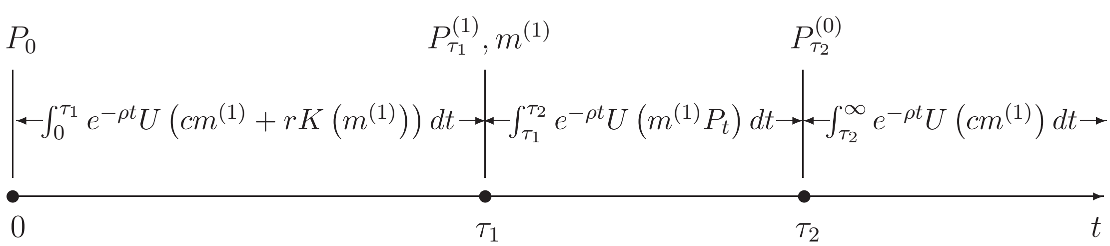


Figure 3.2: Investment under risk aversion with a single abandonment option

3.1.4 Investment with a Single Suspension and a Single Resumption Option

Finally, we allow for a subsequent resumption option at random time, $\tau_3 > \tau_2$. The capacity of the project now depends on $P_{\tau_1}^{(2)}$, i.e., $m^{(2)} \equiv m^{(2)}\left(P_{\tau_1}^{(2)}\right)$, where the output price at which the resumption option is exercised is denoted by $P_{\tau_3}^{(0)}$ as shown in Figure 3.3.

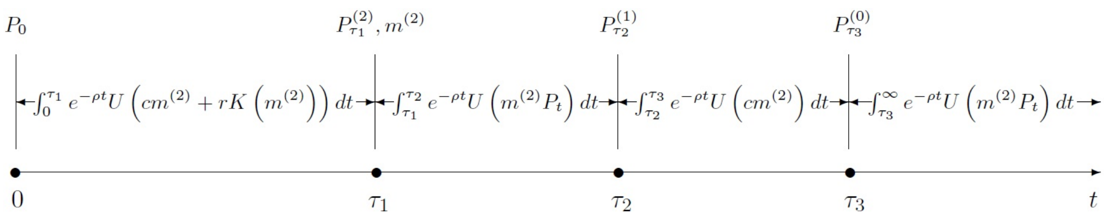


Figure 3.3: Investment under risk aversion with one suspension and one resumption option

Here, the time-zero expected discounted utility of all future cash flows is:

$$\int_0^{\infty} e^{-\rho t} U\left(cm^{(2)} + rK\left(m^{(2)}\right)\right) dt + \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right]\Big[V_1\left(P_{\tau_1}^{(2)}, m^{(2)}\right)$$
$$+ \mathbb{E}_{P_{\tau_1}^{(2)}}\left[e^{-\rho(\tau_2-\tau_1)}\right]\Big[V_2\left(P_{\tau_2}^{(1)}, m^{(2)}\right) + \mathbb{E}_{P_{\tau_2}^{(1)}}\left[e^{-\rho(\tau_3-\tau_2)}\right] V_3\left(P_{\tau_3}^{(0)}, m^{(2)}\right)\Big]\Big] \tag{3.8}$$

where

$$V_3\left(P_{\tau_3}^{(0)}, m^{(2)}\right) = \mathbb{E}_{P_{\tau_3}^{(0)}}\left[\int_0^{\infty} e^{-\rho t}\left[U\left(m^{(2)} P_t\right) - U\left(cm^{(2)}\right)\right] dt\right] \tag{3.9}$$

is the expected utility of the project's cash flows discounted to τ_3. Following the same reasoning, we can extend the model to include complete operational flexibility, i.e., infinite suspension and resumption options.

3.1.5 Methodology

Our methodology to determine the optimal investment threshold and capacity of the project is described in Figure 3.4. Initially, we assume that investment occurs immediately, which implies knowledge of the output price at investment and enables the calculation of the corresponding optimal capacity by maximising the value of the now-or-never investment opportunity. At the optimal capacity choice, the marginal benefit of increasing capacity, $\widetilde{MB}$, equals the marginal cost, $\widetilde{MC}$. This yields the expression relating the initial output price, P_0, to the corresponding optimal capacity, i.e., $\tilde{m}^{(i)^*} \equiv \tilde{m}^{(i)}(P_0)$. We then account for the option to defer investment and maximise the value of the investment opportunity by determining the optimal investment threshold taking into account the inner extremum of optimal capacity choice at investment. The solution to this optimisation problem is obtained by equating the marginal benefit of delaying investment, MB, to the marginal cost, MC. Thus, we obtain the expression relating the optimal investment threshold with the optimal capacity, i.e., $P_{\tau_1}^{(i)^*} \equiv P_{\tau_1}^{(i)}\left(m^{(i)^*}\right)$. Inserting this expression into the condition of optimal capacity choice at τ_1, we obtain the optimal capacity of the project, i.e., $m^{(i)^*} \equiv m^{(i)}\left(P_{\tau_1}^{(i)^*}\right)$. Finally, using $m^{(i)^*}$, we can determine the corresponding optimal investment threshold price, $P_{\tau_1}^{(i)^*}$. If P_0 exceeds $P_{\tau_1}^{(i)^*}$, then we invest immediately and install capacity of size $\tilde{m}^{(i)^*}$. Otherwise, we wait for the threshold price, $P_{\tau_1}^{(i)^*}$, to be reached before investing in a project of size $m^{(i)^*}$

3.2 Analytical Results

3.2.1 Capacity Choice for an Irreversible Investment Opportunity

3.2.1.1 Now-or-Never Investment

Initially, we assume that the firm ignores the option to wait for more information and invests in the project immediately at the initial output price. This assumption implies that, at investment, the output price, P_0, is known, and, for this reason, the firm needs to determine only the corresponding optimal capacity, $\tilde{m}^{(0)^*}$. Hence, the decision-making problem the firm faces when exercising a now-or-never investment opportunity is described by (3.10):

$$\tilde{F}^{(0)}(P_0) \equiv \max_{\tilde{m}^{(0)}} V_1\left(P_0, \tilde{m}^{(0)}\right) \tag{3.10}$$

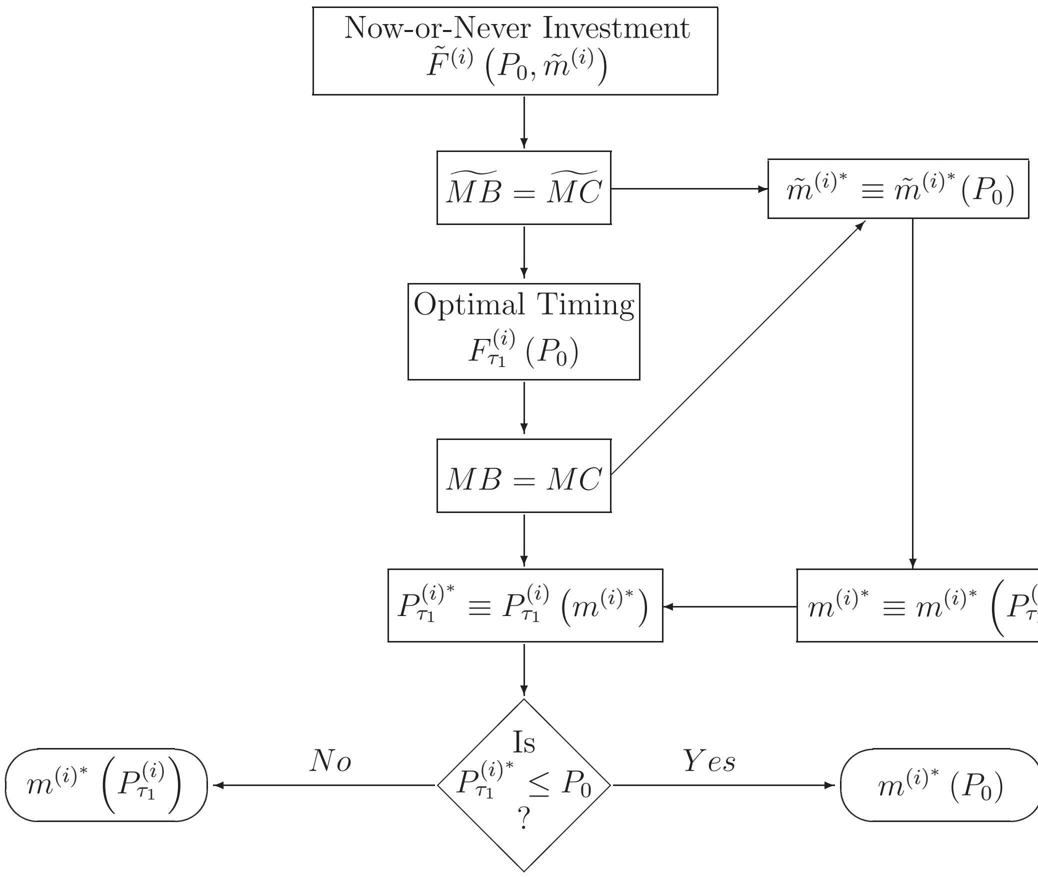


Figure 3.4: Summary of methodology

Using Theorem 9.18 of Karatzas and Shreve (1999) for the CRRA utility function in (3.3), we find that:

$$\mathbb{E}_{P_0} \int_0^\infty e^{-\rho t} U\left(\tilde{m}^{(i)} P_t\right) dt = \mathcal{A} U\left(\tilde{m}^{(i)} P_0\right) \tag{3.11}$$

where $\mathcal{A} = \frac{\beta_1 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} > 0$ and $\beta_1 > 1$, $\beta_2 < 0$ are the solutions to the following quadratic equation for x:

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - \rho = 0 \tag{3.12}$$

Consequently, (3.10) can be written as follows:

$$\tilde{F}^{(0)}\left(P_0\right)=\max_{\tilde{m}^{(0)}}\left\{\mathcal{A}U\left(\tilde{m}^{(0)}P_0\right)-\frac{U\left(c\tilde{m}^{(0)}+rK\left(\tilde{m}^{(0)}\right)\right)}{\rho}\right\} \tag{3.13}$$

At the optimal capacity level, $\tilde{m}^{(0)^*}$, the marginal benefit of a unit increase in the project's capacity evaluated at P_0, $\widetilde{MB}$, has to be equal to the marginal cost, $\widetilde{MC}$, which yields (3.14):

$$\frac{\mathcal{A}{P_0}^{1-\gamma}}{\tilde{m}^{(0)^{*\gamma}}}=\frac{c+rb\lambda\tilde{m}^{(0)^{*\lambda-1}}}{\rho\left[c\tilde{m}^{(0)^*}+rb\tilde{m}^{(0)^{*\lambda}}\right]^{\gamma}} \tag{3.14}$$

The left-hand side of (3.14) is the $\widetilde{MB}$ of increasing capacity, which, due to the property of diminishing marginal utility of the CRRA utility function, is positive and constant in $\tilde{m}^{(0)}$ under risk neutrality but decreasing under risk aversion. The right-hand side of (3.14) represents the $\widetilde{MC}$ of increasing capacity. Notice that, under risk neutrality, the cost function, i.e., $c\tilde{m}^{(0)}+rb\tilde{m}^{(0)^{\lambda}}$, is strictly convex in capacity; however, as risk aversion increases, the concavity of the utility function offsets the convexity of the cost function, which implies that, under risk neutrality, the $\widetilde{MC}$ is strictly increasing in $\tilde{m}^{(0)}$, while as risk aversion increases it decreases initially and then increases. To ensure the existence of the optimal solution to (3.13), the cost function must be strictly convex since, according to Proposition 3.2.1, for $\lambda \downarrow 1$ the $\widetilde{MC}$ curve is steeper than the $\widetilde{MB}$ curve. This implies that a marginal increase in capacity reduces the marginal cost by more than the marginal benefit and, as a result, it is always optimal to install greater capacity. This result reflects an additional limitation of this model since, although it may be optimal to install very large capacity as $\lambda \downarrow 1$, nevertheless, the output price is not affected by such an investment decision as it is assumed to be exogenous. All proofs can be found in the appendix.

Proposition 3.2.1 *For all $\lambda > 1$ the optimisation problem (3.13) has a solution, while for $\lambda \downarrow 1$the solution diverges.*

Although both the $\widetilde{MB}$ and $\widetilde{MC}$ curves shift downward with risk aversion, the former decreases by more since it consists of the risky cash flows. As a result, the marginal utility of the investment's payoff decreases, thereby motivating the installation of smaller capacity. Finally, under risk aversion, increased volatility decreases the $\widetilde{MB}$ by decreasing the expected utility of risky cash flows but leaves the $\widetilde{MC}$ unaffected since the cost is deterministic. Consequently, the marginal utility of the investment's payoff decreases, thereby increasing the incentive to install smaller capacity.

Proposition 3.2.2 *For a now-or-never investment opportunity, the optimal capacity decreases with uncertainty under risk aversion.*

If a firm were to install a capacity level, $\tilde{m}^{(0)}$, such that $\tilde{m}^{(0)} < \tilde{m}^{(0)^*}$, then the expected utility of the investment's payoff will be reduced. As we will illustrate numerically, the relative loss in project value due to fixed capacity diminishes with risk aversion and uncertainty for $\tilde{m}^{(0)} < \tilde{m}^{(0)^*}$ and increases for $\tilde{m}^{(0)} > \tilde{m}^{(0)^*}$. This implies that discretion over capacity becomes relatively less valuable as risk aversion and uncertainty increase when the capacity installed is less than the optimal and more valuable when the capacity installed is greater than the optimal. Intuitively, risk aversion and uncertainty lower the expected utility of the project, thereby diminishing the relative loss in the value of the project when the capacity installed is suboptimal. By contrast, both of these attributes increase the firm's exposure to unfavourable market conditions when the capacity installed exceeds the optimal level.

3.2.1.2 With a Deferral Option

Now, the firm has the option to defer investment, and the value of the investment opportunity is defined as follows:

$$F_{\tau_1}^{(0)}\left(P_0\right) \equiv \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\tilde{F}^{(0)}\left(P_{\tau_1}^{(0)}\right)\right] \tag{3.15}$$

where $\mathcal{S}$ denotes the collection of admissible stopping times of the filtration generated by the price process. Notice that now investment is assumed to take place at τ_1; therefore, the optimal capacity depends on the investment threshold price at τ_1. Since the expected discount factor is $\mathbb{E}_{P_0}\left[e^{-\rho\tau_1}\right] = \left(\frac{P_0}{P_{\tau_1}^{(0)}}\right)^{\beta_1}$ (Karatzas and

Shreve, 1999), the firm's optimisation problem can be written as follows:

$$F_{\tau_1}^{(0)}\left(P_0\right) = \max_{P_{\tau_1}^{(0)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(0)}}\right)^{\beta_1} \left[\mathcal{A}U\left(m^{(0)^*} P_{\tau_1}^{(0)}\right) - \frac{U\left(cm^{(0)^*} + rbm^{(0)^*\lambda}\right)}{\rho}\right] \tag{3.16}$$

Notice that the optimisation problem described by (3.16) considers the inner extremum over capacity choice. Solving this constrained optimisation problem, we obtain the optimal investment threshold price as expressed by (3.17).

Proposition 3.2.3 *For an irreversible investment opportunity, the optimal investment threshold is:*

$$P_{\tau_1}^{(0)^*} = \left(c + rbm^{(0)^*\lambda-1}\right)\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \tag{3.17}$$

An equivalent way of expressing (3.17) is by equating the marginal benefit of delaying investment (MB) to the marginal cost (MC):

$$\left(\frac{P_0}{P_{\tau_1}^{(0)^*}}\right)^{\beta_1}\left[\mathcal{A}\frac{m^{(0)^*1-\gamma}}{P_{\tau_1}^{(0)^*\gamma}} + \frac{\beta_1}{P_{\tau_1}^{(0)^*}}\frac{U\left(cm^{(0)^*} + rbm^{(0)^*\lambda}\right)}{\rho}\right] =$$
$$\frac{\beta_1\mathcal{A}}{P_{\tau_1}^{(0)^*}}\left(\frac{P_0}{P_{\tau_1}^{(0)^*}}\right)^{\beta_1} U\left(m^{(0)^*}P_{\tau_1}^{(0)^*}\right) \tag{3.18}$$

As we will illustrate numerically, a marginal change in capacity impacts the left-hand side of (3.18), which reflects the benefit from allowing the project to start at a higher output price and the benefit from saving on investment and operating cost from waiting longer, by more than the right-hand side, which reflects the opportunity cost of forgone cash flows. As a result, the marginal utility of the investment's payoff when investment is delayed increases with greater capacity, thereby increasing the incentive to postpone investment. According to Corollary 3.2.1, for a low output price, it is worthwhile to postpone investment as the MB is greater than the MC.

Corollary 3.2.1 *The MB curve is steeper than the MC curve at* $P_{\tau_1}^{(0)^*}$.

Notice that (3.17) describes the dependence of the optimal output price on the optimal capacity. Since (3.14) yields the optimal capacity installed at the initial output price, by setting $P_0 = P_{\tau_1}^{(0)^*}$ in (3.14) and inserting the expression of $P_{\tau_1}^{(0)^*}$ from (3.17), we can solve with respect to $m^{(0)^*}$ and, thus, obtain the expression of the optimal capacity. Finally, inserting the resulting expression for $m^{(0)^*}$ into (3.17), we obtain the optimal investment threshold, $P_{\tau_1}^{(0)^*}$. Unlike the case of now-or-never investment, it is now possible to derive closed-form expressions for both the optimal capacity and the optimal investment threshold price.

Proposition 3.2.4 *For an irreversible investment opportunity, the optimal capacity is:*

$$m^{(0)^*} = \left[\frac{c}{rb}\frac{1-\gamma}{\lambda\left(\beta_1+\gamma-1\right)-\beta_1}\right]^{\frac{1}{\lambda-1}}, \quad \lambda > \frac{\beta_1}{\beta_1+\gamma-1} \tag{3.19}$$

According to (3.19), the cost function has to be strictly convex for the solutions to the optimisation problems (3.13) and (3.15) to exist. Notice that with discretion over capacity, risk aversion and volatility influence the optimal sizing and, in turn, the investment timing decisions of the firm. Indeed, we can show that greater risk aversion increases the incentive to install a project with smaller capacity in order to reduce the long-run average cost of investment, i.e., $c + rb{m^{(0)}}^{\lambda-1}$.

Proposition 3.2.5 *The optimal capacity is decreasing in risk aversion.*

While increased risk aversion creates an incentive to install a project with smaller capacity, thereby resulting in a lower investment threshold, it simultaneously creates an incentive to delay investment by lowering the expected utility of the project. According to Proposition 3.2.6, the incentive to reduce the amount of installed capacity in order to decrease the long-run average cost of investment is more profound, thus resulting in the decrease of the optimal investment threshold.

Proposition 3.2.6 *The optimal investment threshold price is decreasing in risk aversion.*

As (3.17) indicates, the optimal investment threshold price is equal to the product of the long-run average cost of investment and a factor that represents the value of waiting to invest. This factor is greater than one, thereby implying that the minimum output price triggering investment is strictly greater than the Marshallian threshold, i.e., $c + rbm^{(0)^{\lambda-1}}$. Since uncertainty increases the value of waiting, the firm is more willing to postpone investment, and, as a result, the required investment threshold increases, thereby raising the corresponding optimal capacity of the project and, in turn, the long-run average cost of investment.

Proposition 3.2.7 *The optimal capacity is increasing in volatility.*

Proposition 3.2.8 *The optimal investment threshold price is increasing in volatility.*

The greater the restrictions the firm faces regarding its flexibility to adjust the capacity of the project, the greater the loss in the value of the investment opportunity. Like in the now-or-never investment case, risk aversion lowers the optimal capacity of the project, thereby diminishing the relative loss in option value when the capacity installed is suboptimal. However, uncertainty has the opposite effect as it increases the investment threshold and the optimal capacity of the project. Intuitively, as uncertainty increases the value of the option to invest, the loss in option value increases with greater uncertainty when the capacity installed is suboptimal and diminishes when the capacity level exceeds the optimal one.

3.2.2 Capacity Choice for an Investment Opportunity with a Single Abandonment Option

3.2.2.1 Now-or-Never Investment

When an embedded option to abandon the project is available, the decision-making problem the firm faces when exercising the investment opportunity immediately is described by (3.20):

$$\tilde{F}^{(1)}\left(P_0\right) = \max_{\tilde{m}^{(1)}} \left\{ V_1\left(P_0, \tilde{m}^{(1)}\right) + F_{\tau_2}^{(0)}\left(P_0\right) \right\} \tag{3.20}$$

The last term, $F_{\tau_2}^{(0)}\left(P_0\right)$, is the maximised value of the option to abandon an active project, which is exercised when the output price is $P_{\tau_2}^{(0)}$. Assuming that the optimal capacity installed at the investment stage is $\tilde{m}^{(1)^*}$, $F_{\tau_2}^{(0)}(P_0)$ is defined as follows:

$$F_{\tau_2}^{(0)}\left(P_0\right) = \max_{P_{\tau_2}^{(0)} \leq P_0} \left(\frac{P_0}{P_{\tau_2}^{(0)}}\right)^{\beta_2} V_2\left(P_{\tau_2}^{(0)}, \tilde{m}^{(1)^*}\right) \tag{3.21}$$

Note that the CRRA utility function is homogeneous of degree $1-\gamma$ in $\tilde{m}^{(i)}$, i.e.:

$$U\left(\tilde{m}^{(i)}x\right) = \tilde{m}^{(i)^{1-\gamma}} U(x) \tag{3.22}$$

This implies that the solution to the optimisation problem described by (3.21), i.e., the optimal abandonment threshold price, is independent of the optimal capacity initially installed since, at abandonment, it equally impacts the firm's expected utility of both the revenues and the cost. Solving the unconstrained optimisation problem, (3.21), yields the optimal abandonment threshold:

$$P_{\tau_2}^{(0)^*} = c\left[\frac{\beta_1+\gamma-1}{\beta_1}\right]^{\frac{1}{1-\gamma}} \tag{3.23}$$

In Chapter 2, it was shown that at each level of risk aversion, increased market uncertainty delays abandonment by increasing its opportunity cost, while at each level of volatility, risk aversion precipitates abandonment by decreasing the expected utility of the project.

Finally, we address the optimisation problem described in (3.20) and determine the optimal capacity corresponding to the initial output price, i.e., $\tilde{m}^{(1)^*}$, for the case of now-or-never investment. Like in Section 3.1, at the optimal capacity level, the $\widetilde{MB}$ of a unit increase in the project's capacity has to be equal to the $\widetilde{MC}$, which yields (3.24):

$$\begin{aligned}\frac{\mathcal{A}{P_0}^{1-\gamma}}{\tilde{m}^{(1)^{*\gamma}}} + \left(\frac{P_0}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \frac{c^{1-\gamma}}{\rho\tilde{m}^{(1)^{*\gamma}}} &= \frac{c + rb\lambda\tilde{m}^{(1)^{*\lambda-1}}}{\rho\left[c\tilde{m}^{(1)^*} + rb\tilde{m}^{(1)^{*\lambda}}\right]^{\gamma}} \\ &+ \left(\frac{P_0}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \frac{\mathcal{A}P_{\tau_2}^{(0)^{*1-\gamma}}}{\tilde{m}^{(1)^{*\gamma}}}\end{aligned} \tag{3.24}$$

Compared to (3.14), both the $\widetilde{MB}$ and $\widetilde{MC}$ have now increased due to the extra terms on each side of (3.24), which are positive and represent the extra marginal benefit and marginal cost from the embedded abandonment option. Since at abandonment, the expected utility of the salvageable operating cost is greater than the expected utility of the forgone cash flows, the $\widetilde{MB}$ increases by more than the $\widetilde{MC}$. Consequently, the abandonment option increases the marginal utility of the investment's payoff, thereby creating an incentive to install a project with greater capacity.

Proposition 3.2.9 *With a single abandonment option, the optimal capacity of the project is greater compared to an irreversible now-or-never investment opportunity, ceteris paribus.*

Consequently, when investing immediately, an abandonment option increases the relative loss in project value due to fixed capacity when $\tilde{m}^{(1)} < \tilde{m}^{(1)^*}$, thus increasing the value of discretion over capacity, and decreases it when $\tilde{m}^{(1)} > \tilde{m}^{(1)^*}$ as it offers downside protection.

3.2.2.2 With a Deferral Option

Now, the decision-making problem the firm faces is to maximise the value of the investment opportunity subject to the constraint of optimal capacity choice at investment:

$$F_{\tau_1}^{(1)}\left(P_0\right) \equiv \max_{P_{\tau_1}^{(1)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(1)}}\right)^{\beta_1} \left[V_1\left(P_{\tau_1}^{(1)}, m^{(1)^*}\right) + F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right)\right] \tag{3.25}$$

Compared to (3.16), the value of the investment opportunity has increased due to the extra term on the right-hand side of (3.25), which is positive and represents the value of the option to abandon the project. Solving the optimisation problem described by (3.25), we obtain the non-linear equation (3.26), which yields the optimal investment threshold price:

$$P_{\tau_1}^{(1)^*} = \left[\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)\left(c+rbm^{(1)^*\lambda-1}\right)^{1-\gamma} + \frac{\rho(\beta_2-\beta_1)(1-\gamma)}{\beta_1 m^{(1)^*1-\gamma}} F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right)\right]^{\frac{1}{1-\gamma}} \quad (3.26)$$

Due to the extra term on the right-hand side of (3.26), which is negative, the optimal investment threshold and, in turn, the corresponding capacity are now lower relative to the case of investment without operational flexibility, thereby implying that the ability to hedge against the possibility of a downturn, through an embedded abandonment option, increases the incentive to invest. Consequently, relative to the case of irreversible investment, discretion over capacity is now less valuable as the option to abandon the project narrows the wedge between the optimal capacity level and the suboptimal capacity level the firm chooses. Intuitively, the embedded option to abandon the project raises the value of the option to invest, thus compensating for the loss in option value when the capacity installed is lower than the optimal one.

3.2.3 Capacity Choice for an Investment Opportunity with a Single Suspension and Resumption Option

3.2.3.1 Now-or-Never Investment

With a single suspension and resumption option, the firm's problem when exercising a now-or-never investment opportunity, is described by (3.27):

$$\tilde{F}^{(2)}(P_0) = \max_{\tilde{m}^{(2)}} \left\{V_1\left(P_0, \tilde{m}^{(2)}\right) + F_{\tau_2}^{(1)}(P_0)\right\} \quad (3.27)$$

The last term, $F_{\tau_2}^{(1)}(P_0)$, is the maximised value of the option to suspend an active project with a subsequent resumption option, which is exercised when the output price is $P_{\tau_2}^{(1)}$. Assuming that the optimal capacity of the project is $\tilde{m}^{(2)^*}$, $F_{\tau_2}^{(1)}(P_0)$ is defined as:

$$F_{\tau_2}^{(1)}\left(P_0\right) \equiv \max_{P_{\tau_2}^{(1)} \leq P_0} \left(\frac{P_0}{P_{\tau_2}^{(1)}}\right)^{\beta_2} \left[V_2\left(P_{\tau_2}^{(1)}, \tilde{m}^{(2)^*}\right) + F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)}\right)\right] \tag{3.28}$$

Here, $P_{\tau_2}^{(1)}$ is the threshold price at which we suspend the investment project. Analogously, we define $F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)}\right)$ to be the maximised value of the option to resume forever a suspended project, and $P_{\tau_3}^{(0)}$ is the threshold price at which we exercise the option to resume the investment project:

$$F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)}\right) = \max_{P_{\tau_3}^{(0)} \geq P_{\tau_2}^{(1)}} \left(\frac{P_{\tau_2}^{(1)}}{P_{\tau_3}^{(0)}}\right)^{\beta_1} V_3\left(P_{\tau_3}^{(0)}, \tilde{m}^{(2)^*}\right) \tag{3.29}$$

Again, the homogeneity of the CRRA utility function implies that the optimal resumption threshold is independent of the capacity of the project.

$$P_{\tau_3}^{(0)^*} = c\left[\frac{\beta_2 + \gamma - 1}{\beta_2}\right]^{\frac{1}{1-\gamma}} \tag{3.30}$$

Intuitively, this happens because at resumption the capacity initially installed will impact equally the expected utility of the revenues and the cost. Since $\left[\frac{\beta_2+\gamma-1}{\beta_2}\right]^{\frac{1}{1-\gamma}} > 1$, we have $P_{\tau_3}^{(0)^*} > c$, which implies that uncertainty and risk aversion drive a wedge between the optimal resumption threshold and the operating cost. The size of this wedge depends on the levels of uncertainty and risk aversion (Chapter 2).

Stepping back to when the investment project is active, we solve sub-problem (3.28) and, thus, determine the critical threshold at which the suspension should be exercised. Applying the FONC, we obtain the following non-linear equation that gives the optimal suspension threshold:

$$P_{\tau_2}^{(1)^*} = c\left[\frac{\beta_1 + \gamma - 1}{\beta_1} + \left(\frac{P_{\tau_2}^{(1)^*}}{P_{\tau_3}^{(0)^*}}\right)^{\beta_1} \frac{(\beta_2 - \beta_1)(1-\gamma)}{\beta_1\beta_2}\right]^{\frac{1}{1-\gamma}} \tag{3.31}$$

Notice that an embedded option to resume operations allows the firm to recover the cash flows it forgoes when exercising a suspension option and creates the incentive to suspend operations at a higher threshold by protecting the firm against a temporary downturn.

Now, we can determine the optimal capacity corresponding to the initial output price by solving the optimisation problem described by (3.27), given $P_{\tau_2}^{(1)^*}$ and $P_{\tau_3}^{(0)^*}$. The solution to this problem, $\tilde{m}^{(2)^*}$, is when the $\widetilde{MB}$ of increasing capacity is equal to the $\widetilde{MC}$, as illustrated by (3.32):

$$\frac{\mathcal{A}{P_0}^{1-\gamma}}{\tilde{m}^{(2)^*\gamma}} + \left(\frac{P_0}{P_{\tau_2}^{(1)^*}}\right)^{\beta_2} \frac{c^{1-\gamma}}{\rho\tilde{m}^{(2)^*\gamma}} + \left(\frac{P_0}{P_{\tau_2}^{(1)^*}}\right)^{\beta_2} \left(\frac{P_{\tau_2}^{(1)^*}}{P_{\tau_3}^{(0)^*}}\right)^{\beta_1} \frac{\mathcal{A}U\left(P_{\tau_3}^{(0)^*}\right)}{\tilde{m}^{(2)^*\gamma}} =$$
$$\left(\frac{P_0}{P_{\tau_2}^{(1)^*}}\right)^{\beta_2} \left[\frac{\mathcal{A}{P_{\tau_2}^{(1)^*}}^{1-\gamma}}{\tilde{m}^{(2)^*\gamma}} + \frac{c + rb\lambda\tilde{m}^{(2)^*\lambda-1}}{\rho\left[c\tilde{m}^{(2)^*} + rb\tilde{m}^{(2)^*\lambda}\right]^{\gamma}} + \left(\frac{P_{\tau_2}^{(1)^*}}{P_{\tau_3}^{(0)^*}}\right)^{\beta_1} \frac{U(c)}{\rho\tilde{m}^{(2)^*\gamma}}\right] \quad (3.32)$$

Compared to (3.24), the $\widetilde{MB}$ and $\widetilde{MC}$ of increasing capacity have increased further due to the extra terms on each side of (3.32) that are positive and represent the extra $\widetilde{MB}$ and $\widetilde{MC}$ from the embedded resumption option. Since at resumption the expected utility of revenues is greater than the expected utility of the operating cost, the marginal utility of the investment's payoff increases further, thereby motivating the installation of greater capacity.

