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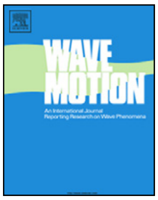

A model of acoustic absorption in fluids based on a continuous distribution of relaxation times

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ABSTRACT

This work extends the quasi-equilibrium relaxation theory of sound absorption in liquids to the case of continuous distribution of relaxation times. Such extension is needed when absorption mechanisms are not confined to the action of viscosity and heat conduction, but are mainly due to the excitation of a large number of internal molecular degrees of freedom. In this case the conventional Navier–Stokes equations are not sufficient to describe the fluid motion, and additional equations are required to model normal relaxation stresses. When relaxation frequencies form a sufficiently dense distribution, as is the case for many biological fluids, it makes sense to consider the limit of continuously distributed relaxation frequencies, in order to obtain the required equation for normal relaxation stresses.

In contrast to its discrete counterparts, the proposed method avoids the use of a potentially infinite number of relaxation equations for a given set of distinct relaxation frequencies. Instead, these are replaced by a single evolution equation of Boltzmann type whose right-hand side is a linear combination of the time derivatives of density and entropy. The rheological functions appearing before these derivatives are expressed in terms of the absorption coefficient. Since the dependence of absorption coefficient on sound frequency is measurable experimentally, these rheological coefficients can be recovered from experimental data.

The key feature of the present study is that a closed system of equations of motion can be formulated directly from absorption measurement data on the basis of the theory proposed for the very wide range of absorption laws that can occur in practice.

As an illustration of the generality of the present method, a number of absorption laws documented in the experimental literature are considered in detail, in order to derive the coefficients of the related systems of equations of motion for these liquids. For example, the methodology based on modelling of acoustic absorption in biologically soft tissue by the employment of fractional derivatives, which has been recently developed in the literature, is shown to be a special case of the proposed theory.

1. Introduction

This paper is concerned with the problem of sound absorption in liquids with complex molecular structure. The main motivation for this study is the need to design high intensity focused ultrasound (HIFU) equipment for the treatment of

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oncological disease. The related medical procedure relies on the ability to convert the energy of an ultrasound beam, which is focused on a tumour, into a thermal field whose temperature is sufficient to denature cellular proteins. The central mechanism involved in this process is ultrasound absorption.

When dissipation due to the relaxation mechanisms associated with the redistribution of the energy of macroscopic motion into the internal degrees of freedom of molecular motion is weak, acoustic losses can be accounted for by conventional viscosities and heat conductivity. The related small amplitude linear theory based on the Navier–Stokes model then gives a quadratic dependency of sound absorption coefficient on frequency, [1–3].

However, for example, in biological tissue the effects of viscosity and heat conduction are small compared to relaxation losses (see [4] and the references therein). Experiments have shown that, for a wide range of frequencies, dependence of absorption on frequency obeys a power law whose exponent is close to unity, rather than the conventional quadratic law [5–8].

For several decades this constituted the major difficulty in formulating the time-domain wave equation needed, for example, for efficient numerical modelling of problems with realistic geometries. A method of constructing such a wave equation was proposed by [9]. In this paper Szabo constructs a dispersion relation with the required properties in the frequency domain by the analysis of related wave operators and causality considerations. A linear wave equation in the time domain was then recovered from the dispersion relation via a Fourier inversion. A modification of Szabo's wave operator, originally obtained in terms of generalized functions, has been proposed in [10], where it was recast as an integro-differential operator where the lossy term was represented by a fractional derivative with respect to the time variable. A generalization of this model, for the case in which attenuation results from the spatial microstructure of the propagation medium, is presented in [11]. In contrast to the time-varying Caputo fractional derivative used originally by them in Ref. [10], in [11] Chen and Holm use a fractional Laplacian, which involves a space-varying fractional derivative. Further development of this approach and its numerical implementation can be found in [12].

A fractal viscoelastic model with an infinite number of alternating nested springs and dash-pots was recently proposed [13] in order to give physical justification for the power law absorption

$$\alpha_{\mathrm{ab}} \sim |\omega|^{\chi}, \tag{1}$$

where ω is the sound frequency. In this approach permissible values of the power law exponent χ were found to be given by the expression

$$\chi = \frac{3}{2} - \frac{1}{2^{N+1}} + \frac{1}{2^{M+1}}, \tag{2}$$

where integer numbers N and M define the depth of recursion of the dampers and the springs, respectively. In the linear limit the fractional wave equation obtained in [12] was recovered. The major drawback of this model, however, is that it does not allow for the power law exponent χ to take arbitrary real values such as, for example, $\chi = \sqrt{3}, \pi/2$, *etc.* Instead, the experimentally measured power law exponent χ must be represented by the nearest value allowed by the right-hand side of the above formula. Consequently, the refinement of the experimental value of χ is not parallelled by the refinement of the governing equation. The choice of numbers N and M may also be non-unique, *i.e.*, different configurations of dash-pots and springs may result in the same law of absorption. For example, $\chi = 1.5$ for any $N = M$.

