A GENERALIZED STRIP METHOD OF REINFORCED CONCRETE SLAB DESIGN

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A THESIS SUBMITTED TO THE UNIVERSITY
OF LONDON FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

1977

DEPARTMENT OF CIVIL AND MUNICIPAL ENGINEERING
UNIVERSITY COLLEGE LONDON

ABSTRACT

This thesis is concerned with the development of the strip method of reinforced concrete slab design to extend and improve its practical use. The existing elastic and plastic methods for reinforced concrete slab design are first reviewed. The fundamentals and conditions for uniqueness of the predicted collapse load of slabs designed by the strip method are examined. A new generalised strip method of reinforced concrete slab design is suggested which overcomes the limitations of the Hillerborg method. An experimental programme of tests on model slabs designed by the new method is described and results are compared with the theory. The relevance of this work in the design of concrete slabs is discussed and recommendations are made for future work.

ACKNOWLEDGEMENTS

I take this opportunity to express my sincere gratitude to Professor K.O. Kemp for providing the facilities required to conduct this research. His guidance and helpfulness throughout the whole project was invaluable.

I should like to thank Messrs. D. Vale, J. Jackson, J. Ford, W. Vines, Y.O. Adepoju and L.T. Cheung for their assistance and advice in preparing the experimental programme.

I gratefully acknowledge the financial support given to me by the Science Research Council.

Finally I should like to thank my wife for having the patience to type a difficult manuscript.

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NOTATIONS

x, y, z	cartesian reference system
D	flexural rigidity of the slab
E	modulus of elasticity
I	moment of inertia
L, ℓ	dimensions of slab
M_x, M_y	normal bending moments acting on unit lengths of the slab in the x and y directions respectively
M_{xy}, M_{yx}	twisting moments acting on unit lengths of the slab in the y and x directions respectively
M_1, M_2	principal moments acting on unit lengths of the slab
m_x, m_y	positive yield moments per unit lengths of the slab in the x and y directions respectively
$-m_x', -m_y'$	negative yield moments per unit length of the slab in the x and y directions respectively
$\dot{K}_x, \dot{K}_y, \dot{K}_{xy}$	plastic curvature rates associated with M_x, M_y and M_{xy} respectively
q	intensity of distributed load
q_c	upper bound on the collapse load intensity
q_x, q_y	intensities of load distributed in the x and y directions respectively
W	concentrated load
W_c	upper bound on the collapse load
W_D	design load
W_M	maximum load applied during the tests
W_T	theoretical failure load

α	load distribution factor
$(\Delta)_{ij}$	vertical deflection of grid (ij)
γ	stiffness ratio beam / slab
θ	clockwise angle from x axis to the normal to a yield line
$(k_x)_{ij}, (k_y)_{ij}$	flexibility matrices for strips X_i and Y_j respectively
Q_x, Q_y	Shear forces per unit length in the y and x directions respectively

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CHAPTER ONE

INTRODUCTION

In reinforced concrete structures, slab systems are of great importance. As in all other structures their design is governed by the general desire to have a safe and satisfactory structure at minimum cost. The designer in consultation with the client will arrive at the required standards of strength, durability, safety and aesthetics. If a value or a cost can be assigned to each of these factors then there exists a total minimum cost of the slab system.

No matter how much money is spent on a structure, its absolute safety cannot be guaranteed. Limit state design philosophy requires an estimate of the probabilities of overloading and the variability of strengths. Currently safeguards are taken against the variations of material strength and applied loads by the use of "Partial safety factors" applied to the "Characteristic" loads and "Characteristic" material strengths. The structure is designed for various combitinations of dead, imposed and wind loads and variable partial safety factors on these loads are used to allow for the probability of joint occurence, the inaccuracies of the theories employed and defects in construction. Currently statistical methods can only be used directly for wind loading (or wave loading) where extensive data is available.

With the help of statistics limit state design aims at a more economical and more reliable design. The usual approach will be to design on the basis of the most critical limit state and then to check that the remaining limit states will not be reached. For most structures the critical state is the state of collapse or the ultimate limit state. If the limit capacity of a slab can be determined by considering its actual behaviour at collapse then the designer is in a position to estimate the true reserve strength available. The state at which the slab system ceases to be serviceable is again important. The size of flexural crack and the deflection of the slab are the main parameters which control the serviceability limit states. Fire resistance or vibrations may determine the usefulness in other cases.

The main purpose of this study is to present a new strip method that can be applied to any reinforced concrete slab design for the limit state of collapse but in which the serviceability limit states are also considered. Firstly the development of elastic plate theory and the popular elastic methods used in reinforced concrete slab design are described in chapter two and their limitations discussed.

Chapter three critically examines the existing plastic methods associated with slab design. The yield criterion, yield line theory, minimum weight design, plastic theorems are briefly discussed and the use of the lower bound theorem in Hillerborg's strip method of slab design described.

Chapter four investigates the uniqueness of the predicted collapse load of uniformly loaded and continuously supported concrete slabs designed by the strip method. The conditions under which they give a unique value of the collapse load are discussed. It is shown that the strip method does not always produce a unique solution on the collapse load.

Chapter five proceeds to formulate a new method known as "The Strip Deflection Method" to cover all types of loading, slab geometry and boundary conditions including partial composite action with supporting beams in the design of reinforced concrete slabs. This method aims at retaining all the advantages and overcoming the restrictions in the Hillerborg approach. It will further ensure that the designer will not depart too far from the working load moment field and thereby ensuring satisfactory serviceability conditions. Using this method the design loads of the slabs will be the unique collapse loads as predicted by yield line theory. Although the effects of membrane action are important, it has been excluded from this study of slabs. In most cases membrane action will enhance the load carrying capacity, therefore the slabs will in practice carry loads above the collapse load predicted by yield line theory.

Point loads on slabs and column supports are two areas where the Hillerborg strip method failed to produce a simple design procedure but can be readily accommodated by the new proposed method. Uniqueness of the predicted collapse load is then affected by the particular choice of the strip layout and the actual position of the loads and columns. Methods of restoring uniqueness for such slabs are given in chapter six.

To establish the validity of the analytical methods in the previous chapters a series of tests were performed on model concrete slabs and are described in chapter seven.

Chapter eight summarises the theoretical and experimental results of the study from which certain conclusions are drawn and reccommendations made for the use of the strip method for the design of reinforced concrete slabs. Suggestions are also made for further research.

Appendix 1 contains the method used to calculate end reactions, fixed end moments and deflections of slab strips with standard boundary conditions. Appendix 11 summarises miscellaneous calculations. Reference to existing literature are numbered after the author in consecutive order in the text and a complete list is given at the end of the thesis.

CHAPTER TWO

ELASTIC METHODS OF SLAB DESIGN

2.1 INTRODUCTION

Flat slabs or plates are important structural elements. Before the introduction of reinforced concrete the use of plates was confined to flat plates, plating for ships and floating decks, hopper bottoms for coal bunkers etc. Today reinforced concrete slabs are almost invariably used for the floor slabs of public and commercial buildings, multistorey housing, bridge decks, tanks and containers.

2.2. DEVELOPMENT OF THE ELASTIC THIN PLATE THEORY

2.2.1 Equation for the deflected surface

A good account of the historical development of the elastic theory of plates is given by Timoshenko and Woinowsky-Krieger (1). A historical summary is also found in the publication by Westergaard and Slater (3). The invention of the high speed electronic computer gave the real impetus to the development of numerical methods in solving complex plate problems. The recent advancements in the design and analysis of plates is given by Szilard (2).

Euler, Bernoulli and Chladni were among the contributors to this subject in the eigtheenth century. Early incentives to the studies of slabs appear to have been an interest in their vibrations, particularly those producing sound. Madame Sophie Germain was the first to obtain a differential equation for the elastic deflected surface. This work which she submitted to the French Academy of Science in 1811 was however found to be in error. Lagrange was one of the judges that examined Madame Germain's work and in the same year stated the classical fourth order partial differential equation that governs the elastic flexure of plates. This equation named after him and with the notations and sign convention given in Ref (1) and Fig (2.1) is

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (2.1)$$

Equation (2.1) can be written in the symbolic form

$$D \Delta \Delta w = q \qquad (2.1.a)$$

where

$$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \qquad (2.2)$$

The plate equation (2.1) is based on the following assumptions. (a) The material of the plate is linear elastic, homogeneous and isotropic. (b) The thickness of the plate is small compared to other dimensions. (c) The deflections are small compared to the plate thickness. (d) Loads are carried normal to the plate surface and the transverse compressive stresses produced by the loads are negligible. (e) Shear deflections are small.

To obtain the solution for any elastic plate problem it is necessary to satisfy simultaneously the partial differential equation (2.1) and the boundary conditions. Since it is a fourth order differential equation, four conditions, two at each boundary are required. The boundary conditions will be determined by either force conditions such as bending moment, shear force, twisting moment or displacement conditions - deflection, slope of the deflected surface at each edge. Poisson showed that along a free edge there are no bending or twisting moments and also no shearing forces. It was later argued by Kirchoff (1850) that three conditions are too many and only two conditions can

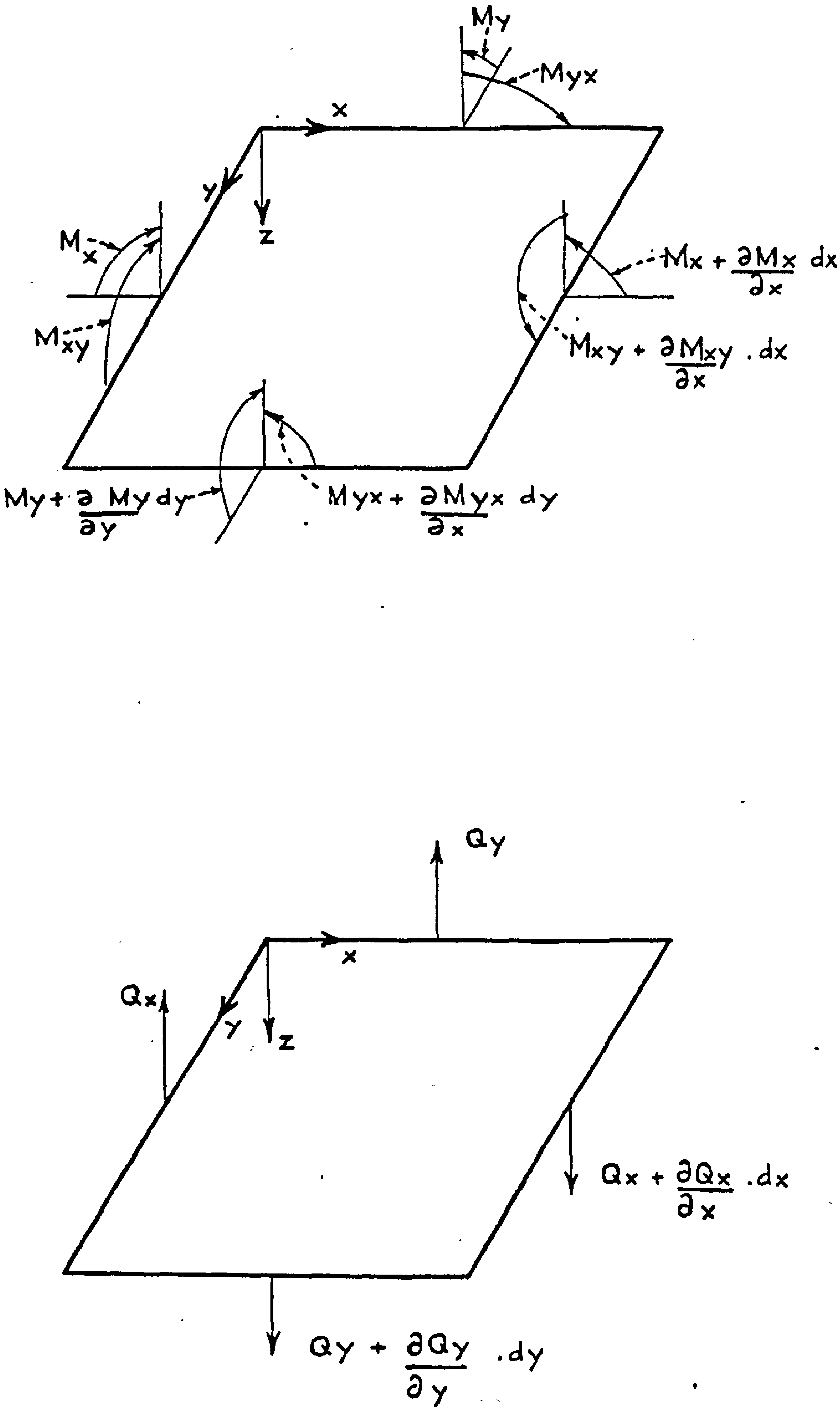


FIG. 2.1 POSITIVE MOMENT AND SHEAR STRESS RESULTANTS

be satisfied. Poisson's requirement dealing with the shearing force and the twisting moment were replaced by one condition namely total transverse shear force. The requirement along a free edge (say $x = a$) is then

$$V_x = \left(Q_x - \frac{\partial M_{xy}}{\partial y} \right)_{x = a} = 0 \qquad (2.3)$$

2. 2. 2 Solutions to Langrange's Equation

(a) Navier's Solution

In 1820 Navier in a paper presented before the French Academy of Science solved the Langrange equation for the case of a rectangular slab. His solution is applicable only to slabs with simply supported edge conditions at the four boundaries. The method which transforms the equations (2.1) into an algebraic equation was based on the use of trigonometric series introduced by Fourier in the same decade. The loading q at any point is represented by

$$q = f(x, y) \qquad (2.4)$$

Where $f(x, y)$ is expressed in the form of a double trigonometric series.

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} . \text{Sin} \frac{m \pi x}{a} . \text{Sin} \frac{n \pi y}{b} \qquad (2.5)$$

The coefficient a_{mn} of the double trigonometric series depends on the type of loading namely uniform, patch or point. The deflection of a plate carrying a uniform load q_o per unit area and with the notations given in Ref (1) is

$$w = \frac{16 q_o}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\text{Sin} \frac{m \pi x}{a} . \text{Sin} \frac{n \pi y}{b}}{mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \qquad (2.6)$$

$$m,n = 1, 3, 5 \ldots\ldots \infty$$

Solutions can be obtained for other types of loading, provided the slab is rectangular and simply supported.

(b) Levy's Solution

Levy's solution is in the form of a single trigonometric series.

$$w = \sum_{m=1}^{\infty} Y_m \sin \frac{m\pi x}{a} \qquad (2.7)$$

This method is applicable to rectangular slabs with two edges simply supported. Equation (2.1) is a linear differential equation and therefore Levy made a further simplification and expressed the deflected surface as

$$w = w_1 + w_2 \qquad (2.8)$$

Where w_1 is a particular solution of equation (2.1) and w_2 is the solution of the homogeneous plate equation

$$\Delta\Delta w = 0 \qquad (2.9)$$

For a uniformly loaded rectangular plate it can be shown that (see Ref (1))

$$w_1 = \sum_{m=1}^{\infty} w_m \sin \frac{m\pi x}{a} \qquad (2\ 10)$$

and

$$w_2 = \sum_{m=1}^{\infty} \left(A_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} . \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a} \qquad (2.11)$$

The method in the form presented by Levy can be applied to rectangular plates when (a) two opposite edges of the plate are simply supported (b) The shape of the loading diagram is the same for all sections perpendicular to these simply supported edges.

These limitations have now been overcome and Levy's method can be applied to all possible combinations of boundary conditions round the periphery of a rectangular slab. Hence Levy's method is more general than Navier's Solution.

(c) Navier's and Levy's Solutions - General comment

The solutions by Navier and Levy offer definite mathematical advantages in that the solution of the fourth order partial differential equation is presented in a series form. Further these methods provided the standard solutions to which the results of many approximate and numerical method subsequently developed can be compared.

Explicit solutions for elastic plates can however only be found for a limited number of cases. For the majority of plate problems such a series solution cannot be found or are tedious to obtain due to excessive computations. For some problems these series may yield mathematically "Exact" solutions with a reasonable number of terms. For many more their rates of convergence do however present difficulties. The double Fourier series in the Navier's solution converges very slowly near the boundaries. Slow convergence is also pronounced for discontinuous loading or concentrated forces. Levy's method based on a single Fourier Series converges more rapidly.

The shear forces and bending moments etc are derived by differentiation from the deflection surface. The rates of convergence of these functions is slower than for deflections and is very poor near the corner of the plate and in the vicinity of concentrated applied loads.

2.3 DEVELOPMENTS OF ELASTIC METHODS

2.3.1 Introduction

The solution of plate problems via a classical series method is limited to simple shapes, loads and boundary conditions. For more general cases with complex shapes, boundaries or loading the analysis by this method becomes tedious or may even be impossible. In such cases numerical and approximate methods are the only approaches that can be employed.

In structural mechanics, numerical methods have been used when rigourous mathametical solutions are unobtainable. The introduction of the digital computer has played a major part in the development and application of numerical methods. Standard programmes for solving large matrix equations are now readily available. Computer oriented numerical methods now in common use are the finite difference and the finite element techniques, a development of the earlier energy method. A popular approximate method that is suitable for computer application and employed for plate problems is the grid analogy approach, where the plate or the slab is approximated to a gridwork of beams.

2.3.2 Finite difference method

Finite difference methods of solving differential equations were known even in the eighteenth century. Here the governing differential equation is replaced by a set of difference quantities at certain selected points. This method was originally applied to beams by considering it as a loaded cable. Later N.J. Nielson and Dr. Marcus developed and applied the finite difference method to slab design.

The aim of the method is to transform the governing Lagrange equation (2.1) into a set of simultaneous linear equations involving the unknown deflections at the mesh points. Difference representations of boundary condition normally require the introduction of fictitious points outside the slab. If the loading on the slab is uniform its value can be directly used but if the load variation is large then a method of averaging is necessary together with a finer mesh.

The finite difference technique is a general numerical method which is easy to understand. In many standard problems the set of simultaneous equations can be solved using a programmable desk top calculator. The accuracy of the method depends on the size of the grid and the manner in which the loads and boundary conditions are represented. Its accuracy can infact be improved by using refined finite difference expressions. The shear forces and bending moments which are needed in the design depend on the second and third derivatives of the deflection function. The accuracy of the derivatives deteriorates with their order and hence the moments and shears will be less accurate than deflections.

2.3.3 Energy Methods

In the preceding section the elastic plate problem has been represented by the partial differential equation, which with the boundary conditions have been solved by a series or a difference method. An alternative approach is based on methods using either the principle of virtual work or the principle of minimum potential energy. Energy methods using the principle of minimum potential energy were first applied to plates by Ritz in 1909. The deflected middle surface of the plate was represented in a series form

$$w\,(x,y) = C_1 f_1\,(x,y) + C_2 f_2\,(x,y) \;.....\; C_n f_n\,(x,y) \qquad (2.12)$$

where each $f\,(x,y)$ must satisfy the boundary conditions. The total strain energy of the plate in bending (Ref (2)) is given by

$$U = \int_{Area} (M_x K_x + M_y K_y + 2 M_{xy} K_{xy})\, dA \qquad (2.13)$$

The change in the potential energy of the external forces is

$$V = - \int_{Area} (p_z \ w) \, dA \qquad (2.14)$$

and the total potential energy of the plate is

$$P = U + V \qquad (2.15)$$

The unknowns C_1 , C_2 , ... C_n are then determined from the principle of minimum potential energy, which states that of all possible deflected forms satisfying the boundary conditions that for which the total potential energy is a minimum satisfies equilibrium.

$$\text{Thus} \quad \frac{\partial P}{\partial C_1} = \frac{\partial P}{\partial C_2} = \ldots\ldots \frac{\partial P}{\partial C_n} = 0 \qquad (2.16)$$

This yields n simultaneous equations from which the unknown coefficients C_i can be calculated. Clearly the solution of a plate problem is reduced to selecting functions f(x,y) and the accuracy of Ritz's method depends on how well these functions are capable of describing the actual deflected surface including the conditions at the boundaries.

2.3.4 Finite Element Methods

Energy methods are widely used for solving structural problems. However the application of Ritz method to complex plate problems was retarded due to the difficulties in selecting proper functions f (x,y) . The finite element technique is a direct development of the Ritz method where these functions (shape functions) are chosen for a smaller region rather than for the entire area of the slab. Hence the finite element method is sometimes called the " Localised Rayleigh - Ritz Method ".

The structure, in this case the plate or the slab is divided into a set of elements which are "joined" at the nodal points. The deflection within each element is defined in terms of generalised coordinates at the nodal points. These generalised coordinates can be the deflection, slopes of the deflected curve, curvatures etc. To satisfy minimum conditions for convergence, continuity should be achieved for all derivatives up to and including one order lower than those contained in the strain energy expression. The expression for the flexural strain energy for plates contain second order derivatives, therefore continuity of deflection and slope of deflected curve are required to define conforming displacement functions.

Using these functions and the derived stiffness matrices for standard shapes of elements an approximate total potential energy of the element is computed. By summing up for all the elements the potential energy of the plate is determined. Similar to the Ritz's method the basic Lagrange equation is not used but the principle of minimum potential energy is invoked to produce a set of simultaneous equilibrium equations from which the generalised coordinates can be found.

Standard finite element programmes are available to solve almost any plate problem. This method is perhaps more computer orientated than the finite difference technique. The accuracy of this method depends on the number of elements, the accuracy of the displacement function, techniques of representing loads and boundary conditions. The preparation of data can be time consuming and is a potential source of human error. At times the method requires the services of computer specialists. Generally the physical understanding of the problem is lost and it is difficult to check the accuracy of the results.

2.3.5 The Grid Analogy

The grid analogy is again a very old concept dating back to the times of Euler and Bernoulli. They attempted to explain the vibration of plates by considering their division into beam strips Danusso extended this idea to the elastic bending of plates in which a continuum form of the plate is approximated by a grid system of beams. The loads are applied at the joints where the beams are connected.

The analogy is achieved by using an appropriate geometry for the grid and selecting equivalent section properties for the individual beam elements inorder to represent a grid of equivalent structural performance to the true plate or slab.

(a) Torsion Grids

In a torsion grid both ends of the beam elements are subjected to shear, torsion and bending moments. At each joint the corresponding displacement consists of rotations about two axes together with a vertical deflection. For any beam element the force vector $[F]$ and the displacement vector $[\Delta]$ are related by

$$[F] = [k][\Delta] \qquad (2.17)$$

Where $[k]$ is the (6x6)stiffness matrix of the beam element.

(b) Torsionless Grids

If the torsion of the beam elements and the corresponding twists at the joints are ignored a torsionless grid is produced. The (6 x 6) element stifness matrix is then reduced to a (4 x 4) matrix. Examples of torsionless grids are therefore easier to solve due to the reduction in the number of unknowns. The torsionless grid is not a commonly used method now in slab design, probably because designers believe that a more realistic solution is obtained by the inclusion of torsion. It will however be shown later using plastic theory that a torsionless grid can be used to produce an exact solution for the collapse load of a reinforced concrete slab, whereas the torsion grid leads only to a lower bound solution.

2.4 DESIGN OF TWO WAY REINFORCED CONCRETE SLABS - AN ENGINEERING APPROACH TO ELASTIC DESIGN

2.4. 1 Introduction

The floor system of buildings often consists of a regular array of rectangular concrete slabs and therefore the design of such continuous panels is of considerable practical importance. The internal panels are in general supported on the four sides by beams or walls except in the case of flat slabs where only column supports are provided. Free edges may occur at the boundaries of external panels. The dead loads on the floors are of course uniformly distributed but it is not yet possible to determine the true nature of the imposed loads. In normal buildings the imposed loads too are also approximated by a distributed load.

The analysis of such systems by the classical elasticity methods previously described is normally too costly and time consuming for the design office. Therefore the use of reasonably accurate simple design methods are required, but their application is usually limited to specific problems. These solutions are presented in the form of empirical formulae, graphs or tabulated coefficients. Some methods that are commonly used and recommended in codes of practice are discussed here.

2.4. 2 Rankine and Grashof's Method

This method which has been popular in codes of practice recommendations assumes a load distribution in two orthogonal directions which are uniform over the entire slab. The loads are carried only by flexure and the twisting moments are ignored. The uniform loads p_x and p_y carried in the respective x and y direction are such that

$$p_x + p_y = p \qquad (2.18)$$

Where p is the total uniform applied load. The actual distributions p_x and p_y are determined by the compatibility of deflections of the centre strips.

ie
$$\frac{5}{384}\frac{p_x L_x^4}{E_x I_x} = \frac{5}{384}\frac{p_y L_y^4}{E_y I_y}$$

and assuming that the flexural rigidity of the strips are equal

$$p_x = \left[\frac{L_y^4}{L_x^4 + L_y^4}\right] p \qquad (2.19)$$

The bending moment of the x and y strips are parabolic with maximum values.

$$M_x = p\,\frac{1}{8}\left[\frac{L_y^4}{L_x^4 + L_y^4}\right] L_x^2 = \beta_x\, pL_x^2$$

$$M_y = p\,\frac{1}{8}\left[\frac{L_x^2 L_y^2}{L_x^4 + L_y^4}\right] L_x^2 = \beta_y\, pL_x^2 \qquad (2.20)$$

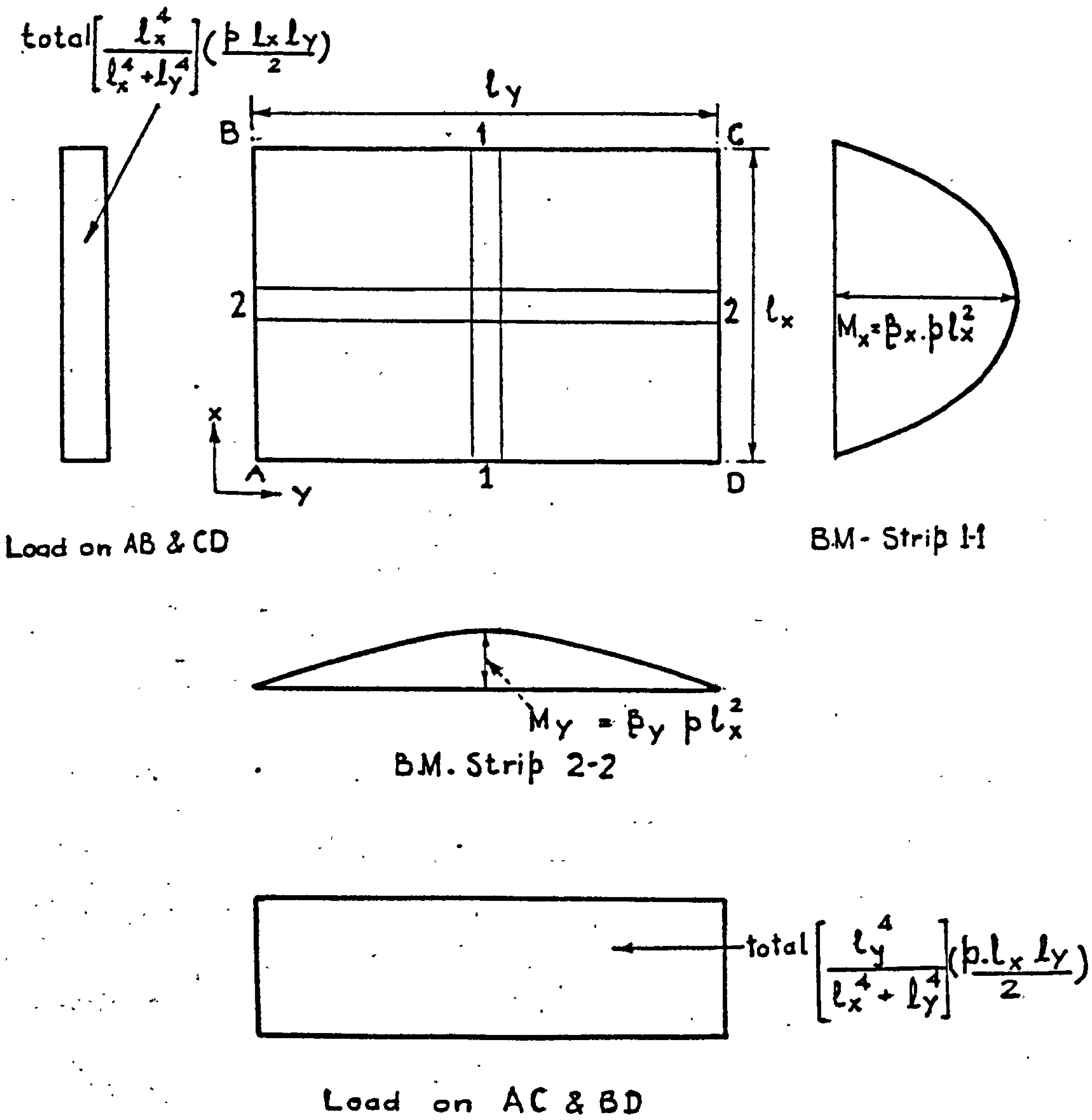


FIG. 2.2 RANKINE AND GRASHOF'S METHOD - SIMPLY SUPPORTED RECTANGULAR SLAB CARRYING A UNIFORMLY DISTRIBUTED LOAD.

Coefficients of β_x and β_y are given in table 16 CP 114 (1969) (4) and table 12 CP 110 (1972) (5). The load transmitted by the slab to the surrounding beams is uniform and is in the ratio $\left[\frac{L_x^4}{L_y^4}\right]$ (see fig. 2.2). This method is normally applicable to rectangular slabs simply supported on four sides and carrying a uniformly distributed load. Rankine and Grashof's method is normally assumed to be an approximate elastic method but it will be shown later to be more closely related to the plastic strip method of slab design and a generalised form of it based on the plastic theory will be developed in Chapter Five.

2.4. 3 Dr. Marcus's Method

This method extends the approximate elastic solution proposed by Rankine and Grashof. In a more general form, equation (2.19) can be written as

$$p_x = \left[\frac{C_x \ L_x^4}{C_y L_y^4 + C_x L_x^4}\right] p \qquad (2.21)$$

Where C_x and C_y depend on the boundary conditions of the x and y strips and these values are based on the elastic beam theory. Dr. Marcus has introduced simple corrections to allow for the assistance given by torsion and the bending moments obtained in this manner agree favourably with those obtained from rigourous analysis based on the elastic plate theory.

2.4. 4 Westergaard's Method

A familiar method for the design of rectangular slabs supported on all four sides and loaded uniformly is the use of coefficients given in table 17 of CP 114 (4). These long standing coefficients were based on the analytical work done by Westergaard and the useful tests performed by Slater (3). Westergaard utilized and extended the results of Nadai, Neilsen, Hencky, Lietz, Mesnager which were based on the Navier, Levy or Ritz methods of solving the elastic plate problem.

For rectangular panels with sides a and b (b less than a) subjected to a uniform load w per unit area, the central and the edge bending moments were expressed as coefficients (of the form M / wb^2). Westergaard plotted these coefficients against the ratio b/a for various combinations of the slab boundary conditions and the shapes of these curves were approximated by simple expressions.

Westergaard (6) realised that these coefficients needed further modification in the light of test results which demonstrated the phenomenon of redistribution of stresses. He stated that with increasing load the stiffness of the material becomes small at the centre and greater near the edges and stresses are redistributed from the centre to the edges. Consideration was also given to the probability of simultaneous loading in the neighbouring panels. Further he divided the slab into a middle strip of half the span and two equal side strips of one fourth the span and proposed design coefficients for all strips. In addition Westergaard suggested coefficients for the bending moments in the supporting beams but these do not take into account the true distribution of loading .

In the current recommendations in CP 114 (4) the slab is divided into a middle strip of three quarters the span and two equal edge strips and bending moment coefficients are given only for the middle strips. Additional torsion reinforcements is required at the corners with simply supported or discontinous edges.

This method has proved to be a useful practical way of designing commonly occuring reinforced concrete slabs but its application is restricted to rectangular slabs carrying a uniformly distributed load.

2.4.5 Concentrated Loads on Slabs

The method of slab design discussed so far in this section are confined to uniformly distributed loading. In structures such as bridges the effects of point loads are important and two methods commonly used in design are due to Pigeaud and Westergaard. A good account of these is given by Rowe (7)

(a) Pigeaud's Method

This method is suitable for central concentrated loads and the results are derived from the Lagrange equation. Pigeaud's curves give values of moment per unit length M_1 and M_2 as functions of u/a and v/b for various ratios of sides a/b . The dimensions of the concentrated loads u and v along the respective sides a and b are determined from the pressure area of the load and assumed to be spread through the thickness at 45° .The maximum moments per unit length M_a and M_b across the sides a and b respectively are given for a Poisson's ratio of 0.15 as

$$M_a = (M_1 + 0.15\,M_2)\ P$$

$$M_b = (M_2 + 0.15\,M_1)\ P \qquad (2.22)$$

Where P is the value of the concentrated load. Pigeaud's method is useful for slabs in which $b < 1.8\,a$. The limitations are that only central loads can be dealt with and the effects of a group of separated concentrated loads cannot be accurately determined. Also, in practice the values of u/a or v/b can be very small, and

M_1 and M_2 cannot be accurately determined.

(b) Westergaard's Method

Westergaard considered the effects of wheel loads on slabs. His analysis was also based on the classical elastic theory for the flexure of slabs. The bending and twisting moments M_x, M_y and M_{xy} at a point (x,y) on the slab is given in terms of hyperbolic and trigonometric functions. Expressions for moments are also given for a group of four equal loads placed at the corners of a rectangle. The induced moments immediately under the concentrated loads are very high and these take into account the finite area of contact of the load and the thickness of the slab.

Westergaards expressions have proved useful in the practical design of bridges and enable the elastic moments to be derived due to a group of separated wheel loads.

2.4. 6 Composite Action - Design of Supporting Beams

In most structures, especially those in reinforced concrete, the slabs and supporting beams are constructed to be monolithic. Therefore they act as a single structural system which provides resistance to the applied loads. This behaviour is usually referred to as composite action.

When the beam centroid is at a different level from the slab neutral axis, full composite action takes place, in which both vertical and horizontal shear forces are transmitted between them. Significant composite action can still take place however when the slab and beam centroids coincide. Now only vertical shear forces are transmitted between them and this interaction has been termed partial composite action.

The theory of partial composite action in elastic slab - beam systems has been well presented by Wood (8). Later Khan and Kemp (9) extended the approach to cover full composite action. Wood has shown how the distribution of reaction between a supporting beam and a slab can undergo remarkable changes for the same applied load depending on the ratio γ , which is the ratio of the flexural rigidity of each beam to the flexural rigidity of half the slab width. In their study on full composite action Khan and Kemp too have concluded that γ is still the dominant parameter governing the load distribution to the beams. Next in importance is the eccentricity of the slab and beam neutral surfaces. However for values encountered in practice the eccentricity factor is not very significant.

Both codes of practice CP 114 (4) and CP 110 (5) recommend a 45^{o} triangular load distribution to be taken by the beam supporting either a square slab or the shorter side of a rectangular slab. The maximum bending moment on the supporting beam corresponding to a uniform load of q_o per unit area is therefore (q_o L^3 / 24). For any other type of loading on the slab there are no guide lines to determine the load distribution on the supporting beams.

Khan and Kemp have analysed numerically a single square panel slab beam fully composite system. They have proposed some simple design rules to predict the maximum slab and beam moments and deflection and the loading on the supporting beams.

Unfortunately in practical design codes the effects of composite action are still largely ignored, mainly due to the complexity of the effects. There is however clearly a need for a simple method of slab design, which will readily incorporate the important effects of interaction between slabs and supporting beams if the design methods are to represent correctly the real physical behaviour.

2.4. 7 Flat Slabs

Flat slabs are peculiar to reinforced concrete construction and consist of slabs supported solely on columns. They offer advantages over the conventional floors supported on beams in providing better head room, economy in shuttering and a clear and unbroken appearence of the underside. To reduce the adverse shear effects from concentrated supports, the construction can have flared column heads and drops in the flat slab. Recommendations for these are to be found in most codes of practice.

The analysis of flat slabs based on solving the Lagrange equation by Levy's method is given in Ref (1) and (2). The flat slab is idealised as a continuous elastic plate supported by a row of columns. Coefficients for the positive moment at mid span, negative moment over the column and the deflection at the centre of the panel are available. This method does not take into account the bending moments induced in the columns, but this can be done approximately by analysing the flat slab and columns as a continuous frame.

An empirical method of flat slab design is given in CP 114 (4) and CP 110 (5). This method imposes further restrictions on the ratio of length to breadth in a panel, variations of length and breadth, number of panels in each direction and the size of drops. Taylor (10) showed that the formula given in equation 31, clause 332, CP 114 of 1959 (4) for the bending moment failed to satisfy the overall equilibrium of the panel and was on the unsafe side. The CP 110 of 1972 (5) acknowledges this error and has increased this coefficient from $\frac{1}{10}$ to $\frac{1}{8}$, to give a moment value;

$$M_o = \frac{n L_2}{8} \left(L_1 - \frac{2 h_c}{3} \right)^2$$

In effect flat slabs have in the past been designed to a lower load factor by the use of CP 114 which may have been partly responsible for their economy and popularity.

2.5 CRITICAL ASSESMENT OF THE VALUE OF ELASTIC METHODS OF SLAB DESIGN.

The analysis of slabs based on the elastic theory gives information on internal forces and deflections but only under working load conditions. The estimate of the load factor by the elastic method is generally conservative since it merely restricts the maximum stress at a point to a permissible value. The elastic methods do not provide real information on the collapse load limit state. With the introduction of the new code CP 110 - 1972 and the limit state design philosophy, various critical states including collapse and serviceability must be considered.

At failure or sometimes even the working load range the fundamental assumptions of the Lagrange equation are not valid. Materials are behaving plastically instead of being linearly elastic. The material of the slab is considered to be isotropic and homogeneous, yet even at the working loads there will be cracking which affects the stiffness of different regions and therefore the true distribution of internal forces within the slab. At higher loads more of the basic assumptions of classical elastic theory become invalid. The deflections may be large compared to the thickness of the slab and the geometry of the slab will be changed significantly. It has been observed that concrete slabs with their low percentage of steel are capable of substantial redistribution of stresses. These effects are not considered in the elastic analysis apart from the redistribution of moments from the centre to the support regions allowed in practical design codes.

Therefore it can be concluded that the classical elastic theory is both complex for the design office purpose and yet not physically real.

Useful approximate elastic methods of slab design are available for a limited number of slab problems such as the regular floors of multistorey buildings. However, the information provided in codes of practice on distribution of loads to the supporting beams is very limited and physically incorrect and these are not applicable to irregular shaped slabs or complex loading patterns.

Numerical methods based on the finite element or finite difference techniques can be employed to determine an elastic solution of any slab problem. These methods give the bending moments M_x , M_y and the torsional moment M_{xy}, but no rational elastic method has yet been developed to determine the reinforcement in the x and y directions which include the torsional moment M_{xy}. Recourse then has to be made to plastic theory as proposed by Wood (11) and this will be discussed in the next chapter.

Generally therefore it may be concluded that although elastic methods have proved useful in the past, their contributions within the limit state design philisophy leaves much to be desired.

CHAPTER THREE

PLASTIC METHODS OF SLAB DESIGN

3. 1. INTRODUCTION

It is accepted that most engineering materials are elastic only at small loads and they undergo inelastic or plastic deformations at higher loads. These deformations, in general are time and temperature dependent, but these effects are not considered here. In analysis an important assumption made about the mechanical properties of the material is that the material is considered to be perfectly plastic, which means it is capable of indefinite strains once the condition of yield have been reached. In the simple plastic theory the effects of elastic deformation and strain hardening are ignored. This simple rigid - perfectly plastic method for the design of structures has the advantage of savings in material, simplicity of calculations and a more realistic prediction of behaviour near collapse.

The mechanical properties of mild steel make it an ideal material to be analysed by the simple plastic method. Plastic methods can also be extended to reinforced concrete sections particularly slabs and beams where the percentage of steel reinforcements are small. This quantity of steel must be small enough to ensure that the failure of the members are dominated by the yielding of steel reinforcement rather than the crushing of the concrete.

3. 2. HISTORICAL BACKGROUND

The foundations of the theory of plasticity were laid in about 1870 by Saint Venant and Levy (12). Saint Venant derived the equation of plane stress and Levy extended this method to cover the three dimensional solid. In 1911 - 1914 Bach and Graf (2) carried out a series of tests on plates and conlcuded that the average bending moment per unit length across the diagonal of a simply supported square plate of side L was $(WL^2/_{24})$ where W is the load per unit area.

The origin of the plastic method of concrete slab design must be accredited to Inglerslev (13) who in 1923 presented a method for calculating the ultimate strength of rectangular reinforced concrete slabs. He observed that the cracks start near the centre and propagates to the corners, indicating that "Stresses have been equalised after the steel has passed the elastic limit when deformation of the slab takes place without corresponding increase in stress ". Ingerslev made the following assumptions; (a) the bending moment is distributed uniformly across the rupture lines, (b) there is no shear at sections where the bending moment is a maximum, (c) each segment of the slab is in equilibrium due to the action of the total working load, upward reaction and bending moment along the rupture (yield) line.

Essentially Ingerslev has stated the principles of the yield line theory using the so called but misleading "equilibrium method". He obtained the correct collapse load for a rectangular slab with simple supports carrying a uniform load. This method can infact be extended to the general case of a rectangular slab with restrained edges and unequal (orthotropic)reinforcements .

The yield line theory in its present form is the result of the work done by Johansen(14) in the 1940 s. His thesis on the yield line theory is an amplification and an extention of the work done by Ingerslev. He proposed two ways of calculating the collapse loads; these are the work method and the equilibrium method.

The yield line theory had a very controversial introduction to the English - speaking countries largely due to the theoretical justification of the "equilibrium method". Extensive research continued following the English translation of Johansen's thesis (14) and publications by Wood (15) and Jones (16). Theoretical work by Kemp, Morley and others in a special publication (17) which appeared in 1965 has resolved the controversy and it is now known that the two methods are valid alternative approaches and give the same prediction of upper bounds to the collapse load.

A powerful alternative to the yield line theory is the strip method suggested by Hillerborg (18). It is essentially a design technique and provides complete information on the distribution of reinforcement required. Hillerborg originally intended this method to produce safe or lower bound results for the collapse load. The predicted collapse load of slabs designed by the strip method will be discussed in detail in chapters 4 and 6 .

3. 3. PLASTIC THEOREMS

For many structures no exact solution can be found for the forces required to cause continuing plastic deformation. However there are methods that are developed to establish two values for this force, one of which is an overestimate, whilst the other is an underestimate. This subject and the theorems of limit analysis were developed in the 1950's by Prager and Hodge (12) and others. The theorems when applied to plate problems can be presented as follows.

(a) Upper bound theorem

Any solution which provides

(1) A kinematically acceptable mechanism and

(11) Satisfies the work equation:- external work done by the loads equal to dissipation of plastic energy, will give an upper bound on the collapse load of the plate. The corresponding collapse load

is therefore either correct or too high and from a design point of view it is an unsafe solution.

(b) Lower bound theorem

Any solution which

(1) Satisfies equilibrium at all internal points and at the boundary of the plate. The equilibrium equation of a plate can be written as (see fig (2.1.))

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -q \qquad (3.1)$$

and (11) The yield criterion (see section 3.4) is not violated at any point, will lead to a lower bound on the collapse load. The calculated collapse load is therefore correct or too low and from a design viewpoint is a safe solution.

(c) Uniqueness theorem

Any solution which simultaneously satisfies the upper and lower bound theorems will give the unique value of the collapse load. The requirements of the plastic theorems can be summarised as

Unique solution	Mechanism	Upper bound
	Work equation	Upper bound
	Equilibrium	Lower bound
	Yield criterion	Lower bound

3.4. YIELD CRITERION FOR REINFORCED CONCRETE SLABS

To apply the plastic theorems to rigid plastic plates it is necessary to define a yield criterion. The yield criterion for orthotropically reinforced concrete slabs is due to Kemp (19) and Morley, and was derived by requiring that in all directions the applied normal moment M_n be equal to or less than the yield normal moment m_n provided by the reinforcements in the slab. Yield occurs when the applied normal moment curve just touches the yield normal moment curve as in Fig (3.1) which can be mathematically defined as

$$m_n = M_n$$

$$\frac{\partial m_n}{\partial \theta} = \frac{\partial M_n}{\partial \theta} \qquad (3.2)$$

For reinforced concrete slabs it is assumed essentially that the reinforcing bars are yielding in uniaxial tension and the plastic bending moment per unit length is given by (see fig 3.3.)

$$m_n = m_x \operatorname{Cos}^2\theta + m_y \operatorname{Sin}^2\theta \quad - \quad (3.3)$$

where m_x and m_y are the yield moments per unit length in the y and x directions and θ is the inclination of the yield line to the y axis. Using the particular bending moment equation (3.3), the second expression of equation (3.2) becomes identical to stating the equality of the yield and applied twisting moments

$$m_{nt} = M_{nt} \qquad (3.4)$$

By eliminating θ from the equation the yield criterion for an orthotropically reinforced slab is defined by

positive yield - $(m_x - M_x)(m_y - M_y) \geqslant M_{xy}^2$ (3.5)

negative yield - $(m_x' + M_x)(m_y' + M_y) \geqslant M_{xy}^2$ (3.6)

These equations each define an elliptical cone and the complete yield surface is shown geometrically by fig (3.2)

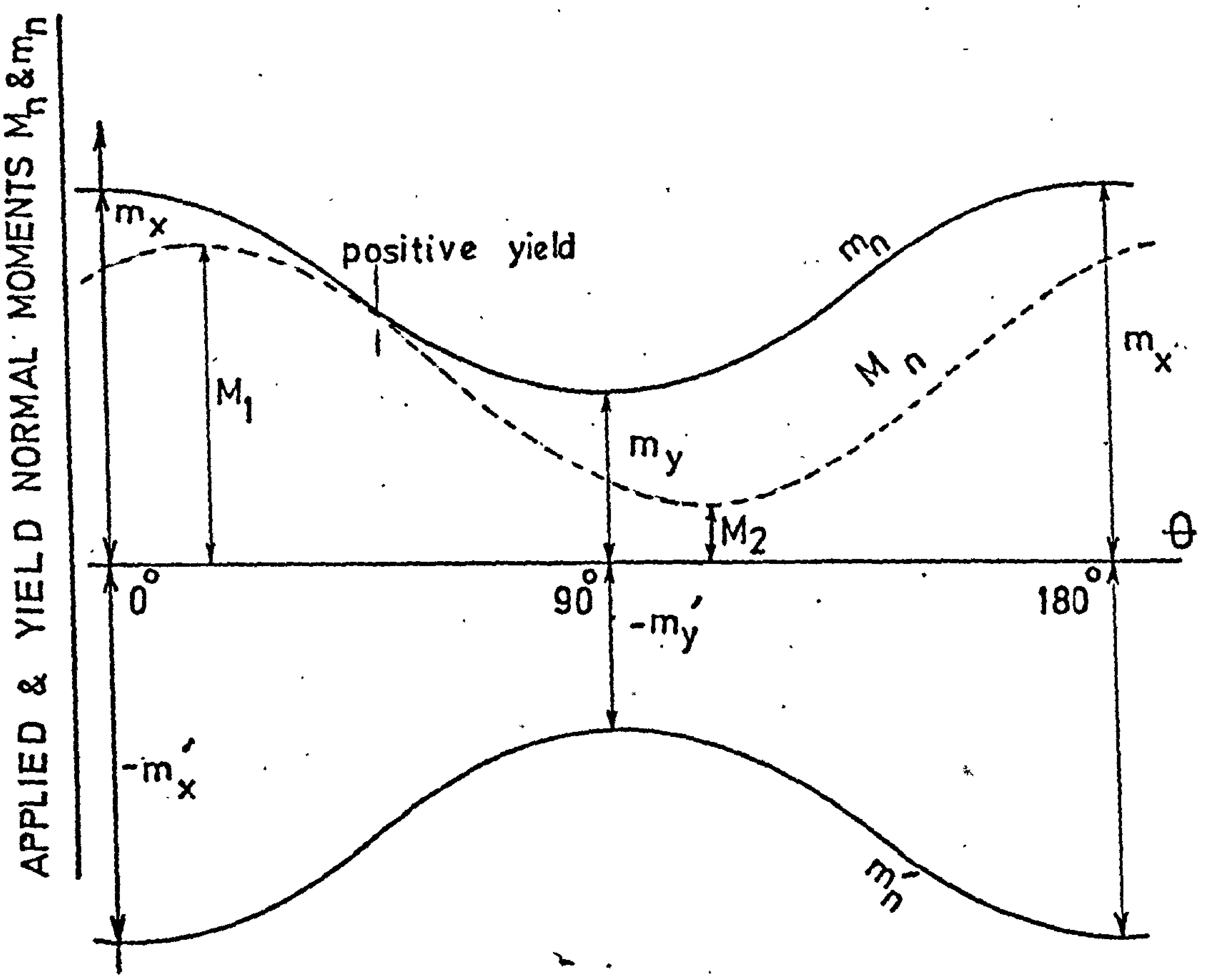


FIG.3.1. VARIATION OF APPLIED & YIELD MOMENTS WITH ORIENTATION

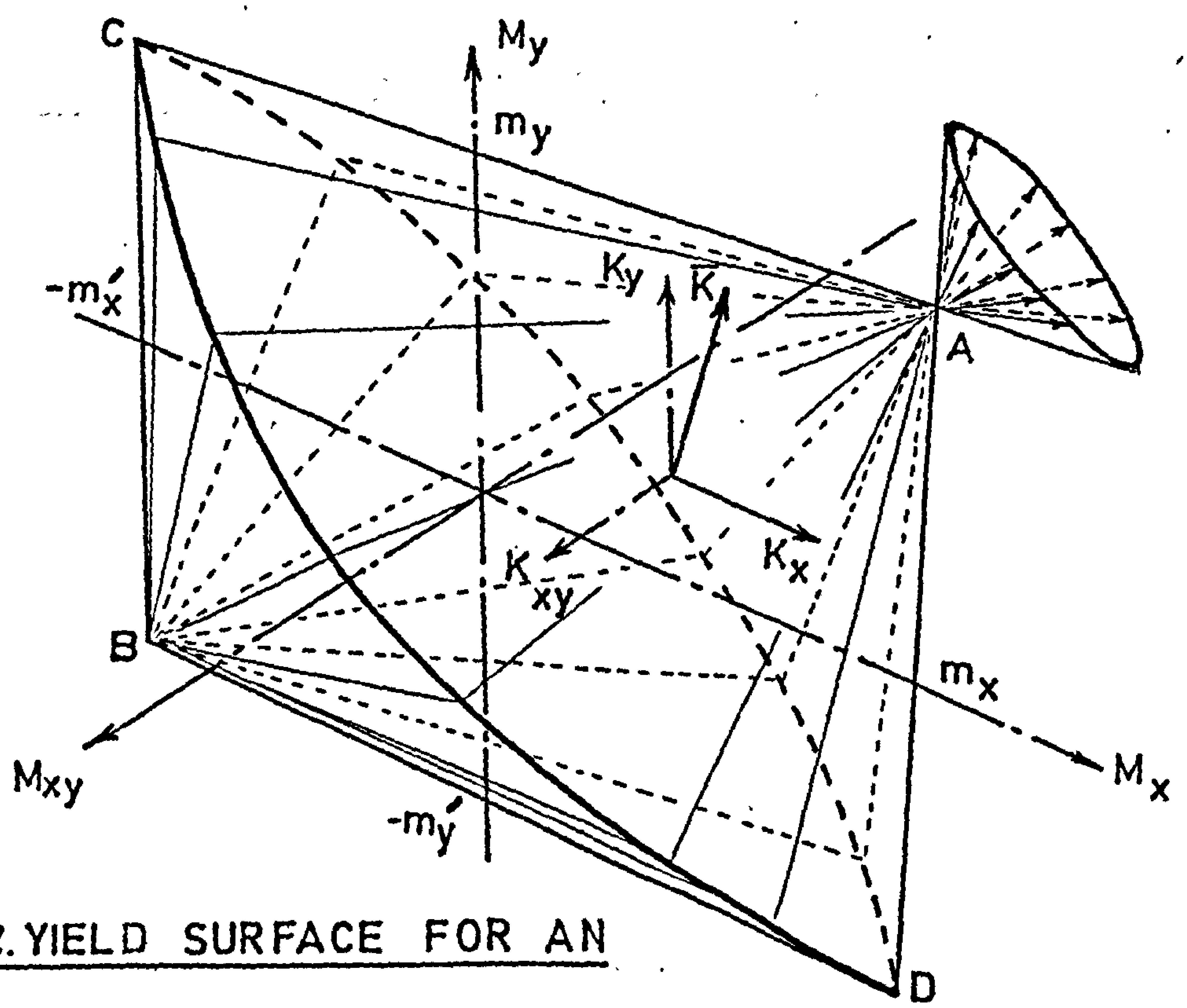


FIG3.2. YIELD SURFACE FOR AN ORTHOTROPICALLY REINFORCED CONCRETE SLAB

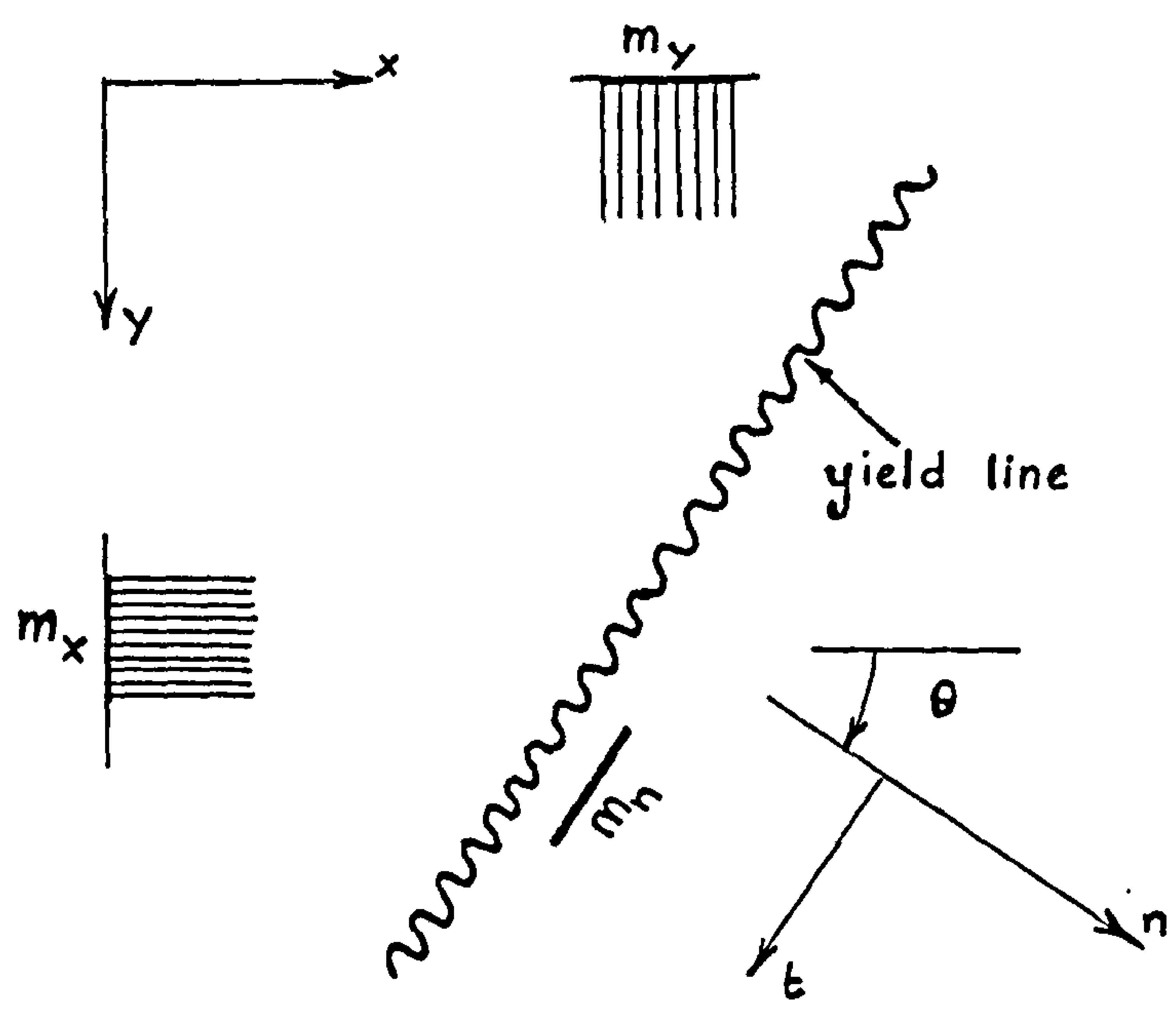


FIG. 3.3. BENDING AND TWISTING MOMENTS ON A YIELD LINE

Kemp (19) has also given the yield criterion for concrete slabs in terms of the principal moments. The yield locus is composed of two sets of hyperbolas. For isotropic reinforcement the yield locus becomes the familiar square or rectangular yield criterion presented by Johansen (14). The corresponding principal moment surfaces are defined by

positive yield $(m_x - M_1)(m_y - M_2) \geqslant 0$ (3.7)

negative yield $(m_x' + M_1)(m_y' + M_2) \geqslant 0$ (3.8)

3.5 PLASTIC DEFORMATIONS

The yield surface of a reinforced concrete slab is defined by equations (3.5) and (3.6) which can be written as

$$F \ (M_x, M_y, M_{xy}, m_x, m_y) = 0 \qquad (3.9)$$

The slab is considered rigid for any stress state within the surface. Unlimited plastic deformation is possible for stress states on the surface F and it is not possible to have any stress state outside F. The slab is rigid until the generalised stresses satisfy equation (3.9) when plastic curvature rates $\dot{K}_x$, $\dot{K}_y$ and $\dot{K}_{xy}$ occur.

According to the plastic potential theory the tensor of curvature rates $\dot{K}_x$, $\dot{K}_y$ and $\dot{K}_{xy}$ corresponding to the generalised stresses M_x, M_y and M_{xy} are

$$\dot{K}_x = \lambda \frac{\partial F}{\partial M_x} \ ; \quad \dot{K}_y = \lambda \frac{\partial F}{\partial M_y} \ \text{and} \ \dot{K}_{xy} = \lambda \frac{\partial F}{\partial M_{xy}} \qquad (3.10)$$

where λ is an arbitrary positive scalar.

Partial differentiation of equation (3.5) gives

$$\dot{K}_x = - \lambda\ (m_y - M_y)$$
$$\dot{K}_y = - \lambda\ (m_x - M_x)$$
$$\dot{K}_{xy} = - \lambda\ M_{xy} \qquad (3.11)$$

The dissipation of energy per unit area of the slab is given by

$$D = M_x \dot{K}_x + M_y \dot{K}_y + 2M_{xy} \dot{K}_{xy} \qquad (3.12)$$

In the principal stress space there are only two generalised stresses M_1 and M_2 and associated with these are the principal curvature rates $\dot{K}_1$ and $\dot{K}_2$, the direction of which are given by

$$\mathrm{Tan}^{-1}\left(\frac{2\dot{K}_{xy}}{\dot{K}_x - \dot{K}_y}\right) = \left(\frac{2\ M_{xy}}{(M_x - M_y) - (m_x - m_y)}\right) \qquad (3.13)$$

Kemp (19) has shown that the curvature rates defined by equations (3.11) and (3.13) are exactly in accordance with a yield line in the direction θ shown in fig (3.1). Thus the plastic flows predicted by the plastic potential theory and those derived by the concepts of yield line theory are identical. The principal curvature the direction of the yield line is therefore zero and the dissipation of energy is given by

$$D = M_n \dot{K}_n \qquad (3.14)$$

It is of particular interest to note the discontinuities in the general yield surface (Fig. 3.2) at the apices of the cones and at the intersection plane of the two cones. The apex points A and B represents the conditions where the yield moments are principal moments and both positive or both negative respectively. At all points on the discontinuity at the intersection plane of the two cones simultaneous negative and positive yield takes place, but the applied principal moments are only yield principal moments at points C and D. In accordance with the plastic potential theory at points A and B, positive and negative yield respectively may occur in any direction. Whereas at points on the intersection plane of the two cones, the directions of positive and negative yield are specifically defined. At points C and D the yield will be in the directions of the principal moments.

The representation of the yield criterion in terms of the normal moments (fig 3.1) has considerable advantages when considering deformations. Points A and B would correspond to the two curves M_n and m_n being entirely coincident. Positive or negative yield is possible in any direction as in fig (3.2). For any point on the intersection plane of the two cones, M_n will touch both the m_n and m'_n curves as in fig (3. 1) and both positive and negative yield occurs and the directions of yield are defined by the angles θ where the two curves touch.

3.6 YIELD LINE THEORY

3.6. 1 INTRODUCTION

The pioneer work in the plastic design of reinforced concrete slabs must be accredited to K. W. Johansen. The English translation of this thesis (14) is a valuable reference for engineers and designers. Johansen developed the yield line theory before the plastic theorems had been published in their present form. The essential assumptions in the yield line theory are:-

(1) The slab is divided into rigid segments by yield lines (so that the elastic deformations are neglected) and is deforming plastically at those yield lines.

(ll) All reinforcing steel is "plastic" along the yield lines

(lll) The moments along the yield lines correspond to the yield normal moment (equation 3.3)

$$m_n = M_x \cos^2\theta + M_y \sin^2\theta$$

It should be noted that although Johansen defined the twisting moment on the yield lines as $M_{nt} = \frac{1}{2}(m_x - m_y) \sin 2\theta$ only the plastic normal moment is used in the analysis. The actual twisting moment on the yield line is strictly undefined. This is effectively equivalent to using tangent planes to the true yield surface defined by equation (3.3)

Johansen goes on to determine the solution for the collapse load using the concept of nodal forces and in a later section developes the alternative virtual work method. Kemp, Morley and others (17) have shown that these two methods are identical if properly used. The Johansen's "equilibrium" method is therefore a misnomer in that it leads to an upper bound and not as its name might imply, a lower bound solution. It is necessary in both approaches to consider the most critical yield line pattern in order to determine the lowest upper bound collapse load.

3.6. 2 Applications of the yield line theory

Strictly yield line theory is not a design method, but a method of analysis. However it has become a popular method with designers and is an approved method in many national design codes.

Yield line theory has also been used to derive the coefficients in table 13 of CP110 (5). These coefficients were first derived assuming a uniform distribution of reinforcement. However to maintain the format of the CP114 (4) recommendations, the steel is then concentrated in the middle strips which are three quarters the width of the slab.

In deriving these coefficients, it was thought desirable, from serviceability viewpoint to maintain the same ratio of positive moment within the span to negative moment at the support as in Table 17 of CP114. Also for the same reason, the relative proportion of short span to long span resisting moment was maintained. Comparing these tables it is clear that in many cases the yield line method requires less material for the same slab problem.

3.6.3 Advantages and limitations of the yield line theory

Experimental evidence has shown that yield line theory is reliable for determining the mode of failure and the ultimate strength of concrete slabs. Although theoretically the method leads to an upper bound solution to the collapse load, in practice, strain hardening and membrane action provide reserves of strength not considered in the theory. The method when compared with elastic solutions is associated with economy of steel.

Yield line theory does not however give much information on how the loads on the slab are transmitted to the supporting beams. For rectangular slabs CP110 has copied the CP114 recommended 45° load distribution to the short side. It has also been suggested (13) that beams carry the imposed load on the segments corresponding to the collapse mechanism. Application of this method is not straight forward with corner fans or point loads.

The critical load has to be obtained by trial and error and in practice a reduction is made (as with corner fans) to cover more critical but more complex mechanisms. Prediction of the mechanism with a combination of loads can be very difficult and the law of superposition is not strictly valid though upper bounds can be obtained.

This method does not give any information on deflections, but in practice these are controlled by specifying ratio of span to depth and choice of the load factors. Another serious disadvantage is that this method does not provide information on the required distribution of reinforcements within the rigid portions of the slab between the yield lines. If due to economy variable or banded reinforcements are provided, the number of possible mechanisms will increase and it becomes difficult to be sure that the most critical mechanism has been found.

3.7. PROVISION OF STEEL IN ACCORDANCE WITH A PREDETERMINED FIELD OF MOMENTS

3.7. 1 Introduction

If a moment field can be derived for a slab which is in equilbrium with the loads, and then reinforcement provided to satisfy the yield criterion at all points then a lower bound solution for the collapse load will be obtained. An elastic moment field is one such equilbrium field which can be determined systematically and which will not depart too far from the moments under working loads. If therefore the correct yield moment field can be provided, a general method of producing lower bound solutions for the slabs is available.

The yield criterion for reinforced concrete slabs has been described in section 3.4. This can be expressed either in principal moment space or in terms of the generalised stress resultants M_x, M_y and M_{xy}. The problem of reinforcing a slab when this moment triad is known is of great practical importance and it has been emphasized previously that this problem has not been solved satisfactorily using elastic theory.

A procedure for placing orthogonal reinforcement in a concrete slab subjected to a single moment triad (M_x, M_y, M_{xy}) was suggested by Hillerborg (21). Wood re-examined Hillerborg ' s work and presented the rules in a slightly different form. Wood's restatements are based on the yield criterion proposed by Kemp (19).

In practice many slabs and particularly bridge decks are subjected to multiple loadings and therefore reinforcements must satisfy the multiple moment triads. An extension of the method was suggested by Kemp (22) which then becomes a problem in non linear programming.

3.7. 2 Provision of steel

If the reinforcement is arranged to follow the paths of the principal moments, the total amount of steel will depend on the sum of the principal moments, $|M_1| + |M_2|$. Although the minimum steel is required when the reinforcement follows the principal moment trajectories, in practice it is more economical to arrange the reinforcements in two directions usually at right angles, decided by the geometry of the slab.

The problem can be stated as given a single moment triad (M_x, M_y, M_{xy}) at a point, find the optimum yield moments m_x and m_y such that the yield criterion is not exceeded at that point, expressed in a mathematical form.

$$(m_x - M_x) (m_y - M_y) \geqslant M_{xy}^2 \qquad (3.15)$$

and ($m_x + m_y$) to be a minimum

With the equality sign introduced, equation (3.15) represents a rectangular hyperbola with asymptotes at $m_x = M_x$ and $m_y = M_y$ as shown in fig (3.4). The reinforcement provided (m_x , m_y) must be selected to lie in the safe region such that the function ($m_x + m_y$) is a minimum. The function ($m_x + m_y$) = constant defines a family of straight lines at 45^o to the axes as shown in fig (3.4).

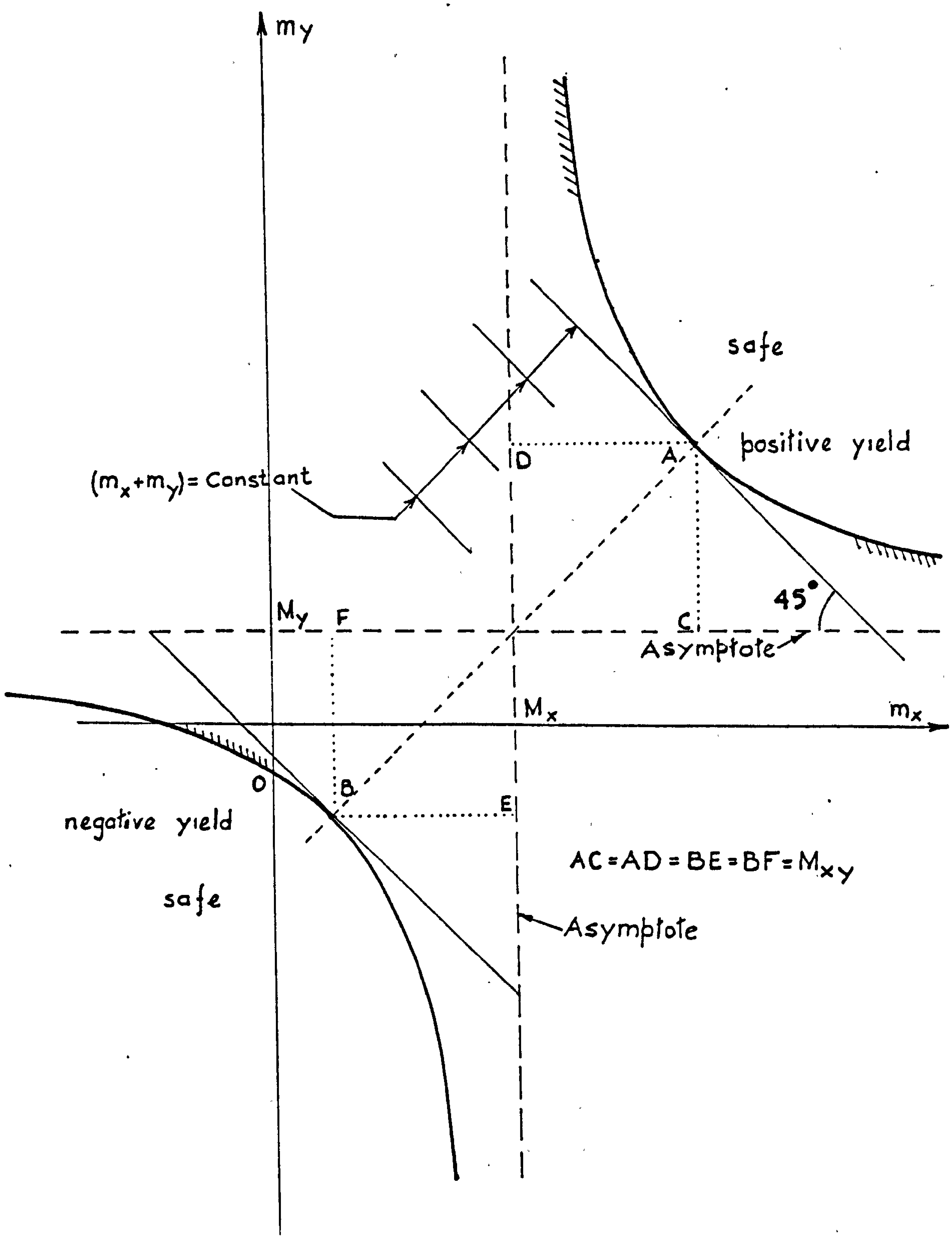


FIG. 3.4 YIELD CURVE FOR ORTHOGONAL REINFORCEMENTS

It can be seen from the figure that for positive yield the stationary minimum safe value of $(m_x + m_y)$ occurs at the point A, where

$$m_x = M_x + M_{xy}$$

$$m_y = M_y + M_{xy} \qquad (3.16)$$

In general it can be shown that the optimum moments are given by

$$m_x = M_x \pm |M_{xy}|$$

$$m_y = M_y \pm |M_{xy}| \qquad (3.17)$$

Where it occurs as in fig (3.4), point B does not lie on the real part of the yield curve, there is no stationary minimum value for the required negative moments i e the point B does not lie in the third quadrant. A least value of $(m_x + m_y)$ for negative yield will however be provided by the point O , where the yield curve cuts the m_y axis. Substituting $m_x = 0$ into the yield criterion (3.15), the required moments are given by

$$m_x = 0$$

$$m_y = M_y - \left| \frac{M_{xy}^2}{M_x} \right| \text{ (negative)} \qquad (3.18)$$

Algebraic expressions for the required yield moment in various situation, which arise have been derived by Wood (11). Depending on the sign and magnitude of M_x , M_y and M_{xy} there are eight different cases and it is necessary to check all eight cases for a single loading.

The graphical presentation of the problem given by Kemp (22) and shown in fig (3.4) is helpful in visualizing the problem and is particularly useful when the problem of multiple triads are considered. Expressed in a mathematical form the problem is to select (m_x , m_y) such that

$$(m_x - M_{xi}) (m_y - M_{yi}) \geqslant M_{xyi}$$

$$i = 1 \text{ to } n \qquad (3.19)$$

$$(m_x + m_y) \text{ is a minimum}$$

This is a problem in non-linear programming, with a linear optimisation function and non-linear constraints which are the yield criteria. In general the optimum yield moments cannot be found with ease and in such circumstances Kemp has suggested ways to compute upper bounds to the yield moments that are close to the optimum.

3.7. 3 Conclusions

This method of slab design has proved popular with designers largely because it is safe and systematic. The optimum yield moments m_x and m_y are chosen such that the yield criteria are not violated and therefore by the plastic theorems will lead to a lower bound on the collapse load. This method is computer orientated and in practice, particularly with multiple loading, may lead to a very poor lower bound. The reinforcement pattern is not banded and often leads to a concentration of steel in regions of high twist. Further the method is not economical with high twist, since $|M_x| + |M_y| = |M_1| + |M_2|$ and the factor $\dfrac{2\,|M_{xy}|}{|M_x| + |M_y|}$ is a direct measure of the excess of steel provided.

3.8 MINIMUM WEIGHT SOLUTIONS

3.8.1. Introduction

It is common in slab designs to provide steel in specified directions over certain regions of the slab. This is perhaps the simplest form of reinforcement and is a suitable starting point for the study of multiple mechanisms which will be associated with reducing reinforcement. The amount of steel that can be saved depends on the work put in by the designer, but may be profitable where designs are repetitive. There is however, a theoretical lower limit to the amount of reinforcement which can be determined for particular slabs by minimum weight concepts.

3.8. 2 Minimum reinforcements in concrete slabs

Morley (23) established sufficient conditions for the minimum reinforcement in concrete slabs. The concrete slab was considered to be of uniform thickness and the effects of membrane forces, shear forces and tensile strength of concrete were neglected. It is envisaged that the mild steel bars were of small diameter compared to the thickness of the slab and were closely spaced. In effect, each steel layer can be replaced by a thin sheet of the same local mean cross sectional area in any desired direction and acting only in uniaxial tension. The variation of the lever arm over the slab is thus neglected.