3.2.3.2 With a Deferral Option

With the discretion to wait, the firm's decision-making problem is described by (3.33):

$$F_{\tau_1}^{(2)}(P_0) \equiv \max_{P_{\tau_1}^{(2)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(2)}}\right)^{\beta_1} \left[V_1\left(P_{\tau_1}^{(2)}, m^{(2)^*}\right) + F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)}\right)\right] \quad (3.33)$$

The further increase in operational flexibility increases the firm's incentive to invest in the project, thereby lowering the required investment threshold and corresponding capacity. This is indicated by the last term on the right-hand side of (3.34), which, compared to the case of investment with a single abandonment option, is a greater negative number since it also consists of the embedded option to resume operations:

$$P_{\tau_1}^{(2)^*} = \left[\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)\left[\left(c+rbm^{(2)^*\lambda-1}\right)^{1-\gamma} + \frac{\rho(\beta_2-\beta_1)(1-\gamma)}{\beta_1 m^{(2)^*1-\gamma}} F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)}\right)\right]\right]^{\frac{1}{1-\gamma}} \quad (3.34)$$

Similar to the case of investment with abandonment, the subsequent option to resume operations increases the value of the investment opportunity further. This, in turn, compensates for the loss in option value when the firm chooses a suboptimal capacity level, thereby making discretion over capacity less valuable.

3.2.4 Capacity Choice for an Investment Opportunity with Complete Flexibility

3.2.4.1 Now-or-Never Investment

By investing immediately in a project with complete operational flexibility, the firm solves the following optimisation problem:

$$\tilde{F}^{(\infty)}(P_0) \equiv \max_{\tilde{m}^{(\infty)}}\left[V_1\left(P_0,\tilde{m}^{(\infty)}\right) + \left(\frac{P_0}{P_{\tau_e}^{(\infty)^*}}\right)^{\beta_2}\left[V_2\left(P_{\tau_e}^{(\infty)^*},\tilde{m}^{(\infty)}\right) + F_{\tau_o}^{(\infty)}\left(P_{\tau_e}^{(\infty)^*}\right)\right]\right] \quad (3.35)$$

The symmetry of the problem suggests that the optimal values of the output prices at which each embedded option is exercised will not be affected by additional flexibility. Thus, by $P_{\tau_e}^{(\infty)}$, where e stands for *even* (i.e., 2,4,6,...), we denote the common threshold at which all suspension options are exercised, and $P_{\tau_o}^{(\infty)}$, where o stands for *odd* (i.e., 3,5,7,...), the common threshold at which all resumption options are exercised. $F_{\tau_o}^{(\infty)}\left(P_{\tau_e}^{(\infty)^*}\right)$ is the option to resume a currently suspended project with infinite resumption and suspension options, given that the current value of the output price is $P_{\tau_e}^{(\infty)}$, and is given by (3.36):

$$F_{\tau_o}^{(\infty)}\left(P_{\tau_e}^{(\infty)}\right) = \max_{P_{\tau_o}^{(\infty)} \geq P_{\tau_e}^{(\infty)}} \left(\frac{P_{\tau_e}^{(\infty)}}{P_{\tau_o}^{(\infty)}}\right)^{\beta_1} F_{\tau_e}^{(\infty)}\left(P_{\tau_o}^{(\infty)}\right) \quad (3.36)$$

where $F^{(\infty)}_{\tau_e}\left(P^{(\infty)}_{\tau_o}\right)$ describes the project's value in an active state.

Following the methodology of McDonald (2006), the value of an operating project activated at $P^{(\infty)}_{\tau_o}$ can be written as follows:

$$\begin{aligned} V_3\left(P^{(\infty)}_{\tau_o}, m^{(\infty)^*}\right) \; &+ \; \left(\frac{P^{(\infty)}_{\tau_o}}{P^{\infty}_{\tau_e}}\right)^{\beta_2} V_2\left(P^{(\infty)}_{\tau_e}, m^{(\infty)^*}\right) \\ &+ \; \left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{\tau_e}}\right)^{\beta_2}\left(\frac{P^{(\infty)}_{\tau_e}}{P^{(\infty)}_{\tau_o}}\right)^{\beta_1} V_3\left(P^{(\infty)}_{\tau_o}, m^{(\infty)^*}\right) + ... \\ = \; \sum_{i=0}^{\infty}&\left\{\left(\frac{P^{(\infty)}_{\tau_e}}{P^{(\infty)}_{\tau_o}}\right)^{\beta_2}\left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{e}}\right)^{\beta_1}\right\}^i \times \\ &\left\{V_3\left(P^{(\infty)}_{\tau_o}, m^{(\infty)^*}\right) + \left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{\tau_e}}\right)^{\beta_2} V_2\left(P^{(\infty)}_{\tau_e}, m^{(\infty)^*}\right)\right\} \end{aligned} \tag{3.37}$$

Since $\sum_{i=0}^{\infty}\left\{\left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{\tau_e}}\right)^{\beta_2}\left(\frac{P^{(\infty)}_{\tau_e}}{P^{(\infty)}_{\tau_o}}\right)^{\beta_1}\right\}^i$ is a geometric series with $\left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{\tau_e}}\right)^{\beta_2}\left(\frac{P^{(\infty)}_{\tau_e}}{P^{(\infty)}_{\tau_o}}\right)^{\beta_1} < 1$, we have:

$$\sum_{i=0}^{\infty}\left\{\left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{\tau_e}}\right)^{\beta_2}\left(\frac{P^{(\infty)}_{\tau_e}}{P^{(\infty)}_{\tau_o}}\right)^{\beta_1}\right\}^i = \frac{1}{1-\left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{\tau_e}}\right)^{\beta_2}\left(\frac{P^{(\infty)}_{\tau_e}}{P^{(\infty)}_{\tau_o}}\right)^{\beta_1}} \tag{3.38}$$

Hence, the firm's problem in an active state is:

$$F^{\infty}_{\tau_e}(P^{(\infty)}_{\tau_o}) = \max_{P^{(\infty)}_{\tau_e} \leq P^{(\infty)}_{\tau_o}} \frac{V_3\left(P^{(\infty)}_{\tau_o}, m^{(\infty)^*}\right) + \left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{\tau_e}}\right)^{\beta_2} V_2\left(P^{(\infty)}_{\tau_e}, m^{(\infty)^*}\right)}{1-\left(\frac{P^{(\infty)}_{\tau_o}}{P^{(\infty)}_{\tau_e}}\right)^{\beta_2}\left(\frac{P^{(\infty)}_{\tau_e}}{P^{(\infty)}_{\tau_o}}\right)^{\beta_1}} \tag{3.39}$$

Substituting (3.39) into (3.36) and using as an initial guess for $P^{(\infty)}_{\tau_e}$ the price at which the option to abandon the investment project is exercised, i.e., $P^{(\infty)}_{\tau_e} = P^{(0)^*}_{\tau_2}$, we obtain an equation that we maximise with respect to $P^{(\infty)}_{\tau_o}$. The estimate of $P^{(\infty)}_{\tau_o}$ we obtain this way is subsequently substituted into (3.39), which we maximise with respect to $P^{(\infty)}_{\tau_e}$. This procedure is iterated until each solution converges. Finally, we return to the optimisation problem described by (3.35) in order to obtain $\tilde{m}^{(\infty)^*}$.

3.2.4.2 With a Deferral Option

Next, we account for the option to defer investment and solve the optimisation problem (3.40), in order to obtain the optimal investment threshold, $P_{\tau_1}^{(\infty)^*}$, and, in turn, the optimal capacity, $m^{(\infty)^*}$:

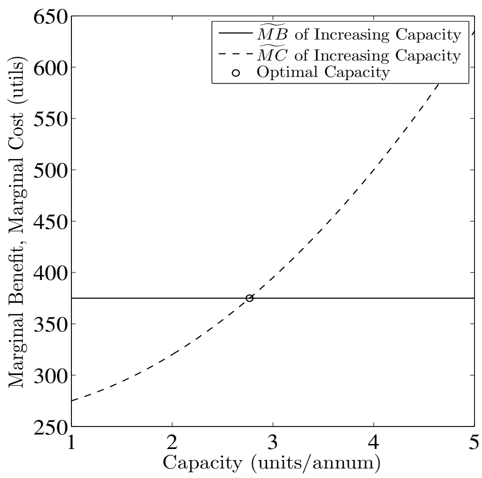
$$F_{\tau_1}^{(\infty)}(P_0) \equiv \max_{P_{\tau_1}^{(\infty)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(\infty)}}\right)^{\beta_1} \Bigg[V_1\left(P_{\tau_1}^{(\infty)}, m^{(\infty)}\right) + \left(\frac{P_{\tau_1}^{(\infty)}}{P_{\tau_e}^{(\infty)^*}}\right)^{\beta_2} \left[V_2\left(P_{\tau_e}^{(\infty)^*}, m^{(\infty)}\right) + F_{\tau_o}^{(\infty)}\left(P_{\tau_e}^{(\infty)^*}\right)\right] \Bigg] \tag{3.40}$$

Using the FONC, $P_{\tau_1}^{(\infty)^*}$ and $m^{(\infty)^*}$ are found numerically using the results for $P_{\tau_o}^{(\infty)^*}$, $P_{\tau_e}^{(\infty)^*}$, and $\tilde{m}^{(\infty)^*}$

3.3 Numerical Results

3.3.1 Capacity Choice for an Irreversible Investment Opportunity

In order to illustrate the impact of risk aversion and uncertainty on the optimal capacity, $m^{(0)^*}$, and optimal investment threshold, $P_{\tau_1}^{(0)^*}$, let $\lambda = 3$, $b = 5$, $r = \rho = 0.05$, $\sigma \in [0, 0.2]$, $\gamma \in [0, 1)$ and $\mu = 0.01$. The initial output price is $P_0 = \$15$, while the operating cost is $c = \$13$. Figure 3.5 illustrates the impact of risk aversion and uncertainty on the marginal benefit and marginal cost of increasing capacity for an irreversible now-or-never investment opportunity. According to the graph on the right, increased risk aversion decreases the $\widetilde{MB}$, which consists of the risky cash flows, more than the $\widetilde{MC}$ and results in the decrease of the marginal utility of the investment's payoff and the installation of smaller capacity. Notice also that the cost of investment and, in turn, the $\widetilde{MC}$ of increasing capacity, is deterministic and, therefore, is not affected by uncertainty both under risk neutrality and risk aversion. On the other hand, the revenues are stochastic and, while uncertainty does not impact the $\widetilde{MB}$ under risk neutrality, thereby leaving the optimal capacity of the project unaffected, under risk aversion, the expected utility of the revenues decreases. Hence, the $\widetilde{MB}$ curve shifts downward with volatility under risk aversion and intersects with the $\widetilde{MC}$ curve at a lower level of capacity, thus illustrating that, under risk aversion, increased volatility decreases the optimal capacity of the project as shown in the graph on the right.



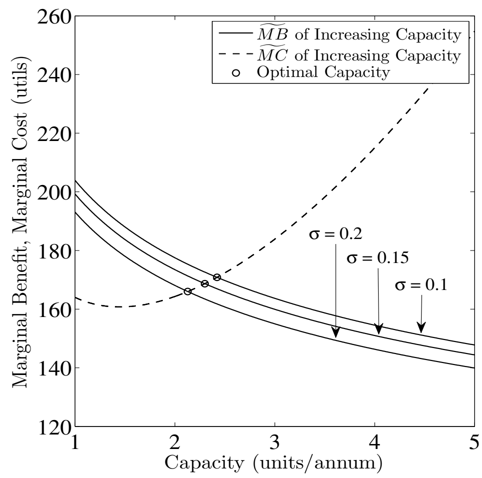


Figure 3.5: Marginal benefit and marginal cost of increasing capacity under risk neutrality for $\sigma = 0.2$ (left) and risk aversion, i.e., $\gamma = 0.2$ (right) for a now-or-never investment opportunity

The impact of risk aversion and uncertainty on the optimal capacity is illustrated in Figure 3.6. Notice that while under risk neutrality the optimal capacity of the project is unaffected by uncertainty, under risk aversion, the optimal capacity decreases. This happens because under risk aversion, uncertainty lowers the expected utility of the investment's payoff, thereby creating an incentive to reduce the amount of installed capacity.

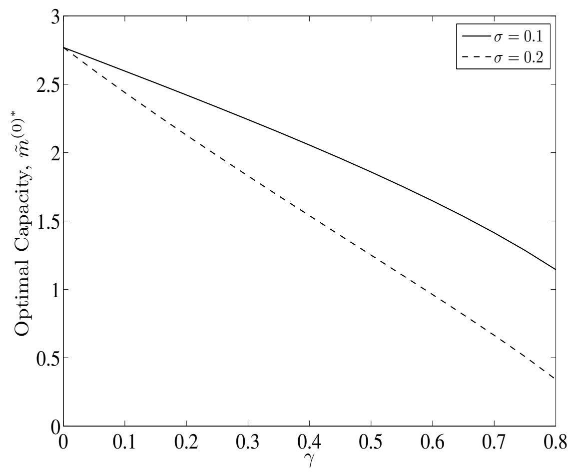


Figure 3.6: Optimal capacity versus risk aversion for an irreversible now-or-never investment opportunity

As Figure 3.7 illustrates, the relative loss in the value of the now-or-never investment opportunity due to fixed capacity, i.e.,

$$\frac{V_1\left(P_0, \tilde{m}^{(0)^*}\right) - V_1\left(P_0, \tilde{m}^{(0)}\right)}{V_1\left(P_0, \tilde{m}^{(0)^*}\right)} = 1 - \frac{V_1\left(P_0, \tilde{m}^{(0)}\right)}{V_1\left(P_0, \tilde{m}^{(0)^*}\right)} \tag{3.41}$$

diminishes for $\tilde{m}^{(0)} < \tilde{m}^{(0)^*}$, becomes zero for $\tilde{m}^{(0)} = \tilde{m}^{(0)^*}$, and then increases for $\tilde{m}^{(0)} > \tilde{m}^{(0)^*}$. This happens because an increase in the amount of installed capacity raises the expected utility of the investment payoff, and, as a result, the discrepancy between the maximised value of the now-or-never investment opportunity, $V_1\left(P_0, \tilde{m}^{(0)^*}\right)$, and the expected utility of the static NPV, $V_1\left(P_0, \tilde{m}^{(0)}\right)$, diminishes as $\tilde{m}^{(0)}$ increases and approaches $\tilde{m}^{(0)^*}$, becomes zero for $\tilde{m}^{(0)} = \tilde{m}^{(0)^*}$, and then increases for $\tilde{m}^{(0)} > \tilde{m}^{(0)^*}$. Notice also that $V_1\left(P_0, \tilde{m}^{(0)^*}\right)$ becomes negative for large values of $m^{(0)}$, since then the expected utility of the cost is larger than that of the revenues. As a result, the relative loss in project value becomes greater than 1. As the left panel illustrates, for $\tilde{m}^{(0)} < \tilde{m}^{(0)^*}$ the relative loss in the value of the now-or-never investment opportunity decreases with increasing risk aversion. This happens because if a firm picks some capacity level $\tilde{m}^{(0)} < \tilde{m}^{(0)^*}$, then increased risk aversion reduces $V_1\left(P_0, \tilde{m}^{(0)}\right)$. By contrast, with discretion over capacity, greater risk aversion lowers $m^{(0)^*}$ as well, thereby reducing $V_1\left(P_0, \tilde{m}^{(0)^*}\right)$ further. Consequently, for $\tilde{m}^{(0)} < \tilde{m}^{(0)^*}$, with increasing risk aversion, the expected utility of the now-or-never investment opportunity decreases by more than that of the static NPV, thereby reducing the discrepancy between $V_1\left(P_0, \tilde{m}^{(0)}\right)$ and $V_1\left(P_0, \tilde{m}^{(0)^*}\right)$. For $\tilde{m}^{(0)} > \tilde{m}^{(0)^*}$, we observe the opposite effect since, due to the decrease of optimal capacity with increasing risk aversion, the wedge between $\tilde{m}^{(0)}$ and $\tilde{m}^{(0)^*}$ increases. Notice that uncertainty has the same impact on the optimal capacity as risk aversion, and, as a result, the relative loss in the value of the now-or-never investment opportunity diminishes with increasing uncertainty for $\tilde{m}^{(0)} < \tilde{m}^{(0)^*}$ and increases for $\tilde{m}^{(0)} > \tilde{m}^{(0)^*}$ as in the right panel.

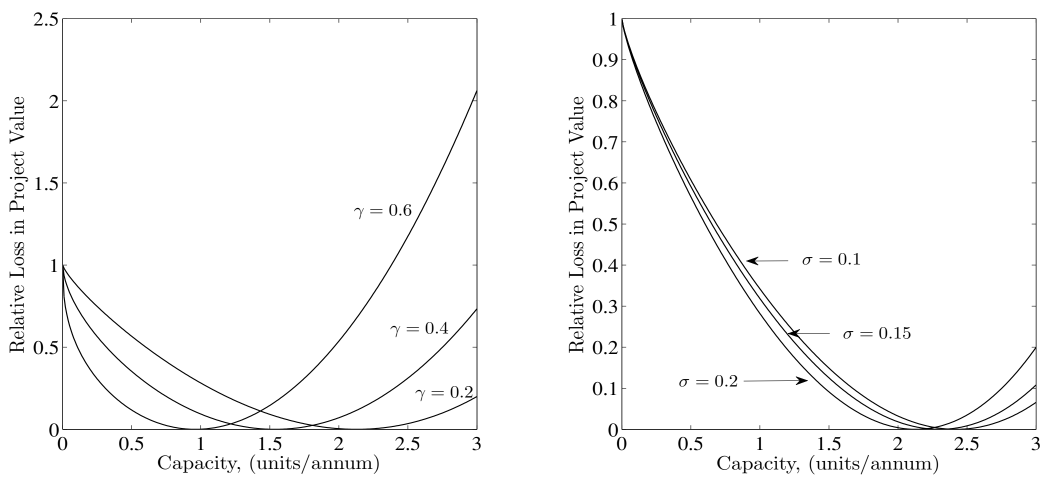


Figure 3.7: Relative loss in project value due to fixed capacity versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$ (right)

Interestingly, when allowing for the option to delay investment, we observe that increased risk aversion lowers the value of the investment opportunity, thereby increasing the required investment threshold while, at the same time, creating an incentive to install a project with smaller capacity in order to reduce the incurred investment cost. In Figure 3.8, the graph on the left shows that the optimal capacity decreases with risk aversion, thereby implying that the incentive to invest at a lower threshold in order to incur a lower sunk and operating cost is more profound than the incentive to delay investment. By contrast, uncertainty increases the investment threshold by increasing its opportunity cost, thereby increasing the optimal capacity. As the graph on the right indicates, the optimal investment threshold price decreases with risk aversion since a lower output price is required to install smaller capacity. On the other hand, uncertainty increases the opportunity cost of investment and, in turn, the required investment threshold price.

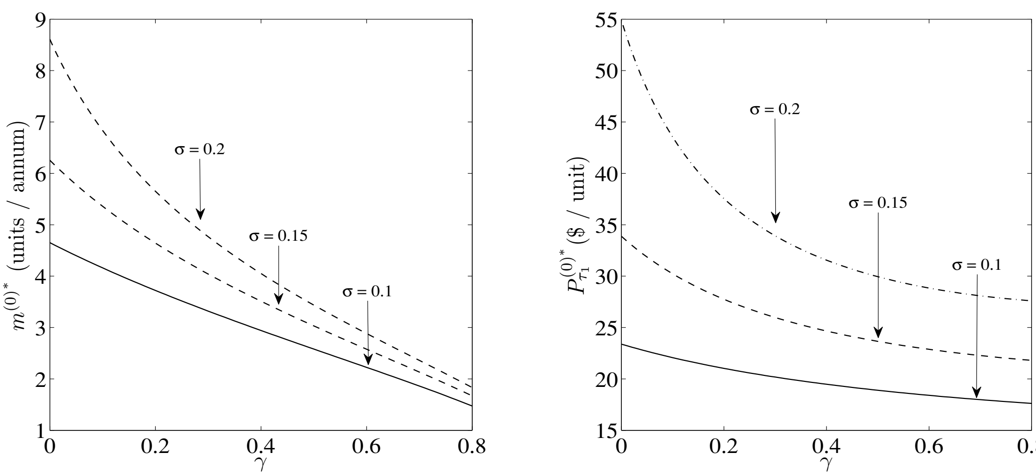


Figure 3.8: Optimal capacity and optimal investment threshold versus risk aversion and uncertainty

In Figure 3.9, the left panel illustrates how risk aversion impacts the marginal benefit and marginal cost of delaying investment. By delaying investment, the firm suffers from forgoing cash flows; however, it benefits from not only allowing the project to start at a higher output price but also saving on sunk and operating cost. According to the left panel, risk aversion decreases the marginal benefit and marginal cost of delaying investment. Since the latter consists exclusively of the risky cash flow it should get affected more, thereby resulting in the increase of the marginal utility of the investment's payoff as well as the increase of the optimal investment threshold. However, with discretion over capacity, the subsequent decrease in the amount of installed capacity due to increased risk aversion causes the MB to decrease by more than the MC. This lowers the expected utility of the investment's payoff and, in turn, the amount of installed capacity. On the other hand, uncertainty increases the investment threshold and increases the amount of installed capacity. As a result, the marginal benefit increases by more than the marginal cost, thereby resulting in the increase of the marginal utility of the investment's payoff and the subsequent increase of the required investment threshold.

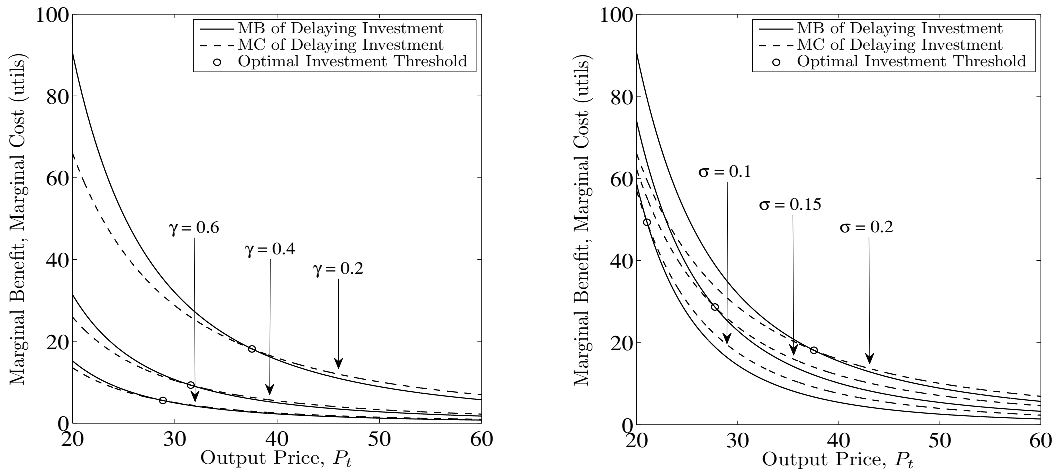


Figure 3.9: Marginal benefit and marginal cost of delaying investment versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$

The impact of volatility and risk aversion on the value of the investment opportunity, $F_{\tau_1}^{(0)}(P_0)$, and the value of the project, $V_1\left(P_t, m^{(0)}\right)$, is illustrated in Figure 3.10. The value of the investment opportunity, evaluated at the initial output price, P_0, increases with uncertainty, thereby raising the required investment threshold, as in the graph on the left. As a result, the optimal capacity increases, thus causing the expected utility of the project to increase more rapidly. By contrast, the graph on the right shows that the value of the investment opportunity decreases with risk aversion. At the same time, risk aversion increases the incentive to install a project with smaller capacity in order to incur lower investment cost, thereby causing the project value to increase more slowly.

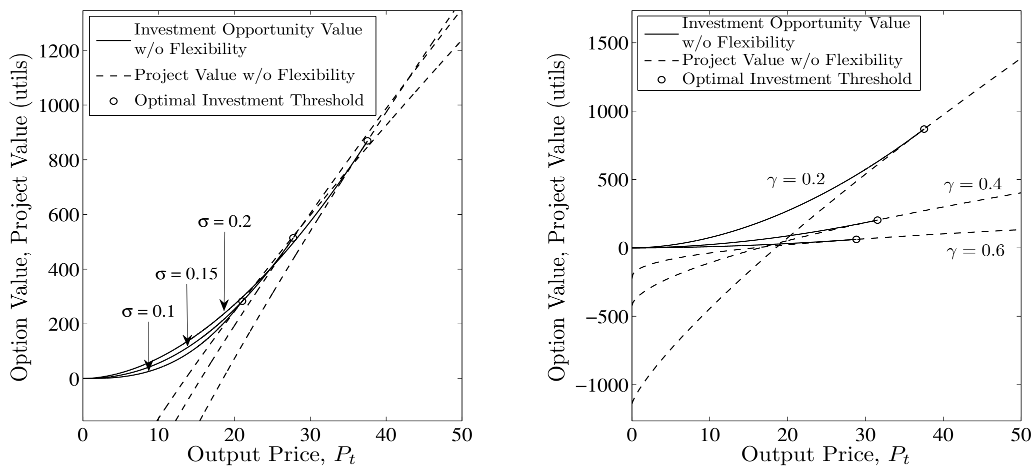


Figure 3.10: Option and project value versus σ for $\gamma = 0.2$ (left) and versus γ for $\sigma = 0.2$ (right)

Figure 3.11 illustrates how risk aversion and uncertainty impact the relative loss in option value, i.e.,

$$\frac{\sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho\tau_1} V_1\left(m^{(0)^*}, P_{\tau_1}^{(0)}\right)\right] - \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho\tau_1} V_1\left(m^{(0)}, P_{\tau_1}^{(0)}\right)\right]}{\sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho\tau_1} V_1\left(m^{(0)^*}, P_{\tau_1}^{(0)}\right)\right]} \tag{3.42}$$

due to fixed capacity. Like in the now-or-never investment case, increased risk aversion reduces the amount of installed capacity and, as a result, with greater risk aversion the relative loss in option value diminishes for $m^{(0)} < m^{(0)^*}$ and increases for $m^{(0)} > m^{(0)^*}$. By contrast, uncertainty now delays investment, thereby increasing the required investment threshold and the corresponding optimal capacity. Consequently, as the graph on the right illustrates, the relative loss in option value increases with uncertainty for $m^{(0)} < m^{(0)^*}$ and diminishes for $m^{(0)} > m^{(0)^*}$. Hence, discretion over capacity becomes relatively less valuable with increasing risk aversion and more valuable with increasing uncertainty when $m^{(0)} < m^{(0)^*}$, while the opposite is observed for $m^{(0)} > m^{(0)^*}$.

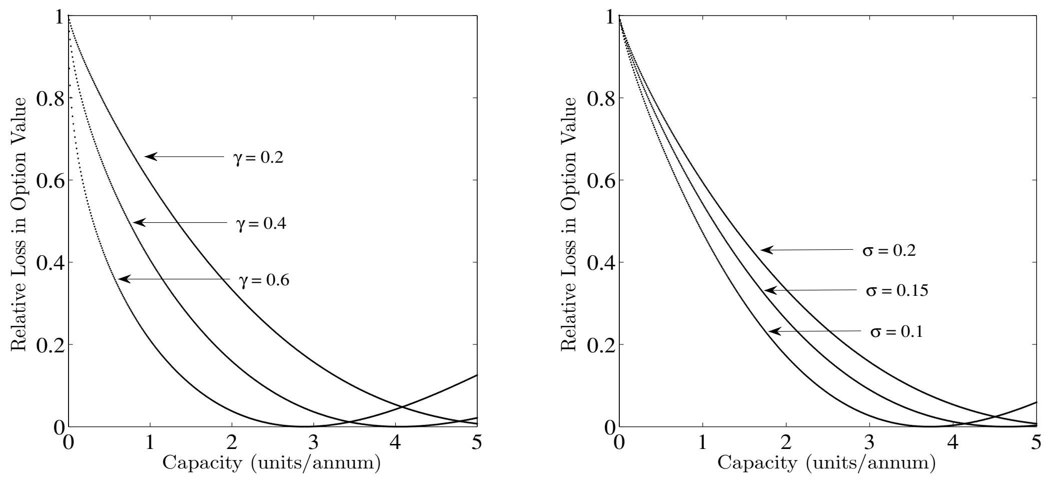


Figure 3.11: Relative loss in option value due to fixed capacity versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$ (right)

3.3.2 Capacity Choice for an Investment Opportunity with a Single Abandonment Option

For the same parameter values as in Section 3.3.1 and assuming that investment takes place immediately, we observe that the marginal utility of the investment's payoff increases when the firm has the additional option to abandon the project, as Figure 3.12 illustrates. This happens because, at abandonment, the expected utility of the salvageable operating cost is greater than the expected utility of the forgone cash flows, and, as a result, the marginal benefit of increasing capacity increases by more than the marginal cost. Consequently, the $\widetilde{MB}$ and $\widetilde{MC}$ curves intersect at a higher capacity level compared to the case of irreversible investment. Notice that under risk neutrality, uncertainty impacts only the extra marginal benefit and extra marginal cost of increasing capacity from the embedded abandonment option and, since $c > P_{\tau_2}^{(0)^*}$, the $\widetilde{MB}$ increases by more than the $\widetilde{MC}$, thereby resulting in higher installed capacity. Under risk aversion, we observe that the increase in capacity is less profound than under risk neutrality because now uncertainty decreases also the expected utility of the risk cash flows, thereby creating an opposing effect that does not allow the $\widetilde{MB}$ to increase as much as under risk neutrality.