These difficulties can be overcome if fractional time derivatives in the relationship between the stress and strain tensors are used (see [14] and the references therein). This makes it possible to account for an arbitrary power law exponent.

However, if the absorption coefficient of a liquid does not obey the power law, as is often the case, say, in the food industry, [15], then a more generic approach may be required. The large variety of absorption laws for different applications in a number of liquids shows the need for a unifying approach to the problem.

The aim of the present work is to formulate a phenomenological theory of sound absorption in liquids from the standpoint of relaxation theory, with a continuous distribution of relaxation frequencies.

The relaxation theory of absorption and dispersion of acoustic waves was proposed for the first time by [16] (see also [17]). There appears to be no comprehensive recent review of the different relaxation theories applied to acoustics, but a detailed treatment of the subject can be found in [18].

The derivation of a linear wave equation for an inhomogeneous medium with absorption dominated by a multiple, but finite, number of relaxation processes was proposed in [4]. Its analysis builds on the extensive previous research in the area summarized in [18,1] and more recently [3]. The authors of [4] employ the irreversible thermodynamic theory used to describe chemical relaxations in an electrolyte.

Experiments carried out in tissues such as liver, and in protein solutions, demonstrate that the major part of the ultrasound absorption arises at a molecular, rather than cellular, or higher, level of organization, [6,5]. There exists a broadband statistical distribution of relaxation frequencies over the four orders of magnitude starting at 0.1 MHz, i.e. over the whole range of frequencies available for observations. Consequently, it is argued in the literature, [19,6,5,3] that continuously distributed relaxations should be used to model relaxation losses, since the assumption of a finite number of relaxation processes is not sufficient to derive a power law of absorption.

In a wider scientific context, the assumption of an infinite number of relaxation processes has been used to derive the power law relationship for a number of physical problems other than acoustics. The details of such models and of those key aspects not discussed here can be found in [20–23].

In this paper we construct such a physical model with a continuous distribution of relaxations. It is based on the phenomenological theory of relaxation processes which have a long history in physics literature and was recently summarized in a monograph in which references to other relevant publications can be found, [24]; also see [25]. The present work is confined to relaxation mechanisms which result from changes in normal stresses. More specifically, we are interested in the local mechanisms of irreversible energy loss caused by uniform compression or expansion of a medium for which all components remain unchanged, rather than the losses caused by friction between different layers of a medium which move with different velocities (for a more detailed discussion of this issue see [26]). No attempt is made to model effects of shear viscosity and heat conduction beyond the conventional Navier–Stokes approach, since this topic goes far beyond the scope of this paper.

Section 2 summarizes the related thermodynamic and relaxation theories used in Section 3 to derive the system of equations with continuously distributed relaxations. Section 4 derives the expressions for the unknown phenomenological coefficients of these systems in terms of the frequency-domain absorption coefficient which may be determined by experiment. This is the main result of the work, since it makes it possible to formulate a closed system of equations of motion in physical time-space domain directly from experimental data for absorption measurements. Section 5 provides examples of application of the proposed theory. In particular, it considers the model of absorption in soft tissue with continuously distributed relaxations. The above mentioned fractional derivative model originally discovered from analysis of dispersion relations, [9], is recovered as a special case of the approach proposed in the present work which, we believe, provides a more fundamental treatment of this problem.

2. Relaxation equations

The basis for the different relaxation theories of absorption is often hidden in the underlying thermodynamic theory assumptions. In the absence of a well developed statistical theory for liquids with complex molecular structure, a number of phenomenological non-equilibrium thermodynamic models has been proposed to date. In this paper we examine the approach which builds upon the original quasi-equilibrium thermodynamic model used by [16,17], in order to derive a closed system of governing equations.

This model assumes that every sufficiently small but macroscopic fluid element is locally in a state of thermodynamic equilibrium. The Gibbs equation is postulated as the definition of entropy s so that for a homogeneous medium

$$T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}v, \tag{3}$$

where $v = 1/\rho$ and e are the specific volume and the internal energy per unit mass, ρ is the density. The pressure, p, and the temperature, T, retain their conventional meanings.

The parameters of the relaxation processes do not enter the entropy Eq. (3), or other thermodynamic relations explicitly. In a quasi-equilibrium approach relaxations are taken into account via an equation of state.