The total volume of steel V_s required over an area A of the slab is given by

$$V_s = \int_A (a_1 + a_2) \, dA \qquad (3.20)$$

where

$$a_1 = \frac{|M_1|}{\sigma_y d} \quad \text{and} \quad a_2 = \frac{|M_2|}{\sigma_y d} \qquad (3.21)$$

M_1 and M_2 are the principal moments per unit length, σ_y and d the yield stress and lever arm respectively. a_1 and a_2 are the thicknesses of the equivalent steel layers in the directions of M_1

and M_2.

Intergrating over the whole area

$$V_s = \frac{1}{\sigma_y d} \int_A \left\{ |M_1| + |M_2| \right\} dA \qquad (3.22)$$

Hence the problem of finding the minimum reinforcement reduces to that of finding the minimum value of V the moment volume where

$$V = \int_A \left\{ |M_1| + |M_2| \right\} dA \qquad (3.23)$$

3.8. 3 Sufficient conditions for minimum weight solutions and applications to slab design

A moment distribution is said to " correspond " if the principal moments M_1 and M_2 and the principal curvatures K_1 and K_2 have the same sign and direction. Morlěy proved that :-

$$\int \left\{ |M_1| + |M_2| \right\} dA \leqslant \int \left\{ |M_1| + |M_2| \right\} dA \qquad (3.24)$$

corresponding field   non corresponding field

The sufficient conditions can be summarised as :-

If for a slab a particular moment distribution 'O' "corresponds" to the displacement field which has

(a) The curvatures $|K_1| = |K_2| = K$ throughout

except in regions where

(b) $|K_1| = K$, $|K_2| \leqslant K$ and $M_2 = 0$ or

(c) $|K_2| = K$, $|K_1| \leqslant K$ and $M_1 = 0$

then that field has a minimum moment volume. The problem of finding such a distribution field is purely geometrical.

In a neutral area where $|K_1| = |K_2| = \pm K$ it is possible that $M_1 \neq 0$ and $M_2 \neq 0$ M_1 and M_2 can be in any

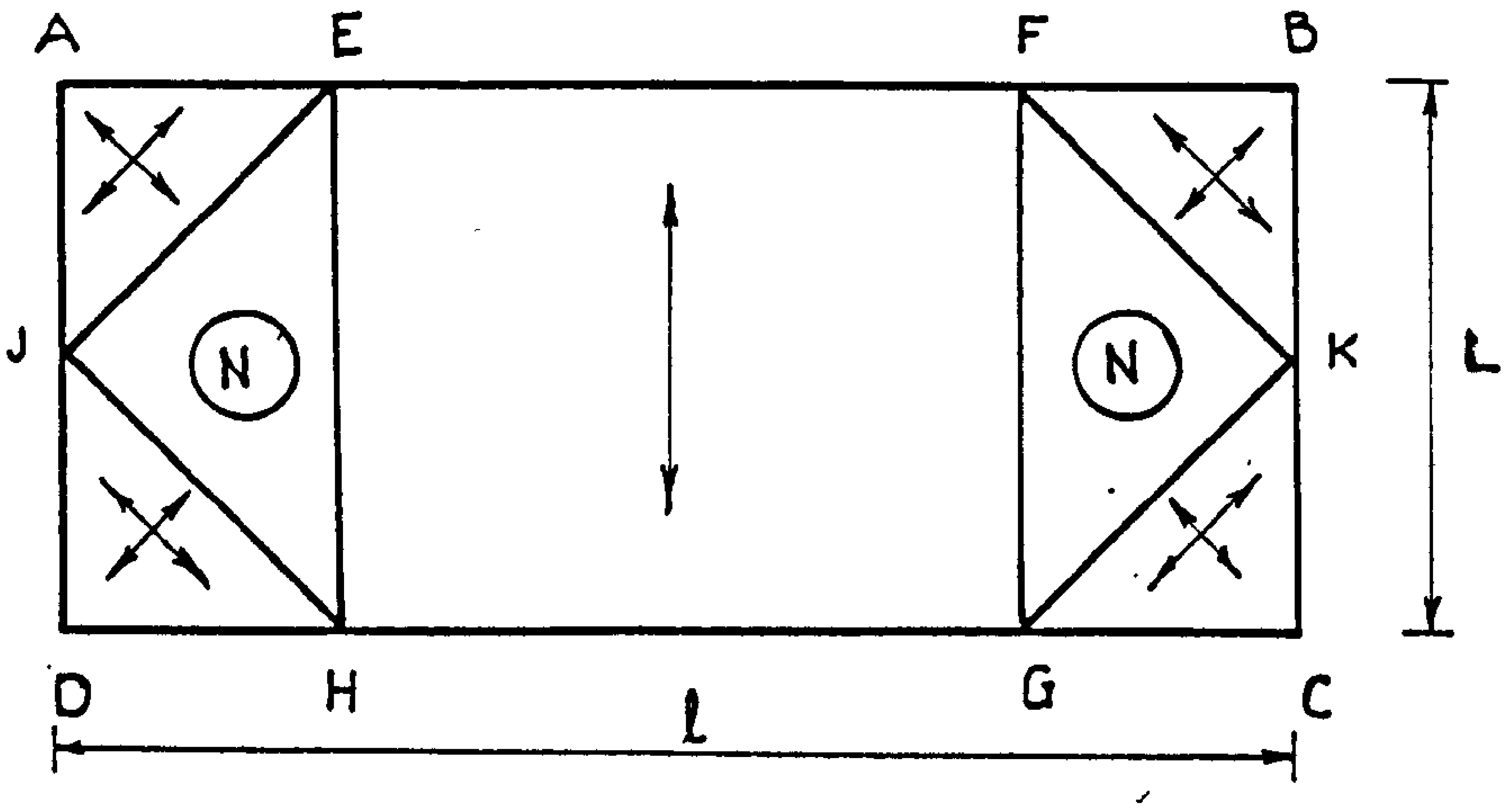


FIG. 3.5 MINIMUM WEIGHT SOLUTION - SIMPLY SUPPORTED RECTANGULAR SLAB.

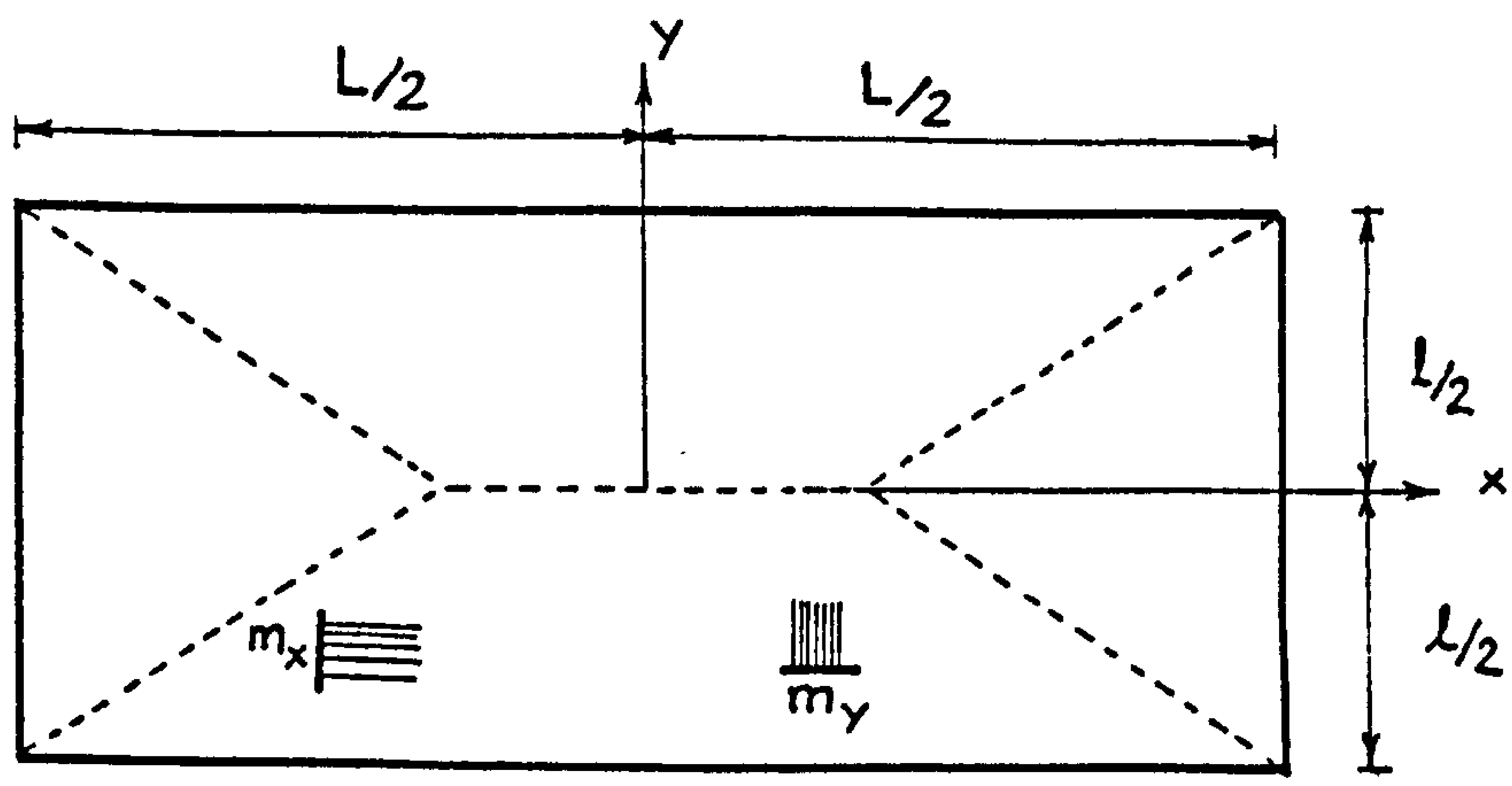


FIG. 3.6 LOWER BOUND SOLUTION - SIMPLY SUPPORTED SLAB CARRYING A UNIFORM LOAD.

direction and the loads too can be distributed in any direction. For a simply supported rectangular slab shown in Fig (3.5), the regions JEH and FKG are such neutral areas.

If $K_1 = -K_2 = \pm K$ the deformation surface is anticlastic and there is less freedom since for correspondence the loads must be distributed in the directions of the principal curvatures. ie for regions such as AEJ loads must be distributed parallel or perpendicular to side EJ as shown though the ratio of such distributions can be arbitary.

In regions where $|K_1| = K$ and $|K_2| \leqslant K$ the moments M_2 must be zero. The region EFGH is such an example and the loads must be carried only in the direction of K_1 (ie EH or FG) and the signs of M_1 and K_1 must be the same.

Fig. 3.5 shows the solution for the slab ABCD and illustrates the three types of displacement fields which are sufficient for a minimum weight solution. The moment volume due to a uniform load q is

$$V_0 = (0.0834\ \ell - 0.0313 L)\ pL^3 \qquad (3.25)$$

which reduces to $0.521\ pL^4$ or $\frac{5}{96}\ pL^4$ for a simply supported square slab.

Morley's methods are associated with no constraints on the reinforcement directions. It is likely that such methods are less applicable to practical problems and a minimum weight solution for reinforcement that is straight and orthogonal over the entire slab may be more important.

Such a method is due to Rozvany and Charratt (24). They derived optimal solutions for straight reinforcing bars in specific directions assuming that the twisting moment $M_{xy} = 0$ and satisfying sufficient conditions. For rectangular slabs simply supported on four sides or simply supported on three sides and free along the fourth their results indicated that the difference between the torsionless optimum solution and the more general absolute minimum solution is very small. If straight curtailed reinforcement is provided on the basis of the elastic moment field and a suitable yield criterion (section 3.7) then V/VO takes a value of 1.596 .

3.8. 4 Comment on minimum weight solutions.

It is important to designers to know the absolute minimum moment volume for a given slab problem. This acts as a standard against which practical designs can be judged. To achieve the minimum it would be necessary to provide very complex layout of reinforcements and the designer must balance the cost of complicated detailing and steel fixing against the saving of material.

The minimum weight solutions do not give any indication of the deflection of the slab or cracking so the serviceability of the slabs may not be satisfactory. Tests on these slabs have shown that they exhibited membrane action to a smaller extent than normal slabs.

3.9. LOWER BOUND SOLUTIONS - CLASSICAL PLASTICITY.

3.9. 1. Introduction.

Another method of obtaining lower bound solutions has been given by Wood (15) following those developed by Prager for the steel plates. However this semi-intuitive method has been more successfully applied to concrete slabs primarily due to the form of yield criterion.

3.9. 2. Applications to rectangular slabs.

Consider a rectangular slab, simply supported and carrying a uniform load Fig.(3.6). The origin of the axes X and Y are at the centre. Intuitively derived normal bending moments M_x and M_y which satisfy the boundary conditions are

$$M_x = m_x \left(1 - \frac{4x^2}{L^2} \right)$$

and $$M_y = m_y \left(1 - \frac{4y^2}{\ell^2} \right) \qquad (3.26)$$

The distribution of the twisting moments is then chosen so that the equilibrium equation (3.1) is satisfied; clearly

$$\frac{\partial^2 Mxy}{\partial x \partial y} = \frac{p}{2} - \frac{4m_x}{L^2} - \frac{4m_y}{\ell^2}.$$

or $$M_{xy} = \left[\frac{p}{2} - \frac{4m_x}{L^2} - \frac{4m_y}{\ell^2} \right] xy + Bx + Cy + A$$

along the centre lines $x = o$ and $y = o$, $M_{xy} = 0$ and A too can be chosen so that

$$M_{xy} = \left[\frac{p}{2} - \frac{4m_x}{L^2} - \frac{4m_y}{\ell^2} \right] xy \qquad (3.27)$$

Equations (3.26) and (3.27) do not indicate how much load can be safely carried by the slab and in order to find this the yield criterion is invoked ;

ie $$(m_x - M_x) (m_y - M_y) \geq M_{xy}^2$$

and the safe solution for the [illegible] distributed load p becomes

$$p \geq \frac{8m_x}{\ell^2} \left[1 + \frac{m_x}{m_y} \frac{\ell}{L} + \frac{m_x}{m_y} \frac{\ell^2}{L^2} \right] \qquad (3.28)$$

Substituting $m_x = M$ and $m_y = \mu M$ and $\frac{\ell}{L} = \lambda$

$$p \geq \frac{8\mu M}{\ell^2} \left[\frac{\lambda^2}{\mu} + \frac{\lambda}{\sqrt{\mu}} + 1 \right] \qquad (3.29)$$

For the same slab it can be proved by yield line theory that the upper bound to the collapse load is

$$p \leqslant \frac{24\mu M}{\ell^2} \times \frac{1}{\left[\sqrt{3 + \frac{\lambda}{\mu^2}} - \frac{\lambda}{\sqrt{\mu}}\right]^2} \qquad (3.30)$$

Kemp (25) has calculated the values of p from equations (3.29) and (3.30) for the range values of λ and μ normally encountered. He observed that the lower and upper bounds agree with a maximum discrepancy of about 1.5%. The lower bound moment field is given by

$$M_x = M\left(1 - \frac{4x^2}{L^2}\right) ; \; M_y = \mu M\left(1 - \frac{4y^2}{\ell^2}\right) \text{ and } M_{xy} = -4\mu M \frac{(xy)}{L\ell}$$

$$(3.31)$$

Although the lower bound solution derived is here close to the unique solution, the classical plasticity approach is not likely to be of great value in practice since such solutions have been found only for a limited number of slab problems with simple geometry and loading.

3.10. HILLERBORG'S STRIP METHOD.

3.10 1. Introduction.

In 1956 Arne Hillerborg presented an equilibrium theory for the design of reinforced concrete slabs. His intention was to present a method that is easy to apply and at the same time gives conservative values for the collapse load. This equilibrium method is referred to as the strip method and Hillerborg presented a simple theory (25) and an advanced method (26). A good description and a critical assessment of this work has been given by Wood and Armer (27).

The equilibrium equation for the slabs is given by equation (3.1) as

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial xy} = -q$$

Hillerborg stated that "If for a certain load q a moment distribution can be found which satisfies the equilibrium equation (3.1) and the boundary conditions and if the limit capacity of the slab is not exceeded at all points, then the value of q is a lower limiting value of the collapse load". This is clearly a restatement of the lower bound theorem. Later it will be shown, rather surprisingly that in most cases the solutions are infact unique.

3.10 . 2 The simple strip method.

This method is applicable to slabs of any shape which are loaded uniformly and supported continuously. The theory assumes that at failure no load is carried by the twisting moments and therefore these are equated to zero at all points of the slab

ie $$M_{xy} = 0 \qquad (3.32)$$

and of course all derivatives of M_{xy} are also zero at all points. The equilibrium equation (3.1) now becomes

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} = -q \qquad (3.33)$$

If $$\frac{\partial^2 M_x}{x^2} = -\alpha q = -q_x \qquad (3.33.\ a)$$

then $$\frac{{}^2 M_y}{y^2} = -(1-\alpha)q = -q_y \qquad (3.33.\ b)$$

and $$q_x + q_y = \alpha q + (1-\alpha)q = q \qquad (3.33.\ c)$$

The division of q can be seen as a distribution of loads in the x and y directions, which leads to the name "STRIP METHOD" and α and $(1 - \alpha)$ define the proportion of the loads carried in the x and y directions respectively. Generally α is so chosen that the load is carried to the nearest support. Normally α is assumed to be such that $0 \leqslant \alpha \leqslant 1$, or more often α is chosen to be 0 or 1 in which case load is transmitted totally in the y or x direction. This has been a serious limitation of the simple strip method though it is possible to allocate any value to α without violating the equilibrium equation (3.1)

Lines of load discontinuity are introduced as shown in fig (3.7). These show the sudden changes in the direction of load dispersion and also gives the load distribution on the supporting beams as shown in fig (3.8)

3.10. 3 Application of the simple strip method.

Fig (3.7) and (3.8) shows the application of this method to rectangular slabs subjected to uniform loads. Fig. (3.9)to (3.11) shows three possible methods of designing a simply supported and uniformly loaded square slab.

The example shown in fig (3.9) is a slab having only one strip each way. Here $\alpha = 1 - \alpha = 0.5$ for the entire slab. This corresponds exactly to the Rankine and Grashof's method discussed in Chapter 2 and the moment volume corresponding to this distribution is $\frac{qa^4}{12}$ ($0.0833\, qa^4$) .

The distribution in accordance with fig (3.10) will require much less reinforcement. The moment volume for this arrangement is $\frac{qa^4}{16}$ ($0.0625\, qa^4$) which is 20% above the absolute minimum.

The division of strips in fig (3.11) is identical to the CP 114 and CP 110 recommendations, the middle strips having a width of three quarters of the span and the edge strips one eighth of the span. This example indicates the provision of banded reinforcement which avoids the varying strips associated with trapezoidal and triangular shapes formed by the load distribution lines. The corresponding moment volume is - $0.0697\, qa^4$.

In each case the supporting beams must be designed to carry the theoretical distribution of load.

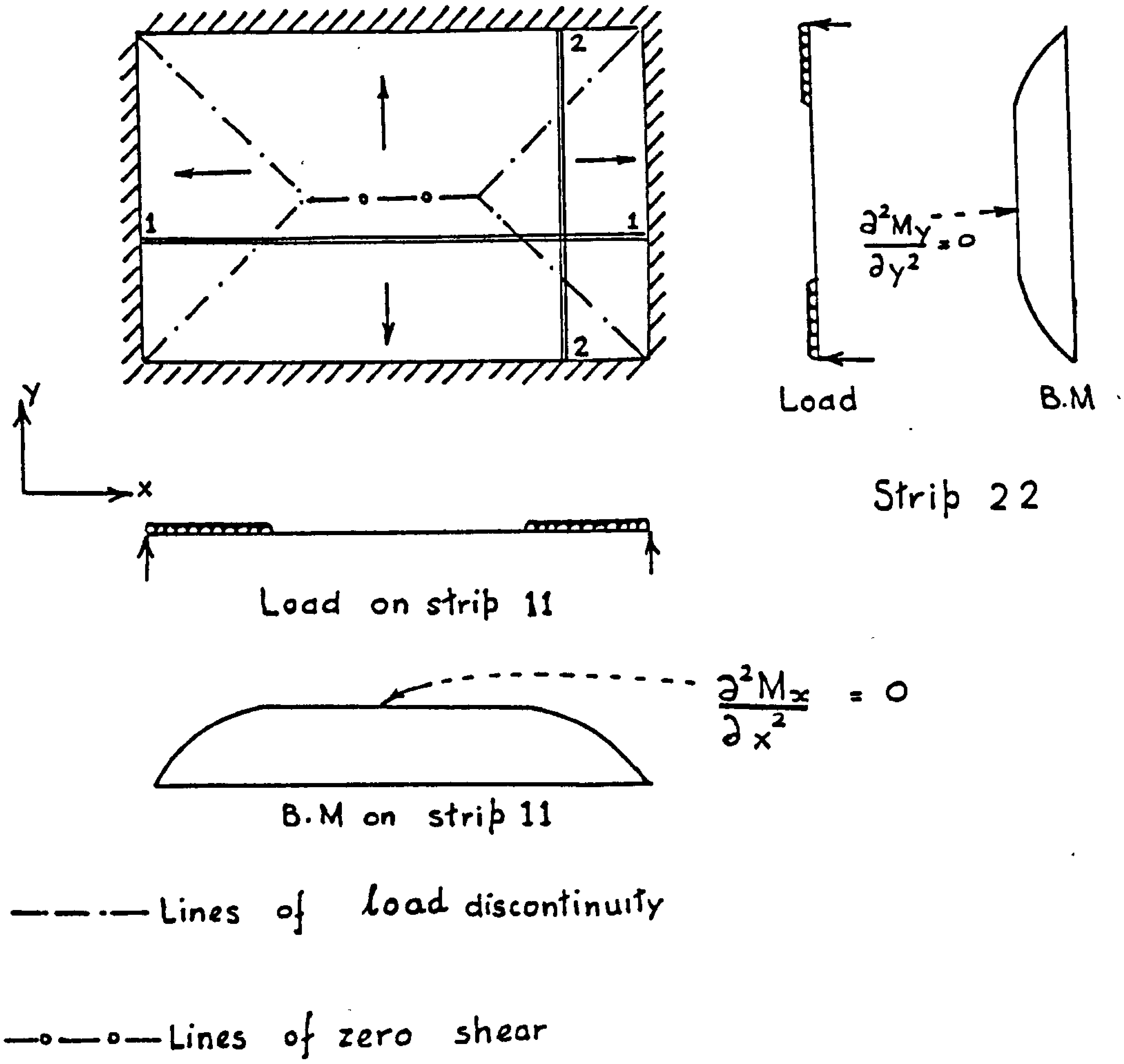


FIG. 3.7 STRIP SOLUTION - DISTRIBUTION OF LOADING AND BENDING MOMENTS IN SLAB STRIPS

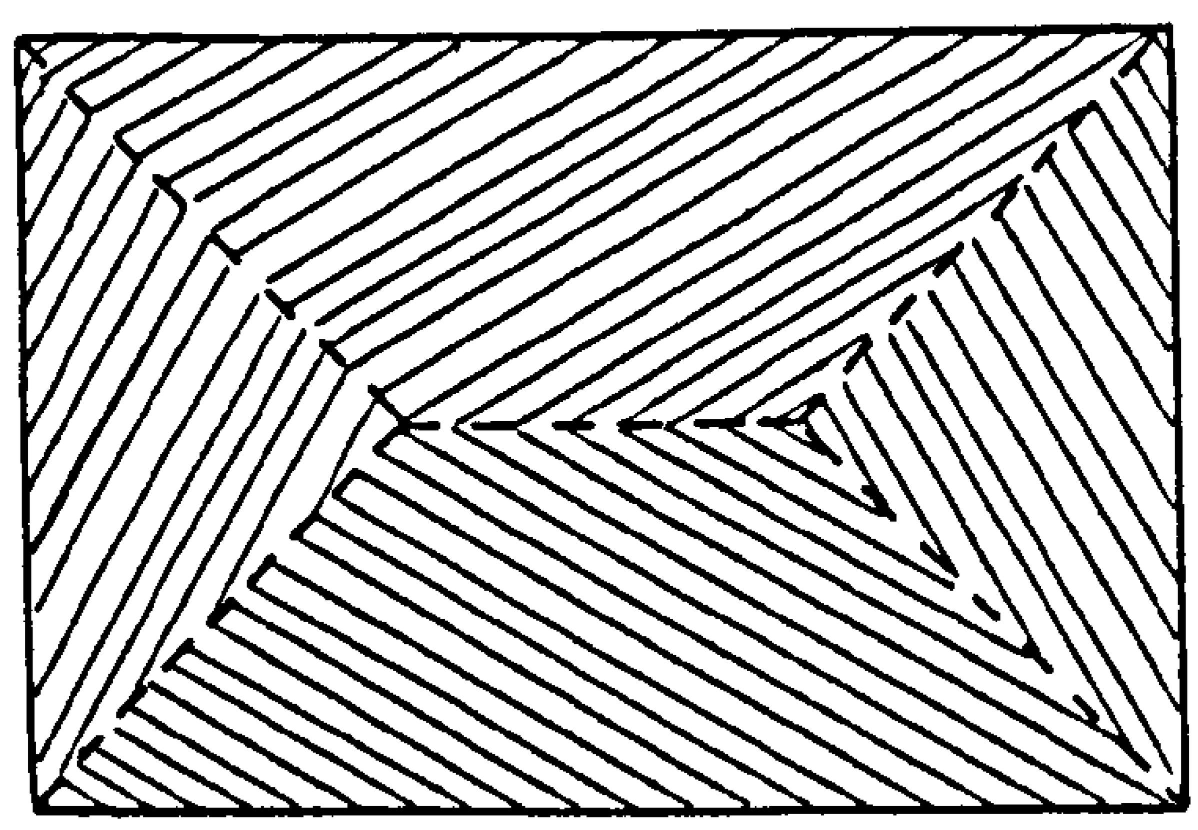


FIG. 3.8 LOAD ON SUPPORTING BEAMS - SLAB IN FIG. 3.7

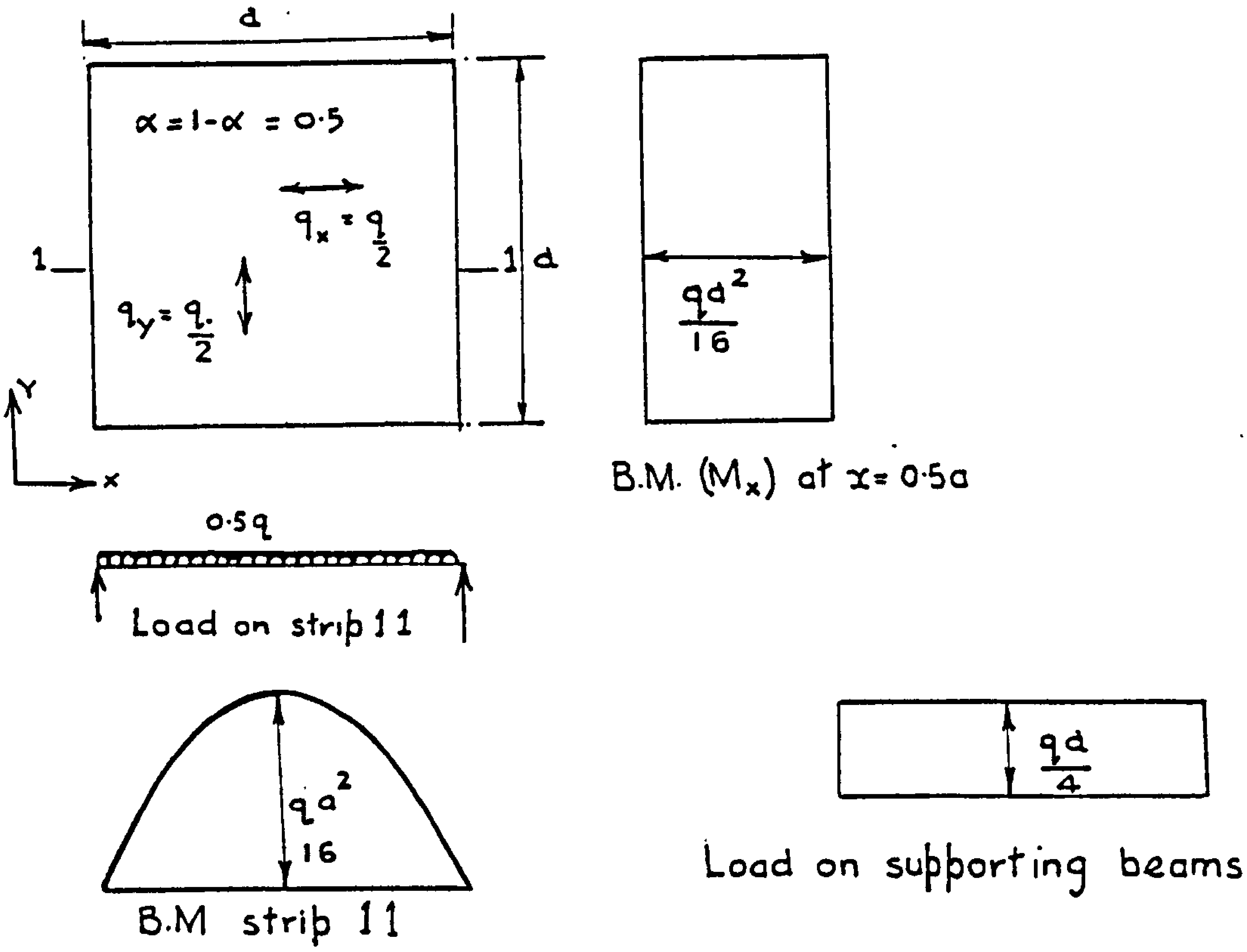


FIG. 3.9 STRIP SOLUTION - SIMPLY SUPPORTED SQUARE SLAB CARRYING A UNIFORM LOAD - ONE STRIP EACH WAY

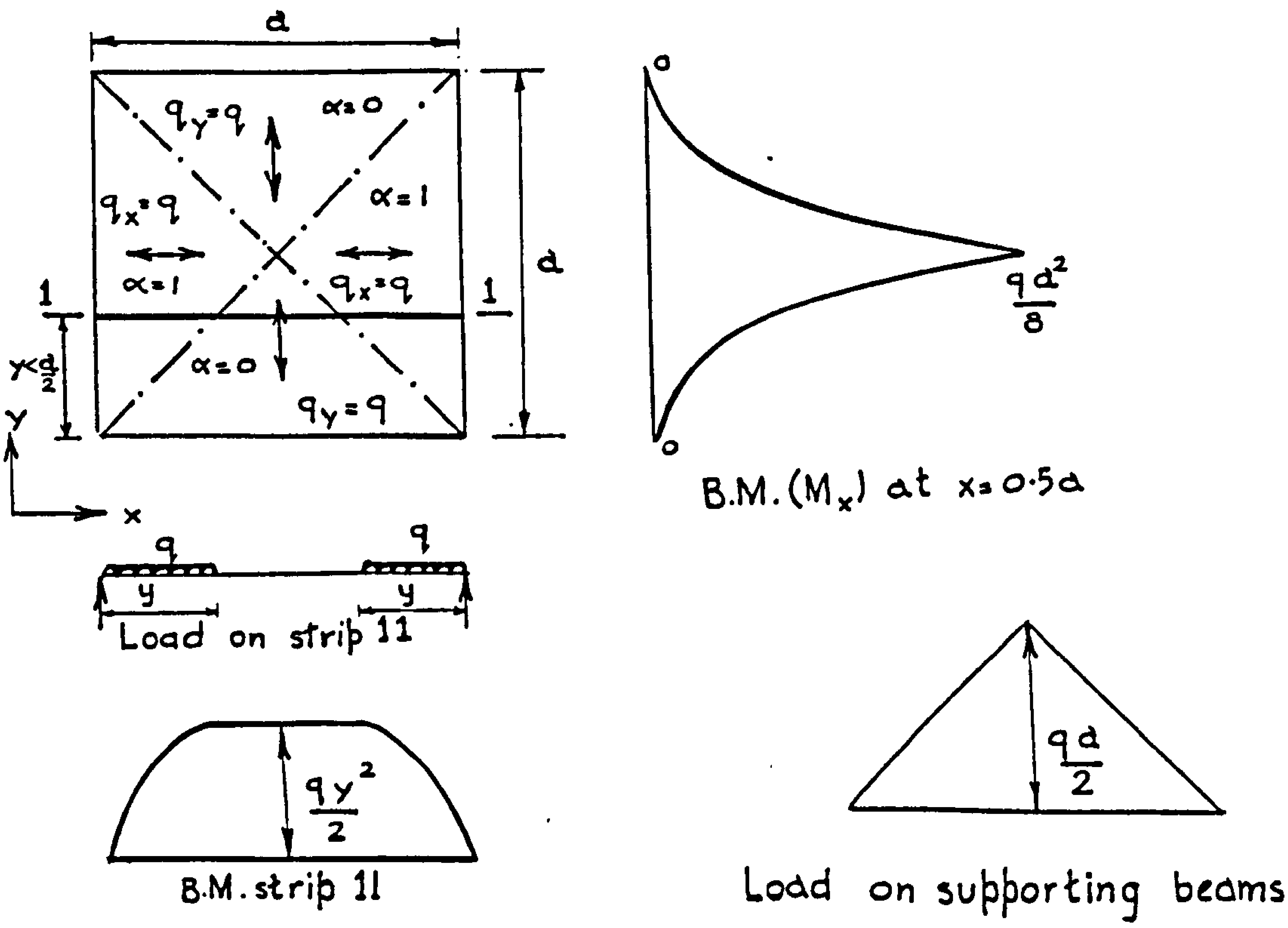


FIG. 3.10 STRIP SOLUTION - SIMPLY SUPPORTED SQUARE SLAB CARRYING A UNIFORM LOAD - DIAGONAL LOAD DIVISION.

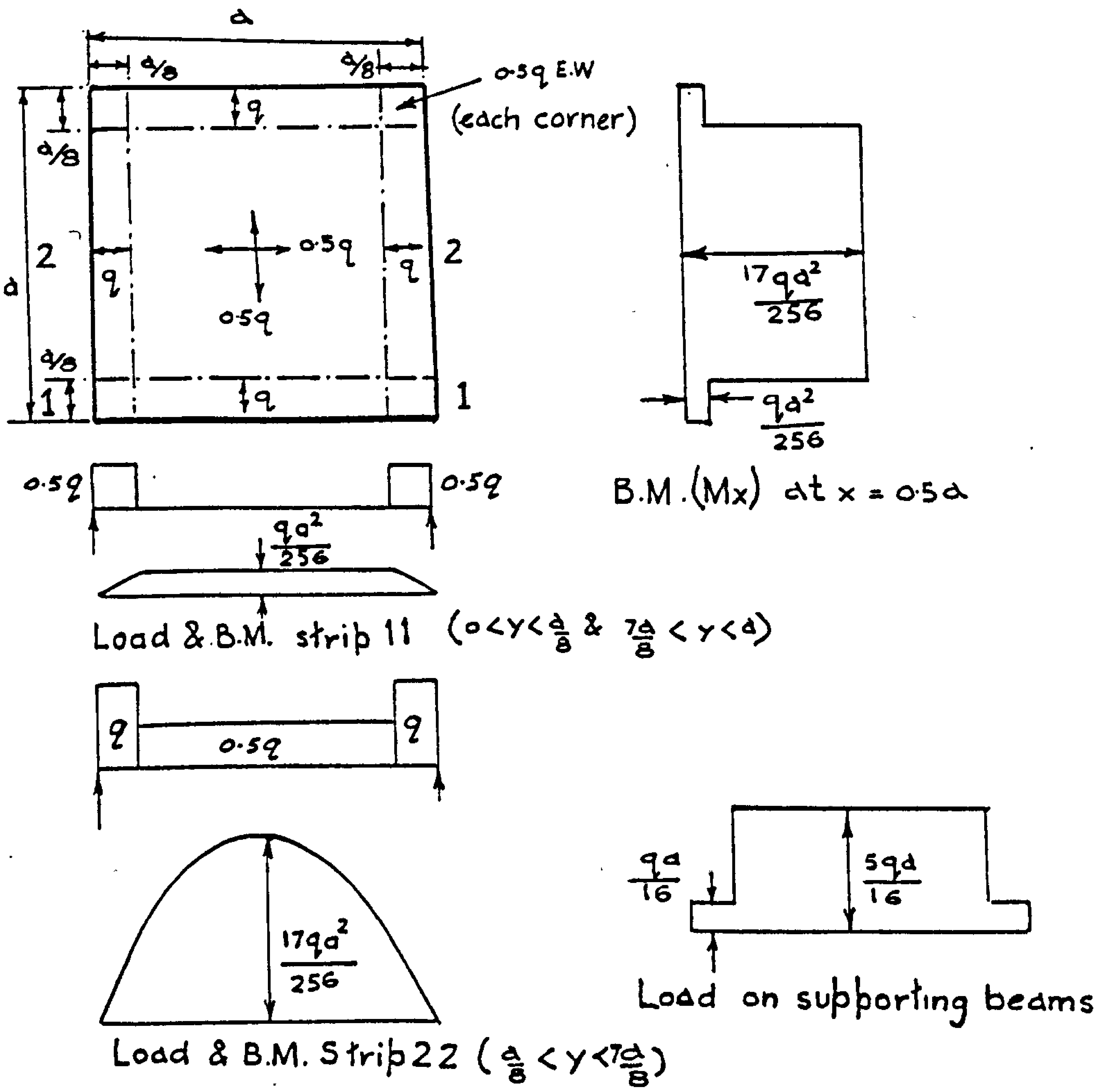


FIG. 3.11 STRIP SOLUTION - SIMPLY SUPPORTED SQUARE SLAB CARRYING A UNIFORM LOAD - SLAB DIVIDED ACCORDING TO CP 114 & CP 110

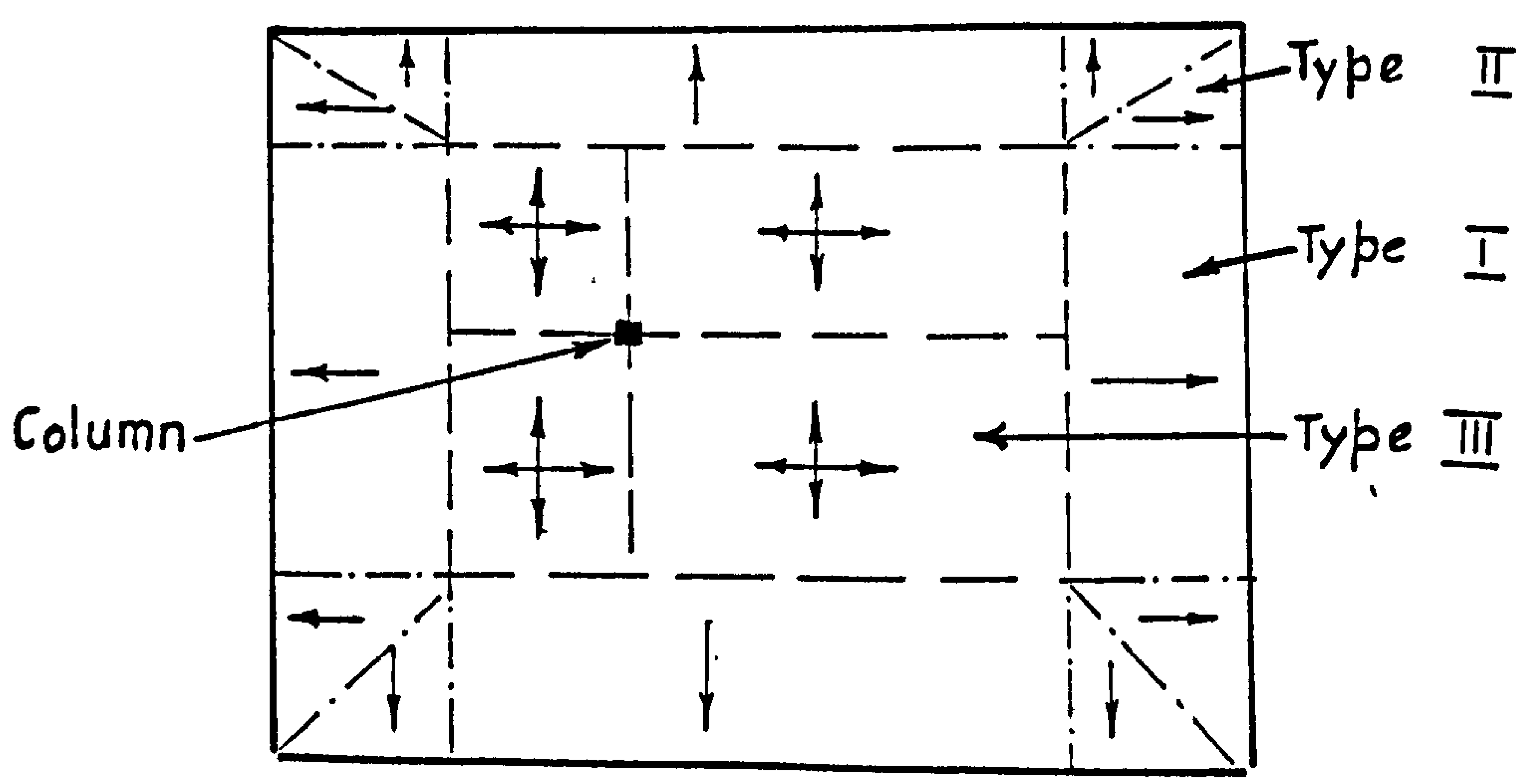


FIG. 3.12 - THREE TYPES OF SLAB ELEMENT

3.10. 4 Comments on the simple strip method.

The simplicity of the method and its advantages are apprarent. This is a method of practical design rather than analysis of a slab with given reinforcement. Design of the supporting beams and the curtailment of the steel presents no problems since full information on loads and moment distributions is available. Discontinuity lines can be chosen to fit bands of reinforcements as in fig (3.11).

The simple method is readily applicable to certain slab problems but the method breaks down with point loads or point supports. At the moment there appears to be no rational way of determining α for the regions which will ensure satisfactory service conditions.

3.10. 5. The advanced strip method.

In 1959 Hillerborg (26) developed a method to overcome the limitations of the simple theory especially the transfer of shears from the strips to a column support. Hillerborg recognised three types of elements generally in slab problems which are shown in fig (3.12).

Type (1) Element - Rectangular in shape with load carried in one direction.

Type (11) Element - Triangular in shape with load carried in one direction.

Type(111) Element - Rectangular in shape, supported at one corner and load carried in two directions.

Type (1) and type (11) elements are similar to those encountered in the simple strip method. These elements can carry both positive and negative reinforcements depending on the nature of the problem.

The analysis of the type III element is more complex and rather difficult to develop rationally. Hillerborg uses a radial stress field together with primary and secondary load actions to transfer the loads from the element to the column. Finally he achieves his solution by proposing a set of rules for reinforcing the element.

Hillerborg has devoted considerable efforts to overcome the problem of point supports by the use of type III elements. Nevertheless the simplicity of the strip method is lost and this approach is not satisfactory as a practical design procedure. The proof given by Hillerborg is for the case of a uniform distribution of load within the element and it will be increasingly difficult to find a suitable stress field for any other type of loading.

The reinforcement pattern has been intuitively derived to satisfy the overall equilibrium of the element only and it will not be possible to argue that the advanced method will always give lower bound solutions for the collapse load.

3.10. 6 Wood and Armer's Alternative Treatment of Type III Elements.

Wood and Armer (27) critically examined Hillerborg's strip method and suggested an alternative approach. They used the classical plasticity method to derive a more systematic and comprehensible type (III) moment field. For the type (III) elements shown in fig (3.13). Wood and Armer observed that the moment field

$$M_x = \frac{pa^2}{8}\left(1 - \frac{4x^2}{a^2}\right)$$

$$M_y = \frac{pb^2}{8}\left(1 - \frac{4y^2}{b^2}\right)$$

and $$M_{xy} = -\frac{p}{2}xy \qquad (3.34)$$

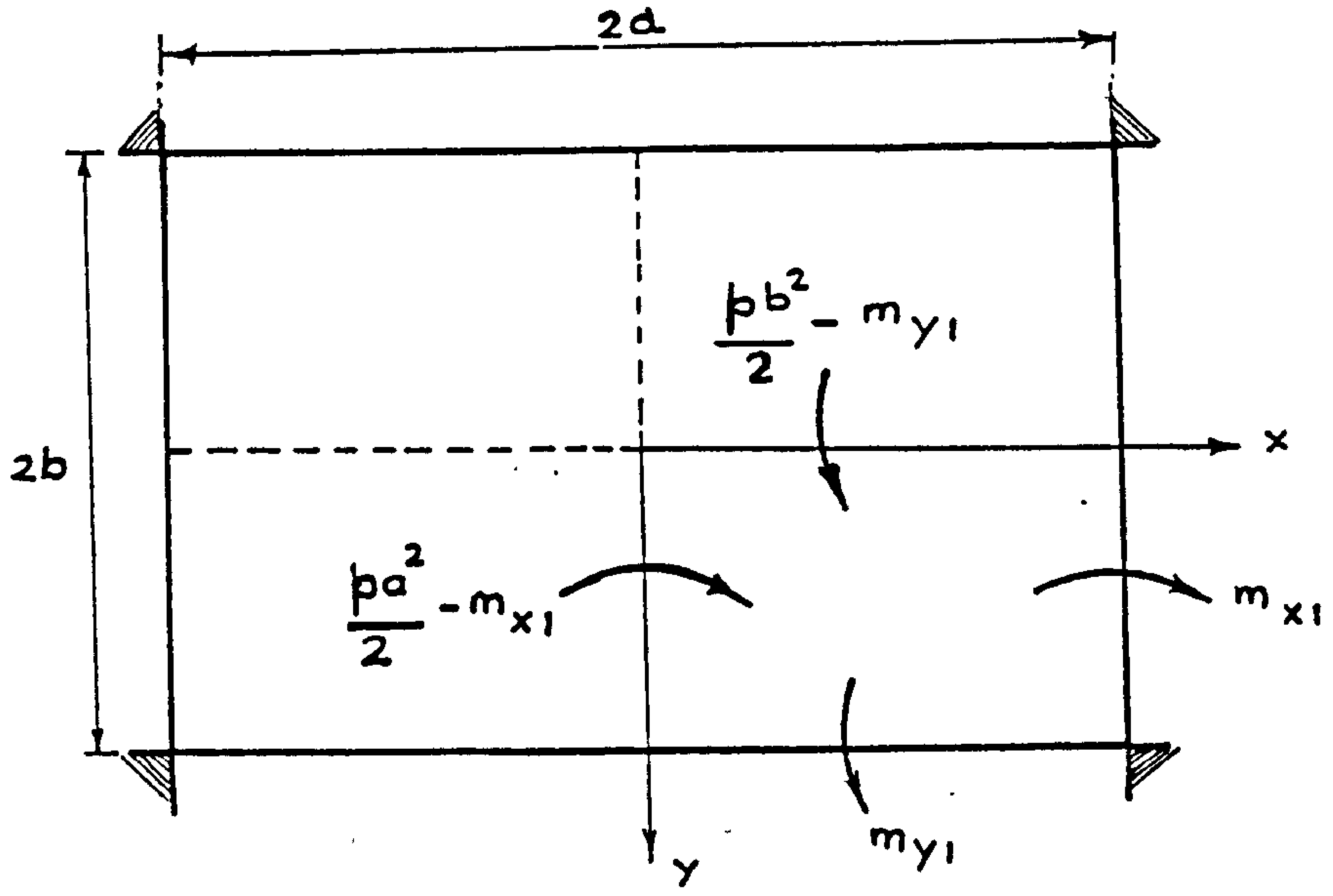


FIG. 3.13 ALTERNATIVE TREATMENT OF TYPE III ELEMENT.

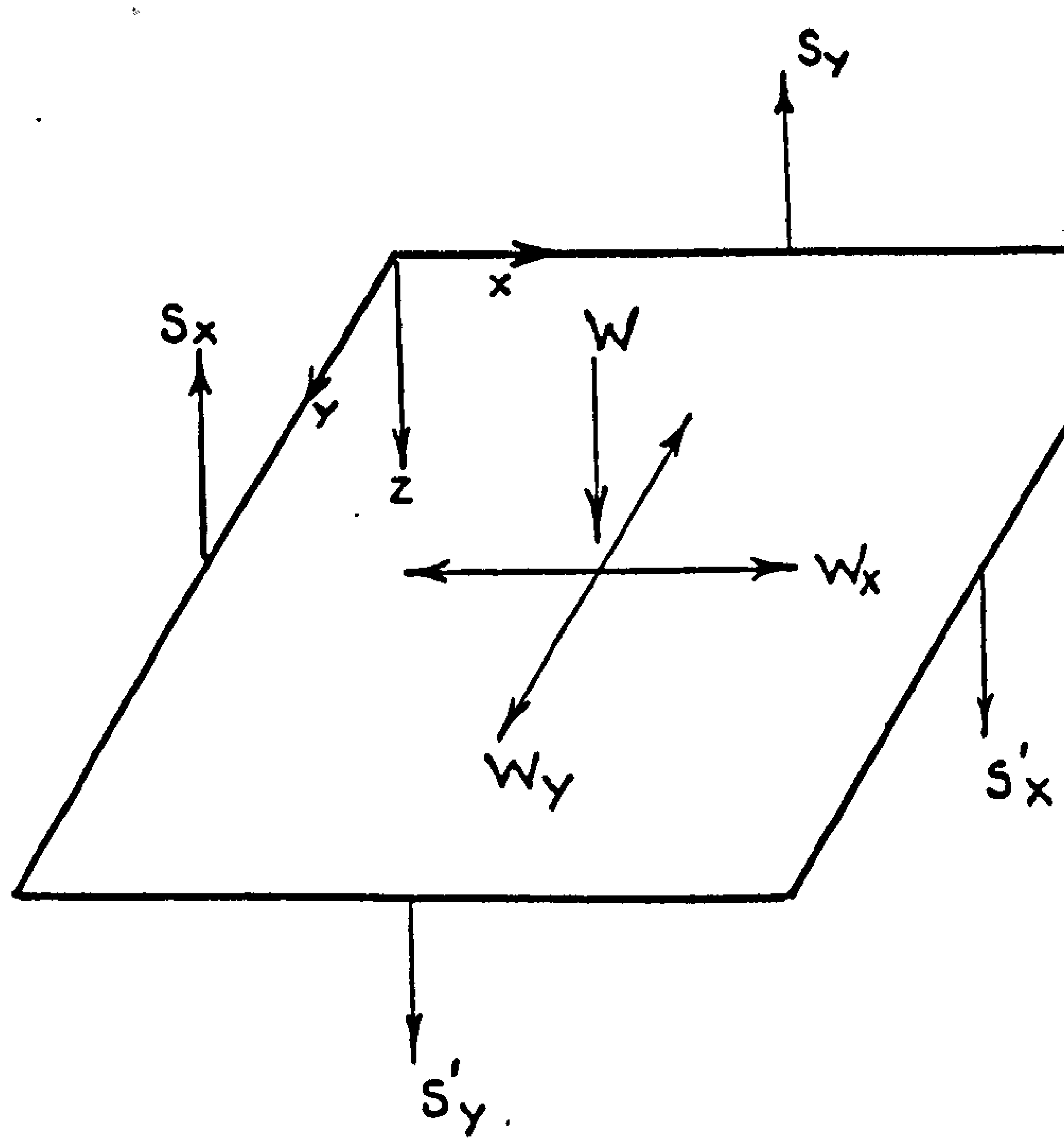


FIG. 3.14 VERTICAL SHEAR FORCES ON AN ELEMENT OF SLAB.

satisfy the equilibrium equation and boundary conditions. M_x, M_y and M_{xy} are known therefore at all points and reinforcement can be provided in accordance with section (3.7)

If however the type (III) element is internal, then a moment field with negative moments along the two boundaries containing the column and positive moment along the other two boundaries is required. This can be achieved by adding uniform negative moments m_{x1} and m_{y1} over the whole area. The corresponding moment field is then

$$M_x = \frac{pa^2}{8}\left(1 - \frac{4x^2}{a^2}\right) - m_{x1}$$

$$M_y = \frac{pb^2}{8}\left(1 - \frac{4y^2}{b^2}\right) - m_{y1}$$

$$M_{xy} = \frac{p}{2}\,xy \qquad (3.35)$$

and m_{x1} and m_{y1} will not alter the equilibrium equation. This method introduces the twisting moment and therefore is strictly not a strip method. Further it will not be possible to find a suitable stress distribution for all types of loading.

Wood and Armer have also suggested the use of strong bands to carry point loads instead of the type (III) element. They have remarked that this method is successful in tests, but there are no reliable rules for determining a width of band which will ensure satisfactory service conditions nor is there any information on the reinforcement required outside the bands.

3.11 KEMP'S MODIFICATIONS TO HILLERBORG'S STRIP METHOD.

In 1971 Kemp (28) published an extension of the Hillerborg's strip method to deal with concentrated loads and supports which maintained the concepts of the simple strip method of assuming the twisting moment $M_{xy} = 0$ everywhere. For an intensity of loading q per unit area the equilibrium of the vertical forces is

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q \qquad (3.36)$$

where Q_x and Q_y are the vertical shears per unit length along the y and x directions respectively as shown in Fig (2.1). Kemp used this shear equilibrium equation rather than the moment equilibrium equation (3.1) and derived local distributions of loading. His method can be illustrated by referring to a finite orthogonal element of slab shown in Fig (3.14). The equation of vertical equilibrium for the element is

$$(S_x' - S_x) + (S_y' - S_y) = -W \qquad (3.37\ a)$$

if W is zero ie if the slab element is unloaded then the equilibrium requirement is

$$(S_x' - S_x) = -(S_y' - S_y) \qquad (3.37\ b)$$

which means for such an unloaded element the interaction of forces between orthogonal strips must be equal and opposite, one strip being loaded and the other supported by the same pressure. Further Kemp emphasised the quantities $(S_x' - S_x)$ or $(S_y' - S_y)$ need not be zero, as it is normally assumed in the simple strip method.

As in the simple strip method W is divided into two components W_x and W_y transmitted in the x and y directions respectively. W, W_x and W_y are considered to be distributed uniformly over the slab element so that

$$(S_x' - S_x) = -W_x$$

$$(S_y' - S_y) = -W_y$$

then $(W_x + W_y) = W \qquad (3.38)$

W_x and W_y determines the local load distribution factor α and clearly no restrictions are imposed on the individual values.

Therefore this is a method where the designer chooses a load distribution pattern for the vertical shears rather than the individual loads. Kemp illustrated generally how this method can be extended to cover any shape of slab,boundary condition or loading. With complexity of shape and loading this method becomes tedious and is then quite difficult to assign realistic values for shears. This method pays no attention to service conditions and like Hillerborg's strip method could lead to unsatisfactory solutions in the hands of inexperienced designers. The concept of local load distribution is however a key to generalising the Hillerborg strip method.

CHAPTER FOUR.

UNIQUENESS OF THE COLLAPSE LOADS OF SLABS DESIGNED BY THE STRIP METHOD.

4.1. INTRODUCTION

The essence of the strip method of slab design is that the applied load is distributed in two orthogonal directions x and y and the twisting moment M_{xy} is set equal to zero at all points. The load is carried by pure bending on the strips in the x and y directions so that $\frac{\partial^2 M_x}{\partial x^2} = -\alpha q$ and $\frac{\partial^2 M_y}{\partial y^2} = -(1 - \alpha)q$, where α is the chosen load distribution factor. The slab problem is thereby reduced to analysing beam strips.

The strip method can be considered to be derived from the "Lower bound solutions via classical plasticity" described in section (3.9), whereby any solution to the equilibrium equation (3.1.) which satisfies the boundary condition and the yield criterion may be used for the safe design of reinforced concrete slabs. Infact when Hillerborg (18) first proposed this method his intention was to produce lower bound solutions for the collapse loads. He specifically stated that "If for a certain load q, a moment distribution can be found which satisfies the equilibrium equation and the edge conditions and if the slab can take up this moments at all points, then the value of q is lower limiting value of the collapse load".

More recently Wood and Armer (27) have critically examined the strip method and concluded that, when reinforcements are provided in accordance with the slab strip moments, Hillerborg's method provides an exact solution with an unlimited number of simultaneous modes.

4.2. WOOD AND ARMER'S PROOF ON UNIQUENESS.

Curiosity about the question of uniqueness arose when Wood and Armer analysed a layout of yield lines corresponding to the load distribution lines shown in fig (3.7) for a rectangular slab carrying a uniform load. Later they investigated the square slab shown in fig (3.10) with reinforcements placed exactly in accordance with the applied moment field. By yield line analysis of the slab it was concluded that all possible modes gave identical collapse loads equal to the design load. Their observations were valid for the problems considered but it will be shown to be true only for the particular types of moment field encountered.

In their mathematical proof Wood and Armer established that for a Hillerborg stress field in equilibrium with the applied loads the dissipation of internal energy is equal to the work done by the loads. Their proof is based on the assumption that the applied and the yield normal moments are identical at all points and in all directions. The Hillerborg method therefore satisfies both the upper and lower bound theorems and Wood and Armer have concluded that the collapse load for all possible mechanisms is unique.

With the twisting moment M_{xy} set to zero at all points, the stress states on the yield surface lie on the locus defined by the intersection of the two cones and the vertical plane through axis M_x and M_y as shown in fig (3.2). If reinforcement is provided exactly in accordance with the theoretical applied moment field, then the stress state is at one of the four points A,B,C and D. Points A and B are at the apex of the two cones and points C and D lie on the line of intersection of the two bases. The proof given by Wood and Armer which is based on the assumption of identical applied and yield normal moments is true only when the stress fields are at points A and B.

It will be shown that when the stress field is at points C or D the applied and yield normal moments coincide only in two orthogonal directions. Therefore, regions with such stress fields can only have positive or negative yield lines in these specified directions. Under such restricted conditions unique solutions can be found only if kinematically admissible mechanisms can be formed from permissible yield lines. The number of such possible mechanisms is obviously limited and it is not true in general to state that there will be an unlimited number of simultaneous modes.

If such mechanisms do not exist then the strip method will lead to a lower bound on the collapse load as anticipated by Hillerborg.

4.3. MOMENT FIELDS IN THE SIMPLE STRIP METHOD.

4.3. 1. Introduction.

Since the twisting moment M_{xy} has been set equal to zero at all points, the principal moments (M_x and M_y) are in the direction of the x and y reinforcements. If reinforcements are provided exactly in accordance with the calculated moment field, the applied principal moments will be equal to the yield principal moments (m_x and m_y) at all points. Clearly the yield criterion will be satisfied everywhere, but the manner in which the yield criterion is satisfied will depend on the particular moment field. There are three categories of principal moment field to be considered.

(a) With both principal moments positive.

(b) With both principal moments negative.

(c) With one principal moment positive, one negative.

4.3. 2. Both principal moments positive. (Positive yield lines).

In this case reinforcements will be provided in two orthogonal directions (x and y) in the bottom face only. If reinforcement is provided exactly in accordance with the strip solution then $m_x = M_x$, $m_y = M_y$ and $-m'_x = -m'_y = 0$.

The variation of the applied and yield normal moments with orientation at a typical point in this moment field is shown in fig (4.1). Because the applied and yield principal moments are of the same sign, magnitude and direction, the applied normal moment M_n is equal to the yield normal moment m_n in all directions. Thus positive yield can occur at all points of the slab in any direction. An alternative approach is to examine the double cone yield surface given in fig (3.2). The moment field is at the singular point A, the apex of the positive yield cone. By using plastic potential theory outlined in section (3.5) the curvature rate vector can act in any direction at A, so that yield lines can occur in any direction.

Thus positive yield lines may occur in any direction at all points in a positive - positive moment field and be consistent with the strip solution moment field. If this moment field covers the entire slab, all kinematically admissible mechanisms composed of positive yield lines only will have a collapse load equal to the strip method design ultimate load. In general there will be an infinity of such mechanisms. The moment field in the slab examples analysed by Wood and Armer (27) were precisely of this type and in these restricted circumstances their conclusions are valid.

<u>4.3.3. Both principal moments negative.(negative yield lines).</u>

If precisely the calculated reinforcement is provided in accordance with the strip solution then $m_x = m_y = 0$; $-m'_x = M_x$ and $-m'_y = M_y$. The yield conditions will be identical with those for positive - positive moment field but for the change of sign and the two normal moment curves M_n and m_n are again coincident.

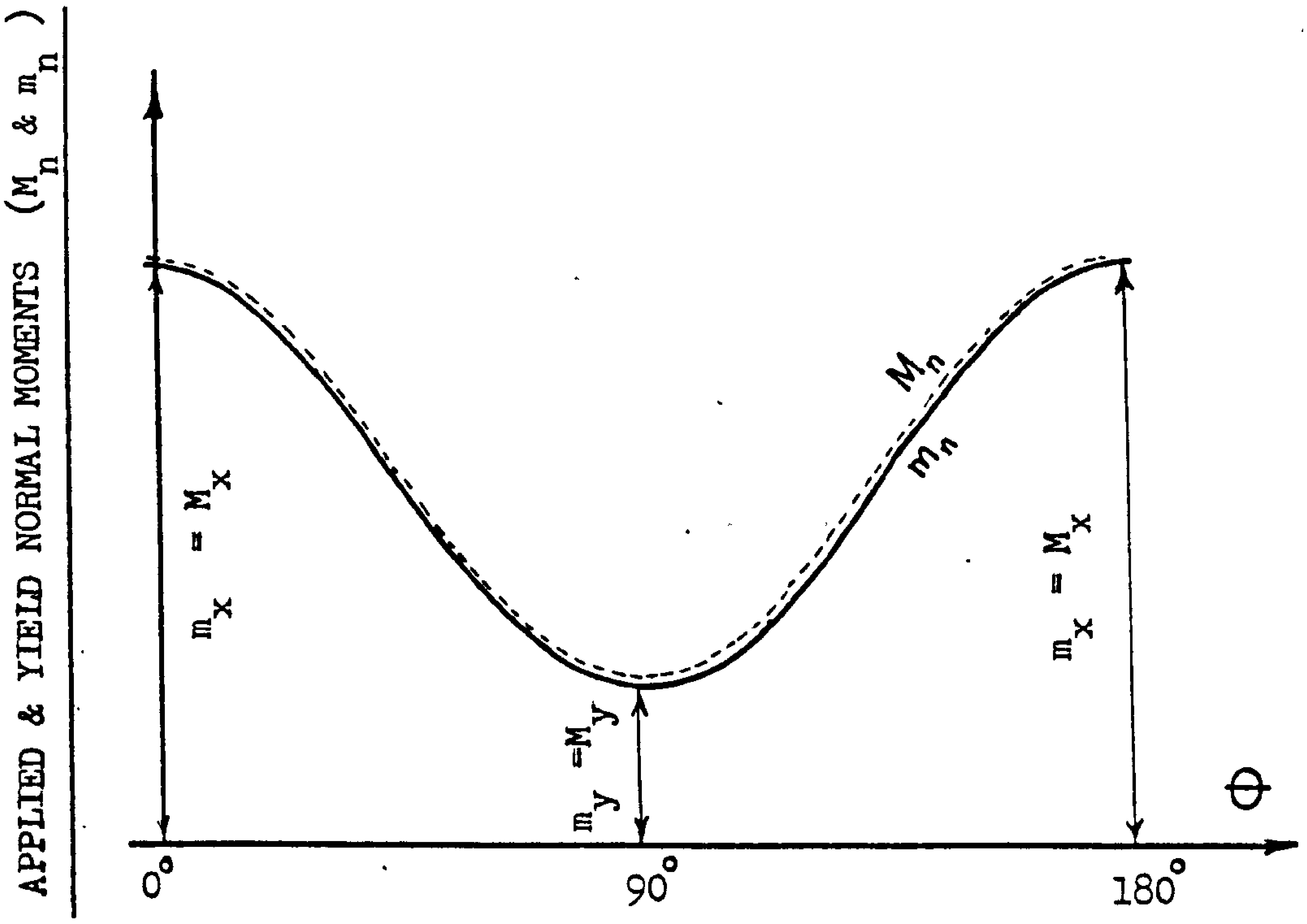


FIG. 4.1 YIELD CONDITIONS IN A POSITIVE - POSITIVE MOMENT FIELD

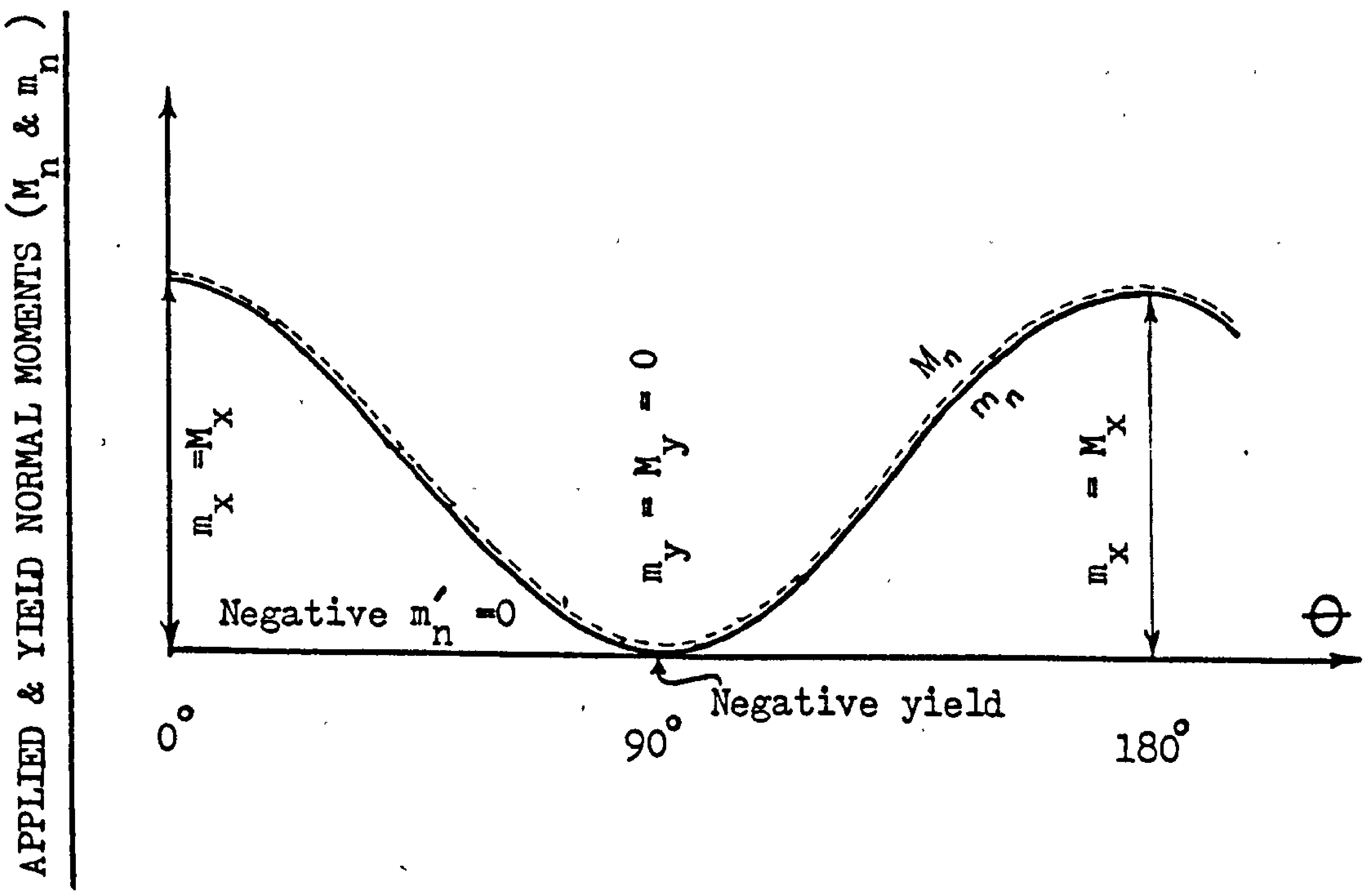


FIG. 4.2 YIELD CONDITIONS IN A POSITIVE-ZERO MOMENT FIELD

The position on the yield surface fig (3.2) is now at the apex of the negative cone B and negative yield can occur in any direction at all points.

All kinematically admissible mechanisms composed of negative yield lines only will produce a unique solution. If this negative - negative moment field covers the entire slab then there will be an infinite number of simultaneous collapse mechanisms. The collapse load calculated from any such mechanism will be identical to the strip method design load.

4.3. 4. Negative yield lines in a positive - positive moment field or positive yield lines in a negative- negative moment field.

It is also necessary to consider the restrictions on negative yield lines in a positive - positive moment field. From the normal moment curve fig.(4.1) it can be seen that for a unique solution negative yield lines are not permissible except in special cases fig. (4. 2). This is only possible when one of the principal moments is zero and the negative yield line direction must be the reinforcement direction along which the principal moment is zero. Similarly positive yield lines in a negative - negative moment field will only be consistent with a unique solution along a line of zero principal moment which must be a reinforcement direction. These are severe restrictions on the permissible yield lines for unique solutions.

4.3. 5. One principal moment positive, one negative.

If reinforcement is provided exactly in accordance with the calculated moment field, there will be bottom reinforcement in one direction and top reinforcement in the orthogonal direction. The strip solution moment field can be either.

(a) $m_x = M_x$; $m_y = 0$; $m'_x = 0$ and $-m'_y = M_y$ (Fig. 4.3) or

(b) $m_x = 0$; $m_y = M_y$; $-m'_x = M_x$ and $m'_y = 0$

The applied and yield normal moment curves are tangential at only two positions, so the conditions for plastic flow are restrictive as illustrated in fig.(4.3) positive yield can occur only at orientation $\theta = 0$ and negative yield only at $\theta = \pi/2$

Thus in a positive - negative moment field the only yield lines consistent with the unique solution are positive yield lines normal to the positive reinforcement and negative yield lines normal to the negative reinforcement. If straight strips and straight reinforcements are used then it follows that the yield lines must be straight in regions of positive-negative moment field.

The same conclusions can be drawn by examining the yield surface. The moment state is at one of the two points C and D on the yield surface(fig.(3.2)) where the vertical plane through the M_x and M_y axes and the intersection plane of the two cones interesct. The curvature rate vector must be in a vertical plane to satisfy the normality rules and depending on the sign of M_x and M_y the permissible curvature rates are $+\dot{K}_x$ and $-\dot{K}_y$ or $-\dot{K}_x$ and $+\dot{K}_y$.

The effects of these restrictions on yield lines upon the uniqueness of the strip solutions containing positive and negative moment fields do not seem to have been considered previously.

4.3. 6. Rules for yield lines consistent with a unique solution.

For a slab reinforced exactly in accordance with the calculated moment field, the strip solution will give the unique value of collapse load, if a kinematically admissible collapse mechanism is possible in which the yield lines satisfy the following rules.