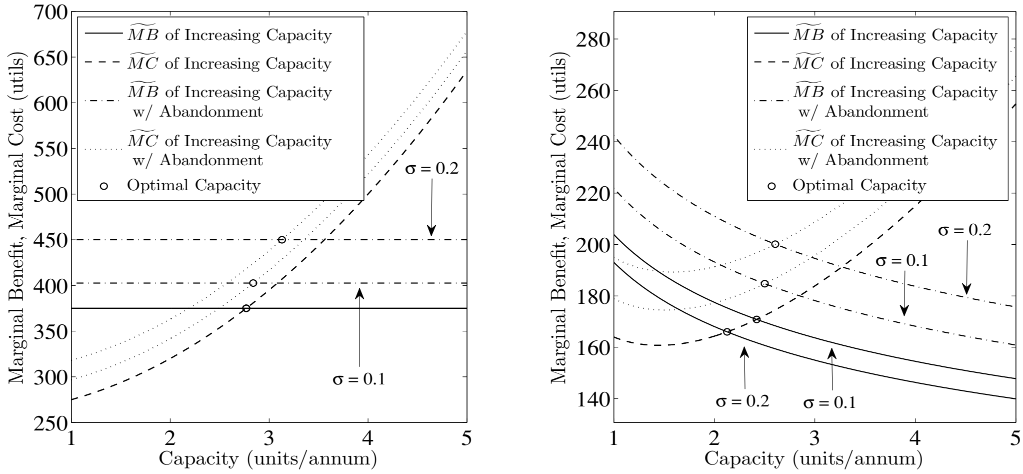


Figure 3.12: $\widetilde{MB}$ and $\widetilde{MC}$ of increasing capacity under risk neutrality (left) and risk aversion, i.e., $\gamma = 0.2$ (right), for a now-or-never investment opportunity with an embedded abandonment option

Unlike the case of irreversible investment, the optimal capacity of the project increases with uncertainty under low risk aversion and decreases with uncertainty under high risk aversion, as in Figure 3.13. Notice that under risk neutrality, uncertainty increases the value of the embedded abandonment option without affecting the value of the active project. Thus, the expected utility of the investment's payoff and, in turn, the incentive to install greater capacity increases. By contrast, under risk aversion, uncertainty lowers the value of the active project and this effect becomes more profound and dominates for high levels of risk aversion.

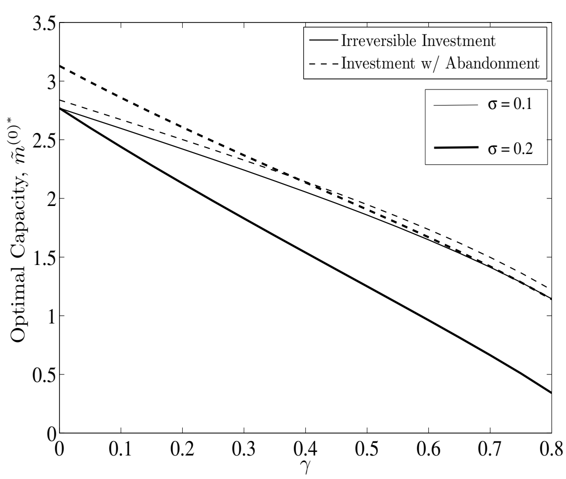


Figure 3.13: Optimal capacity versus risk aversion for a now-or-never investment opportunity with an abandonment option

According to Figure 3.14, the relative loss in project value due to fixed capacity has increased for $\tilde{m}^{(1)} < \tilde{m}^{(1)^*}$ and $\tilde{m}^{(0)} < \tilde{m}^{(0)^*}$, since now the firm misses the optimal capacity by more than in the case of irreversible investment, and decreased for $\tilde{m}^{(1)} > \tilde{m}^{(1)^*}$ and $\tilde{m}^{(0)} > \tilde{m}^{(0)^*}$, since the abandonment option provides downside protection. This implies that, with an abandonment option, discretion over capacity becomes more valuable relative to the case of irreversible investment when, in both cases, the amount of installed capacity is less than the optimal one and less valuable when it is greater. Like in Section 3.3.1, risk aversion diminishes the relative loss in project value for $\tilde{m}^{(1)} < \tilde{m}^{(1)^*}$ and increases it for $\tilde{m}^{(1)} > \tilde{m}^{(1)^*}$ as it increases the incentive to install less capacity. By contrast, uncertainty increases the optimal capacity of the project by increasing the value of the embedded abandonment option. Thus, the relative loss in project value, which equals 1 for $\tilde{m}^{(1)} = 0$, increases for $\tilde{m}^{(1)} < \tilde{m}^{(1)^*}$ and decreases for $\tilde{m}^{(1)} > \tilde{m}^{(1)^*}$. However, the impact of uncertainty under risk aversion is less profound compared to the risk neutrality case due to the simultaneous decrease of the expected utility of the active project.

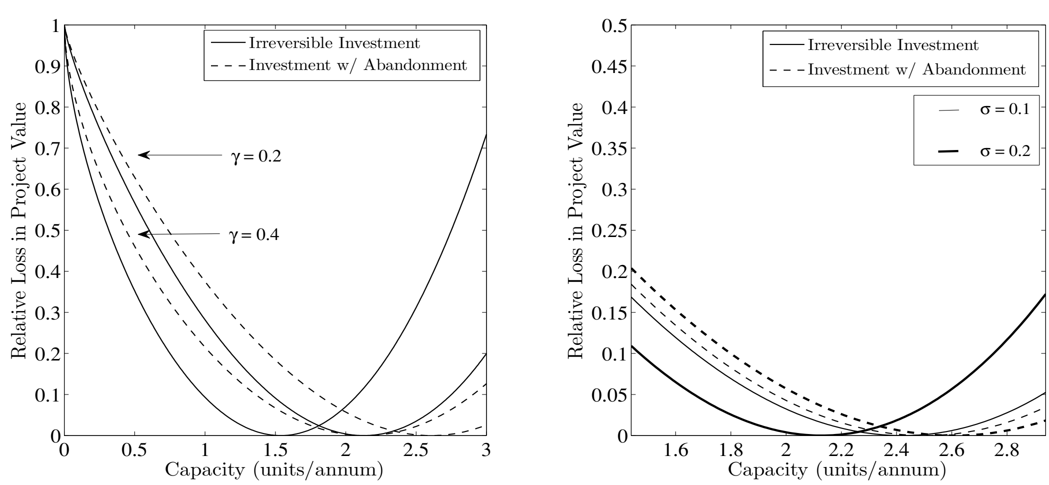


Figure 3.14: Relative loss in project value due to fixed capacity versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$ (right)

An embedded option to abandon an active project in case the output price drops increases the value of the investment opportunity and decreases the required investment threshold. Figure 3.15 shows the impact of the embedded abandonment option on the value of the project and the value of the option to invest, as well as the relative increase in option value evaluated at P_0 due to the embedded abandonment option. In the graph on the left, for $\sigma = 0.2$ and $\gamma = 0.2$, we observe that the embedded abandonment option increases the value of the investment opportunity as well as the value of the active project, thereby increasing the firm's incentive to invest in the project. The graph on right shows that the relative increase in option value due to the embedded abandonment option is more significant for higher levels of uncertainty and risk aversion. This implies that as risk aversion and uncertainty increase, the option to abandon an active project becomes more valuable.

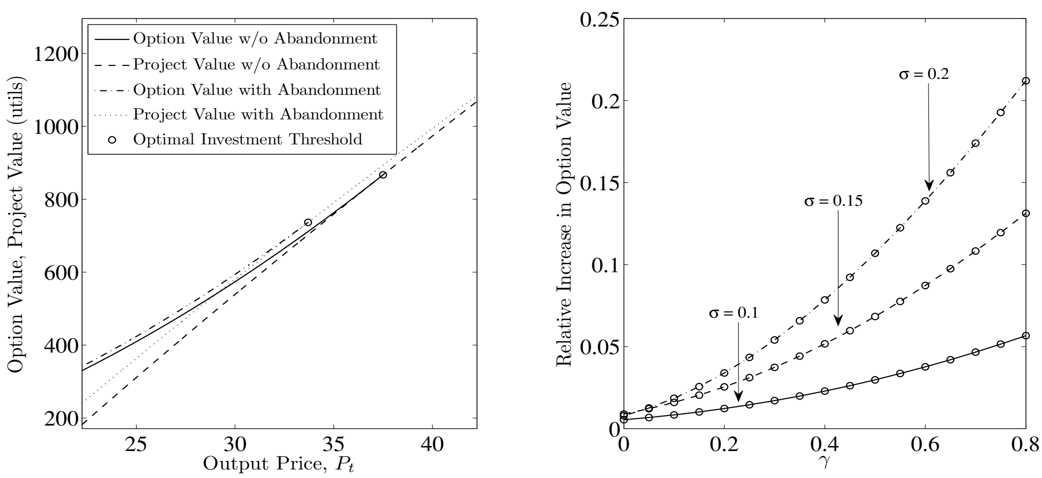


Figure 3.15: Impact of abandonment option on the option to invest and the value of the project

Like in Section 3.3.1, both the optimal investment threshold and the optimal capacity decrease with risk aversion and increase with volatility, as in Figure 3.16. The graph on the left indicates that the optimal investment threshold has decreased compared to the case of irreversible investment because the embedded abandonment option increases the value of the investment opportunity, thereby increasing the incentive to invest. Although at a common investment threshold, an embedded abandonment option increases the project's optimal capacity, allowing for the option to delay investment, increased operational flexibility facilitates investment, thus leading to the installation of a project with smaller capacity, as the graph on the right indicates.

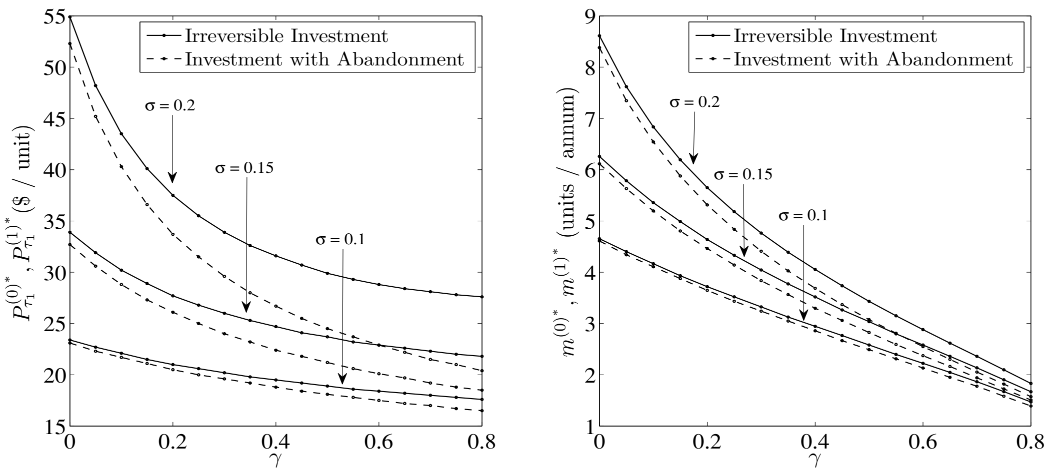


Figure 3.16: Optimal investment threshold and optimal capacity versus risk aversion and uncertainty

According to (3.17), since the embedded abandonment option facilitates investment, thereby lowering the required investment threshold and the corresponding optimal capacity, discretion over capacity becomes less valuable when $m^{(1)} < m^{(1)^*}$ and more valuable when $m^{(1)} > m^{(1)^*}$, as illustrated in Figure 3.17. Like in Section 3.3.1, increased risk aversion reduces the optimal capacity of the project and, as a result, the relative loss in option value diminishes for $m^{(1)} < m^{(1)^*}$ and increases for $m^{(1)} > m^{(1)^*}$. On the other hand uncertainty has the opposite effect as it delays investment thereby increasing the required investment threshold and the corresponding optimal capacity.

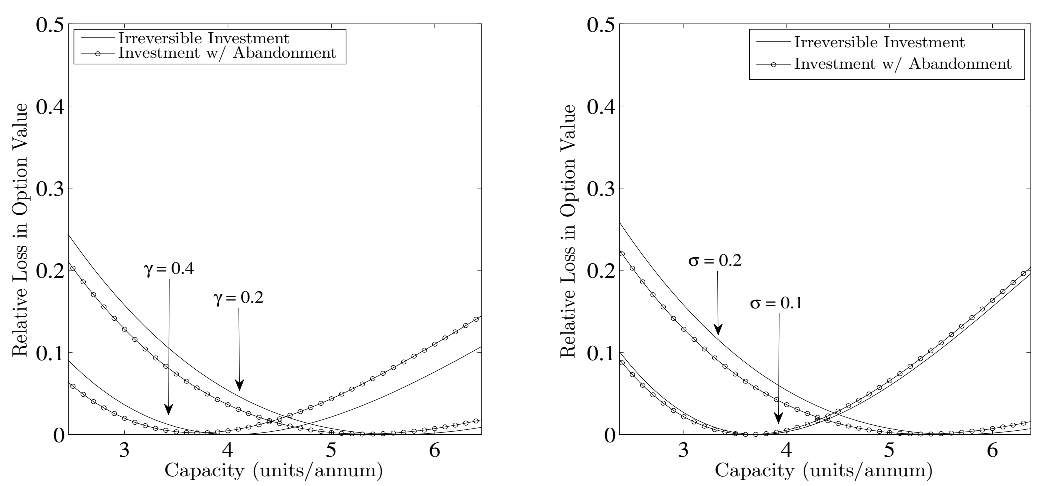


Figure 3.17: Relative loss in option value due to fixed capacity versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$ (right)

3.3.3 Capacity Choice for an Investment Opportunity with a Single Suspension and Resumption Option

In Figure 3.18, we illustrate the impact of the additional resumption option on the $\widetilde{MB}$ and $\widetilde{MC}$ of increasing capacity for the now-or-never investment case. Since, at resumption, the expected utility of the revenues is greater than that of the operating cost, the impact of the embedded resumption option on the $\widetilde{MB}$ of increasing capacity is more profound. As a result, the embedded resumption option increases the marginal utility of the investment payoff and, in turn, the optimal capacity.

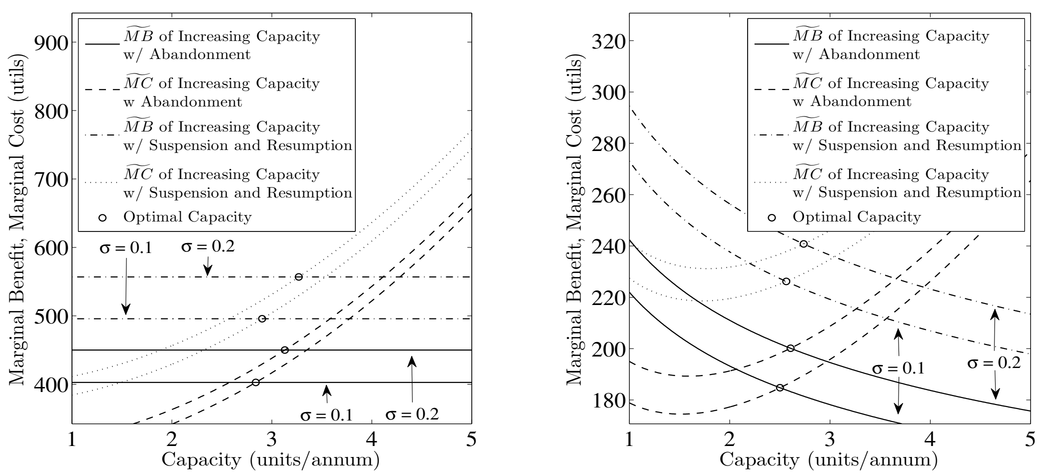


Figure 3.18: $\widetilde{MB}$ and $\widetilde{MC}$ of increasing capacity under risk neutrality (left) and risk aversion, i.e., $\gamma = 0.2$ (right), for a now-or-never investment opportunity with a single suspension and resumption option

Similar to the case of investment with abandonment, the optimal capacity of the project increases with uncertainty under low risk aversion and decreases under high risk aversion as in Figure 3.19. However, we observe now that uncertainty increases the optimal capacity for a larger range of values of the risk aversion parameter. This happens because as operational flexibility increases, the impact of uncertainty on the embedded options becomes more profound than on the value of the active project.

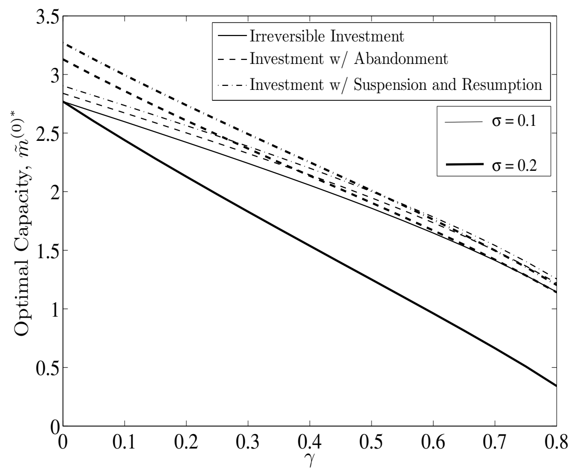


Figure 3.19: Optimal capacity versus risk aversion for a now-or-never investment opportunity with a single suspension and resumption option

As a resumption option increases the optimal capacity of the project further, relative to the case of investment with abandonment, discretion over capacity is now more valuable when, in both cases, the amount of installed capacity is lower than the optimal one and relatively less valuable when it is greater. Like in Section 3.3.2, increased risk aversion lowers the amount of installed capacity, and, as a result, the relative loss in project value diminishes for $\tilde{m}^{(2)} < \tilde{m}^{(2)^*}$ and increases for $\tilde{m}^{(2)} > \tilde{m}^{(2)^*}$. On the other hand, uncertainty has the opposite effect as it raises the optimal capacity and, as a result, the relative loss in project value increases for $\tilde{m}^{(2)} < \tilde{m}^{(2)^*}$ and diminishes for $\tilde{m}^{(2)} > \tilde{m}^{(2)^*}$.

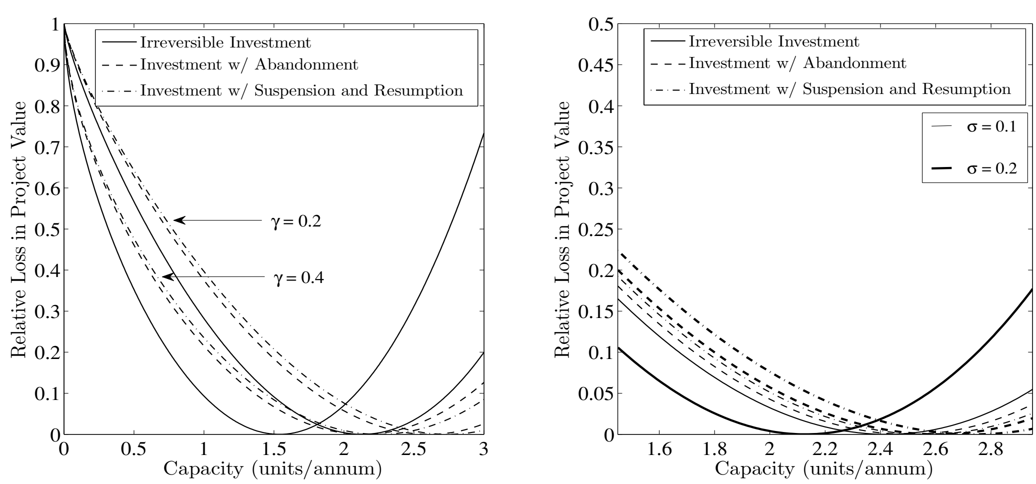


Figure 3.20: Relative loss in project value due to fixed capacity versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$ (right)

An embedded option to resume a currently suspended project increases the value of the option to invest as well as the value of the active project further. Greater operational flexibility increases the firm's incentive to invest, thereby lowering the required investment threshold further. This is illustrated in Figure 3.21, where the graph on the left indicates the further increase of the value of the investment opportunity as well as the further decrease of the optimal investment threshold for $\sigma = 0.2$ and $\gamma = 0.2$. The graph on the right shows that the impact of the embedded resumption option on the value of the option to invest is more profound for higher levels of uncertainty and risk aversion.

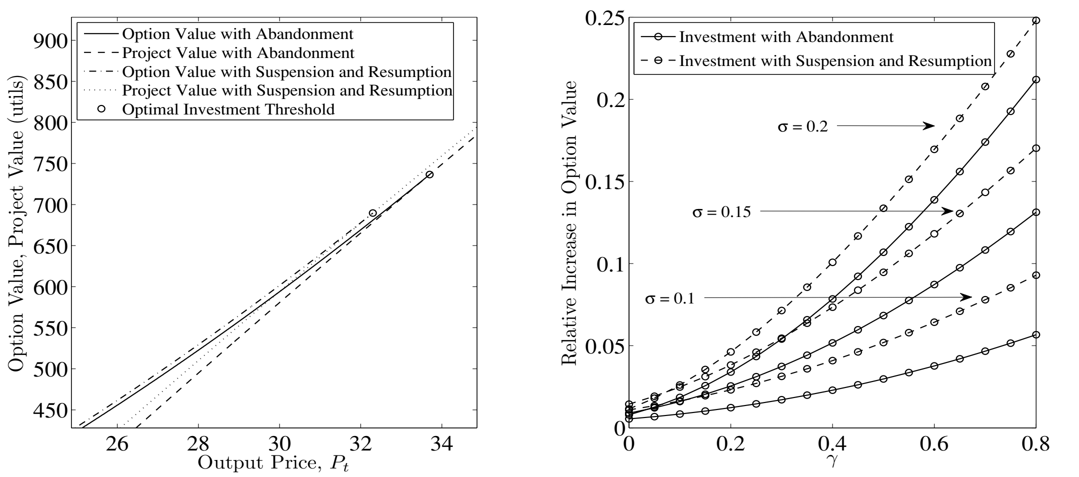


Figure 3.21: Impact of resumption option on the project value and the value of the option to invest

As Figure 3.22 indicates, relative to the case of investment with abandonment, discretion over capacity becomes less valuable for $m^{(2)} < m^{(2)^*}$ and $m^{(1)} < m^{(1)^*}$, and more valuable for $m^{(2)} > m^{(2)^*}$ and $m^{(1)} > m^{(1)^*}$, as the option to resume operations lowers the required investment threshold and the corresponding optimal capacity further. Again, risk aversion diminishes the relative loss in option value for $m^{(2)} < m^{(2)^*}$ and increases it for $m^{(2)} > m^{(2)^*}$ by reducing the optimal capacity, while increasing uncertainty has the opposite effect by increasing the optimal capacity.

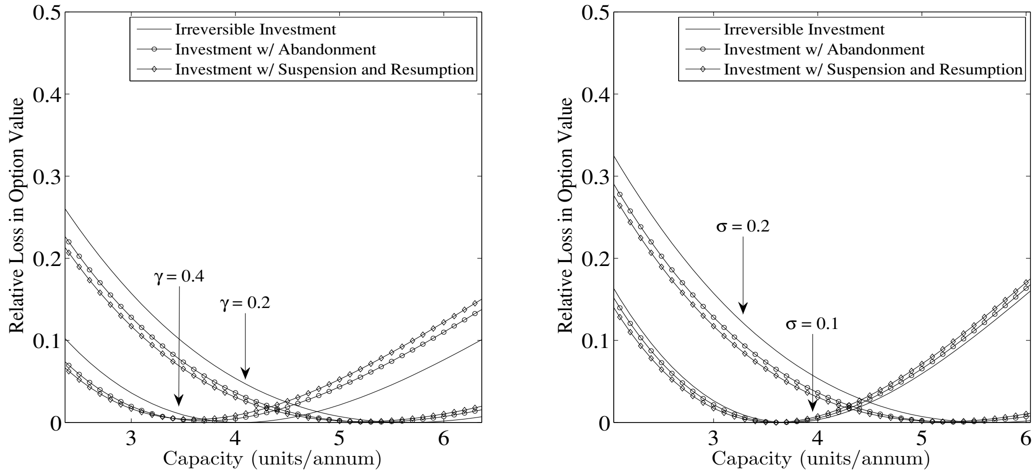


Figure 3.22: Relative loss in option due to fixed capacity value versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$ (right)

Like the case of investment with a single abandonment option, an embedded resumption option increases the value of the investment opportunity further, thereby increasing the firm's incentive to invest. As a result, the optimal investment threshold and the corresponding optimal capacity decrease further, as illustrated in Figure 3.23. Again, with discretion over capacity, risk aversion creates an incentive to reduce the cost of investment, thereby resulting in a reduced optimal capacity as well as a lower optimal investment threshold.

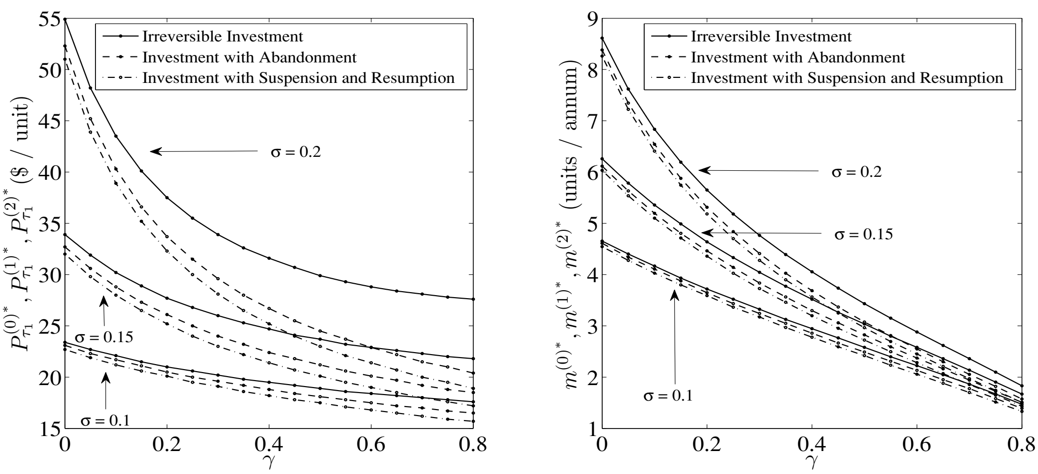


Figure 3.23: Optimal investment threshold and optimal capacity versus risk aversion and uncertainty

3.3.4 Capacity Choice under Complete Operational Flexibility

With complete operational flexibility, the expected utility of the investment's payoff increases further, thereby increasing the incentive to install more capacity when investing immediately, as shown in Figure 3.24. This, in turn, increases the value of discretion over capacity, relative to the cases of investment with finite flexibility, for $\tilde{m}^{(\infty)} < \tilde{m}^{(\infty)*}$ and $\tilde{m}^{(2)} < \tilde{m}^{(2)*}$ and decreases it for $\tilde{m}^{(\infty)} > \tilde{m}^{(\infty)*}$ and $\tilde{m}^{(2)} > \tilde{m}^{(2)*}$. Again, like in Sections 3.3.2 and 3.3.3, risk aversion reduces the relative loss in project value for $\tilde{m}^{(\infty)} < \tilde{m}^{(\infty)*}$ by lowering the amount of installed capacity, while uncertainty has the opposite effect by increasing the optimal capacity of the project.

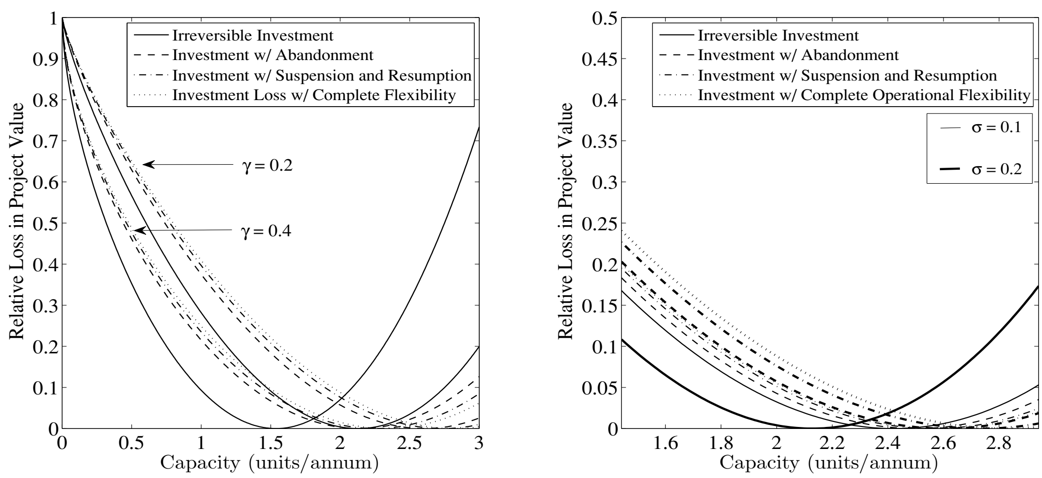


Figure 3.24: Relative loss in project value due to fixed capacity versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$ (right)

Infinite suspension and resumption options increase the value of the investment opportunity further creating an even greater incentive to invest. As Figure 3.25 illustrates, under complete operational flexibility the value of the option to invest and the value of the active project is greater compared to the case of investment with a single suspension and resumption option. Consequently, the increased incentive to invest leads to a further decrease of the optimal investment threshold.

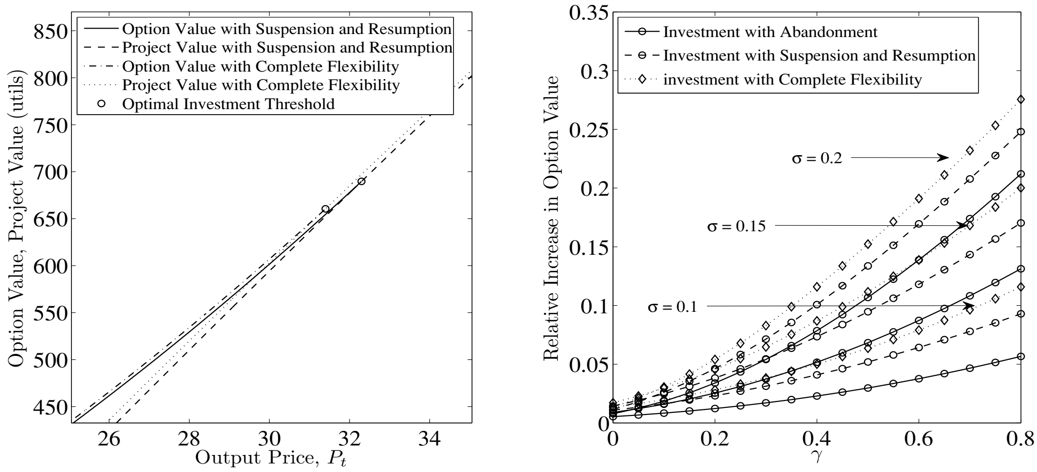


Figure 3.25: Option value and project value versus P_t under complete operational flexibility

Similar to Sections 3.3.2 and 3.3.3, the relative loss in option value due to fixed capacity increases further when the capacity installed is lower than the optimal and diminishes when it is greater as shown in Figure 3.26. The impact of risk aversion and uncertainty is the same as in Sections 3.3.2 and 3.3.3 since the former reduces the optimal capacity by decreasing the expected utility of the project and the latter increases it by delaying investment.