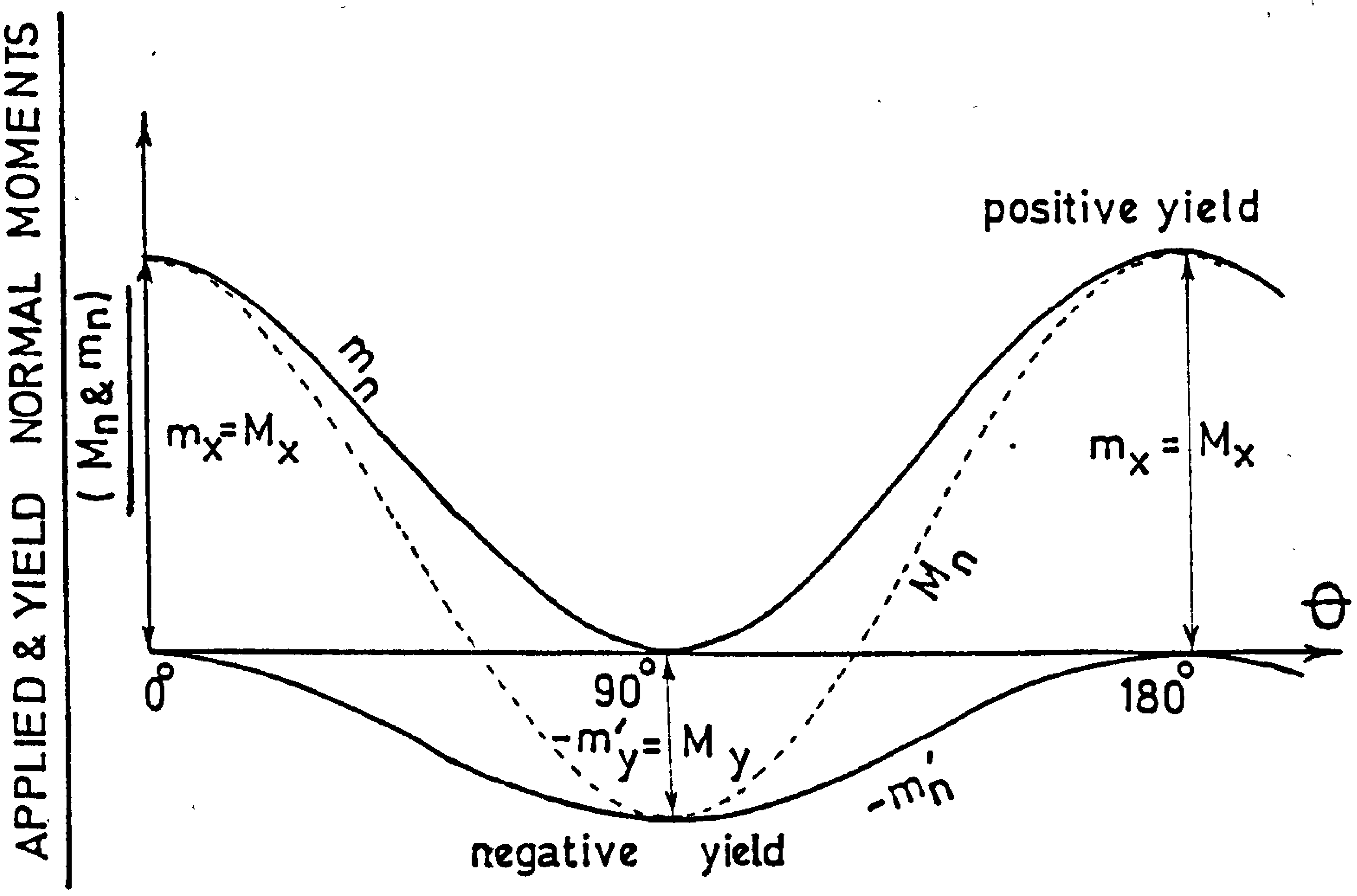


FIG4.3 YIELD CONDITIONS IN A POSITIVE-NEGATIVE MOMENT FIELD

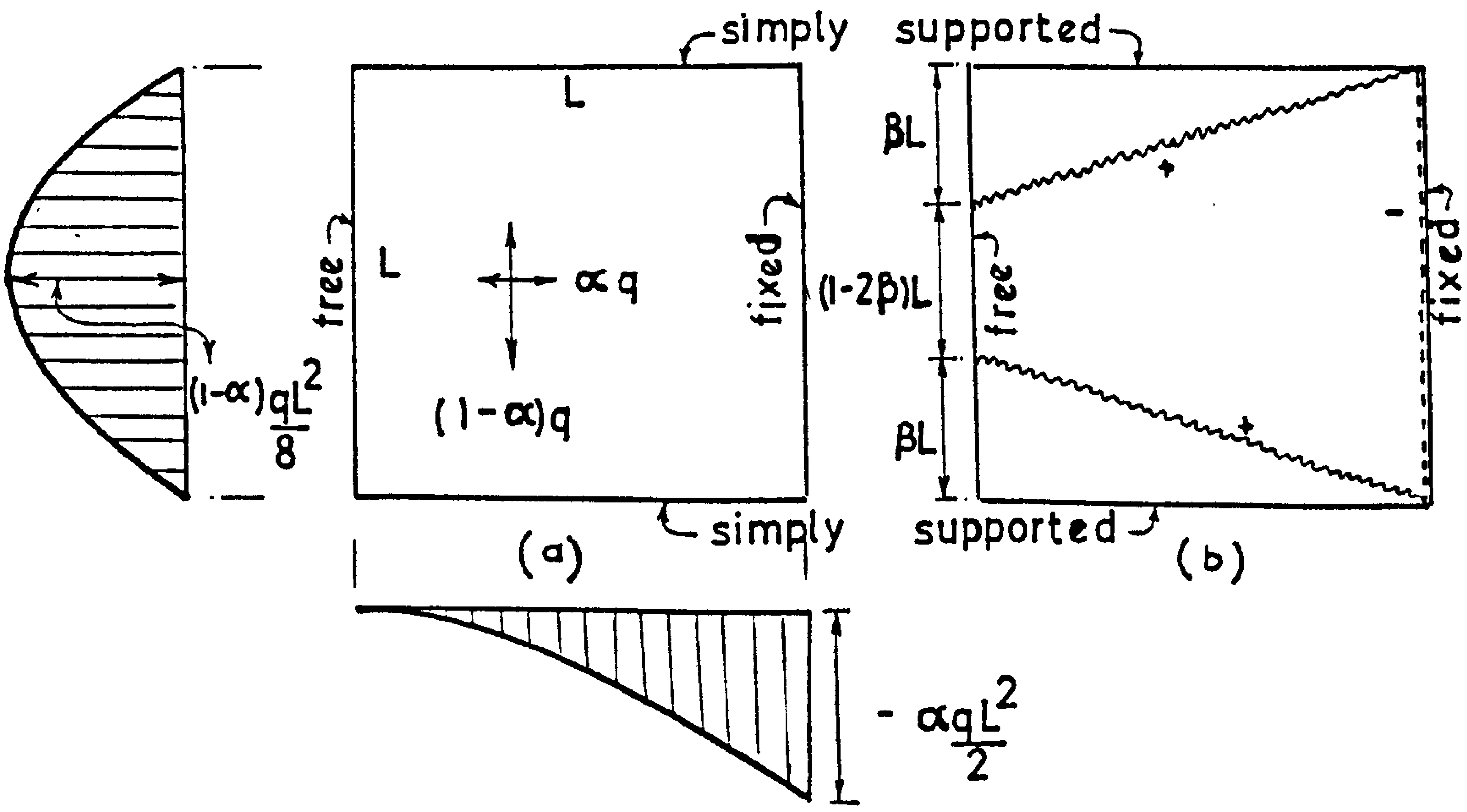


FIG 4.4. EXAMPLE OF A POSITIVE-NEGATIVE MOMENT FIELD

1. Positive - Positive moment field.

Positive yield lines may act in any position and in any direction. Negative yield lines are only allowed in a reinforcement direction along which the principal moment is zero.

2. Negative - Negative moment field.

Negative yield lines may act in any position and in any direction. Positive yield lines are only allowed in a reinforcement direction along which the principal moment is zero.

3. Positive - Negative moment field.

The only yield lines allowed are positive yield lines normal to the positive reinforcement and negative yield lines normal to the negative reinforcement.

If the entire slab is covered by a positive - positive or negative - negative moment field, then there can be an infinity of simultaneous collapse mechanisms. The calculated collapse load from any such mechanism will be identical to the strip solution design load.

In general a slab designed by the strip method will contain combinations of these three types of moment field and it does not seem possible to argue generally that a yield line pattern consistent with a unique solution can be found. It will however be shown in the examples presented that it is extremely difficult to find a problem, (atleast with distributed loading), where one mechnaism consistent with the unique solution cannot be found.

4.4. EXAMPLES .

EXAMPLE 4.4. 1. Positive - Negative Moment field

Consider the square slab in Fig.(4.4.a). It is simply supported on two opposite sides, fixed on the third and free on the fourth.

Lower bound solution.

The slab is designed to carry an ultimate load q per unit area, which is distributed in the two directions such that $q_x = \alpha q$ and $q_y = (1 - \alpha) q$. Further α is chosen such that $0 < \alpha < 1$ and the strip solution leads to a positive, negative moment field over the whole slab.

$$M_x = - \alpha \frac{qx^2}{2} \quad - \text{negative}$$

$$M_y = (1 - \alpha) q \frac{y}{2} (L-y) \quad - \text{positive}$$

Upper bound solution.

Consider the family of collapse mechanisms shown in Fig (4.4.b) defined by parameter β ($\beta \leqslant 0.5$). If the centre of the free edge moves through a unit vertical distance, the upper bound to the collapse load q_c can be calculated by considering the external work and plastic energy dissipated.

$$\text{External Work} = q_c (1-2\beta) L^2 \cdot \frac{1}{2} + q_c \, 2\beta L^2 \cdot \frac{1}{3} = q_c \frac{L^2}{6} (3-2\beta)$$

By making use of the vector method the internal work done

$$D = \alpha \frac{qL^2}{2} \cdot L \cdot \frac{1}{L} + 2 \cdot \int_0^L My \frac{1}{\beta L} \cdot dx$$

since $\frac{dy}{dx} = \frac{\beta L}{L} = \beta$

$$\int_0^L My \frac{1}{\beta L} \cdot dx = \int_0^{\beta L} \frac{My}{\beta^2 L} \cdot dy = \frac{(1-\alpha)q}{2\beta^2 L} \int_0^{\beta L} y(L-y)\, dy$$

$$= (1-\alpha) q \frac{L^2}{12} (3-2\beta)$$

$$\therefore D = \alpha q \frac{L^2}{2} + (1-\alpha) q \frac{L^2}{6} (3-2\beta)$$

$$= \frac{qL^2}{6} (3-2\beta + 2\beta\alpha)$$

Equating external and internal work gives

$$q_c \leqslant q\left(1+\frac{2\beta\alpha}{3-2\beta}\right) = q\ K$$

If $\alpha \neq 0$; K varies with β but is greater than unity for all values of $\beta > 0$.

There is not a stationary minimum value of K but the least value of K is obtained when $\beta = 0$, and then $K = 1$. The solution is then apparently unique. If $\beta = 0$ the positive and negative yield lines are consistent with the rules postulated for a positive - negative moment field, but the mechanism is not strictly kinematically admissible. The mechanism becomes valid with an infinitely small β. So that in this example, the strip method does not strictly give a unique solution for the collapse load and certainly not an infinite number of collapse mechanisms. It can however be stated that the unique solution is approached as $\beta \rightarrow 0$.

EXAMPLE 4.4. 2. Positive - Negative moment field "special case".

Consider the slab shown in fig. (4.5.a) in the shape of a isoscelese right angled triangle. The base A B length 2 L is free and the other two sides fixed. The X strips are considered to be simply supported and the Y strips are cantilevers. The moment field shown in Fig (4.5.b.) calculated by the strip method is designed for a collapse load q per unit area with distribution factor α

M_x the moment in the x direction is positive whereas M_y is negative. However a closer examination will reveal that M_x along edges AC and CB is zero and therefore what appears to be a positive-negative moment field is infact composed of three moment fields negative-zero along AC and CB and negative-positive at all other points. According to the rules postulated it is possible to have negative yield lines in any direction in a negative - zero

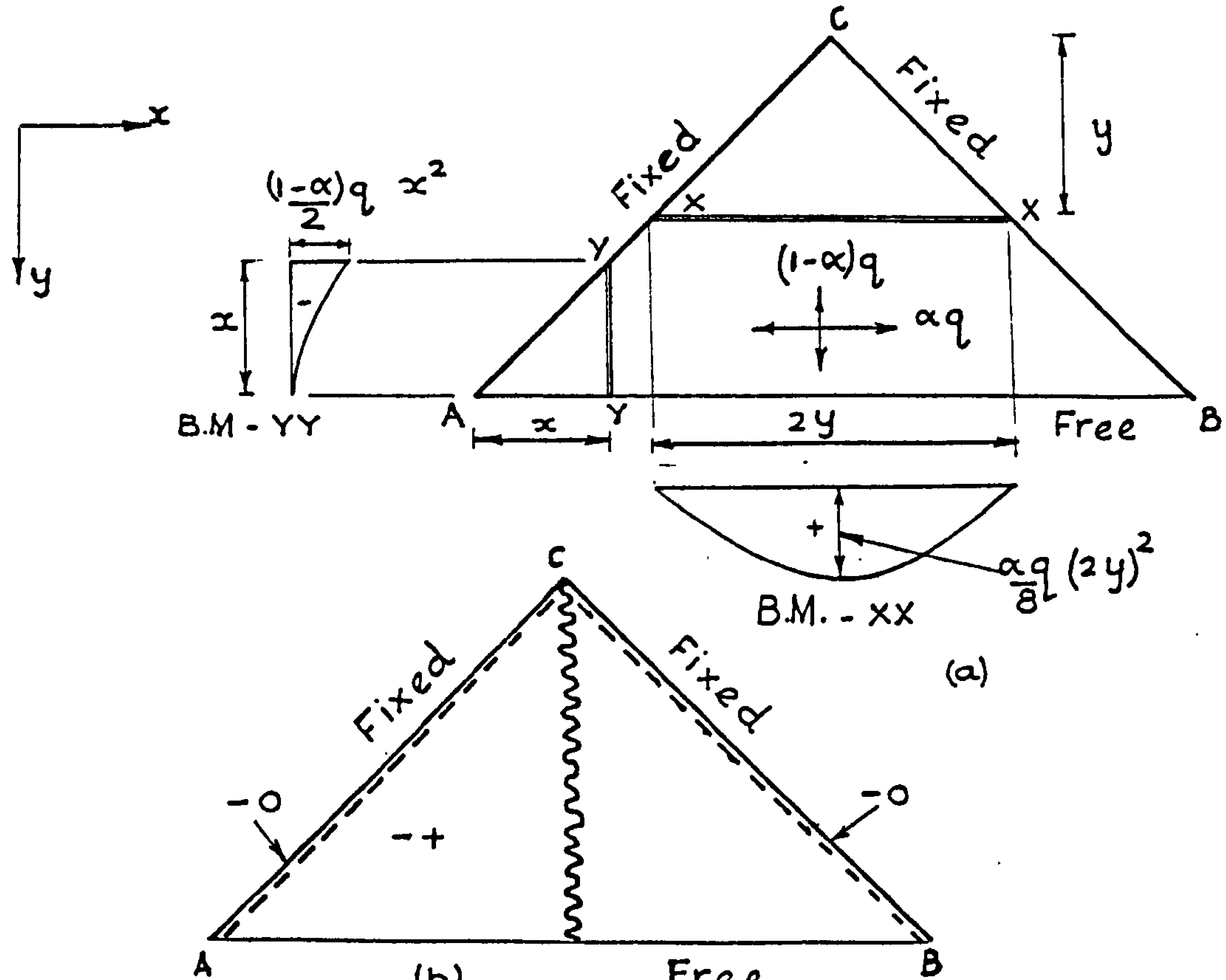


FIG. 4.5 EXAMPLE OF A POSITIVE - NEGATIVE MOMENT FIELD-'SPECIAL CASE'

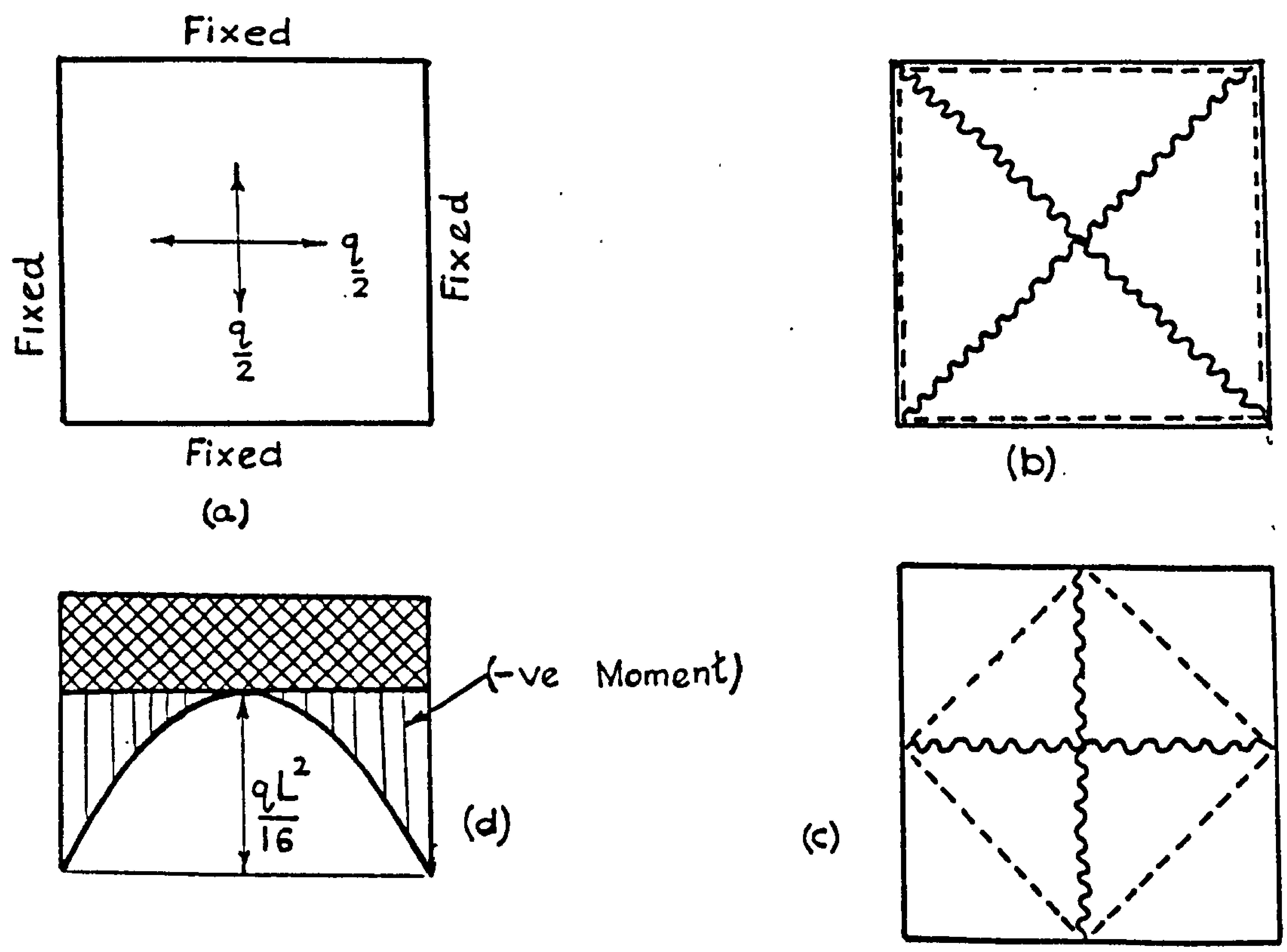


FIG. 4.6 EXAMPLE OF A NEGATIVE-NEGATIVE MOMENT FIELD

moment field such as along AC and CB. The mechanism shown in Fig (4.5. b.) is therefore consistent with a unique solution which is confirmed by analysis. If the centre of AB moves through unit distance and q_c is the upper bound collapse load

$$\text{External work } E = q_c \frac{L^2}{3}$$

$$\text{Internal work done } D = 2\int_0^L (1-\alpha)\, q\, \frac{x^2}{2}\, \frac{1}{L}\, dx + \int_0^L \frac{\alpha q\, y^2}{2} \cdot \frac{2}{L}\, dy$$

$$= (1-\alpha)\, q\, \frac{L^2}{3} + \alpha q\, \frac{L^2}{3} = q\, \frac{L^2}{3}$$

Equating internal and external work gives $q_c = q$. The solution is therefore unique but there is only one consistent mechanism.

EXAMPLE 4.4. 3. Negative - Negative moment field.

Fig (4.6. a) shows a square slab fixed on all edges and designed to carry a uniformly distributed load q which is divided equally in the x and y directions and M_{xy} is set to zero. Further, the negative moments for the strips are chosen as shown in Fig. (4.6. d) so that the moment along the centre lines is zero. The moment field is negative - negative except along the centre lines where it is negative - zero and given by

$$M_x = q\, \frac{x}{4}\, (L-x) - q\, \frac{L^2}{16}$$

$$M_y = q\, \frac{y}{4}\, (L-y) - q\, \frac{L^2}{16}$$

Consider first the collapse mechanism shown in fig (4.6. b). The negative yield lines are consistent with the a unique solution but the positive yield lines do not coincide with the direction of zero principal moment. The mechanism therefore can be predicted to lead to an upper bound on the collapse load. Equating the internal and external work the upper bound on the collapse load q_c is

$$q_c \leqslant \frac{3}{2} q$$

so that $\frac{3}{2} q \geqslant q_c \geqslant q$

In the alternative mechanism shown in Fig (4.6. c) the positive and negative yield lines are consistent with the prescribed rules and therefore the mechanism can be predicted to lead to the unique value of the collapse load. For unit central deflection

$$E = q_c \frac{L^2}{6}$$

$$D = 4\int_0^{L/2} M_x dy \frac{2}{L} + 4\int_0^{L/2} M_y \, dx \frac{2}{L}$$

Now $\frac{dy}{dx} = 1$ on the negative yield lines so

$$D = 4\int_0^{L/2} \left[\frac{q L^2}{16} - q \frac{x}{4} (L-x) \right] \frac{2}{L} . \; dx + \int_0^{L/2} \left[\frac{qL^2}{16} - \frac{qy}{4} (L-y) \right] . \frac{2}{L} dy$$

or $D = q \frac{L^2}{6}$

Equating internal and external work gives $q_c = q$. The strip method does therefore, give a unique solution for the collapse load, but there appears to be only one possible mechanism associated with a unique solution.

It is also of interest to consider what happens if the negative support moment is chosen to be greater (numerically) than $q \frac{L^2}{16}$ as shown in fig (4.6.d). The moment field would then be negative - negative everywhere with no zero principal moments. There would then appear to be no valid mechanism that will produce yield lines which would lead to a unique solution. All kinematically admissible mechanisms would lead to an upper bound on the collapse load. In this case therefore the strip solution must be accepted as a lower bound solution only.

The reason for this becomes obvious on examination of Fig (4.6. d). It is evident that more negative reinforcement is provided than required, the excess being equal to the cross hatched area. A unique solution is still possible by the strip method provided this excess moment is zero.

A similar example of excess moment is a square slab on simple supports in which the uniformly distributed load q is distributed as + 2 q in one direction and - q in the other. A unique solution can be found for any distribution α q, (1- α) q for which $1 \geqslant \alpha \geqslant 0$, but with $\alpha = 2$ the solution obtained is only a lower bound on the collapse load again due to excess moments.

EXAMPLE 4.4. 5.

The example shown in Fig (4.7) illustrates the application of the uniqueness rules to determine a consistent mechanism. The slab is designed by the strip method and leads to regions of positive - positive, positive - negative and negative - negative moment fields. In the positive-positive region P Q R S there is an infinity of permissible layouts of positive yield lines. However at the corners and in the positive - negative regions near the fixed boundaries the yield lines indicated appear to be the only ones consistent with the unique solution.

4.5. YIELD LINES AND MODES OF FAILURE IN MINIMUM WEIGHT DESIGNS.

4.5. 1. Introduction.

The strip method normally restricts the whole area of the slab to have at most two reinforcement directions. Steel can be placed in these orthogonal directions in either one or both faces of the slab. In minimum weight designs this constraint is relaxed and there are a number of reinforcement directions depending on the geometry of

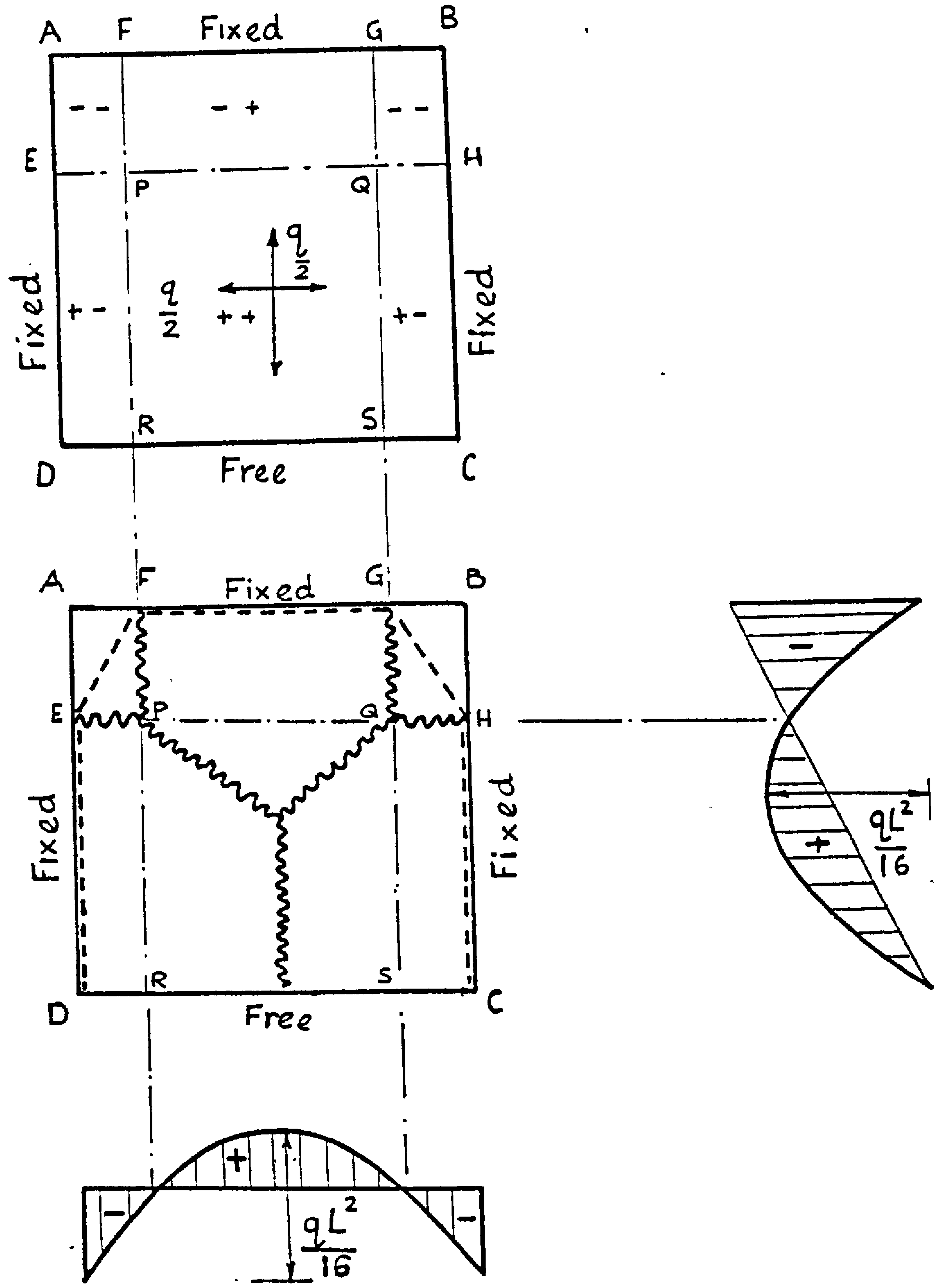


FIG. 4.7 CONSISTENT MECHANISM ASSOCIATED WITH A UNIQUE COLLAPSE LOAD

the slab and the nature of boundaries. Nevertheless there are similarities in the moment field obtained in the two methods and the same rules must be applied to determine consistent mechanisms.

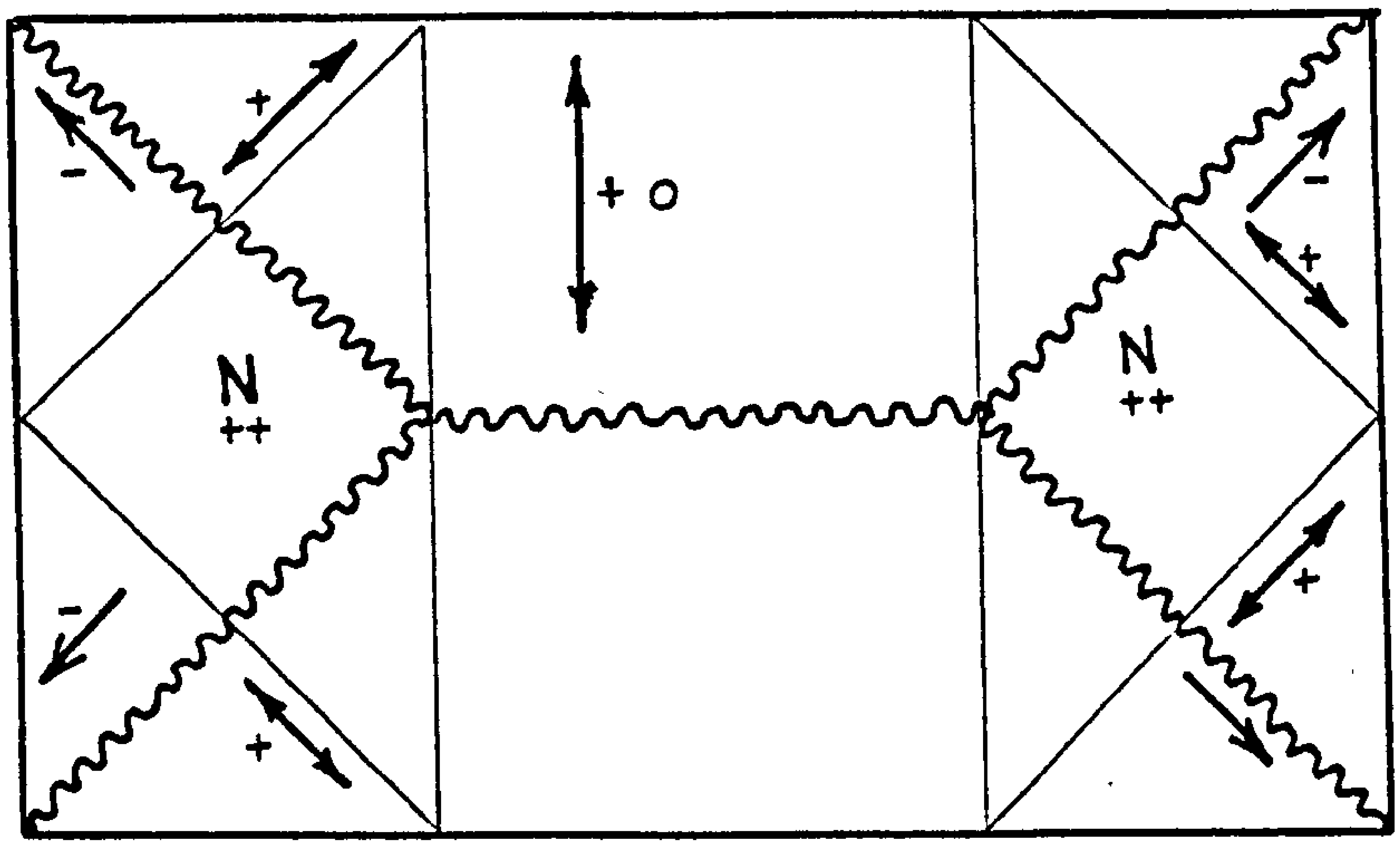
The neutral zones with a spherical deformation surface discussed in section 3.8 are identical to positive - positive (or negative - negative) moment fields. For the simply supported rectangular slab in Fig. (3.5) regions such as AEJ at the corner resemble the positive - negative moment field with principal positive and negative moments parallel and perendicular to EJ respectively. The central portion EFGH is similar to a positive -zero moment field with the principal positive moment in the direction of EH. In slabs with built in edges, regions of negative - zero moment field can be found near the fixed boundaries and the corresponding negative moment will be in a direction normal to the fixed edge.

It appears that researchers on minimum weight solutions have overlooked the collapse behaviour of the slabs. The optimum slab is considered to be yielding simultaneously at all regions and it is generally assumed that there are an unlimited number of simultaneous modes all providing an exact solution. The rules developed in section 4.3.6. shows this to be untrue.

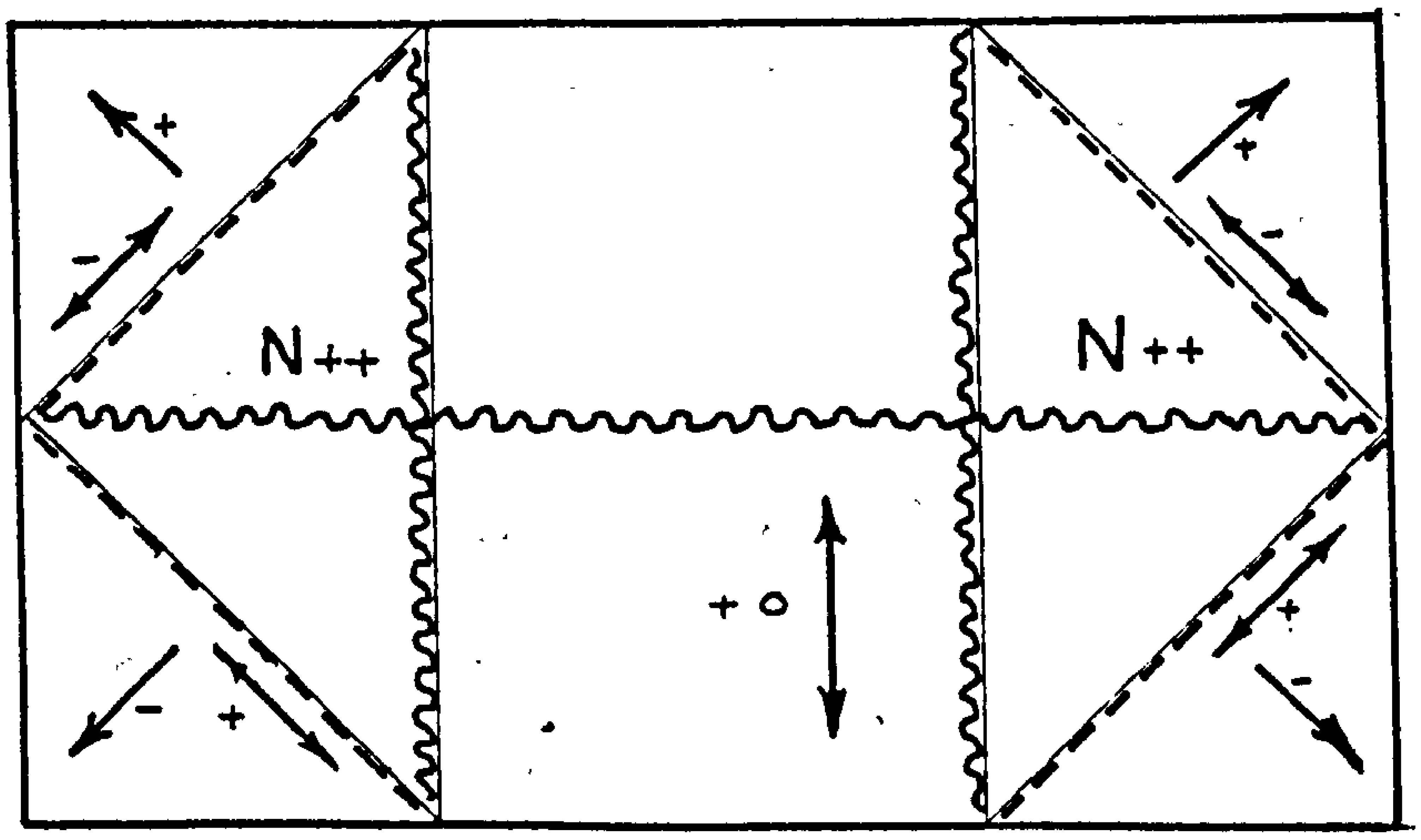
4.5. 2 Applications to minimum weight solutions.

Figs. (4.8) to (4.12) show five familiar minimum weight solutions. These examples are due to Morley (23), Rosvany and Adidam (29), Lowe and Melchers (30), (31). The notations on the figures is given in Ref. (29), (30) and (31).

For the simply supported rectangular slab shown in Fig (4.8. a) the reinforcement directions for the corner triangles are parallel and normal to the bisector of the right angles. Positive yield lines must therefore form along the bisector of the corners and this rules out the possibility of corner fans. Although positive yield lines can occur in



(a) WITH NO CORNER FANS



(b) WITH CORNER FANS

N - Neutral zone

→ -ve Curvature

↔ +ve Curvature

FIG. 4.8 SIMPLY SUPPORTED RECTANGULAR SLAB - MECHANISMS CONSISTENT WITH THE MINIMUM WEIGHT SOLUTION

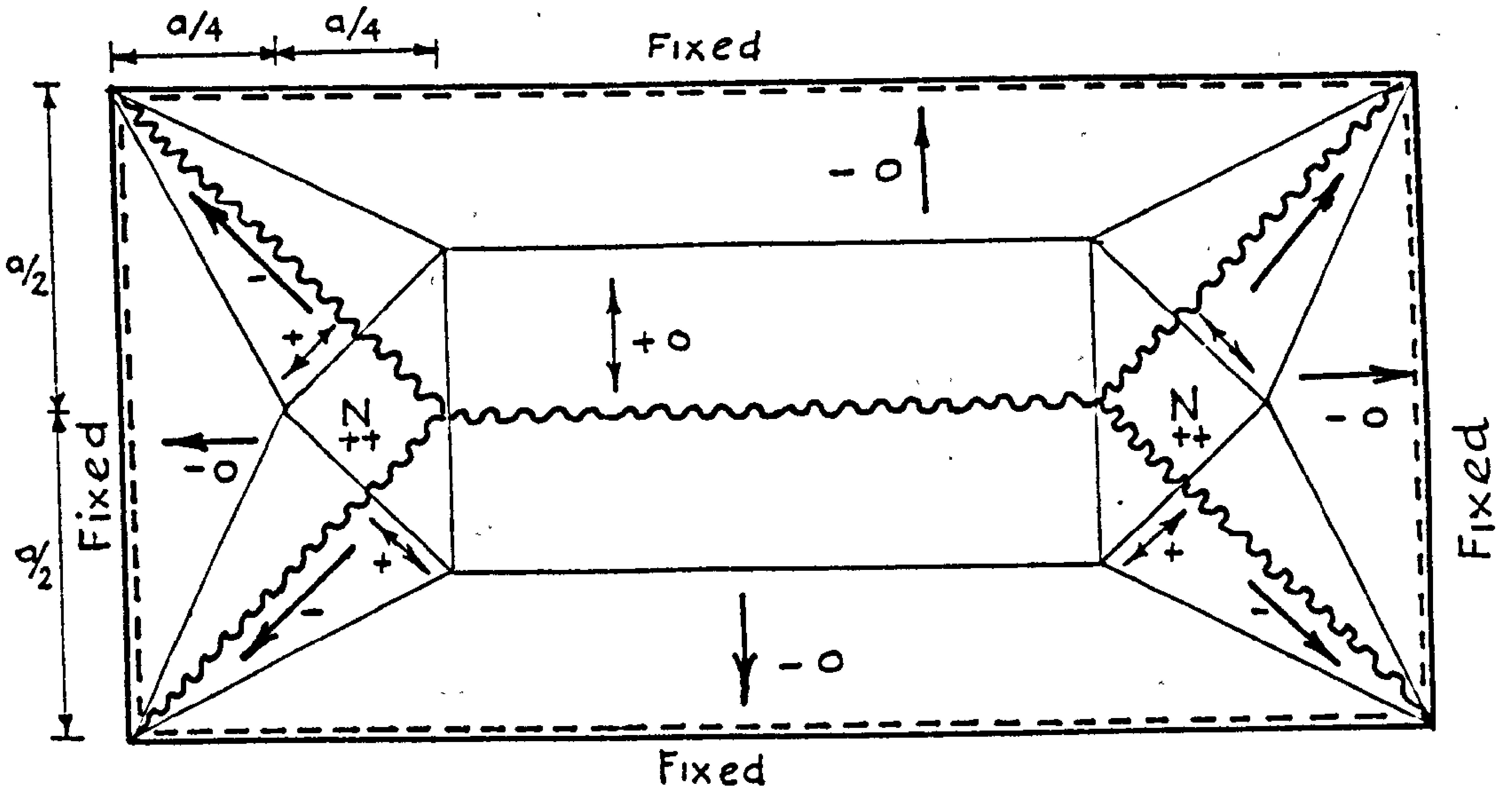


FIG. 4. 9 CLAMPED RECTANGULAR SLAB - MECHANISM CONSISTENT WITH THE MINIMUM WEIGHT SOLUTION

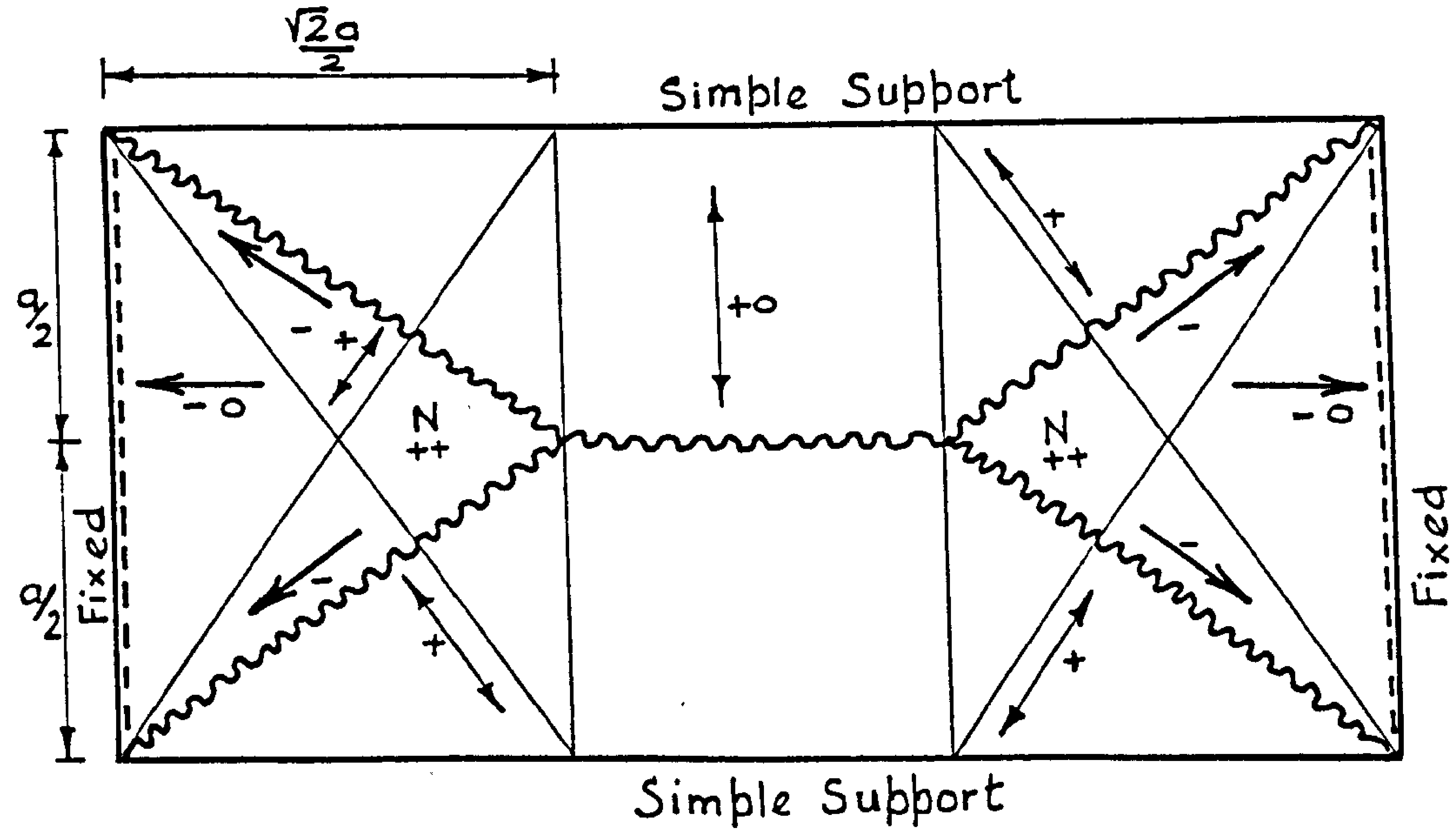


FIG. 4.10 RECTANGULAR SLAB, TWO SIDES FIXED, OTHER TWO SIDES SIMPLY SUPPORTED - MECHANISM CONSISTENT WITH THE MINIMUM WEIGHT SOLUTION

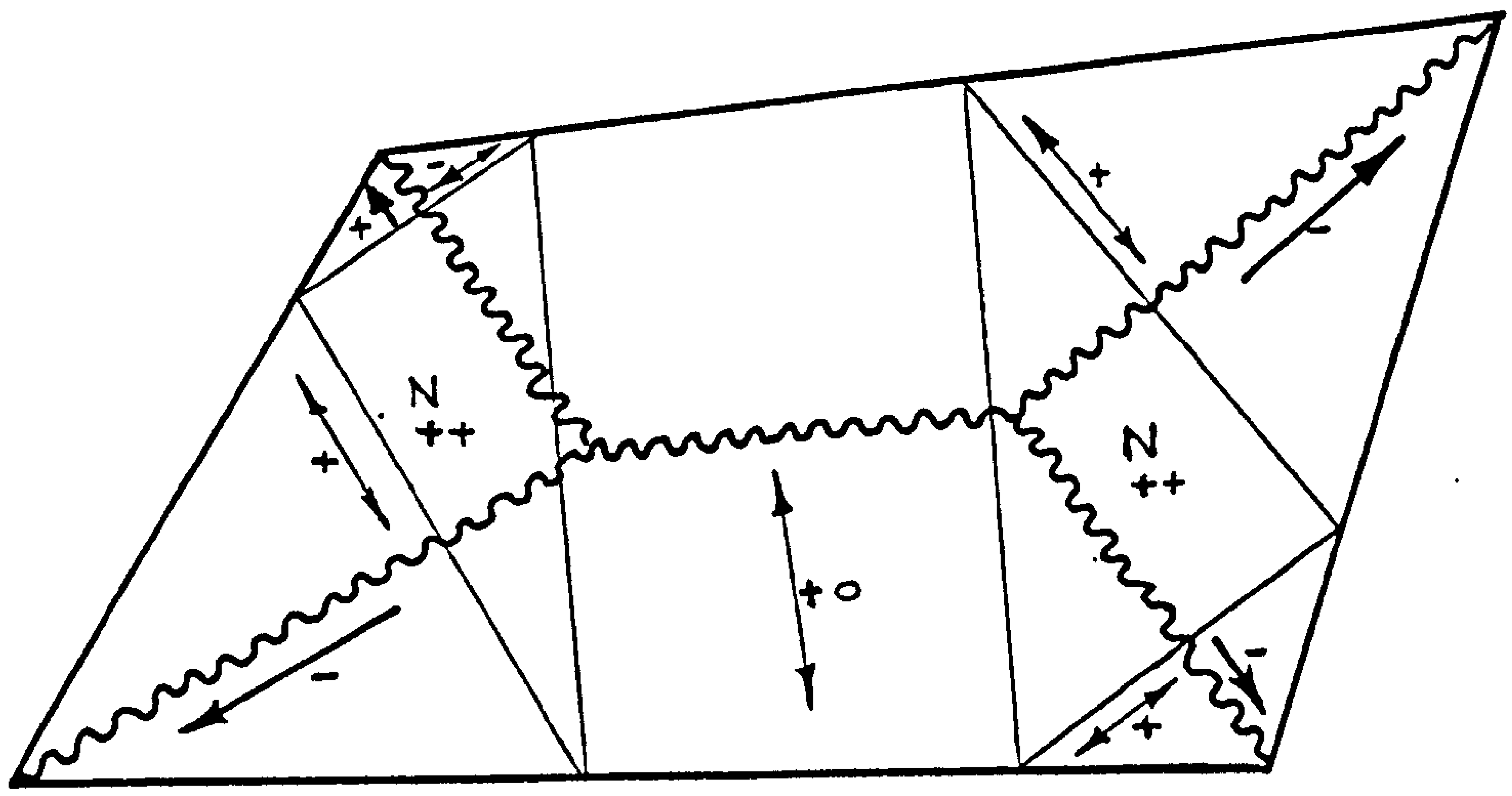


FIG. 4.11 TYPICAL SIMPLY SUPPORTED SLAB, MECHANISM CONSISTENT WITH MINIMUM WEIGHT SOLUTION - NO CORNER FANS

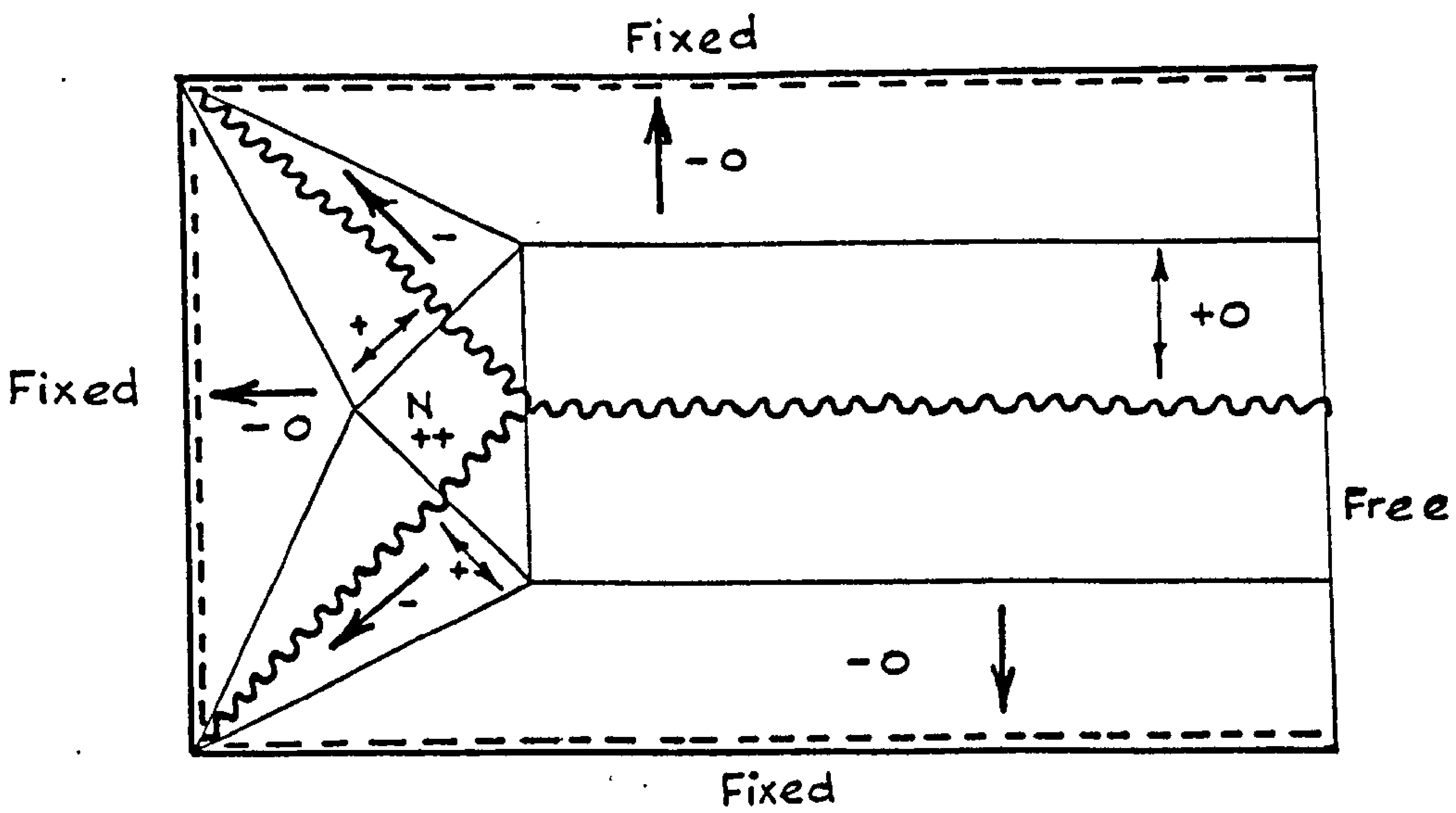


FIG. 4.12 RECTANGULAR SLAB, THREE SIDES FIXED ONE FREE. MECHANISM CONSISTENT WITH THE MINIMUM WEIGHT SOLUTION

any direction within the neutral zone and in the central area with a positive - zero moment field, taken the slab as a whole, the mechanism shown in Fig (4.8.a) appears to be the one with positive yield lines only and consistent with the rules set out in section (4.3). If for the same example the possibilities of corner fans with negative yield lines are considered then the mechanism shown in Fig (4.8.b) seems to be the only other permissible alternative.

The same can be concluded for the other four slabs. The mechanisms shown in Fig (4.9) to (4.12) seems to be the only kinematically acceptable patterns without corner fans which are consistent with a unique solution for the collapse load.

4.6. CONCLUSIONS

The discussion in this chapter was aimed at correcting and clarifying the mathematical proof for uniqueness and multiple mechanisms given by Wood and Armer. For slabs designed by the strip method, although the principal applied moment and the principal yield moment can be made equal at all points, it is however not generally possible to make the applied and yield normal moments equal in all directions. If in the slab as a whole the applied and yield normal moments are identical at all points in every direction (positive - positive or negative - negative moment field) then there exists a unique solution with an unlimited number of simultaneous modes.

In general a slab designed by the strip method will not only have regions where the two normal moments coincide. Rules have been derived for the postulation of yield lines for such slabs with a combination of moment fields. In view of these rules, some of which are very restrictive, it is not possible to argue that the strip method will always lead to a unique solution. However it must be admitted that it is remarkably difficult to find a practical example of a slab (with distributed load) designed by the strip method for which there is not at least one collapse mechanism consistent with these rules and therefore leading to a unique collapse load. Examples have been presented where the strip method can be demonstrated to approach the unique solution as a limiting case or to be a lower bound solution. In the latter case the load distributions could be altered to obtain uniqueness.

The derived rules for uniqueness were also applied to some minimum weight designs. In all cases there appear to be only a limited number of such mechanisms consistent with the unique solution for collapse load. In the absence of corner fans there seems to be only one mechanism.

The slabs considered in this chapter were confined to distributed types of loading. Point, patch loads and point supports will be considered in Chapter Six.

CHAPTER FIVE

STRIP DEFLECTION - A GENERALISED METHOD OF REINFORCED CONCRETE SLAB DESIGN.

5.1 INTRODUCTION

The current elastic and plastic methods of reinforced concrete slab design were discussed in Chapters 2 and 3 with comments on their merits and limitations. Limit state methods are well established for the design of reinforced concrete structures and for slabs the critical limit state for design is usually the state of collapse. Design is therefore commonly based on this state with checks made on cracks, deflections and any other serviceability condition where necessary.

An ideal method of slab design should be easily understood, simple in computation, applicable to any shape of slab, boundary condition and loading system. Ideally the method should give the unique value of the collapse load. In addition it should provide information about the total moment field, shears, reactions, deflections etc., which for good serviceability condition should not depart too far from the working load conditions. The site conditions too need recognition and simple banded layouts of reinforcement will be helpful and economical in steel fixing. Ideally the total quantity of steel must be as close as possible to the minimum weight solution. Not surprisingly none of the methods discussed so far satisfy all these requirements.

The purpose of this chapter is to present a generalised approach to the strip method of slab design, which aims at retaining all its attactiveness and eliminating its disadvantages. This method intends to accommodate point loads, point supports, free edges, to cover the design of any slab system. It will also ensure that the designer will not depart too far from the working load moment fields, shears reactions and thereby ensure that serviceability is satisfied

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5.2. THE CHOICE OF THE LOAD DISTRIBUTIONS.

5.2.1. Introduction.

There are two main features in the simple strip method which have prevented its more general application. Firstly, the designer chooses a load distribution q_x and q_y to be constant over an extensive area of the slab. The second is that the distribution factor α is chosen to be between 0 and 1 and usually the extreme values 0 or 1 are selected, since the designer at present has no way of intuitively selecting values of α outside this range. Due to these factors it is possible to depart far from working load moment fields and so serviceability may not be satisfied.A method of overcoming some of these restrictions was suggested by Kemp (28) in which the shear forces were distributed over a grid area rather than choosing load distributions. This approach has lifted some restrictions on α,but except for simple problems it is too tedious and could lead to unsatisfactory service conditions.

5.2.2. Elastic load distributions - Uniformly loaded slabs.

Consider the simply supported square slab shown in Fig (5.1). It is an easy task to design this slab by the strip method. It is however curious to find out what are the realistic values of the load distributions at any point of the slab. The basic equilibrium equation in plate theory is(Fig. 2.1.).

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

which is satisfied by

$$\frac{\partial Q_x}{\partial x} = -\alpha q = -q_x$$

and

$$\frac{\partial Q_y}{\partial y} = -(1-\alpha)q = -q_y \qquad (5.1)$$

Navier's method of solving elastic plates can be used to determine

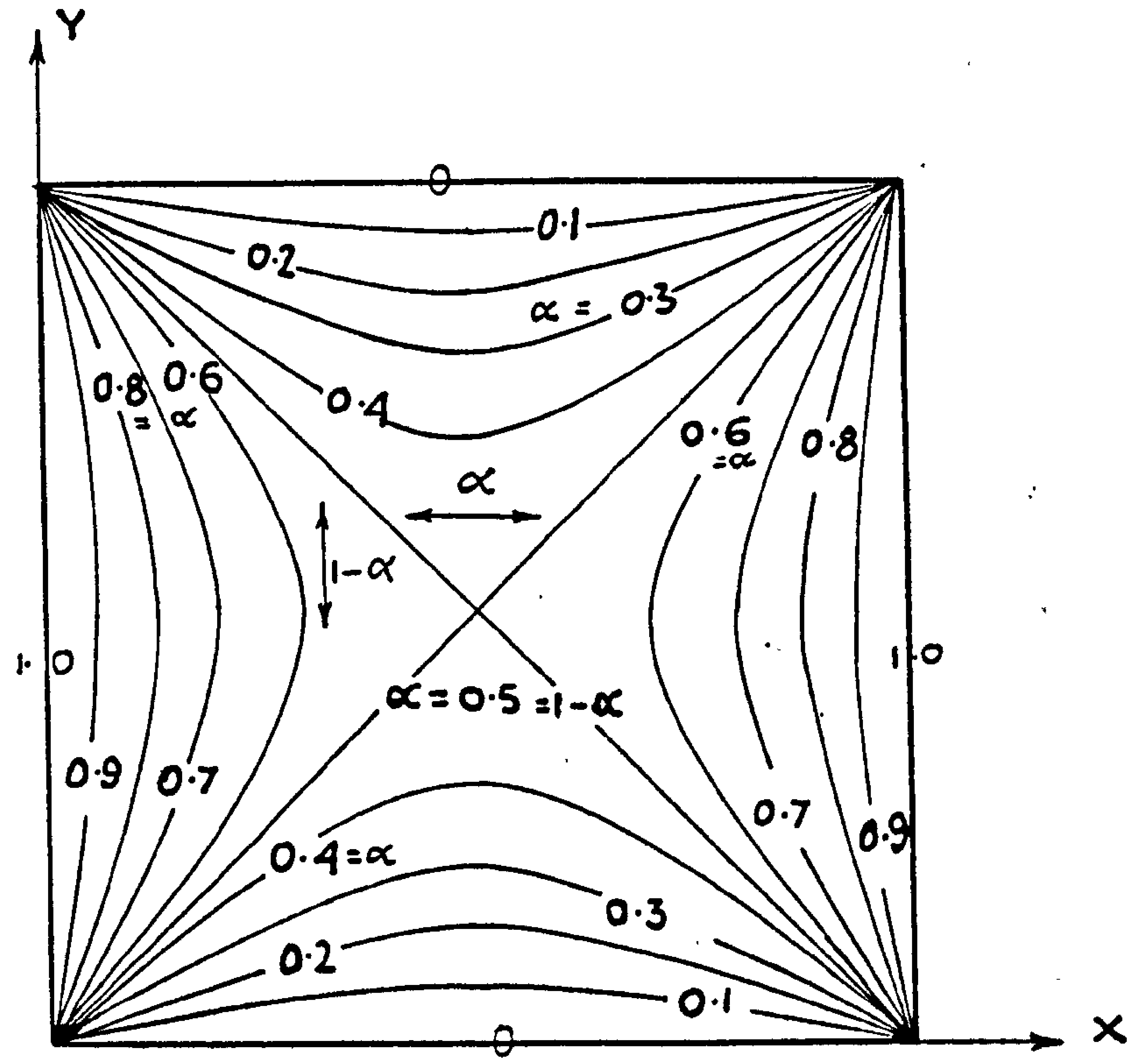


FIG (5.1) SIMPLY SUPPORTED SQUARE SLAB CARRYING A UNIFORMLY DISTRIBUTED LOAD - VARIATION OF LOAD DISTRIBUTION FACTOR (α)

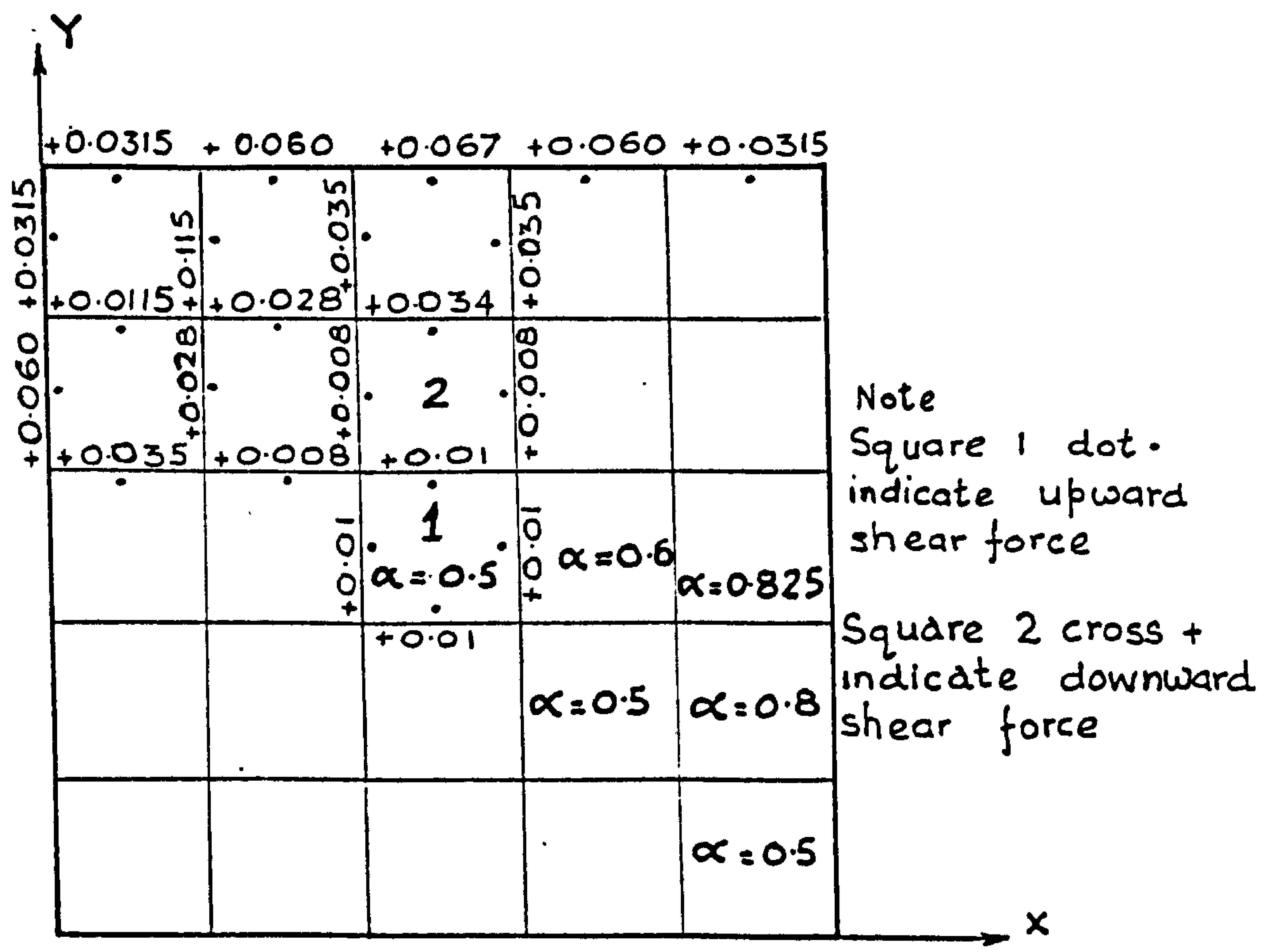


FIG (5.2) VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR SLAB SHOWN IN FIG (5.1) WITH FIVE EQUAL STRIPS IN EACH DIRECTION shown. The load distribution factor α, in each grid area are also shown and these agree very closely with values for similar positions in Fig (5.1). Being a symmetrical problem α is 0.5 along the diagonals and for this particular example α varies between 0 and 1.0 with the extreme values along the boundaries.

5.2.3 Elastic load distributions - concentrated loads.

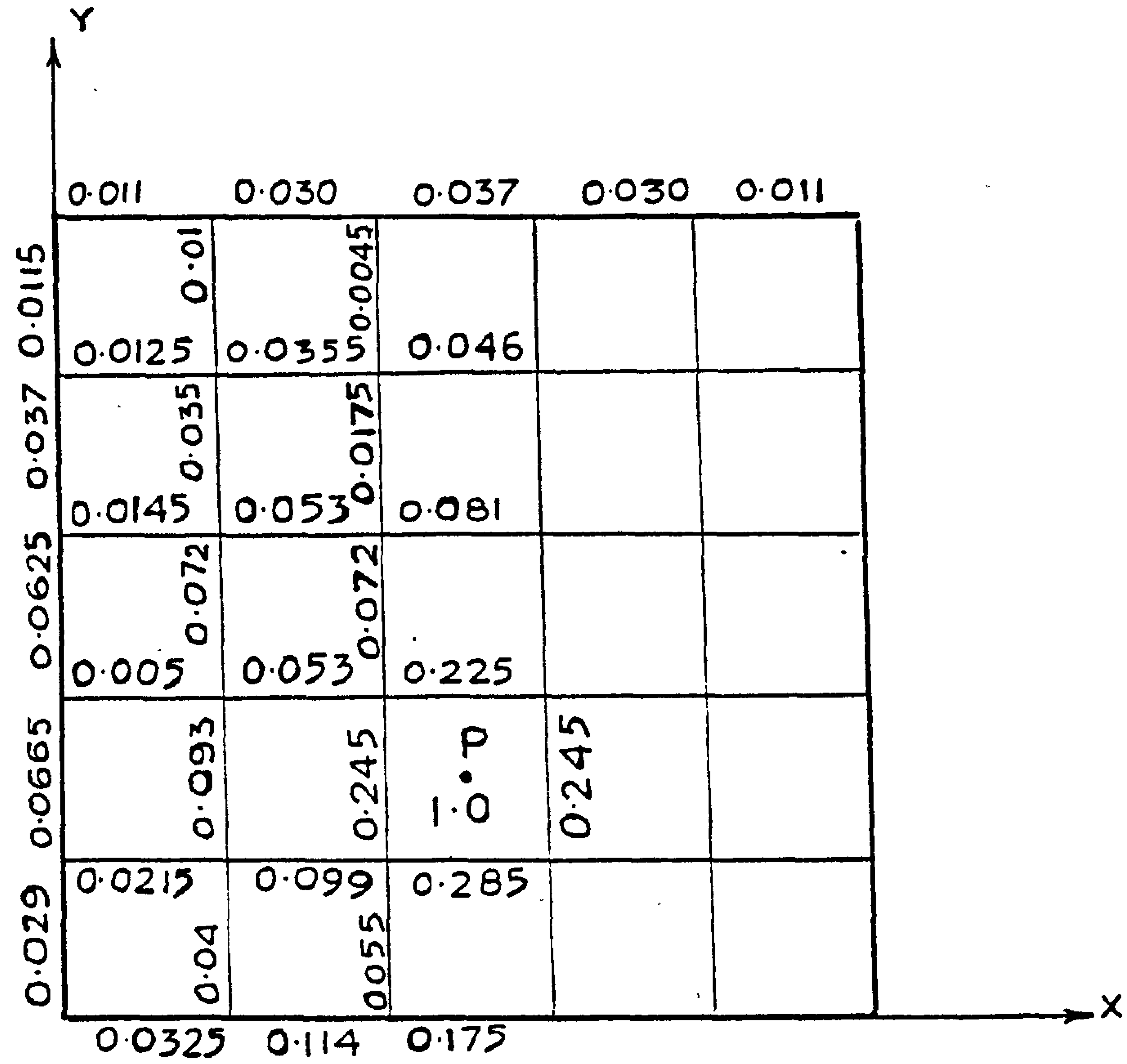
Consider the same simply supported square slab, this time carrying a unit concentrated load at the point P ($x = 0.5$, $y = 0.3$) as in Fig. 5.3. Navier's method of solving slabs with concentrated loads can be used to calculate the elastic load distribution factors α and $(1 - \alpha)$ in the directions x and y . These factors correspond to values of $-\frac{\partial Q_x}{\partial x}$ and $-\frac{\partial Q_y}{\partial y}$ respectively. The intensity of loading at all points other than P is zero and at P it becomes infinite. Elastic calculations show that values of $\frac{\partial Q_x}{\partial x}$ and $\frac{\partial Q_y}{\partial y}$ are equal and opposite at all points other than P as equilibrium requires.

Fig (5.3) shows the vertical shear force distribution pattern with a division of strips identical to the example in Fig (5.2). Here too the individual values were obtained by integrating the shear intensities along each side of the grid.

5.2.4. Comment.

The example shown in Fig (5.1) shows the variations of the load distribution factor over the area of the slab. Values vary substantially from element to element although in the simple strip method extreme values are chosen over large regions.

The method of determining the shear force distribution pattern as illustrated in Fig (5.2) and (5.3) is similar to the one suggested by Kemp (28). Effects of torsions are ignored and the strips can be designed on the basis of the distribution patterns.



Coods of Load P (x=0·5 y=0·3)

For all grids the position and the magnitude of the downward shear forces are shown.

FIG. (5.3) VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR A SIMPLY SUPPORTED SQUARE SLAB CARRYING A POINT LOAD

This elastic method is limited to simple slab problems and even for these the procedure is very tedious. Therefore a general method needs to be developed to systematically determine the shear force distribution or the load distribution factors.

5.3 THE STRIP DEFLECTION METHOD

5.3.1. The basic principles.

Consider the rectangular slab shown in Fig 5.4. The slab is divided into m strips parallel to the x- axis which will be referred to as X - strips. Similarly there are n strips parallel to the y - axis. The slab is therefore divided into (m x n) grid areas and the key assumption is that the load distribution (q) is uniform over each grid area, but of course its value can vary from grid to grid. For any grid area (ij) the load distribution $(q)_{ij}$ is divided into two components $(q_x)_{ij}$ and $(q_y)_{ij}$ the respective load distributions transmitted in the x and y direction and it follows from equilibrium that

$$(q_x)_{ij} + (q_y)_{ij} = (q)_{ij} \qquad (5.4)$$

To determine the load distributions (q_x or q_y) we examine the elastic deflections of the slabs strips in the x and y directions when they are loaded with the unknown (q_x) and (q_y) respectively. That is at each intersection point of the centre lines of the X and Y strips we insist that the elastic deflection $(\Delta_x)_{ij}$ of the X-strip is equal to the elastic deflection $(\Delta_y)_{ij}$ of the Y strip. Hence for the strip X_i the deflection $(\Delta_x)_{ij}$ at point P is given by

$$(\Delta_x)_{ij} = \begin{bmatrix}(k_x)_{i1} & \cdots & (k_x)_{ij} & \cdots & (k_x)_{in}\end{bmatrix} \begin{bmatrix}(q_x)_{i1} \\ \vdots \\ (q_x)_{ij} \\ \vdots \\ (q_x)_{in}\end{bmatrix}$$

$$\text{or } (\Delta_x)_{ij} = \sum_{j=1}^{\infty} (k_x)_{ij} \cdot (q_x)_{ij} \qquad (5.5)$$

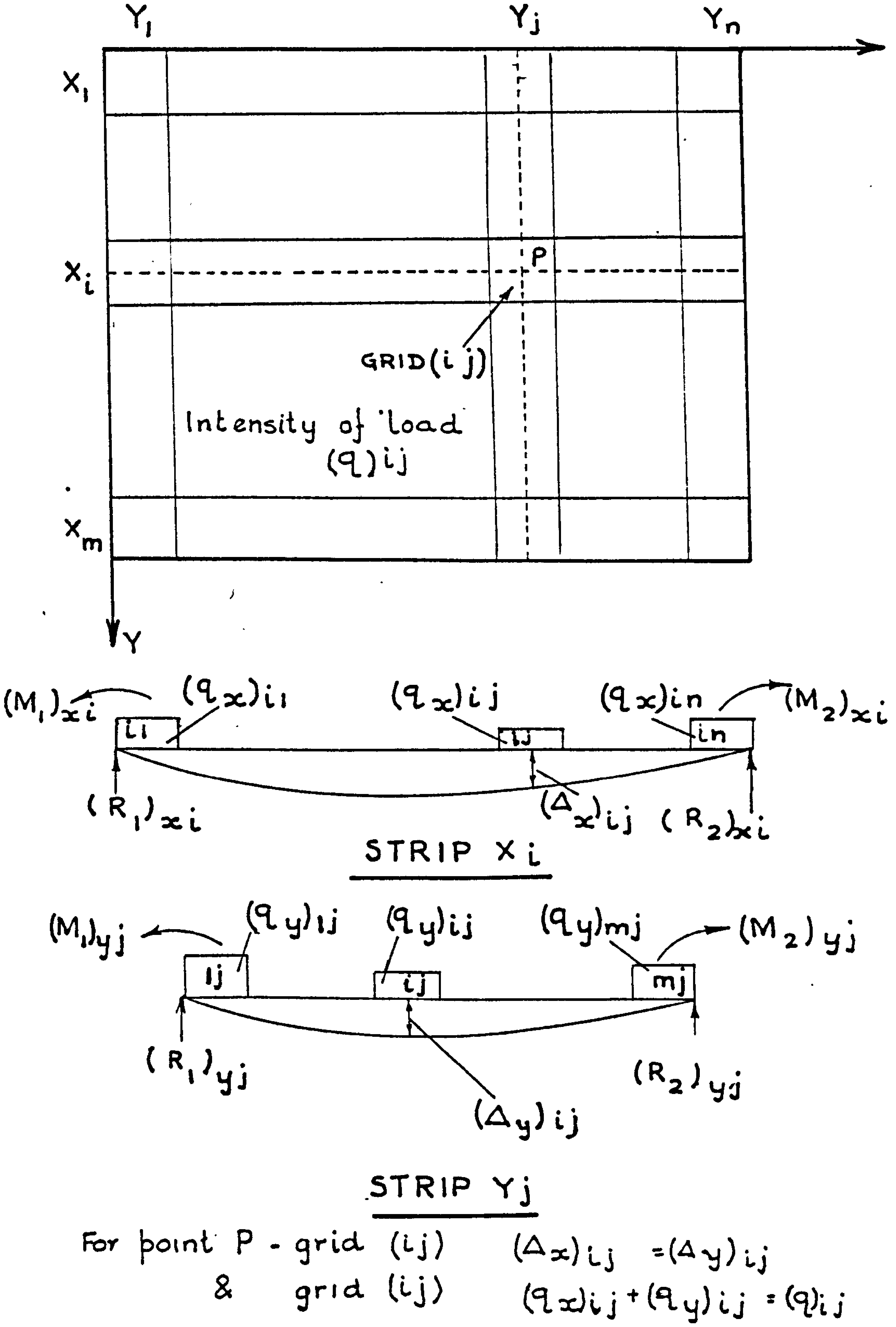


FIG (5.4) FUNDAMENTALS AND NOTATIONS OF THE STRIP DEFLECTION METHOD where $(k_x)_{ij}$ represents the flexibility coefficients for the strip X_i .