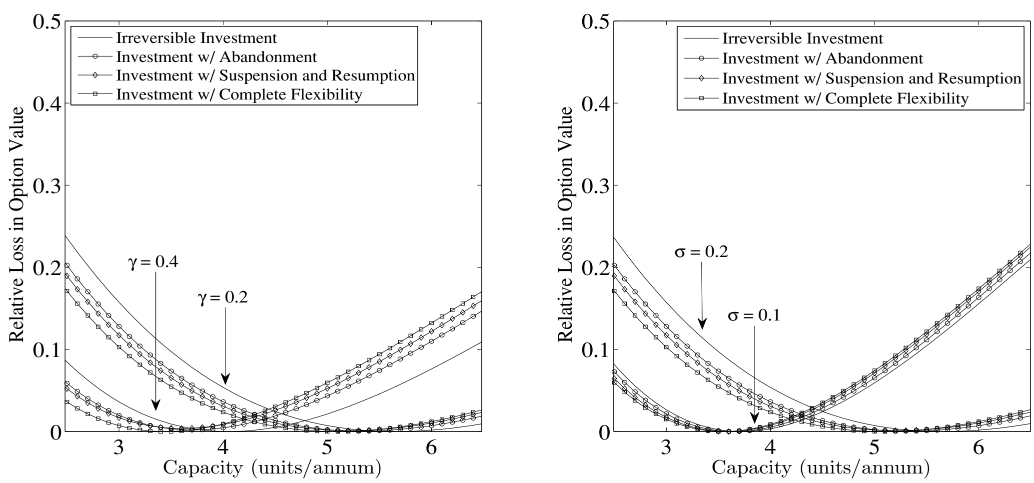


Figure 3.26: Relative loss in option value due to fixed capacity versus γ for $\sigma = 0.2$ (left) and versus σ for $\gamma = 0.2$ (right)

Like in Sections 3.3.2 and 3.3.3, by increasing the level of flexibility the value of the investment opportunity increases, thereby motivating the firm to require a lower investment threshold. This, in turn, leads to the further decrease of the project's optimal capacity as Figure 3.27 illustrates.

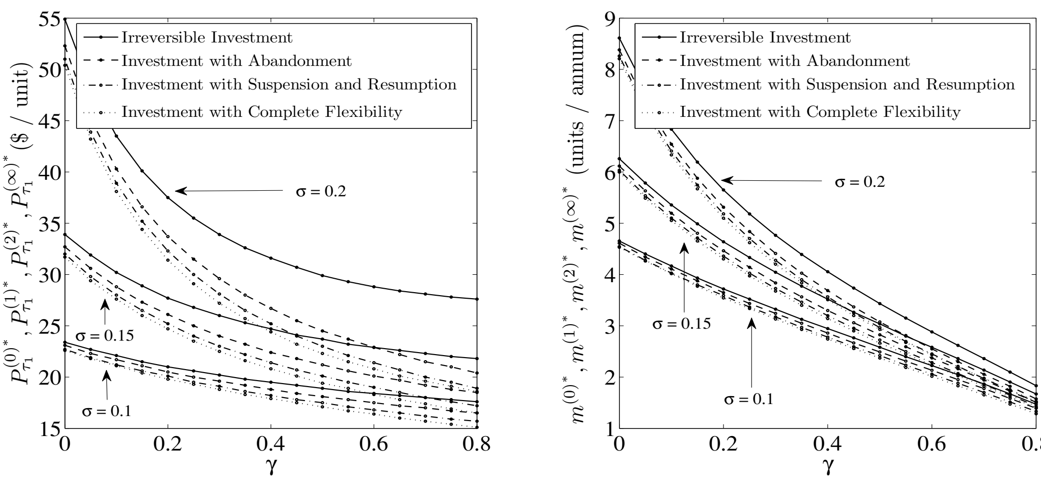


Figure 3.27: Optimal investment threshold and optimal capacity versus risk aversion, under complete operational flexibility

3.4 Conclusions

In a world of increasing economic uncertainty, the need to examine the interaction between risk aversion and operational flexibility, so as to provide optimal investment, operational, and sizing decisions, is of great essence. The main contribution of this chapter is that it illustrates how risk aversion can impact the optimal investment strategy in a very different way than what the traditional literature has so far indicated. We show that risk aversion facilitates investment (unlike the case with fixed capacity) and reduces the optimal capacity of the project. Furthermore, uncertainty now affects the now-or-never optimal capacity in a non-monotonic manner that is not captured by Dangl (1999). We also find that, with the option to delay investment, discretion over capacity becomes more important with increasing uncertainty and less important with higher risk aversion when the level of capacity installed is lower than the optimal one. When investing immediately in a project without operational flexibility, discretion over capacity is more important with increasing risk aversion and uncertainty, however, with operational flexibility, the impact of uncertainty on the value of discretion over capacity is ambiguous. Finally, we illustrate how operational flexibility increases the value of the investment opportunity and, in turn, the incentive to invest, thereby resulting in the decrease of the optimal capacity.

So far, we have examined how the optimal investment, operational, and sizing decisions of a single firm are affected by risk aversion, uncertainty, and operational flexibility without taking into account the presence of other firms that may want to enter the market. This is the objective of Chapter 4, where we analyse how duopolistic competition impacts the entry decision of firms in pre-emptive and non-pre-emptive duopoly settings.

Chapter 4

Duopolistic Competition under Risk Aversion and Uncertainty

Due to the deregulation of many sectors of the economy, decision rules for managing capital projects should consider not only uncertainty in the underlying variables but also competition in the output market. For example, in Europe, ever since the euro was introduced, there has been an increase in competition in sectors such as transport, energy, and telecommunications, which only a decade ago were the preserve of state monopolies. Furthermore, the ongoing process of mergers and takeovers as well as legislation against monopolies justifies the existence of and development toward more competitive markets. Indicative of this situation is the new partnership between Nokia and Microsoft. This alliance was the result of tough competition due to which Nokia lost its leadership in the area of smartphone operating system shipments to Android (and, in turn, market share to rivals such as Google and Apple) and risk aversion due to costs of financial distress (The Wall Street Journal, 2011). Another example is from the energy sector where the natural gas industry is undergoing significant changes as European legislation regarding competition is forcing gas companies to restructure their business and make room for new entrants, thus leading to increased competition (Independent Energy Review, 2010).

Canonical real options theory finds particular application in such sectors as it facilitates the analysis of capital budgeting decisions by accounting for the flexibility embedded in them. However, treatment of such decision-making problems via canonical real options theory has mainly been under monopoly or perfect competition. Moreover, recent work that considers a duopolistic setting has assumed risk neutrality. In this chapter, we extend the traditional real options approach to strategic decision making under uncertainty by examining how duopolistic competition affects the entry of a risk-averse firm. We consider two identical firms that are risk averse and hold an option each to invest in a project that yields stochastic revenues. The firms face the same output market, and, as a result, investment decisions of one firm impact the revenues of both firms. We begin by analysing the monopolistic case and then extend this framework by adding one more firm assuming either a pre-emptive or a non-pre-emptive setting. In the pre-emptive duopoly, both firms have the incentive to invest in order to obtain the leader's advantage, while in the non-pre-emptive duopoly, the role of the leader is assigned exogenously. For each setting, we analyse the impact of uncertainty and risk aversion on the optimal investment timing decisions of the two competing firms and examine the degree to which the presence of a competitor impacts the entry of a risk-averse firm. Hence, the contribution of this chapter is threefold. First, we develop a theoretical framework for analysing investment under uncertainty and risk aversion for a monopoly as well as pre-emptive and non-pre-emptive duopolies in order to derive closed-form expressions where possible for the optimal investment thresholds. Second, we quantify the degree to which competition impacts the strategic investment decisions of a risk-averse rival. Finally, we provide managerial insights for investment decisions and relative firm values under each setting based on analytical and numerical results.

We proceed by formulating the problems in Section 4.1. In Section 4.2, we solve the problems and analyse the impact of uncertainty and risk aversion on the optimal investment timing decisions of the two competing firms in each setting. In Section 4.3, we provide numerical examples for each case in order to examine the effects of volatility and risk aversion on the optimal investment timing decisions and quantify the degree to which the entry of the risk-averse firm is affected by the presence of a rival. We also illustrate the interaction between risk aversion and uncertainty and present managerial insights to enable more informed investment decisions. Section 4.4 concludes by summarising the results and offering directions for future research.

4.1 Problem Formulation

4.1.1 Assumptions and Notation

Assume that each firm i, $i = 1, 2$, can incur an investment cost, K, in order to start a project that produces output forever with no operating cost. Time is continuous and denoted by t, and the revenue received from the project at time $t \geq 0$ is $R_t = P_t D(Q_t)$ (\$/annum). Here, Q_t denotes the number of firms in the industry, i.e., $Q_t = 0, 1, 2$, and $D(Q_t)$ is a strictly decreasing function reflecting the quantity demanded from each firm per annum. We assume that the price per unit of the project's output, P_t, follows a GBM:

$$dP_t = \mu P_t dt + \sigma P_t dZ_t, \ P_0 > 0 \tag{4.1}$$

where $\mu \geq 0$ is the growth rate of P_t, $\sigma \geq 0$ is the volatility of P_t, and dZ_t is the increment of the standard Brownian motion. Also, we denote by $r \geq 0$ the risk-free discount rate and by $\rho \geq \mu$ the subjective discount rate. Let τ_i^j be the time at which firm j, $j = \ell, f$ (denoting leader or follower, respectively), enters the industry given market structure $i = m, p, n$ (denoting monopoly, pre-emptive duopoly, or non-pre-emptive duopoly, respectively), i.e.,

$$\tau_i^j \equiv \min\left\{t \geq 0 : P_t \geq P_{\tau_i^j}\right\} \tag{4.2}$$

where $P_{\tau_i^j}$ is the corresponding output price. Finally, we denote by $F_{\tau_i^j}(P_0)$ the expected value of firm j's investment opportunity under market structure i that is exercised at time τ_i^j and by $V_i^j(P_0)$ the expected NPV of firm j given the initial output price, P_0.

In order to account for risk aversion, we assume that the preferences of both firms are described by an identical increasing and concave utility function, $U(\cdot)$. As a result, our analysis can accommodate a wide range of utility functions, such as HARA, CARA, and CRRA utility functions. In our analysis, we apply a CRRA utility function as in Hugonnier and Morellec (2007) defined as follows:

$$U(P_t) = \begin{cases} \frac{P_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \geq 0 \ \& \ \gamma \neq 1 \\ \ln(P_t) & \text{if } \gamma = 1 \end{cases} \tag{4.3}$$

4.1.2 Monopoly

We begin by formulating the problem for the case of monopoly, where a single firm starts a perpetually operating project at a random time τ_m^j. Up to time τ_m^j, the monopolist invests K in a risk-free bond and earns an instantaneous cash flow of rK per time unit with utility $U(rK)$ discounted at her subjective rate of time preference, $\rho > \mu$. At τ_m^j, when the output price is $P_{\tau_m^j}$, the monopolist swaps this risk-free cash flow for a risky one, $P_t D(1)$, with utility $U(P_t D(1))$ as illustrated in Figure 4.1.

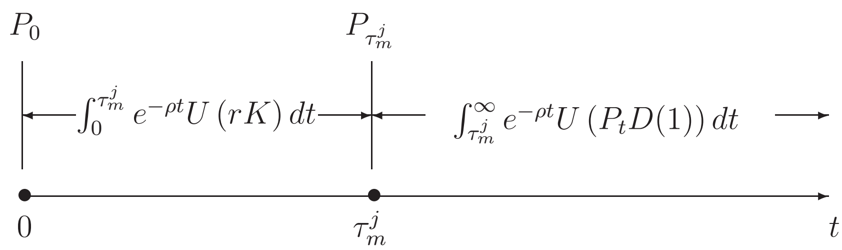


Figure 4.1: Investment under risk aversion for a monopoly

The conditional expected utility of the cash flows discounted to time $t = 0$ is:

$$\mathbb{E}_{P_0}\left[\int_0^{\tau_m^j} e^{-\rho t} U(rK)\, dt + \int_{\tau_m^j}^{\infty} e^{-\rho t} U(P_t D(1))\, dt\right] = \int_0^{\infty} e^{-\rho t} U(rK)\, dt + \mathbb{E}_{P_0}\left[e^{-\rho \tau_m^j}\right] V_m^j\left(P_{\tau_m^j}\right) \qquad (4.4)$$

where,

$$V_m^j\left(P_{\tau_m^j}\right) = \mathbb{E}_{P_{\tau_m^j}}\left[\int_0^{\infty} e^{-\rho t}\left[U(P_t D(1)) - U(rK)\right] dt\right] \qquad (4.5)$$

is the expected utility of the project's cash flows discounted to τ_m^j, and the monopolist's objective is to maximise the discounted expected utility of the project's cash flows, i.e., $\mathbb{E}_{P_0}\left[e^{-\rho \tau_m^j}\right] V_m^j\left(P_{\tau_m^j}\right)$. Here, $\mathbb{E}_{P_0}$ denotes the expectation operator, which is conditional on the initial value of the price process.

4.1.3 Duopoly

4.1.3.1 Pre-Emptive Duopoly

We extend the previous framework by adding one more firm to the industry. Since here the roles of the leader and the follower are defined endogenously, the two firms are fighting for the leader's position, and, therefore, each one of them runs the risk of pre-emption. The firm that enters the market first is the leader, and the firm that enters second is the follower as shown in Figure 4.2.

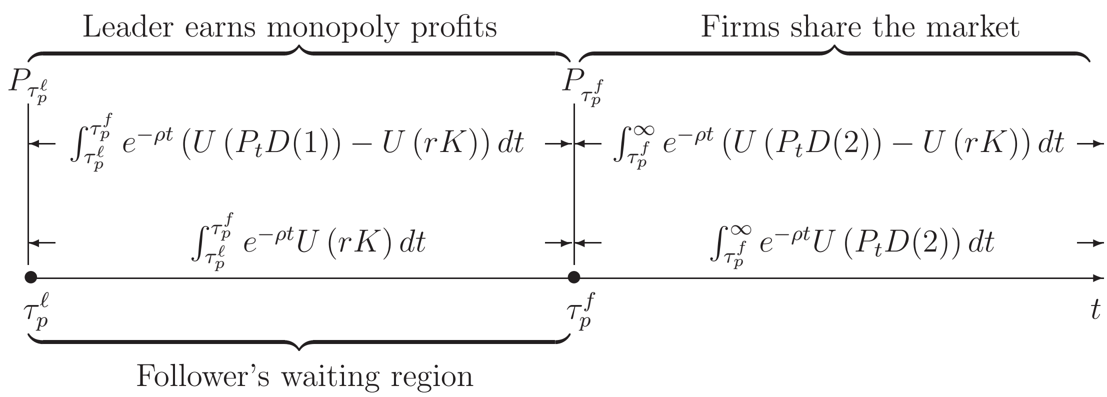


Figure 4.2: Investment under risk aversion for a pre-emptive duopoly

Consequently, the conditional expected utility of all future cash flows of the follower discounted to $t = \tau_p^\ell$ is:

$$\mathbb{E}_{P_{\tau_p^\ell}}\left[\int_{\tau_p^\ell}^{\tau_p^f} e^{-\rho t}U\left(rK\right)dt + \int_{\tau_p^f}^{\infty} e^{-\rho t}U\left(P_t D(2)\right)dt\right] = \int_{\tau_p^\ell}^{\infty} e^{-\rho t}U\left(rK\right)dt + \mathbb{E}_{P_{\tau_p^\ell}}\left[e^{-\rho\left(\tau_p^f-\tau_p^\ell\right)}\right] V_p^f\left(P_{\tau_p^f}\right) \quad (4.6)$$

where,

$$V_p^f\left(P_{\tau_p^f}\right) = \mathbb{E}_{P_{\tau_p^f}}\left[\int_0^{\infty} e^{-\rho t}\left[U\left(P_t D(2)\right)-U\left(rK\right)\right]dt\right] \quad (4.7)$$

is the expected utility of the project's cash flows discounted to τ_p^f, and, like the monopoly case, the scope of the pre-emptive follower is to maximise the discounted to τ_p^ℓ expected utility of the project's cash flows, i.e., $\mathbb{E}_{P_{\tau_p^\ell}}\left[e^{-\rho\left(\tau_p^f-\tau_p^\ell\right)}\right] \times$

$V_p^f\left(P_{\tau_p^f}\right)$.

Next, the conditional expected utility of all future cash flows of the leader discounted to $t = \tau_p^\ell$ is:

$$\begin{aligned} V_p^\ell\left(P_{\tau_p^\ell}\right) &= \mathbb{E}_{P_{\tau_p^\ell}}\left[\int_{\tau_p^\ell}^{\tau_p^f} e^{-\rho t}\left[U(P_t D(1)) - U(rK)\right] dt \right. \\ &\qquad \left. + \int_{\tau_p^f}^{\infty} e^{-\rho t}\left[U(P_t D(2)) - U(rK)\right] dt\right] \\ &= V_m^j\left(P_{\tau_p^\ell}\right) + \mathbb{E}_{P_{\tau_p^\ell}}\left[e^{-\rho\left(\tau_p^f - \tau_p^\ell\right)}\right] \times \\ &\qquad \mathbb{E}_{P_{\tau_p^f}}\left[\int_0^{\infty} e^{-\rho t}\left[U(P_t D(2)) - U(P_t D(1))\right] dt\right] \end{aligned} \tag{4.8}$$

Notice that up to time τ_p^f, the leader enjoys monopolistic profits as in (2.4), while after the entry of the follower the two firms share the market, as illustrated in Figure 4.2. This implies that, although up to time τ_p^f the leader is alone in the market, her value function does not correspond to that of a monopolist since the future entry of the follower reduces the expected utility of the leader's profits. This reduction is reflected by the second term on the right-hand side of (4.8), which is negative since $D(2) < D(1)$.

4.1.3.2 Non-Pre-Emptive Duopoly

Here, the roles of the leader and the follower are defined exogenously. Consequently, the future cash flows of the leader are discounted to time $t = 0$. Since the follower considers entry into the market assuming that the leader has already invested, the future cash flows of the follower are discounted to τ_n^ℓ as illustrated in Figure 4.3. The conditional expected utility of the follower's cash flows is the same as in the pre-emptive case but discounted to τ_n^ℓ, i.e.,

$$\int_{\tau_n^\ell}^{\infty} e^{-\rho t} U(rK)\, dt + \mathbb{E}_{P_{\tau_n^\ell}}\left[e^{-\rho\left(\tau_n^f - \tau_n^\ell\right)}\right] V_n^f\left(P_{\tau_n^f}\right) \tag{4.9}$$

where $V_n^f(\cdot) = V_p^f(\cdot)$ and the objective of the follower is to maximise the expression $\mathbb{E}_{P_{\tau_n^\ell}}\left[e^{-\rho\left(\tau_n^f - \tau_n^\ell\right)}\right] \times V_n^f\left(P_{\tau_n^f}\right)$.

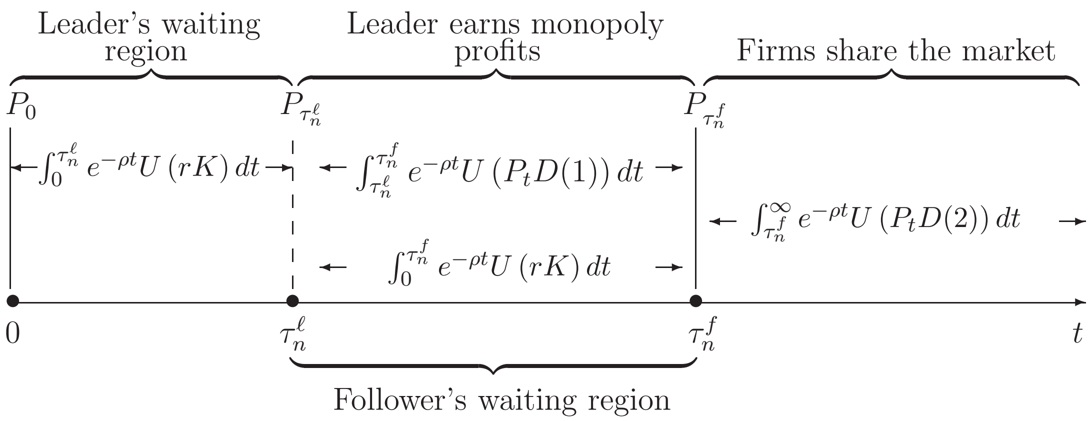


Figure 4.3: Investment under risk aversion for a non-pre-emptive duopoly

The leader now knows that she has the right to enter the market first and, therefore, does not run the risk of pre-emption. As a result, the expected utility of the leader's future cash flows discounted to $t = 0$ is:

$$\begin{aligned}
&\mathbb{E}_{P_0}\left[\int_0^{\tau_n^\ell} e^{-\rho t}U(rK)\,dt + \int_{\tau_n^\ell}^{\tau_n^f} e^{-\rho t}U(P_tD(1))\,dt + \int_{\tau_n^f}^{\infty} e^{-\rho t}U(P_tD(2))\,dt\right] \\
&= \int_0^{\infty} e^{-\rho t}U(rK)\,dt + \mathbb{E}_{P_0}\left[e^{-\rho\tau_n^\ell}\right]V_p^\ell\left(P_{\tau_n^\ell}\right) \qquad (4.10)
\end{aligned}$$

where $V_p^\ell(\cdot)$ is defined as in (4.8). Here, the objective of the leader is to maximise $\mathbb{E}_{P_0}\left[e^{-\rho\tau_n^\ell}\right]V_p^\ell\left(P_{\tau_n^\ell}\right)$.

4.2 Analytical Results

4.2.1 Monopoly

In this case, there is a single firm in the market that contemplates investment without the fear of pre-emption from the entry of a competitor. Consequently, the firm has the option to delay investment until the output price hits the optimal threshold, $P_{\tau_m^{j*}}$, that will trigger investment. Hence, for $P_0 \leq P_{\tau_m^{j*}}$, (4.11) indicates the value of the monopolist's investment opportunity:

$$
\begin{aligned}
F_{\tau_m^j}(P_0) &= \sup_{\tau_m^j \in \mathcal{S}} \mathbb{E}_{P_0}\left[\int_{\tau_m^j}^{\infty} e^{-\rho t}\left[U\left(P_t D(1)\right) - U\left(rK\right)\right] dt\right] \\
&= \sup_{\tau_m^j \in \mathcal{S}} \mathbb{E}_{P_0}\left[e^{-\rho \tau_m^j}\right] V_m^j\left(P_{\tau_m^j}\right) \qquad (4.11)
\end{aligned}
$$

Here, $\mathcal{S}$ denotes the collection of admissible stopping times of the filtration generated by the price process. Using Theorem 9.18 of Karatzas and Shreve (1999) for the CRRA utility function in (4.3), we find that the expression in (4.5) can be simplified using the following:

$$
\mathbb{E}_{P_0}\int_0^{\infty} e^{-\rho t} U\left(P_t\right) dt = \mathcal{A} U\left(P_0\right) \qquad (4.12)
$$

where $\mathcal{A} = \frac{\beta_1\beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} > 0$, and $\beta_1 > 1$, $\beta_2 < 0$ are the solutions for x to the following quadratic equation:

$$
\frac{1}{2}\sigma^2 x(x-1) + \mu x - \rho = 0 \qquad (4.13)
$$

By using the fact that the expected discount factor is $\mathbb{E}_{P_0}\left[e^{-\rho\tau_m^j}\right] = \left(\frac{P_0}{P_{\tau_m^j}}\right)^{\beta_1}$ (Karatzas and Shreve, 1999) and applying the strong Markov property along with the law of iterated expectations, (4.11) can be written as follows:

$$
F_{\tau_m^j}(P_0) = \max_{P_{\tau_m^j} \geq P_0}\left(\frac{P_0}{P_{\tau_m^j}}\right)^{\beta_1}\left[\mathcal{A} U\left(P_{\tau_m^j} D(1)\right) - \frac{U(rK)}{\rho}\right] \qquad (4.14)
$$

Solving the unconstrained optimisation problem (4.14), we obtain the optimal investment threshold, $P_{\tau_m^{j*}}$, for the monopolist:

$$
P_{\tau_m^{j*}} = \frac{rK}{D(1)}\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \qquad (4.15)
$$

According to (4.15), uncertainty and risk aversion drive a wedge between the optimal investment threshold and the amortised investment cost. Indeed, it can be shown that higher risk aversion increases the required investment threshold by decreasing the expected utility of the investment's payoff, while increased uncertainty delays investment by increasing the value of waiting. All proofs can be found in the appendix.

Proposition 4.2.1 *Uncertainty and risk aversion increase the optimal investment threshold.*

4.2.2 Symmetric Pre-Emptive Duopoly

We solve this dynamic game backward by first assuming that the leader has just entered the market. The value of the follower at $\tau_p^\ell < \tau_p^f$ is indicated in (4.16):

$$\begin{aligned} F_{\tau_p^f}(P_{\tau_p^\ell}) &= \sup_{\tau_p^f \geq \tau_p^\ell} \mathbb{E}_{P_{\tau_p^\ell}} \left[e^{-\rho \tau_p^f} \right] V_p^f \left(P_{\tau_p^f} \right) \\ &= \max_{P_{\tau_p^f} \geq P_{\tau_p^\ell}} \left(\frac{P_{\tau_p^\ell}}{P_{\tau_p^f}} \right)^{\beta_1} V_p^f \left(P_{\tau_p^f} \right) \end{aligned} \tag{4.16}$$

Solving the unconstrained optimisation problem described by (4.16), we obtain the optimal threshold, $P_{\tau_p^{f*}}$, that triggers the entry of the follower:

$$P_{\tau_p^{f*}} = \frac{rK}{D(2)} \left(\frac{\beta_2 + \gamma - 1}{\beta_2} \right)^{\frac{1}{1-\gamma}} \tag{4.17}$$

Notice that since $D(2) < D(1)$, we have $P_{\tau_p^{f*}} > P_{\tau_m^{j*}}$, i.e., the optimal entry threshold of the pre-emptive follower is higher than that of the monopolist. Intuitively, this happens because the follower requires compensation for losing the first-mover advantage. After the critical threshold, $P_{\tau_p^{f*}}$, is hit, the value of the follower is the discounted expected utility of the project's cash flows, as indicated by (4.7).

Assuming that the follower chooses the optimal policy, the value function of the leader for $P_{\tau_p^\ell} \leq P_t < P_{\tau_p^{f*}}$, i.e., when the leader is alone in the market, is:

$$\begin{aligned}
V_p^{\ell}(P_t) &= \mathbb{E}_{P_t}\left[\int_0^{\tau_p^{f^*}} e^{-\rho t}\left(U(P_t D(1)) - U(rK)\right) dt \right. \\
&\qquad\qquad \left. + \int_{\tau_p^{f^*}}^{\infty} e^{-\rho t}\left(U(P_t D(2)) - U(rK)\right) dt\right] \\
&= \mathcal{A}U\left(P_t D(1)\right) - \frac{U\left(rK\right)}{\rho} \\
&\qquad\qquad + \left(\frac{P_t}{P_{\tau_p^{f^*}}}\right)^{\beta_1} \mathcal{A}U\left(P_{\tau_p^{f^*}}\right)\left[D(2)^{1-\gamma} - D(1)^{1-\gamma}\right]
\end{aligned} \tag{4.18}$$

For $P_t \geq P_{\tau_p^{f^*}}$, the two firms share the market and, as a result, the value function of the leader is the same as the follower's.

As we show in Proposition 4.2.2, under a large discrepancy in market share, there exists a finite output price at which the pre-emptive leader's value function is maximised. Otherwise, the pre-emptive leader's value function is strictly increasing. Intuitively, a higher output price simultaneously increases the expected discounted utility of cash flows and facilitates the follower's entry. With a higher loss in market share, the impact of the latter effect dominates.

Proposition 4.2.2 *The value function of the pre-emptive leader is concave, and its maximum value is obtained prior to the entry of the pre-emptive follower provided that:*

$$D(2) < D(1)\left(\frac{\beta_1 + \gamma - 1}{\beta_1}\right)^{\frac{1}{1-\gamma}} \tag{4.19}$$

In order to determine the leader's optimal investment threshold, we need to consider the strategic interactions between the leader and the follower. Let $P_{\tau_p^{\ell^*}}$ denote the threshold price at which a firm is indifferent between becoming a leader or a follower. Recall that in the pre-emptive setting both firms want to enter first in order to obtain the leader's advantage. However, for $P_t < P_{\tau_p^{\ell^*}}$, the follower has not entered the market, and a firm would be better off being the follower since then $V_p^{\ell}(P_t) < F_{\tau_p^f}(P_t)$, while for $P_t > P_{\tau_p^{\ell^*}}$, a firm is better off being a leader since then $V_p^{\ell}(P_t) > F_{\tau_p^f}(P_t)$. Hence, it must be the case that $V_p^{\ell}\left(P_{\tau_p^{\ell^*}}\right) = F_{\tau_p^f}\left(P_{\tau_p^{\ell^*}}\right)$

for entry, a condition that is found numerically by solving the following equation:

$$\mathcal{A}U\left(P_{\tau_p^{\ell*}}D(1)\right)-\frac{U\left(rK\right)}{\rho}\quad+\quad\left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_p^{f*}}}\right)^{\beta_1}\mathcal{A}U\left(P_{\tau_p^{f*}}\right)\left[D(2)^{1-\gamma}-D(1)^{1-\gamma}\right]=$$
$$\left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_p^{f*}}}\right)^{\beta_1}\left[\mathcal{A}U\left(P_{\tau_p^{f*}}D(2)\right)-\frac{U(rK)}{\rho}\right]\quad(4.20)$$

Solving (4.20) for $P_{\tau_p^{\ell*}}$, we obtain the entry threshold of the leader that denotes the output price at which a firm is indifferent between becoming a leader or a follower. Indeed, as we show in Proposition 4.2.3, the optimal entry threshold of the pre-emptive leader is lower than that of the monopolist. This happens because the risk of pre-emption deprives the leader of the option to postpone investment, thereby lowering the required investment threshold.