Similarly for the strip Y_j the deflection $(\Delta_y)_{ij}$ at point P is given by

$$(\Delta_y)_{ij} = \sum_{i=1}^{m} (k_y)_{ij} \cdot (q_y)_{ij} \qquad (5.6)$$

The deflection equation for the point P is

$$(\Delta_x)_{ij} = (\Delta_y)_{ij}$$

ie $$\sum_{j=1}^{n} (k_x)_{ij} \cdot (q_x)_{ij} = \sum_{i=1}^{m} (k_y)_{ij} \cdot (q_y)_{ij} \qquad (5.7)$$

or substituting for $(q_y)_{ij}$ from equation (5.4) we have

$$\sum_{j=1}^{n} (k_x)_{ij} \cdot (q_x)_{ij} = \sum_{i=1}^{m} (k_y)_{ij} \cdot \left[(q)_{ij} - (q_x)_{ij} \right] \qquad (5.8)$$

This procedure can be applied to each of the grid areas and there will be (m x n) independent deflection equations exactly equal to the number of unknown load distributions $(q_x)_{ij}$. The plate problem is therefore reduced to solving a set of simultaneous linear equations. In some practical examples it may be more convenient to consider the total imposed load $(W)_{ij}$ rather than the distributed load $(q)_{ij}$ and the corresponding governing equations will be

$$(W_x)_{ij} + (W_y)_{ij} = (W)_{ij} \qquad (5.9)$$

and $$\sum_{j=1}^{n} (C_x)_{ij} \cdot (W_x)_{ij} = \sum_{i=1}^{m} (C_y)_{ij} \left[(W)_{ij} - (W_y)_{ij} \right] \qquad (5.10)$$

The total loads $(W)_{ij}$, $(W_x)_{ij}$ and $(W_y)_{ij}$ are also assumed to be distributed uniformly over the grid area (ij).

5.3.2 An Example of the Strip Deflection Method

The method is best illustrated by reference to a simple slab example. Fig (5.5) shows a rectangular slab simply supported on two short sides, fixed and free along the two long sides and carrying a uniform load q. The slab is divided into four equal strips in each direction and in this example all X - strips are of length L_x and all Y- strips are of length L_y. The first assumption is that in each of the 16 rectangular grid areas the imposed loads (q or W) and the corresponding distributions in the x and y directions are uniform over a particular grid area (see Fig. (5.5) grid 32).

To determine the individual load distributions we examine the elastic deflections of the slab strips in the x and y directions. In this example X- strips are all simply supported and Y- strips are all cantilevers and therefore all strips are statically determinate. The deflections can be calculated by using simple beam theory. Fig (5.5) also shows how these deflections are equated for the point A at the centre of the grid (3.2).

End reactions, fixed end moments and deflections for the following types of slab strip are given in Appendix 1.

(a) Both ends simply supported

(b) Both ends built in

(c) One end built in one end simply supported (proped cantilever)

(d) One end free one end built in (cantilever)

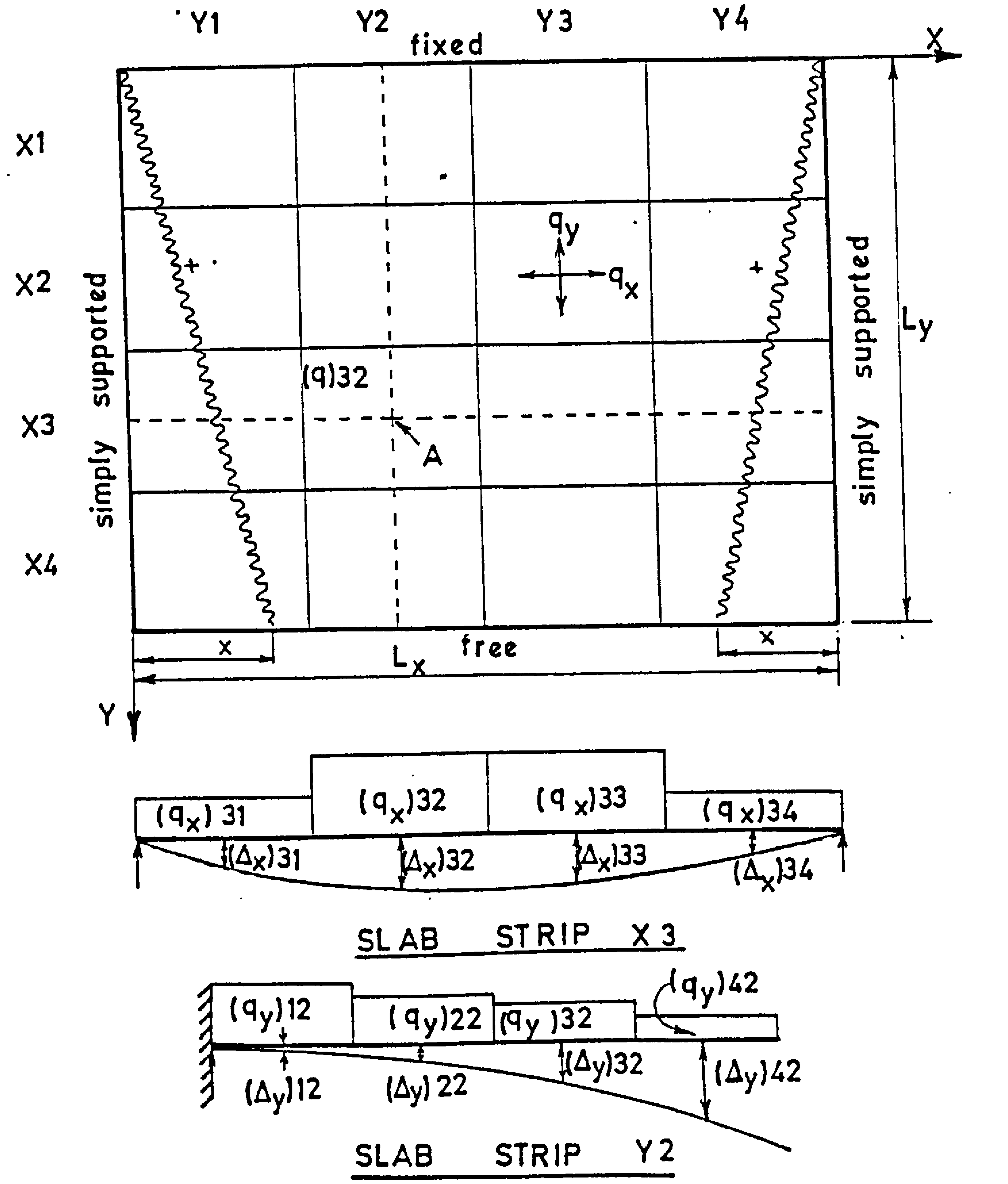
It is of importance to mention that due account of the width of strips must be taken in the flexural rigidity in the computation of deflections at required points.

It is assumed that $L_x : L_y = 1.4$ and the total imposed load which is distributed uniformly over the slab is 140 units. The total load on each of the 16 grid areas is therefore 8.75 units. Due to symmetry strips Y3 and Y4 are identical to strips Y1 and Y2 respectively. The details of the analysis are given in Appendix 2. The eight values of W_y are considered the basic unknowns and the resulting set of eight simultaneous equations are shown here and have been solved using a standard matrix inversion programme. The solutions are given in Table (5.1). It is evident that the loads are carried towards the nearest support and more towards the fixed support than the simple support. The values for load distribution factors vary widely. That is for the centre grid near the fixed edge the values are outside the usual range of 0 to 1.0. Also shown are the fixed end moments and end reactions for the X and Y strips.

The values of the load distributions in the x and y directions are also shown in Fig (5.6) together with the vertical shear forces acting at the boundaries of the grids. The vertical shear force acting at the extreme edge of each strip give the reaction at the support and of course these values are zero at the free edge.

The moments, shear forces in both x and y directions can now be calculated by statics. The moments are uniform across the width of any slab strip and so the required reinforcement will be in simple banded layouts. In this symmetrical example there will be four different bands in the x direction and two different bands in the y direction.

If the slab is now reinforced according to the strip deflection method, the simply supported X strips will carry positive reinforcements and the cantilever Y strips will carry negative reinforcements. Consider the family of yield lines shown in Fig (5.5) defined by parameter x. Analysis shows that unique collapse load is only approached as x tends to zero. A similar example was illustated in Chapter 4. For this slab the positive yield lines are consistent with the rules postulated in the Chapter 4 only when x is zero. Here too the mechanism is strictly not kinematically admissible but becomes valid with an infinitely small value of x.



For Point A in Grid 32 ; $(\Delta_x)32 = (\Delta_y)32$

For Grid 32 , $(q)32 = (q_x)32 + (q_y)32$

or

$(W)32 = (W_x)32 + (W_y)32$

FIG 5.5 EXAMPLE ILLUSTRATIVE OF THE FUNDAMENTALS OF THE STRIP DEFLECTION METHOD

$$\begin{bmatrix} 811.9464 & 16.0 & 28.0 & 40.0 & 330.3776 & 0.0 & 0.0 & 0.0 \\ 20.0 & 915.9464 & 216.0 & 324.0 & 0.0 & 330.3776 & 0.0 & 0.0 \\ 36.0 & 220.0 & 1307.9464 & 800.0 & 0.0 & 0.0 & 330.3776 & 0.0 \\ 52.0 & 332.0 & 804.0 & 2179.9464 & 0.0 & 0.0 & 0.0 & 330.3776 \\ 330.3776 & 0.0 & 0.0 & 0.0 & 151.1912 & 16.0 & 28.0 & 40.0 \\ 0.0 & 330.3776 & 0.0 & 0.0 & 20.0 & 225.1912 & 216.0 & 324.0 \\ 0.0 & 0.0 & 330.3776 & 0.0 & 36.0 & 220.0 & 647.1912 & 800.0 \\ 0.0 & 0.0 & 0.0 & 330.3776 & 52.0 & 332.0 & 804.0 & 1519.1912 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \\ W_7 \\ W_8 \end{bmatrix} = \begin{bmatrix} 9958.1475 \\ 9958.1475 \\ 9958.1475 \\ 9958.1475 \\ 4176.5395 \\ 4176.5395 \\ 4176.5395 \\ 4176.5395 \end{bmatrix}$$

RESULTING SET OF EQUATIONS FOR THE

ILLUSTRATIVE EXAMPLE SHOWN IN FIG.5.5

	Strip Deflection Method	No - Torsion Grid Method
W_1	8.844 (1.011)	8.817 (1.008)
W_2	7.757 (0.887)	7.689 (0.879)
W_3	4.826 (0.552)	4.889 (0.559)
W_4	1.320 (0.151)	1.495 (0.171)
W_5	7.462 (0.853)	7.535 (0.861)
W_6	3.652 (0.417)	3.923 (0.448)
W_7	1.712 (0.196)	1.815 (0.207)
W_8	0.502 (0.057)	0.506 (0.058)
$(R_x)_1$	1.194	1.148
$(R_x)_2$	6.091	5.888
$(R_x)_3$	10.962	10.796
$(R_x)_4$	15.678	15.499
$(R_y)_1$	13.328	13.779
$(R_y)_2$	22.747	22.890
$(M_y)_1$	3.8115	3.990875
$(M_y)_2$	8.185625	8.34925

Load distribution factors are shown within brackets.

Table 5.1 Solution To The Illustrative Example Shown In Fig.(5.5)

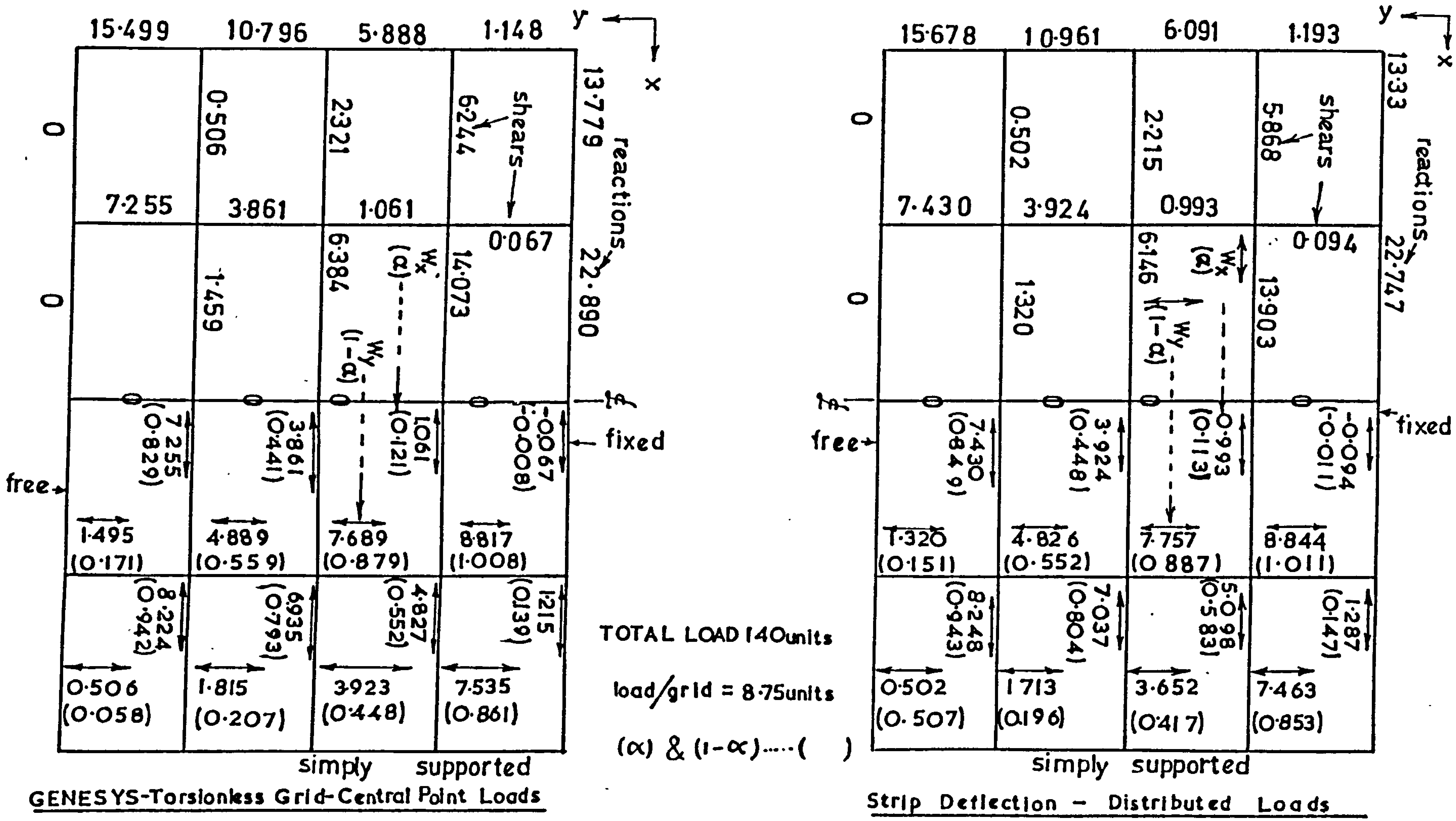


FIG (5.6) SOLUTION TO THE SLAB EXAMPLE SHOWN IN FIG (5.5)

5.4 COMPARISON OF STRIP DEFLECTION METHOD WITH OTHER METHODS OF CONCRETE SLAB DESIGN.

5.4.1 Rankine and Grashof's Method

The Rankine and Grashof's method which is still recommended in the Codes of Practice CP110 and CP114 can be seen as a special case of the strip deflection method with just one strip in each direction and is restricted to simply supported slabs carrying a uniform load. As discussed in Chapter 2 this method assumes a single distribution of load for the entire slab based on the elastic deflections of the centre strip and it ensures that equilibrium ($q_x + q_y = q$) is satisfied at all points.

If reinforcement is provided exactly in accordance with the calculated moments in each directions then the resultant moment field is positive-positive. Rankine and Grashof's method can be shown to provide the unique solution for all possible mechanisms composed of positive yield lines only. This perhaps explain its sucess over the years.

5.4.2. Grid analogy Method

A strong similarity exists between the strip deflection method and the old established grid analogy approach to slab design. Before proceeding to discuss further applications it is helpful to compare the two approaches, since this will show how we can utilise existing computer programmes for grid works to solve problems by the strip deflection method.

In the grid analogy method the slab is divided into an intersecting set of beam strips to form an equivalent grid system. If these beam strips are orthogonal and assumed to be torsionless we have a very similar system to the one proposed in the strip deflection method. The only important difference is that in the grid analogy method the load interaction takes place only at the intersection point, where as in the strip deflection method a uniform interaction is assumed to occur over the area of the grid rectangle.

In both cases the load interactions can be determined by equating the vertical deflection at the grid intersection points. The difference between the results obtained by the two approaches will obviously decrease as the grid size is reduced. Since we only require load distributions reasonably close to the elastic working load conditions there is no reason why we should not use the grid analogy programmes for calculating these load interactions.

FRAME ANALYSIS / 1 a subsystem of the GENESYS computer programme was used to solve the grids. It must be mentioned that a small positive value of torsional rigidity must be assumed for each member otherwise the programme will not work. The subsystem can accommodate any slab geometry, boundary condition including elastic deflections, elastic rotations or permanent settlements. The output gives a print out of the bending moments, shear forces at the internal nodes and reactions, end moments at the supports. For four or more strips in each direction the difference between the results from the strip deflection method and the above values which are based on point loads applied at the corresponding grid intersection points is insignificant.

The load distributions, reactions and shear values for the slab discussed in section (5.3.2) and calculated by the grid method are compared and shown alongside with the strip deflection results in Fig (5.6) and Table (5.1). The assumed equivalent grid is described in Appendix 2. Point loads of value identical to the total distributed load within each element area were applied at the corresponding intersection points.

However in calculating the statical moments in each beam strip the load interaction must be strictly assumed to be distributed over the grid area, otherwise the equilibrium equation $q_x + q_y = q$ will not be satisfied at all points of the slab and uniqueness of the collapse load will be lost. This example too provides a positive-negative moment field for the entire slab. With uniform load interaction over the grid areas, for the family of yield lines shown in Fig(5.5) the unique solution is again approached as the value of x tends to zero confirming the rules set out in Chapter 4.

We therefore have the surprising conclusion that the grid analogy method can be used to produce a completely generalised strip method of slab design. Using an orthogonal grid system of torsionless beams to calculate the load interaction and distributing these over the grid areas to calculate the bending moments, shear forces etc. The solution will in general give a unique collapse load. In retrospect it is fascinating to note that the efforts expended in the past to include the torsion component in the equivalent grid system can now be seen to have the effect of changing a unique solution to a lower bound solution.

West (32) has recently proposed the use of grid method to analyse slabs and bridge decks arguing that it has the advantages of being universally applicable, easy for the engineer to visualize and prepare data for the analysis, cheaper computation cost especially compared with the finite element method and that the agreement between the analysis and experimental observations is encouraging. Now that the relationship between the grid analogy method and the plastic strip method are established the reason for its success become clear.

5.5 APPLICATIONS OF THE STRIP DEFLECTION METHOD

The application of this method to uniformly loaded slabs of any shape will be discussed here. The next Chapter will be devoted exclusively to slabs with point loads and point supports.

5.5.1. Flexible Supports - Partial composite action

Where the slab is supported at its boundaries by flexible beams, the strip delfection method readily allows some composite action to be taken into account. The basic equilibrium equations are unaltered but the deflection equation $(\Delta_x)_{ij} = (\Delta_y)_{ij}$ takes into account the deflections at the boundary. For point P in Fig 5.7.

$$\text{Strip } X_i \quad (\Delta_x)_{ij} = \delta_a + (\delta_b - \delta_a)\frac{x}{L_x} + (\delta_x)_{ij}$$

$$\text{and strip } Y_j \quad (\Delta_y)_{ij} = \delta_c + (\delta_d - \delta_c)\frac{y}{L_y} + (\delta_y)_{ij}$$

For compatibility $\delta_a + (\delta_b - \delta_a)\frac{x}{L_x} + (\delta_x)_{ij}$

$$= \delta_c + (\delta_d - \delta_c)\frac{y}{L_y} + (\delta_y)_{ij} \qquad (5.11)$$

where δ_a, δ_b, δ_c, δ_d are the deflections at points A,B,C,D which are at the ends of strips X_i and Y_j.
$(\delta_x)_{ij}$ and $(\delta_y)_{ij}$ are the deflection at point P below the line joining AB and CD respectively.

For each intersection between a slab strip and a supporting beam there will be one additional unknown reaction. However by considering the deflection at the supporting beam, it will be possible to write one additional equation to give the same number of equations as there are unknowns which are the load distributions plus slab-beam interactions.

To illustrate the effects of flexible supports consider a square slab side L carrying a uniform load shown in Fig (5.8 a). The slab is supported by four identical edge beams of flexural stiffness (EI). It is divided into five equal strips in each direction and the beams are supported at the four corners. The total load is assumed to be 100 units and therefore the uniform load on each of

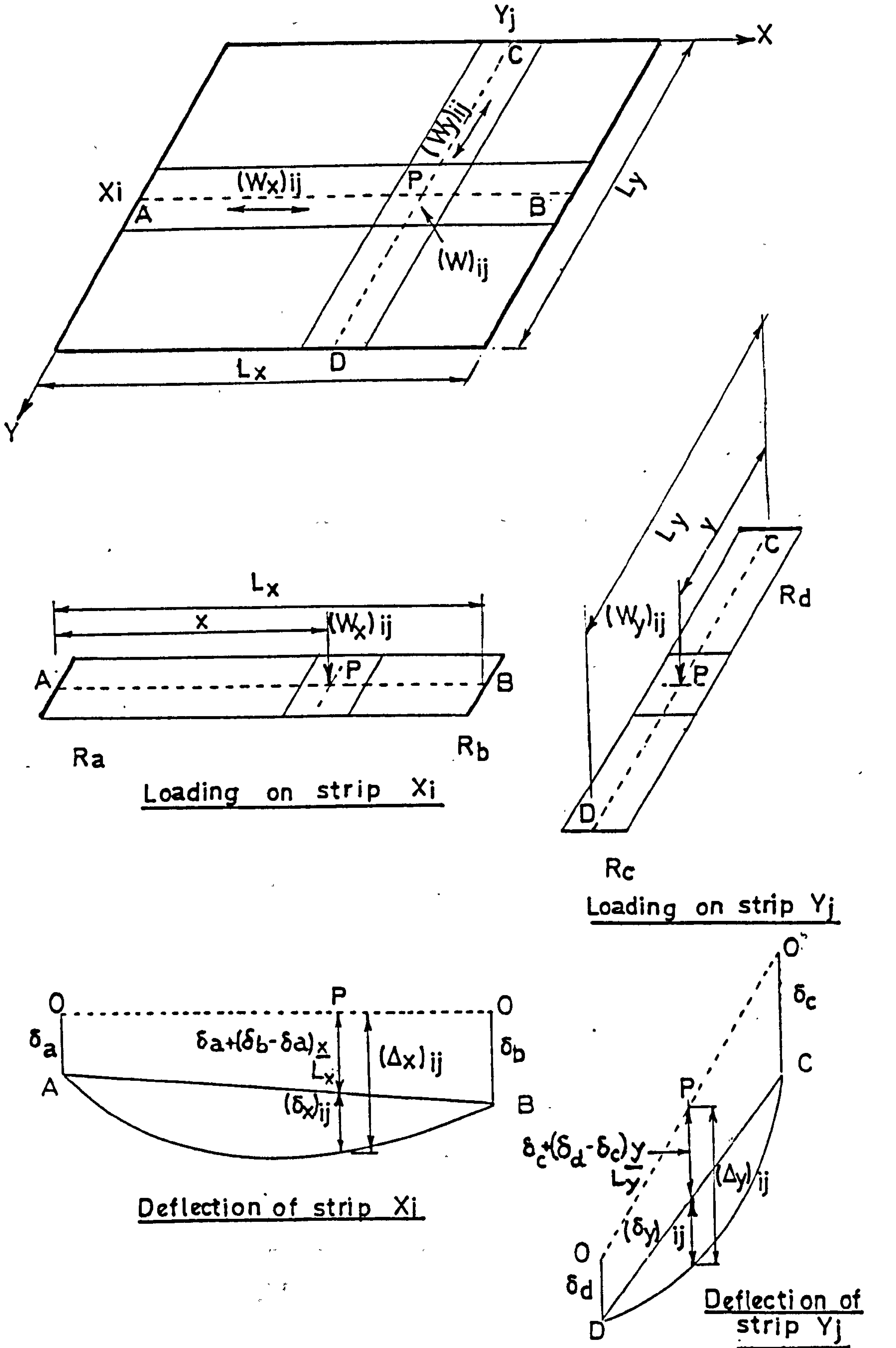
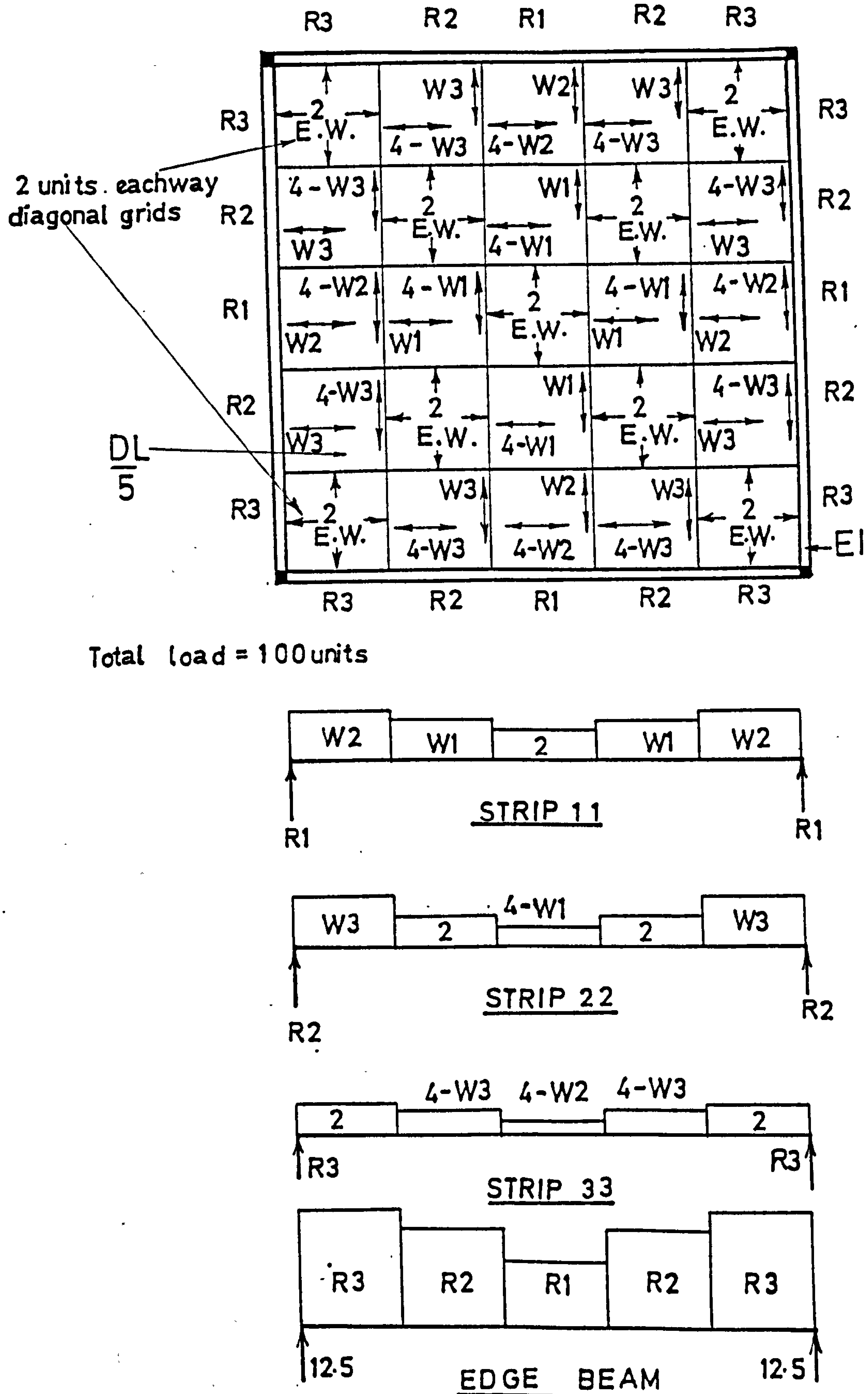
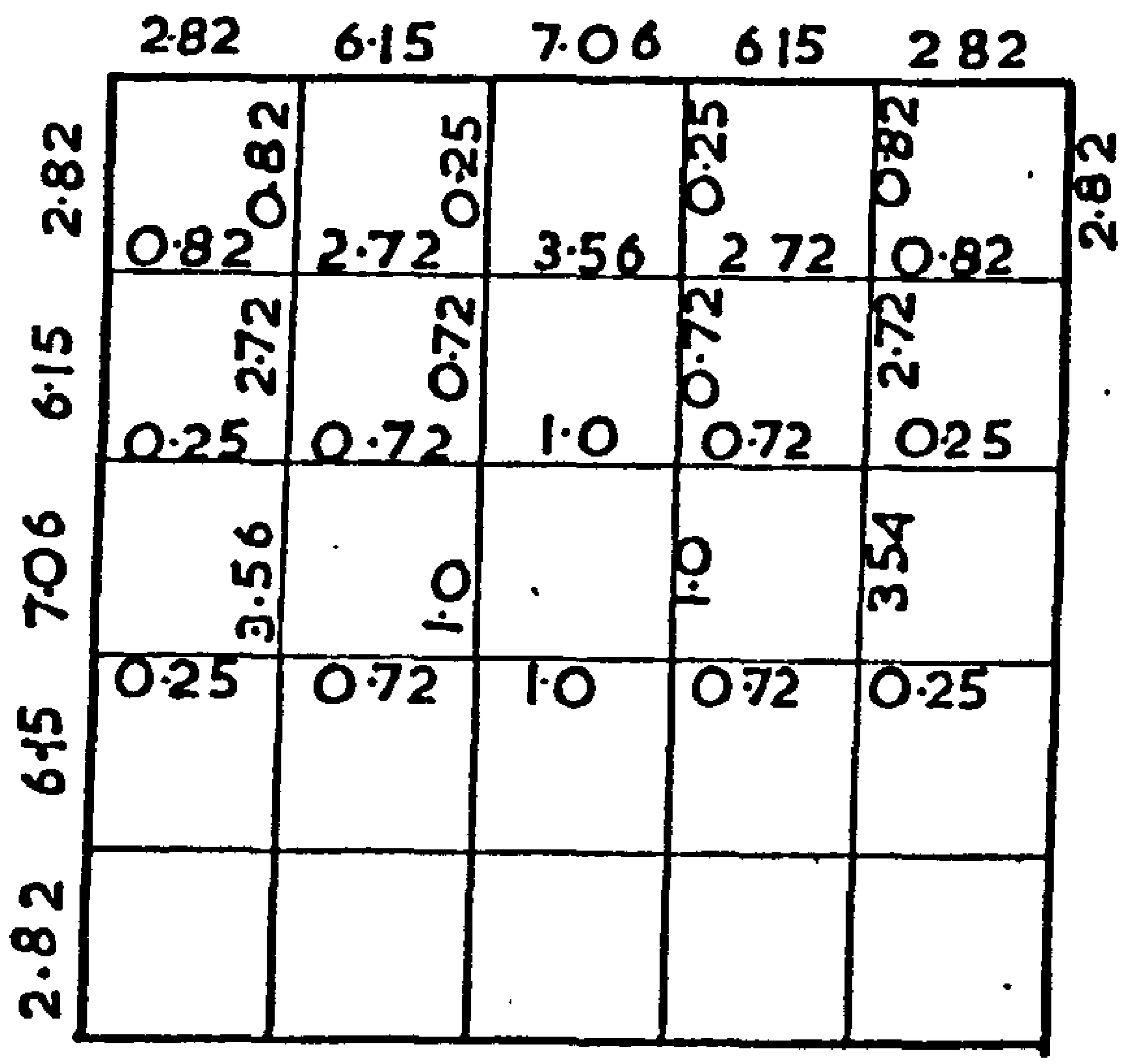


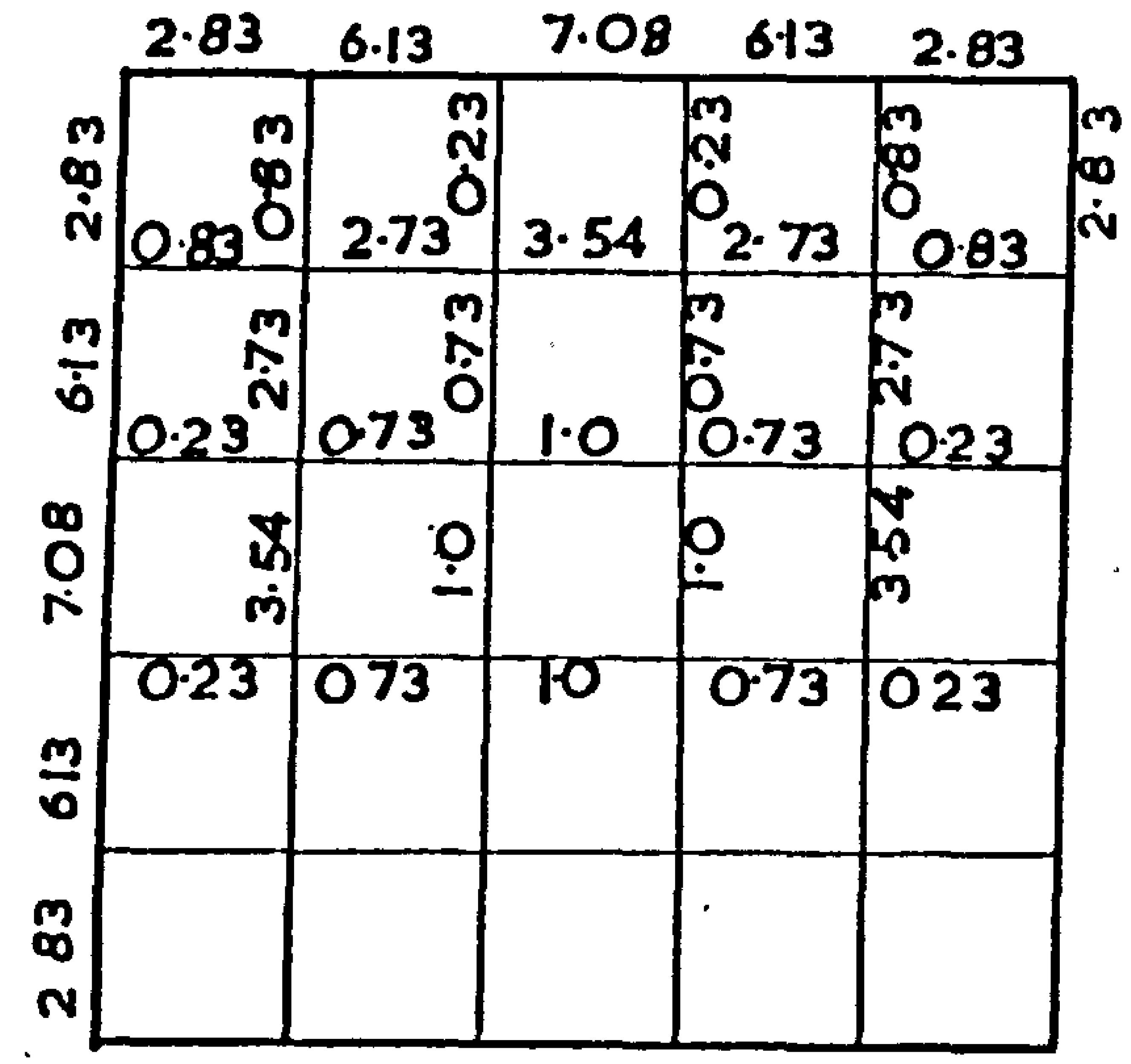
FIG (5.7) COMPATIBILITY OF DEFLECTIONS FOR SLABS ON FLEXIBLE SUPPORTS



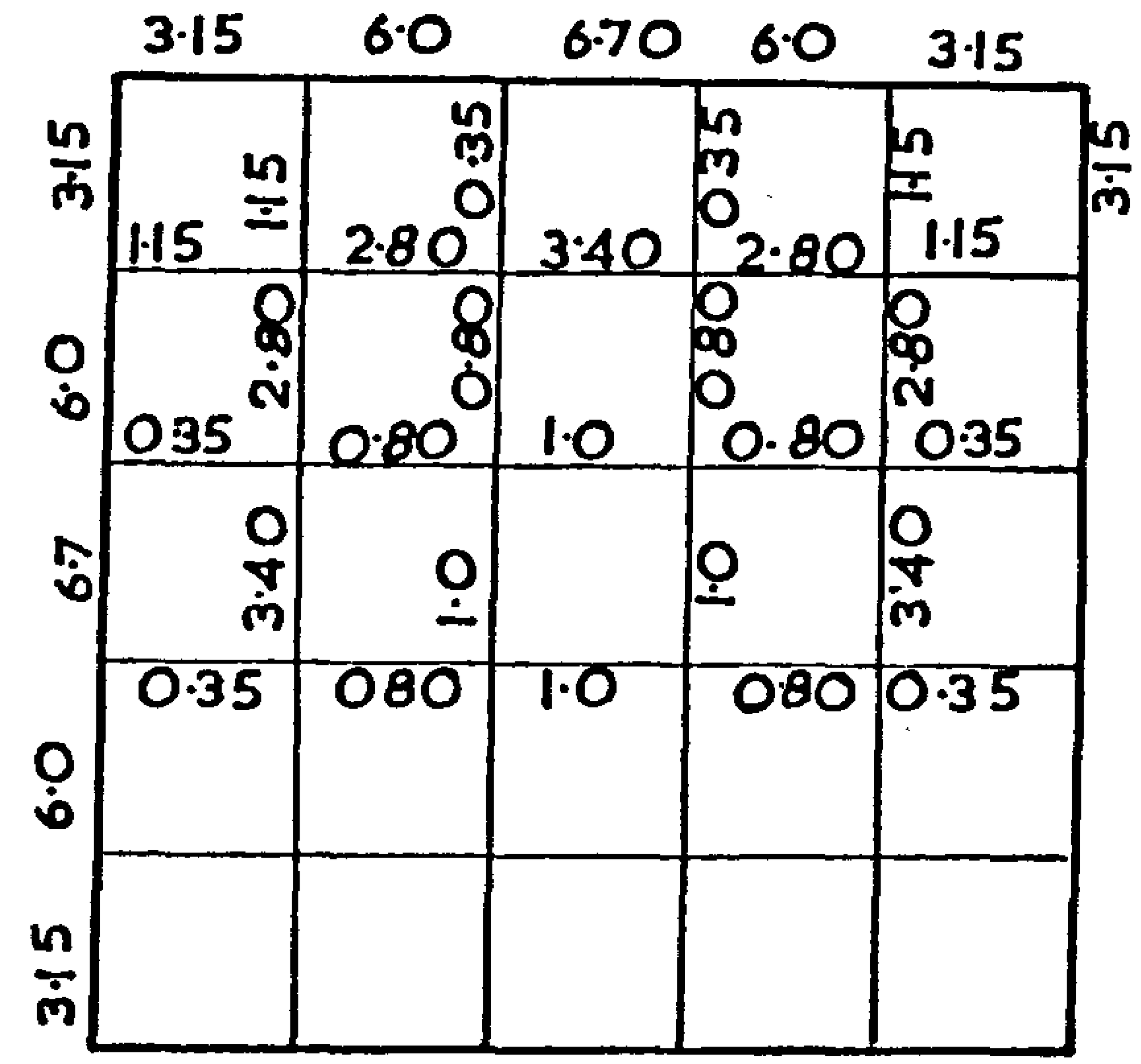
FIG(5.8.a.)LOAD DISTRIBUTION;LOADING ON STRIPS & EDGE BEAM SQUARE SLAB ON FLEXIBLE BEAMS CARRYING A U.D.L.



(a) STRIP DEFLECTION



(b) GENESYS - TORSIONLESS GRID



(c) EXACT ELASTIC (FIG. 5.2)

Total load 100 units

Load / grid 4 units

FIG(5.8.b) VERTICAL SHEAR FORCE DISTRIBUTION PATTERN; SQUARE SLAB IN FIG (5.8) the grids is four units. Due to symmetry the load distribution on each of the diagonal grids is 2 units each way and the number of unknowns is reduced to 6 (W_1, W_2, W_3, R_1, R_2, R_3). The reactions (R_1, R_2 and R_3) between the slab strips and the supporting beams are again assumed to be spread uniformly across the slab strips widths.

For a given stiffness of edge beam EI and of slab strip, their ratio ϵ is defined by

$$\epsilon = \frac{\text{Stiffness of each strip}}{\text{Stiffness of each edge beam}} = \left(D\,\frac{L}{5}\right) / (E\,I)$$

where D is the stiffness of unit width of slab. The stiffness factor γ used by Wood (8) is

$$\gamma = \frac{\text{Stiffness of beam}}{\text{Stiffness of half width of slab}} = \frac{(E\,I)}{\left(D\,\frac{L}{2}\right)} = \frac{0.4}{\epsilon}$$

Detail calculations for this slab are given in Appendix 2, but the final set of equations based on equilibrium and deflection compatibility for a given value of ϵ are in matrix form

$$\begin{bmatrix} 2 & 2 & 0 & -2 & 0 & 0 \\ -1 & 0 & 2 & 0 & -2 & 0 \\ 0 & -1 & -2 & 0 & 0 & -2 \\ 281.55 & 61 & 73 & 25.125\epsilon & 36.075\epsilon & 12\,\epsilon \\ 61 & 147.75 & 195 & 86.125\epsilon & 134.0\epsilon & 47.875\epsilon \\ 36.5 & 97.5 & 184.06 & 61.0\,\epsilon & 97.925\epsilon & 35.875\epsilon \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} -2. \\ -8. \\ -16. \\ 685.5 \\ 1343.5 \\ 879.7 \end{bmatrix} \qquad (5.12)$$

In the derivation of the above equations it was implied that only vertical shear forces were transmitted between the slab and the edge beam. This assumption is similar to the partial composite action discussed in Chapter 2. The Equations (5.12) were solved for four values of γ and the results for the distribution of load on the supporting beams were compared with values obtained by Wood (8). The corresponding values for a unit imposed load are shown in Table 5.2 and Fig 5.9.

γ	R_1	R_2	R_3
∞	0.0706	0.0615	0.0282
2.0	0.0592	0.0555	0.0399
1.0	0.05	0.05	0.05
1/3	0.0278	0.0310	0.0801

TABLE 5.2

Wood's (8) results were based on solving Lagranges plate equation by the finite difference method. Effects of Poissons ratio were neglected and the centriods of slab and edge beams were assumed to coincide.Fig. (5.9) shows the values of beam slab reaction R obtained by Wood which includes a concentrated corner reaction and the arrows show the direction of this reaction on the beam. The corner reaction does not appear in the strip deflection method where the effects of torsion are ignored.

$\gamma = 1$ can be instantly recognised as Wood's twistless case and the strip deflection results are then in complete agreement with Wood's values with the corner reaction R = 0. Also, in equation (5.12) $W_1 = W_2 = W_3 = 0.02$ which means that the load is equally distributed in two orthogonal directions at all points of the slab and the bending moment, shear force diagrams for all slab strips are identical.

For values of $\gamma > 1$ the beams will carry more load at the centre of the span. For values of $\gamma < 1$ the load on the beams will diminish at the centre and correspondingly increase near the support.

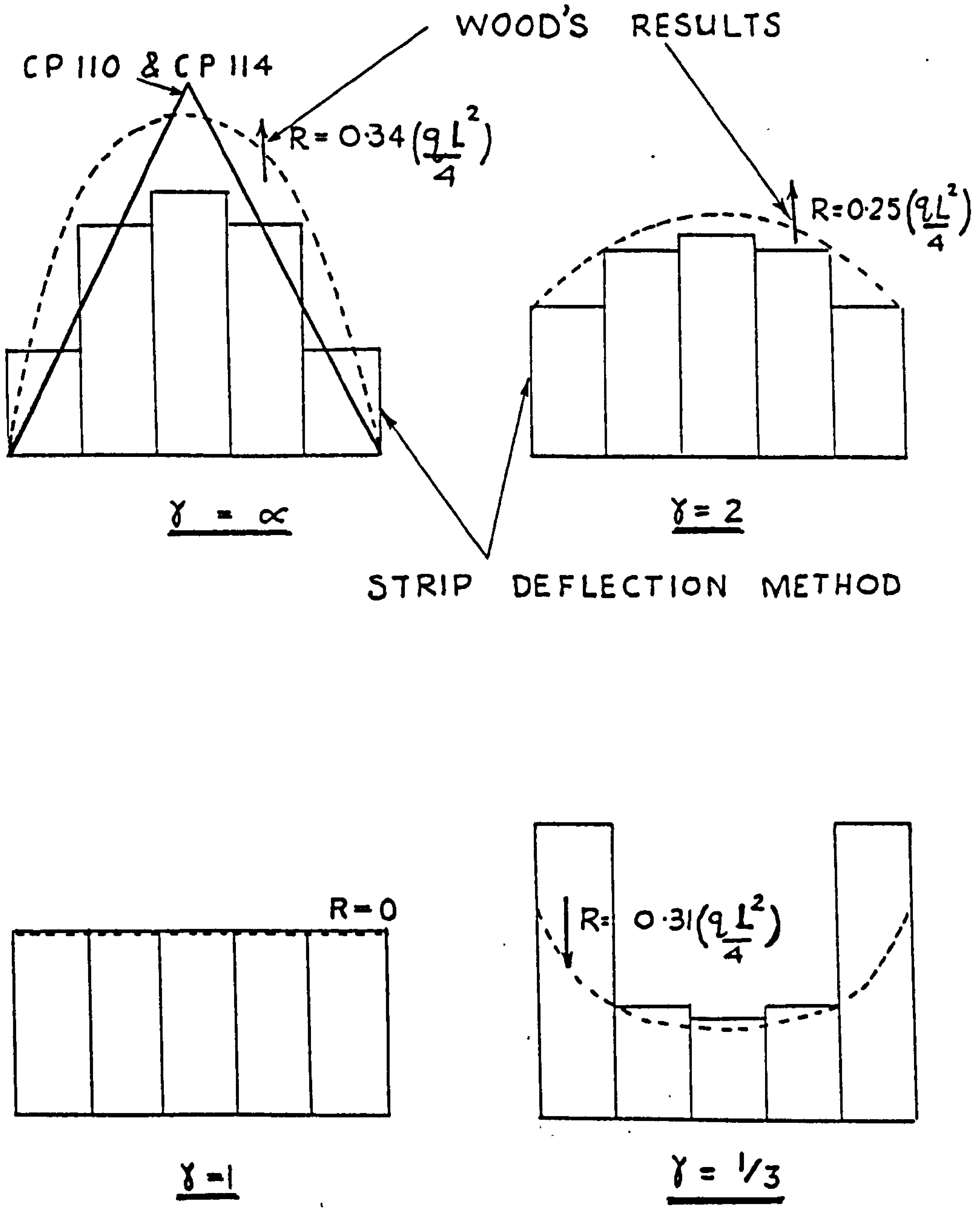


FIG (5.9) SQUARE SLAB SUPPORTED ON FOUR IDENTICAL BEAMS, CARRYING A UNIFORMLY DISTRIBUTED LOAD - DISTRIBUTION OF LOAD ON THE SUPPORTING BEAMS

Both codes of practice CP110 and CP114 recommend a 45° triangular load distribution to be taken by the beams of a square panel or the shorter sides of a rectangle, irrespective of the stiffness of the supporting beam. The maximum bending moment at the centre of the beam due to this distribution is $qL^3 / 24$. Corresponding values obtained by the strip deflection method are less than this value for all values of γ . A comparison of values of maximum beam moments predicted by different methods for rigid beams ($\gamma = \infty$) is shown in Table 5.3.

Method		Maximum Beam Bending Moment (multiple of $qL^3 /_{24}$)
Code of Practice	114, 110	1.0
Wood (8)		1.203
Timoshenko (1)		1.207
Strip deflection		0.892

TABLE 5.3. Comparison of maximum beam bending moments ($\gamma = \infty$)

Woods and Timoshenko's values for the maximum bending moment are greater than the code of practice values and this is entirely because the load applied to each beam is in excess of one quarter of the total load. In their elastic analysis this is due to concentrated corner reactions R/2 at each end. In the strip method and the code of practice the load carried by each beam is exactly one quarter of the total load. However the strip deflection distribution is more uniform than the 45° triangle and the corresponding beam moment is less.

Finally let us consider a square slab carrying a uniform load in which one supporting beam is more flexible than the other three. In Fig (5.10.a) the stiffness values (γ) are (2, 2, 2, 1/4) and in Fig. (5.10.b) corresponding values are ($\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, 1/16). It can be

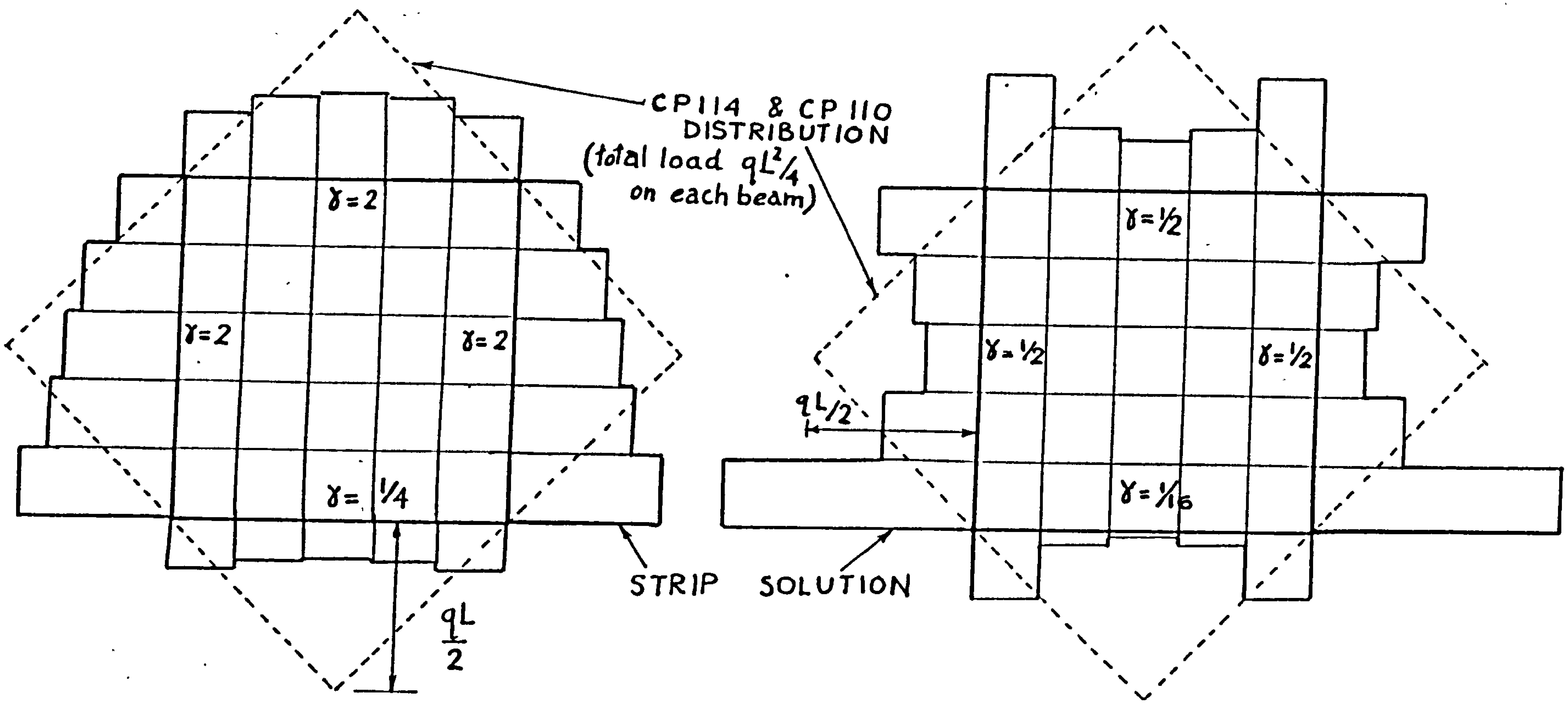


FIG (5.10) DISTRIBUTION OF LOADS ON BEAMS SUPPORTING A SQUARE SLAB CARRYING A UNIFORMLY DISTRIBUTED LOAD - ONE BEAM MORE FLEXIBLE THAN THE OTHERS seen that the slab beam interactions are very different from the code of practice recommendations. For these examples the maximum bending moment on the supporting beams varies between 0.16 $(qL^3 / _{24})$ and 0.78 $(qL^3 / _{24})$. The strip deflection method will clearly provide a close approximation to the actual load distribution on supporting beams than given in current codes of practice and in most cases will result in a saving in material.

For the distribution of loads shown in Fig (5.9) and Fig (5.10) modes of failure which include simultaneous yielding in the slab and supporting beams were checked. For all cases the moment field within the slabs is positive-positive and for the beams it is positive over its entire length. For combined mechanisms with positive yield lines within the slab and positive hinges in the beams the strip deflection method design load is identical to the collapse load. It can therefore be concluded that the rules for uniqueness set out in Chapter 4 can now be extended to composite systems as well.

For the square slab on simple supports (ie $\gamma = \alpha$) the 6 equations in Egn. (5.12) reduces to 3 which can be easily solved. Fig (5.8.b) shows the vertical shear force diagram for this slab. Shown alongside in Fig (5.8.b) are the corresponding results obtained from the torsionless grid method. The loading, shear force and bending moments at corresponding points show remarkable similarities. The moment field for the slab in both designs is positive - positive. For the slab designed by the strip method the collapse load is again identical to the the design load for all mechanisms with positive yield lines, as expected from the rules in Chapter 4. Unique results can also be obtained when the load distributions obtained by the no torsion grid method are uniformly distributed over the grid area.

5.5.2 Comparison of Reinforcement quantities.

Economy of steel is clearly of considerable importance in

Method	λ x100%
1) Minimum Weight Solution	100
2) Strip deflection Method	
3 Strips each way	144.7
4 ,, ,,	139.2
5 ,, ,,	137.0
10 ,, ,,	133.0
3) Hillerborg strip method	
4 Strips each way	160.5
5 ,, ,,	159.9
Minimum possible with continuously variable reinforcement	120
4) Rankine and Grashof's Method	158.8
5) Yield line theory uniform isotropic reinforcement, no top steel $M_p = pL^2/22$	174.5
6) Elastic Moment field, reinforced to satisfy yield criteria and straight reinforcements	159.6
7) CP 110- Four edges discontinous with torsion reinforcements at each corner	164.5
8) CP 114- Four edges discontinous with torsion reinforcements at each corner	146.9

TABLE 5.4 COMPARISON OF MOMENT VOLUMES FOR A SQUARE SLAB ON SIMPLE SUPPORTS CARRYING A UNIFORM LOAD.

the design of such commonly occuring structural members as reinforced concrete slabs. It is therefore instructive to compare the steel quantities required by the strip deflection method with those demanded by other currently used design methods. For a square slab, side L on simple supports carrying a uniformly distributed load q per unit area the minimum weight solution (see Chapter 3) gives a moment volume $V_{min} = 5/96\ qL^4$. For any other design the moment volume can be expressed as $V = \lambda(\frac{5}{96} qL^4)$ and Table 5.4 compares the values of λ obtained by various design methods.

It can be seen that the strip deflection method compares well with the other methods for steel quantity. The moment volume decreases with the number of strips and with five strips in each direction it is 37% above the absolute minimum. It is about 15% less than the Hillerborg strip method with five strips, the elastic moment field method and Rankine and Grashof's method., 27% less than the yield line theory solution. When compared with the coefficients given CP110 and CP114, the strip deflection method claims economies of steel of 20% and 7.2% respectivety, however these figures do not take into account the minimum reinforcement required in the edge strips so that the total economies will be greater.

5.6. TREATMENT OF OTHER BOUNDARY CONDITIONS

When the slab strips are cantilevers or simply supported as in the examples discussed so far they are statically determinate. Other boundary conditions can however arise and these will now be considered.

5.6.1 Free edges.

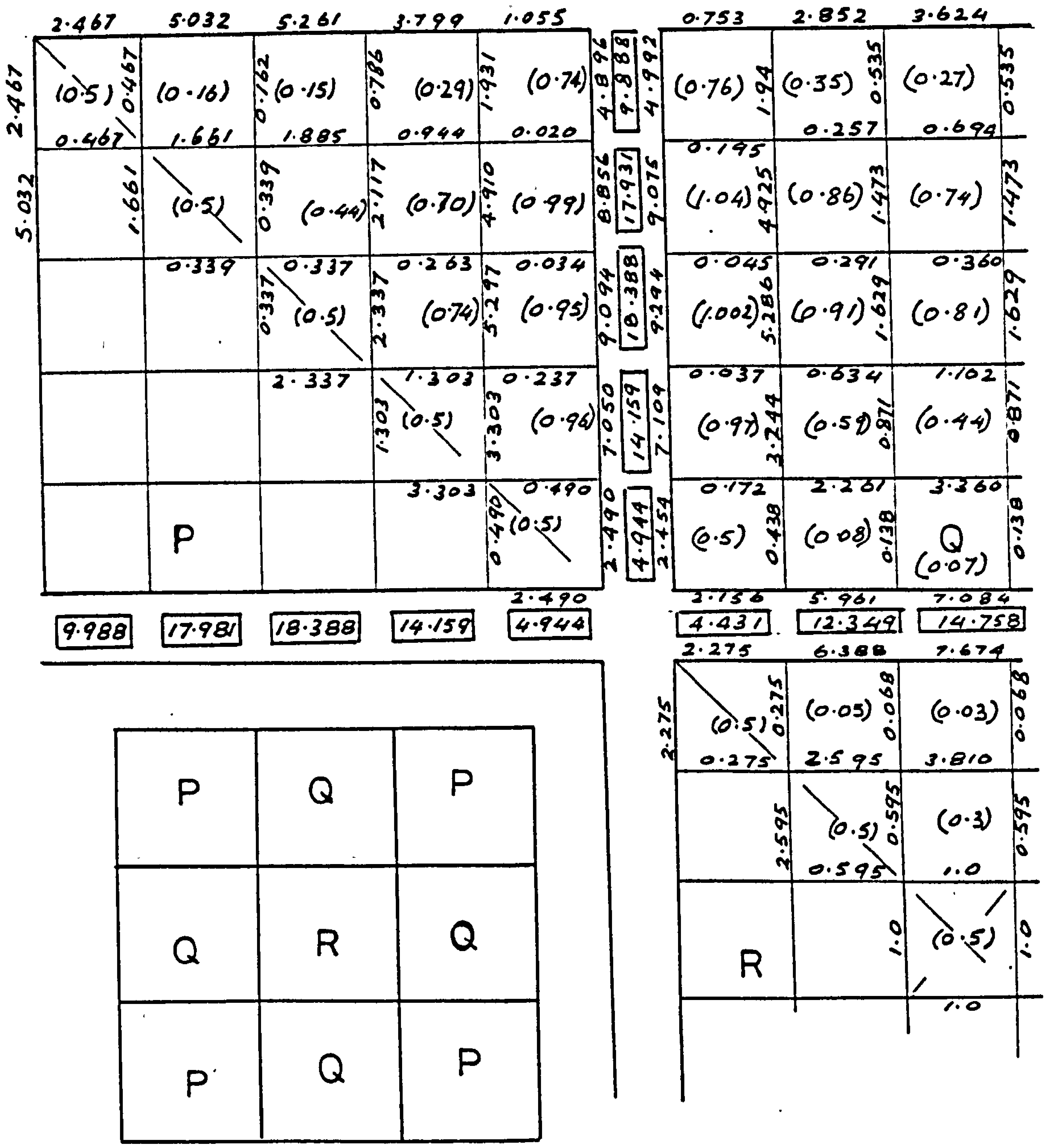
Free edges can be readily accommodated by the strip deflection method. Three situations can arise where free edges occur, a slab strip may be free at one end and either (a) fixed (b) simply supported (c) free at the other end. In the first case the slab strip is a cantilever and deflections may be calculated directly as in illustrative example already discussed with no change in the number of unknowns.

In each of the other two cases the deflection of the slab at the free edge will introduce one additional unknown. However, there will be one corresponding additional equation of statics for the strip at each free edge. With one simply supported edge and one free edge the slab strip must be in moment equilibrium under the unknown load distributions. With both edges free there will be a moment and a vertical force equilibrium equation and the number of unknowns will again be equal to the number of independent liner equations.

5.6.2 Fixed or Continous edges.

If the boundaries of the slab are fixed or continous, the slab strips are effectively statically indeterminate beams. The strip deflection method can be used to analyse such slabs. Consider the slab shown in Fig (5.11 a). It is continous over three equal spans bothways and is simply supported at the outer edges thus forming a (3 x 3) panel. The entire slab is divided into fifteen strips each way and it is assumed that the total uniformly distributed load on each panel is 100 units. Equilibrium and deflection compatibility equations can be written to each grid and these in general will take into account the support conditions. If the loading and the support conditions are symmetrical the number of unknown load distributions (q_x or q_y) will be 28.

Certainly it is more convenient here to utilize a torsionless grid programme. The strip layout is replaced by equivalent grid and at each internal intersection a point load of 4 units is applied . Fig (5.11 b) shows the vertical shear force distribution pattern and the load distribution factor α when all support nodes are inelastic.



(a)

KEY DIAGRAM

(b)

FIG (5.11 a) VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR A SLAB CONTINUOUS OVER THREE SPANS BOTHWAYS AND CARRYING A UNIFORMLY DISTRIBUTED LOAD

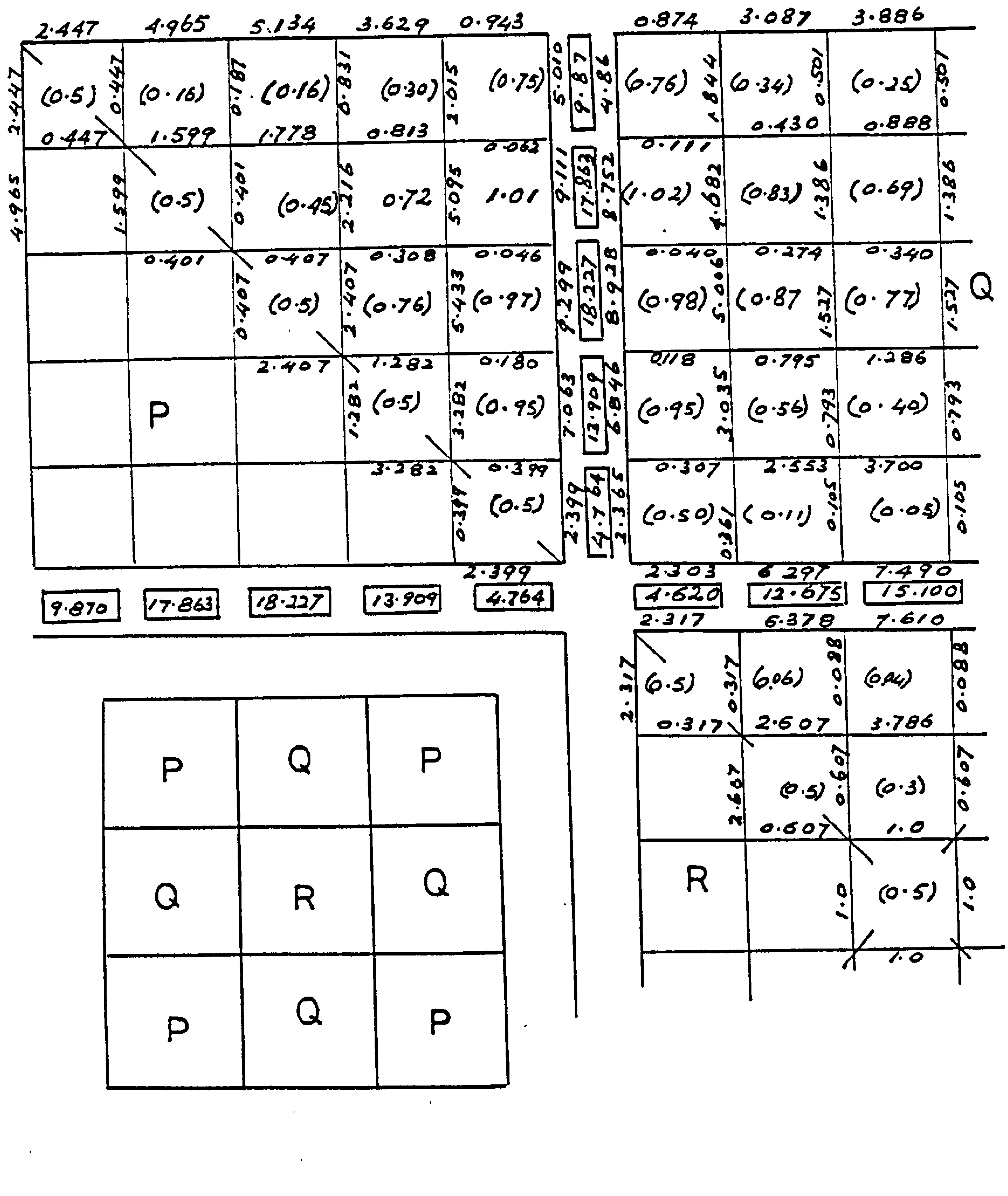


FIG (5.11 c) VERTICAL SHEAR FORCE PATTERN FOR THE CONTINOUS SLAB CONSIDERING EACH PANEL SEPARATELY

Values of α for the grids on the diagonal are 0.5 and for some areas near the continous supports α takes values outside the normal range of 0 to 1.0. Figures shown within the rectangles are the values of reactions at the continous supports.

Although the slab is continous both ways and has nine panels a closer look will reveal the existance of only three types of panels with different edge conditions. The panel R is similar to an internal panel continous over all four edges. Panel Q resembles one that is continous over three edges and discontinous over one edge and the panel P has two adjacent edges continous and the other two edges discontinuous.

The slab can be assumed to be an assembly of panels P,Q and R as shown in the key diagram and Fig (5.11.c) shows the vertical shear force distribution pattern for the whole slab so obtained. The reactions at the support nodes and the load distribution factors for the grids are also shown. Figures (5.11.b) and (5.11.c) shows that there is remarkable agreement between the results obtained by the two procedures. The second has advantages in that it considers one panel at a time and the number of equations can be small in comparison and in general can be solved using a small computer. This method of panel assembly is particularly useful in the design of floor systems of buildings which consists of regular array of rectangular panels.

The designer has considerable freedom in the choice of the strip layout. Fig (5.12) shows an internal rectangular panel ($l_y : l_x = 1.4$) and the chosen strip layout is identical to the one that is currently recommended in CP110 or CP114. Clearly the edges of panels are considered to be fully fixed. Also shown are the load distribution factors, bending moment for the middle strips calculated by the grid method. Shown alongside are the bending moment diagram for the middle strips in an internal panel from a design according to CP110. Clearly considerable saving of material can be achieved by providing steel according to

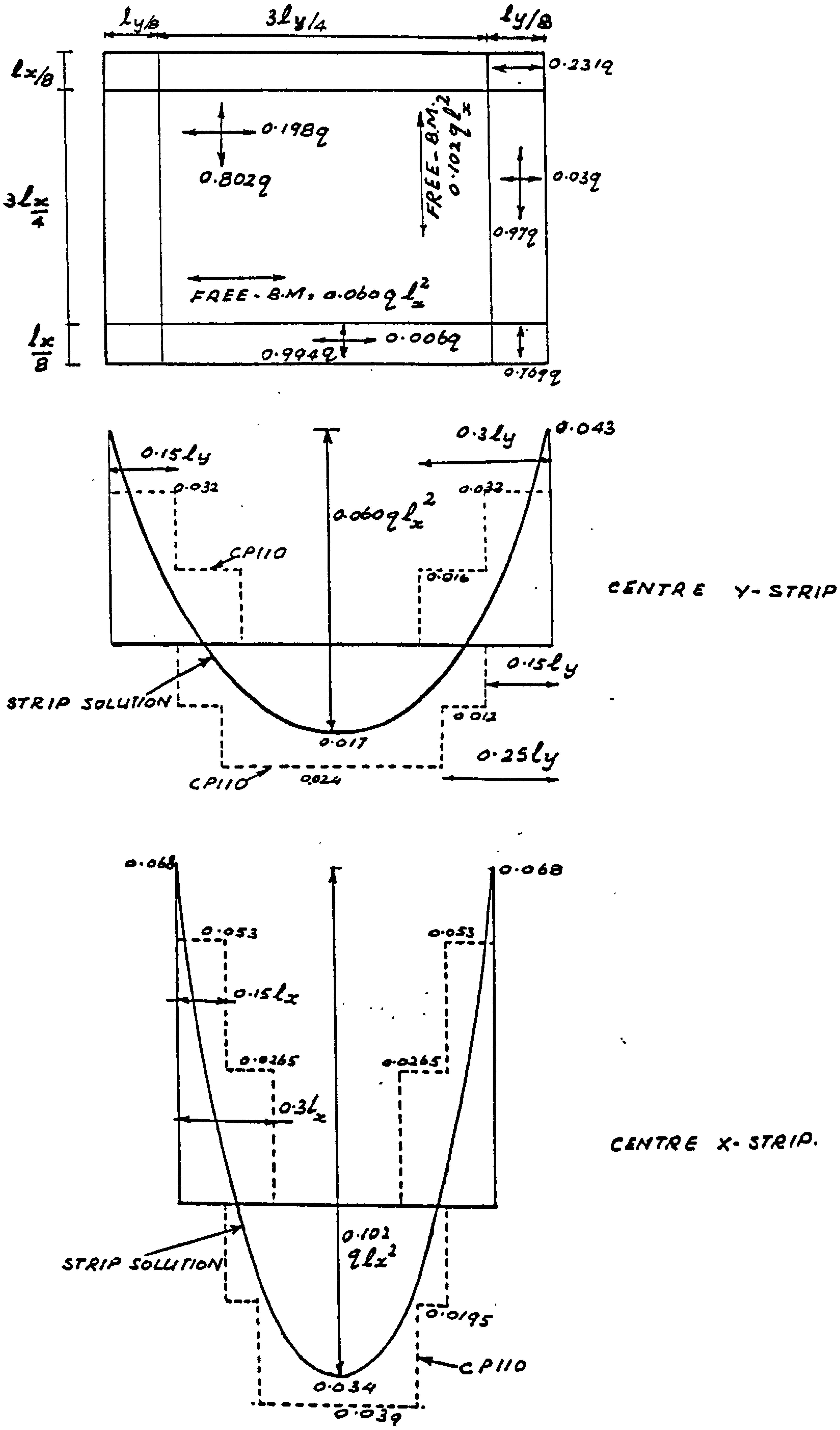


FIG (5.12) UNIFORMLY LOADED RECTANGULAR PANEL FIXED ON ALL SIDES

$L_y : L_x = 1.4$

the strip deflection method.

Therefore it can be concluded that although it appears to be rigourous to analyse a continous slab as a single unit this procedure is scarcely worth while. It is much simplar to analyse each panel as a single unit. For many practical examples it will be possible to assemble the slab system with panels having standard edge conditions.

5.7 SKEW SLABS.

The strip deflection method can be readily applied to skew slabs carrying uniformly distributed loads. Fig (5.13 a) shows a skew slab with two opposite edges simply supported and the other two edges free. A layout of the strips must be chosen such that they are parallel and perpendicular to the simple supports. Due to the presence of triangular shaped elements the Y- strips such as P Q R S are trapeziodal and all X - strips are in the shape of a parallelogram.

In the strip deflection method we only deal with rectangular strips and these must be replaced by ones that are of the same width. The procedure is shown in Fig (5.13.b) and the assumed strips for strips Y3 Y10 and X3 are

Actual strip		Assumed Strip	Equivalent Grid
Y_3 -	PQRS	PQGS	AB
Y_{10}	TUVW	THVW	CD
X_3	RSTU	RGUH	BC

The triangular elements at the boundaries are replaced by rectangular ones of same width and overall length. Hence for points within these triangular elements equilibrium will only be satisfied approximately. In the provision of reinforcements within the triangle. the variation of strip width is taken into account. The procedure will

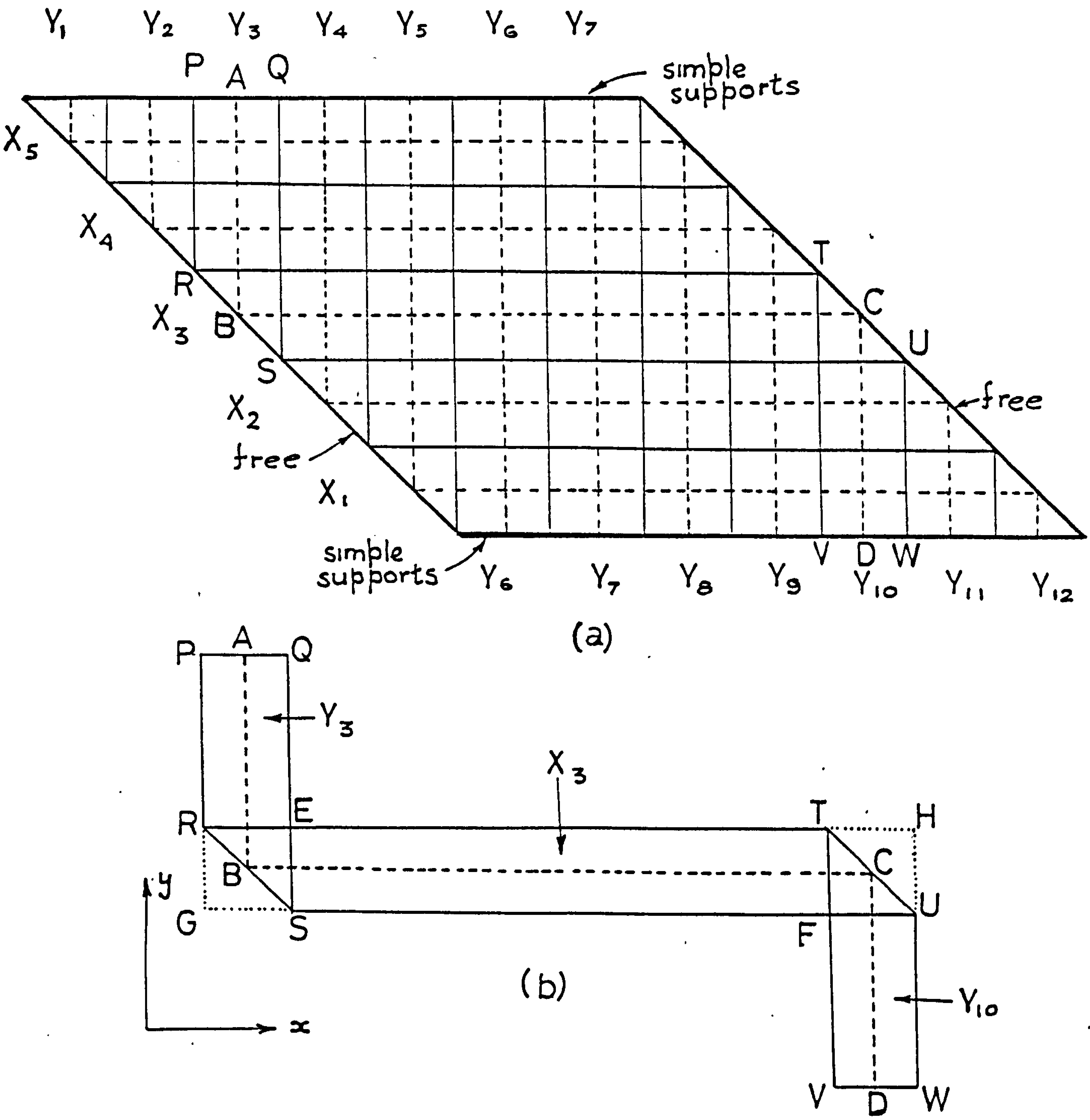


FIG (5.13) APPLICATION OF STRIP DEFLECTION METHOD, APPROXIMATIONS OF STRIPS AND GRIDS FOR SKEW SLABS not affect the equilibrium conditions for rectangular elements within the slab. Clearly the equivalent grid so produced comprises of members which are of the same mean length.

To determine the load distributions elastic deflections can be equated at the intersection points along centre lines of the orthogonal strips. For triangular elements (RES) these correspond to mid points (B) of the hypotenuse (RS).

The skew slab can again be solved using the torsionless grid method and a grid consisting of equivalent torsionless beams positioned along the centre lines of the orthogonal strips is used. For the slab shown in Fig (5.13) the equivalent grid is shown with the dotted lines. The load interactions takes place at the grid

intersection points and for triangular elements (RES) these correspond to mid points (B) of the hypotenuse (RS).

Fig (5.14) shows the vertical shear force distribution pattern obtained by the grid method for a skew slab simply supported on two opposite edges with the following properties.

Simply supported side	- b	=	10 units
skew length	- l_ϕ	=	$\sqrt{99}$ = 9.9 units
clear span	- l_x	=	7 units
ratio	-b:l_ϕ	=	1.01
angle of skew		=	45^o

Uniform load - one unit per unit area.

Also shown are the loading on strips X_5 and Y_6. The shear forces and loadings are rounded off to the second decimal and therefore there can be very small out of balance moment at the free edges and simple supports.

In skew slabs an extremely high bearing reaction occurs near the obtuse angled corner. This may be many times as high as the reaction of corresponding right angled slabs and in this example over 57% of the total load is concentrated over the strip width nearest to this corner. The other noticeable feature is that the load distribution factors on the grid nearest to this corner are (α = - 9.7) and (1 - α = 10.7). These two values lie widely outside the assumed range for the load distribution factors in the simple strip method.

The other factor that is of importance is to check the collapse load of the skew slab. The positive yield line at the centre of the slab and shown in Fig (5.14) conforms with the uniqueness rules set out in Chapter Four. A numerical analysis shows that the collapse load is within 0.2% of the design load. The slight variation from uniqueness can be attributed to the fact that equilibrium is satisfied approximately at the triangular elements near the free edge.

In the above example rigid supports have been assumed but the effects of flexible supports in skew slabs is by no means negligible. The effects of this flexibility is to reduce the bearing reaction and the bending moment in the regions of the obtuse angled corner. Rusch and Hergenroder (32) have used model tests to determine the bearing reactions of skew slabs. These were measured at 10 equally spaced points along the support with highly sensitive instruments. The models which were made of a special low setting plaster had the following geometry.

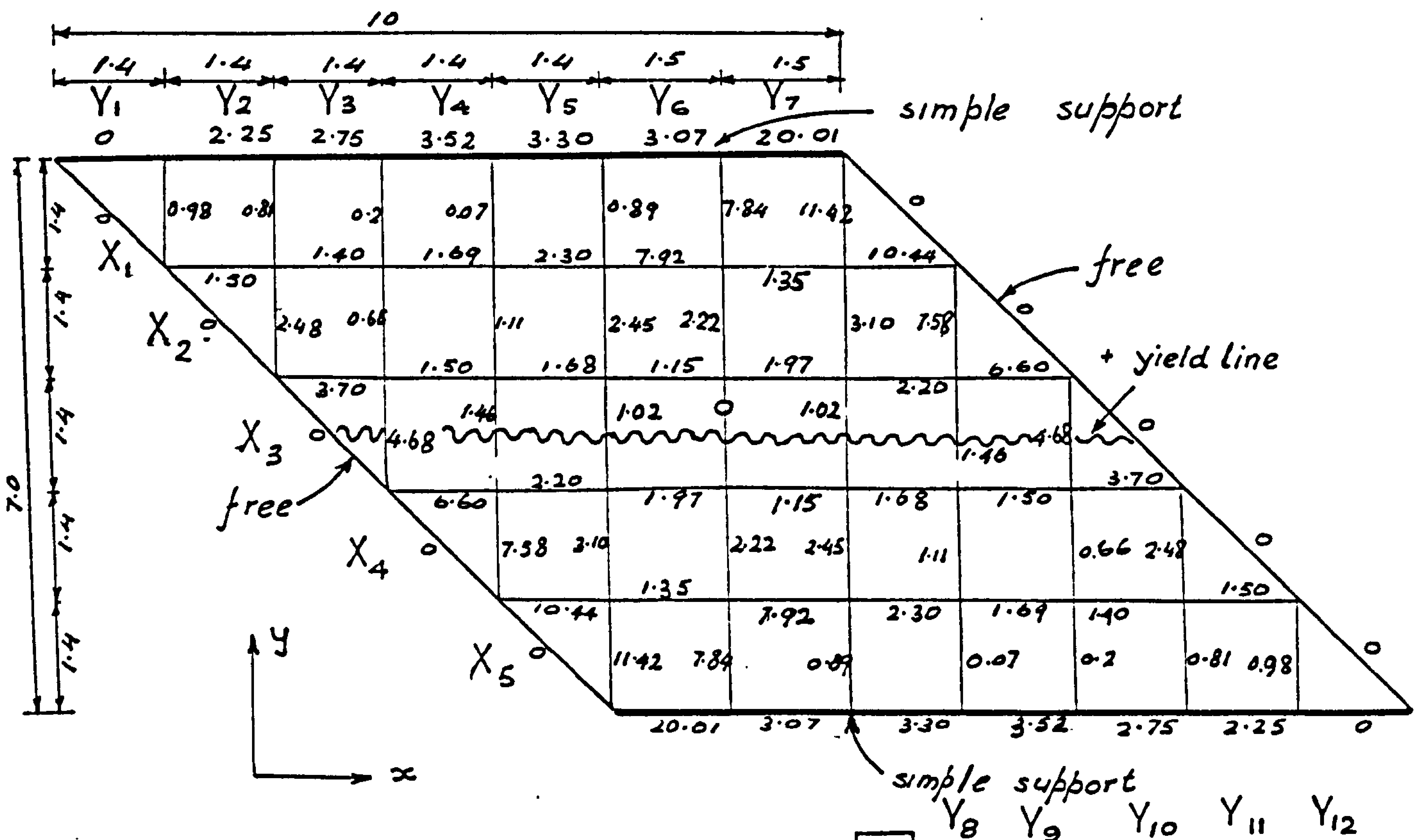
angle of skew ϕ = 30°

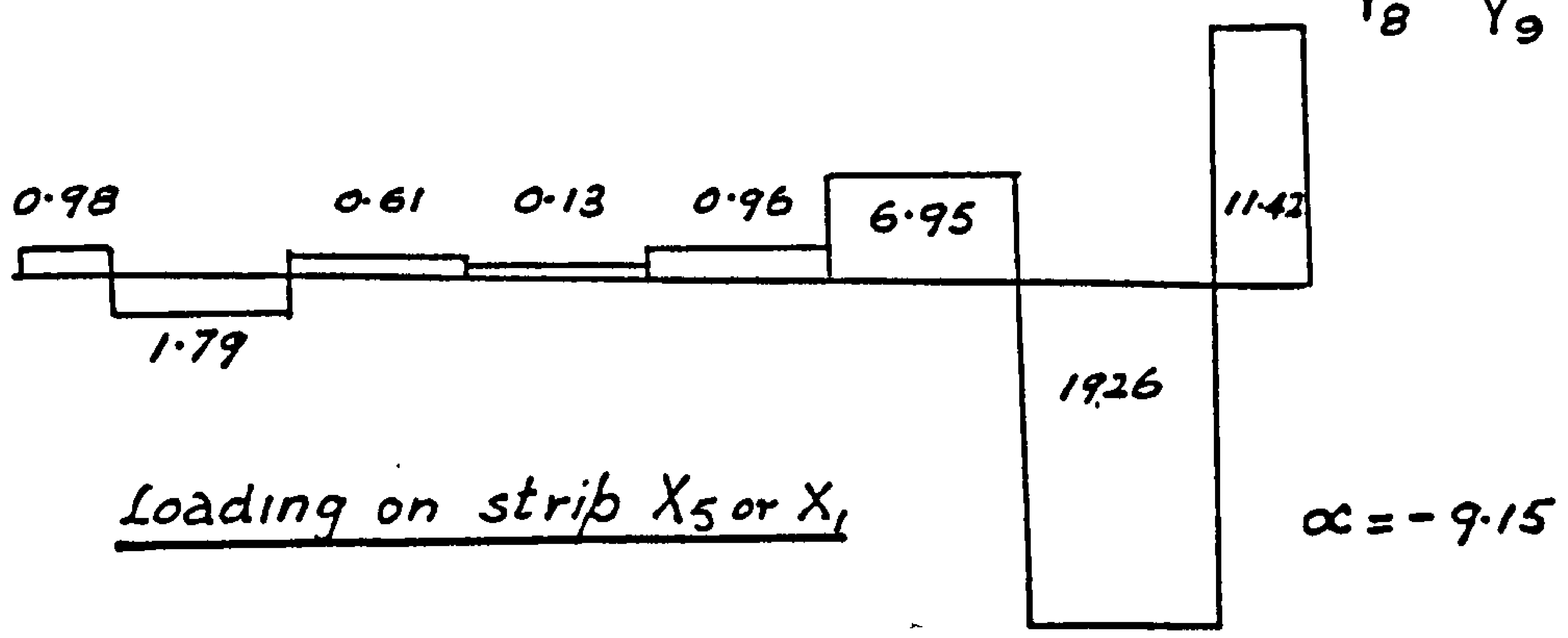
ratio b: l_{ϕ} = 1.2

clear span = 500 mm

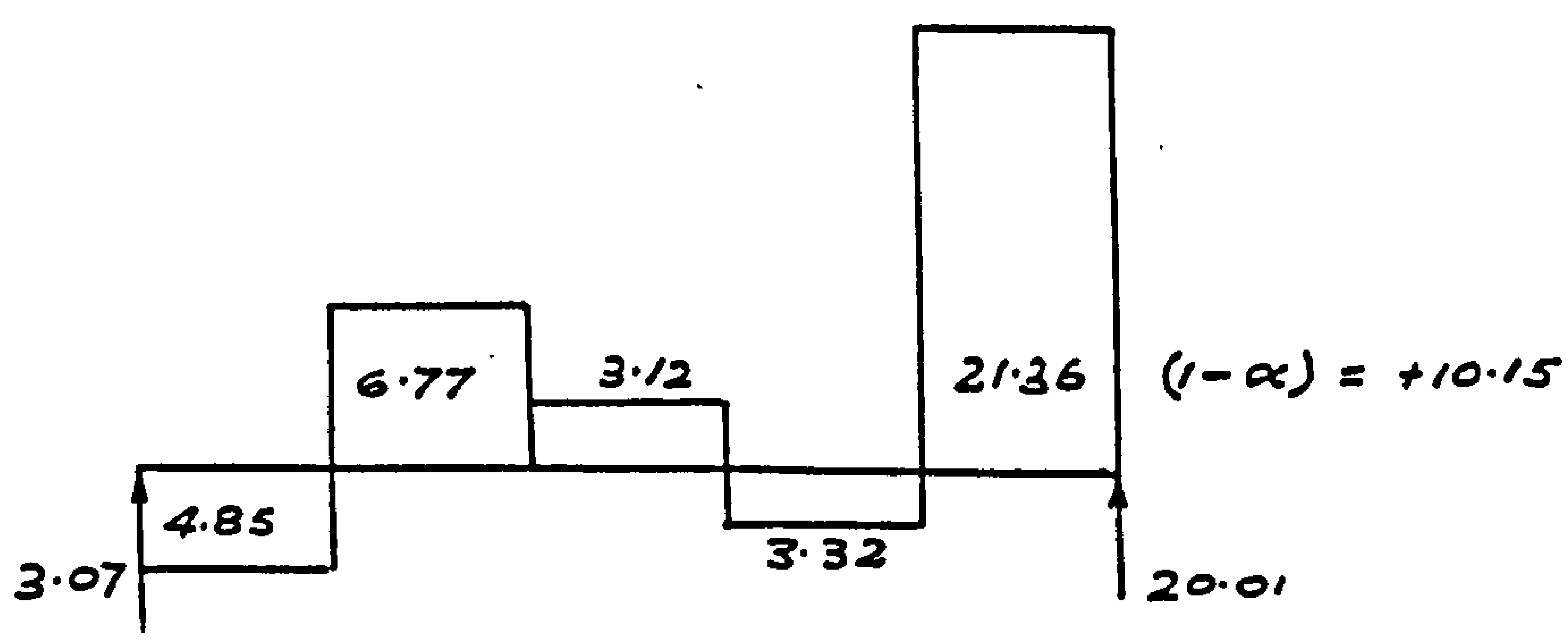
slenderness ratio l_x : d = 25

The spring constant c for the bearings were chosen such that $l_x^2 \cdot \frac{c}{k}$ = 370 . Where k is the flexural rigidity per unit width of the slab :- $\frac{E\,d^3}{12\,(1-\mu^2)}$.



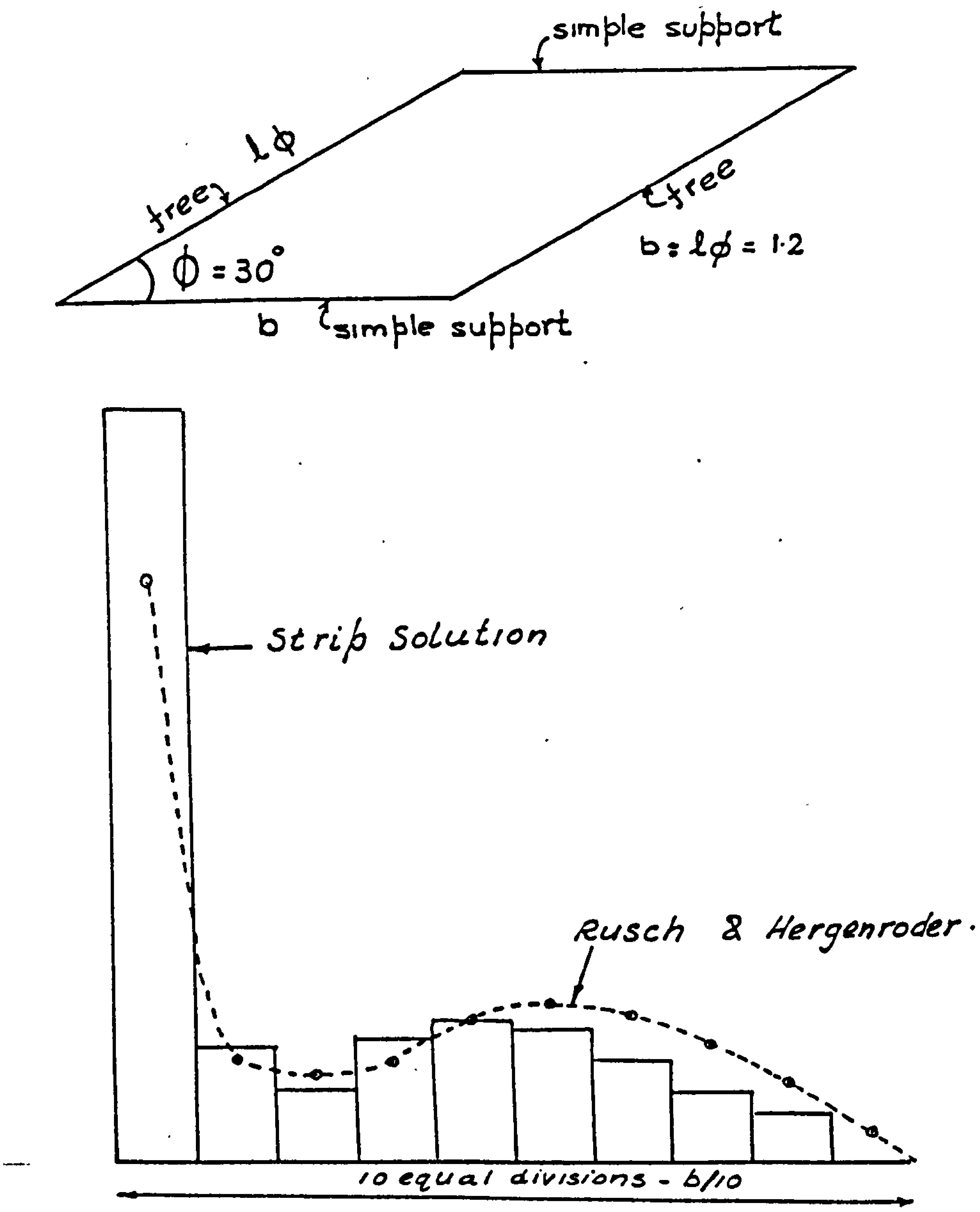


Loading on strip X_5 or X_1



Loading on strips Y_6 or Y_7

FIG (5.14) SOLUTION TO A SKEW SLAB SIMPLY SUPPORTED ON TWO OPPOSITE EDGES AND THE OTHER TWO EDGES FREE CARRYING A UNIFORMLY DISTRIBUTED LOAD



Rusch & Hergenroder - 0·276 0·065 0·056 0·065 0·093 0·103 0·096 0·075 0·051 0·031

Strip Solution - 0·486 0·076 0·046 0·078 0·092 0·084 0·065 0·044 0·022 0·0

(above values - fractions of half total uniform load)

FIG (5.15) DISTRIBUTION OF REACTION FOR A UNIFORMLY LOADED SKEW SLAB

Fig (5.15) shows the distribution of reaction as a fraction of half the total load on the slab corresponding to a uniform load. Also shown are the values obtained by the torsionless grid method for a similar slab. The agreement between the experimental results and the theoritical values is very clear and therefore this method can be recommended to determine the bearing reactions of all slabs.

5.8 CONCLUSIONS

A generalised strip method of concrete slab design has been presented in which the load distribution over finite regions of the slab are determine systematically by ensuring compatibility of elastic deflections of orthogonal slab strips. It has been shown that all types of distributed loading, slab shape and boundary conditions including partial composite action with supporting beams can be accommodated. Point loads, point supports and patch loads will be considered in the next chapter.

The designer has considerable freedom in choosing the strip layout, but whatever the choice the method ensure that the resulting load distribution will not depart too far from elastic working load conditions and should therefore ensure adequate serviceability. The method provides full information about bending moments, shear forces and loading at all points. The resulting layout of reinforcement is orthogonal and banded and the total amount of reinforcement required compares favourably with other methods of design commonly used. With uniformly distributed load the method leads to a unique solution for the collapse load in all the cases which have been considered, except for skew slabs and even here it is very close to the unique. Therefore the strip deflection method appears to offer a unified collapse limit state approach to the design of all slab type structures which is simple, safe and economical.

For many designs the calculations can be carried out on a simple desk calculator but the method has been shown to be closely related to the torsionless grid analogy method and so existing computer programmes for grid work can be readily used where they are available. The GENESY'S system which has been used for all the grid analysis presented is very flexible and any type of slab geometry or support condition can be accommodated. The only difference between the strip deflection method and the torsionless grid approach is the assumption made about the load distribution between the slab strips. In most practical cases the difference between the bending moment fields will be insignificant but for a unique solution for the collapse load the load interactions should be assumed strictly to be uniform over a grid area.

CHAPTER SIX

STRIP METHOD OF SLAB DESIGN WITH POINT LOADS AND POINT SUPPORTS.

6.1. Introduction.

Application of the strip method of slab design with distrubed loads was discussed in the earlier chapters. When the applied load q is uniform over a certain area, then q can be distributed in the x and y directions such that $q_x + q_y = q$. The value of q can vary from region to region and for each such region there is an equilibrium equation. When equilibrium conditions are satisfied at all points of the slab and at the boundaries, the strip method will produce a safe solution and in most cases of distributed loaded slabs it is possible to find a unique solution.

Consider the arrangement of strips in the rectangular slab shown in Fig (6.1.). The slab carries a single patch load W on its centre grid. If the dimensions of this patch load fits that of the centre grid then the problem is the same as that just considered. However it may not always be practicable to have a layout of strips to fit the positions and dimensions of all applied patch loads. Where the loaded area does not coincide with a grid area it is not possible to satisfy equilibrium and therefore a safe solution is not produced.

A similar situation can arise due to a point support or a column and Hillerborg's type 3 element was an attempt to overcome this problem. This load distribution element has the function of a distributing concentrated load over the grid area, but the moment field within this element is complicated and has to include torsional moments.

Wood and Armer (27) in their alternative treatment of the type 3 element suggest the use of strong bands, together with strong

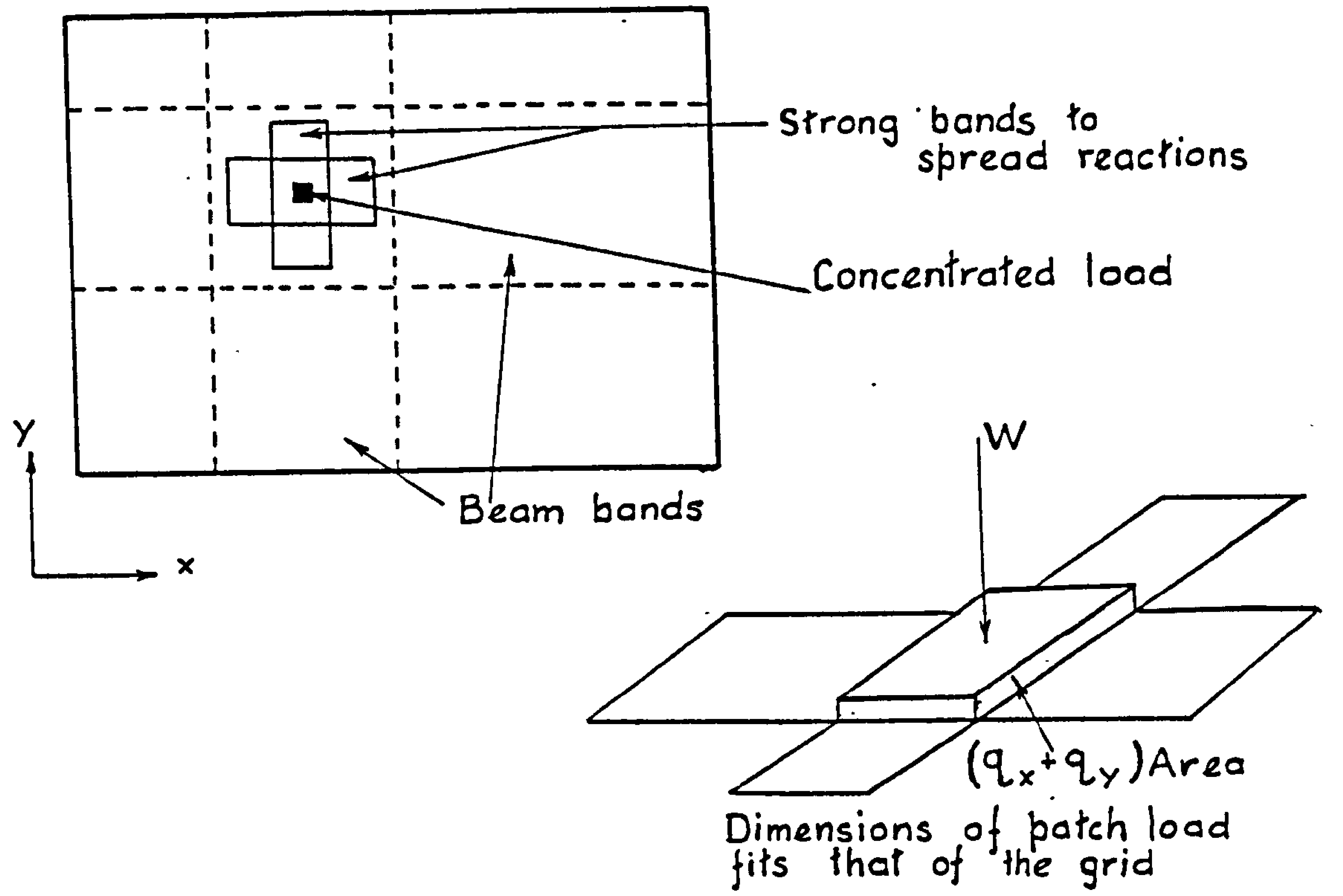


FIG (6.1) USE OF STRONG BANDS, FOR TRANSMITTING CONCENTRATED LOADS TO SUPPORTS

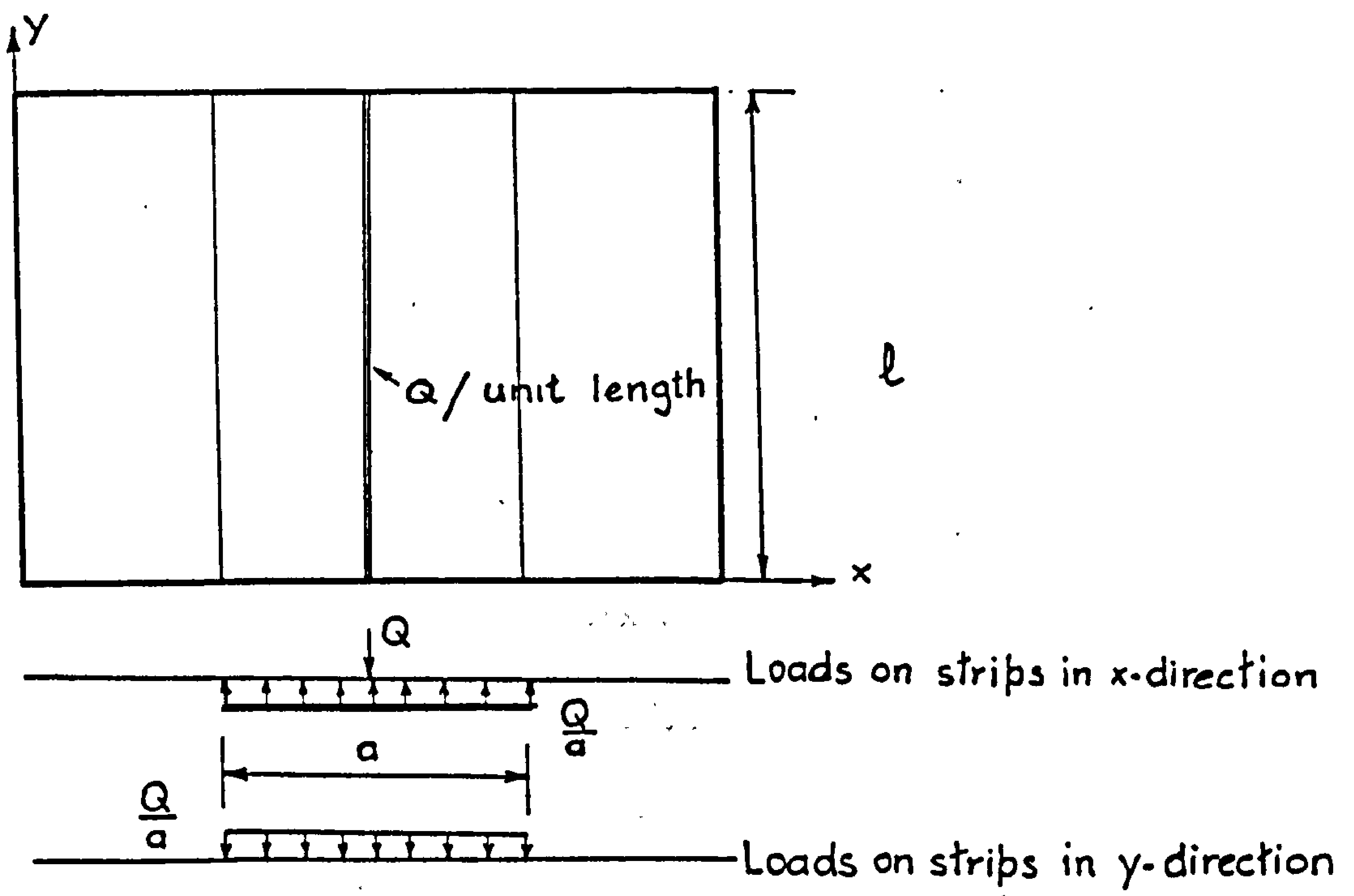


FIG (6.2) USE OF SPREADER FOR A SLAB WITH A LONG NARROW LOAD strips of short length to spread the column reaction see Fig (6.1.a). Unfortunately the authors did not give rational methods of designing such strips and no information is available on the reinforcement in the unloaded regions.

Hillerborg (34) also solved the simply supported slab carrying a narrowly distributed line load Q per unit length as shown in Fig (6.2). The load Q was carried by strips in the x direction, which in turn was carried by strips in the y direction. Hillerborg suggested a suitable choice of 'a' the length over which the load is spread. The maximum design moments per unit length are $m_x = \frac{Q.a}{8}$(Sagging) and $m_y = \frac{Ql^2}{8a}$ (Sagging). Hillerborg's procedure clearly ensures that equilibrium is satisfied at all points and does so without the use of torsional moments. Hillerborg has devoted considerable efforts and ingenuity to generalise the simple strip method, but surprisingly he didnot develop this simple concept of spreader systems to distribute concentrated load in more general problems. It is the purpose of this chapter to illustrate how a particular layout of strips can affect the uniqueness of the strip method. Point loads, patch loads and columns will be dealt with in detail and in general these can occur within the slab, along an edge or at a corner. In each case recommendations will be given to ensure that the strip method will provide a safe solution with respect to collapse.

6. 2. STRIP SYSTEMS REQUIRED TO PRODUCE UNIQUE COLLAPSE LOAD.

6.2. 1. Corner and edge columns.

In the strip method the load distribution over a grid area is assumed to be uniform. With an external column however the designer may sometimes be inclined to consider the column reaction to be concentrated at an edge and thereby the bending moment diagram of the strips is affected. The effect of this assumption on the collapse load of the slab will be discussed here.

Consider the square slab supported at the four corners and carrying a central point load as shown in Fig (6.3). The centre strip containing the point load is of the same zero width as the load thus eliminating any load distribution errors. The edge strips are each of width p L which is different from the width of column q L.

One quarter of the load is taken by each of the columns and Fig (6.3.b) shows the three possible assumption about the distribution of reaction. In assumption (1) the reaction is concentrated over the outer edge of the slab and therefore the effective span of the column strip is L. (2) Assumes a uniform distribution of reaction over the column grid and the effective span is $L(1-p)$. If designed according to assumption (3) the reaction is assumed to be concentrated at the inner edge of the column grid and the effective span is further reduced to $L(1-2p)$. Due to the difference in span in each of the methods the bending moment in the support strip is different which will affect the collapse load of the design. The moment field for the entire slab is positive - positive and the collapse load is checked for the diagonal and central line mechanisms.

<u>Assumption (1)</u> column reaction concentrated at the outer edge

a) <u>diagonal collapse mode</u>

$$E = Wc \times 1$$

$$\text{and} \quad D = 4M \times 1/\frac{L}{2}(1-2q) = \frac{8M}{L(1-2q)}$$

where M is the total moment accross the diagonal

$$\text{ie} \quad M = 2\frac{W}{8} \times \frac{p}{2}L + \frac{W}{4}\left(\frac{L}{2} - \frac{pL}{2}\right) = \frac{WL}{8}$$

equating $E = D$ gives $Wc \leqslant W/(1-2q)$

b) <u>Central line collapse mode.</u>

$$E = Wc$$

$$\text{and} \quad D = M \times 2\theta \quad \text{where} \quad \theta = \frac{2}{L(1-2q)}$$

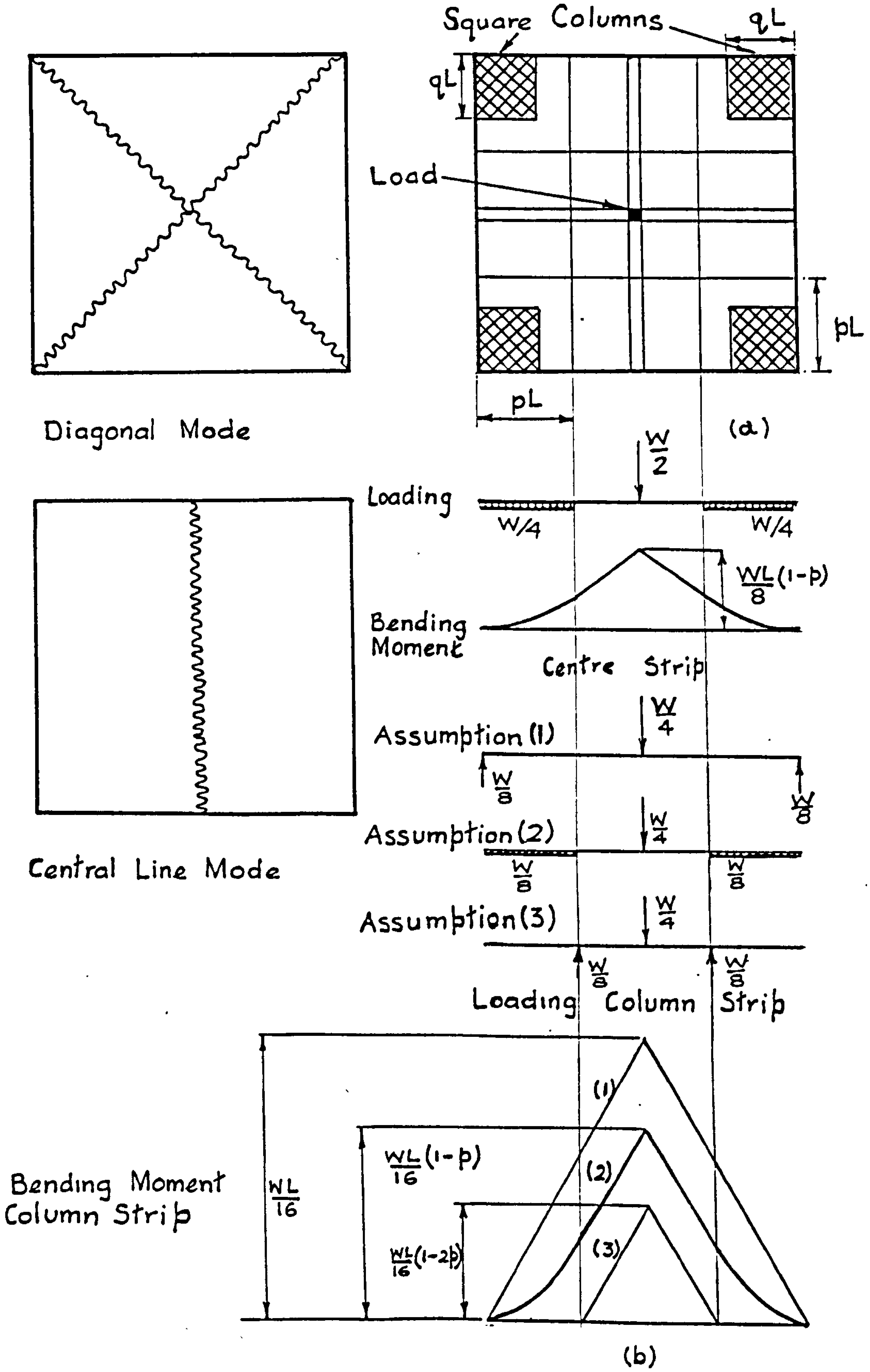


FIG (6.3) SQUARE SLAB ON FOUR CORNER SUPPORTS CARRYING A CENTRAL POINT LOAD and $M = \frac{WL}{8} (1 - p) + 2 \frac{WL}{16} = \frac{WL}{4} (1 - \frac{p}{2})$

equating E and D gives

$$Wc \leqslant \frac{W (1 - \frac{p}{2})}{(1 - 2q)} \qquad (6 . 1)$$

The central line mechanism will therefore govern and when $p > 4q$ then $W_c < W$. It is therefore possible to have a support strip width upto four times without producing an unsafe solution

<u>Assumption (2)</u> Column reaction uniformly spread over column grid

(a) <u>Diagonal Collapse Mode</u>

$$M = \frac{WL}{8} (1 - p) + 2 \frac{W.p.L}{16} \times \frac{1}{3} = \frac{WL}{8} (1 - \frac{2}{3} p)$$

Similarly $E = Wc$ and $D = \frac{8M}{L(1-2q)}$

$$\therefore Wc < \frac{W (1 - 2/3 p)}{(1 - 2 q)}$$

(b) <u>Central line collapse mode</u>

$$M = \frac{WL}{8} (1 - p) + \frac{WL}{8} (1 - p) = \frac{WL}{4} (1 - p)$$

clearly $Wc \times 1 \leqslant \frac{W}{4} (1 - p) (\frac{4}{1 - 2q})$

$$\text{or} \quad Wc \leqslant W \frac{(1 - p)}{(1 - 2q)} \qquad (6 . 2)$$

Clearly the central line collapse mode governs and uniform distribution of support reaction over the column grid area will thus enable a designer to choose a strip width up to twice the column width and ensure a safe solution.

<u>Assumption (3)</u> column reaction concentrated at the inner edge of the column grid.

Diagonal collapse mode

$$M = \frac{WL}{8} (1 - p)$$

equating E and D as above gives

$$Wc \leqslant W \frac{(1 - p)}{(1 - 2q)}$$

Central line collapse mode

$$M = \frac{WL}{8}(1 - p) + 2\frac{WL}{16}(1 - 2p)$$

$$= \frac{WL}{4}\left(1 - \frac{3p}{2}\right)$$

$$D = M \times \frac{4}{L}(1 - 2q) \quad \text{and} \quad E = Wc$$

therefore $W_c \leqslant W \dfrac{(1 - 3p/2)}{(1 - 2q)}$

Again the central line mode governs and $W_c = W$ when $p = 4q/3$

This example shows how the collapse load can be influenced by the boundary assumption. Contrary to the observations in chapter four only one collapse mode governs the ultimate load of the slab, which is due to the fact that equilbrium is not satisfied in the corner grid. The critical collapse mode for all cases is the central line mode and for each assumption a particular layout of strips will ensure uniqueness.

It is recommended that the column reactions are distributed uniformly over the grid areas rather than being concentrated at an edge. Under this condition a choice of strip width twice as large as the column width will give the unique collapse load and can be used for practical designs.

To prove why the above assumptions lead to a unique solution let us consider the rectangular slab carrying an eccentric point load as shown in Fig 6.4. The load strip is of zero width thus eliminating any load distribution errors and the loading in this strip is shown in Fig (6.4.) The reaction R_2 is spread uniformly over the column strip which is of width p.L. If this uniformly distributed reaction is replaced by a single concentrated force R_2 of the same value at the centre of the column strip, then for points outside the column strip the shear forces and bending moments remain unchanged. For a strip width (pL) exactly equal to twice the column width (2x q L)

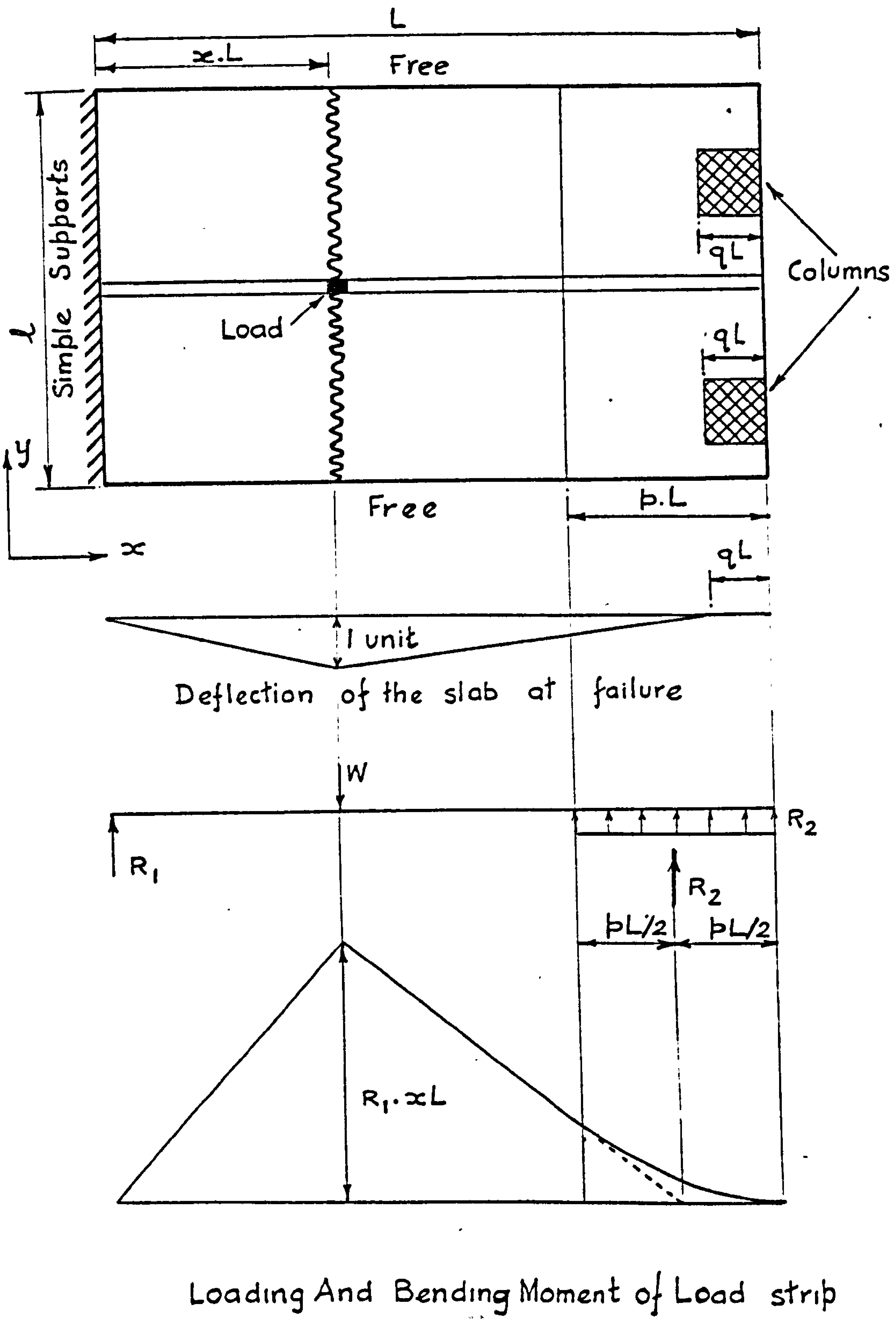


FIG (6.4) RECTANGULAR SLAB SIMPLY SUPPORTED ON ONE EDGE AND ON TWO COLUMNS CARRYING AN ECCENTRIC POINT LOAD the line of action of the column reaction R will coincide with the inner edge of the column. At failure with a centre line mechanism the deflection at this edge is zero and for such mechanisms this assumption will therefore produce unique results for the collapse load. For the slab shown in Fig (6.4)this can be again verified by analysis. For the central line mechanism

$$E = W_c \times 1$$

and $$D = M \times \theta$$

where $$M = W \left(\frac{2 - p - 2x}{2 - p}\right) . L . x$$

and $$\theta = \left[\frac{1}{L.x.} + \frac{1}{L-Lx - ql}\right] = \frac{1 - q}{Lx\,(1 - q - x)}$$

Equaling E and D

$$W_c \leqslant W \frac{(2 - p - 2x)\,(1 - q)}{(2 - p)\,(1 - x - q)}$$

clearly if the column strip width (p) is equal to twice the column width (2 x q) this mode of failure will give a unique solution for all values of x.

The uniqueness assumption can be extended to slabs subjected to uniform loads as well and can be illustrated by the example shown in Fig (6.5). The rectangular slab ($L \times \ell$) is supported by four columns size ($q_1 L \times q_2 \ell$) at the corners. The X strips are of width $p_1 L$, ($1 - 2p_1$) L, p_1 L and the Y strips are similarly $p_2\,\ell$, $(1-2p_2)\ell$, $p_2\ell$. The layout of the strips and the loading, bending moments for the X strips are shown in Fig. (6.5). In the design it is assumed that the column reaction is uniformly distributed over the corner grid ($p_1 L \times p_2 \ell$). Collapse is governed by the central line mechanism for which

$$E = (1 - 2q_1)\frac{Wc}{2} - 2q_1\left(\frac{2q_1}{1 - 2q_1}\right)\frac{Wc}{2}$$

$$= \frac{Wc}{2\,(1 - 2q_1)}(1 - 4q_1)$$

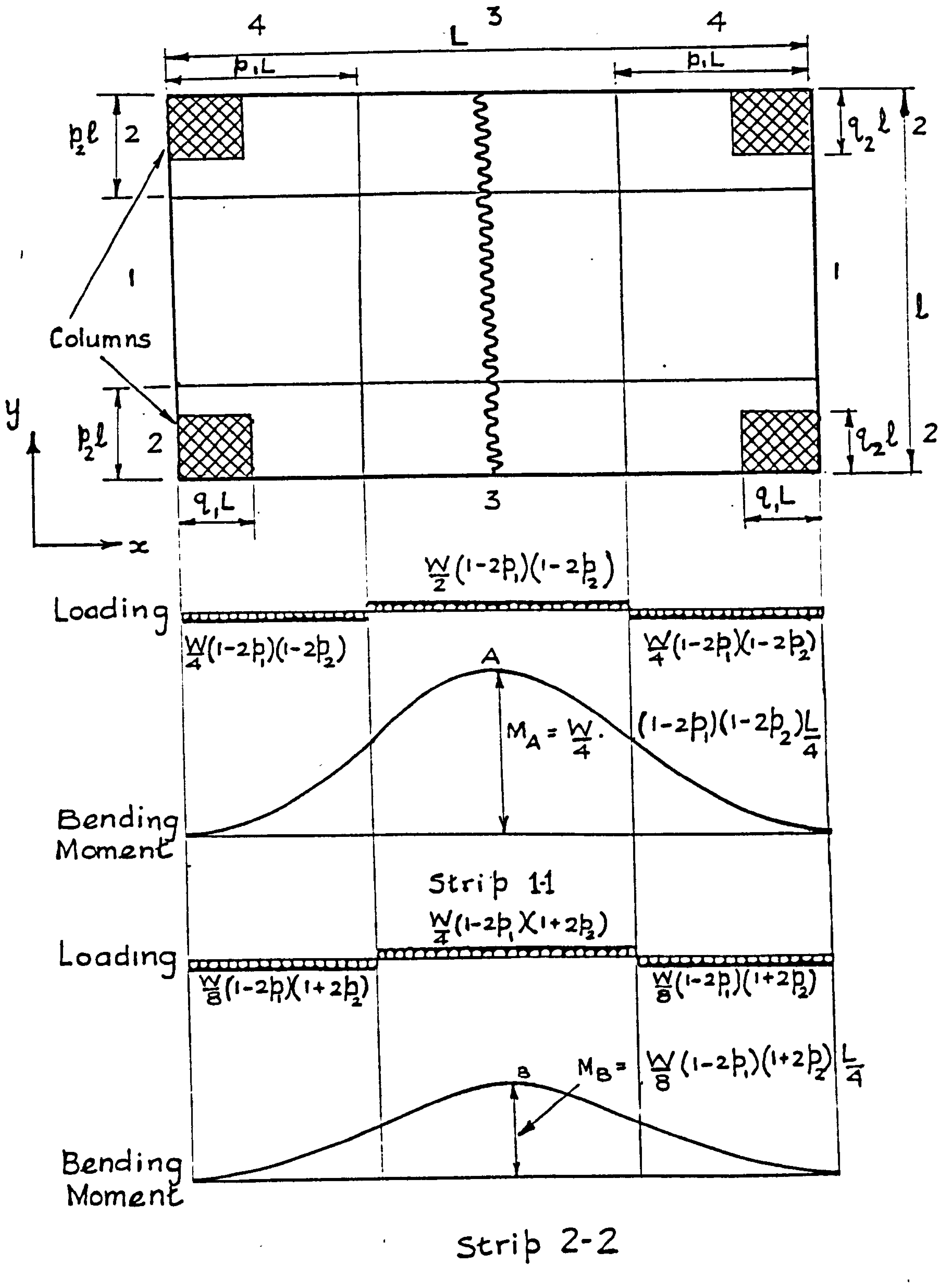


FIG (6.5) RECTANGULAR SLAB ON FOUR CORNER SUPPORTS CARRYING A UNIFORM LOAD

Where Wc is the total uniform load on the slab at failure and

$$D = M \times 2\theta$$

where M is the total moment across the yield line

$$M = \frac{WL}{16}(1 - 2p_1)(1 - 2p_2) + \frac{2WL}{32}(1-2p_1)(1+2p_2)$$

$$= \frac{WL}{8}(1 - 2p_1)$$

and $\theta = \dfrac{2}{L(1 - 2q_1)}$

Equating E and D

$$W_c \leqslant W \frac{(1 - 2p_1)}{(1 - 4q_1)}$$

Results are unique when the column strip width (p_1 L) equals twice the column width (2 x q_1 L).

Similar results can be derived by considering the Y strips, then $W_c = W$ when $p_2 = 2q_2$. For convenience it was assumed that the load distribution factor within the centre grid was half and the yield line passes through the middle of the slab. The results will not be changed for any other distribution nor for any other yield line within the centre grid.

These examples confirm that the designer can if support is assumed to be distributed uniformly over the column grid, uses a strip layout such that the width of the strip adjacent to the edge is twice the width of the column and obtain a safe solution. This recommendation has been proved for corner columns and columns at an edge.

6.2.2 Point and patch loads

For point and patch loads equilibrium must be satisfied at all points within the loaded grids and for such slabs reinforced according to the calculated moment field the strip method will give unique vlaues of collapse load if a kinematically admissible collapse mechanism is possible in which the yield lines satisfy the rules postulated in Chapter Four.

The application of the strip method to a square slab carrying a central point load W is shown in Fig (6.6). The slab is supported by a point support at each corner. A possible vertical shear force distribution for the slab is also shown, together with the loading and bending moment diagrams for strips A A, B B, and C C. The edge strips C C containing the column is of the same zero width as the columns. The width of the strips A A and that of patch load are identical.

The moment field of the entire slab is positive-positive and for all valid mechanisms with positive yield lines only the collapse load will be identical to the design load. This is so for any value of x (see Fig. 6.6) ie any load distribution can be assumed.

What is interesting is to compute the moment volume for this slab when the diamensions of the patch loads are zero. ie for a central point load

Moment volume of the slab $= \sum$(4.Strip CC + 4.Strip BB + 2.Strips AA)

$$= 4\left[2\int_0^{L/2} (0.125Wy - \frac{xy^2}{L})dy\right] + 4\left[2\int_0^{L/2} xydy\right]$$

$$+ 2\left[2\int_0^{L/2}(0.25\ Wy - \frac{2xy}{L} + 2xy^2)\ dy\right]$$

$$= 2W \int_0^{L/2} y.dy$$

$$= \frac{WL^4}{4}$$

For a square slab supported at the four corners it is easy to postulate a neutral(spherical) region with $|K_1| = |K_2| = K$ for the entire slab surface. For such a slab loads can be distributed in any direction (see section 3.8). The strip method provides one possible load distribution and therefore the corresponding moment volume is a minimum.

6.3. ERRORS IN COLLAPSE LOAD DUE TO EQUILIBRIUM NOT SATISFYING AT ALL POINTS.

Conditions under which the strip method produces the unique

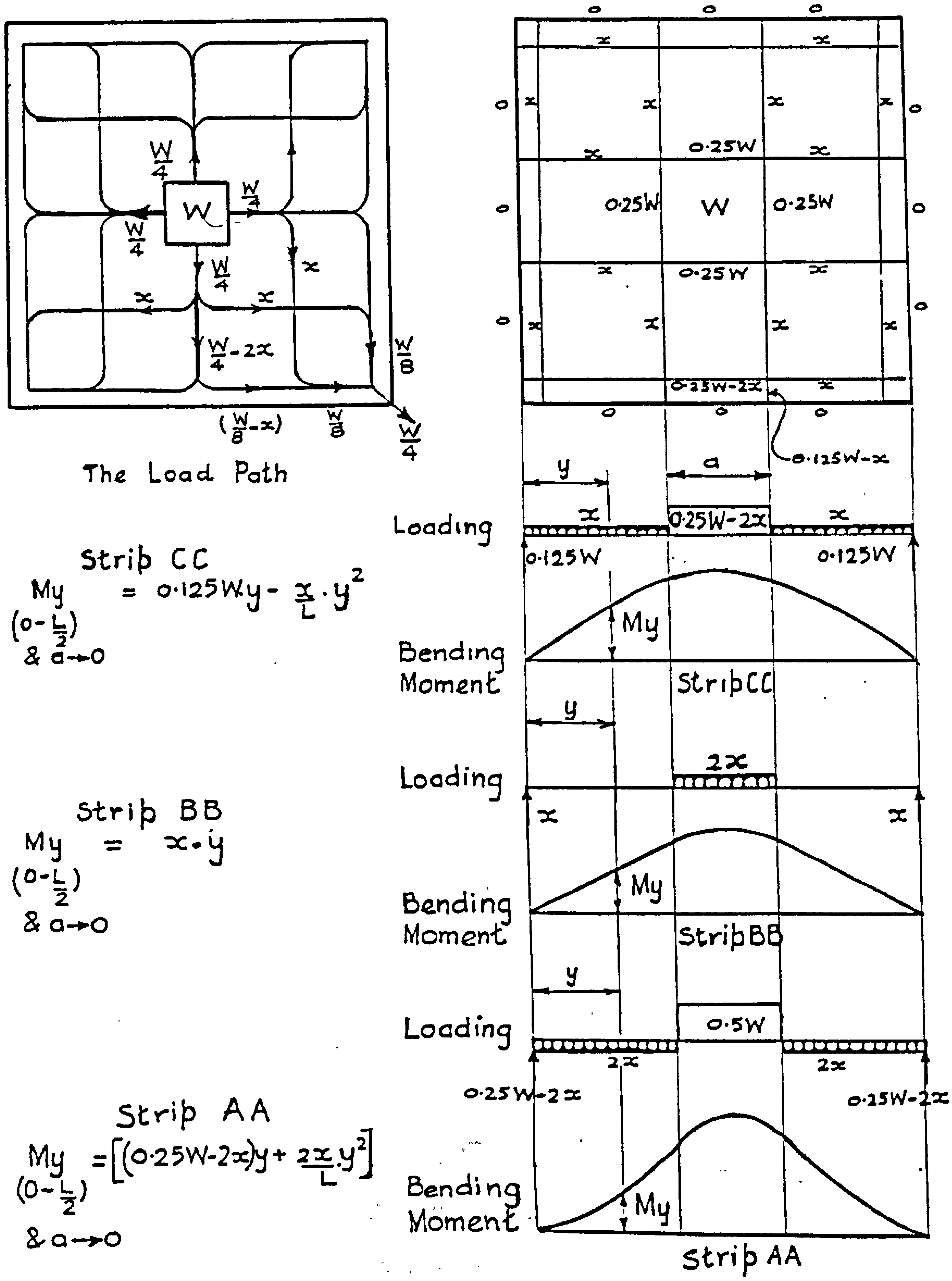


FIG (6.6) SQUARE SLAB ON FOUR CORNER SUPPORTS CARRYING A CENTRAL POINT LOAD collapse load for (a) external and edge columns (b) patch or point loads were discussed in the earlier section. It may not always be practicable to choose a layout that will give unique results. It is therefore of interest to know the errors associated with a choice of a strip layout.

6.3. 1 Simply supported rectangular slab with a central load - Unequal load and load strip widths.

The simply supported rectangular slab of side L and λL is symmetrically divided into three strips each way such that the widths of the centre strips are aL and λaL as shown in Fig (6.7). The central load has dimensions bL and λbL respectively. The designer assumes that the load W is spread uniformly over the central grid area and the effects of this on the collapse loads needs to be determined.

Also shown in the figure is a possible vertical shear force distribution and the loadings, and bending moment diagram for a centre strip. The collapse load W_c is dtermined for a diagonal mechanism by the vector method

$$\text{External work } E = \frac{W_c}{3} \; (3 - 2b)$$

$$\text{Internal work } D = 2\,(m_x)\,\frac{1}{\lambda L/_2} + 2\,(m_y).\,\frac{1}{L/_2}$$

$$= 4\left(\frac{m_x}{\lambda L} + \frac{m_y}{L}\right) \qquad (6.3)$$

Where m_x and m_y are the average moments over the central grid for the central x and y strips respectively

$$\text{clearly } m_x = \frac{W\lambda L}{4}\left[\left(\frac{1-a}{2}\right) + \frac{2\,(a)}{3\,.\,4}\right]$$

$$= \frac{W\lambda L}{24}\;(3 - 2a)$$

$$\text{similarly } m_y = \frac{W L}{24}\;(3 - 2a)$$

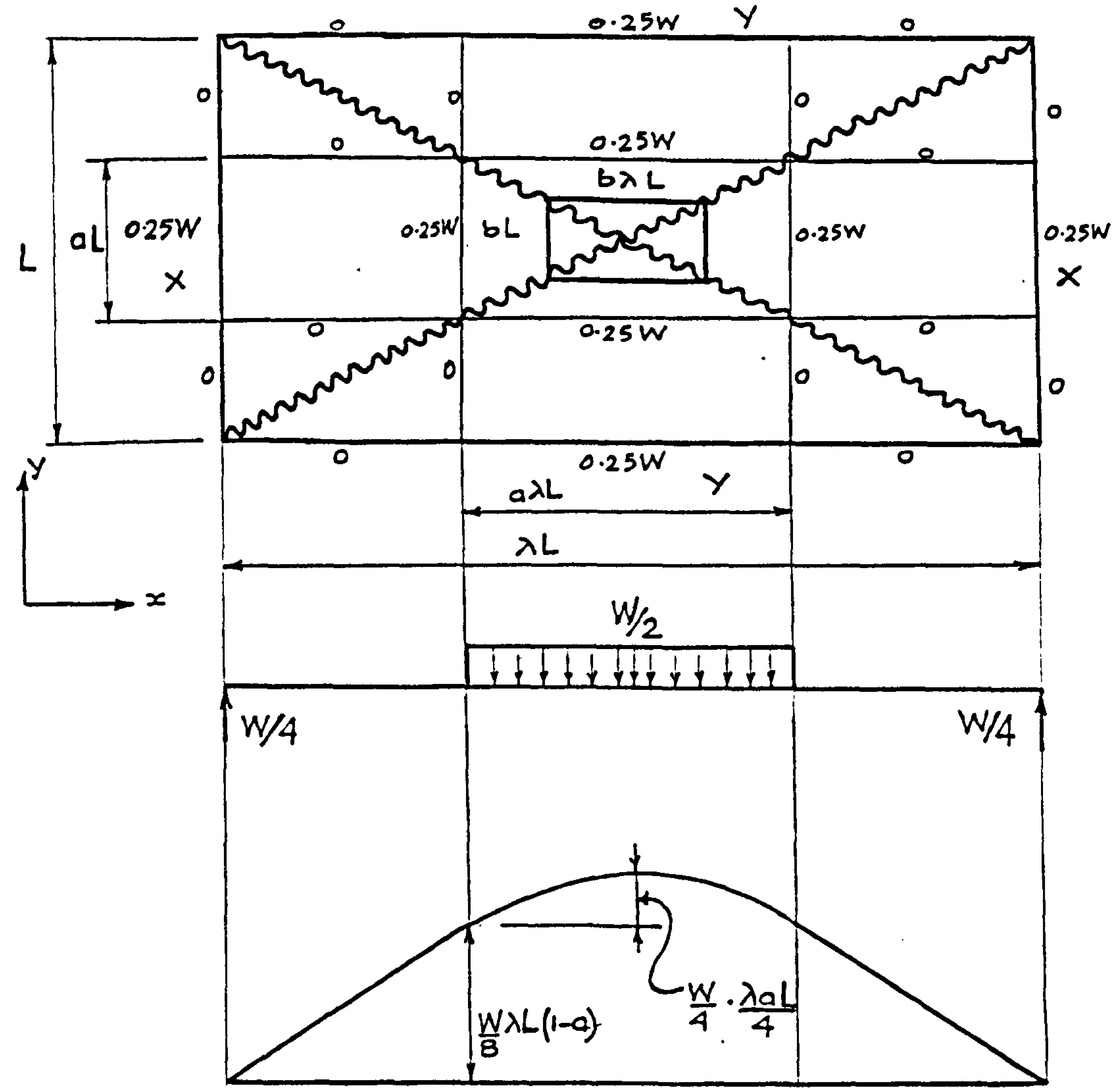


FIG (6.7) SIMPLY SUPPORTED RECTANGULAR SLAB, CARRYING A CENTRAL LOAD, LOAD WIDTH AND LOAD STRIP WIDTHS UNEQUAL

Hence $D = \frac{W}{3}(3-2a)$

Equaling E and D we get

$$\frac{Wc}{W} \leqslant \frac{(3-2a)}{(3-2b)}$$

Clearly equilibrium is satisfied when $a = b$ and then the results are unique. The least value of Wc/W is 1/3 ie when $a = 1$ and $b = 0$ (point load). Further when equilibrium is not satisfied the collapse load can be above or below the design load depending on the values of a and b . It is relevent to note that the loads are considered "flexible" in calculating the external work E ie the deflection of the slab at any point is identical to the deflection of the load at the same point.

6.3. 2 Unequal column widths and column strip widths for corner and edge columns.

The conditions for uniqueness for edge and corner columns were discussed in section 6.2.1. For the example shown in Fig. (6.3) the column reaction is concentrated at the outer edge, the collapse load Wc is given by Eqn. (6.1).

$$\frac{Wc}{W} \leqslant \frac{(1 - p/2)}{(1 - 2q)}$$

If it is assumed that the column reaction is spread over the column grid area then Wc calaulated from Eqn (6.2) is

$$\frac{Wc}{W} \leqslant \frac{(1 - p)}{(1 - 2q)}$$

Where p and q represent the widths of the column strip and the actual width of the column respectively and W is the design load. The collapse load could be above or below W depending on the values of p and q .

6. 4. USE OF SPREADER SYSTEMS TO SATISFY EQUILIBRIUM - INTERNAL LOADS AND INTERNAL COLUMNS .

A method of obtaining uniqueness for the ultimate load of slabs supported by columns at the corners or edges has been established. It is necessary to choose width equal to twice the dimension of the column for the edge strip and in most practical problems this width is sufficeint to accommodate the required reinforcement.

For loads however it has been concluded uniqueness can be achieved by insisting that the dimensions of the loaded grid are identical to the widths of load. When the load dimensions are large it is possible to choose such a layout which will accommodate the required reinforcement.

When the dimensions of the load are small it may not be practicable to accommodate this reinforcement. Supposing as in Fig (6.7) the chosen strip layout is such that the centre grid is larger than the dimensions of the load, the question therefore is how can equilibrium be maintained and thereby ensure uniqueness.

In order to do this a series of load distributions as originally suggested by Hillerborg (34) for the slab in Fig. (6.2.) can be employed.

(a) The central patch load W is distributed half in each of the x and y and directions.

(b) Each half of the load is spread along a band of the same width as the load and length equal to the corresponding dimension of the grid.

(c) The band loads are then transformed into a patch load such the the dimension of the patch load coincide with that of the grid.

(d) Finally this patch load acts on the grid of the same area.

6.4. 1 Internal loads.

These steps are illustrated in Fig. (6.8). Clearly the transformation of the patch load to a uniform load over the grid area will effectively change the moment field within the grid. But this procedure will not change the moment field elsewhere which are produced by step (d) alone.

The ultimate load of the slab can be calculated by the yield line theory. For the same diagonal mechanism the external work E remains unchanged as (see Fig 6.7.)

$$E = \frac{Wc}{3} (3 - 2b)$$

The internal work D is given by (Equation 6.3)

$$D = 4 \left(\frac{m_x}{\lambda L} + \frac{m_y}{L} \right)$$

and the value of m_x in this example is

$$m_x = \frac{W \lambda L}{24} (3-2a)$$

The increase in m_x due to the spreader load systems can be shown to be

$$\frac{\lambda(a - b) M_1 + b \lambda (M_1 + 2M_2)}{3a} + \frac{(M_1 + 2M_2)}{3}$$

$$= \frac{2}{3a} \left[M_1 a + M_2 a + M_2 b \right]$$

Now $M_1 = \frac{W \lambda L}{16a} (a - b)^2$ and $M_2 = \frac{W \lambda L}{16} (a-b)$

The increase on $m_x = \frac{WL \lambda}{12} (a-b)$

and the new value of $m_x = \frac{WL \lambda}{24} (3-2a) + \frac{WL \lambda}{12} (a-b)$

$$= \frac{WL \lambda}{24} (3 - 2b)$$

Similarly the value of $m_y = \frac{WL}{24} (3 - 2b)$

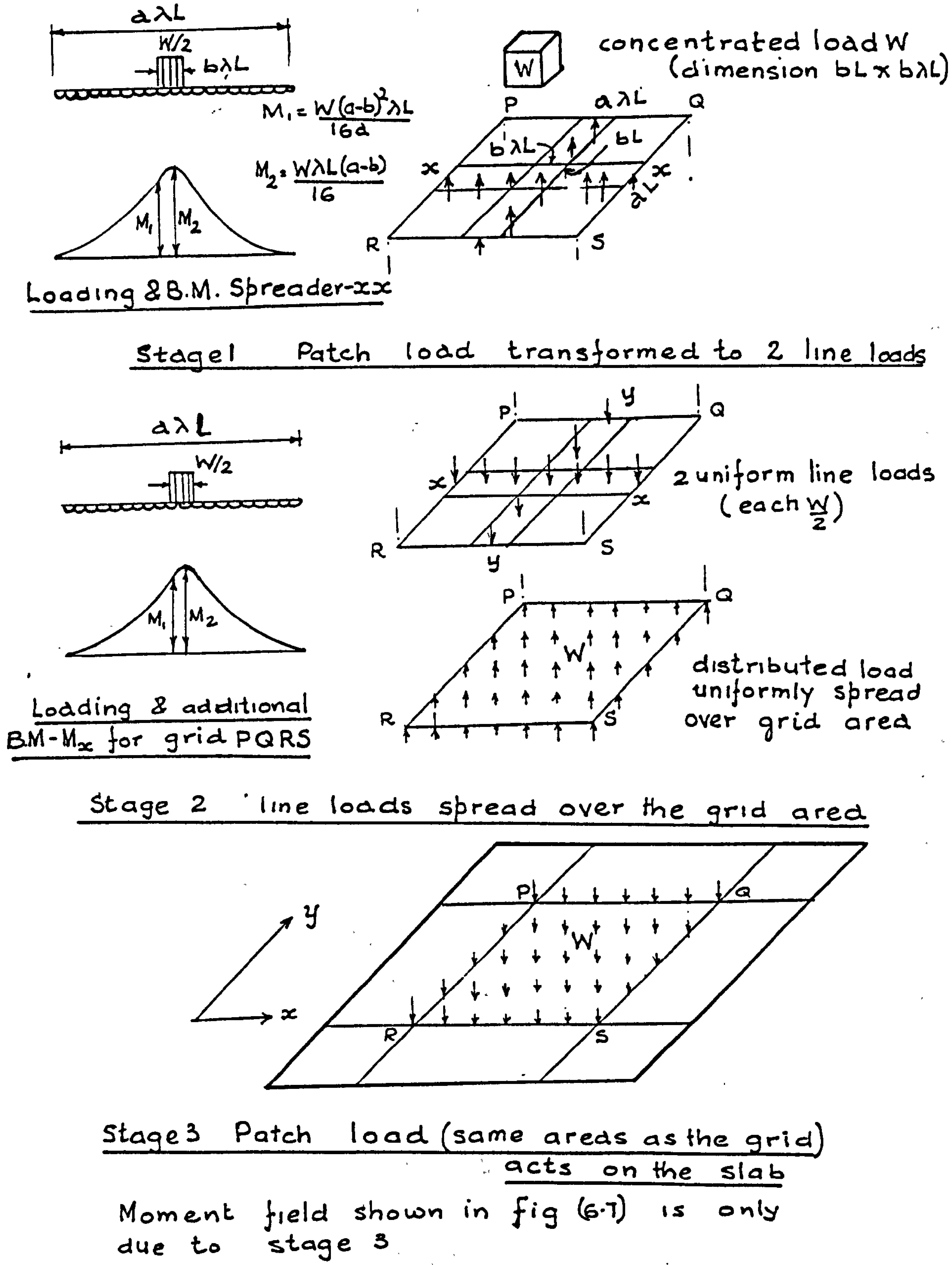


FIG. (6.8) USE OF SPREADERS AND THE STAGES IN THE TRANSFORMATION OF THE CONCENTRATED LOAD TO A UNIFORM PATCH LOAD FOR THE SLAB EXAMPLE SHOWN IN FIG (6.7).

On equating E and D the collapse load W_c becomes identical to W the design load. Since equilibrium and yield conditions are satisfied and the principal moments are positive everywhere there is a multiple of collapse mechanisms all leading to the same value of collapse load.

6. 4. 2. Internal Columns

The use of spreader systems which ensure local equilibrium can now be extended to slabs with internal columns. The use of torsion free spreader systems will provide unique solutions if it is possible to postulate a yield line mechanism in accordance with the rules given in Chapter Four.

Fig. (6.9) illustrates a square slab of side 2L, simply supported along the four edges and carrying a uniform load of q per unit area. In addition the slab is supported by a square column of side 2x at the centre. For simplicity the load distribution factor α (and $1 - \alpha$) is assumed to be 0.5 everywhere. The moment fields are chosen to be positive - positive all over except for the centre area of side 2a, where the moment field is negative - negative. This is attained by properly choosing the values of a, R_A , R_B and R_C such that the bending moment for the strip AA along the edges of the central grid are zero.

Taking moments about the edge of the centre grid

$$R_A (L - a) - qa \left(\frac{L - a}{2} \right)^2 = 0$$

$$\therefore R_A = qa \left(\frac{L - a}{2} \right)$$

Further $2R_A + R_C = qa \; 2L$

$$\therefore R_C = q a (a + L)$$

For the entire slab $4R_A + 8R_B + 2R_C = 4qL^2$

$$\text{and } R_B = \frac{q}{2} L (L - a)$$

Max. moment for strip BB $= qL^2 \left(\frac{L - a}{4} \right)$

To maintain equilibrium within the centre grid the column reaction $2R_C$ is distributed by two spreader strips each carrying R_C and the bending moment in the spreader is negative.