Proposition 4.2.3 *The pre-emptive leader's optimal entry threshold is lower than that of the monopolist.*

Although increased risk aversion raises the required investment threshold by decreasing the expected utility of the investment's payoff, the loss in the value of the leader due to the entry of the follower, evaluated at $P_{\tau_p^{\ell*}}$, relative to that of the monopolist is not affected by risk aversion. Intuitively, the value of the leader at $P_{\tau_p^{\ell*}}$ equals the value of the follower's investment opportunity. Since both the follower and the monopolist hold a single option each to enter the market, increased risk aversion poses a proportional decrease in the option value of the follower relative to the monopolist.

Proposition 4.2.4 *The loss in the pre-emptive leader's value relative to the monopolist's value of investment opportunity at the pre-emptive leader's optimal entry threshold price is not affected by risk aversion.*

We next investigate how this ratio changes with uncertainty. In Figure 4.4, the horizontal lines represent the utility of the instantaneous revenues the leader receives over time under low uncertainty, σ, and under high uncertainty, σ'. As we will illustrate numerically, increased uncertainty raises the required entry threshold of the follower by more than that of the leader. This results in the increase of the expected utility of the leader's profits, represented by the shaded area of Figure 4.4, since, under higher uncertainty, she enjoys monopoly profits for longer time and the loss in the leader's expected utility due to the entry of the follower is not significant enough to offset it. In fact, this result is enhanced when the discrepancy in market share is large, since the greater $D(1)$ is, the greater the pre-emptive leader's incentive to invest will be as then the first-mover advantages are greater. Notice also that as greater uncertainty raises the required entry threshold of the follower, the leader's instantaneous revenues cannot drop below the level corresponding to σ' for $t \geq \tau_p^{f'}$.

Proposition 4.2.5 *The relative discrepancy between the value of the pre-emptive leader and the monopolist at the pre-emptive leader's optimal entry threshold price diminishes with increasing uncertainty.*

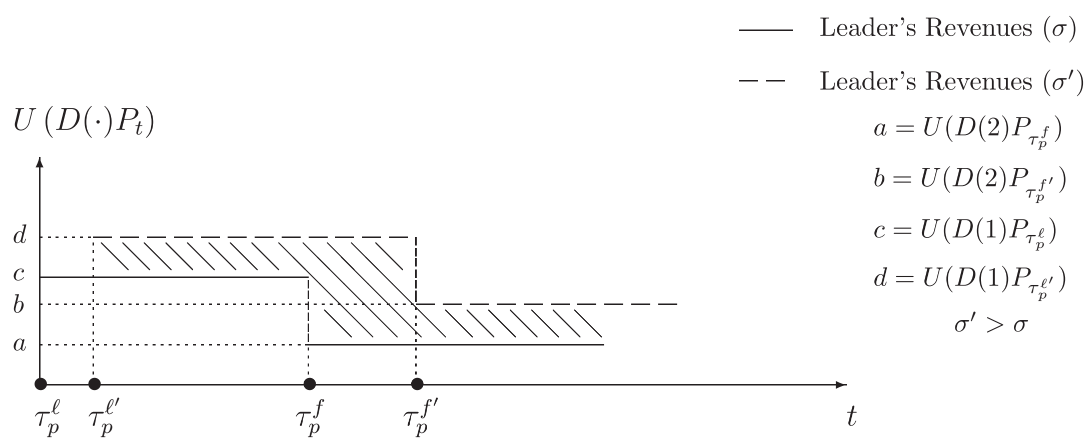


Figure 4.4: Incremental change in pre-emptive leader's instantaneous revenues due to increased uncertainty

4.2.3 Symmetric Non-Pre-Emptive Duopoly

In the non-pre-emptive setting, the roles of the leader and the follower are defined exogenously, and, as a result, both firms have the option to delay their entry into the market as the risk of pre-emption is eliminated. The follower's value function and entry threshold are unchanged from the pre-emptive case since she will still enter the market considering that the leader is already there. Hence, the follower's value of investment opportunity at τ_n^ℓ is:

$$F_{\tau_n^f}\left(P_{\tau_n^\ell}\right) = \max_{P_{\tau_n^f} \geq P_{\tau_n^\ell}} \left(\frac{P_{\tau_n^\ell}}{P_{\tau_n^f}}\right)^{\beta_1} V_n^f\left(P_{\tau_n^f}\right) \tag{4.21}$$

Since the non-pre-emptive leader has discretion over investment timing, her value of investment opportunity is described by:

$$\begin{aligned} F_{\tau_n^\ell}(P_0) &= \max_{P_{\tau_n^\ell} \geq P_0} \left(\frac{P_0}{P_{\tau_n^\ell}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_n^\ell} D(1)\right) - \frac{U(rK)}{\rho} + \right. \\ &\left. + \left(\frac{P_{\tau_n^\ell}}{P_{\tau_n^{f*}}}\right)^{\beta_1} \mathcal{A}U\left(P_{\tau_n^{f*}}\right)\left[D(2)^{1-\gamma} - D(1)^{1-\gamma}\right]\right] \end{aligned} \tag{4.22}$$

The solution to the optimisation problem (4.22) yields the optimal entry threshold of the non-pre-emptive leader:

$$P_{\tau_n^{\ell*}} = \frac{rK}{D(1)} \left(\frac{\beta_2 + \gamma - 1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \tag{4.23}$$

Notice that by delaying entry, the leader suffers from forgoing cash flows but benefits from temporarily delaying the entry of the follower. At the same time, allowing the project to start at a higher output price yields a higher NPV but then the leader enjoys monopoly revenues for less time. As it is shown in the appendix, the marginal benefit and marginal cost corresponding to the entry of the follower cancel.

Proposition 4.2.6 *The optimal entry threshold of the non-pre-emptive leader is the same as that of the monopolist.*

Notice that the leader's option to invest consists of the expected utility of the immediate payoff reduced by an amount corresponding to the expected loss in utility due to the entry of the follower. After the leader has entered the market and prior to the entry of the follower, i.e., for $P_{\tau_n^\ell} \leq P_t < P_{\tau_n^{f*}}$, the leader receives monopolistic profits with expected utility described by (4.24):

$$\mathcal{A}U\left(P_t D(1)\right) - \frac{U(rK)}{\rho} + \left(\frac{P_t}{P_{\tau_n^{f*}}}\right)^{\beta_1} \mathcal{A}U\left(P_{\tau_n^{f*}}\right)\left[D(2)^{1-\gamma} - D(1)^{1-\gamma}\right] \tag{4.24}$$

According to (4.24), although the leader is alone in the industry, the expected utility of her profits do not correspond to those of a monopolist since the potential entry of a rival reduces the expected utility of the leader's profits. Finally, after the follower's entry, i.e., for $t \geq \tau_n^f$, the two firms share the industry, thereby making equal profits, and their value is simply the discounted expected utility of the project's cash flows.

In the non-pre-emptive framework, the value of the leader would be the same as the monopolist's if it were not for the potential entry of the follower that reduces the expected utility of the leader's profits. However, the reduction in the leader's value of investment opportunity due to the potential entry of the follower decreases with risk aversion. This happens because risk aversion delays the entry of the follower, thereby reducing the expected loss in the option value of the leader. Consequently, the relative discrepancy between the leader's value of investment opportunity and the monopolist's diminishes with increasing risk aversion, thereby reducing the relative loss in the value of the non-pre-emptive leader.

Proposition 4.2.7 *The loss in the value of the investment opportunity for the non-pre-emptive leader relative to that of a monopolist at the pre-emptive leader's optimal entry threshold price decreases with risk aversion.*

According to Proposition 4.2.8, depending on the discrepancy in market share, uncertainty may increase or decrease the relative loss in the value of the investment opportunity for the non-pre-emptive leader relative to that of a monopolist. Notice that the value of the non-pre-emptive leader consists of the value of the monopolistic investment opportunity and the expected loss in project value due to the entry of the follower. Both of these components increase with uncertainty; however, for the latter, the impact of uncertainty becomes less profound as the discrepancy in market share diminishes. As a result, under low discrepancy in market share, the impact of uncertainty on the non-pre-emptive leader's value of monopolistic investment opportunity dominates, thereby making her better off. By contrast, under large discrepancy in market share, increased uncertainty causes the loss in project value to increase faster than the value of the investment opportunity, thereby making the non-pre-emptive leader worse off.

Proposition 4.2.8 *The discrepancy between the non-pre-emptive leader's value of investment opportunity and the monopolist's at the pre-emptive leader's optimal entry threshold price increases with uncertainty if:*

$$\left(\frac{D(1)}{D(2)}\right)^{\beta_1} > e, \;\; e \simeq 2.718 \tag{4.25}$$

In Figure 4.5, the instantaneous revenues of the leader are represented by the solid line for low uncertainty, σ, and by the broken line for high uncertainty, σ'. Here, unlike the pre-emptive setting, the leader has the option to delay entry into the market. Notice that a large discrepancy in market share implies a greater first-mover advantage but also leads to a greater loss in the value of the leader upon the entry of the follower, which becomes more profound with higher uncertainty. However, increased uncertainty also raises the value of the leader's investment opportunity, thereby creating an opposing effect. According to Proposition 4.2.8, under small discrepancy in market share, the increase in option value due to increased uncertainty, represented by the shaded area between τ_n^{ℓ} and $\tau_n^{\ell'}$ in Figure 4.5, offsets the loss in the leader's revenues due to the entry of the follower, thereby reducing the discrepancy between the value of the monopolist and the leader. The opposite result is observed if the discrepancy in market share is large, since then the loss in the leader's revenues is more profound than the increase in the value of her investment opportunity. This happens because a higher first-mover advantage reduces the required entry threshold of the leader. Consequently, the increase in the value of the investment opportunity is less profound, and as higher uncertainty impacts the loss in project value by more, the non-pre-emptive leader becomes worse off.

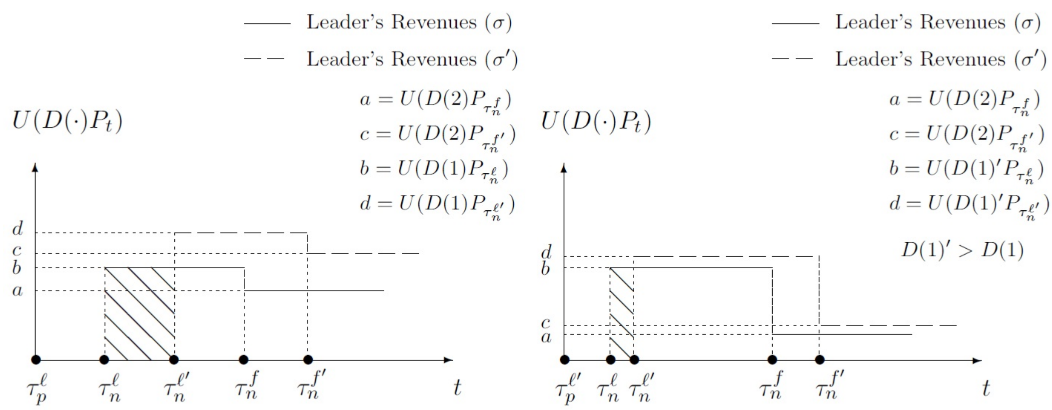


Figure 4.5: Incremental change in non-pre-emptive leader's instantaneous revenues due to increased uncertainty under low discrepancy in market share (left) and large discrepancy (right)

4.3 Numerical Results

4.3.1 Pre-Emptive Duopoly

In order to examine the impact of risk aversion and uncertainty on the entry of the pre-emptive leader and follower, we assume the following parameter values: $\gamma \in [0, 1)$, $\sigma \in [0.1, 0.5]$, $\mu = 0.01$, $r = \rho = 0.04$, $K = \$100$, $c = \$0$, $D(0) = 0$, $D(1) = 1.5$ or 3, and $D(2) = 1$. Figure 4.6 illustrates the impact of uncertainty on the value of the pre-emptive leader and follower under risk aversion. First, we observe that the leader's entry threshold is lower than the monopolist's. This happens due to pre-emption since the leader does not have the option to defer investment and, as a result, the risk of pre-emption reduces the required investment threshold. On the other hand, the required investment threshold of the pre-emptive follower is higher than that of the monopolist since the former requires compensation for losing the first-mover advantage. According to the graph on the right, uncertainty increases the value of waiting, thereby raising the required investment threshold and delaying the entry of the follower. This, in turn, increases the time interval in which the leader enjoys monopoly profits and diminishes the relative discrepancy between the value of the pre-emptive leader and that of the monopolist.

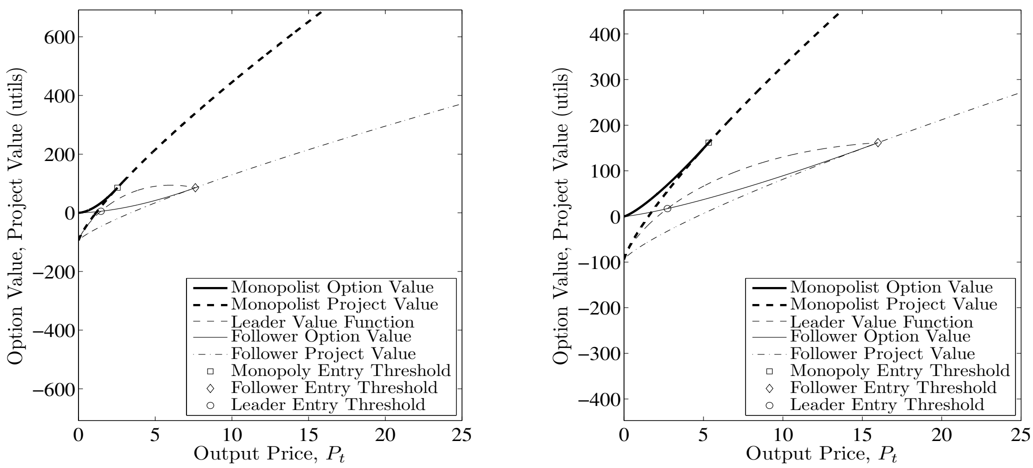


Figure 4.6: Project and investment opportunity value of monopolist, pre-emptive leader, and follower for $\sigma = 0.2$ (left) and $\sigma = 0.4$ (right) under risk aversion ($\gamma = 0.2$) for $D(1) = 3$

Figure 4.7 illustrates the impact of risk aversion on the value of the pre-emptive leader and follower. According to the graph on the right, increased risk aversion reduces the expected utility of the investment's payoff for both the leader and the monopolist, thereby raising their required investment thresholds. Furthermore, it seems that the impact of risk aversion on the pre-emptive leader's value is greater than on the follower's value. Consequently, the two curves intersect at a higher output price, thereby indicating that the output price at which a firm is indifferent between becoming a leader or a follower increases with higher risk aversion.

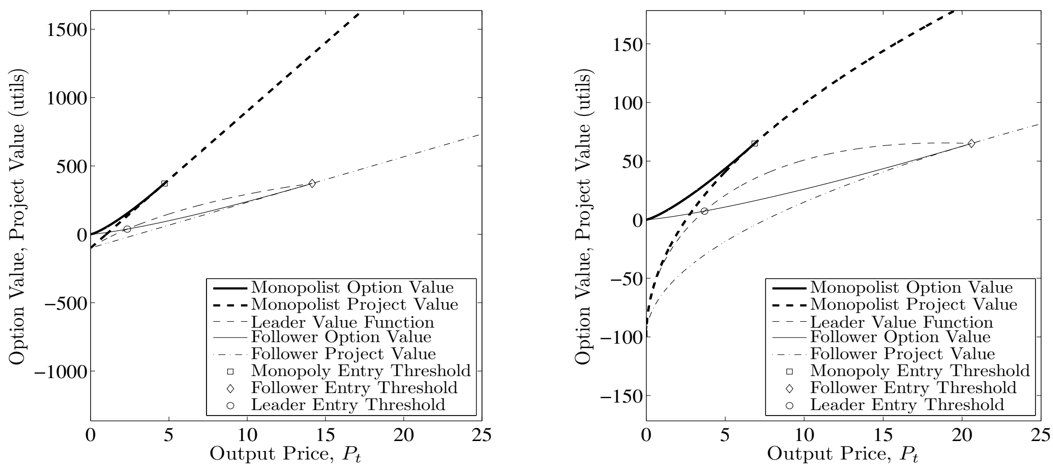


Figure 4.7: Investment opportunity and project value of monopolist, pre-emptive leader, and follower under risk neutrality (left) and risk aversion ($\gamma = 0.5$) (right) for $\sigma = 0.4$ and $D(1) = 3$

4.3.2 Non-Pre-Emptive Duopoly

In the non-pre-emptive duopoly, the roles of the leader and the follower are pre-assigned, and, as a result, both firms have the option to postpone their entry into the market. According to Figure 4.8, the optimal entry threshold of the non-pre-emptive follower is the same as in the pre-emptive case since the follower will still enter the market considering that the leader has already invested. Notice also that, the optimal entry threshold of the non-pre-emptive leader is the same as the monopolist's, and, as a result, the required investment threshold of the non-pre-emptive leader is higher than that in the pre-emptive scenario. Although the optimal entry threshold is the same for the monopolist and non-pre-emptive leader, the investment opportunity value of the latter is lower than that of the former since the potential entry of the follower reduces the expected utility of the leader's profits. As the graph on the right illustrates, increased uncertainty raises the value of waiting, which, in turn, postpones investment in all cases, thereby increasing the required investment thresholds.

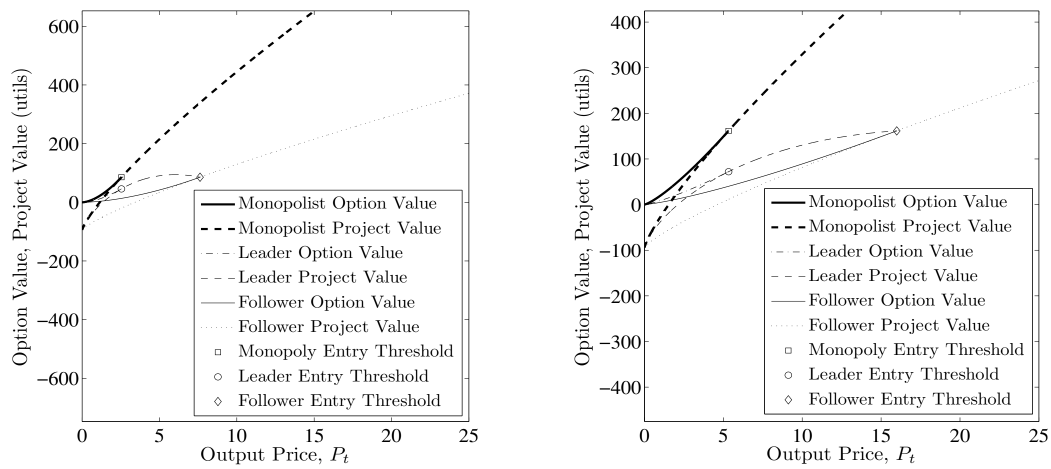


Figure 4.8: Project and investment opportunity value for non-pre-emptive leader and follower for $\sigma = 0.2$ (left) and $\sigma = 0.4$ (right) under risk aversion ($\gamma = 0.2$) for $D(1) = 3$

Figure 4.9 illustrates the impact of risk aversion on the optimal entry thresholds of the monopolist and the non-pre-emptive leader and follower. As indicated in the graphs, higher risk aversion reduces the expected utility of the investment's payoff in all cases, thereby raising the required investment thresholds.

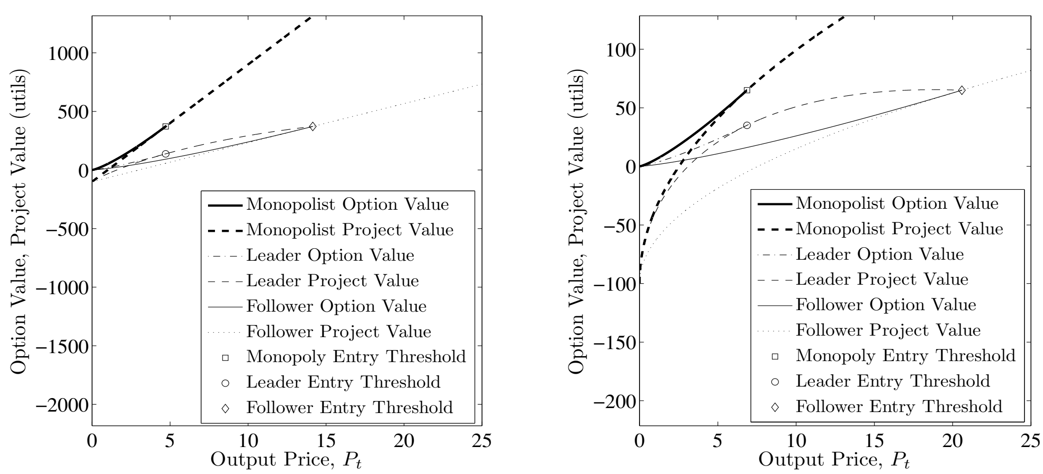


Figure 4.9: Project and investment opportunity value for non-pre-emptive leader and follower under risk neutrality (left) and risk aversion ($\gamma = 0.5$) (right) for $\sigma = 0.4$ and $D(1) = 3$

4.3.3 Sensitivity Analysis

As the left panel in Figure 4.10 illustrates, all entry thresholds increase with volatility as greater uncertainty implies greater value of waiting and are higher with risk aversion as it delays investment both for the leader and the follower by decreasing the expected utility of the project's cash flows. Proposition 4.2.6 is illustrated by the fact that the leader's optimal investment threshold is the same as the monopolist's. Also, higher first-mover advantages represented by greater $D(1)$ result in the decrease of the required entry thresholds of the pre-emptive and non-pre-emptive leader as illustrated in the graph on the right.

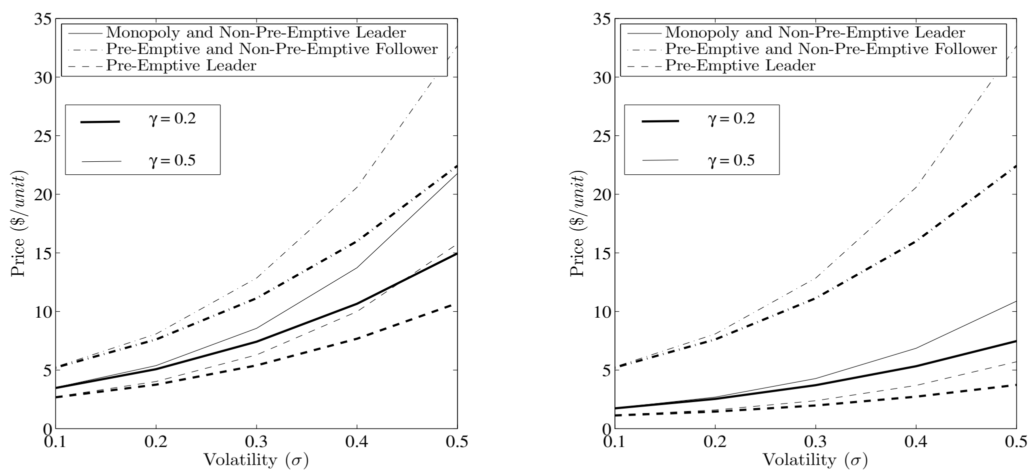


Figure 4.10: Optimal entry thresholds for $D(1) = 1.5$ (left) and $D(1) = 3$ (right)

In order to compare the pre-emptive and non-pre-emptive leader's values to the monopolist's, we evaluate both at the pre-emptive leader's optimal entry threshold, i.e., at $P_{\tau_p^{\ell*}}$. According to the graph on the left in Figure 4.11, increased uncertainty diminishes the relative loss in the pre-emptive leader's value function, i.e.,

$$\frac{F_{\tau_m^j}(P_{\tau_p^{\ell*}}) - V_p^{\ell}(P_{\tau_p^{\ell*}})}{F_{\tau_m^j}(P_{\tau_p^{\ell*}})} \tag{4.26}$$

thereby reducing the discrepancy between the pre-emptive leader's value and the monopolist's value of investment opportunity. This happens because uncertainty postpones the entry of the follower, thus allowing the pre-emptive leader to enjoy monopoly profits longer. Notice that the impact of uncertainty is more profound when the discrepancy in market share is low since then the expected loss due to the follower's entry is smaller.

Uncertainty increases the discrepancy in the non-pre-emptive leader's value of investment opportunity, i.e.,

$$\frac{F_{\tau_m^j}(P_{\tau_p^{\ell *}}) - F_{\tau_n^{\ell}}(P_{\tau_p^{\ell *}})}{F_{\tau_m^j}(P_{\tau_p^{\ell *}})} \tag{4.27}$$

if the discrepancy in market share is small, i.e., $\left(\frac{D(1)}{D(2)}\right)^{\beta_1} < e$, as in the graph on the left. Intuitively, this happens because under low discrepancy in market share, the increase in the non-pre-emtpive leader's value of investment opportunity due to increased uncertainty is greater than the expected loss due to the entry of the follower. However, if the discrepancy is large, then the increase in option value is less profound with higher uncertainty due to higher first-mover advantages and, as a result, cannot offset the expected loss from the follower's entry, which is now greater.

Furthermore, risk aversion does not affect the relative loss in the value of the leader for the pre-emptive duopoly setting, but it makes the loss in value relatively less for the leader in a non-pre-emptive duopoly setting due to delayed entry of the follower. Notice that at $P_{\tau_p^{\ell *}}$, the value function of the pre-emptive leader is the same as the option value of the pre-emptive follower. As a result, the impact of risk aversion on the value of the pre-emptive leader at $P_{\tau_p^{\ell *}}$ is the same as that on the value of the follower's investment opportunity at the same output price. Since the follower's investment opportunity value differs from the monopolist's only with respect to the market share, risk aversion impacts the values of the follower and the monopolist proportionally.

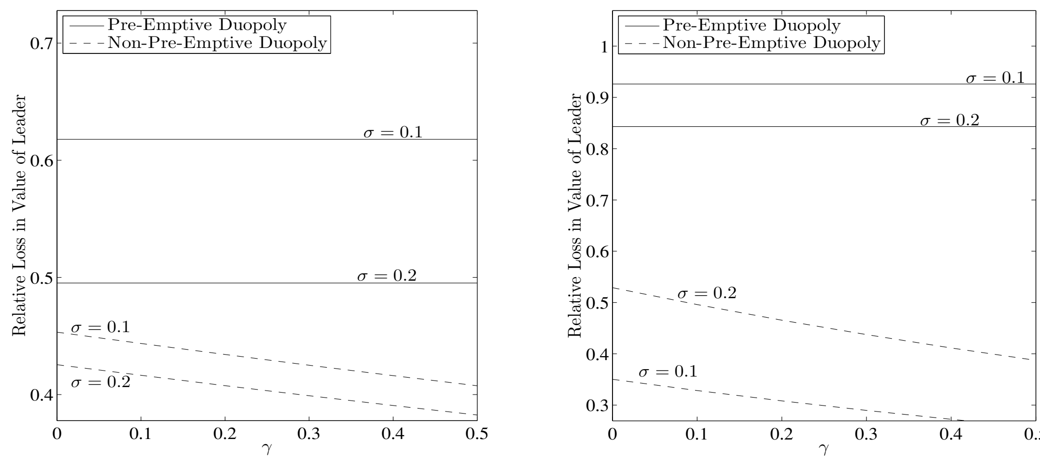


Figure 4.11: Relative loss in value of the pre-emptive and non-pre-emptive leader for $D(1) = 1.5$ (left) and $D(1) = 3$ (right)

4.4 Conclusions

In Chapter 4, we develop a utility-based framework in order to examine the impact of risk aversion and uncertainty on the optimal investment timing decisions of a firm that faces competition. The analysis is motivated both by the increasing competition resulting from the deregulation of many sectors of the economy such as energy, telecommunications, transport, etc, and the fact that attitudes towards the risk arising from the potential entry of a rival may impact investment decisions of a firm. The combination of these two factors creates the need to incorporate risk aversion into the real options framework, in order to analyse strategic aspects of decision making under uncertainty.

We find that, under the fear of pre-emption, higher uncertainty reduces the relative loss in the value of the leader due to competition by delaying the entry of the follower. However, in the non-pre-emptive setting, the impact of uncertainty is ambiguous and depends on the discrepancy in market share. If the discrepancy is large, the non-pre-emptive leader's relative loss in value increases with uncertainty since then the impact of the follower's entry is more profound and offsets the increase in the leader's value of investment opportunity. By contrast, under low discrepancy in market share, higher uncertainty makes the non-pre-emptive leader better off as the increase in the value of investment opportunity is greater than the expected loss in value due to competition. Interestingly, the relative loss in the pre-emptive leader's value is not affected by risk aversion, while the non-pre-emptive leader becomes better off with greater risk aversion as it delays the entry of the follower.

Since Chapter 4 considers the case where the two competing firms exhibit the same level of risk aversion, a potential extension is to relax this assumption and consider an asymmetric duopoly in terms of different levels of risk aversion for each firm. Furthermore, potentially useful insights could be extracted by allowing for operational flexibility like in Chapters 2 and 3. Directions for future research may also include the application of a different stochastic process, e.g., arithmetic Brownian motion, or the study of other aspects of the real options literature, such as the time to build or capacity sizing, under the same framework.

Chapter 5

Summary and Conclusions

Real options analysis adapts valuation techniques developed for financial options to non-financial settings. Thus, it addresses the flexibility and uncertainty present in most capital budgeting decisions by analysing capital projects as a series of options. The bulk of real options models are based on the assumptions of market completeness and risk neutrality, which were inherited by corporate finance, thereby ignoring the impact of risk aversion on investment decisions. However, decision makers often exhibit risk aversion either due to costs of financial distress or due to risk that cannot be diversified, which occurs for example in R&D projects with technical risk that is idiosyncratic. The deregulation of many sectors of the economy has resulted in greater competition and uncertainty, thus shifting the focus to knowledge-based sectors such as R&D that become increasingly important. Consequently, in an era of increasing economic uncertainty and deregulation the need to develop a utility-based framework in order to account for risk aversion becomes essential.

The objective of this thesis is to bridge the gaps in real options theory so that it may be more suitable not only for decision making but also for risk assessment. We begin by developing a utility-based framework in order to analyse how the optimal investment decisions are affected by risk aversion and uncertainty. In order to quantify the impact of risk and operational flexibility, we assume that the project offers suspension and resumption options that can be exercised at any time. We proceed by assessing how the flexibility to adjust capacity in the presence of risk aversion impacts the value of an option to invest, thereby extending the traditional real options approach to investment under uncertainty with discretion over capacity by allowing for risk aversion, through a CRRA utility function, and operational flexibility in the form of suspension and resumption options. Finally, in order to explore strategic aspects of decision making under uncertainty, we determine how duopolistic competition affects the entry of risk-averse investors and how the value of a firm under two different oligopolistic frameworks varies with risk aversion and uncertainty. Here, we will summarise the results of this thesis, discuss its limitations and offer directions for further research.