The mechanism shown in Fig (6.9) with positive and negative yield lines conforms to the rules stated in Chapter Four for unique solutions. The external work E for a unit deflection along the edges of the centre grid and zero deflection at the column and at the supports is given by

$$E = q_c \left[(L-a)^2 \frac{4}{3} + 2a\,(L-a)\,\frac{4}{3} + \frac{2}{3}\,(a-y)^2 4 + 2y\,(a-y)\,\frac{4}{2} \right]$$

$$= \frac{4}{3} q_c \left[L^2 + aL - ay - y^2 \right]$$

The total dissipation of energy D is given as a sum of

(1) For positive moments - mean moment $= q\,\frac{(L-a)^2}{12}\,(2L + a)$

Dissipation of Energy $= q\frac{(L-a)^2}{12}(2L + a)\,\frac{8}{L-a} = \frac{2}{3}\,q\,(L-a)\,(2L + a)$

(11) Negative moment

spreader $= q \left(\frac{a + L}{4} \right) (a - y)^2$

centre grid $= \frac{qa}{2}\,(a-y)\,(L-y).\,\frac{2y}{2a} + \frac{qa}{12}\,(a-y)\,(3L-a-2y).2\frac{(a-y)}{2a}$

$$= q\,(a-y)\left[y\,\frac{(L-y)}{2} + \frac{(a-y)}{12}\,(3L-a-2y) \right]$$

Total negative moment $= q\frac{(a-y)}{12}\,(2a^2 - 4y^2 + 6La - 4ya)$

Dissipation of Energy $= q\frac{(a-y)}{12}\,(2a^2 - 4y^2 + 6La - 4ya)\,\frac{4}{a-y}$

and total dissipation of energy $= \frac{q}{3}\left[(2a^2 - 4y^2 + 6La - 4ya) + 2(2L^2 - La - a^2) \right]$

$$= \frac{4q}{3}\,(L^2 + aL - ay - y^2)$$

The external work done E and the dissipation of internal energy D are equal and the strip method design load is the unique

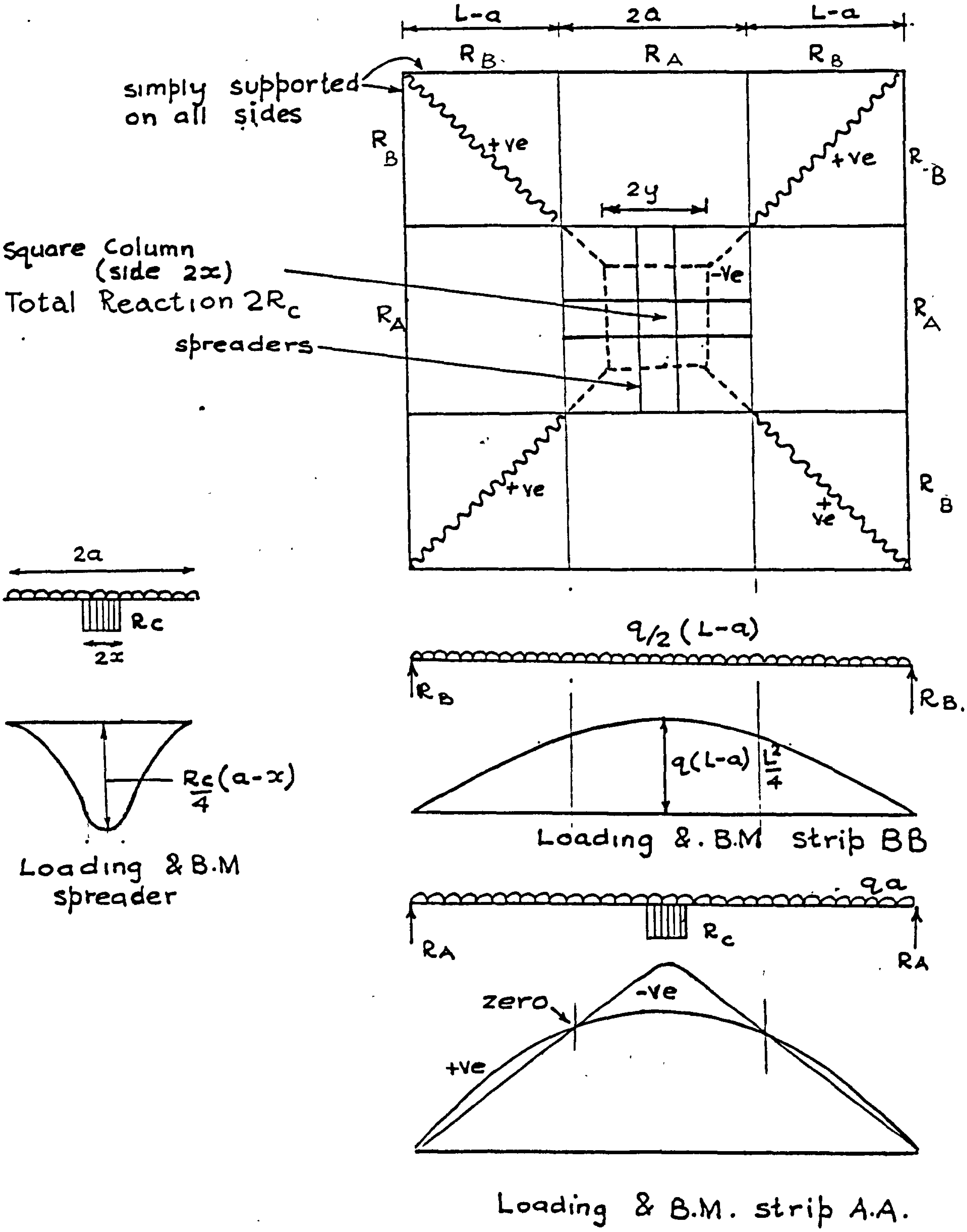


FIG (6.9) SQUARE SLAB CARRYING A UNIFORMLY DISTRIBUTED LOAD, SIMPLY SUPPORTED ON ALL SIDES AND AT THE CENTRE BY A SQUARE COLUMN collapse load as predicted. For this example there are multiple collapse mechanisms all giving the same unique failure load.

Therefore we arrive at a very important conclusion for internal point (patch) load or internal supports. A strip layout is chosen such that the load or the column is at the centre of an internal grid. The forces are distributed within the internal grid area by a spreader system which ensures that equilibrium is satisfied at all points of the slab.

The slab is then designed for a moment field which is the sum of that due to the uniform distribution of load or reaction over the internal grid area and that due to the spreader system within the grid area.

In such cases of slab design it is possible to draw a collapse pattern that conforms to the rules set out in chapter four and for such mechanisms the collapse load will be identical to the strip method design load.

6.5 USE OF SPREADER SYSTEMS - EDGE AND CORNER LOADS.

When the load or the column is within the slab it is possible to choose a strip layout such that the load or the column is at the middle of a grid. However when the load or column is at an edge or a corner the layout of the spreader system is more restricted.

6.5.1. Edge Loads.

Consider the rectangular slab shown in Fig (6.10). The slab is simply supported along the two long edges of length λL and carries a point load at the middle of one of the other two free edges of length L. The slab is diveded into two equal strips along the x direction. Due to the position of the load the spreader system SS is restricted to span only in the y direction. The spreader system, the loading and bending moment diagrams for each strip are also shown. The load is first spread to produce a uniform line load in the y direction,

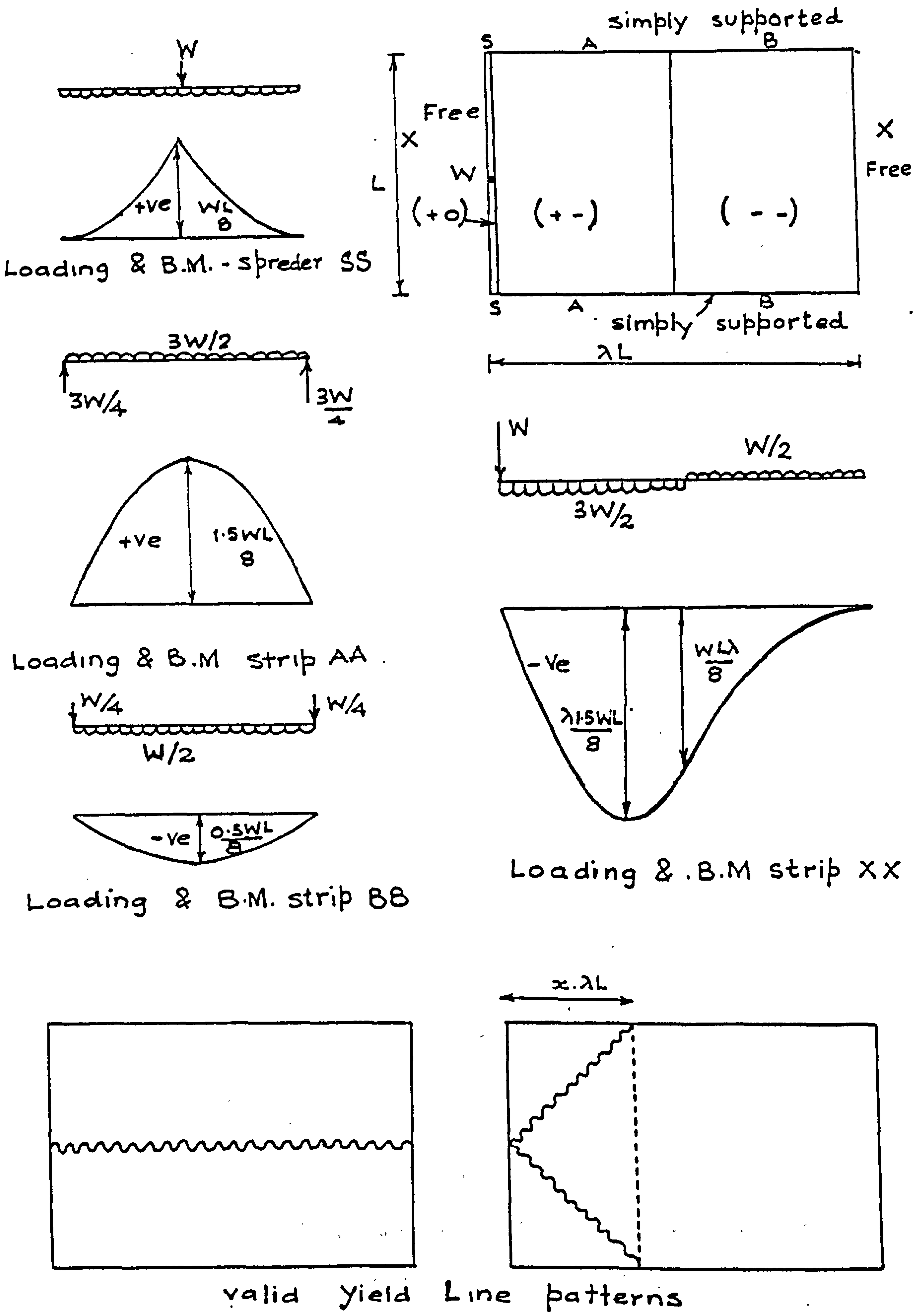


FIG (6.10) RECTANGULAR SLAB SIMPLY SUPPORTED ALONG THE TWO LONG EDGES AND CARRYING A POINT LOAD AT THE MIDDLE OF ONE OF THE OTHER TWO FREE EDGES.

which in turn produces a concentrated line edge load on the x strips. In order to produce overall equilibrium the strip AA is assumed to carry a load of $\left(\frac{3W}{2}\right)$ downwards and the strip BB a load of $\left(\frac{W}{2}\right)$ upwards each uniformly distributed. The bending moments along the strip AA and spreader SS are positive, and those along strips BB and XX are negative. The collapse load of the slab is calculated by

(a) Central line mechanism (parallel to simple supports at the middle of the slab).

$$\text{External Work} \quad E = Wc\,.1$$

$$\text{Internal Work} \quad D = \left[\frac{WL}{8} + \frac{WL}{8} \times 1.5\right]\frac{4}{L}$$

$$= 1.25\,W$$

Equating E and D

$$Wc \leqslant 1.25\,W$$

(b) For the yield line pattern shown in Fig (6.10b)

The External Work $E = W_c \times 1$

Dissipation of internal Energy D

$$= \left(\frac{WL}{8}\right) \times \frac{4}{L} + \left(\frac{2}{3}\,1.5\frac{WL}{8}\right) \cdot \left(\frac{x\,\lambda\,L}{0.5\,\lambda\,L}\right).\frac{4}{L} + W\,\lambda\,L(x-1.5x^2).\frac{1}{x\lambda L}$$

$$= \frac{W}{2} + Wx + W - 1.5Wx = W\left[\frac{3}{2} - \frac{x}{2}\right]$$

Equating E and D

$$W_c \leqslant W\left[\frac{3}{2} - \frac{x}{2}\right]$$

This solution is valid for $0 < x < 0.5$ and the minimum value of W_c is equal to 1.25 W, when $x = 0.5$. It can be shown that the collapse load W_c is greater than 1.25W when $x > 0.5$. Therefore for this particular layout of strips a lower bound solution is produced. This is because it is not possible to postulate a yield line mechanism in accordance with the rules for uniqueness given in chapter four.

It is however possible to obtain a unique solution for this slab problem by altering the strip layout. As shown in Fig(6.11), strips AA and BB are now of width zero and λL respectively. Consequently the total load carried by strip AA is W and that by BB is zero.

The bending moment for spreader system SS is unaltered and the parabolic moment diagram for strip AA has a maximum value of $\frac{WL}{8}$. Strips XX and BB will not cater for any bending moment. The combined loading and the bending moment of SS and AA are also shown. The moment field in AA, SS are positive - zero and in BB, XX it is zero - zero. Under these conditions there will be a multiple of collapse patterns all producing the collapse load identical to the strip method design load.

To illustrate the effects of spreader system in a more general moment field consider the square slab shown in Fig (6.12) carrying a point load W at the middle of the free edge. The side opposite this free edge is built in and the other two are simply supported. The load is distributed along the free edge by a spreader system SS. The positive moment in the spreader system has a maximum value of $\frac{WL}{8}$. The line load so produced acts at the edge of the slab producing a negative principal moment all over the slab with a maximum value of W per unit width. The principal moment normal to the simple supports is zero for the entire slab. Thus the moment field for the slab is negative - zero and for the spreader system SS is positive - zero.

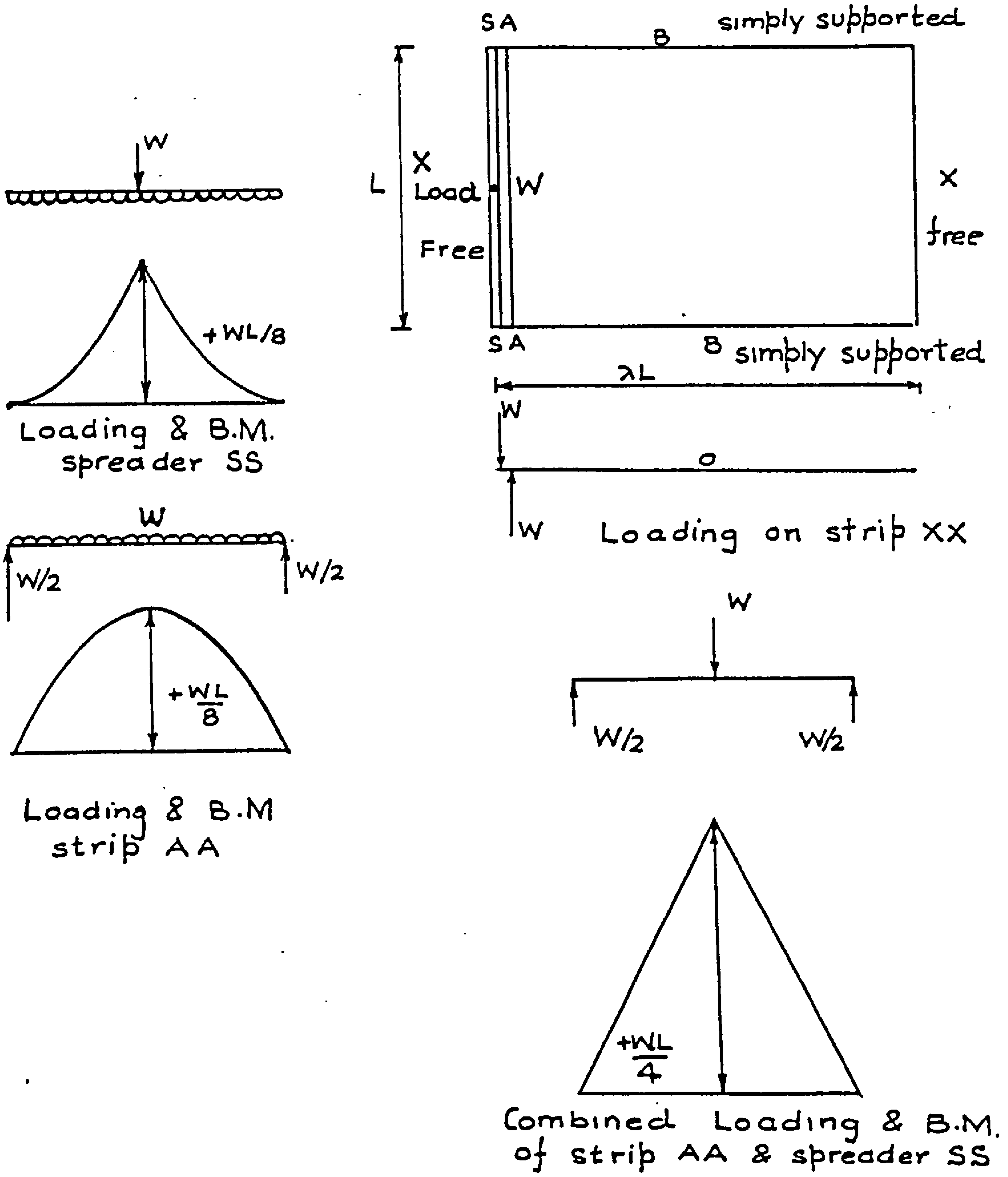
For the yield line pattern shown in Fig (6.12)

External work $E = W_c \times 1$

Internal Energy $D = \frac{W}{L} \cdot \frac{x^2}{2} \cdot \frac{1}{x} + Wy \cdot \frac{1}{y}$

$$= W\left(1 + \frac{x}{L}\right)$$

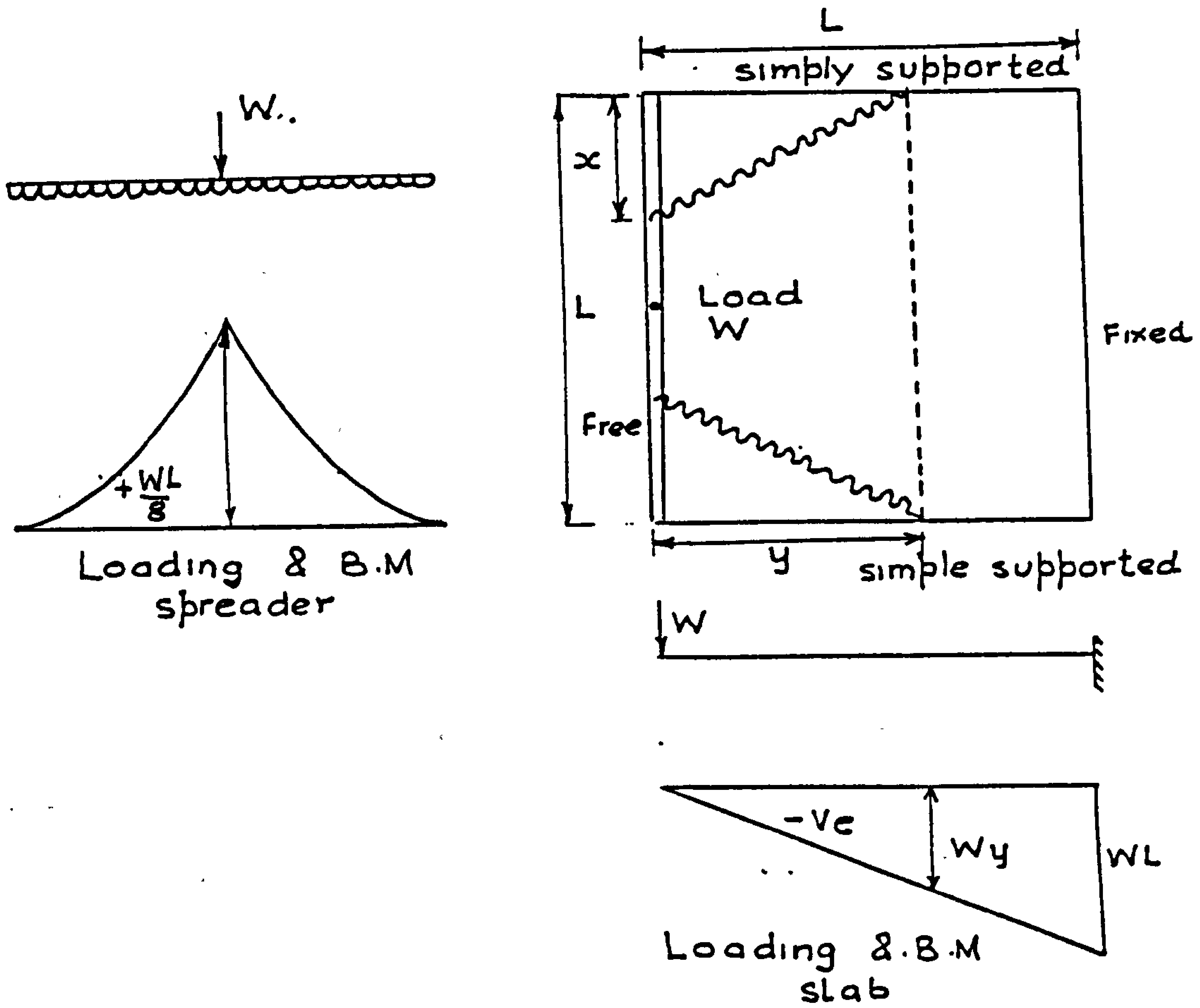
The minimum value of $W_c = W$ is when $x = 0$. Similar to the example in Fig. (4.4.) the collapse pattern agrees with the conditions for



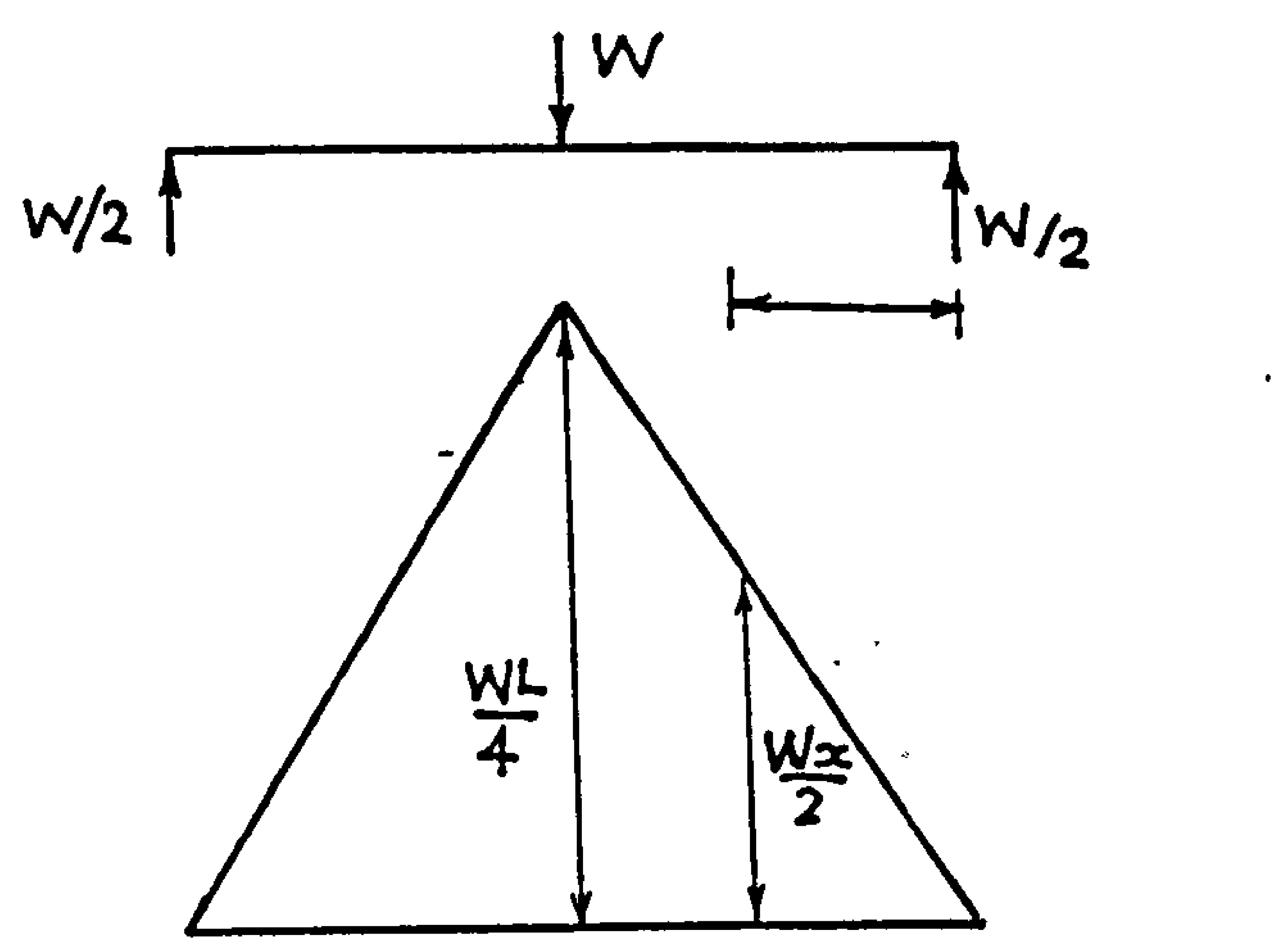
Note :- (i) There is no bending in strip XX

(ii) Strip B.B carries no Load, hence no bending

FIG (6.11) ALTERNATIVE STRIP LAYOUT TO PRODUCE UNIQUE SOLUTIONS FOR THE SLAB EXAMPLE SHOWN IN FIG (6.10).



(a)



(b) Alternative design

FIG (6.12) UNIQUE SOLUTIONS FOR A SLAB CARRYING A POINT LOAD AT THE MIDDLE OF THE FREE EDGE. THE SIDE OPPOSITE THIS EDGE IS BUILT AND THE OTHER SIDES ARE SIMPLY SUPPORTED uniqueness although the pattern is not strictly kinematically admissible.

Alternatively if the spreader is designed to take the entire bending moment ie the spreader has a triangular bending moment diagram with a maximum value of $\frac{WL}{4}$ at the centre as shown in Fig (6.12.b) and the remainder of the slab is free of moment, then from the previous arguments it can be concluded that there will be an infinite number of collapse mechanisms all producing the unique collapse load.

6.5.2. Corner Loads.

To conclude the special treatment of loads consider the square slab of side L shown in Fig 6.13. Two adjacent edges are built in and the remaining two edges are free. The slab carries a point load at the corner formed by the two free edges. Equilibrium is maintained through two spreader strips SS, each carrying one half of the load W. The strips AA and BB are of width L/2 and Fig (6.13) also shows the loading and bending moment diagrams for the strips AA, BB and spreader SS. The yield line pattern shown is one which conforms to the uniqueness rules given in Chapter 4, and analysis of this mechanism confirms that a unique solution is obtained.

$$\text{External Work} = W_c x 1$$

$$\text{Dissipation of internal Energy} = \frac{2WL}{16} \times \frac{2}{L} + 2.3\,\frac{WL}{16} \cdot \frac{2}{L}$$

$$= W$$

which shows that the collapse and design loads are identical.

6.6. APPLICATION OF THE STRIP DEFLECTION METHOD OR NO TORSION GRID METHOD FOR SLAB DESIGN WITH POINT LOADS AND POINT COLUMNS.

It is now possible to extend the strip deflection method described in chapter five to cover point loads and point columns. To obtain safe solutions with internal columns and point or patch loads will in general require spreader systems which ensure that local equilibrium is maintained However with edge or corner columns safe

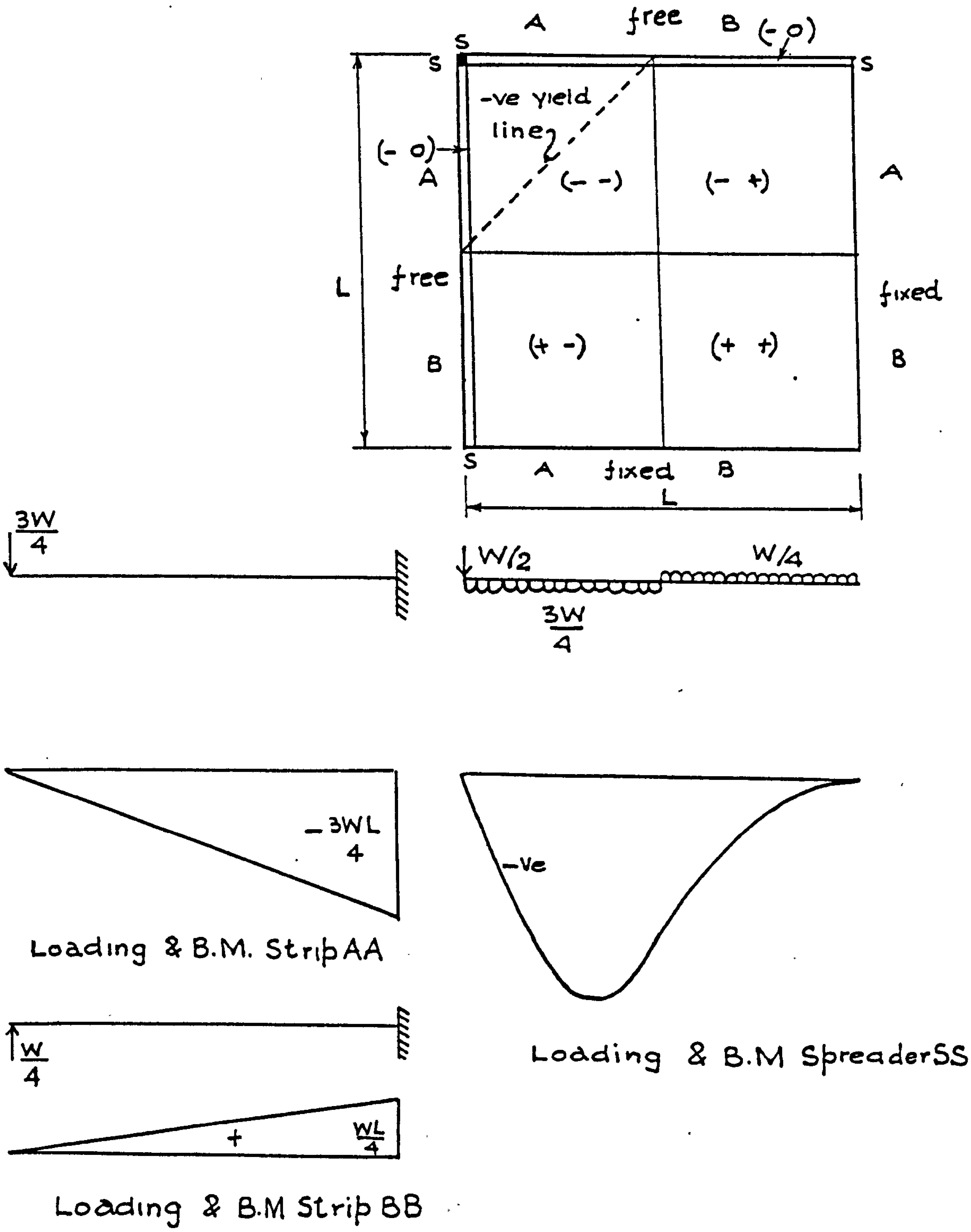


FIG (6.13) UNIQUE SOLUTIONS FOR A SLAB FIXED ALONG TWO ADJACENT EDGES AND CARRYING A POINT LOAD AT THE CORNER FORMED BY THE OTHER TWO FREE EDGES.

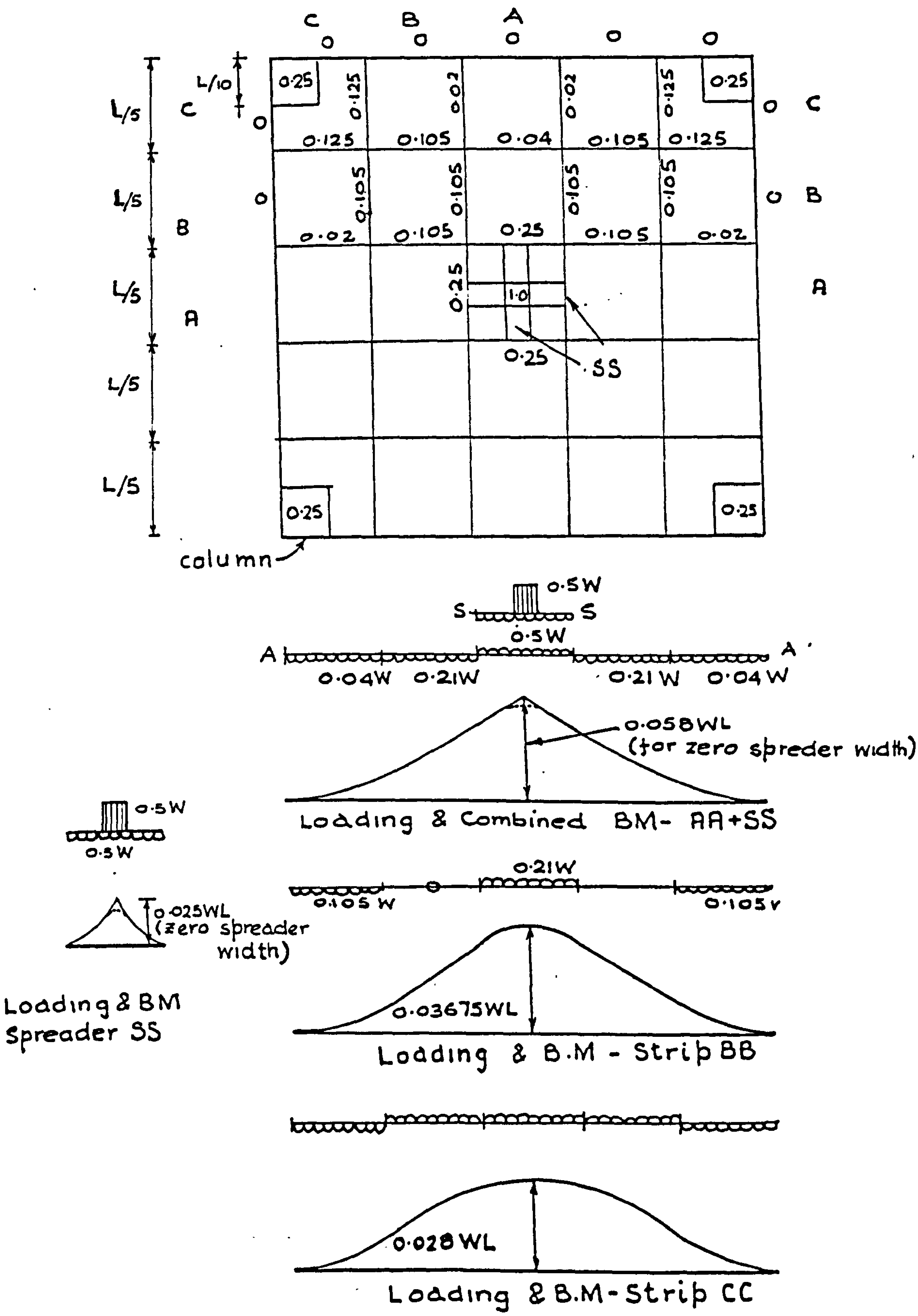


FIG (6.14) APPLICATION OF STRIP DEFLECTION METHOD FOR A SQUARE SLAB SUPPORTED AT THE FOUR CORNERS AND CARRYING A CENTRAL LOAD.

solutions can be obtained by choosing the edge strip containing the column to have a width equal to twice the dimension of the column and assuming that the reaction is spread uniformly over the column grid.

To illustrate the application of the strip deflection consider the square slab side L carrying a central patch load W as shown in Fig (6.14). The slab is supported by four identical square columns assumed to be of side (L/10). The first step would be to choose a column strip of width $2 \times \frac{L}{10} = \frac{L}{5}$ and assume that the designer decided to choose a strip layout with five equal strips each way.

The vertical shear force distribution diagram shown in Fig (6. 14) was obtained by assuming a uniform spread of column reaction over the centre grid. Safe conditions prevail over the column grid area and equilibrium conditions are maintained over the rest of the slab except in the central loaded area. The spreader system SS transforms the patch load first to a line load and next spreads it over the grid area. The loadings and bending moments in the slab strips and the spreader are also shown and these must be added where relevent. Analysis of this slab again will show that the design load and the collapse load are identical.

6.7. CONCLUSIONS.

A generalised method of designing slabs with patch loads and patch columns has been presented. Generally local equilibrium must be satisfied at all points including the boundaries for the strip method to produce safe solutions. It is not always practicable to have a layout of strips to fit the position and dimensions of columns and loads and thereby to satisfy equilibrium. Procedures have been derived to produce spreader systems for loads and internal columns. These spreaders will ensure that equilibrium is satisfied and the collapse load of such slabs will be unique or lower bound depending on the possibility of postulating a yield line pattern which conforms to the rules given in chapter four.

With edge or corner columns however safe solutions can be obtained by choosing a strip layout such that the width of the column strip is twice the dimension of the column and assuming that the column reaction is spread uniformly over the column grid. In most practical problems this width is sufficient to accommodate the required reinforcement.

If the above conditions are not fulfilled, the collapse load and the proposed strip method design loads will differ. The ratio will depend on the chosen strip layout and an extreme example was presented where the collapse load was only a third of the design load.

CHAPTER SEVEN

EXPERIMENTAL TESTS ON REINFORCED CONCRETE SLABS

7.1 INTRODUCTION

In the earlier chapters the theory of both the strip method and the strip deflection method were presented in a manner suitable for practical design. In order to investigate some aspects of the theory and design recommendations, tests were carried out on nine square reinforced concrete slabs. The testing procedure and comparison of experimental results with the corresponding analytical predictions will be presented in this chapter.

During the loading of the slab, whenever possible the following items were recorded.

(1) Deflection of the slab under the applied load.

(11) The loads corresponding to the first visible crack and the developement of the cracks.

(111) Mechanism of failure and a photograph of the pattern of the slab.

(1V) Maximum applied load.

Because of factors such as casting, curing, handling and the size of the testing rig, the dimensions of all slabs was fixed at 800 mm x 800 mm x 50 mm. The overall span to depth ratio was 16. The length and breadth of the slabs were such that they were about 1:6 scale models and Clark (35) has shown that with such models it is not possible to draw wholly reliable conclusions regarding the crack widths in full scale slabs. Therefore it was realised at the outset of this investigation that the primary aim was to check the ultimate load behaviour rather than the serviceability conditions although some information is provided on the latter.

7.2 MATERIALS

(a) Concrete

In all the slabs a medium workability mix was used. The design of the mix was based on Table 5 CP114 (4). For a 19 mm (3/4 in) maximum aggregate size the mix had the following propotions.

Water : cement ratio	0.5
Coarse aggregate : cement ratio	1.8
Sand : cement ratio	1.3
Total aggregate : cement ratio	3.1

Preliminary tests carried out on 12, 152.4 mm (6 in) cubes and made from the specified mix gave 28 day crushing strength of just over 40N/mm^2. Therefore this strength (40 N/mm^2) was used in the design of slabs and for the provision of steel reinforcements.

The aggregates were cleaned, washed and dried for 24 hours before use. The three 152.4mm (6 in) cubes cast with the slab were cured with the slab for 28 days in water. The slab and the cubes were tested on the same day. Table 7.1 gives details of cube strength of the concrete for each test.

(b) Reinforcements

The overall depth of the slabs was 50mm. The cover to the reinforcement was 12.5mm (1/2 in). Severe restrictions had to be placed on the diameter of the bars in order to keep the effective depth of the slab as large as possible and reasonably constant in both directions of the two directions of span. Further a smaller diameter bar had the advantage of a smaller bond length so 2.38mm diameter bright mild steel bars were considered suitable.

The stress - strain characteristics of this steel was determined using a Hounsfield Tensometer. Friction grips were used to apply the tensile force which was measured directly from a mercury column.

TEST NO	AGE AT TEST (DAYS)	CUBE STRENGTH N/mm^2	AVERAGE CUBE STRENGTH N/mm^2
1	28	44.1 44.5 44.2	44.3
2	28	37.2 38.6 38.9	38.2
3	28	37.8 38.6 39.3	38.6
4	28	43.9 44.7 44.7	44.4
5	28	43.1 43.6 42.5	43.1
6	28	43.0 42.9 42.0	42.6
7	28	44.8 42.1 44.8	43.9
8	28	38.8 40.3 41.7	40.3
9	28	41.8 42.5 42.0	42.1

TABLE 7.1 Properties of Concrete in the test slabs

The extension of 50.8mm (2 in) length under test was measured using a mechanical extensometer. Fig (7.1) shows the average of four stress - strain curves. The bright mild steel does not show a definite yield point but does have a long yield plateau. It was decided to take the yield stress of steel as the 0.2% proof stress, which has the value 615 N/mm^2. The strain corresponding to this yield point is 0.00496.

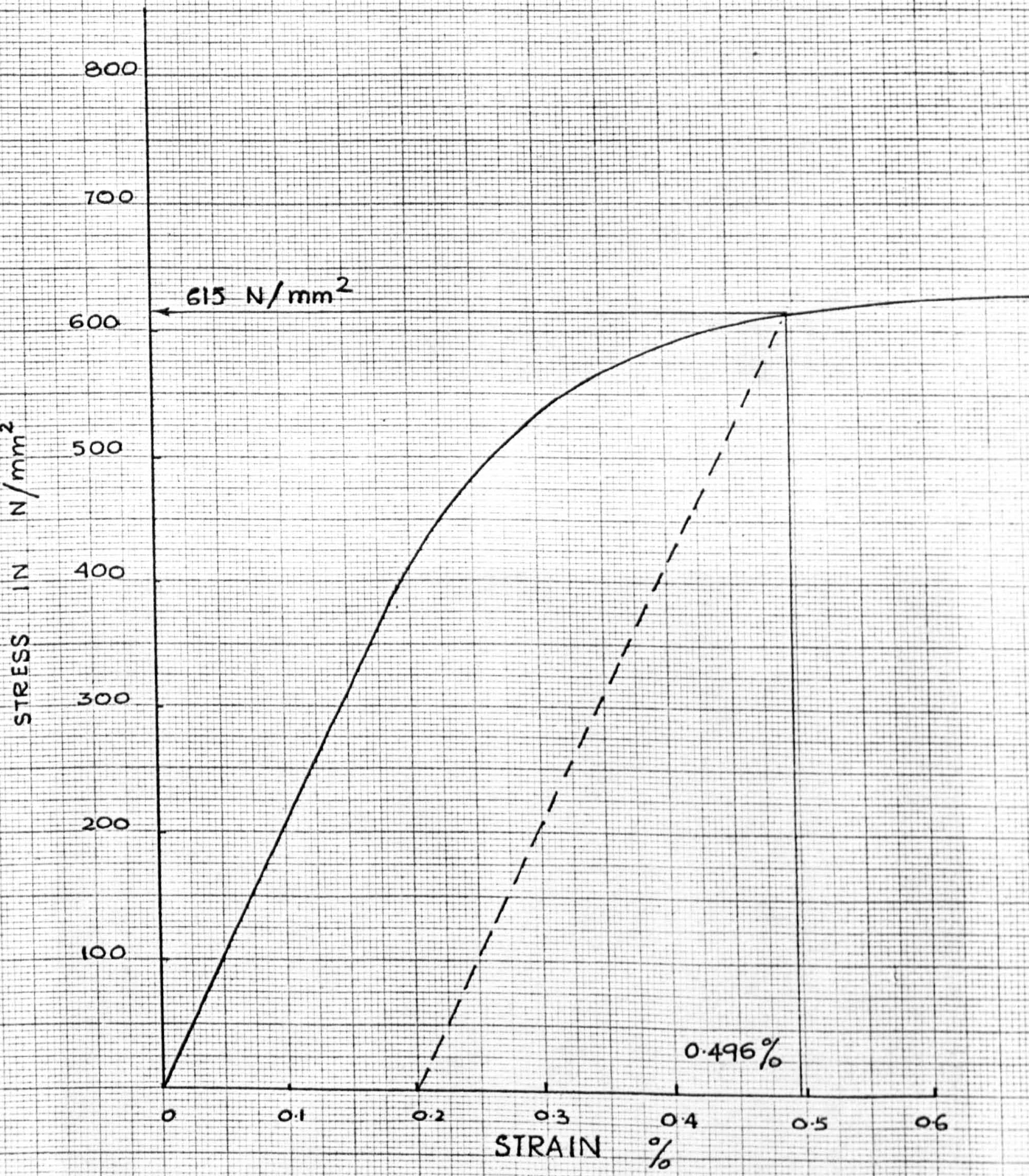
7.3 CASTING AND TESTING OF SLABS

The slabs were cast on the mould shown in Photo (7.1). The bottom of the mould was covered with Fablon to prevent water entering the wooden mould and to produce a smooth under surface on the slab. 12.5mm (1/2 in) plastic cover blocks were used to position the steel reinforcements. The concrete was mixed for 2 minutes, placed carefully in the mould and compacted well on the vibrating table.

The general arrangement for the tests is shown in Photo (7.2). The slabs were tested vertically to permit easy observation of cracks as they developed during the experiment. The undersides of the slabs were covered with an even coating of white emulsion to facilitate the observation of these cracks.

Each corner of slab in Tests Nos. 1 to 8 was supported against transverse displacement by a 25mm diameter steel ball as shown in Fig (7.2 a). In plane movements were allowed by the use of two steel plates placed between the steel ball and the concrete slab. The mating surfaces of the plates and the surface of the ball seat were well greased. Slab No 9 was simply supported at two edges. The support sketched in Fig (7.2b) has a 25mm diameter bar and a 45^{o} - V groove. A 6.3mm thick steel plate was plastered to the back of the slab and the mating surfaces were well greased to facilitate the in plane movements.

In all tests loads were applied horizontally using an hydraulic jack and a 50 mm (2 in) diameter ball seat to spread the load.



AVERAGE STRESS/STRAIN CURVE FOR REINFORCING BARS

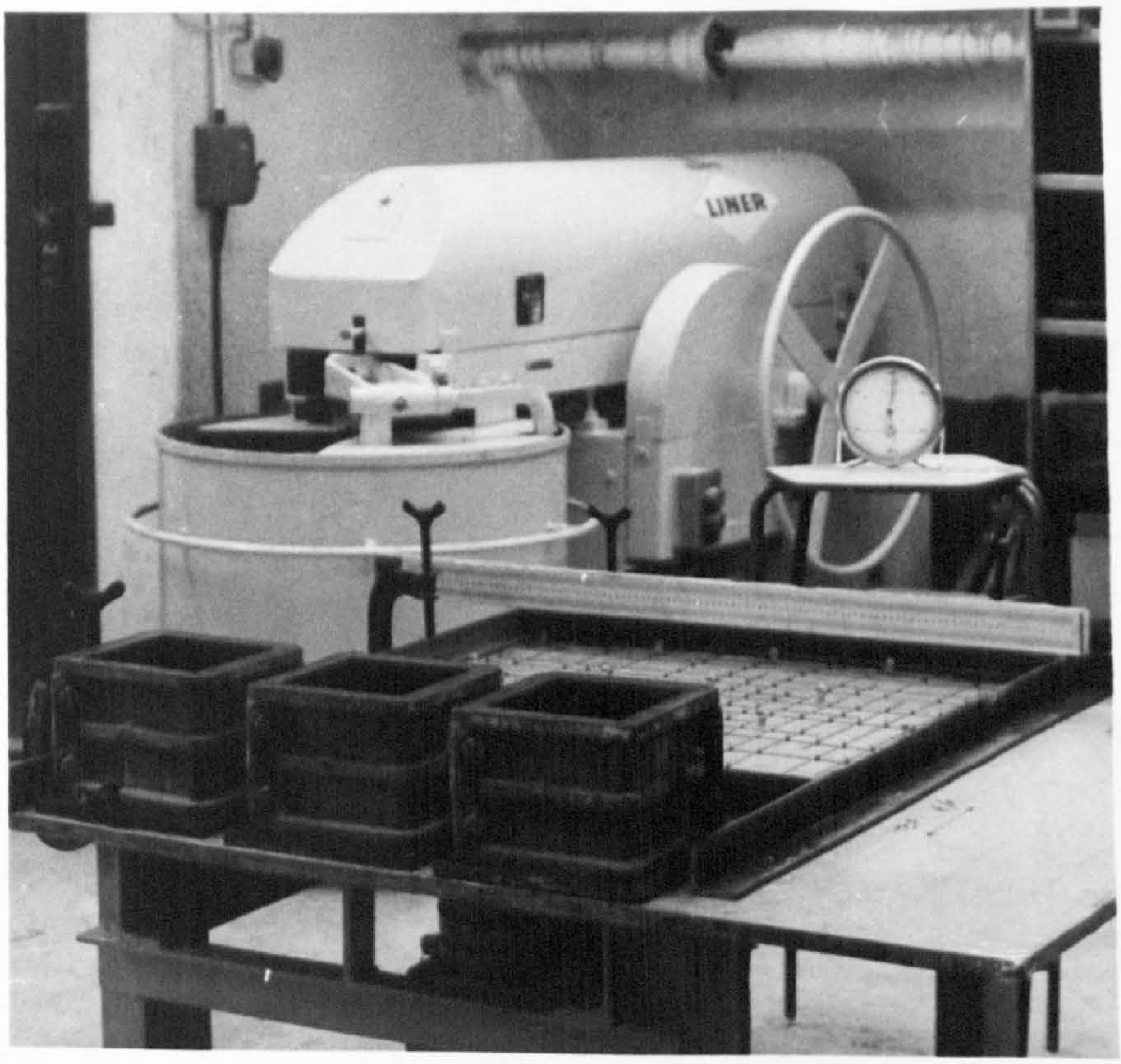


Photo 7.1.

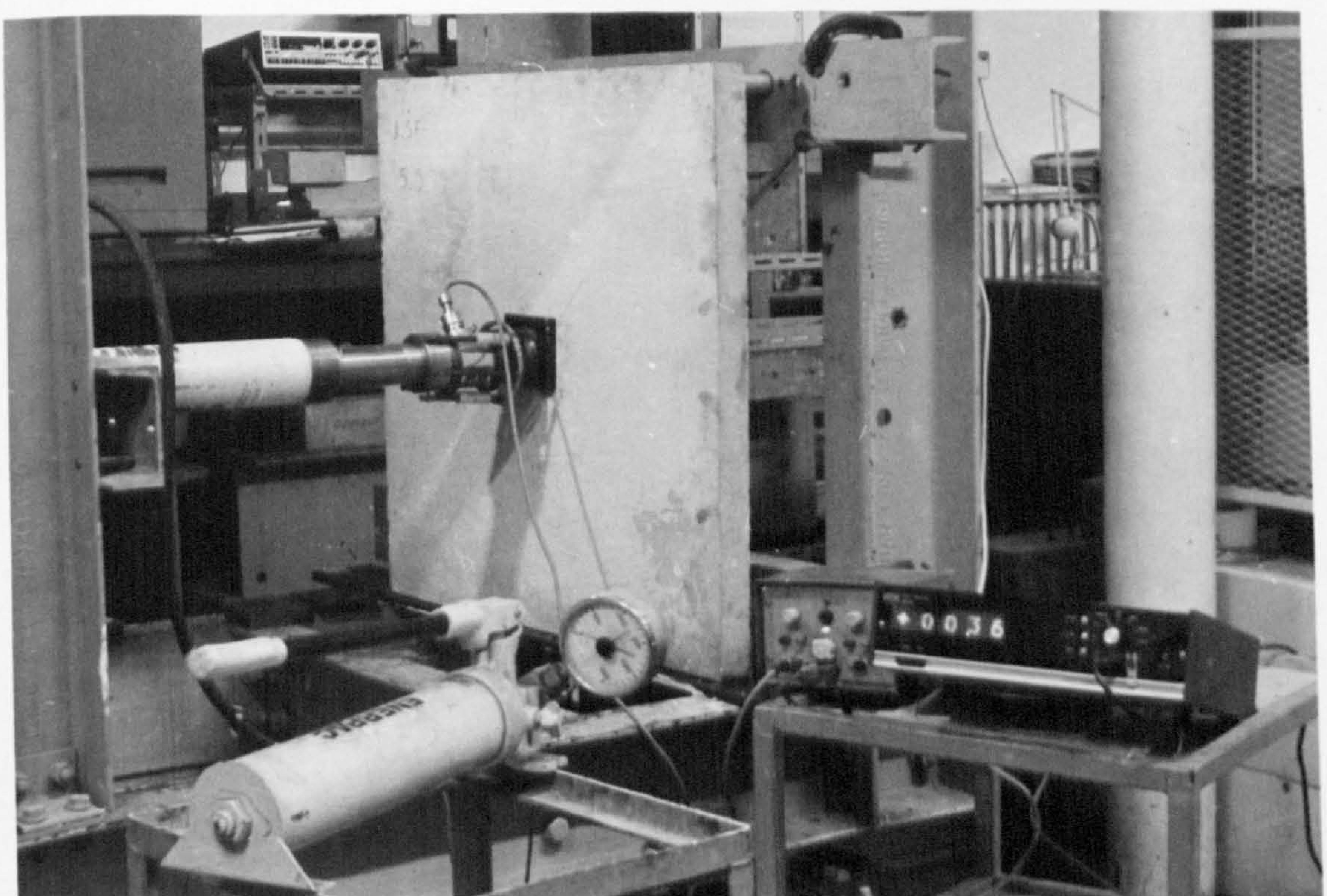
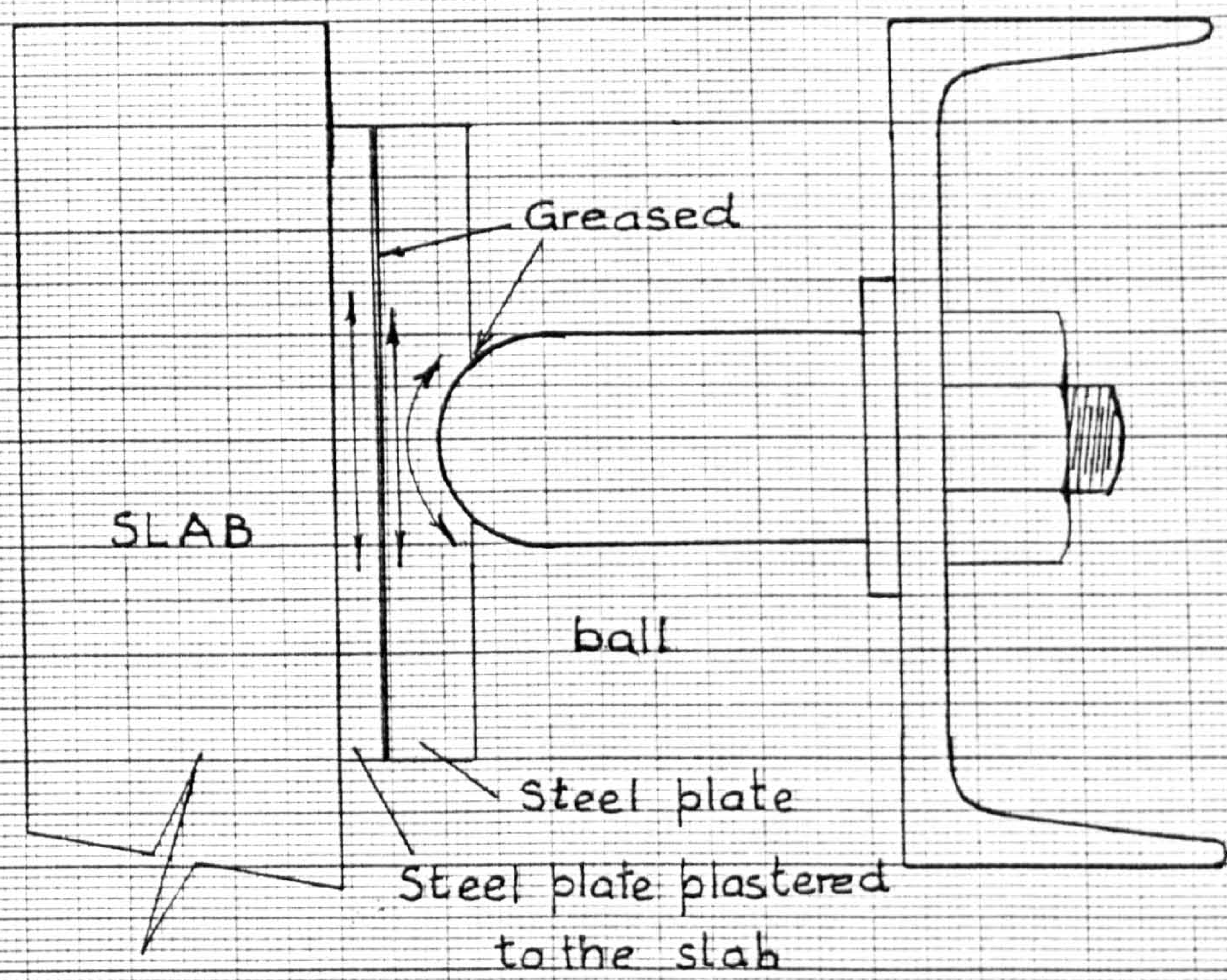
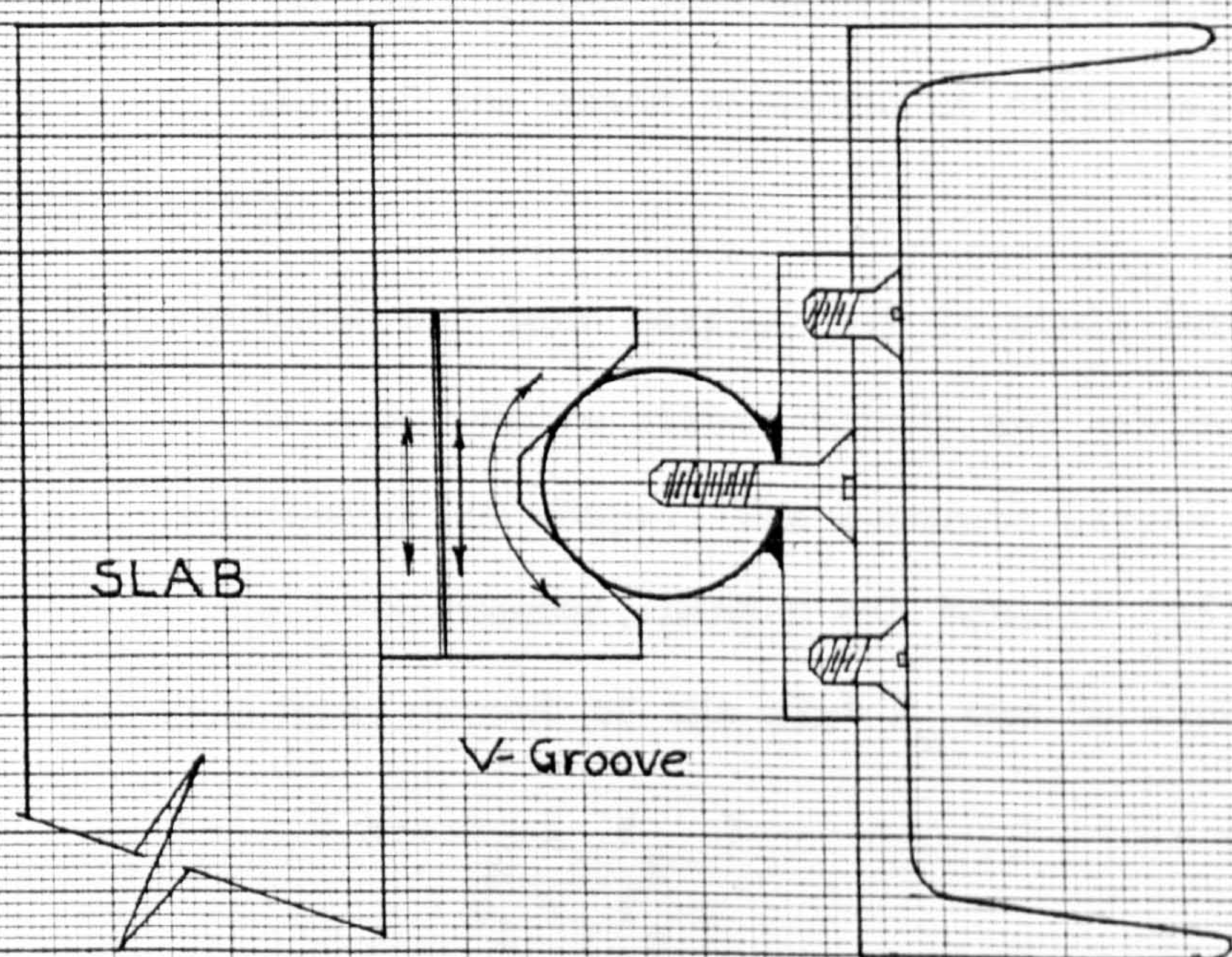


Photo 7.2



(a) BALL SUPPORT CAPABLE OF SLIDING AND ROTATION



(b) SIMPLE SUPPORT CAPABLE OF SLIDING AND ROTATION

FIG. (7.2) SUPPORTS

Slabs in Tests Nos.1-5 were subjected to a patch load and a piece of wood of the required dimensions was glued to the front of the slab as a local spreader. The loading arrangement can be seen in Photo 7.2.

The applied load was measured using a load cell and a digital volt meter. The load cell, the voltage stabliser and the digital voltmeter were accurately calibrated before the test programme and the same set was used for all the tests. The deflections of the slab under the applied load were measured using dial gauges. The readings of the digital voltmeter and and the dial gauge were recorded at desired intervals. At each stage of loading the underside of the slab was examined using a magnifying glass for any cracks under the illumination supplied by spot lamp.It was not intended to measure the crack widths whilst loading but the first visible crack were of the order of 0.03mm width.

The slabs were loaded until the load passed the peak value. All cracks visible on the surface were marked using a felt pen and photographs were taken of the crack patterns at collapse.

7.4 DESIGN OF SLABS

Much research has been carried out to study the characteristics of the concrete compressive stress blocks and various methods are available to approximate its shape. The concrete stress distribution adopted here is due to Hognestad, Hanson and McHenry (36) and the corresponding yield principal moment of the slab M_u can be expressed as

$$M_u = A_s f_y \left(d - \frac{k_2 A_s f_y}{k_1 k_3 f_c b} \right) \qquad (7.1)$$

where

A_s is the area of tension reinforcement in width b

f_y is the yield stress of reinforcements = 615N/mm^2

f_c is the crushing strength of concrete

b is the width of slab strip

d is the effective depth to the reinforcement.

Factors k_1, k_2, k_3 define the magnitude and position of the resultant compressive force. The variation of these factors with the cylinder compressive strength (f_c) is given in (36).

Initially the value of cube strength = 40 N/mm^2 has been assumed to determine the steel reinforcement in each slab strip. The design load (W_D) and the theoretical failure load given in Table (7.2) are however based on the average crushing strength of cubes cast with each slab. The relationship between the theoretical failure load and the design load will be discussed in detail for each test in the next section.

7.5 DETAILS OF SLAB TESTS.

7.5. 1 Unique design by the strip deflection method

(a) TEST No 1. (Slab No 12)

This test was aimed at investigating the behaviour of a slab designed to give a unique collapse load. The slab has five strips of widths 100 mm, 250 mm, 100 mm, 250 mm, and 100 mm each way and was designed by the strip deflection method. The strip layout, the vertical shear force distribution pattern and the position of supports are shown in Fig (7.3).

In chapter 6.2 it was shown that by assuming a uniform spread of column reaction over the column grid area and choosing a column strip width equal to twice the column width it was possible to obtain a unique solution. Accordingly in the test a column strip width of 100mm and column width of 50mm were chosen. The errors due to loads were eliminated by choosing a square central grid of dimension 100mm x 100mm and gluing a piece of wood of same dimensión so as to distribute the applied

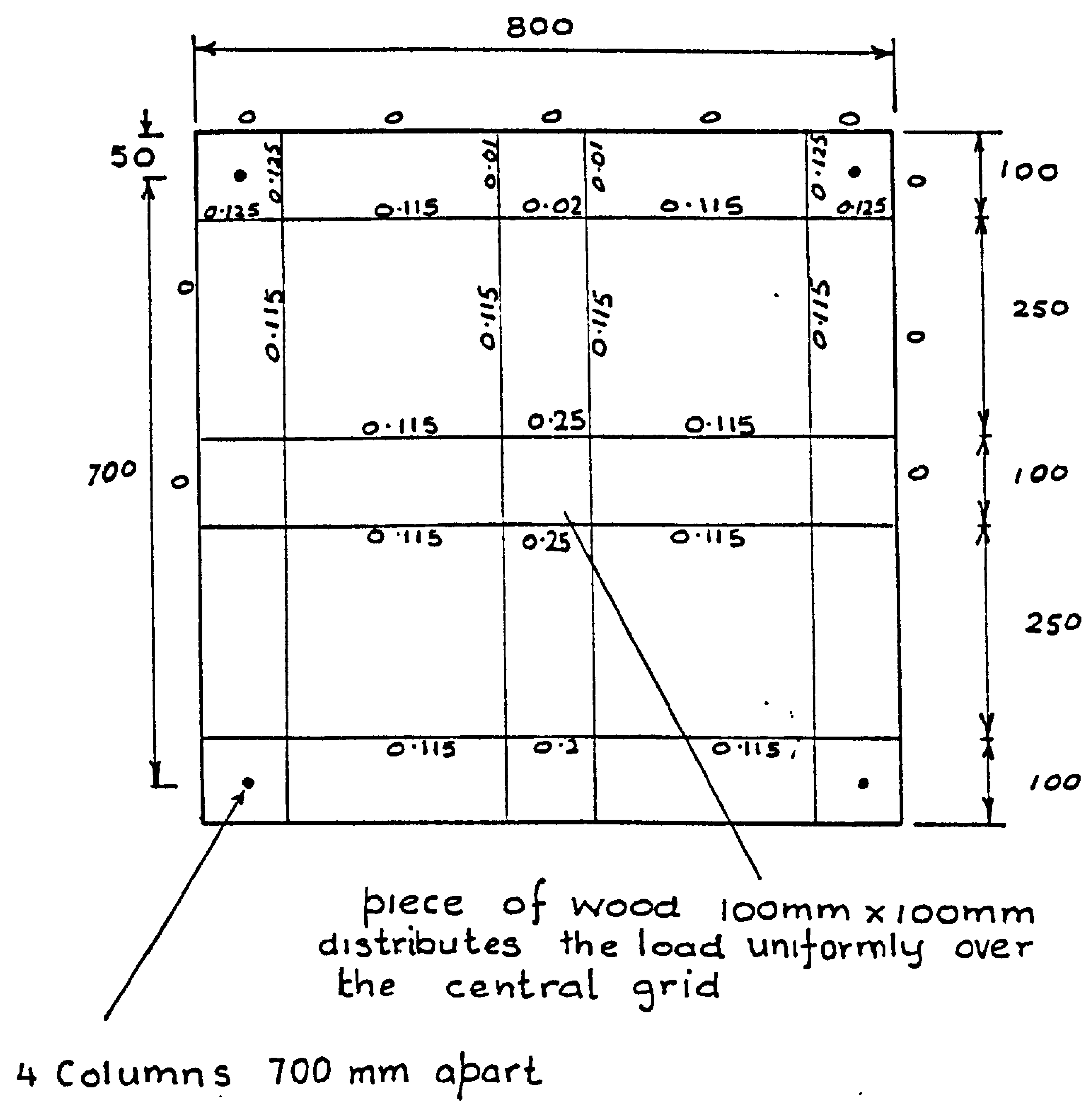


FIG 7.3 STRIP LAYOUT, POSITION OF COLUMNS AND VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR TEST 1.

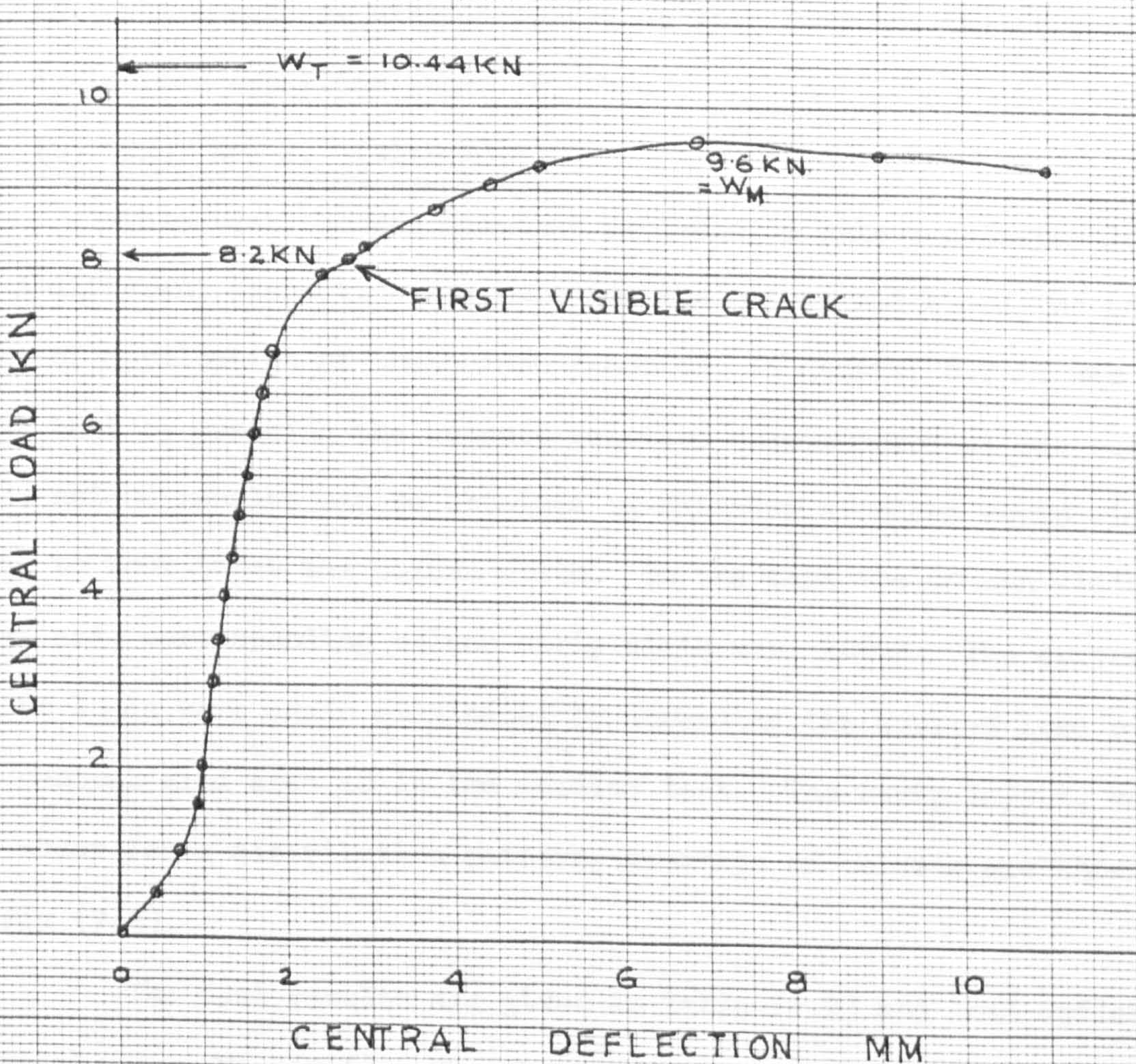


FIG 7.4 LOAD DEFLECTION CURVE FOR TEST 1

load uniformly over the grid area.

The results of the tests are given in Table (7.2) and load deflection curve in Fig (7.4). The slab failed by a central yield line mode. The theoretical failure load (W_T) was 10.44KN and the maximum applied load W_M was 9.6 KN (92% W_T). The first crack appeared at 8.2 KN (78.5% W_T) and the slab has deflected 2.7mm (span ÷ 300) at this stage.

The collapse mode agreed with the one predicted. The difference between the theoretical and experimental maximum loads is within the accuracy obtainable. If partial safety factors of 1.15 and 1.5 are allowed for steel and loads the combined factor is 1.725. Then the working load for the slab can be established as (W_T ÷ 1.725) or 6.05 KN. There were no visible cracks at this load and the corresponding deflection was 1.6mm (span ÷ 500).

(b) TEST No. 2. (Slab No.5)

This test too was aimed at producing a unique collapse load but a different assumption was made for the column width and the assumed point of application of the column reaction. In Chapter 6.2 it was shown that by assuming the support reaction concentrated at the outer edge, unique results can be obtained by choosing a column strip width equal to four times the column width. In the test the column width was 50 mm and corresponding column strip width was 200mm. The errors due to loads were again eliminated by spreading the applied load over the central grid area as in the earlier test. The strip layout, vertical shear force distribution pattern, position of columns are shown in Fig (7.5).

The test results are given in Table (7.2) and load deflection curve shown in Fig (7.6). The theoretical failure load (W_T) and maximum applied load (W_M) were 12.03KN and 11.7KN (W_M = 97.3% W_T) respectively.