5.1 Optimal Investment under Operational Flexibility, Risk Aversion, and Uncertainty

In Chapter 2, we extend the real options approach with operational flexibility, i.e., a situation in which the decision maker has infinitely many embedded options to suspend or resume the project, to account for risk aversion. We introduce risk aversion on part of the decision maker via a CRRA utility-of-wealth function analogous to Hugonnier and Morellec (2007). We solve this problem backwards by first assuming that the project is active and has a single abandonment option at its disposal. This enables us to find its expected value and exercise threshold by solving the analogous optimal stopping problem. By then including this abandonment option in the payoff of the original investment opportunity, we solve the problem of investment in a project with a single abandonment option, i.e., we obtain the investment and abandonment thresholds along with the expected value of the investment opportunity. Finally, by extending this methodology to incorporate infinitely many suspension and resumption options, we analyse a project with complete operational flexibility.

Our results indicate that operational flexibility facilitates investment and operational decisions by increasing the likelihood of investment, suspension, and resumption of the investment project. Furthermore, we show that risk aversion may increase the incentive for decision makers to delay the investment and resumption of the investment project and accelerates their decision to abandon it. Moreover, we describe how an environment of increasing uncertainty may affect the optimal investment policy and lead to hysteresis. Also, we provide insights regarding the behaviour of the optimal suspension threshold when the level of risk aversion changes. Interestingly, numerical results indicate that increased risk aversion may facilitate the abandonment of a project while delaying its temporary suspension prior to permanent resumption. Finally, we demonstrate how operational flexibility becomes more valuable as risk aversion increases and the economic environment becomes more volatile.

In order to quantify further the degree to which the investor's risk is hedged through operational flexibility, risk measures such as value-at-risk and conditional value-at-risk for the canonical real options investment problem could be developed. Such risk measures quantify the market risk of a project which to date has been applied only numerically to real options. Furthermore, in contrast to the standard real options approach, it could be particularly insightful to determine not only the expected value of the option to invest but also its moment-generating function.

5.2 The Value of Capacity Sizing under Risk Aversion and Operational Flexibility

In many capital projects, the investor also holds discretion over capacity, e.g., the size of a factory. Dangl (1999) uses a continuous cost function first to find the optimal size of the active investment as a function of the price by maximising the expected NPV. He then obtains the expected value of the option to invest and the investment threshold price by using value-matching and smooth-pasting conditions with the expected NPV function. In the third chapter, we extend this approach by accounting for risk aversion on part of the investor via a CRRA utility-of-wealth-function. Specifically, by first assuming that the project value evolves according to a GBM, we work backwards by solving for the capacity size of the project that maximises the discounted expected utility of the project's cash flows. Next, we substitute the utility-maximising capacity size, which is a function of the project's cash flows, into the expression for the project value and then solve the analogous optimal stopping problem. The resulting expected option value, investment threshold price, and capacity size are then compared with the ones under the risk-neutral assumption of Dangl (1999) to gauge the extent to which risk aversion affects sizing decisions.

The objective in Chapter 3 is to assess how valuable discretion over the capacity of the project is under risk aversion and operational flexibility. In contrast to a project without scalable capacity, we find that, with the option to delay investment, increased risk aversion facilitates investment and decreases the required investment threshold price by reducing the amount of installed capacity. We also find that, when investing immediately, the relative loss in project value due to installation of suboptimal capacity diminishes with risk aversion and uncertainty. With operational flexibility, discretion over capacity becomes more valuable when exercising a now-or-never investment opportunity and less valuable when the option to defer investment is available when the capacity installed is suboptimal.

One of the limitations of this model arises from the particular choice of a cost function that is increasing and strictly convex, thereby implying that the average cost is increasing. Due to this property, this model is best suited for analysing projects that exhibit diseconomies of scale, e.g., renewable energy power plants. This choice results from the assumption of an exogenous output price, which is an additional limitation of the model since investment decisions do not affect future prices. This limitation is particularly obvious when investment in very large capacity is optimal. Therefore, it would be interesting to examine the implications of relaxing this assumption by allowing for the price to depend on the capacity installed. Although useful insights regarding the robustness of the model could be obtained through the application of different utility functions or alternative stochastic processes, nevertheless, it would be interesting to generalise the results presented in the second and third chapters of the thesis so that they are independent of the type of the utility function. This requires the verification of the results simply assuming that the output price follows a GBM and that the decision maker's preferences are described by an increasing and concave utility function. Furthermore, other aspects of the real options literature, e.g., the time-to-build problem, may be examined under the same framework.

5.3 Duopolistic Competition under Risk Aversion and Uncertainty

A monopolist typically defers entry into an industry as both price uncertainty and the level of relative risk aversion increase. The former attribute may be present in most deregulated industries, while the latter may be relevant for reasons of market incompleteness or the presence of technical uncertainty. By contrast, it has been shown that the presence of a rival hastens entry under risk neutrality in certain frameworks. In the first two chapters of this thesis, we have taken the perspective of a price-taking firm that holds a perpetual option to investment in a project with infinite lifetime and wants to maximise the expected utility of future profits. In Chapter 4, we assume a duopolostic setting and analyse the strategic interactions of decision making under uncertainty and risk aversion. Specifically, we examine how duopolistic competition affects the entry decisions of risk-averse firms and explore how the impact of competition on the value of a firm under two different oligopolistic frameworks, i.e., pre-emptive and non-pre-emptive duopoly, varies with risk aversion and uncertainty.

We show that the entry threshold of the non-pre-emptive leader is the same as that of the monopolist under both risk neutrality and risk aversion. Moreover, we illustrate how all entry thresholds increase with volatility as greater uncertainty implies greater value of waiting and are higher than the thresholds under risk neutrality. Also, the value of the leader in a duopoly relative to the value of a monopolist at the pre-emptive leader's optimal entry threshold indicates that the non-pre-emptive duopoly leader is hurt less than the pre-emptive duopoly leader. In the non-pre-emptive duopoly setting, if the discrepancy between the market share of the leader and the follower is large, then the non-pre-emptive leader's relative loss in value increases with uncertainty since the impact of the follower's entry is more profound and offsets the increase in the leader's value of investment opportunity. By contrast, under low discrepancy in market share, higher uncertainty makes the non-pre-emptive leader better off as the increase in the value of investment opportunity is greater than the expected loss in value due to competition. Furthermore, risk aversion does not affect the loss in the value of the leader for the pre-emptive duopoly setting, but it makes the loss in value relatively less for the leader in a non-pre-emptive duopoly setting.

Since the analysis presented in Chapter 4 is restricted to the case where both firms exhibit the same level of risk aversion, this framework can be extended to account for the case of asymmetric duopoly where the levels of risk aversion are different. Furthermore, by analysing the case where each firm has discretion over the capacity of the project will also provide further insights and allow for comparisons with the results of the third chapter.

Appendix A

Proofs of the Propositions of Chapter 2

Proposition 2.2.1: *Under a CRRA utility-of-wealth function, the expected utility of a perpetual GBM discounted to P_0 is given by (2.10):*

$$\mathbb{E}_{P_0} \int_0^\infty e^{-\rho t} U(P_t) dt = \mathcal{A} U(P_0) \tag{A.1}$$

where $\mathcal{A} = \frac{\beta_1 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} > 0$ and $\beta_1 > 1$, $\beta_2 < 0$ are the solutions to the following quadratic equation:

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - \rho = 0 \tag{A.2}$$

Proof: We want to derive the analytical expression of the following expectation:

$$\mathbb{E}_{P_0} \int_0^\infty e^{-\rho t} U\left(P_t\right) dt \tag{A.3}$$

where P_t follows GBM:

$$dP_t = \mu P_t dt + \sigma P_t dZ_t \tag{A.4}$$

Solving for P_t we have:

$$P_t = P_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right\} \tag{A.5}$$

Inserting (A.5) into (A.3), the expression of the expectation we want to determine becomes

$$\mathbb{E}_{P_0} \int_0^\infty e^{-\rho t} U\left(P_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right\}\right) dt \tag{A.6}$$

Following the steps of Karatzas and Shreve (1999) p.146, the expectation (A.6) can be written as:

$$\begin{aligned}
&\mathbb{E}_{P_0} \int_0^\infty e^{-\rho t} U\left(P_t\right) dt = \\
&\frac{2}{\sigma^2(\beta_1 - \beta_2)} \left\{P_0^{\beta_2} \int_0^{P_0} x^{-\beta_2 - 1} U\left(x\right) dx + P_0^{\beta_1} \int_{P_0}^\infty x^{-\beta_1 - 1} U\left(x\right) dx\right\}
\end{aligned} \tag{A.7}$$

Applying a CRRA utility function we have:

$$\begin{aligned}
\mathbb{E}_{P_0} &\int_0^\infty e^{-\rho t} U\left(P_t\right) dt \\
&= \frac{2}{\sigma^2(\beta_1 - \beta_2)} \left\{P_0^{\beta_2} \int_0^{P_0} \frac{x^{-\beta_2 - \gamma}}{1-\gamma} dx + P_0^{\beta_1} \int_{P_0}^\infty \frac{x^{-\beta_1 - \gamma}}{1-\gamma} dx\right\} \\
&= -\frac{2}{\sigma^2} \frac{1}{(1-\beta_1-\gamma)(1-\beta_2-\gamma)} U(P_0) \\
&= \mathcal{A} U(P_0)
\end{aligned} \tag{A.8}$$

where $\mathcal{A} = \frac{\beta_1 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)}$.

Proposition 2.2.2: *Under a CRRA utility-of-wealth function, the optimal investment threshold is:*

$$P_{\tau_1}^{(0)^*} = (c + rK)\left[\frac{\beta_2 + \gamma - 1}{\beta_2}\right]^{\frac{1}{1-\gamma}} \tag{A.9}$$

Proof: The optimisation problem is described by (A.10):

$$F_{\tau_1}^{(0)}(P_0) \quad = \quad \max_{P_{\tau_1}^{(0)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(0)}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_1}^{(0)}\right) - \frac{U(c + rK)}{\rho}\right] \tag{A.10}$$

The FONC for the unconstrained optimisation problem (A.10) may be expressed as follows:

$$\frac{\beta_2}{1 - \beta_2 - \gamma} P_{\tau_1}^{(0)^* \, 1-\gamma} + (c + rK)^{1-\gamma} = 0 \tag{A.11}$$

Solving with respect to $P_{\tau_1}^{(0)^*}$, we obtain the following expression for the optimal investment threshold:

$$P_{\tau_1}^{(0)^*} = (c + rK)\left(\frac{\beta_2 + \gamma - 1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \tag{A.12}$$

This is equivalent to the expression of the optimal investment threshold in Hugonnier and Morellec (2007). Indeed, according to Hugonnier & Morellec:

$$P_{\tau_1}^{(0)^*} = (c + rK)\left[\frac{\beta_1}{\beta_1 + \gamma - 1}\frac{\Delta}{\rho}\right]^{\frac{1}{1-\gamma}} \tag{A.13}$$

where,

$$\Delta = \rho + (\gamma - 1)\left(\mu - \frac{1}{2}\sigma^2\gamma\right) \tag{A.14}$$

For the two expressions to match, the following equality must be hold.

$$
\begin{aligned}
& \frac{\beta_1}{\beta_1+\gamma-1}\frac{\Delta}{\rho} = \frac{\beta_2+\gamma-1}{\beta_2} \\
\Leftrightarrow\ & \frac{\beta_1}{\beta_1+\gamma-1}\left(1+\frac{\gamma-1}{\rho}\left(\mu-\frac{1}{2}\sigma^2\gamma\right)\right) = \frac{\beta_2+\gamma-1}{\beta_2} \\
\Leftrightarrow\ & \beta_1\beta_2\left(1+\frac{\gamma-1}{\rho}\left(\mu-\frac{1}{2}\sigma^2\gamma\right)\right) = (\beta_1+\gamma-1)(\beta_2+\gamma-1) \\
\Leftrightarrow\ & \beta_1\beta_2\left(1+\frac{\gamma-1}{\rho}\left(\mu-\frac{1}{2}\sigma^2\gamma\right)\right) = \beta_1\beta_2+\beta_1\gamma-\beta_1+\beta_2\gamma+\gamma^2-\gamma-\beta_2-\gamma+1 \\
\Leftrightarrow\ & \beta_1\beta_2\frac{\gamma-1}{\rho}\left(\mu-\frac{1}{2}\sigma^2\gamma\right) = \beta_1(\gamma-1)+\beta_2(\gamma-1)+\gamma(\gamma-1)-(\gamma-1) \\
\Leftrightarrow\ & \frac{\beta_1\beta_2}{\rho}\left(\mu-\frac{1}{2}\sigma^2\gamma\right) = \beta_1+\beta_2+\gamma-1 \qquad \text{(A.15)}
\end{aligned}
$$

Since

$$
\beta_1 = \frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\rho}{\sigma^2}} \qquad \text{(A.16)}
$$

$$
\beta_2 = \frac{1}{2}-\frac{\mu}{\sigma^2}-\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\rho}{\sigma^2}} \qquad \text{(A.17)}
$$

it follows from (A.15) that:

$$
\begin{aligned}
& \frac{\beta_1\beta_2}{\rho}\left(\mu-\frac{1}{2}\sigma^2\gamma\right) = \beta_1+\beta_2+\gamma-1 \\
\Leftrightarrow\ & -\frac{\frac{2\rho}{\sigma^2}}{\rho}\left(\mu-\frac{1}{2}\sigma^2\gamma\right) = 1-\frac{2\mu}{\sigma^2}+\gamma-1 \\
\Leftrightarrow\ & -\frac{2\mu}{\sigma^2}+\gamma = -\frac{2\mu}{\sigma^2}+\gamma \qquad \text{(A.18)}
\end{aligned}
$$

which is true.

Proposition 2.2.3: *The objective function is strictly concave at* $P_{\tau_1}^{(0)^*}$ *iff* $\gamma < 1$.
Proof: The objective function evaluated at the critical value, $P_{\tau_1}^{(0)^*}$, is the fol-

lowing:

$$F_1^{(0)}\left(P_{\tau_1}^{(0)^*}\right) = \left(\frac{P_0}{P_{\tau_1}^{(0)^*}}\right)^{\beta_1}\left[\mathcal{A}\frac{P_{\tau_1}^{(0)^*\,1-\gamma}}{1-\gamma} - \frac{(c+rK)^{1-\gamma}}{\rho(1-\gamma)}\right] \tag{A.19}$$

Differentiating the objective function with respect to $P_{\tau_1}^{(0)^*}$ twice yields the following result:

$$\begin{aligned}\frac{\partial^2 F_1^{(0)}\left(P_{\tau_1}^{(0)^*}\right)}{\partial P_{\tau_1}^{(0)^*\,2}} &= (1+\beta_1)\beta_1\left(\frac{P_0}{P_{\tau_1}^{(0)^*}}\right)^{\beta_1}\left(\frac{1}{P_{\tau_1}^{(0)^*}}\right)^2\left[\mathcal{A}\frac{P_{\tau_1}^{(0)^*\,1-\gamma}}{1-\gamma} - \frac{(c+rK)^{1-\gamma}}{\rho(1-\gamma)}\right] \\ &+ \beta_1\left(\frac{P_0}{P_{\tau_1}^{(0)^*}}\right)^{\beta_1}\left(-\frac{1}{P_{\tau_1}^{(0)^*}}\right)\mathcal{A}P_{\tau_1}^{(0)^*\,-\gamma} \\ &- \left(\frac{P_0}{P_{\tau_1}^{(0)^*}}\right)^{\beta_1}\mathcal{A}(\beta_1+\gamma)P_{\tau_1}^{(0)^*\,-\gamma-1}\end{aligned} \tag{A.20}$$

The SOSC requires that $\frac{\partial^2 F_1^{(0)}\left(P_{\tau_1}^{(0)^*}\right)}{\partial P_{\tau_1}^{(0)^*\,2}} < 0$. Simplifying the above expression yields:

$$\frac{\partial^2 F_{\tau_1}^{(0)}\left(P_{\tau_1}^{(0)^*}\right)}{\partial P_{\tau_1}^{(0)^*\,2}} < 0 \quad \Leftrightarrow \quad \frac{2\beta_1+\gamma+1}{1-\gamma} > 0 \tag{A.21}$$

Notice that the numerator is positive, which implies that for the inequality to hold the denominator needs to be positive as well. Hence, the SOSC is satisfied if and only if $0 \leq \gamma < 1$.

Corollary 2.2.1: *The MB curve is steeper than the MC curve at* $P_{\tau_1}^{(0)^*}$.

Proof: We will show that $\left|\frac{\partial MB}{\partial P_{\tau_1}^{(0)}}\right| > \left|\frac{\partial MC}{\partial P_{\tau_1}^{(0)}}\right|_{P_{\tau_1}^{(0)} \equiv P_{\tau_1}^{(0)*}}$.

$$\beta_1 \left(\frac{P_0}{P_{\tau_1}^{(0)*}}\right)^{\beta_1} \frac{1}{P_{\tau_1}^{(0)*}} \left[\frac{\beta_1\beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} P_{\tau_1}^{(0)*-\gamma} + \frac{\beta_1}{P_{\tau_1}^{(0)*}} \frac{U(c+rK)}{\rho}\right] +$$
$$\left(\frac{P_0}{P_{\tau_1}^{(0)*}}\right)^{\beta_1} \left[\frac{\beta_1\beta_2\gamma}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} P_{\tau_1}^{(0)*-\gamma-1} + \frac{\beta_1}{P_{\tau_1}^{(0)*2}} \frac{U(c+rK)}{\rho}\right] >$$
$$\beta_1 \left(\frac{P_0}{P_{\tau_1}^{(0)*}}\right)^{\beta_1} \frac{1}{P_{\tau_1}^{(0)*}} \frac{\beta_1^2\beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} \frac{P_{\tau_1}^{(0)*-\gamma}}{1-\gamma} +$$
$$\left(\frac{P_0}{P_{\tau_1}^{(0)*}}\right)^{\beta_1} \frac{\beta_1^2\beta_2\gamma}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} \frac{P_{\tau_1}^{(0)*-\gamma-1}}{1-\gamma} \tag{A.22}$$

Simplifying (A.22) and substituting for $P_{\tau_1}^{(0)*}$ leads to the following result:

$$2\beta_1 + \gamma + 1 > 0 \tag{A.23}$$

This is true since, $\beta_1 > 1$ and $0 \leq \gamma < 1$

Proposition 2.2.4: *The optimal investment threshold is increasing with risk aversion.*

Proof: Differentiating the optimal investment threshold, $P_{\tau_1}^{(0)*}$, with respect to γ yields:

$$\begin{aligned} P_{\tau_1}^{(0)*} &= (c+rK)\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \\ \Rightarrow \frac{\partial P_{\tau_1}^{(0)*}}{\partial \gamma} &= P_{\tau_1}^{(0)*} \frac{\partial}{\partial \gamma} \ln\left[(c+rK)\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}}\right] \end{aligned} \tag{A.24}$$

Since $P_{\tau_1}^{(0)*} > 0$, we only need to determine the sign of $\frac{\partial}{\partial \gamma} \ln\left[(c+rK)\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}}\right]$.

Hence,

$$\frac{\partial \ln P_{\tau_1}^{(0)^*}}{\partial \gamma} > 0 \quad \Leftrightarrow \quad \ln\left[\frac{1-\beta_2-\gamma}{-\beta_2}\right] > 1 - \frac{-\beta_2}{1-\beta_2-\gamma} \tag{A.25}$$

We now set $x = \frac{-\beta_2}{1-\beta_2-\gamma} > 0 \Rightarrow \frac{1}{x} = \frac{1-\beta_2-\gamma}{-\beta_2}$. Consequently, we need to show that:

$$-\ln x > 1 - x \quad \Leftrightarrow \quad \ln x < x - 1 \tag{A.26}$$

The equality $\ln x = x - 1$ holds for $\gamma = 1$, which is not considered here. To show that inequality (A.26) holds, we first need to show that:

$$e^x \geq 1 + x, \qquad \forall x \in \mathbb{R} \tag{A.27}$$

For all $x \in \mathbb{R}$, we assume a $\nu \in \mathbb{N}$ such that $\nu > -x$, i.e. $\nu + x > 0$. Then, $1 + \frac{x}{\nu} > 0$, and so we have $\left(1 + \frac{x}{\nu}\right)^\nu \geq 1 + \nu\frac{x}{\nu} = 1 + x$ from Bernoulli's inequality. Finally, we have:

$$e^x = \lim_{\nu\to\infty}\left(1 + \frac{x}{\nu}\right)^\nu \geq \lim_{\nu\to\infty}(1+x) = 1 + x \Rightarrow e^x \geq 1 + x \;\; \forall x \in \mathbb{R} \tag{A.28}$$

Thus, we have shown that $e^x \geq 1 + x$, $\forall x \in \mathbb{R}$. Hence, assuming that $x > 0$ and using $\ln x$ instead of x, we have:

$$e^{\ln x} = x \geq 1 + \ln x \Rightarrow \ln x \leq x - 1 \tag{A.29}$$

Proposition 2.2.5: *The optimal investment threshold is increasing with volatility.*
Proof: Since $\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}$, substituting into the expression of

the optimal investment threshold we have:

$$P_{\tau_1}^{0*} \quad = \quad (c+rK)\left(1+\frac{\gamma-1}{\frac{1}{2}-\frac{\mu}{\sigma^2}-\sqrt{\left[\frac{\mu}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2\rho}{\sigma^2}}}\right)^{\frac{1}{1-\gamma}} \tag{A.30}$$

Differentiating with respect to σ^2 yields:

$$\begin{aligned}\frac{\partial P_{\tau_1}^{(0)^*}}{\partial\sigma^2} = c\left(\frac{1}{1-\gamma}\right)&\left[1+\frac{\gamma-1}{\frac{1}{2}-\frac{\mu}{\sigma^2}-\sqrt{\left[\frac{\mu}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2\rho}{\sigma^2}}}\right]^{\frac{1}{1-\gamma}-1}\\ \times&\left[-\frac{(\gamma-1)\left(\frac{\mu}{\sigma^4}-\left(\frac{1}{2\sqrt{\left[\frac{\mu}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2\rho}{\sigma^2}}}\right)\left\{2\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)\left(-\frac{\mu}{\sigma^4}\right)-\frac{2\rho}{\sigma^4}\right\}\right)}{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}-\sqrt{\left[\frac{\mu}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2\rho}{\sigma^2}}\right)^2}\right]\end{aligned} \tag{A.31}$$

Note that:

$$2\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)\left(-\frac{\mu}{\sigma^4}\right)-\frac{2\rho}{\sigma^4}<0 \Leftrightarrow -\frac{\mu^2}{\sigma^2}-\frac{2\rho-\mu}{2}<0 \tag{A.32}$$

which is true. Hence, the last term in (A.31) is positive. Since the rest of the factors in (A.31) are positive, we conclude that $\frac{\partial P_{\tau_1}^{(0)^*}}{\partial\sigma^2}>0$.

Proposition 2.2.6: *The objective function is strictly concave at* $P_{\tau_2}^{(0)^*}$ *iff* $\gamma<1$.
Proof: The objective function evaluated at $P_{\tau_2}^{(0)^*}$ takes the following expression,

$$F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right) \quad = \quad \left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2}\left[\frac{c^{1-\gamma}}{\rho(1-\gamma)}-\mathcal{A}\frac{P_{\tau_2}^{(0)*\,1-\gamma}}{1-\gamma}\right] \tag{A.33}$$

Differentiating twice with respect to $P_{\tau_2}^{(0)^*}$ yields the following,

$$\begin{aligned}
\frac{\partial^2 F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right)}{\partial P_{\tau_2}^{(0)^*\,2}} \quad = \quad & \beta_2(\beta_2+1)\left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2}\left(\frac{1}{P_{\tau_2}^{(0)^*}}\right)^2\left[\frac{c^{1-\gamma}}{\rho(1-\gamma)}-\mathcal{A}\frac{P_{\tau_2}^{(0)^*\,1-\gamma}}{1-\gamma}\right] \\
& +\beta_2\left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2}\left(\frac{1}{P_{\tau_2}^{(0)^*}}\right)\mathcal{A}P_{\tau_2}^{(0)^*\,-\gamma} \\
& +\left(\frac{P_{\tau_1}^{(1)}}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2}\mathcal{A}(\beta_2+\gamma)P_{\tau_2}^{(0)^*\,-\gamma-1}
\end{aligned} \tag{A.34}$$

Simplifying the above expression, we have the following result:

$$\frac{\partial^2 F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)}\right)}{\partial P_{\tau_2}^{(0)^*\,2}} < 0 \Leftrightarrow \gamma < 1 \tag{A.35}$$

Hence, the objective function is concave at $P_{\tau_2}^{(0)^*}$ if and only if $\gamma < 1$.

Proposition 2.2.7: *The optimal abandonment threshold is increasing with risk aversion.*

Proof: Following the same steps as in Proposition 2.2.4 we have:

$$\frac{\partial P_{\tau_2}^{(0)^*}}{\partial \gamma} > 0 \quad \Leftrightarrow \quad \ln\left[\frac{\beta_1+\gamma-1}{\beta_1}\right] > 1 - \frac{\beta_1}{\beta_1+\gamma-1} \tag{A.36}$$

We now set $x = \frac{\beta_1}{\beta_1+\gamma-1}$. Thus, we need to show that $\ln x < x - 1$ which we have already shown in Proposition 2.2.4 that holds.

Proposition 2.2.8: *The optimal abandonment threshold is decreasing in volatility.*

Proof: The optimal abandonment threshold is given by the following equation:

$$P_{\tau_2}^{(0)^*} \quad = c\left[\frac{\beta_1+\gamma-1}{\beta_1}\right]^{\frac{1}{1-\gamma}} \tag{A.37}$$

Substituting for β_1, (A.37) takes the following expression:

$$P_{\tau_2}^{(0)^*} \quad = c\left[1+\frac{\gamma-1}{\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left[\frac{\mu}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2\rho}{\sigma^2}}}\right]^{\frac{1}{1-R}} \tag{A.38}$$

Differentiating (A.38) with respect to σ^2 we have:

$$\frac{\partial P_{\tau_2}^{(0)^*}}{\partial \sigma^2} = c\left(\frac{1}{1-\gamma}\right)\left[1+\frac{\gamma-1}{\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left[\frac{\mu}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2\rho}{\sigma^2}}}\right]^{\frac{1}{1-\gamma}-1}$$
$$\times\left[-\frac{(\gamma-1)\left(\frac{\mu}{\sigma^4}+\left(\frac{1}{2\sqrt{\left[\frac{\mu}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2\rho}{\sigma^2}}}\right)\left\{2\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)\left(-\frac{\mu}{\sigma^4}\right)-\frac{2\rho}{\sigma^4}\right\}\right)}{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left[\frac{\mu}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2\rho}{\sigma^2}}\right)^2}\right] \tag{A.39}$$

Notice that in (A.39):

$$2\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)\left(-\frac{\mu}{\sigma^4}\right)-\frac{2\rho}{\sigma^4}<0 \Leftrightarrow -\frac{\mu^2}{\sigma^2}-\frac{2\rho-\mu}{2}<0 \tag{A.40}$$

The latter is true, and, therefore, the last term of (A.39) is negative. Hence, since the first three factors are positive, we conclude that $\frac{\partial P_{\tau_2}^{(0)^*}}{\partial \sigma^2}<0$.

Proposition 2.2.9: *The optimal investment threshold price when a single abandonment option is available is less than that with an irreversible investment opportunity, ceteris paribus.*

Proof: The FONCs that provide the optimal investment thresholds for the case of irreversible investment and investment with a single abandonment option are:

$$\frac{\beta_2}{1-\beta_2-\gamma}P_{\tau_1}^{(0)^*1-\gamma}+(c+rK)^{1-\gamma}=0 \tag{A.41}$$
$$\frac{\beta_2}{1-\beta_2-\gamma}P_{\tau_1}^{(1)^*1-\gamma}+(c+rK)^{1-\gamma}+\frac{\beta_2-\beta_1}{\beta_1}\rho(1-\gamma)F_{\tau_2}^{(0)}(P_{\tau_1}^{(1)^*})=0 \tag{A.42}$$

Subtracting the two equations, we have:

$$P_{\tau_1}^{(0)*\,1-\gamma} - P_{\tau_1}^{(1)*\,1-\gamma} = (1-\beta_2-\gamma)F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)^*}\right) \tag{A.43}$$

Since $\beta_2 < 0$ and $F_{\tau_2}^{(0)}(P_{\tau_1}^{(1)^*}) > 0$ the quantity on the right-hand side is positive. Hence:

$$P_{\tau_1}^{(0)*\,1-\gamma} - P_{\tau_1}^{(1)*\,1-\gamma} > 0 \;\;\Rightarrow\;\; P_{\tau_1}^{(0)^*} > P_{\tau_1}^{1^*} \tag{A.44}$$

Proposition 2.2.10: *The optimal resumption threshold is increasing with risk aversion.*
Proof: Similar to Proposition 2.2.4.