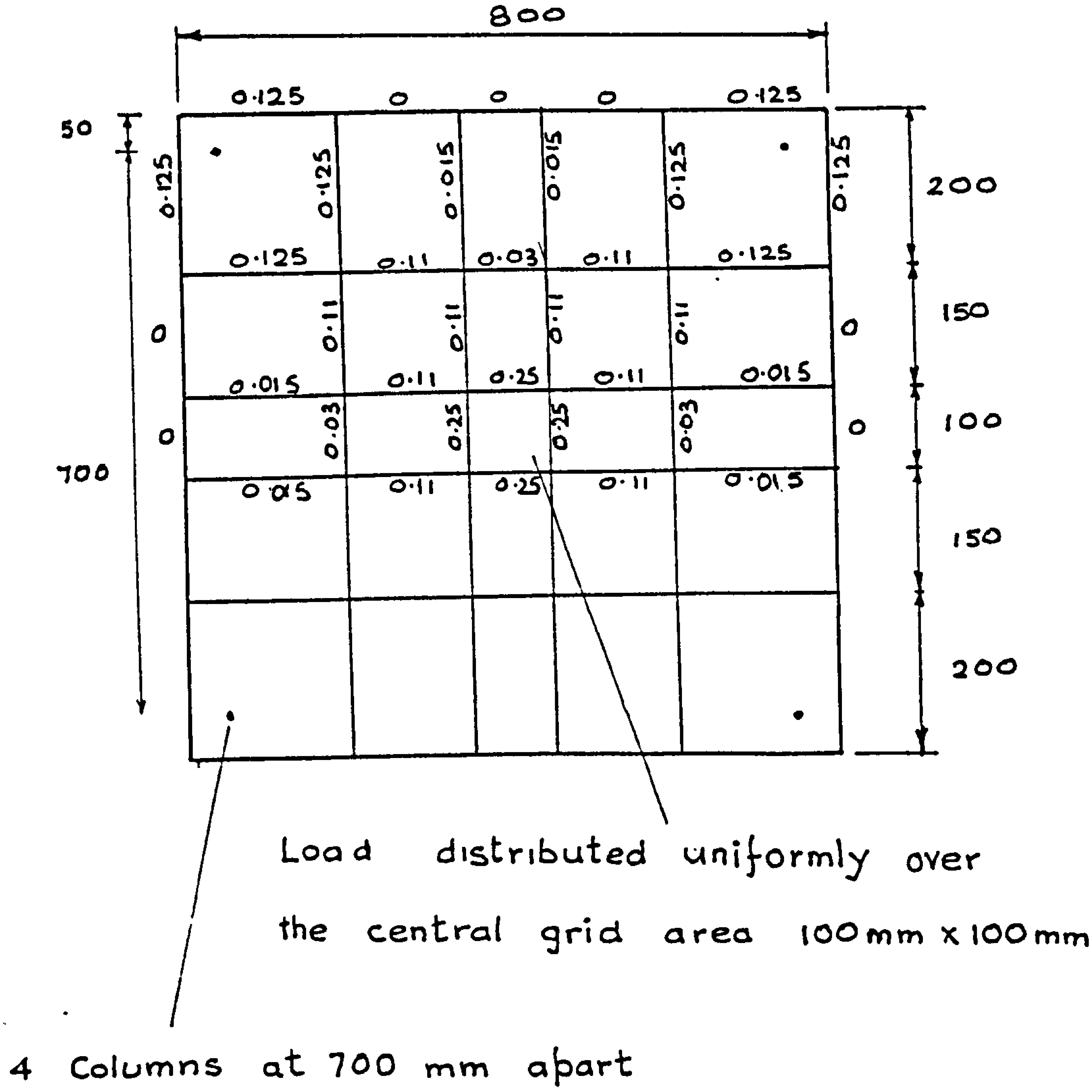


FIG 7.5 STRIP LAYOUT, POSITION OF COLUMNS AND VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR TEST 2.

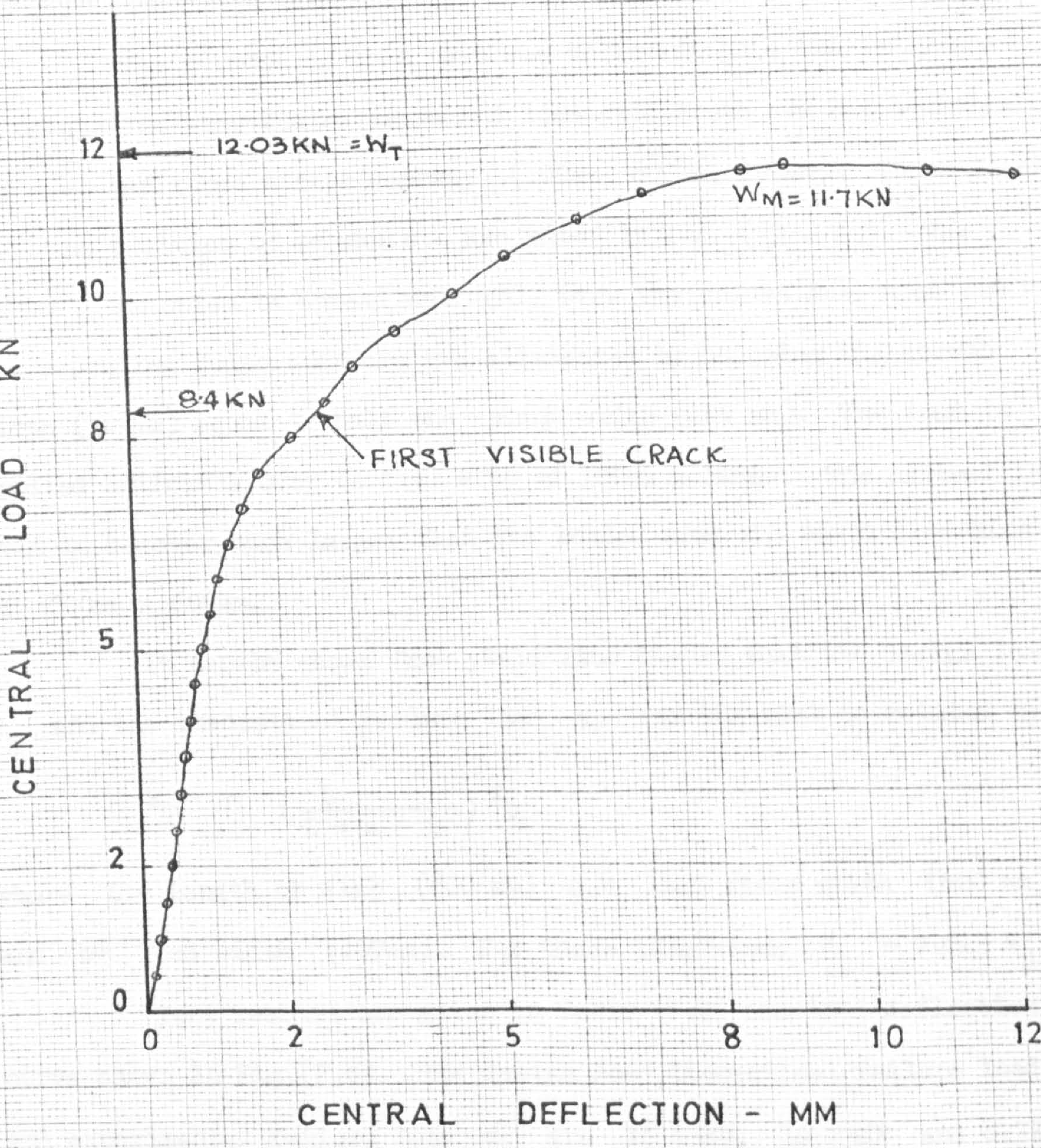


FIG 7.6 LOAD DEFLECTION CURVE FOR TEST 2

Clearly these values agree remarkably. The first crack appeared at 8.4KN (70% W_T) and the corresponding deflection was 2.6mm (span ÷ 308). The slab failed by a central yield line mode the one predicted by the design and the crack pattern at failure is shown in Photo (7.3). The working load of the slab can be established as 6.98KN (W_T ÷ 1.725). There were no visible cracks at this load and the corresponding deflection was 1.5mm (span ÷ 533)

7.5. 2 Errors due to load assumptions

TEST No. 3 (Slab No.7)

The purpose of this test was to investigate the influence on uniqueness when loaded areas and load grid areas are not identical. The slab has three strips eachway (100mm, 600mm, 100mm). These together with the position of columns etc are shown in Fig. (7 . 7). The column reaction is spread uniformly over the column grid area and corresponding support errors are eliminated by choosing the column width (50 mm) equal to half the strip width (100 mm). The load is applied uniformly over central area of 100mm x 100mm. The central load grid is however much larger than the loaded area and has a dimension of 600mm x 600mm.

It can be shown from yield line theory that the design load W_D and the theoretical failure load W_T for the slab is related by

$$W_T = \left(\frac{L}{L + a - b}\right) W_D$$

where L = length of slab (800 mm) a = load strip width (600 mm) and b= load width (100mm). For these dimensions W_T = 0.615 W_D.

Test results are given in Table (7.2) and load deflection curve shown in Fig (7.8). The design load, theoretical failure load and experimental failure load W_m are 16.63KN, 10.23KN and 8.7KN (85% W_T). The ratio $W_M : W_D$ obtained in the experiment was 0.523

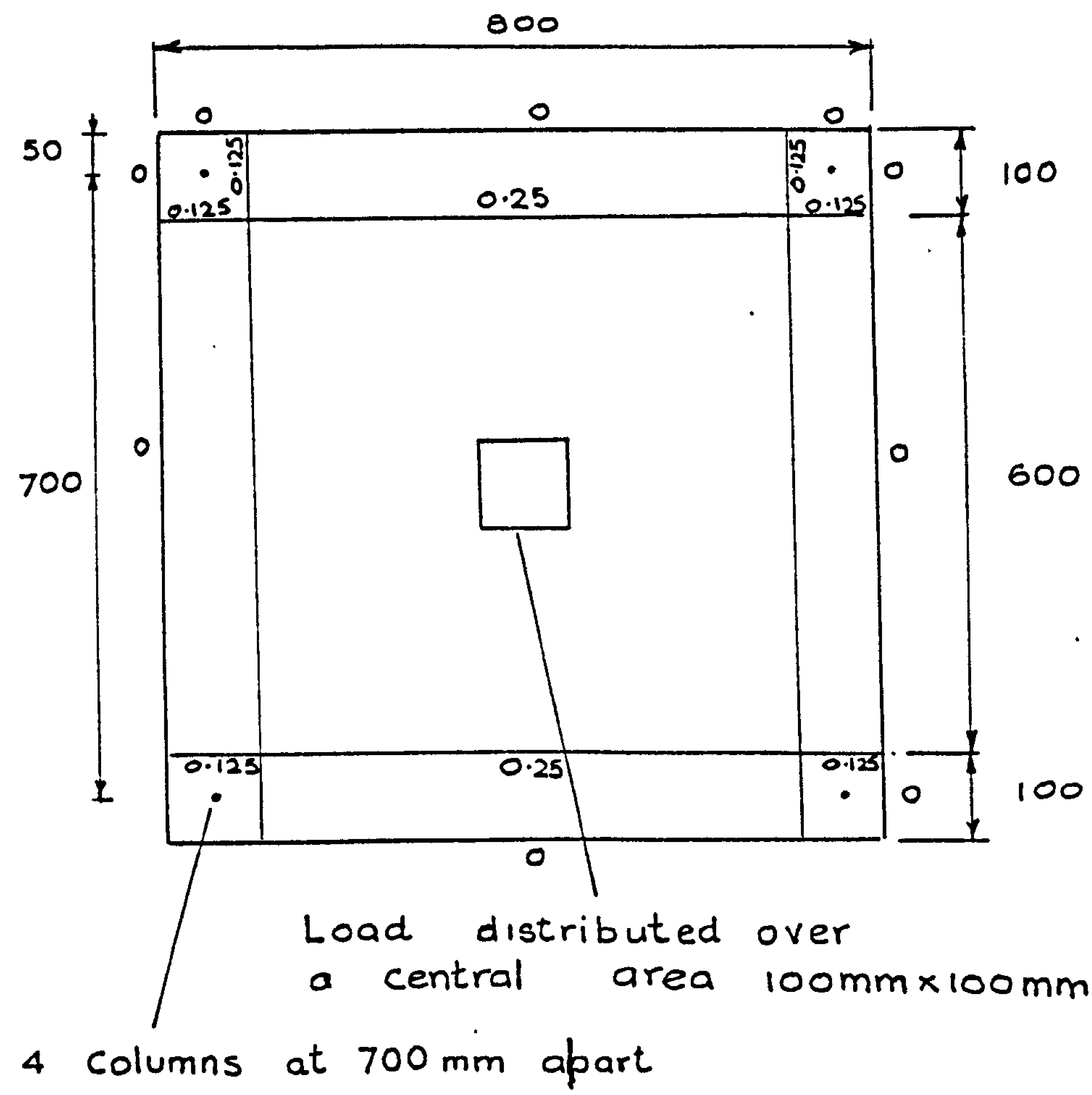


FIG 7.7.STRIP LAYOUT, POSITION OF COLUMNS AND VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR TEST 3

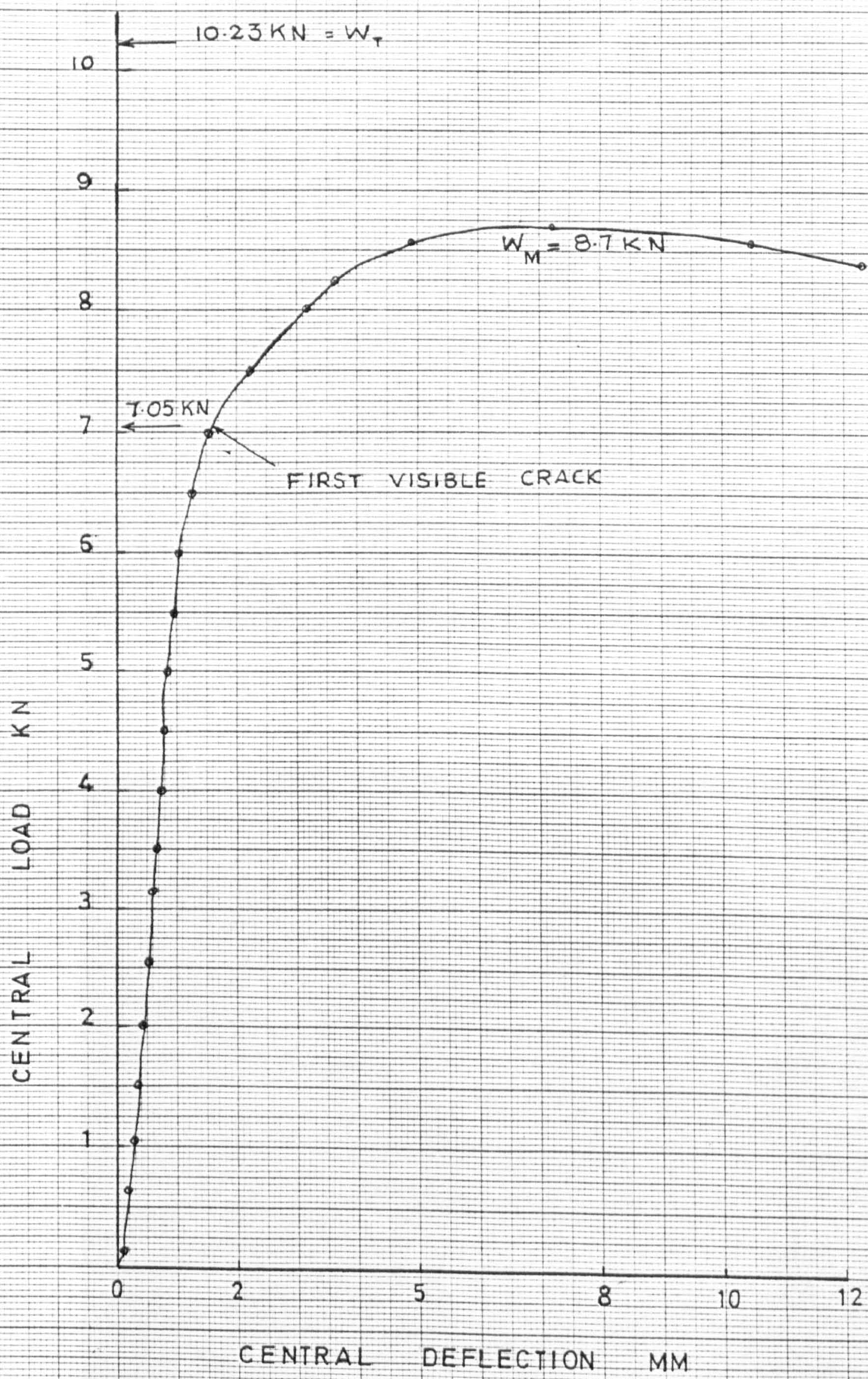


FIG 7.8 LOAD DEELECTION CURVE FOR TEST 3

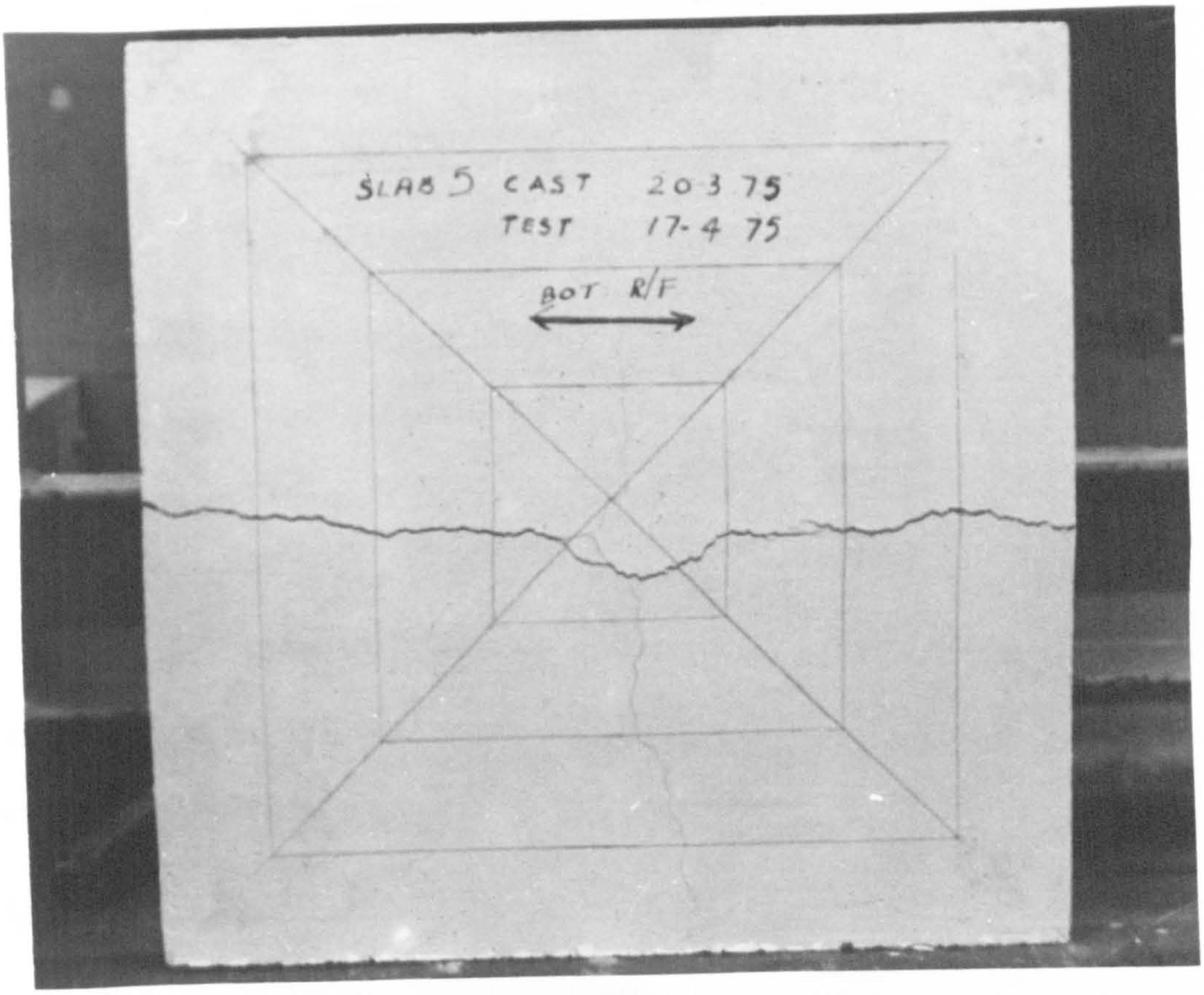


Photo 7.3

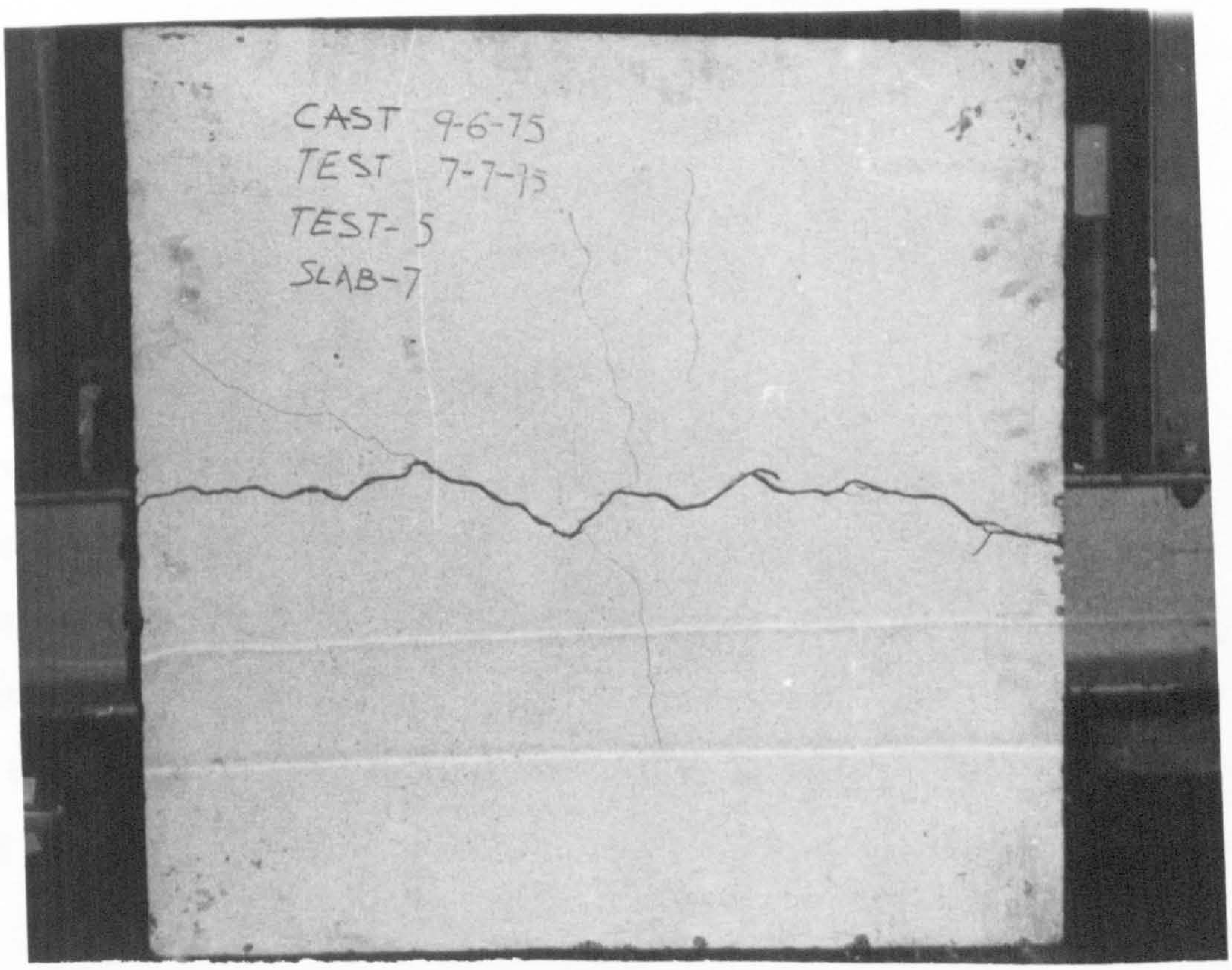


Photo 7.4

which is about 15% below the expected ratio . The first crack appeared at a load of 7.05 KN (69% W_T) and at this stage the slab has deflected 1.6mm (span ÷ 500) photo 7.4 shows the central yield line pattern.

7.5. 3 Errors due to support assumptions

The next two tests were designed to investigate errors produced in the collapse load when the column strip width and column width differ from the conditions required for uniqueness. In both tests the central grid is 100mm x 100mm and a piece of wood of these dimensions glued to the front of the slab distribute the applied load uniformly to eliminate any uniqueness errors due to applied load.

(a) TEST No. 4 (Slab No. 11)

Fig (7.9) shows the strip layout position of column etc for this test. The column reaction is assumed to be spread over the column grid area. If the column strip width (3.50mm) expressed as a ratio to the span (800mm) is p and similarly the column width (50mm) as a factor q then it was shown in Chapter 6.2 that the design load W_D and the theoretical collapse load W_T are related by

$$W_T = \left(\frac{1 - p}{1 - 2q}\right) W_D$$

substituting for p and q gives $W_T = 0.643\ W_D$. Test results are given in Table (7.2) and load deflection curve shown in Fig (7.10). The design load W_D, theoretical collapse load W_T and the experimental maximum applied load W_M are 10.60KN, 6.81KN and 7.40KN (108.7% W_T). The ratio $W_M : W_D$ obtained in the experiment is 0.698 (about 8.7% above the expected ratio). The first crack appeared at a load of 6.7KN (98.3% W_T) and the corresponding deflection of the slab being 2.4mm. (span ÷ 330). The slab failed by a central yield line mode as anticipated.

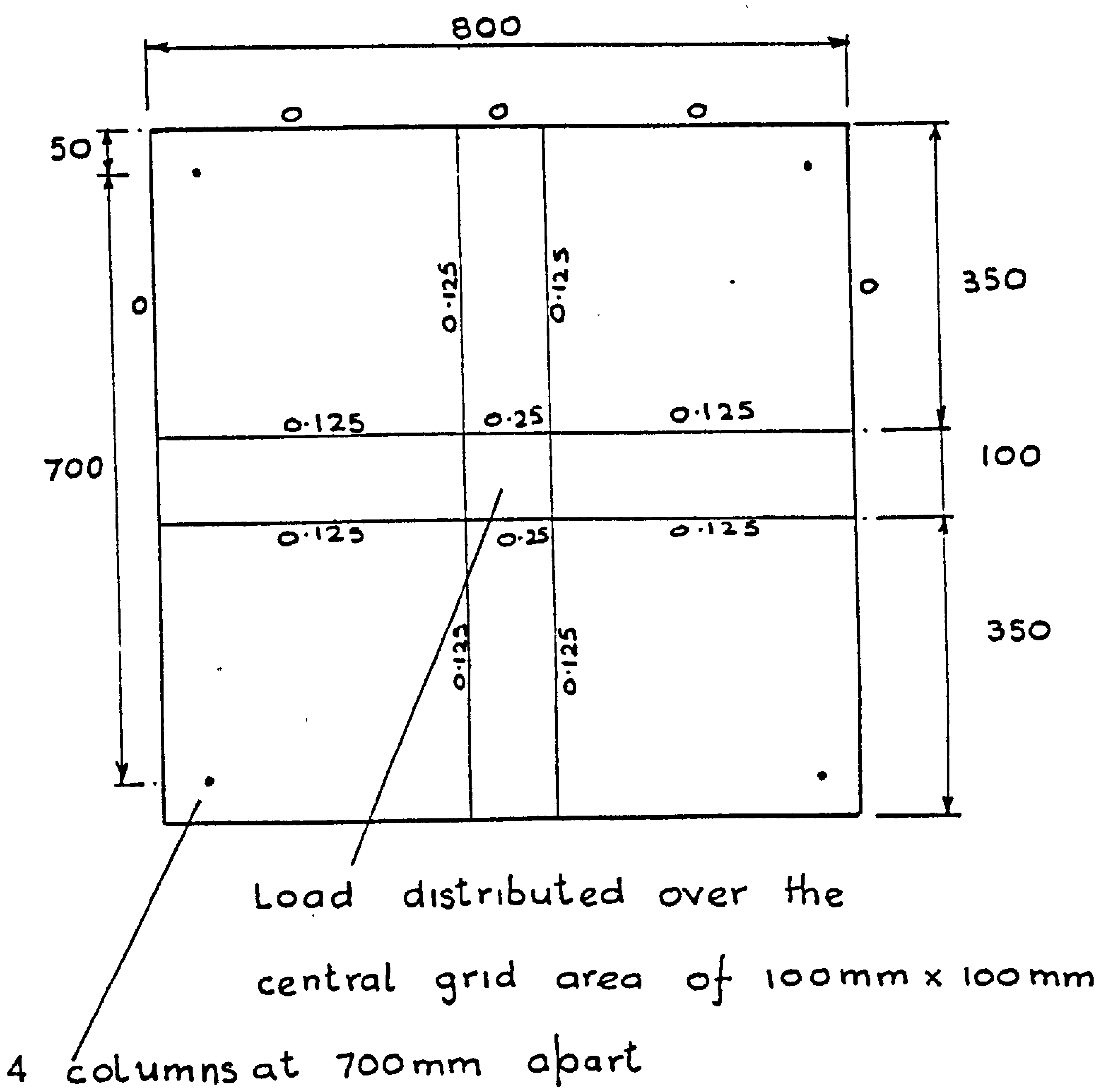


FIG 7.9 STRIP LAYOUT, POSITION OF COLUMNS AND VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR TEST 4

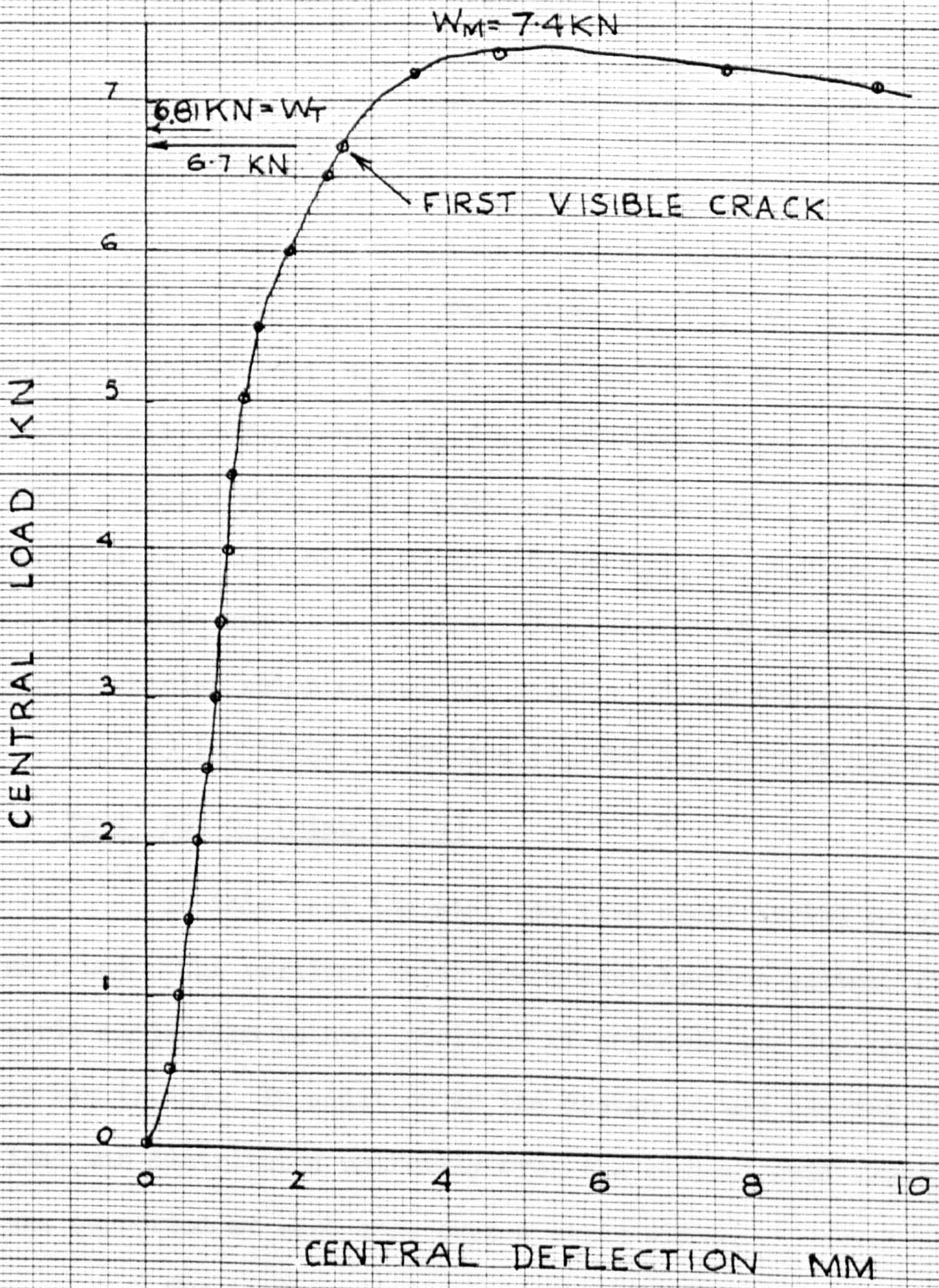


FIG 7.10 LOAD DEFLECTION CURVE FOR TEST 4

(b) TEST No. 5 (Slab No. 6)

Fig. (7.11) shows the strip layout, position of column and the vertical shear force distribution pattern for this design. For this slab it was assumed that the column reaction is concentrated at the outer edge of the slab. For factors p and q defined in the last test it can be shown that design load W_D and the theoretical collapse load W_T are related by (Chapter 6.2),

$$W_T = \frac{1-p/2}{1-2q} W_D$$

the values of p and q used are 350/800 and 20/800 and therefore $W_T = 0.822 W_D$.

Test results are given in Table (7.2) and the load deflection curve shown in Fig (7.12). The values of W_D, W_T and the experimental maximum applied load W_M were 8.89 KN, 7.31 KN and 7.05 KN (96.4% W_T) respectively. The ratio $W_M : W_D$ obtained in the experiment being 0.793 compared with 0.822 predicted by the theory. The first crack appeared at a load of 5.95 KN (81.4% W_T), the deflection of the slab at this stage was 1.4 mm (Span÷570). Photo 7.5 shows the central yield line mechanism which caused the failure of the slab.

(7.5.4) (a) Small errors due to load and support assumptions

TEST NO. 6 (Slab No. 4)

In this test the slab has five equal strips of width 160 mm each way. The vertical shear force distribution pattern obtained by the strip deflection method, position of supports etc for the slab is shown in Fig (7.13). In addition in the design it was assumed that (1) the load was distributed uniformly over the central grid area (11) the slab was supported at each edge over a strip width.

The test conditions were however different in that (1) the load was applied directly through the ball seat (11) the slab was

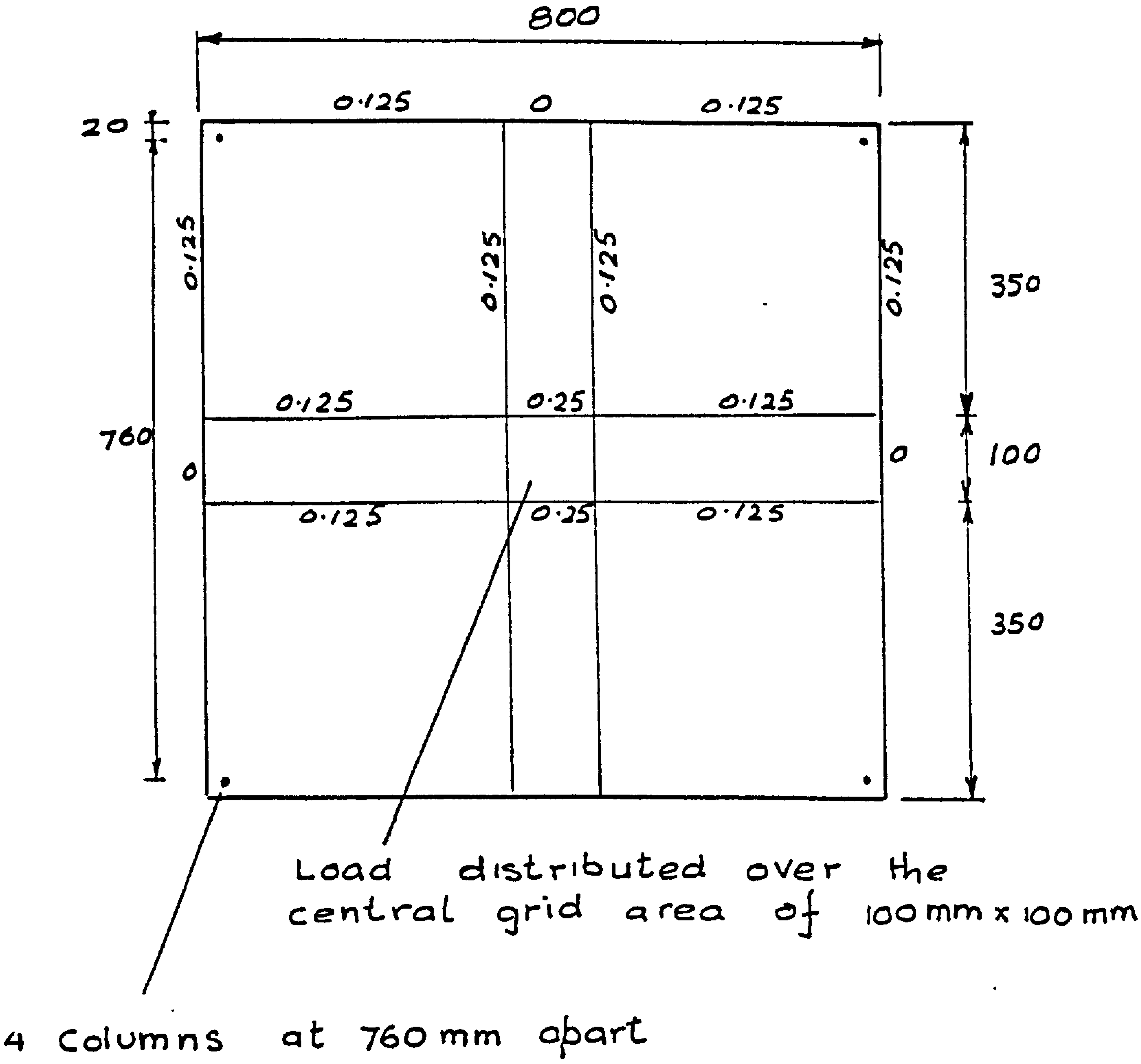


FIG 7.11 STRIP LAYOUT, POSITION OF COLUMNS AND VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR TEST 5

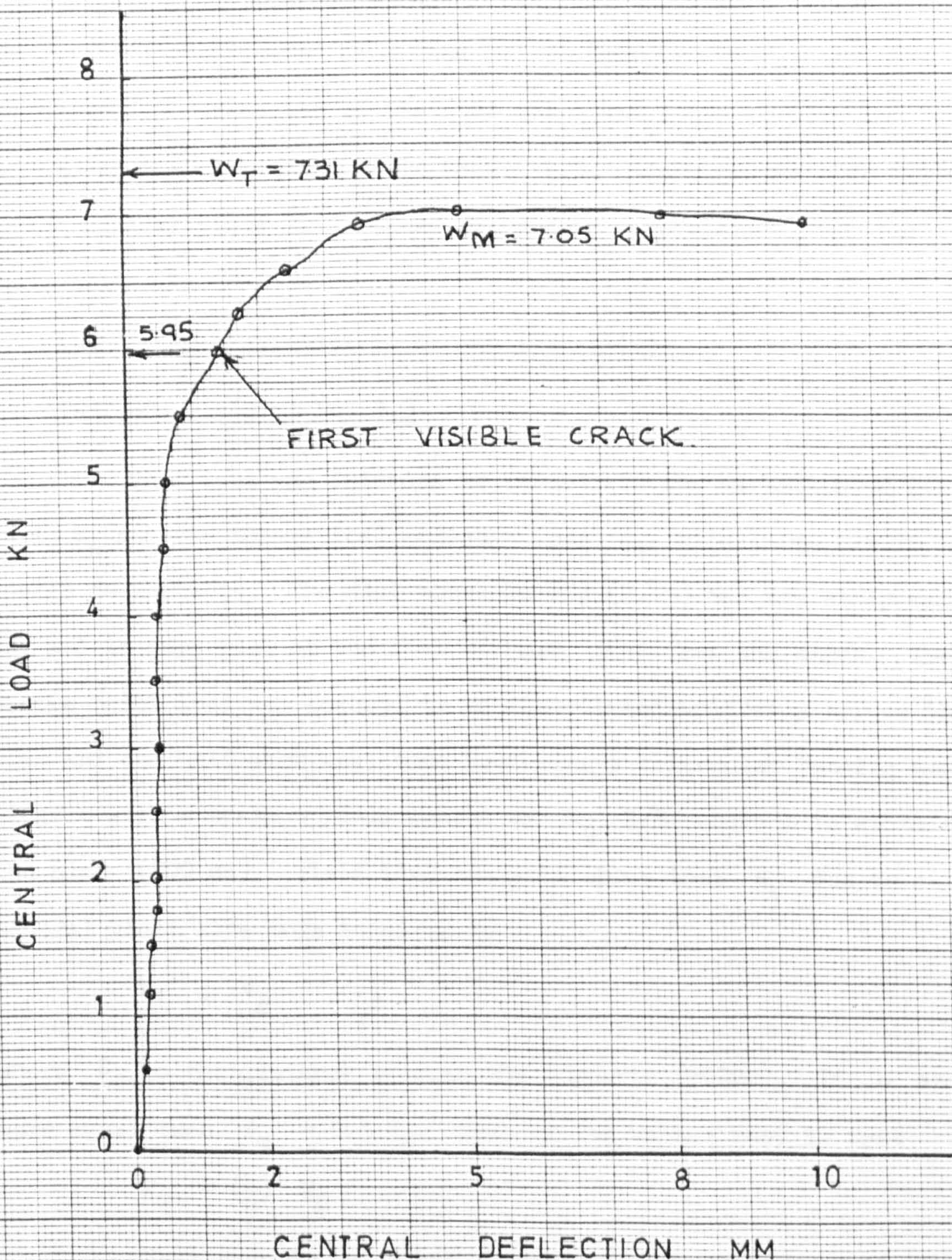


FIG 7.12 LOAD DEFLECTION CURVE FOR TEST 5

supported at four points 50mm from the edges. It so happens that the support errors were favourable and the load errors were adverse.

Test results are given in Table 7.2 and Fig (7.14) shows the load deflection curve. The design load (W_D) of the slab and the theoretical failure load W_T were 14.37 KN and 13.15 KN respectively. The maximum applied load was 14.40 KN (109.5% WT). The first crack appeared at a load of 10.05KN (76.4% W_T) and the corresponding deflection of the slab was 1.4mm (span ÷ 570). The failure of the slab is shown in Photo 7.6. Although the slab failed by the central yield line mechanism, cracks were visible over the whole slab area.

(b) Large errors due to design assumptions

TEST NO. 7 (Slab No. 3.)

This test was intended to illustrate the variation of design load W_D and the theoretical collapse load W_T with large errors in design assumptions. The slab shown in Fig (7.15) has one strip each way and is supported at the four corners 50mm from the edges. The centre point load was assumed to be distributed uniformly across the slab and one half each way, thus producing a triangular bending moment diagram with a maximum yield moment of W_D L/8 where L is width of the slab (800 mm). Yield line analysis shows that W_T due to a central yield line mode is given by

$$W_T = \frac{W_D}{2} \left(\frac{L}{\ell}\right)$$

where ℓ is the distance between the supports (700mm). If L and ℓ were equal then $W_T = 0.5\, W_D$. However for values used in test

$$W_T = \left(\frac{4}{7}\right) W_D$$

Test results are given in Table 7.2 and Fig 7.16 shows the load deflection curve. The values of W_D and W_T and the

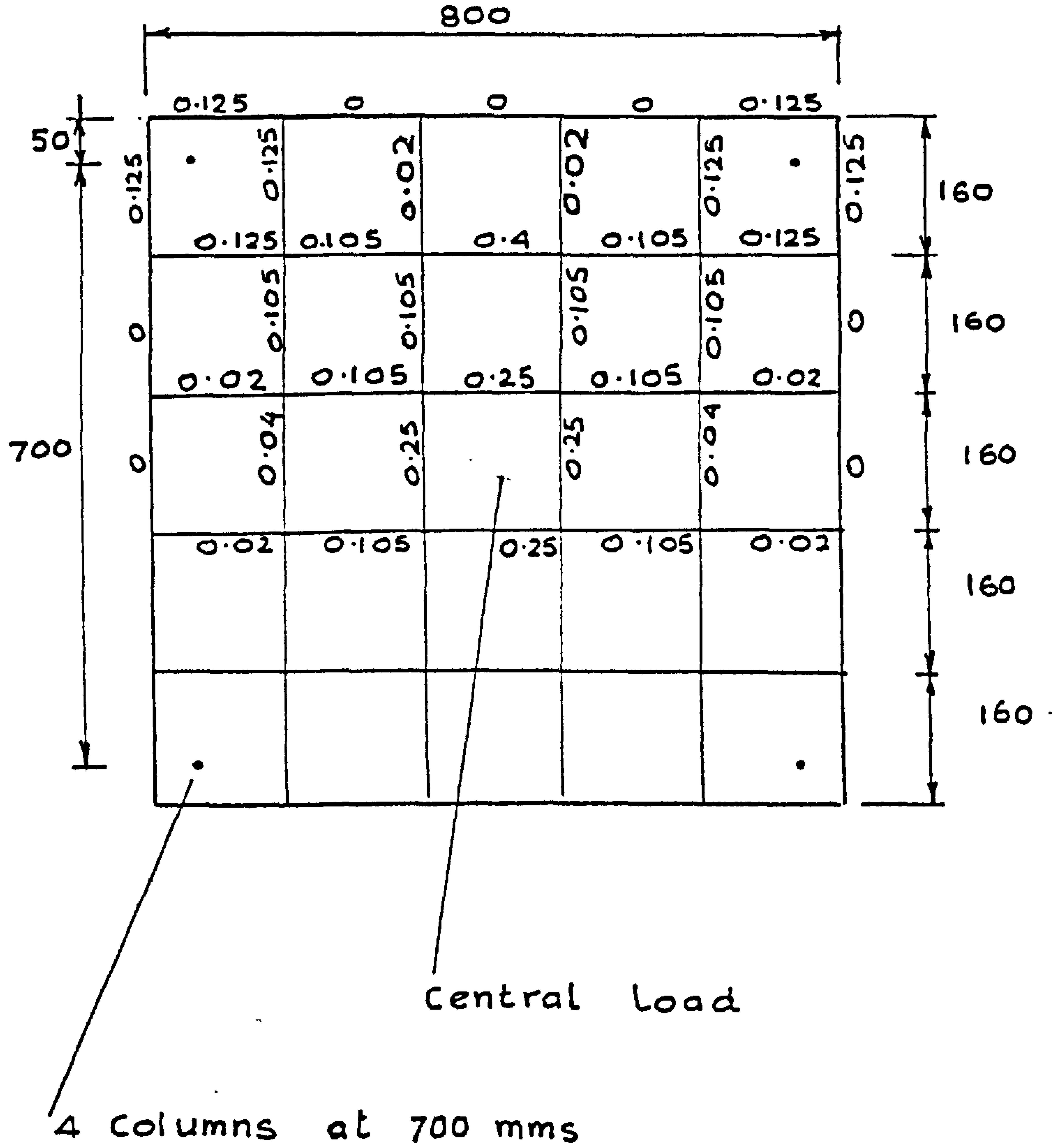


FIG 7.13 STRIP LAYOUT, POSITION OF COLUMNS AND VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR TEST 6

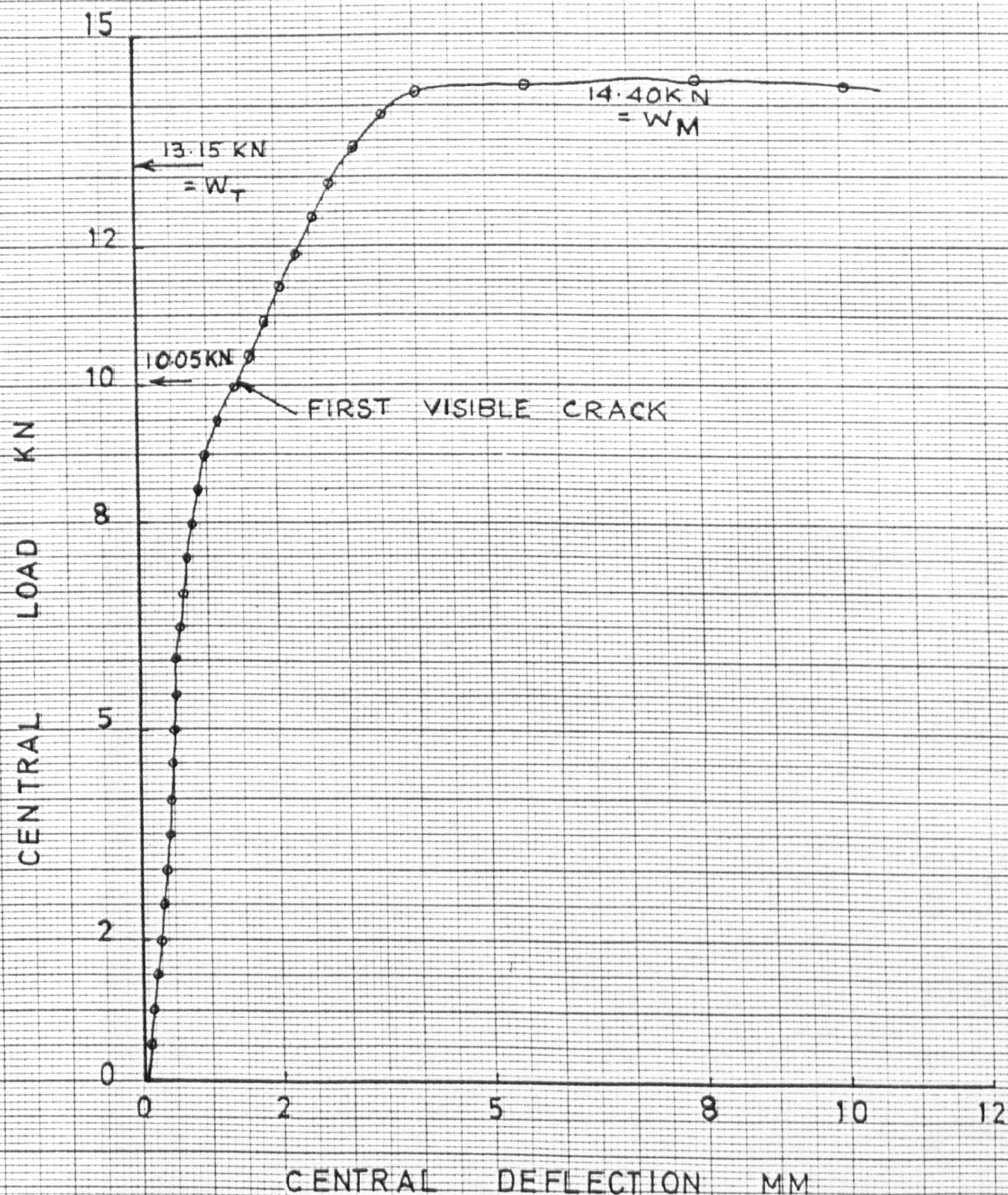


FIG 7.14 LOAD DEFLECTION CURVE FOR TEST 6

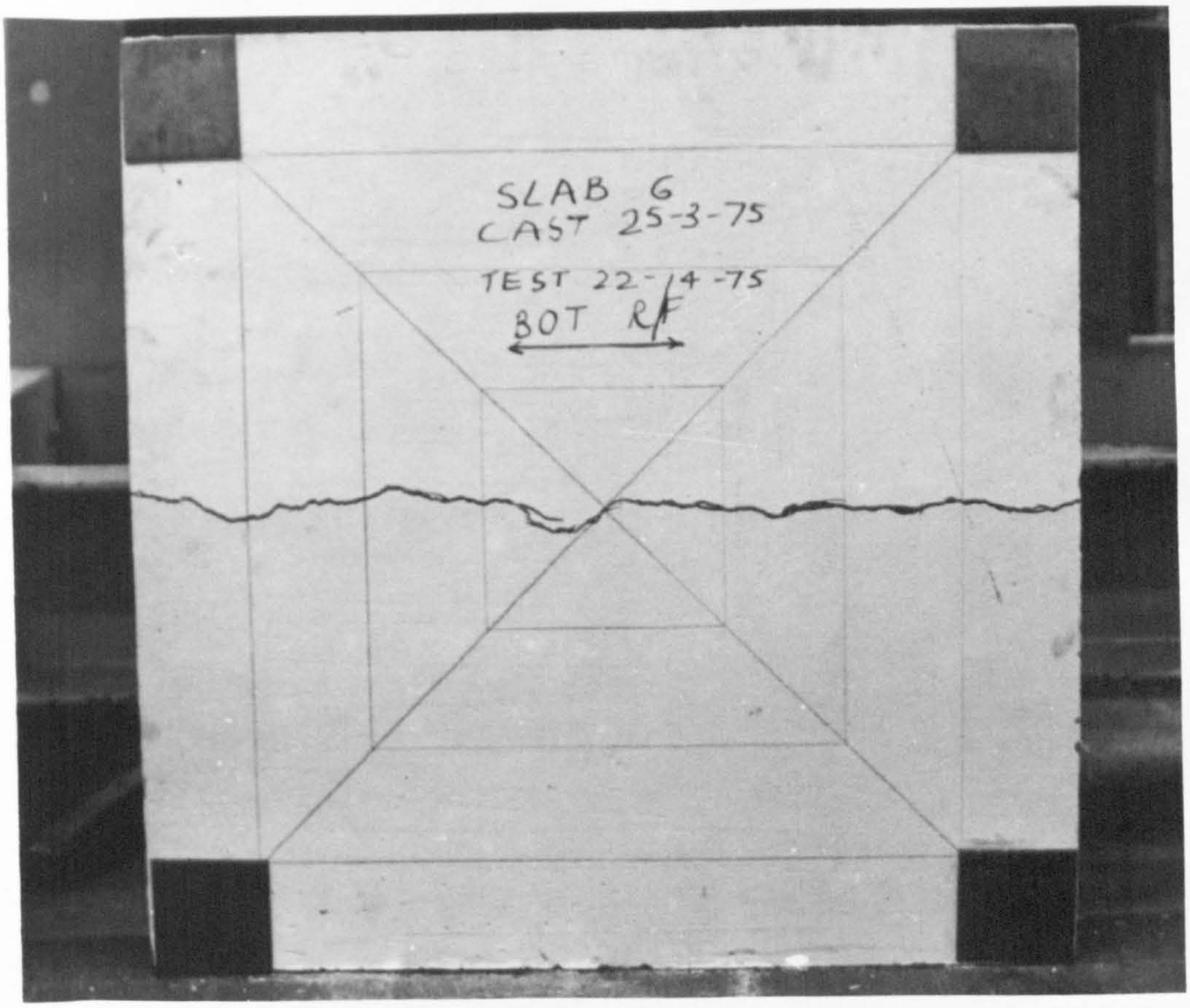


Photo 7.5

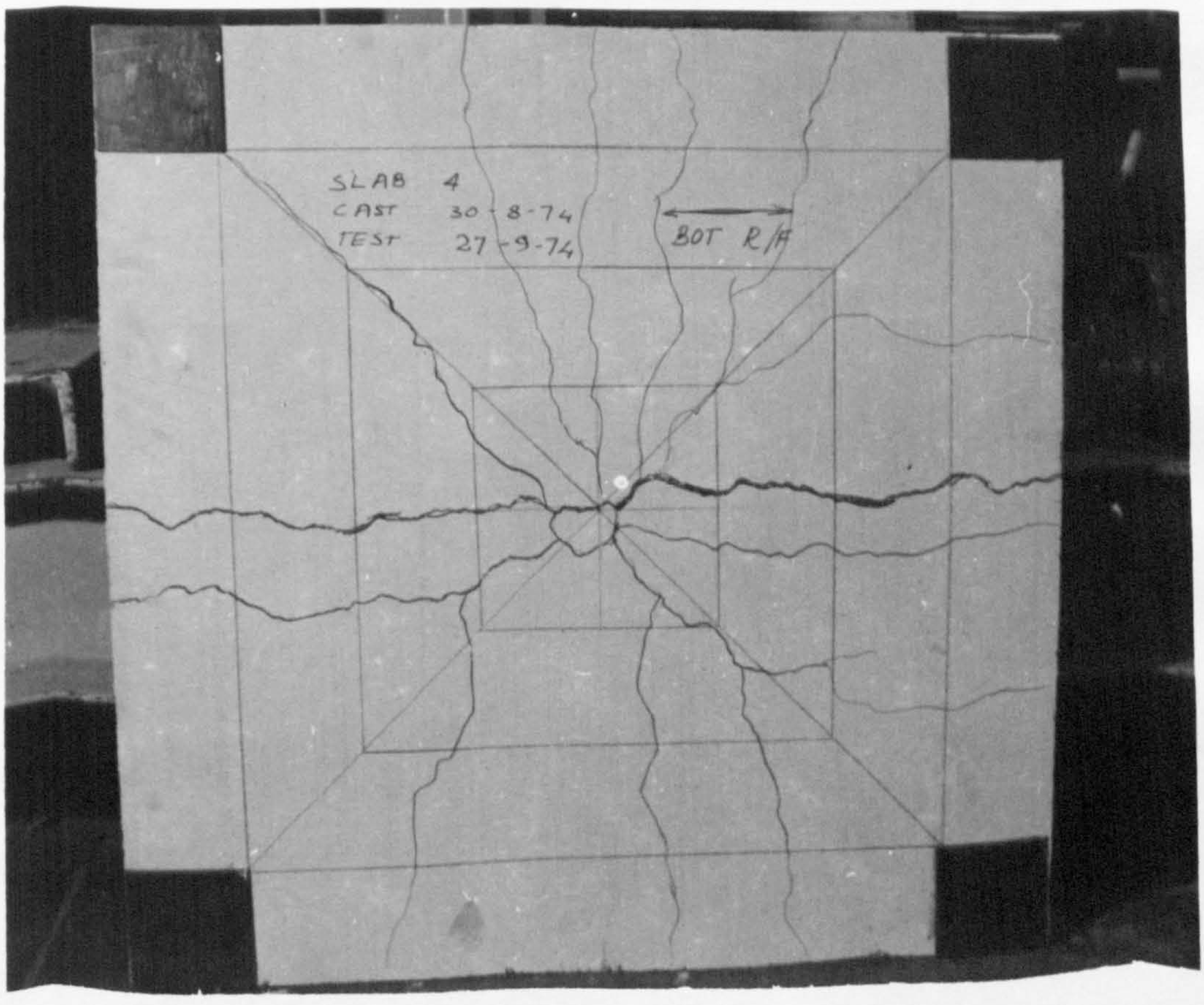


Photo 7.6

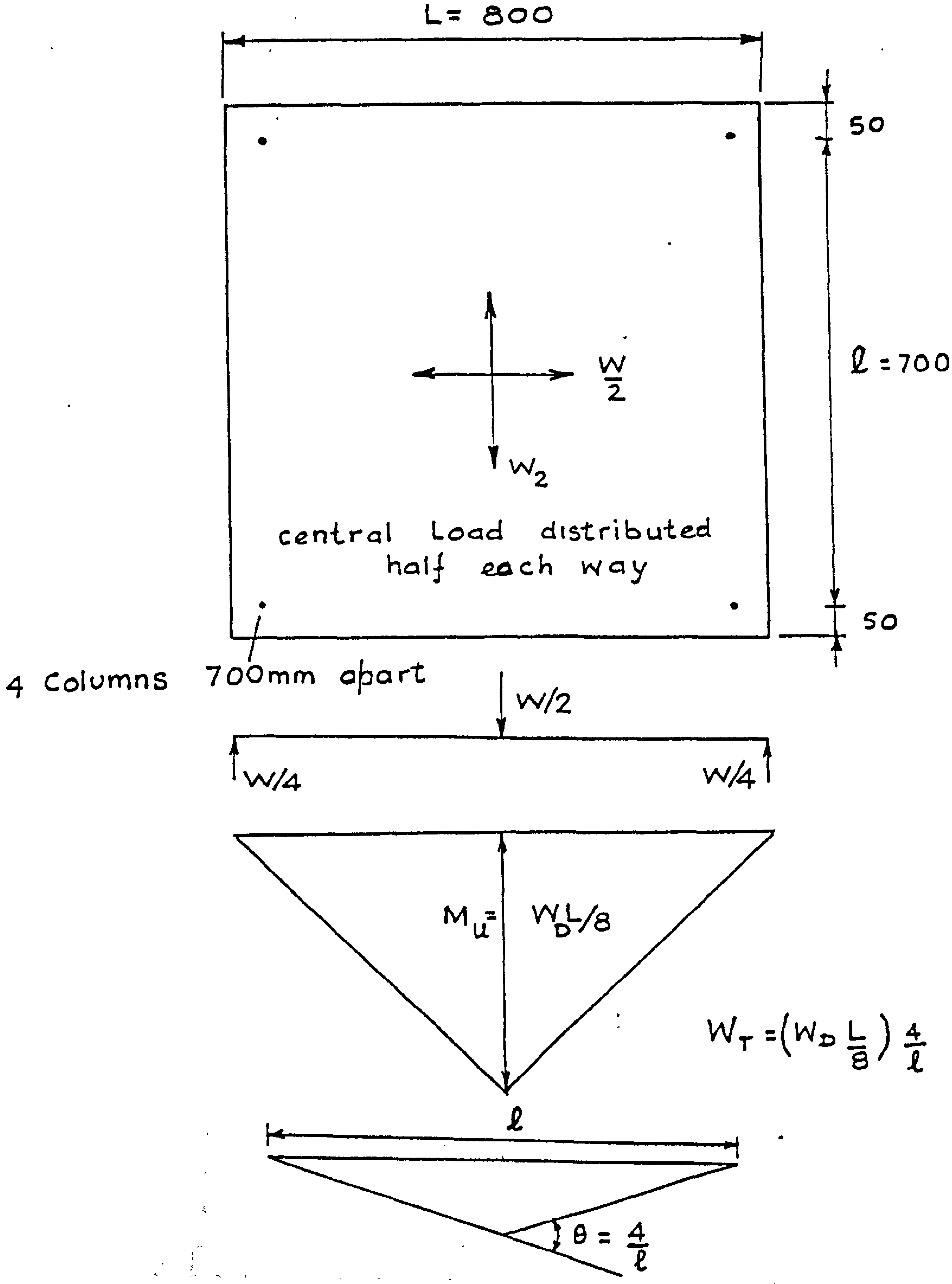


FIG 7.15 DISTRIBUTION OF LOAD, MOMENT FIELD AND POSITION OF COLUMNS FOR TEST 7

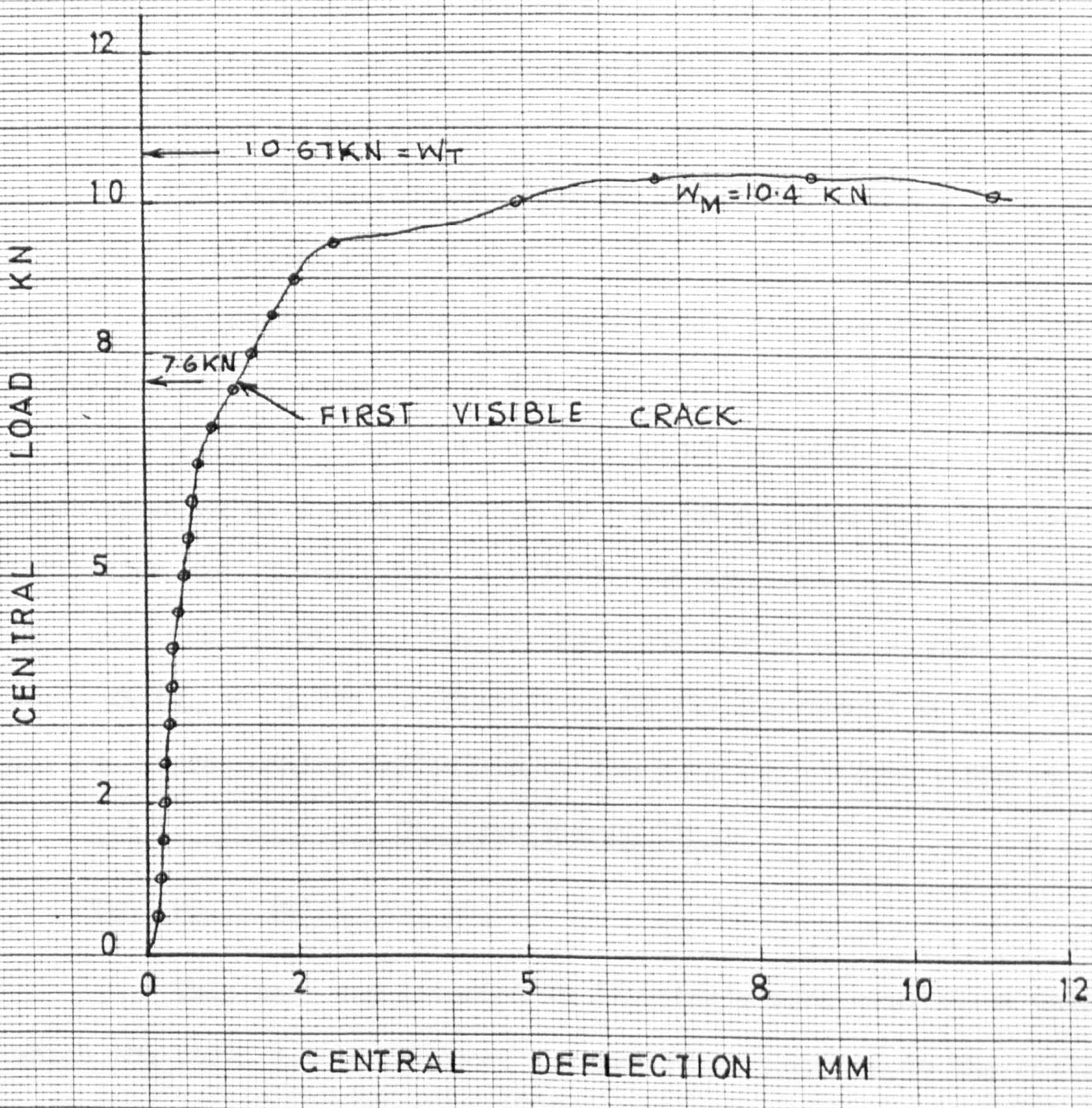


FIG 7.16 LOAD DEFLECTION CURVE FOR TEST 7

maximum experimental load were 18.67KN, 10.67KN and 10.4 KN (97.5% W_T). The agreement between W_M and W_T is remarkable. The first crack appeared when the applied load was 7.6 KN (71.2% WT) and the deflection of the slab at this stage was 1.3mm (span ÷ 600)

7.5.5. Use of spreader system

TEST NO. 8 (Slab No. 8)

This test was devised to investigate the use of spreader system to produce unique solutions. Without these spreaders the slab is identical to the slab in Test 7. In order to accomodate the required reinforcements, the width of all spreaders were chosen as 100 mm. The position of columns, spreader system and the vertical shear force pattern is shown in Fig (7.17).

The test results are given in Table 7.2 and the load deflection curve shown in Fig (7.18).

The design load and the theoretical collapse load for this unique design are equal (18.57 KN).

The maximum applied load was 18.20 KN (98% W_T). The first crack appeared at a load of 15.0 KN (80.8% W_T) and the corresponding deflection was 3.7mm (span ÷ 216). If appropriate safety factors were used for design then the working load can be established as 10.7 KN (W_D ÷ 1.725). There were no visible cracks at this load and the corresponding deflection of the slab was 1.75mm (span ÷ 457)

7.5.6. Positive - Negative moment field

TEST NO. 9 (Slab No. 10)

This test was intended to illustrate some conditions under which the strip method produces a lower bound solution to the collapse load. The layout of the strips and the position of the simple supports are shown in Fig (7.19). The moment field and the bending moment in strips is similar to the slab shown in Fig (6.10). The moment fields

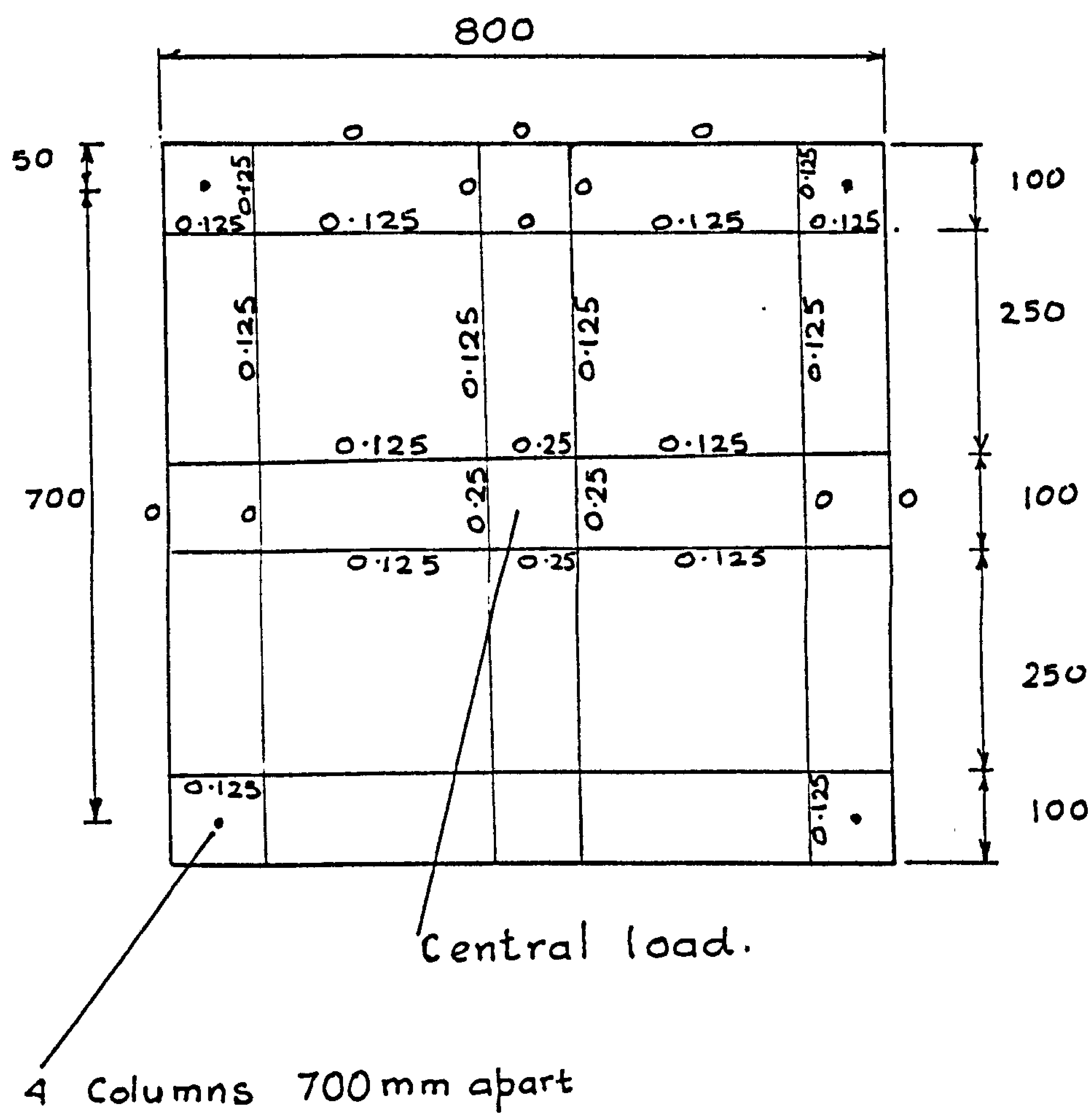


FIG 7.17 STRIP LAYOUT, POSITION OF COLUMNS AND VERTICAL SHEAR FORCE DISTRIBUTION PATTERN FOR TEST 8

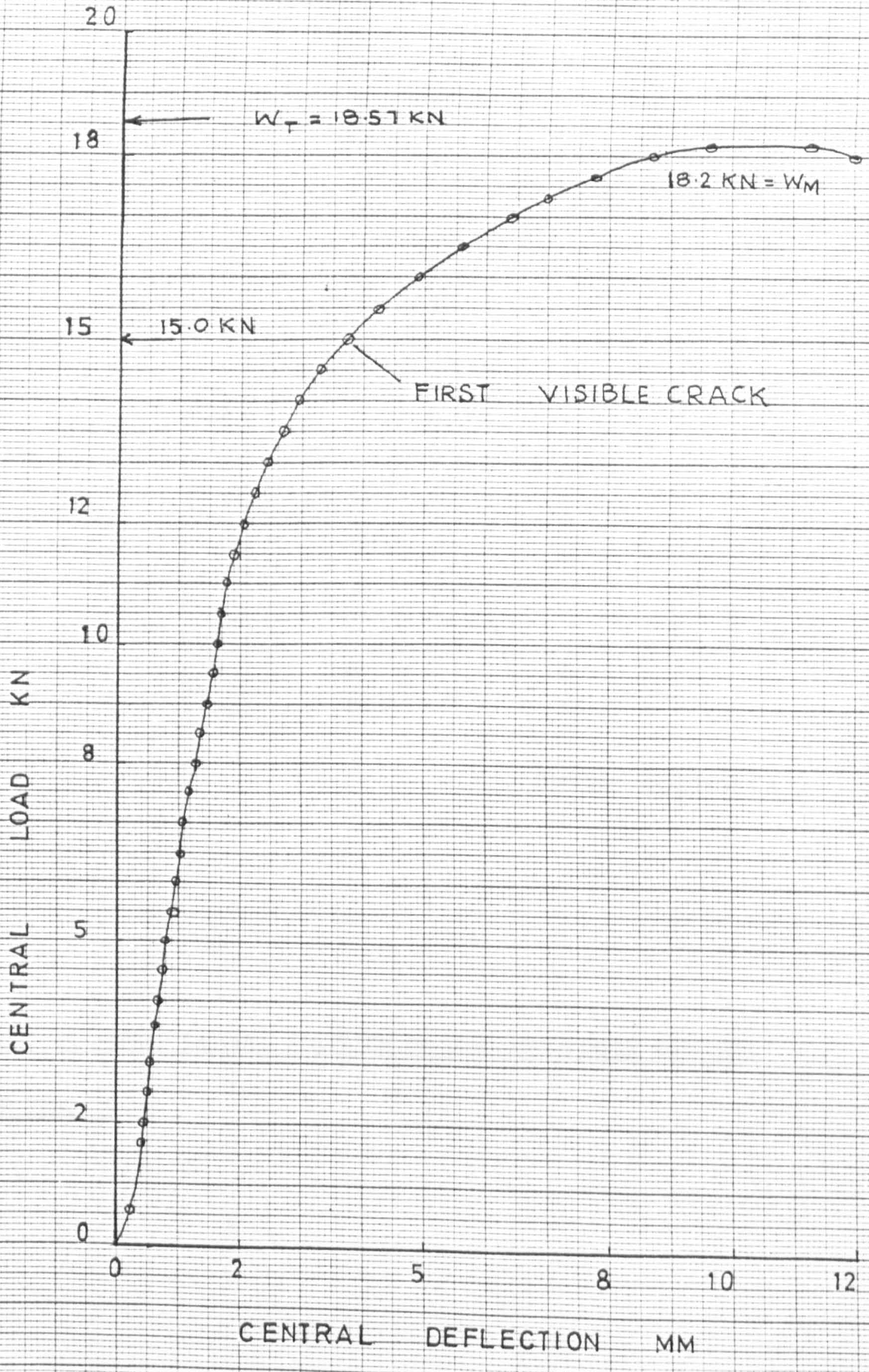


FIG 7.18 LOAD DEFLECTION CURVE FOR TEST 8

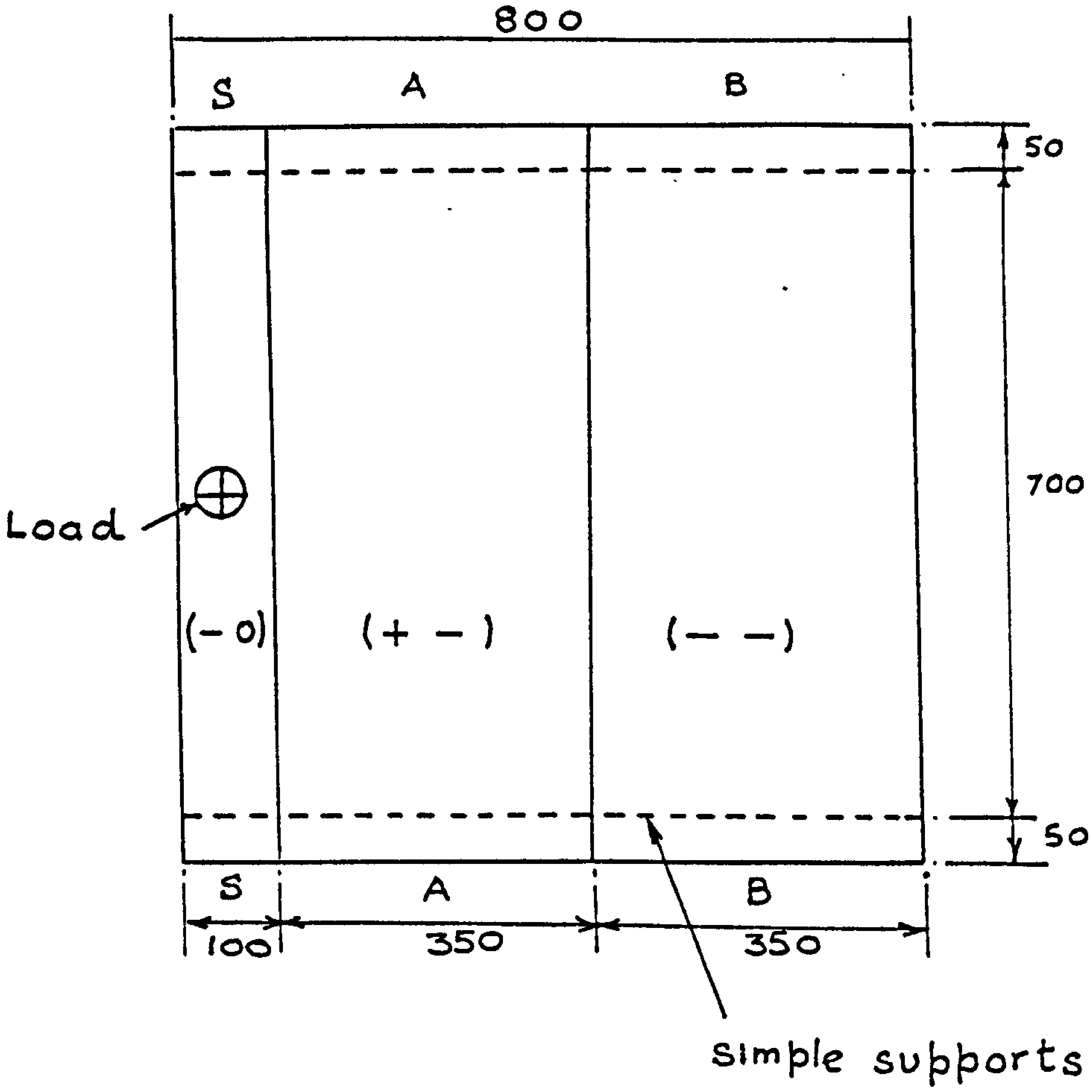


FIG 7.19 STRIP LAYOUT, POSITION OF LOAD AND SIMPLE SUPPORTS FOR TEST 9

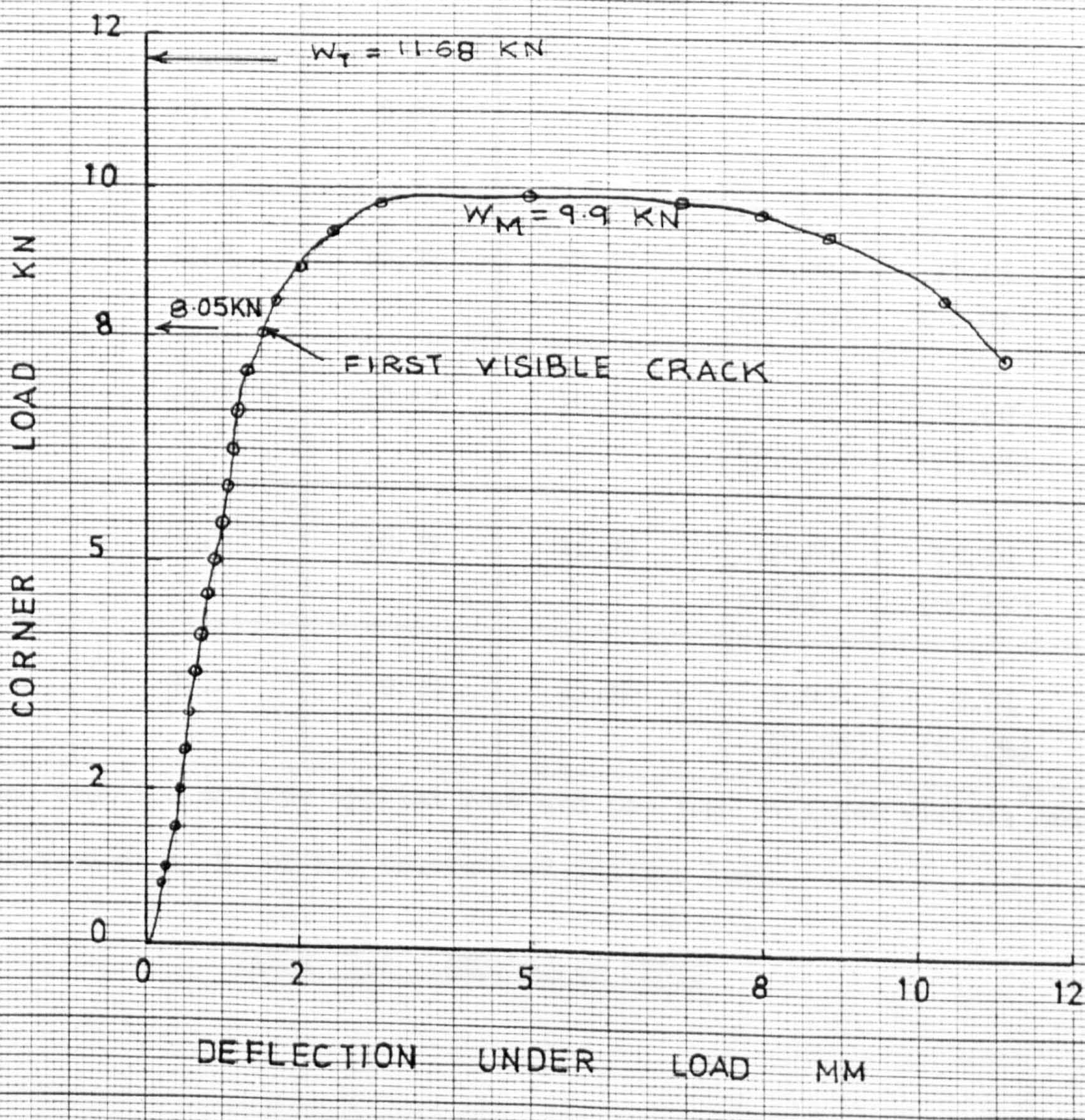


FIG 7.20 LOAD DEFLECTION CURVE FOR TEST 9

TEST NO.	AVERAGE CUBE STRENGTH N/mm^2	DESIGN LOAD (WD) KN	THEORETICAL FAILURE LOAD (WT) KN	MAXIMUM APPLIED LOAD (WM) KN	LOAD AT FIRST VISIBLE CRACK KNS
1	44.3	10.44	10.44	9.6 (92%)	8.2 (78.5%)
2	38.2	12.03	12.03	11.7 (97.3%)	8.4 (70 %)
3	38.6	16.63	10.23	8.7 (85 %)	7.05 (69 %)
4	44.4	10.60	6.81	7.4 (108.7%)	6.7 (98.3%)
5	43.1	8.89	7.31	7.05 (96.4%)	5.95 (81.4%)
6	42.6	14.37	13.15	14.40 (109.5%)	10.05 (76.4%)
7	43.9	18.67	10.61	10.40 (97.4%)	7.60 (71.2%)
8	40.3	18.57	18.57	18.20 (98.%)	15.10 (80.8%)
9	42.1	9.34	11.68	9.90 (84.8%)	8.05 (69.2%)

For each test the maximum applied load and the load at first visible crack is expressed as a % of the theoretical failure load.

TABLE 7.2 RESULTS OF TESTS

for strips SS, AA and BB are (positive - zero), (positive - negative) and (negative - negative) respectively. Theoretically the slab can fail by two yield line modes and the collapse load W_T determined by analysis for both is 1.25 x design load (W_D).

The load was applied at the centre of the spreader SS and edges of the strip BB had a tendency to lift from the simple supports. Any lifts were prevented by the use of smooth bars and G clamps.

Test results are given in Table 7.2 and the load deflection curve shown in Fig (7.20). W_D and W_T were 9.34 KN and 11.68 KN respectively. The maximum experimental load W_M was 9.9 KN (84.8% W_T). The ratio $W_M : W_D$ was 1.06. The first crack recorded at a load of 8.05KN (69.2% W_T) and the deflection of the slab at this stage was 1.6mm. The slab failed by a central yield line.

7.6 SUMMARY OF RESULTS AND CONCLUSIONS

These tests were performed with a main objective of assessing the ultimate strengths of slab designed by both the strip method and the strip deflection method. In all tests slabs sustained loads very close to the theoretical collapse load calculated by the yield line theory. The chosen support system therefore must have reduced the effects of membrane action to a minimum. For the nine tests the average maximum applied load (W_M) : theoretical collapse load (W_T) was 96.6% and the extreme values varied between 109. % and 85 % . These values are well within the results that can be expected for concrete slabs. All slabs failed by a well defined mechanism, the mechanism predicted by the yield line analysis.

Although the information on serviceability condition was not the primary aim, however some interesting conclusions can be drawn on the position of the first crack and deflections. These positions are shown in the load deflection curves and this position corresponds to an average load of 77.2% W_T and a minimum value of 69.2% W_T. If partial safety factors of 1.15 and 1.5 are allowed for steel and loads then a working load can be established as 57.9% W_T $\left(\frac{WT}{1.15 \times 1.5}\right)$. For all slabs there were no visible cracks at this load. The deflection of the slab corresponding to the first visible crack was about 1.6 mm ie a span to deflection ratio of 500.

Therefore the general acceptability of the theory is thus demonstrated especially in the areas of ultimate strength or uniqueness of collapse load. The concepts of equilibrium, spreader systems or recommendations relating to corner support strip widths can now be used in the design of concrete slabs. In the event of deviation the design load will differ from the collapse load and the design can still predict the variations.

The information on serviceability was encouraging. However large scale tests not less than half size are needed to examine the behaviour of slabs at working loads. Armer (37) has shown that the performance of slabs designed by the strip method with extremes of load distribution factors over extensive areas was satisfactory in terms of deflection and cracking. The following topics are suggested for incorporation in the large scale test programme.

(a) Influence of strip layout on serviceability.

(b) Use of spreader system for internal columns and loads.

(c) Use of strip width equal to twice the column width for external and corner columns.

(d) Behaviour of slabs in (b) and (c) above at service loads.

There is little doubt that the ultimate load condition of these large scale slabs will be satisfactory.

CHAPTER EIGHT

SUMMARY CONCLUSIONS AND RECOMMENDATIONS

8.1 INTRODUCTION

The study presented of the available literature on the design and analysis of reinforced concrete slabs shows how restricted they are in their application and illustrates the limitations on their reliability and efficiency. Although the assumptions in the elastic plate theory are not strictly valid, such methods are still adopted for the design of slabs. Due to the fact that maximum stresses are restricted to permissible values, the performance of these slabs under working load conditions has usually been satisfactory and the estimate of load factor is generally conservative which is probably the reason for the continuing use of elastic methods.

The other commonly used method in the design and analysis of reinforced concrete slabs is the yield line theory. According to the plastic theorems it is known that this theory will lead to an unsafe solution with respect to the collapse load, although in practice strain hardening effects and membrane action which are not considered in the theory will provide a reserve of strength. The method when compared with an elastic solution for the same design is generally associated with economy of materials. The yield line theory however does not give information on the load distribution on the supporting beams and distribution of reinforcement on the rigid regions between the yield lines. Serviceability of slabs designed by the yield line theory is controlled by specifying ratios of span to depth together with the choice of load factor. In the application of yield line theory to complex slab systems particularly with a combination of loading there is a risk of incorrect choice of collapse mode leading to a reduction in the load factor.

A powerful alternative to the elastic methods and the yield line theory is the strip method first proposed by Hillerborg. This design procedure aims at a lower bound solution to the collapse load by satisfying the equilibrium conditions and the yield criterion at all points. When the loading on the slab and its supports are both distributed the strip method of design is simple and versatile. When point loads or supports occur no satisfactory general treatment has been previously produced. A number of researchers have devoted considerable time and effort to overcome these limitations, but in all cases the simplicity and the appeal of the simple strip method has been lost. Test on slabs (37) designed by the strip method using extreme values of load distribution factors over extensive areas have shown that the method provides a safe solution for the ultimate load and at working loads the performance of the slabs in terms of deflections and cracking is satisfactory.

8.2 UNIQUENESS OF COLLAPSE LOAD

When Hillerborg proposed the strip method his intention was to produce a lower bound solution for the collapse load. Later Wood and Armer (27) concluded that when reinforced precisely in accordance with the strip moments, "Hillerborg's method provides an exact solution with an unlimited number of collapse modes". Their proof was however based on an assumption that the applied normal moment at any point in any direction be identical with the yield normal moment. This assumption has been shown here to be true only when both applied principal moment and the yield principal moment have the same sign and magnitude. In general slabs designed by the strip method will not only have such moment fields. Taking into account the sign of the principal moments rules have been derived for the postulation of the yield lines in a mechanism to be consitent with the unique solution. These are:-

(1) Both principal moments positive, positive - positive moment field:- Positive yield lines may act in any position and in any direction. Negative yield lines are only allowed in a reinforcement direction along which the principal moment is zero.

(2) Both principal moments negative, negative - negative moment field:- Negative yield lines may act in any position and in any direction. Positive yield lines are only allowed in a reinforcement direction along which the principal moment is zero.

(3) One principal moment positive and the other negative, positive - negative moment field :- The only yield lines allowed are positive yield lines normal to the positive reinforcement and negative yield lines normal to the negative reinforcement.

Unique solutions can therefore be found only when a valid mechanism can be formed in accordance with these rules. The number of such mechanisms are obviously limited and if at least one possible mechanism does not exist the strip method will then lead to a lower bound on the collapse load. However it must be concluded that it is remarkably difficult to find a practical example of a slab carrying a distributed load and designed by the strip method for which there is not atleast one such collapse mechanism consistent with these rules and therefore leading to the unique collapse load. No general proof is yet available for this curious occurence of uniqueness for slabs designed by the strip method, when only the lower bound theorem is satisfied intentionally and further research is required on this problem.

The rules for uniqueness has also been applied to optimum solutions for slabs. In some examples there appears to be only one consistent mechanism. When corner fans occur in the mechanism their form is dictated by the rules which has been postulated.

8.3 THE STRIP DEFLECTION METHOD

The strip deflection method has been proposed as a generalised strip method of reinforced concrete slab design which is easily understood and applicable to any shape of slab and boundary conditions. For many slabs the computation is simple and standard problems can be solved using desk calculators commonly found in design offices. The design is based on the critical limit state of collapse. The method has been shown to produce a safe solution and with distributed loads the method leads invariably to a unique solution. Effects of strain hardening and membrane action has been excluded from this study and in most cases these factors are known to enhance the carrying capacities of the slabs. The designer has considerable freedom in choosing the strip layout, but whatever the choice, the load distributions over finite areas of the slab are determined systematically by ensuring compatibility of elastic deflections of orthogonal slabs strips. The method therefore ensures that the load distributions will not depart too far from the elastic working load conditions and therefore it can be concluded that the strip deflection method provides better serviceability conditions than the simple strip method which itself has been shown to be generally satisfactory in tests.

In addition the strip deflection method provides full information on loading, shear forces and bending moments at all points of the slab. The resultant layout of reinforcement is orthogonal and banded, the total quantity of reinforcement compares favourably with other design methods commonly used and frequently can approach minimum weight design. It also accomodates partial composite action with supporting beams and provides a closer approximation to the actual load distributions on the supporting beams than those in the current codes of practice. The uniqueness rules discussed in the earlier section can be extended to slab - beam systems and for such consistent combined mechanisms the collapse load will be identical to the design load.

The strip deflection method has been shown to be closely related to the torsionless grid analogy method. The only difference between these is the assumption made regarding the load distribution. In most practical problems with a sufficient number of strips or equivalent beams the difference between the two moment fields is insignificant. However for a unique solution the load interactions must strictly be assumed to be uniformly spread over the corresponding grid areas. Therefore existing computer programmes for grid systems can be readily used. The 'Genesys' computer system which has been used here is very flexible and any type of slab geometry, loading and support conditions can be readily accommodated.

Therefore the strip deflection method offers a unified collapse limit state approach to the design of all slabs which is simple, safe and economical. Its attractiveness as a slab design method appear to be greater than that of any existing method and it is hoped that these will become recognised by the profession and the method become a recommended procedure in design codes.

8.4. POINT SUPPORTS AND POINT LOADS

For slabs with point supports or point loads it is again true to say that if equilibrium and the yield conditions are satisfied the strip method will lead to a safe solution. Unlike the case of distributed supports and loads it is not possible to conclude that the collapse load will always be identical to the design load. For a particular layout of strips it may not always be possible to postulate a yield line mechanism consistent with the uniqueness rules.

With edge or corner columns safe solutions can be obtained by choosing a strip layout such that the width of the column strip is twice the dimension of the column and assuming that the column reaction is spread uniformly over the column grid area. In most practical problems this width is sufficient to accommodate the required reinforcements.

It may not always be possible to have a strip layout which satisfies equilibrium at all points and in such cases the solution may well be an upper or lower bound on collapse load. Depending on the design assumptions and the choice of strip layout the ratio of the two loads (W_C / W_D) can be alarmingly low. However procedures have been developed using spreader systems for internal columns and loads to overcome this problem. The slab is then designed for a moment field which is the sum that is due to the uniform distribution of load or internal column reaction over the corresponding grid area and that due to the spreading of the load within the grid area (spreader systems).

In all cases with spreader systems the design load will be a safe solution on the collapse load and in many cases the results will be unique.

8.5 EXPERIMENTAL WORK

It has already been pointed out that to establish the strip deflection method as a common design method for reinforced concrete slabs would require further testing particularly of large scale slabs. This also applies for the special recommendations for slabs with point loads and point supports mentioned in the earlier section. Such a programme of tests including slabs of different shapes, edge conditions and loading was outside the scope of present work. Further it was realised at the outset of the investigation that the primary aim was to check the ultimate behaviour of the slabs rather than the serviceability.

The theoretical failure loads of the slabs were determined by the yield line theory. Corner supports used in most of the tests reduced the effects of membrane actions and it was not surprising that the actual collapse load agreed very well with the theoretical values. Although the model tests cannot be considered wholly reliable for providing information on serviceability and cracking they did show that first visible cracks of about 0.03 mm width occured at about 77% of the failure load.

8.6 FUTURE WORK

The areas of uncertainity in both the theory and experimental work have been pointed out throughout this thesis but it seems worth while to summarise them at this point and to point out the need for further research studies.

In the theoretical field it would be interesting to investigate the possibility of establishing a general proof of uniqueness for slabs designed by the strip method. Also in its present form the strip deflection method does not take into account the effects of membrane action and research might usefully be directed towards incorporating this important strengthening phenomenon.

There appears to be only one slab example where the strip deflection method could fail to provide a design solution. This is the case of a square slab supported at the two diagonal corners and loaded equally at the other two diagonal corners. This is equivalent to the application of pure torsion along all the boundaries and if the strips are chosen parallel to the boundaries no load can be carried since the twisting moments are not equal to zero. However a solution seems to be possible if the strips are chosen to lie in the diagonal directions. Therefore there is a need to investigate the behaviour of slabs designed by the strip deflection method where torsional moments predominte.

The need for further testing of large scale slabs to study the serviceability conditions under working loads has been emphasised. There is little doubt that the ultimate load conditions will be satisfactory and from the tests that have been carried out on large scale slabs designed by the simple strip method in which extreme load distributions are assumed it seems very likely that no real serviceability problems will be encountered using strip deflection method. The programme of tests should cover the following

(a) Influence of the strip layout on serviceability.

(b) Slabs with internal columns.

(c) Slabs with openings.

(d) L shaped and skew slabs.

(e) Use of spreader systems for internal columns and loads.

(f) Use of edge strip width equal to twice the column width for slabs with columns at an edge or a corner.

APPENDIX I

SLAB STRIPS WITH STANDARD BOUNDARY CONDITIONS

Methods of calculating the elastic end reactions, fixed end moments and deflections at required points are presented here. These values are calculated for strips having the following boundary conditions.

(a) Slab strips simply supported at both ends

(b) Cantilever slab strips

(c) Slab strips fixed at one end and simply supported at the other

(d) Slab strips fixed at both ends

Some of these results are available in standard design hand books or elementary theory of structures text books. To illustrate a uniform slab strip of length L and carrying four patch loads W_1, W_2, W_3 and W_4 each of length (L/4) is considered. The end reaction, fixed end moment and the deflection at the centre of each patch load are calculated for the above boundary conditions.

It is of importance to note that in the strip deflection method it is normal to encounter strips of different widths. Care must be therefore taken to insert the correct values of second moment of area (I). In all calculations value of the elastic modulus E is kept constant.

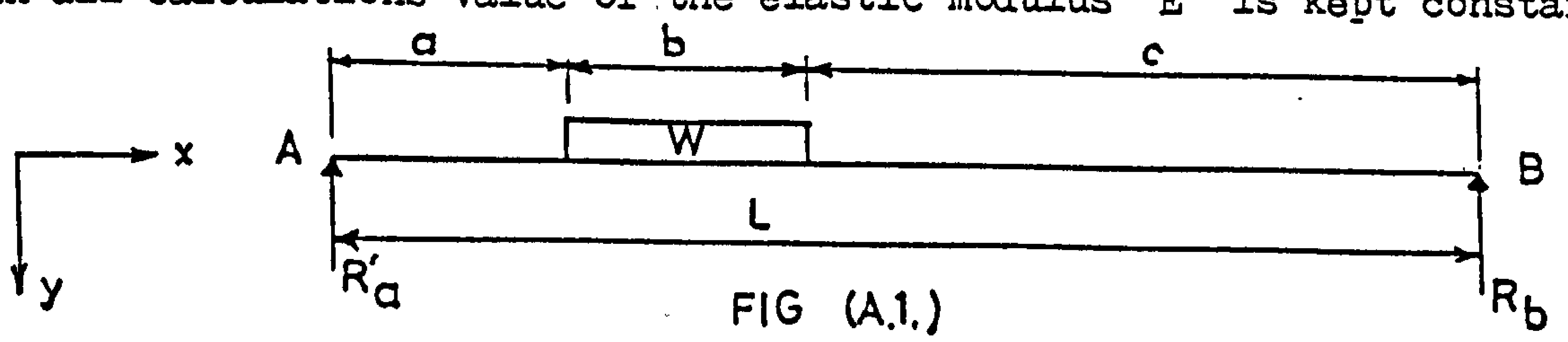


FIG (A.1.)

(a) Slab strips simply supported at both ends

Fixedend moments $M_a = M_b = 0$

Boundary Conditions $y = 0 \quad ; \quad x = 0$

$y = 0 \quad ; \quad x = L$

Clearly reactions $R_a = \frac{W}{L}\left(\frac{b}{2} + c\right)$ and $R_b = \frac{W}{L}\left(\frac{b}{2} + a\right)$

It can be shown that the deflection

$$y = \frac{W}{24EI\ b}\left[\frac{b(b+2c)(2L^2-b^2-2bc-2c^2)x}{L} - \frac{4b(b+2c)x^3}{L} + \{x-a)^4\} - [(x-a-b)^4]\right]$$

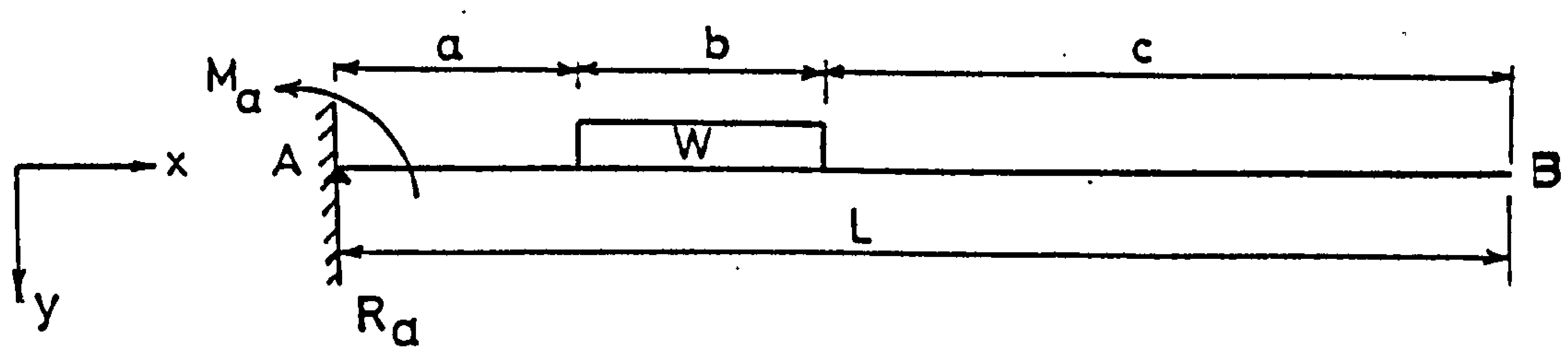


Fig (A.2)

(b) <u>Cantilever slab strips</u>

Clearly $R_a = W$ and $M_a = W\ (a+\frac{b}{2})$

Boundary Conditions $\frac{dy}{dx} = 0$ at $x=0$

$y = 0$ at $x=0$

$$y = \frac{W}{24EI\ b}\left[6b(2a+b)\ x^2 - 4bx^3 + [(x-a)^4] - [(x-a-b)^4]\right]$$

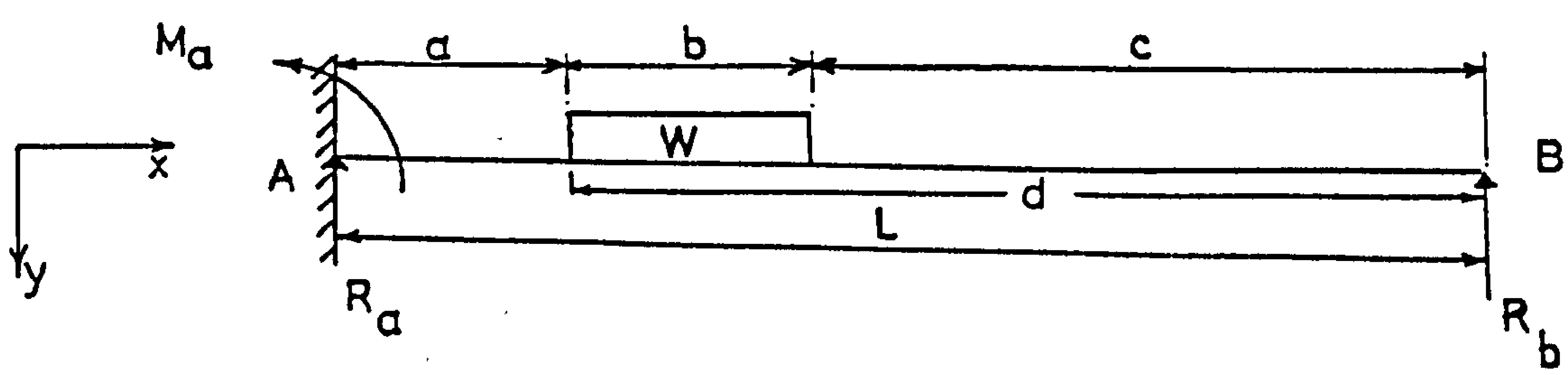


Fig (A.3)

(c) <u>Slab strips fixed at one end and simply supported at the other</u>

Boundary conditions $y=0$ at $x=0$

$y=0$ at $x=L$

and $\frac{dy}{dx}=0$ at $x=0$

$$R_a = r_a + \left|\frac{M_a}{L}\right| \quad \text{and} \quad R_b = r_b - \left|\frac{M_a}{L}\right|$$

Where r_a and r_b are the simple support reactions

$$M_a = \frac{W}{8L^2\ b}(d^2 - c^2).(2L^2 - c^2 - d^2) \qquad \text{(Hogging)}$$

$$\text{Then} \quad y = \frac{W}{24EI\ b}\left[12bM_a x^2 - 4bR_a x^3 + \left[(x-a)^4\right] - \left[(x-a-b)^4\right]\right]$$

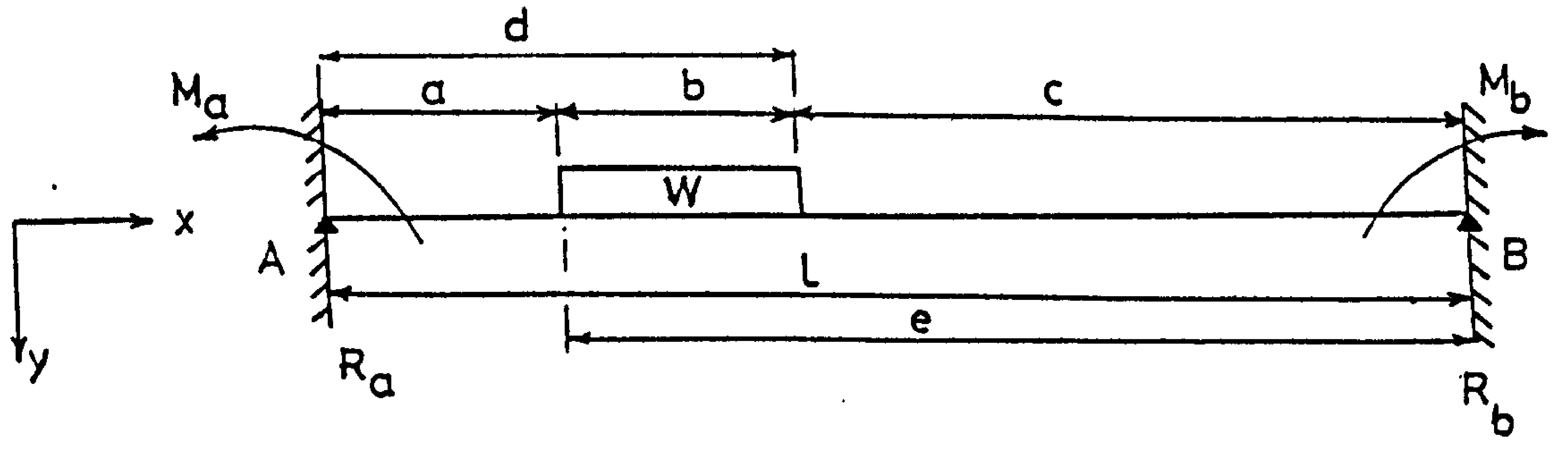


Fig (A.4)

(d) <u>Slab strips fixed at both ends</u>.

Boundary conditions $y=0$ at $x=0$ and $y=0$ at $x=L$

$\frac{dy}{dx}=0$ at $x=0$ and $\frac{dy}{dx}=0$ at $x=L$

$$R_a = r_a + \frac{M_a - M_b}{L}$$

$$R_b = r_b + \frac{M_b - M_a}{L}$$

Where r_a and r_b are the simple support reactions

$$M_a = \frac{W}{12L^2\ b}\left[e^3 (4L - 3e) - c^3 (4L - 3c)\right] \qquad \text{(Hogging)}$$

$$M_b = \frac{W}{12L^2\ b}\left[d^3 (4L - 3d) - a^3 (4L - 3a)\right] \qquad \text{(Hogging)}$$

$$\text{Then} \quad y = \frac{W}{24EI\ b}\left[12.b.M_a x^2 - 4.b.R_a .x^3 + \left[(x-a)^4\right] - \left[(x-a-b)^4\right]\right]$$

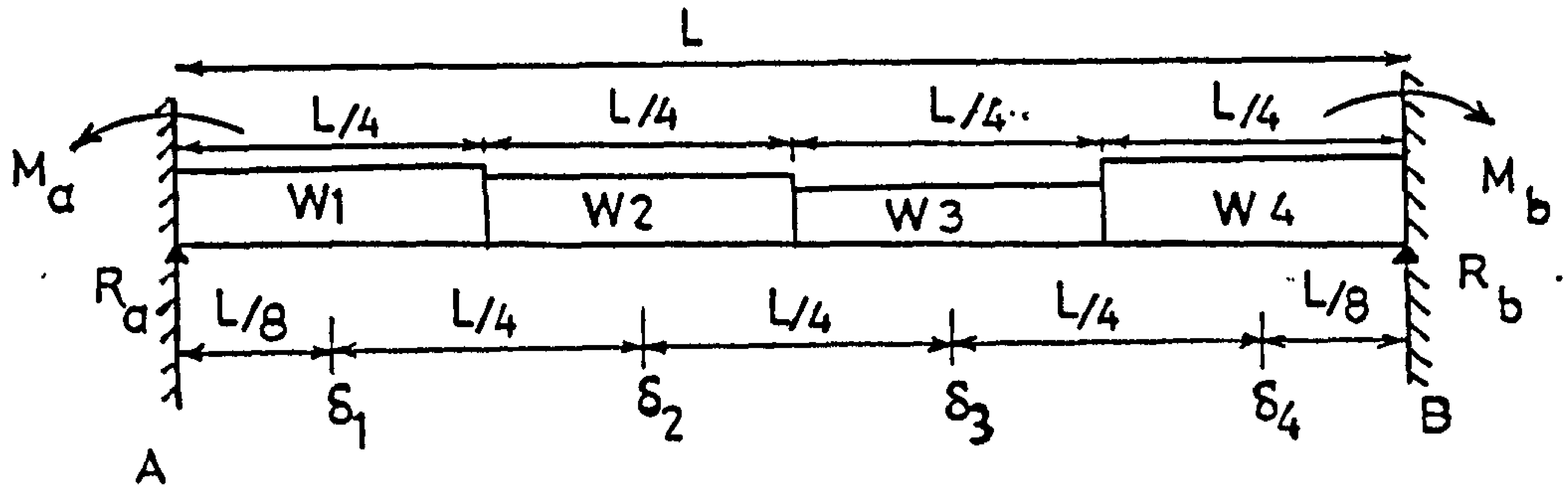


Fig (A.5)

Slab strip carrying four patch loads of equal length.

End reactions fixed end moments and deflections at the centre of patch loads.

(a) Slab strips simply supported at both ends

$R_a = 0.875\, W_1 + 0.625\, W_2 + 0.375\, W_3 + 0.125\, W_4$

$R_b = 0.125\, W_1 + 0.375\, W_2 + 0.625\, W_3 + 0.875\, W_4$

$M_a = 0$

$M_b = 0$

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \frac{L^3}{6144\ EI} \begin{bmatrix} 23.0 & 46.25 & 39.75 & 15.25 \\ 46.25 & 109.0 & 101.25 & 39.75 \\ 39.75 & 101.25 & 109.0 & 46.25 \\ 15.25 & 39.75 & 46.25 & 23.0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}$$

(b) Cantilever slab strip (End A Fixed, End B Free)

$$R_a = W_1 + W_2 + W_3 + W_4$$

$$R_b = 0$$

$$M_a = L\ (0.125W_1 + 0.375W_2 + 0.625W_3 + 0.875W_4)$$

$$M_b = 0$$

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \frac{L^3}{6144\ EI} \begin{bmatrix} 4.25 & 16 & 28 & 40 \\ 20 & 108.25 & 216 & 324 \\ 36 & 220 & 500.25 & 800 \\ 52 & 332 & 804 & 1372.25 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}$$

(c) Slab strips fixed at one edge (A) and simply supported at the other (B)

$$R_a = \frac{1}{8}\ (7.765625W_1 + 6.484375W_2 + 4.265625W_3 + 1.484375W_4)$$

$$R_b = \frac{1}{8}\ (0.234375W_1 + 1.515625W_2 + 3.734375W_3 + 6.515625W_4)$$

$$M_a = \frac{L}{512}(49W_1 + 95W_2 + 81W_3 + 31W_4)$$

$$M_b = 0$$

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \frac{L^3}{6144 \times 256\ EI} \begin{bmatrix} 743 & 1865 & 1671 & 649 \\ 2285 & 9379 & 10725 & 4131 \\ 2091 & 10245 & 14539 & 6725 \\ 817 & 4191 & 6737 & 3935 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}$$

(d) Slab Strips fixed at both edges.

$$R_a = \frac{1}{8} \left(7.5625W_1 + 5.4375W_2 + 2.5625W_3 + 0.4375W_4 \right)$$

$$R_b = \frac{1}{8} \left(0.4375W_1 + 2.5625W_2 + 5.4375W_3 + 7.5625W_4 \right)$$

$$M_a = \frac{L}{768} \left(67W_1 + 109W_2 + 67W_3 + 13W_4 \right)$$

$$M_b = \frac{L}{768} \left(13W_1 + 67W_2 + 109W_3 + 67W_4 \right)$$

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \frac{L^3}{6144 \times 64EI} \begin{bmatrix} 163 & 349 & 227 & 45 \\ 425 & 1591 & 1305 & 279 \\ 279 & 1305 & 1591 & 425 \\ 45 & 227 & 349 & 163 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}$$

Matrices for cantilever slab strips and slab strips fixed on both edges have been used in the illustrative example shown in Fig 5.6

In Chapter 5 the effects of partial composite action and the load distribution on supporting beams was illustrated by using a square slab with five equal strips each way. It is therefore of use to establish the deflection matrix for the slab strip shown in Fig (A.6)

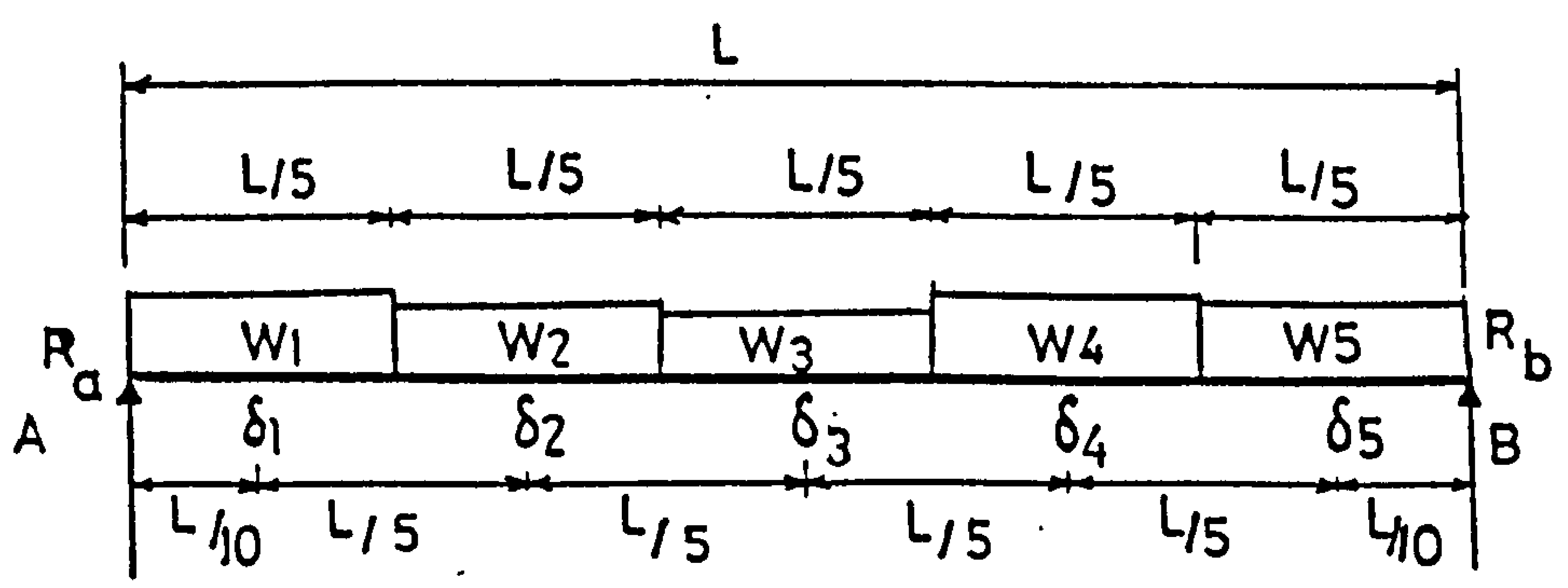


Fig (A.6) Simply supported slab strip carrying five patch loads of equal length.

$R_a \quad 0.9\, W_1 + 0.7\, W_2 + 0.5\, W_3 + 0.3\, W_4 + 0.1\, W_5$

$R_b \quad 0.1\, W_1 + 0.3\, W_2 + 0.5\, W_3 + 0.7\, W_4 + 0.9\, W_5.$

$M_a = 0$

$M_b = 0$

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{bmatrix} = \frac{L^3}{6000\ EI} \begin{bmatrix} 15.425 & 34.3 & 36.5 & 26.7 & 9.7 \\ 34.3 & 86.025 & 97.5 & 72.9 & 26.7 \\ 36.5 & 97.5 & 122.625 & 97.5 & 36.5 \\ 26.7 & 72.9 & 97.5 & 86.025 & 34.3 \\ 9.7 & 26.7 & 36.5 & 34.3 & 15.425 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{bmatrix}$$

APPENDIX II

DETAILS OF CALCULATION - STRIP DEFLECTION AND TORSIONLESS GRID METHOD

Consider the illustrative example described in Chapter (5.3.2). Due to the symmetry there are only eight unknowns. These are taken as W_y for each grid and denoted as W_1 to W_8. The total distributed load on each grid is 8.75 units. Consider the grid (32) formed by strip X_3 and strip Y_2 and shown in Fig (A.7).

Clearly $\quad W_y = W_3$

and equilibrium requires $W_x = 8.75 - W_3$

The W_x values for the other grid can be similarly determined. The fixed end moments and the end reactions can again be calculated as follows.

$$(R_x)_1 = 17.5 - W_1 - W_5$$

$$(R_x)_2 = 17.5 - W_2 - W_6$$

$$(R_x)_3 = 17.5 - W_3 - W_7$$

$$(R_x)_4 = 17.5 - W_4 - W_8$$

$$(R_y)_1 = W_5 + W_6 + W_7 + W_8$$

$$(R_y)_2 = W_1 + W_2 + W_3 + W_4$$

Check:- $2\left[(R_x)_1 + (R_x)_2 + (R_x)_3 + (R_x)_4 + (R_y)_1 + (R_y)_2\right]$

$= 140$ Units $=$ total imposed load

$$(M_y)_1 = L_y\,(0.125\,W_5 + 0.375\,W_6 + 0.625\,W_7 + 0.875\,W_8)$$

$$(M_y)_2 = L_y\,(0.125\,W_1 + 0.375\,W_2 + 0.625\,W_3 + 0.875\,W_4)$$

For this example $L_x = 1.4$ and $L_y = 1.0$ Units

In order to calculate the individual values of W_1 to W_8, the elastic deflections (Δ_x) and (Δ_y) of the respective X and Y strips are equated. For the grid (32) these can be easily

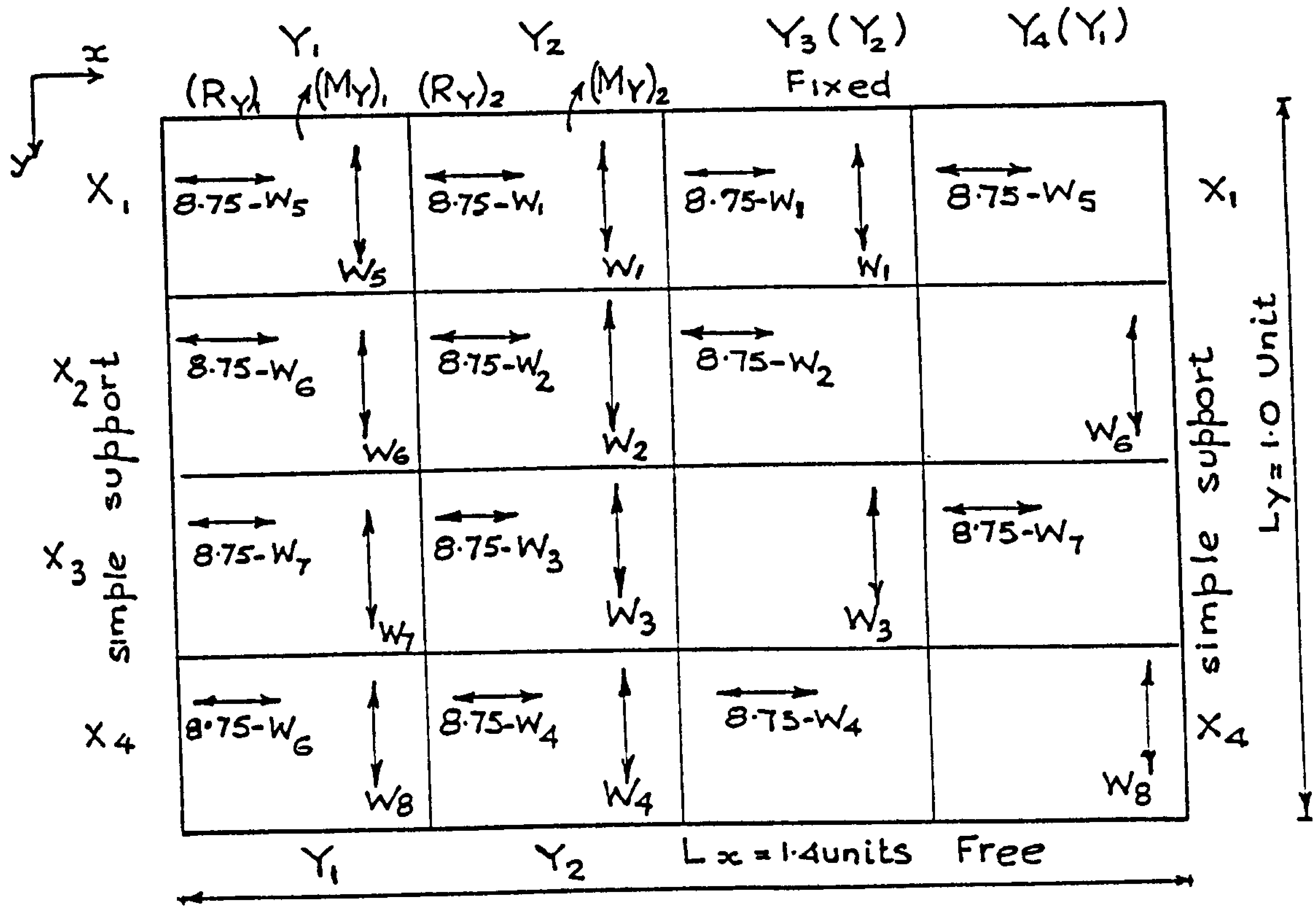


FIG (A.7) EXAMPLE ILLUSTRATIVE OF THE APPLICATION OF THE STRIP DEFLECTION METHOD

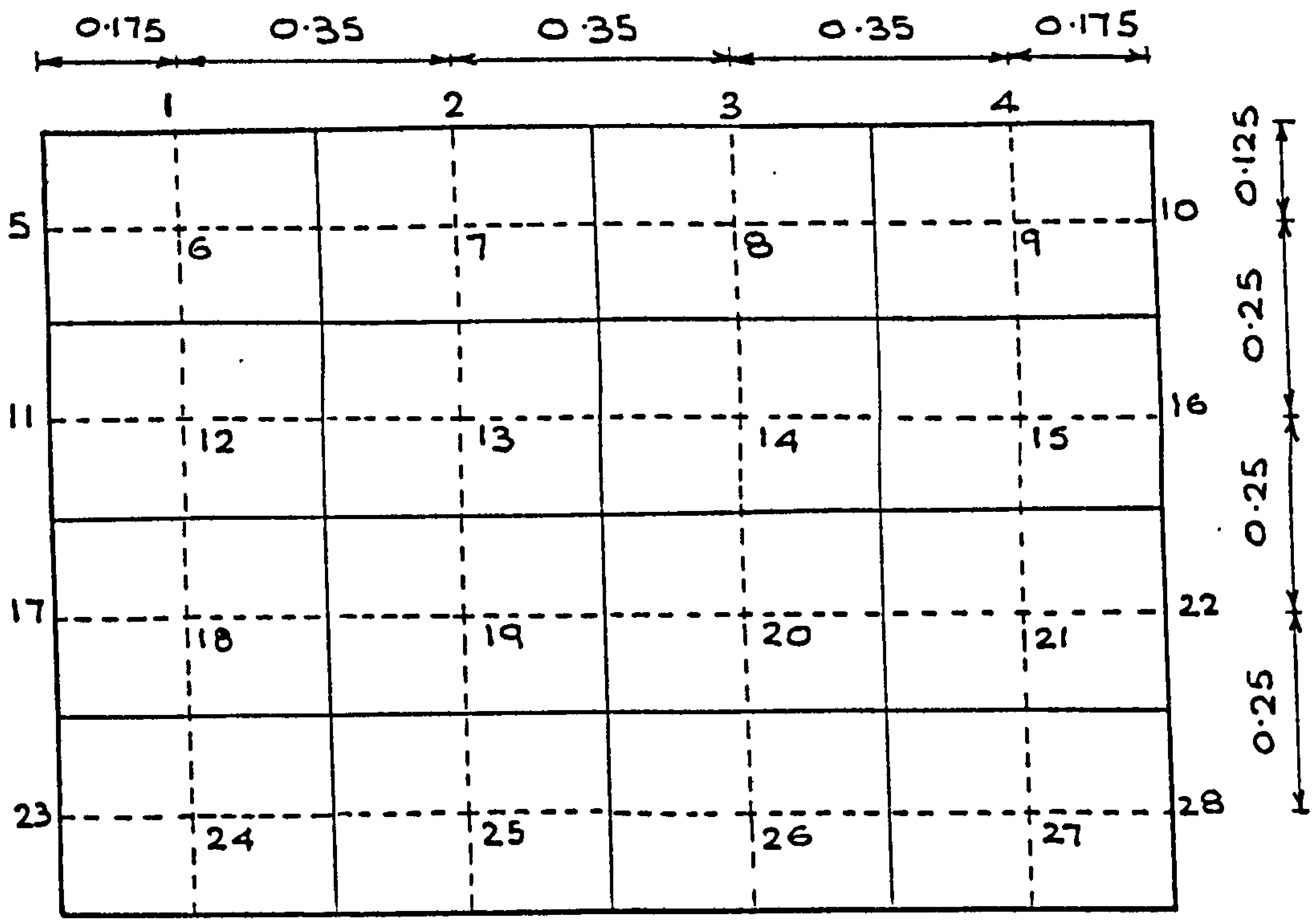


FIG (A.8) EQUIVALENT GRID FOR THE SLAB SHOWN IN FIG (A.7) calculated using the matrices given in Appendix I. Inserting the appropriate values of lengths and stiffness.

$$(\Delta_x)_{32} = \frac{(1.4)^3}{6144\ EI} (46.25 \quad 109 \quad 101.25 \quad 39.75) \begin{bmatrix} 8.75 - W_7 \\ 8.75 - W_3 \\ 8.75 - W_3 \\ 8.75 - W_7 \end{bmatrix}$$

and
$$(\Delta_y)_{32} = \frac{(1)^3}{6144\ E(1.4I)} (36 \quad 220 \quad 500.25 \quad 800) \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}$$

Equating $(\Delta_x) = (\Delta_y)$ for each grid the following set of 8 equations are obtained.

$$3.8416[\,86 \quad (8.75 - W_5) + 210.25 \quad (8.75 - W_1)] = 4.25\,W_1 + 16\,W_2 + 28\,W_3 + 40\,W_4$$

$$3.8416[\,86 \quad (8.75 - W_6) + 210.25 \quad (8.75 - W_2)] = 20\,W_1 + 108.25\,W_2 + 216\,W_3 + 324\,W_4$$

$$3.8416[\,86 \quad (8.75 - W_7) + 210.25 \quad (8.75 - W_3)] = 36\,W_1 + 200\,W_2 + 500.25\,W_3 + 800\,W_4$$

$$3.8416[\,86 \quad (8.75 - W_8) + 210.25 \quad (8.75 - W_4)] = 52\,W_1 + 332\,W_2 + 804\,W_3 + 1372.25\,W_4$$

$$3.8416[\,38.25 \quad (8.75 - W_5) + 86. \quad (8.75 - W_1)] = 4.25\,W_5 + 16\,W_6 + 28\,W_7 + 40\,W_8$$

$$3.8146[\,38.25 \quad (8.75 - W_6) + 86. \quad (8.75 - W_2)] = 20\,W_5 + 108.25\,W_6 + 216\,W_7 + 324\,W_8$$

$$3.8416[\,38.25 \quad (8.75 - W_7) + 86. \quad (8.75 - W_3)] = 36\,W_5 + 200\,W_6 + 500.25\,W_7 + 800\,W_8$$

$$3.8416[\,38.25 \quad (8.75 - W_8) + 86. \quad (8.75 - W_4)] = 52\,W_5 + 332\,W_6 + 804\,W_7 + 1372.25\,W_8$$

The eight simultaneous equations can be rearranged in a matrix form :-

$$\begin{bmatrix} 811.9464 & 16.0 & 28.0 & 40.0 & 330.3776 & 0.0 & 0.0 & 0.0 \\ 20.0 & 915.9464 & 216.0 & 324.0 & 0.0 & 330.3776 & 0.0 & 0.0 \\ 36.0 & 220.0 & 1307.9464 & 800.0 & 0.0 & 0.0 & 330.3776 & 0.0 \\ 52.0 & 332.0 & 804.0 & 2179.9464 & 0.0 & 0.0 & 0.0 & 330.3776 \\ 330.3776 & 0.0 & 0.0 & 0.0 & 151.1912 & 16.0 & 28.0 & 40.0 \\ 0.0 & 330.3776 & 0.0 & 0.0 & 20.0 & 225.1912 & 216.0 & 324.0 \\ 0.0 & 0.0 & 330.3776 & 0.0 & 36.0 & 220.0 & 647.1912 & 800.0 \\ 0.0 & 0.0 & 0.0 & 330.3776 & 52.0 & 332.0 & 804.0 & 1519.1912 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \\ W_7 \\ W_8 \end{bmatrix} = \begin{bmatrix} 9958.1475 \\ 9958.1475 \\ 9958.1475 \\ 9958.1475 \\ 4176.5395 \\ 4176.5395 \\ 4176.5395 \\ 4176.5395 \end{bmatrix}$$

The simultaneous equations can be easily solved using a standard programme. The resulting load distributions, end reactions and fixed end moments are given in Table A 1.

To solve the same slab by the torsionless grid method, the grid shown in Fig (A.8) is used. The X and Y strips are replaced by equivalent beams of the same length and flexural rigidity. The flexural rigidity of the beams are propotional to the widths of the corresponding strips and the torsional rigidity of all beams should be zero. However a very small value must be inserted for the torsional rigidity in using the Genesys Frame - Analysis programme.

The equivalent grid is

STRIP	EQUIVALENT BEAMS
X_1	5, 6, 7, 8, 9, 10
X_2	11, 12, 13, 14,15, 16
X_3	17, 18, 19, 20, 21, 22
X_4	23, 24, 25, 26, 27, 28
Y_1	1, 6, 12, 18, 24
Y_2	2, 7, 13, 19, 25
Y_3	3, 8, 14, 20, 26
Y_4	4, 9, 15, 21, 27

To simulate similar boundary conditions the grid is held fixed at points 1,2,3,4 and simply supported at 5, 10, 11, 16, 17, 22, 23, 28. Point loads of 8.75 units are applied normal to the grid surface at each of the 16 internal grid points - 6, 7, 8, 9, 12, 13, 14, 15, 18, 19, 20, 21, 24, 25, 26, 27. The results are given in Table A 1 and compared with those from the Strip deflection method.

	Strip Deflection Method	No - Torsion Grid Method
W_1	8.844 (1.011)	8.817 (1.008)
W_2	7.757 (0.887)	7.689 (0.879)
W_3	4.826 (0.552)	4.889 (0.559)
W_4	1.320 (0.151)	1.495 (0.171)
W_5	7.462 (0.853)	7.535 (0.861)
W_6	3.652 (0.417)	3.923 (0.448)
W_7	1.712 (0.196)	1.815 (0.207)
W_8	0.502 (0.057)	0.506 (0.058)
$(R_x)_1$	1.194	1.148
$(R_x)_2$	6.091	5.888
$(R_x)_3$	10.962	10.796
$(R_x)_4$	15.678	15.449
$(R_y)_1$	13.328	13.779
$(R_y)_2$	22.747	22.890
$(M_y)_1$	3.8115	3.990875
$(M_y)_2$	8.185625	8.34925

Load distribution factors are shown within brackets.

Table (A.1) Comparison of results from the strip deflection method and torsionless grid method for the slab example shown in Fig (5.5)

Bending moment and shear forces at all points of the slab can now be calculated. However it is of interest to ascertain the collapse load of this slab which has one principal moment positive and the other negative at all points. Consider the yield line pattern shown in (Fig.5.5) defined by parameter x.

$$\text{External work done} = W_c \left[\frac{x}{2} \times 4 \times \frac{100}{3} + (1.4 - 2x) \frac{100}{2} \right]$$

$$= W_c \left[70 - \frac{100\, x}{3} \right]$$

Dissipation of Internal energy D is given by

$$W_D \left[2(M_{y_1} + M_{y_2}).1 + \frac{1}{8}(R_{x1} \cdot x + R_{x2} \cdot 3x + R_{x3} .5x + R_{x4} .7x) \frac{1}{x} \right.$$
$$\left. \frac{-x^2}{2x0.35 \times 64} \left\{ (8.75 - W_5) + 9(8.75 - W_6) + 25(8.75 - W_7) + 49(8.75 - W_8) \right\} \frac{2}{x} \right]$$

Substituting for M_s , R_s and W_s

$$D = W_D \ (70 - 28 . 0 \ x)$$

Clearly the collapse load and the design load are identical when $x = 0$

Consider also the square alsb of side L supported on four identical flexible beams discussed in Chapter 5.5 and shown in Fig(A.9). The slab is divided into five equal strips bothways and carries a total load of 100 units. The loading on each of the 25 small grids is 4 Units and due to symmetry there are only 6 unknowns (W_1, W_2, W_3, R_1, R_2, R_3) which are also assumed to be uniformly distributed. Also shown are the loading on strips and supporting beams.From equilibrium.

$$2W_1 + 2W_2 - 2R_1 = -2$$
$$-W_1 + 2W_3 - 2R_2 = -8$$
$$-W_2 - 2W_3 - 2R_3 = -16 \qquad (7.1)$$

Check $(4R_1 + 8R_2 + 8R_3)$ = 100 Units = Total imposed load

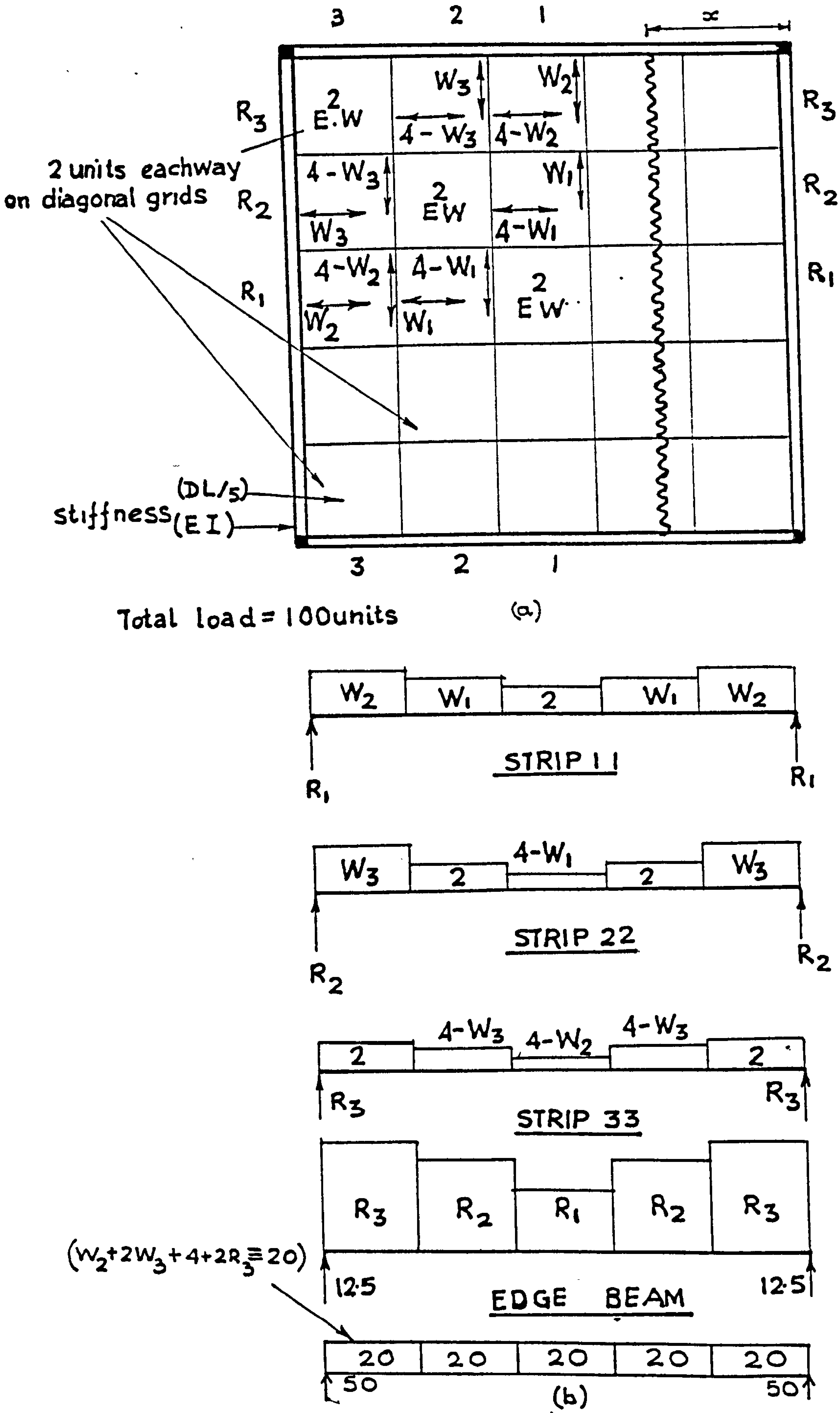


FIG (A.9) SQUARE SLAB SUPPORTED ON FOUR FLEXIBLE BEAMS

3 more equations can be written by considering the elastic deflections for points A,B and C. The edge beams being flexible the respective (Δ_x) and (Δ_y) takes into account the deflection at the boundary. Using the matrices given in Appendix 1 for point B,

$$(\Delta_x)_B = \frac{L^3}{6000(EI_s)}(73,195,122.625)\begin{vmatrix} 2 \\ 4-W_3 \\ 4-W_2 \end{vmatrix} + \frac{L^3}{6000(EI_B)}(25.125,61,36.5)\begin{vmatrix} R_3 \\ R_2 \\ R_1 \end{vmatrix}$$

and

$$(\Delta y)_B = \frac{L^3}{6000(EI_s)}(25.125,61,36.5)\begin{vmatrix} W_2 \\ W_1 \\ 2 \end{vmatrix} + \frac{L^3}{6000(EI_B)}(73,195,122.625)\begin{vmatrix} R_3 \\ R_2 \\ R_1 \end{vmatrix}$$

Where (EI_s) and (EI_B) are the stiffness of the slab strips and supporting beams respectively.

also $\frac{(EI_B)}{(EI_s)} = \epsilon$

The deflection compatibility equation can be simplyfied as

$$61W_1 + 147.75W_2 + 195W_3 + 86.125\epsilon R_1 + 134\epsilon R_2 + 47.875\epsilon R_3 = 1343.5$$

Two more equations can be written for points A and C and the three equilibrium and three deflection can be rearranged in a matrix form.

$$\begin{bmatrix} 2 & 2 & 0 & -2 & 0 & 0 \\ 1 & 0 & 2 & 0 & -2 & 0 \\ 0 & -1 & -2 & 0 & 0 & -2 \\ 281.55 & 61 & 73 & 25.125\epsilon & 36.075\epsilon & 12.0\epsilon \\ 61 & 147.75 & 195 & 86.125\epsilon & 134.0\epsilon & 47.875\epsilon \\ 36.5 & 97.5 & 184.05 & 61.0\epsilon & 97,925\epsilon & 35.875\epsilon \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ -16 \\ 685.5 \\ 1343.5 \\ 879.7 \end{bmatrix}$$

The solution of the six simultaneous equations are given in Table A.2. To determine the collapse load of this slab beam composite system consider the mechanism shown in Fig (A.9) defined by the parameter x, consisting of positive hinges in the supporting beams and positive yield lines in the slab.

$$\text{External Work done } E = W_c \quad 100.\tfrac{1}{2} = 50\, W_c$$

At collapse internal energy will be dissipated along the slab strips and at the beam hinges. In order to determine the value of D let us consider the gross loading on to the beams and strips along the x directions. Fig (A.9.b) clearly shows that this loading (100 units) is distributed uniformly along the length.

$$D = W_D \quad 50.x - \frac{100}{2L} x^2 \left(\frac{1}{x} + \frac{1}{L-X}\right)$$

$$= W_D \, .50$$

Clearly the design load is identical to the collapse load for all values of x. This can be proved for all mechanisms with positive yield lines and positive beam hinges. Results will not be changed even if the supporting beams are all of different stiffness

The equivalent torsionless grid is shown in Fig (A.10). The internal beams eg 2, 9, 16, 23, 30, 37, 44 represent the strips and external beams such as 1, 2, 3, 4, 5, 6, 7 represent the supporting edge beams. Corresponding flexural rigidities must be assigned for the internal and edge beams. The grid is simply supported at the four corners. 1, 7, 43, 49. The results obtained from the Genesys computer programme are given in Table A 2 and compared with those from the strip deflection method.

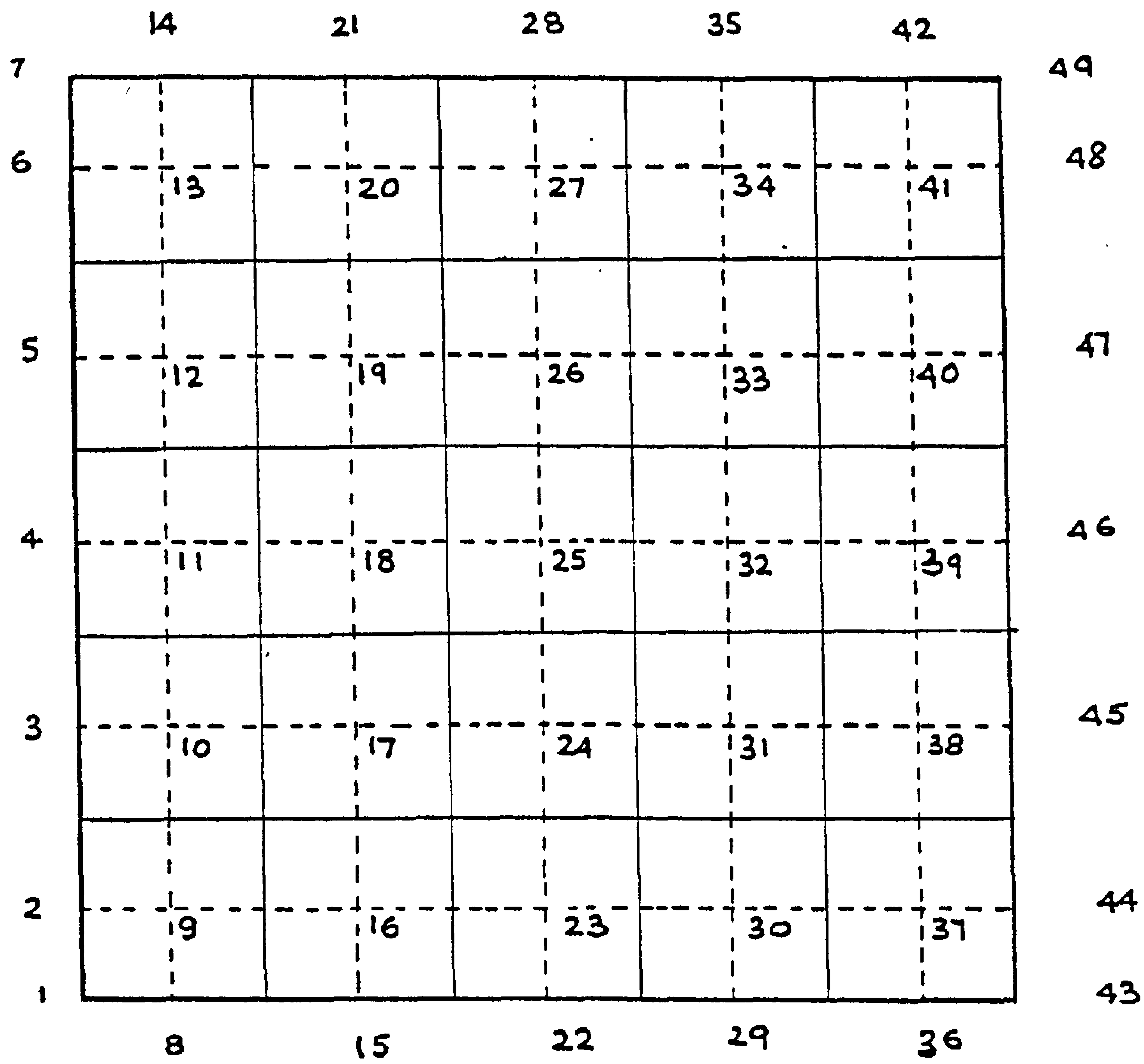


FIG (A.10) EQUIVALENT GRID FOR THE SLAB SHOWN IN FIG (A.9)

	STRIP DEFLECTION METHOD		TORSIONLESS GRID METHOD	
W_1	0.0256	(0.64)	0.0254	(0.63)
W_2	0.0350	(0.87)	.0354	(0.88)
W_3	0.0343	(0.86)	.0340	(0.85)
R_1	0.0706		0.0708	
R_2	0.0615		0.0613	
R_3	0.0282		0.0282	

The above results are for the case γ = infinity and unit imposed load. Values of load distribution factors are given within brackets. For varying beam /slab stiffnesses the values of the reactions given by the strip deflection method are;

γ	R_1	R_2	R_3
∞	0.0706	0.0615	0.0282
2.0	0.0592	0.0555	0.0399
1.0	0.05	0.500	0.05
1/3	0.0278	0.0310	.0801

Table (A.2) Solution to the square slab supported on flexible beams - Chapter 5.5

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