Proposition 2.2.11: *The optimal suspension threshold is higher than the optimal abandonment one.*
Proof: Comparing the two FONCs that provide the optimal abandonment and optimal suspension thresholds, we have:

$$\frac{1}{1-\beta_1-\gamma}P_{\tau_2}^{(0)*\,1-\gamma} + \frac{c^{1-\gamma}}{\beta_1} = 0 \tag{A.45}$$

$$\frac{1}{1-\beta_1-\gamma}P_{\tau_2}^{(1)*\,1-\gamma} + \frac{c^{1-\gamma}}{\beta_1} - \frac{\rho(\beta_1-\beta_2)(1-\gamma)}{\beta_1\beta_2}F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)^*}\right) = 0 \tag{A.46}$$

By subtracting the two equations, we have:

$$P_{\tau_2}^{(1)*\,1-\gamma} - P_{\tau_2}^{(0)*\,1-\gamma} = \frac{\rho(\beta_1-\beta_2)(1-\beta_1-\gamma)(1-\gamma)}{\beta_1\beta_2}F_{\tau_3}^{(0)}\left(P_{\tau_2}^{(1)^*}\right) \tag{A.47}$$

Since $\beta_1 > 1$, $\beta_2 < 0$ and $F_{\tau_3}^{0}\left(P_{\tau_2}^{(1)^*}\right) \geq 0$ quantity on the right-hand side is positive. Therefore,

$$P_{\tau_2}^{(1)*\,1-\gamma} - P_{\tau_2}^{(0)*\,1-\gamma} > 0 \;\;\Rightarrow\;\; P_{\tau_2}^{(1)^*} > P_{\tau_2}^{(0)^*} \tag{A.48}$$

Proposition 2.2.12: *The optimal investment threshold price when a single suspension and a single resumption option is available is lower than that with an investment opportunity with a single abandonment option.*

Proof: The FONCs that yield the optimal investment thresholds are:

$$\frac{\beta_2}{1-\beta_2-\gamma} P_{\tau_1}^{(1)^*\,1-\gamma} + (c+rK)^{1-\gamma} + \frac{\beta_2-\beta_1}{\beta_1}\rho(1-\gamma)F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)^*}\right) = 0 \quad \text{(A.49)}$$

$$\frac{\beta_2}{1-\beta_2-\gamma} P_{\tau_1}^{(2)^*\,1-\gamma} + (c+rK)^{1-\gamma} + \frac{\beta_2-\beta_1}{\beta_1}\rho(1-\gamma)F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)^*}\right) = 0 \quad \text{(A.50)}$$

Subtracting these two equations, we have:

$$P_{\tau_1}^{(1)^*\,1-\gamma} - P_{\tau_1}^{(2)^*\,1-\gamma} = \frac{(1-\beta_2-\gamma)(\beta_1-\beta_2)}{\beta_1\beta_2} \times \rho(1-\gamma)\left[F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)^*}\right) - F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)^*}\right)\right] \quad \text{(A.51)}$$

Since the option to suspend operations with the embedded option to resume them permanently, $F_{\tau_2}^{(1)}\left(P_{\tau_1}^{(2)^*}\right)$, is greater than the abandonment option, $F_{\tau_2}^{(0)}\left(P_{\tau_1}^{(1)^*}\right)$, the right-hand side of (A.49) is negative indicating that:

$$P_{\tau_1}^{(1)^*\,1-\gamma} - P_{\tau_1}^{(2)^*\,1-\gamma} > 0 \Rightarrow P_{\tau_1}^{(1)^*} > P_{\tau_1}^{(2)^*} \quad \text{(A.52)}$$

Appendix B

Proofs of the Propositions of Chapter 3

Proposition 3.2.1: *For all $\lambda > 1$ the optimisation problem (3.13) has a solution, while for $\lambda \downarrow 1$the solution diverges.*
Proof: According to (3.14), the marginal benefit of increasing capacity is:

$$\begin{aligned} \widetilde{MB} &= \frac{\mathcal{A}{P_0}^{1-\gamma}}{{\tilde{m}^{(0)}}^{\gamma}} > 0, \quad \forall \tilde{m}^{(0)} > 0 \\ \Rightarrow \frac{\partial \widetilde{MB}}{\partial \tilde{m}^{(0)}} &= -\frac{\gamma \mathcal{A}{P_0}^{1-\gamma}}{{\tilde{m}^{(0)}}^{\gamma+1}} < 0, \quad \forall \tilde{m}^{(0)} > 0 \end{aligned} \tag{B.1}$$

The marginal cost of increasing capacity is:

$$\begin{aligned} \widetilde{MC} &= \frac{c + rb\lambda {\tilde{m}^{(0)}}^{\lambda-1}}{\rho \left[c\tilde{m}^{(0)} + rb{\tilde{m}^{(0)}}^{\lambda}\right]^{\gamma}} > 0, \quad \forall \tilde{m}^{(0)} > 0 \\ \Rightarrow \frac{\partial \widetilde{MC}}{\partial \tilde{m}^{(0)}} &= \frac{rb\lambda(\lambda-1){\tilde{m}^{(0)}}^{\lambda-2}}{\rho \left(c\tilde{m}^{(0)} + rb{\tilde{m}^{(0)}}^{\lambda}\right)^{\gamma}} - \frac{\gamma \left(c + rb\lambda{\tilde{m}^{(0)}}^{\lambda-1}\right)^2}{\rho \left(c\tilde{m}^{(0)} + rb{\tilde{m}^{(0)}}^{\lambda}\right)^{\gamma+1}} \end{aligned} \tag{B.2}$$

Regarding the sign of $\frac{\partial \widetilde{MC}}{\partial \tilde{m}^{(0)}}$ we have:

$$\begin{aligned}
\frac{\partial \widetilde{MC}}{\partial \tilde{m}^{(0)}} > 0 \quad \Leftrightarrow \quad & \frac{rb\lambda(\lambda-1)\tilde{m}^{(0)^{\lambda-2}}}{\rho\left(c\tilde{m}^{(0)} + rb\tilde{m}^{(0)^{\lambda}}\right)^{\gamma}} - \frac{\gamma\left(c + rb\lambda\tilde{m}^{(0)^{\lambda-1}}\right)^2}{\rho\left(c\tilde{m}^{(0)} + rb\tilde{m}^{(0)^{\lambda}}\right)^{\gamma+1}} > 0 \\
\Leftrightarrow \quad & rb\lambda(\lambda-1)\tilde{m}^{(0)^{\lambda-2}}\left(c\tilde{m}^{(0)} + rb\tilde{m}^{(0)^{\lambda}}\right) - \gamma\left(c + rb\lambda\tilde{m}^{(0)^{\lambda-1}}\right)^2 > 0 \\
\Leftrightarrow \quad & rb\lambda(\lambda-1)c + r^2b^2\lambda(\lambda-1)\tilde{m}^{(0)^{\lambda-1}} - \frac{\gamma c^2}{\tilde{m}^{(0)^{\lambda-1}}} - \gamma r^2b^2\lambda^2\tilde{m}^{(0)^{\lambda-1}} \\
& -2c\gamma rb\lambda > 0 \\
\Leftrightarrow \quad & (\lambda-1)c + rb(\lambda-1)\tilde{m}^{(0)^{\lambda-1}} - \frac{\gamma c^2}{rb\lambda\tilde{m}^{(0)^{\lambda-1}}} - \gamma rb\lambda\tilde{m}^{(0)^{\lambda-1}} \\
& -2\gamma c > 0 \\
\Leftrightarrow \quad & (\lambda-1-2\gamma)c + rb(\lambda-1-\gamma\lambda)\tilde{m}^{(0)^{\lambda-1}} - \frac{\gamma c^2}{rb\lambda\tilde{m}^{(0)^{\lambda-1}}} > 0 \\
\Leftrightarrow \quad & (\lambda-1-2\gamma)crb\lambda\tilde{m}^{(0)^{\lambda-1}} + r^2b^2\lambda(\lambda-1-\gamma\lambda)\tilde{m}^{(0)^{2(\lambda-1)}} \\
& -\gamma c^2 > 0 \qquad \text{(B.3)}
\end{aligned}$$

Setting $x = \tilde{m}^{(0)^{\lambda-1}}$, (B.3) can be rewritten as

$$\theta_1 x^2 + \theta_2 x + \theta_3 > 0 \qquad \text{(B.4)}$$

where $\theta_1 = r^2b^2\lambda(\lambda-1-\gamma\lambda)$, $\theta_2 = rb\lambda(\lambda-1-2\gamma)c$, $\theta_3 = -\gamma c^2$. Since $\Delta = \theta_2^2 - 4\theta_1\theta_3 > 0$ and $\theta_1 > 0$, we conclude that (B.4) is true for $x > \rho_1$, where ρ_1 is the positive root of (B.4). Hence, $\widetilde{MC}$ is increasing with $\tilde{m}^{(0)}$ for $\tilde{m}^{(0)} > \rho_1^{\frac{1}{\lambda-1}}$. Since $\widetilde{MB}$ is constant under risk neutrality and strictly decreasing under risk aversion while $\widetilde{MC}$ is strictly increasing for $\tilde{m}^{(0)} > \rho_1^{\frac{1}{\lambda-1}}$ and, for $\tilde{m}^{(0)} \to 0$, $\widetilde{MB} > \widetilde{MC}$ we conclude that $\widetilde{MB}$ and $\widetilde{MC}$ curves intersect at a single point that denotes the optimal capacity of the project.

As $\lambda \downarrow 1$, we have:

$$\begin{aligned} \widetilde{MC} &= \frac{c + rb}{\rho \left[c\tilde{m}^{(0)} + rb\tilde{m}^{(0)}\right]^{\gamma}} \\ \Rightarrow \quad \frac{\partial \widetilde{MC}}{\partial \tilde{m}^{(0)}} &= \frac{-\gamma \left(c + rb\right)^2}{\rho \left(c\tilde{m}^{(0)} + rb\tilde{m}^{(0)}\right)^{\gamma+1}} < 0, \quad \forall \tilde{m}^{(0)} > 0 \end{aligned} \tag{B.5}$$

Thus, the $\widetilde{MB}$ and $\widetilde{MC}$ are positive and decreasing in $\tilde{m}^{(0)}$. Hence, for $\lambda \downarrow 1$ we have:

$$\begin{aligned} \widetilde{MB} > \widetilde{MC} \quad &\Leftrightarrow \quad \frac{\mathcal{A}{P_0}^{1-\gamma}}{{\tilde{m}^{(0)}}^{\gamma}} > \frac{c + rb}{\rho \left[c\tilde{m}^{(0)} + rb\tilde{m}^{(0)}\right]^{\gamma}} \\ &\Leftrightarrow \quad \mathcal{A}{P_0}^{1-\gamma} > \frac{(c + rb)^{1-\gamma}}{\rho} \\ &\Leftrightarrow \quad \mathcal{A}U(P_0) > \frac{U(c + rb)}{\rho} \end{aligned} \tag{B.6}$$

which is true. As the marginal cost of increasing capacity is always less than the marginal benefit, it is always optimal to install greater capacity.

Proposition 3.2.2: *For a now-or-never investment opportunity, the optimal capacity decreases with uncertainty under risk aversion.*

Proof: When the firm invests in the project immediately, i.e., at P_0, the value of the optimal capacity is determined by equating the $\widetilde{MB}$ of increasing capacity to the $\widetilde{MC}$. This is described by (3.14), which we re-write here:

$$\frac{\mathcal{A}{P_0}^{1-\gamma}}{{\tilde{m}^{(0)}}^{\gamma}} = \frac{c + rb\lambda{\tilde{m}^{(0)}}^{\lambda-1}}{\rho \left[c\tilde{m}^{(0)} + rb{\tilde{m}^{(0)}}^{\lambda}\right]^{\gamma}} \tag{B.7}$$

Notice that while under risk neutrality the $\widetilde{MB}$ is constant, i.e.:

$$\widetilde{MB} = \frac{P_0}{r - \mu} \tag{B.8}$$

under risk aversion, we observe that the $\widetilde{MB}$ is a strictly decreasing and convex function of $\tilde{m}^{(0)}$:

$$\frac{\partial \widetilde{MB}}{\partial \tilde{m}^{(0)}} = -\frac{\gamma \mathcal{A} {P_0}^{1-\gamma}}{{\tilde{m}^{(0)}}^{\gamma+1}} < 0, \quad \forall \tilde{m}^{(0)} > 0 \tag{B.9}$$

$$\frac{\partial^2 \widetilde{MB}}{\partial {\tilde{m}^{(0)}}^2} = \frac{\gamma(\gamma+1) \mathcal{A} {P_0}^{\gamma-1}}{{\tilde{m}^{(0)}}^{\gamma+2}} > 0, \quad \forall \tilde{m}^{(0)} > 0 \tag{B.10}$$

On the other hand,

$$\frac{\partial \widetilde{MC}}{\partial \tilde{m}^{(0)}} = \frac{rb\lambda(\lambda-1){\tilde{m}^{(0)}}^{\lambda-2} - \gamma \left(c\tilde{m}^{(0)} + rb{\tilde{m}^{(0)}}^{\lambda}\right)^{-1} \left(c + rb\lambda {\tilde{m}^{(0)}}^{\lambda-1}\right)^2}{\rho \left(c\tilde{m}^{(0)} + rb{\tilde{m}^{(0)}}^{\lambda}\right)^{\gamma}} \tag{B.11}$$

Notice that under risk neutrality, i.e., for $\gamma = 0$,

$$\frac{\partial \widetilde{MC}}{\partial \tilde{m}^{(0)}} = \frac{rb\lambda(\lambda-1){\tilde{m}^{(0)}}^{\lambda-2}}{\rho} > 0, \quad \forall \tilde{m}^{(0)} > 0 \tag{B.12}$$

and

$$\frac{\partial^2 \widetilde{MC}}{\partial \tilde{m}^{(0)2}} = \frac{rb\lambda(\lambda-1)(\lambda-2){\tilde{m}^{(0)}}^{\lambda-3}}{\rho} > 0, \quad \forall \tilde{m}^{(0)} > 0 \tag{B.13}$$

Hence, under risk neutrality, the cost function is strictly increasing and convex. By introducing a utility-based framework and applying the CRRA utility function, we perform a concave transformation of the cost function. In fact, increasing risk aversion diminishes the convexity of the cost function and increases its concavity. Consequently, for small values of $\tilde{m}^{(0)}$, the $\widetilde{MC}$ decreases with increasing risk aversion and then increases. Without loss of generality, we assume that the initial output price is such that exercising a now-or-never investment opportunity yields positive profits. Since the $\widetilde{MB}$ is strictly decreasing and the $\widetilde{MC}$ eventually increases, the $\widetilde{MB}$ and $\widetilde{MC}$ curves intersect at a single point that denotes the optimal capacity of the project. Notice now that:

$$\frac{\partial \mathcal{A}}{\partial \sigma^2} < 0 \Rightarrow \frac{\partial \widetilde{MB}}{\partial \sigma^2} < 0 \tag{B.14}$$

Hence, volatility decreases the expected utility of the revenues for $\gamma > 0$. Since the cost of investment is non-stochastic, the $\widetilde{MC}$ is independent of σ^2, as volatility increases the $\widetilde{MB}$ and $\widetilde{MC}$ curves will intersect at a lower level of capacity, which indicates that the optimal capacity of the project decreases with volatility when the firm exercises a now-or-never investment opportunity.

Proposition 3.2.3: *For an irreversible investment opportunity, the optimal investment threshold is:*

$$P_{\tau_1}^{(0)}\left(m^{(0)^*}\right) = \left(c + rbm^{(0)^*\lambda-1}\right)\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \tag{B.15}$$

Proof: In the case of investment without operational flexibility, the value of the option to invest is given by:

$$F_{\tau_1}^{(0)}(P_0) = \max_{P_{\tau_1}^{(0)} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{(0)}}\right)^{\beta_1} \left[\mathcal{A}U\left(m^{(0)^*}P_{\tau_1}^{(0)}\right) - \frac{U\left(cm^{(0)^*} + rbm^{(0)^*\lambda}\right)}{\rho}\right] \tag{B.16}$$

This optimisation problem considers the inner extremum over capacity choice, which is described by (B.7) and may be expressed as:

$$\mathcal{A}m^{(0)^*-\gamma}P_{\tau_1}^{(0)1-\gamma} - \frac{1}{\rho}\left(cm^{(0)^*} + rbm^{(0)^*\lambda}\right)^{-\gamma}\left(c + rb\lambda m^{(0)^*\lambda-1}\right) = 0 \tag{B.17}$$

Notice that since investment takes place at τ_1, we have substituted P_0 with $P_{\tau_1}^{(0)}$. Also, since the capacity of the project is a function of the output price at investment, we have $m^{(0)\prime} \equiv \frac{\partial m^{(0)}\left(P_{\tau_1}^{(0)}\right)}{\partial P_{\tau_1}^{(0)}}$. The FONC for the optimisation problem described by (B.16) is:

$$\frac{\partial F_{\tau_1}^{(0)}(P_0)}{\partial P_{\tau_1}^{(0)}} = 0 \Leftrightarrow$$
$$\underbrace{\left[\mathcal{A} m^{(0)*^{-\gamma}} P_{\tau_1}^{(0)^{1-\gamma}} \frac{1}{\rho}\left(cm^{(0)*} + rbm^{(0)*^{\lambda}}\right)^{-\gamma}\left(c + rb\lambda m^{(0)*^{\lambda-1}}\right)\right]}_{=0} m^{(0)*\prime} -$$
$$\frac{\beta_1}{P_{\tau_1}^{(0)}}\left[\mathcal{A}\frac{\left(m^{(0)*}P_{\tau_1}^{(0)}\right)^{1-\gamma}}{1-\gamma} - \frac{\left(cm^{(0)*} + rbm^{(0)*^{\lambda}}\right)^{1-\gamma}}{\rho(1-\gamma)}\right] +$$
$$\mathcal{A} P_{\tau_1}^{(0)^{-\gamma}} m^{(0)*^{1-\gamma}} = 0 \tag{B.18}$$

Solving (B.18) with respect to $P_{\tau_1}^{(0)}$ we have:

$$\frac{\partial F_{\tau_1}^{(0)}(P_0)}{\partial P_{\tau_1}^{(0)}} = 0 \quad \Leftrightarrow \quad P_{\tau_1}^{(0)}\left(m^{(0)*}\right) = \left(c + rbm^{(0)*^{\lambda-1}}\right)\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \tag{B.19}$$

Corollary 3.2.1 The MB curve is steeper than the MC curve at $P_{\tau_1}^{(0)*}$.
Proof: See Corollary 2.2.1 in Chapter 2.
Proposition 3.2.4 *For an irreversible investment opportunity, the optimal capacity is:*

$$m^{(0)*} = \left[\frac{c}{rb}\frac{1-\gamma}{\lambda(\beta_1+\gamma-1)-\beta_1}\right]^{\frac{1}{\lambda-1}}, \ \lambda > \frac{\beta_1}{\beta_1+\gamma-1} \tag{B.20}$$

Proof: Inserting the expression for $P_{\tau_1}^{(0)}\left(m^{(0)*}\right)$ described by (B.19) into (B.17), we have:

$$\mathcal{A} m^{(0)*^{-\gamma}}\left(c + rbm^{(0)*^{\lambda-1}}\right)^{1-\gamma}\left(\frac{\beta_2+\gamma-1}{\beta_2}\right) - \frac{1}{\rho}\frac{c + rb\lambda m^{(0)*^{\lambda-1}}}{\left(cm^{(0)*} + rbm^{(0)*^{\lambda}}\right)^{\gamma}} = 0$$
$$\Leftrightarrow \quad \frac{1-\gamma}{\beta_1+\gamma-1}c + rbm^{(0)*^{\lambda-1}}\left[\frac{\beta_1(1-\lambda)-\lambda(\gamma-1)}{\beta_1+\gamma-1}\right] = 0 \tag{B.21}$$

Hence, the expression for the optimal capacity size is:

$$m^{(0)^*} = \left[\frac{c}{rb}\frac{1-\gamma}{\lambda(\beta_1+\gamma-1)-\beta_1}\right]^{\frac{1}{\lambda-1}} \tag{B.22}$$

For the closed-form expression of $m^{(0)^*}$, and, in turn, $P_{\tau_1}^{(0)^*}$ to hold we need $\lambda(\beta_1+\gamma-1)-\beta_1 > 0$; otherwise, we will obtain negative values for the optimal investment threshold and optimal capacity. This condition implies that $\lambda > \frac{\beta_1}{\beta_1+\gamma-1}$. Notice that $\beta_1+\gamma-1 < \beta_1$ and, as a result, $\frac{\beta_1}{\beta_1+\gamma-1} > 1$. Consequently, for our results to hold, we need to choose λ larger than $\frac{\beta_1}{\beta_1+\gamma-1}$, which implies that the cost function $cm^{(i)} + rK\left(m^{(i)}\right)$ must be significantly convex.

Proposition 3.2.5: *The optimal capacity is decreasing in risk aversion.*
Proof: Partially differentiating (B.22) with respect to γ, we have:

$$\frac{\partial m^{(0)^*}}{\partial\gamma} = \frac{1}{\lambda-1}\left[\frac{c}{rb}\frac{1-\gamma}{\lambda(\beta_1+\gamma-1)-\beta_1}\right]^{\frac{2-\lambda}{\lambda-1}} \times \frac{c}{rb}\frac{-\beta_1(\lambda-1)}{\left(\lambda(\beta_1+\gamma-1)-\beta_1\right)^2} \tag{B.23}$$

Note that the last term on the right-hand side of (B.23) is negative, and, as a result, we have $\frac{\partial m^{(0)^*}}{\partial\gamma} < 0$.

Proposition 3.2.6: *The optimal investment threshold price is decreasing in risk aversion.*
Proof: Solving (B.19) with respect to $m^{(0)^*}$ we have:

$$m^{(0)}\left(P_{\tau_1}^{(0)^*}\right) = \left[\frac{1}{rb}\left[\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{-\frac{1}{1-\gamma}} P_{\tau_1}^{(0)^*} - c\right]\right]^{\frac{1}{\lambda-1}} \tag{B.24}$$

By substituting the expression for $P_{\tau_1}^{(0)^*}$ from (B.19) into (B.24) we have:

$$m^{(0)}\left(P_{\tau_1}^{(0)^*}\right) = \left[\frac{c}{rb}\frac{1-\gamma}{\lambda\left(\beta_1+\gamma-1\right)-\beta_1}\right]^{\frac{1}{\lambda-1}} \tag{B.25}$$

Hence,

$$\left[\frac{1}{rb}\left[\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{-\frac{1}{1-\gamma}} P_{\tau_1}^{(0)^*}-c\right]\right]^{\frac{1}{\lambda-1}} = \left[\frac{c}{rb}\frac{1-\gamma}{\lambda(\beta_1+\gamma-1)-\beta_1}\right]^{\frac{1}{\lambda-1}} \tag{B.26}$$

Notice also that:

$$\begin{aligned}\frac{\partial m^{(0)}\left(P_{\tau_1}^{(0)^*}\right)}{\partial P_{\tau_1}^{(0)^*}} &= \frac{1}{\lambda-1}\left[\frac{1}{rb}\left[\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{-\frac{1}{1-\gamma}} P_{\tau_1}^{(0)^*}-c\right]\right]^{\frac{1}{\lambda-1}-1}\times \\ &\quad \frac{1}{rb}\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{-\frac{1}{1-\gamma}} \end{aligned} \tag{B.27}$$

As a result,

$$\frac{\partial m^{(0)}\left(P_{\tau_1}^{(0)^*}\right)}{\partial P_{\tau_1}^{(0)^*}} > 0 \Leftrightarrow P_{\tau_1}^{(0)^*} > \left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}} c \tag{B.28}$$

which according to (B.19) is true. From this, we infer that $m^{(0)}\left(P_{\tau_1}^{(0)^*}\right)$ is a monotonically increasing function of $P_{\tau_1}^{(0)^*}$. Also, according to Proposition 3.2.5, $m^{(0)^*}$ is decreasing with risk aversion. From these results, we conclude that the optimal investment threshold price is decreasing with risk aversion. This is true because if the optimal investment threshold increased with higher risk aversion, then, due to the monotonicity of $m^{(0)}\left(P_{\tau_1}^{(0)^*}\right)$, the corresponding optimal capacity would be greater, but this would contradict Proposition 3.2.5.

Proposition 3.2.7: *The optimal capacity is increasing in volatility.*
Proof: We now differentiate (B.20) with respect to σ:

$$\frac{\partial m^{(0)^*}}{\partial \sigma} = \frac{1}{\lambda-1}\left[\frac{c}{rb}\frac{1-\gamma}{\lambda(\beta_1+\gamma-1)-\beta_1}\right]^{\frac{2-\lambda}{\lambda-1}} \times \frac{c}{rb}\frac{-\frac{\partial\beta_1}{\partial\sigma}(1-\gamma)(\lambda-1)}{(\lambda(\beta_1+\gamma-1)-\beta_1)^2} \tag{B.29}$$

Since $\frac{\partial\beta_1}{\partial\sigma} < 0$, the right-hand side of (B.29) is positive, and, hence, we have

$\frac{\partial m^{(0)^*}}{\partial \sigma} > 0$.

Proposition 3.2.8: *The optimal investment threshold price is increasing in volatility.*

Proof: Substituting the expression for $m^{(0)^*}$ from (B.22) into (B.19) we obtain the expression of the optimal investment threshold:

$$P_{\tau_1}^{(0)^*} = \left(c + \frac{c(1-\gamma)}{\lambda(\beta_1+\gamma-1)-\beta_1}\right)\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \tag{B.30}$$

Let $\alpha = \left(\frac{\beta_2+\gamma-1}{\beta_2}\right)^{\frac{1}{1-\gamma}}$; From Chapter 2 we have that $\frac{\partial \alpha}{\partial \sigma^2} > 0$. Also, according to Proposition 3.2.7, the optimal capacity of the project is increasing in volatility. Hence, the first factor on the right-hand side of (B.30) is also increasing in volatility.

Proposition 3.2.9: *With a single abandonment option, the optimal capacity of the project is greater compared to an irreversible now-or-never investment opportunity, ceteris paribus.*

Proof: This result follows from the properties of the $\widetilde{MB}$ and the $\widetilde{MC}$ of increasing capacity, described in Proposition 3.2.2, and the fact that at abandonment the expected utility of the revenues is greater than the expected utility of the forgone cash flows. The expressions of the $\widetilde{MB}$ and the $\widetilde{MC}$ are:

$$\widetilde{MB} = \frac{\mathcal{A}U\left(P_0\right)}{\tilde{m}^{(1)^\gamma}} + \left(\frac{P_0}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \frac{U(c)}{\rho \tilde{m}^{(1)^\gamma}} \tag{B.31}$$

$$\widetilde{MC} = \frac{\left(c + rb\tilde{m}^{(1)^{\lambda-1}}\right)^{-\gamma}\left(c + rb\lambda\tilde{m}^{(1)^{\lambda-1}}\right)}{\rho(1-\gamma)\tilde{m}^{(1)^\gamma}} + \left(\frac{P_0}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \frac{\mathcal{A}U\left(P_{\tau_2}^{(0)^*}\right)}{\tilde{m}^{(1)^\gamma}} \tag{B.32}$$

Notice that the last terms on the right-hand sides of (B.31) and (B.32) represent the extra $\widetilde{MB}$ and $\widetilde{MC}$ from the embedded abandonment option, respectively. As their analytic expressions indicate, the former is the discounted expected utility of the salvageable operating cost divided by a monotonic function of the capacity, while the latter is the discounted expected utility of the forgone cash flows divided by the same function of the capacity. Since, at abandonment, the optimal abandonment threshold is strictly less than the operating cost, as (A.37) indicates, the extra $\widetilde{MB}$ from abandonment is greater than the corresponding extra $\widetilde{MC}$:

$$\left(\frac{P_0}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \frac{U(c)}{\rho \tilde{m}^{(1)^\gamma}} > \left(\frac{P_0}{P_{\tau_2}^{(0)^*}}\right)^{\beta_2} \frac{\mathcal{A}U\left(P_{\tau_2}^{(0)^*}\right)}{\tilde{m}^{(1)^\gamma}}, \quad \forall \tilde{m}^{(1)} > 0 \tag{B.33}$$

Hence, the $\widetilde{MB}$ from increasing capacity increases more than the $\widetilde{MC}$. As a result, the $\widetilde{MB}$ and $\widetilde{MC}$ curves intersect at a higher level of capacity, thereby indicating the increase of the optimal capacity due to the embedded abandonment option.

Appendix C

Proofs of the Propositions of Chapter 4

Proposition 4.2.1: *Uncertainty and risk aversion increase the optimal investment threshold.*
Proof: See Propositions 2.2.4 and 2.2.5 in Chapter 2.

Proposition 4.2.2: *The value function of the pre-emptive leader is concave and its maximum value is obtained prior to the entry of the pre-emptive follower provided that:*

$$D(2) < D(1)\left(\frac{\beta_1+\gamma-1}{\beta_1}\right)^{\frac{1}{1-\gamma}} \tag{C.1}$$

Proof: The value of the pre-emtpive leader is:

$$\begin{aligned} V_p^{\ell}(P_t) = \mathcal{A}U\left(P_t D(1)\right) \quad &- \quad \frac{U\left(rK\right)}{\rho} \\ &+ \quad \left(\frac{P_t}{P_{\tau_p^{f*}}}\right)^{\beta_1} \mathcal{A}U(P_{\tau_p^{f*}})\left[D(2)^{1-\gamma} - D(1)^{1-\gamma}\right] \end{aligned} \tag{C.2}$$

Differentiating (C.2) with respect to P_t we have:

$$\begin{aligned}\frac{\partial V_p^\ell(P_t)}{\partial P_t} &= \mathcal{A}D(1)^{1-\gamma}P_t{}^{-\gamma} \\ &+ \beta_1\left(\frac{P_t}{P_{\tau_p^{f*}}}\right)^{\beta_1}\frac{1}{P_t}\mathcal{A}U(P_{\tau_p^{f*}})\left[D(2)^{1-\gamma}-D(1)^{1-\gamma}\right] \qquad \text{(C.3)}\end{aligned}$$

Hence,

$$\frac{\partial V_p^\ell(P_t)}{\partial P_t}=0 \Rightarrow P_t = P_{\tau_p^{f*}}\left\{\frac{\beta_1}{1-\gamma}\left[1-\left(\frac{D(2)}{D(1)}\right)^{1-\gamma}\right]\right\}^{\frac{1}{1-\beta_1-\gamma}} \qquad \text{(C.4)}$$

Notice that $\frac{1}{1-\beta_1-\gamma}<0$. Hence, for (C.4) to be valid we must have:

$$1-\left(\frac{D(2)}{D(1)}\right)^{1-\gamma}>0 \Leftrightarrow \left(\frac{D(2)}{D(1)}\right)^{1-\gamma}<1 \Leftrightarrow D(2)<D(1) \qquad \text{(C.5)}$$

which is true. In order to show that the value of the pre-emptive leader obtains a maximum, we partially differentiate (C.3) with respect to P_t.

$$\begin{aligned}\frac{\partial^2 V_p^\ell(P_t)}{\partial P_t{}^2} &= \mathcal{A}D(1)^{1-\gamma}(-\gamma)P_t{}^{-\gamma-1} \\ &+\beta_1(\beta_1-1)P_t{}^{\beta_1-2}\left(\frac{1}{P_{\tau_p^{f*}}}\right)^{\beta_1}\mathcal{A}U(P_{\tau_p^{f*}})\left[D(2)^{1-\gamma}-D(1)^{1-\gamma}\right] \qquad \text{(C.6)}\end{aligned}$$

As both terms in (C.6) are negative, we have $\frac{\partial^2 V_p^\ell(P_t)}{\partial P_t{}^2}<0$ for all $P_t \in \left[P_{\tau_p^\ell}, P_{\tau_p^{f*}}\right)$. Finally, we will derive the condition under which the output price at which $V_p^\ell(P_t)$

becomes maximised is lower than the optimal entry threshold of the follower:

$$\begin{aligned}
\left\{\frac{\beta_1}{1-\gamma}\left[1-\left(\frac{D(2)}{D(1)}\right)^{1-\gamma}\right]\right\}^{\frac{1}{\beta_1+\gamma-1}} &> 1 \\
\Leftrightarrow \frac{\beta_1}{1-\gamma}\left[1-\left(\frac{D(2)}{D(1)}\right)^{1-\gamma}\right] &> 1 \\
\Leftrightarrow 1-\left(\frac{D(2)}{D(1)}\right)^{1-\gamma} &> \frac{1-\gamma}{\beta_1} \\
\Leftrightarrow D(2) &< \left(\frac{\beta_1+\gamma-1}{\beta_1}\right)^{\frac{1}{1-\gamma}} D(1) \qquad \text{(C.7)}
\end{aligned}$$

Notice that $\frac{\beta_1+\gamma-1}{\beta_1} < 1$. This implies that in order for the value function of the pre-emptive leader to decrease prior to the entry of the follower, the discrepancy in market share must be significantly large.

Proposition 4.2.3: *The pre-emptive leader's entry threshold is lower than that of the monopolist.*
Proof: First, notice that the follower's value of investment opportunity is:

$$F_{\tau_p^f}\left(P_t\right) = \left(\frac{P_t}{P_{\tau_p^{f*}}}\right)^{\beta_1} V_p^f\left(P_{\tau_p^{f*}}\right) \qquad \text{(C.8)}$$

Hence

$$\frac{\partial F_{\tau_p^f}\left(P_t\right)}{\partial P_t} = \beta_1 P_t^{\beta_1-1}\left(\frac{1}{P_{\tau_p^{f*}}}\right)^{\beta_1} V_p^f\left(P_{\tau_p^{f*}}\right) > 0,\ \forall P_t \in \left[P_{\tau_p^\ell}, P_{\tau_p^{f*}}\right) \qquad \text{(C.9)}$$

and

$$\frac{\partial^2 F_{\tau_p^f}\left(P_t\right)}{\partial P_t^2} = \beta_1(\beta_1-1) P_t^{\beta_1-2}\left(\frac{1}{P_{\tau_p^{f*}}}\right)^{\beta_1} V_p^f\left(P_{\tau_p^{f*}}\right) > 0,\ \forall P_t \in \left[P_{\tau_p^\ell}, P_{\tau_p^{f*}}\right) \text{(C.10)}$$

Thus, the value of the follower's investment opportunity is convex and strictly increasing from zero. Second, from Proposition 4.2.2, we know that the pre-emptive leader's value function is strictly concave in P_t starting from a negative value. Consequently, for $P_t < P_{\tau_p^{f*}}$ the two value functions intersect at most once. In order to show that the pre-emptive leader's entry threshold is lower than that of the monopolist, we will evaluate the pre-emptive leader's value and the pre-emptive follower's value of investment opportunity at the monopolist's entry threshold. The objective is to prove that at the monopolist's optimal entry threshold, the value of the pre-emptive leader is greater than the value of the pre-emptive follower's investment opportunity, i.e.,

$$\begin{aligned} \mathcal{A}U\left(P_{\tau_m^{j*}} D(1)\right) \quad - \quad & \frac{U\left(rK\right)}{\rho} + \left(\frac{P_{\tau_m^{j*}}}{P_{\tau_p^{f*}}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_p^{f*}} D(2)\right) - \mathcal{A}U\left(P_{\tau_p^{f*}} D(1)\right)\right] > \\ & \left(\frac{P_{\tau_m^{j*}}}{P_{\tau_p^{f*}}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_p^{f*}} D(2)\right) - \frac{U(rK)}{\rho}\right] \end{aligned} \tag{C.11}$$

Substituting for $P_{\tau_m^{j*}}$ and $P_{\tau_p^{f*}}$ we have:

$$\begin{aligned} & 1-\gamma+\beta_1\left(\frac{D(2)}{D(1)}\right)^{\beta_1}\left[1-\left(\frac{D(1)}{D(2)}\right)^{1-\gamma}\right] > \left(\frac{D(2)}{D(1)}\right)^{\beta_1}(1-\gamma) \\ \Leftrightarrow \quad & (1-\gamma)\left(\frac{D(1)}{D(2)}\right)^{\beta_1} - \beta_1\left(\frac{D(1)}{D(2)}\right)^{1-\gamma} > 1-\beta_1-\gamma \end{aligned} \tag{C.12}$$

The last inequality can be written as follows:

$$b - a + ax^b - bx^a > 0 \tag{C.13}$$

where $a = 1-\gamma < 1$, $b = \beta_1 > 1$, and $x = \left(\frac{D(1)}{D(2)}\right) > 1$. Since $b-a = \beta_1+\gamma-1 > 0$, in order to show (C.13), we need to show that $ax^b - bx^a > 0$. For this reason, let:

$$f(x) = ax^b - bx^a \tag{C.14}$$

Notice that:

$$f'(x) = ab\left(x^{b-1} - x^{a-1}\right) \tag{C.15}$$

Since $b > a \Rightarrow f'(x) > 0$. Notice also that:

$$f''(x) = ab\left((b-1)x^{b-2} - (a-1)x^{a-2}\right) > 0 \tag{C.16}$$

which implies that $f(x)$ is increasing and convex. Also:

$$f'(x) = 0 \Rightarrow x = 1 \quad and \quad f(1) = 0 \tag{C.17}$$

As a result, the minimum value of $f(x)$ is at $x = 1$ and is equal to $f(1) = 0$. Thus,

$$f(x) > f(1) = 0 \Rightarrow ax^b - bx^a > 0, \quad \forall\ x > 1 \tag{C.18}$$

Therefore, at the entry threshold of the monopolist, the value function of the pre-emptive leader is greater than the follower's value of investment opportunity. Notice also that:

$$P_t \to 0 \Rightarrow V_p^{\ell}(P_t) < F_{\tau_p^f}(P_t) \tag{C.19}$$

Since, according to Proposition 4.2.2, the maximum that the value of the pre-emptive leader can obtain in $\left[P_{\tau_p^{\ell}}, P_{\tau_p^{f*}}\right)$ is global, this implies that, $\forall P_t : P_t < P_{\tau_m^{j*}}$, $\exists$ at most one price $P_{\tau_p^{\ell*}} : F_{\tau_p^f} = V_p^{\ell}$. Hence, from (C.18) and (C.19), we conclude that $P_{\tau_p^{\ell*}} < P_{\tau_m^{j*}}$.

Proposition 4.2.4: *The loss in the pre-emptive leader's value relative to the monopolist's value of investment opportunity at the pre-emptive leader's optimal entry threshold price is unaffected by risk aversion.*

Proof: In order to show that the relative loss in value is unaffected by risk aversion, we consider the following ratio:

$$\frac{V_p^\ell\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)} \tag{C.20}$$

Notice that $F_{\tau_m^j}(\cdot)$ is given by (4.11), which we re-write here for $P_0 = P_{\tau_p^{\ell*}}$:

$$F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right) = \left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_m^{j*}}}\right)^{\beta_1}\left[\mathcal{A}U\left(P_{\tau_m^{j*}}D(1)\right) - \frac{U(rK)}{\rho}\right] \tag{C.21}$$

Similarly, the expression for $V_p^\ell(\cdot)$ evaluated at $P_{\tau_p^{\ell*}}$ is given by:

$$\begin{aligned} V_p^\ell\left(P_{\tau_p^{\ell*}}\right) &= \mathcal{A}U\left(P_{\tau_p^{\ell*}}D(1)\right) - \frac{U(rK)}{\rho} \\ &+ \left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_p^{f*}}}\right)^{\beta_1}\mathcal{A}U\left(P_{\tau_p^{f*}}\right)\left(D(2)^{1-\gamma} - D(1)^{1-\gamma}\right) \end{aligned} \tag{C.22}$$

Notice also that for $P_{\tau_p^{\ell*}}$, the equality $V_p^\ell(P_{\tau_p^{\ell*}}) = F_{\tau_p^f}(P_{\tau_p^{\ell*}})$ holds, i.e.:

$$\begin{aligned} \mathcal{A}U\left(P_{\tau_p^{\ell*}}D(1)\right) - \frac{U(rK)}{\rho} + \left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_p^{f*}}}\right)^{\beta_1}\mathcal{A}U\left(P_{\tau_p^{f*}}\right)\left(D(2)^{1-\gamma} - D(1)^{1-\gamma}\right) = \\ \left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_p^{f*}}}\right)^{\beta_1}\left[\mathcal{A}U\left(P_{\tau_p^{f*}}D(2)\right) - \frac{U(rK)}{\rho}\right] \end{aligned} \tag{C.23}$$

Substituting the expressions for $P_{\tau_p^{\ell*}}$ and $P_{\tau_m^{j*}}$ from (4.15) and (4.17) into (C.21) and (C.23), respectively, we have:

$$F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right) = \left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_m^{j*}}}\right)^{\beta_1}\left[\frac{1-\gamma}{\beta_1+\gamma-1}\right]\frac{U(rK)}{\rho} \tag{C.24}$$

and

$$V_p^\ell\left(P_{\tau_p^{\ell*}}\right)=\left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_p^{f*}}}\right)^{\beta_1}\left[\frac{1-\gamma}{\beta_1+\gamma-1}\right]\frac{U\left(rK\right)}{\rho} \tag{C.25}$$

By cancelling the $P_{\tau_p^{\ell*}}$ term and substituting for $P_{\tau_m^{j*}}$ and $P_{\tau_p^{f*}}$, we have:

$$\frac{V_p^\ell\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)}=\left(\frac{D(2)}{D(1)}\right)^{\beta_1} \tag{C.26}$$

As a result, the relative loss in the value of the pre-emptive leader is constant and, for this reason, is unaffected by risk aversion.

Proposition 4.2.5: *The relative discrepancy between the value of the pre-emptive leader and the monopolist at the pre-emptive leader's optimal entry threshold price diminishes with increasing uncertainty.*
Proof: According to (C.26), the relative value of the pre-emptive leader compared to that of a monopolist is:

$$\frac{V_p^\ell\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)}=\frac{V_p^\ell\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)}=\left(\frac{D(2)}{D(1)}\right)^{\beta_1} \tag{C.27}$$

Partially differentiating (C.27) with respect to σ, we have:

$$\frac{\partial}{\partial\sigma}\left\{\left(\frac{D(2)}{D(1)}\right)^{\beta_1}\right\}=\left(\frac{D(2)}{D(1)}\right)^{\beta_1}\ln\left(\frac{D(2)}{D(1)}\right)\frac{\partial\beta_1}{\partial\sigma} \tag{C.28}$$

Notice that since $\frac{\partial\beta_1}{\partial\sigma} < 0$ and $\ln\left(\frac{D(2)}{D(1)}\right) < 0$, we have:

$$\frac{\partial}{\partial\sigma}\left[\frac{V_p^{\ell}\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)}\right] > 0 \tag{C.29}$$

This implies that with increasing uncertainty, the loss in the value of the pre-emptive leader relative to the monopolist's diminishes.

Proposition 4.2.6: *The optimal entry threshold of the non-pre-emptive leader is the same as that of the monopolist.*

Proof: Given the initial output price, P_0, and assuming that the follower has chosen the optimal policy, the non-pre-emptive leader's entry problem is described by (C.30):

$$\begin{aligned} F_{\tau_n^{\ell}}(P_0) \quad = \quad & \max_{P_{\tau_n^{\ell}} \geq P_0}\left\{\left(\frac{P_0}{P_{\tau_n^{\ell}}}\right)^{\beta_1}\left[\mathcal{A}U\left(P_{\tau_n^{\ell}}D(1)\right) - \frac{U(rK)}{\rho}\right.\right. \\ & \left.\left. + \left(\frac{P_{\tau_n^{\ell}}}{P_{\tau_n^{f*}}}\right)^{\beta_1}\left[\mathcal{A}U\left(P_{\tau_n^{f*}}\right)\left(D(2)^{1-\gamma} - D(1)^{1-\gamma}\right)\right]\right]\right\} \end{aligned} \tag{C.30}$$

Partially differentiating (C.30) with respect to $P_{\tau_n^{\ell}}$ yields:

$$\begin{aligned} \frac{\partial F_{\tau_n^{\ell}}}{\partial P_{\tau_n^{\ell}}} = & \beta_1\left(\frac{P_0}{P_{\tau_n^{\ell}}}\right)^{\beta_1}\left(-\frac{1}{P_{\tau_n^{\ell}}}\right)\left[\mathcal{A}U\left(P_{\tau_n^{\ell}}D(1)\right) - \frac{U(rK)}{\rho}\right] + \\ + \quad & \left(\frac{P_0}{P_{\tau_n^{\ell}}}\right)^{\beta_1}\mathcal{A}D(1)^{1-\gamma}P_{\tau_n^{\ell}}^{-\gamma} \\ + \quad & \beta_1\left(\frac{P_0}{P_{\tau_n^{\ell}}}\right)^{\beta_1}\left(-\frac{1}{P_{\tau_n^{\ell}}}\right)\left(\frac{P_{\tau_n^{\ell}}}{P_{\tau_n^{f*}}}\right)^{\beta_1}\left[\mathcal{A}U\left(P_{\tau_n^{f*}}\right)\left(D(2)^{1-\gamma} - D(1)^{1-\gamma}\right)\right] \\ + \quad & \beta_1\left(\frac{P_0}{P_{\tau_n^{\ell}}}\right)^{\beta_1}\left(\frac{P_{\tau_n^{\ell}}}{P_{\tau_n^{f*}}}\right)^{\beta_1}\left(\frac{1}{P_{\tau_n^{\ell}}}\right)\left[\mathcal{A}U\left(P_{\tau_n^{f*}}\right)\left(D(2)^{1-\gamma} - D(1)^{1-\gamma}\right)\right] \end{aligned} \tag{C.31}$$

Rearranging (C.31) in order to equate the marginal benefit of delaying investment to the marginal cost yields (C.32). The first term on the left-hand side of (C.32) corresponds to the reduction in marginal cost due to saved investment cost, while the second term is the marginal benefit from starting the project at a higher output price. The third term reflects the marginal benefit from delaying investment, which postpones the entry of the follower. The first term on the right-hand side of (C.32) corresponds to the marginal cost of forgone cash flows due to postponed investment, while the second term reflect the marginal cost from enjoying monopoly profits for less time.

$$
\begin{aligned}
&\beta_1 \left(\frac{P_0}{P_{\tau_n^\ell}}\right)^{\beta_1} \left(\frac{1}{P_{\tau_n^\ell}}\right) \frac{U(rK)}{\rho} + \left(\frac{P_0}{P_{\tau_n^\ell}}\right)^{\beta_1} \mathcal{A}D(1)^{1-\gamma} P_{\tau_n^\ell}^{-\gamma} + \\
&\beta_1 \left(\frac{P_0}{P_{\tau_n^\ell}}\right)^{\beta_1} \left(-\frac{1}{P_{\tau_n^\ell}}\right) \left(\frac{P_{\tau_n^\ell}}{P_{\tau_n^{f*}}}\right)^{\beta_1} \left[\mathcal{A}U\left(P_{\tau_n^{f*}}\right)\left(D(2)^{1-\gamma} - D(1)^{1-\gamma}\right)\right] \\
= \;\; &\beta_1 \left(\frac{P_0}{P_{\tau_n^\ell}}\right)^{\beta_1} \left(\frac{1}{P_{\tau_n^\ell}}\right) \mathcal{A}U\left(P_{\tau_n^\ell} D(1)\right) - \\
&\beta_1 \left(\frac{P_0}{P_{\tau_n^\ell}}\right)^{\beta_1} \left(\frac{P_{\tau_n^\ell}}{P_{\tau_n^{f*}}}\right)^{\beta_1} \left(\frac{1}{P_{\tau_n^\ell}}\right) \left[\mathcal{A}U\left(P_{\tau_n^{f*}}\right)\left(D(2)^{1-\gamma} - D(1)^{1-\gamma}\right)\right] \qquad \text{(C.32)}
\end{aligned}
$$

Notice that the marginal benefit from postponing investment cancels with the marginal cost from enjoying monopoly profits for less time, and, thus, we obtain (C.33):

$$
\begin{aligned}
\frac{\partial F_{\tau_n^\ell}}{\partial P_{\tau_n^\ell}} = 0 \;\; \Leftrightarrow \;\; &\beta_1 \left(\frac{P_0}{P_{\tau_n^\ell}}\right)^{\beta_1} \left(-\frac{1}{P_{\tau_n^\ell}}\right) \left[\mathcal{A}U\left(P_{\tau_n^\ell} D(1)\right) - \frac{U(rK)}{\rho}\right] + \\
&+ \left(\frac{P_0}{P_{\tau_n^\ell}}\right)^{\beta_1} \mathcal{A}D(1)^{1-\gamma} P_{\tau_n^\ell}^{-\gamma} = 0 \\
\Leftrightarrow \;\; &P_{\tau_n^{\ell*}} = \frac{rK}{D(1)} \left(\frac{\beta_2 + \gamma - 1}{\beta_2}\right)^{\frac{1}{1-\gamma}} \qquad \text{(C.33)}
\end{aligned}
$$

From (C.33), we see that the optimal investment threshold for the non-pre-emptive leader is the same as the monopolist's.

Proposition 4.2.7: *The loss in the value of the investment opportunity for the non-pre-emptive leader relative to that of a monopolist at the pre-emptive leader's optimal entry threshold price decreases with risk aversion.*
Proof: The relative loss in the non-pre-emptive leader's value is:

$$\frac{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right) - F_{\tau_n^\ell}\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)} = 1 - \frac{F_{\tau_n^\ell}\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)} \tag{C.34}$$

Recall that prior to investment, the non-pre-emptive leader's value of investment opportunity at $P_{\tau_p^{\ell*}}$ is:

$$\begin{aligned} F_{\tau_n^\ell}\left(P_{\tau_p^{\ell*}}\right) &= \left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_n^{\ell*}}}\right)^{\beta_1}\left[\mathcal{A}U\left(P_{\tau_n^{\ell*}}D(1)\right) - \frac{U(rK)}{\rho}\right. \\ &\quad \left. + \left(\frac{P_{\tau_n^{\ell*}}}{P_{\tau_n^{f*}}}\right)^{\beta_1}\left[\mathcal{A}\frac{P_{\tau_n^{f*}}^{1-\gamma}}{1-\gamma}\left(D(2)^{1-\gamma} - D(1)^{1-\gamma}\right)\right]\right] \end{aligned} \tag{C.35}$$

Notice that the expression of the monopolist's value of investment opportunity, $F_{\tau_m^j}(\cdot)$, evaluated at $P_{\tau_p^{\ell*}}$ is given by (C.36):

$$F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right) = \left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_m^{j*}}}\right)^{\beta_1}\left[\mathcal{A}U\left(P_{\tau_m^{j*}}D(1)\right) - \frac{U(rK)}{\rho}\right] \tag{C.36}$$

Hence,

$$1-\frac{F_{\tau_n^\ell}\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)} = -\frac{\left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_n^{\ell*}}}\right)^{\beta_1}\left(\frac{P_{\tau_n^{\ell*}}}{P_{\tau_n^{f*}}}\right)^{\beta_1}\left[\mathcal{A}P_{\tau_n^{f*}}^{1-\gamma}\left(D(2)^{1-\gamma}-D(1)^{1-\gamma}\right)\right]}{\left(\frac{P_{\tau_p^{\ell*}}}{P_{\tau_m^{j*}}}\right)^{\beta_1}\left[\mathcal{A}\left(P_{\tau_m^{j*}}D(1)\right)^{1-\gamma}-\frac{(rK)^{1-\gamma}}{\rho}\right]}$$

$$= \left(\frac{D(2)}{D(1)}\right)^{\beta_1}\frac{\mathcal{A}\left(\frac{rK}{D(2)}\right)^{1-\gamma}\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)\left(D(1)^{1-\gamma}-D(2)^{1-\gamma}\right)}{\mathcal{A}\left(\frac{rK}{D(1)}\right)^{1-\gamma}\left(\frac{\beta_2+\gamma-1}{\beta_2}\right)D(1)^{1-\gamma}-\frac{(rK)^{1-\gamma}}{\rho}}$$

$$= \left(\frac{D(2)}{D(1)}\right)^{\beta_1}\frac{\beta_1}{1-\gamma}\left[\left(\frac{D(1)}{D(2)}\right)^{1-\gamma}-1\right] \tag{C.37}$$

Partially differentiating (C.34) with respect to γ yields:

$$\frac{\partial}{\partial\gamma}\left[1-\frac{F_{\tau_n^\ell}\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_n^{\ell*}}\right)}\right] = \left(\frac{D(2)}{D(1)}\right)^{\beta_1}\frac{\beta_1}{1-\gamma}\times \left\{\frac{\left(\frac{D(1)}{D(2)}\right)^{1-\gamma}-1}{1-\gamma}-\left(\frac{D(1)}{D(2)}\right)^{1-\gamma}\ln\frac{D(1)}{D(2)}\right\} \tag{C.38}$$

According to (C.38),

$$\frac{\partial}{\partial\gamma}\left[1-\frac{F_{\tau_n^\ell}\left(P_{\tau_n^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_n^{\ell*}}\right)}\right]\leq 0 \Leftrightarrow \frac{\left(\frac{D(1)}{D(2)}\right)^{1-\gamma}-1}{1-\gamma}-\left(\frac{D(1)}{D(2)}\right)^{1-\gamma}\ln\frac{D(1)}{D(2)}\leq 0$$

$$\Leftrightarrow \left(\frac{D(1)}{D(2)}\right)^{1-\gamma}\left[1-\ln\left(\frac{D(1)}{D(2)}\right)^{1-\gamma}\right]-1\leq 0 \tag{C.39}$$

Setting $x = \left(\frac{D(1)}{D(2)}\right)^{1-\gamma} > 0$ we have:

$$\begin{aligned} x - 1 - x\ln x \leq 0 \quad &\Leftrightarrow \quad x\,(1 - \ln x) \leq 1 \\ &\Leftrightarrow \quad 1 - \ln x \leq \frac{1}{x} \\ &\Leftrightarrow \quad 1 + \ln\frac{1}{x} \leq \frac{1}{x} \\ &\Leftrightarrow \quad -\ln\frac{1}{x} \geq 1 - \frac{1}{x} \end{aligned} \tag{C.40}$$

Setting $\frac{1}{x} = y$ we have:

$$-\ln y \geq 1 - y \Leftrightarrow \ln y \leq y - 1 \tag{C.41}$$

which is true.

Proposition 4.2.8: *The discrepancy between the non-pre-emptive leader's value of investment opportunity and the monopolist's at the pre-emptive leader's optimal entry threshold price increases with uncertainty if:*

$$\left(\frac{D(1)}{D(2)}\right)^{\beta_1} > e$$

Proof: According to (C.37), the relative loss in option value of the non-pre-emptive leader is:

$$1 - \frac{F_{\tau_n^\ell}\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)} = \left(\frac{D(2)}{D(1)}\right)^{\beta_1} \frac{\beta_1}{1-\gamma}\left[\left(\frac{D(1)}{D(2)}\right)^{1-\gamma} - 1\right] \tag{C.42}$$

Partially differentiating with respect to σ we have:

$$\frac{\partial}{\partial\sigma}\left[1-\frac{F_{\tau_n^\ell}\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)}\right] = \left[\left(\frac{D(1)}{D(2)}\right)^{1-\gamma}-1\right]\left(\frac{D(2)}{D(1)}\right)^{\beta_1}\times \left\{\frac{\frac{\partial\beta_1}{\partial\sigma}}{1-\gamma}+\frac{\partial\beta_1}{\partial\sigma}\ln\left(\frac{D(2)}{D(1)}\right)\frac{\beta_1}{1-\gamma}\right\} \tag{C.43}$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\left[1-\frac{F_{\tau_n^\ell}\left(P_{\tau_p^{\ell*}}\right)}{F_{\tau_m^j}\left(P_{\tau_p^{\ell*}}\right)}\right]>0 &\Leftrightarrow \frac{\frac{\partial\beta_1}{\partial\sigma}}{1-\gamma}+\frac{\partial\beta_1}{\partial\sigma}\ln\frac{D(2)}{D(1)}\frac{\beta_1}{1-\gamma}>0\\
&\Leftrightarrow \ln\left(\frac{D(2)}{D(1)}\right)^{\beta_1}<-1\\
&\Leftrightarrow \left(\frac{D(2)}{D(1)}\right)^{\beta_1}<e^{-1}\\
&\Leftrightarrow \left(\frac{D(1)}{D(2)}\right)^{\beta_1}>e
\end{aligned} \tag{C.44}$$

References

Bar-Ilan, A and W Strange (1996), "Investment Lags," *The American Economic Review* 86(3): 610–622.

Black, F, and M Scholes (1973), "The Pricing of Options and Corporate Liabilities," *The Journal of Political Economy* 81(3): 637–654

Bouis, R, KJM Huisman, and PM Kort (2009), "Investment in Oligopoly under Uncertainty: The Accordion Effect," *International Journal of Industrial Organisation* 27: 320–331

Brennan, MJ and ES Schwartz (1985), "Evaluating Natural Resource Investments," *Journal of Business* 58(2): 135–157.

Dangl, T (1999), "Investment and Capacity Choice under Uncertain Demand," *European Journal of Operational Research* 117: 415–428.

Décamps, J–P, T Mariotti, and S Villeneuve (2006), "Irreversible Investment in Alternative Projects," *Economic Theory* 28: 425–448.

Deng, S-J, B Johnson, and A Sogomonian (2001), "Exotic Electricity Options and The Valuation of Electricity Generation and Transmission Assets," *Decision Support Systems* 30: 383–392.

Dixit, AK (1993), "Choosing Among Alternative Discrete Investment

Projects under Uncertainty," *Economic Letters* 41: 265–268.

Dixit, AK and RS Pindyck (1994), *Investment under Uncertainty*, Princeton University Press, Princeton, NJ, USA.

Financial Times (2008), "Honda Quits F1 as Financial Crisis Bites," 4 December.

Fudenberg, D and J Tirole (1985), "Preemption and Rent Equalisation of New Technology," *The Review of Economic Studies* 52: 382–401.

Gollier, C, D Proult, F Thais, and G Walgenwitz (2005), "Choice of Nuclear Power Investments under Price Uncertainty: Valuing Modularity," *Energy Economics* 27: 667–685.

Grenadier, S (2002), "Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms," *Review of Financial Studies* 15(3): 691–721.

Grenadier, S (1996), "The Strategic Exercise of Options: Development Cascades and Overbuilding in Real Estate Markets," *Journal of Finance* 51(5): 1653–1679.

Hagspiel, V, K Huisman, and P Kort (2010), "Production Flexibility and Investment under Uncertainty," working paper, Tilburg University, The Netherlands.

Henderson, V (2007), "Valuing the Option to Invest in an Incomplete Market," *Mathematics and Financial Economics* 1: 103–128.

Henderson, V and D Hobson (2002), "Real Options with Constant Relative Risk Aversion," *Journal of Economic Dynamics & Control* 27: 329–355.

Hugonnier, J and E Morellec (2007), "Real Options and Risk Aversion," working paper, HEC Lausanne, Lausanne, Switzerland.

Huisman, KJM and PM Kort (1999), "Effects of Strategic Interactions on the Option Value of Waiting," working paper, Tilburg University, Tilburg, The Netherlands.

Huisman, KJM and PM Kort (2009), "Strategic Capacity Investment under Uncertainty," working paper, Tilburg University, Tilburg, The Netherlands.

Independent Energy Review (2010), "Turbulent Times for European Gas Market," 19 October.

Karatzas, I and S Shreve (1999), *Methods of Mathematical Finance*, Springer Verlag, New York, NY, USA.

Lund, D (2005), "How to Analyse the Investment-Uncertainty Relationship in Real Option Models," *Review of Financial Economics* 14: 311–322

Majd, S and RS Pindyck (1987), "Time to Build, Option Value, and Investment Decisions," *Journal of Financial Economics* 18(1): 7–27.

Malchow-Møller, N and BJ Thorsen (2005), "Repeated Real Options: Optimal Investment Behaviour and a Good Rule of Thumb," *Journal of Economic Dynamics & Control* 29: 1025–1041

McDonald, RL (2006), *Derivatives Markets*, 2nd edition, Addison-Wesley, Boston, MA, USA.

McDonald, RL and DS Siegel (1985), "Investment and Valuation of Firms When There is an Option to Shut Down," *International Economic Review* 26(2): 331–349.

McDonald, RL and DS Siegel (1986), "The Value of Waiting to Invest," *The Quarterly Journal of Economics* 101(4): 707–728.

Merton, RC (1969), "Lifetime Portfolio Selection under Uncertainty: The Continuous–Time Case," *The Review of Economics and Statistics* 51(3): 247–257.

Merton, RC (1973), "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41(5): 867–887.

Pawlina, G, and PM Kort (2006), "Real Options in an Asymmetric Duopoly: Who Benefits from Your Competitive Disadvantage?" *Journal of Economics & Management Strategy* 15(1): 1–35.

Paxson, D and H Pinto (2005), "Rivalry under Price and Quantity Uncertainty," *Review of Financial Economics* 14: 209–224.

Shackleton, M, A Tsekrekos, and R Wojakowski (2004), "Strategic Entry and Market Leadership in a Two-Player Real Options Game," *Journal of Banking and Finance* 28: 179–201.

Smets, F (1993), "Essays on Foreign Direct Investment," *PhD thesis*, Yale University.

Smit, H, and L Trigeorgis (2006), "Real Options and Games: Competition, Alliances and Other Applications of Valuation and Strategy," *Review of Financial Economics* 15: 95–112.

Takashima, R, M Goto, H Kimura, and H Madarame (2008),"Entry into the Electricity Market: Uncertainty, Competition, and Mothballing Options," *Energy Economics* 30: 1809–1830.

The Economist (2010), "High-Speed Railroading," 22 June.

The Wall Street Journal (2010), "Green Energy Rush Hit by Headwinds," 21 April.

The Wall Street Journal (2011), "Can Nokia and Microsoft Salvage Each Others' Phone Businesses," 11 February.

Thijssen, J, KJM Huisman, and PM Kort (2002), "Symmetric Equilibrium Strategies in Game Theoretic Real Option Models," working paper, Tilburg University, Tilburg, The Netherlands.

Weeds, H (2002), "Strategic Delay in a Real Options Model of R&D Competition," *Review of Economic Studies* 69: 729–747.

Williams, J (1993), "Equilibrium and Options on Real Assets," *Review of Financial Studies* 6(4): 825–850.