Apart from the usual thermodynamic parameters such as ρ and s each elementary macroscopic volume of a non-equilibrium medium is characterized by a set of internal parameters $\xi_k(t)$, where k is a positive number. Each parameter $\xi_k(t)$ describes the related k-th internal molecular relaxation process. During this process the parameter ξ_k recovers its equilibrium value $\xi_k^o = \xi_k^o(\rho, s)$. The difference $\xi_k(t) - \xi_k^o$ characterizes the departure of the medium from equilibrium and is assumed to be small. The time derivative of the parameter $\xi_k(t)$ is then expanded in a power series in $(\xi_k - \xi_k^o)$. Since at equilibrium $\mathrm{d}\xi_k/\mathrm{d}t$ must be zero and $\xi_k(t)$ must remain finite at all times, this results in the relaxation equation (for details see [17])

$$\frac{\mathrm{d}\xi_k}{\mathrm{d}t} = -\frac{1}{\tau_k}(\xi_k - \xi_k^o) + O((\xi_k - \xi_k^o)^2), \tag{4}$$

where τ_k is the relaxation time. In general, τ_k depends on the temperature T and pressure p. Here we assume that the relaxation parameters ξ_k can be chosen so that linear interaction terms of the form

$$\sum_{j\neq k} \frac{1}{\tau_{kj}}(\xi_j - \xi_j^o) \tag{5}$$

which, in general, can appear on the right-hand side of the Taylor series expansion (4) are removed via the diagonalization procedure, so that only the k-th term of this sum is present. Hence, if direct interaction between any two relaxation processes j and k takes place, it must be a higher order effect.

The equations governing the behaviour of the normal stresses in the presence of relaxation processes can be obtained using the methodology developed in [16,17,2,27].

For the k-th relaxation process, we define the pressure increment $p_k^r(\rho, s, \xi_k)$ over the equilibrium pressure $p(\rho, s)$. For small perturbations of the relaxation parameter ξ_k from its equilibrium value ξ_k^o the pressure increment $p_k^r(\rho, s, \xi_k)$ can be expanded into the power series in $(\xi_k - \xi_k^o)$, and with the same accuracy as for the relaxation equation (4) we can write

$$p_k^r(\rho, s, \xi_k) = a_k(\rho, s)(\xi_k - \xi_k^o) + O((\xi_k - \xi_k^o)^2), \tag{6}$$

where $a_k(\rho, s)$ is an unknown coefficient. The zeroth order term is absent in (6), since $p_k^r = 0$ at equilibrium.

Since the theory is linear in $(\xi_k - \xi_k^o)$, the total pressure p_Σ is equal to the quasi-equilibrium pressure $p(\rho, s)$ plus the sum of pressure increments due to each relaxation process

$$p_\Sigma = p(\rho, s) + \sum_{(k)} p_k^r(\rho, s, \xi_k). \tag{7}$$

We also introduce the relaxation pressure $p_k(\rho, s, \xi_k)$ of the k-th relaxation process, that is, the pressure which would exist in the absence of all other relaxation processes:

$$p_k(\rho, s, \xi) = p(\rho, s) + p_k^r(\rho, s, \xi). \tag{8}$$

Taking the time derivative of (6) we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p_k^r = m_\rho^k \frac{\mathrm{d}\rho}{\mathrm{d}t} + m_s^k \frac{\mathrm{d}s}{\mathrm{d}t} + a_k \frac{\mathrm{d}\xi_k}{\mathrm{d}t}. \tag{9}$$

Since $a_k(\xi_k - \xi_k^o) = p_k(\rho, s, \xi) - p(\rho, s)$, here

$$m_\rho^k \equiv \frac{\partial}{\partial \rho}\{a_k(\xi_k - \xi_k^o)\}_s = \left(\frac{\partial p_k}{\partial \rho} - \frac{\partial p}{\partial \rho}\right)_s, \tag{10}$$

$$m_s^k \equiv \frac{\partial}{\partial s}\{a_k(\xi_k - \xi_k^o)\}_\rho = \left(\frac{\partial p_k}{\partial s} - \frac{\partial p}{\partial s}\right)_\rho. \tag{11}$$

The right-hand side of (10) is the difference of the square of the local speed of sound in the medium when only the k-th relaxation process is excited, and the square of the local speed of sound in the absence of relaxation phenomena, respectively. As shown in [17], according to Le Chatelier's principle, $m_\rho^k > 0$. The right-hand side of (11) is proportional to the difference between the rate of temperature change per unit volume expansion when only the k-th relaxation process is excited and the rate of temperature change per unit volume expansion in the absence of relaxation phenomena.

It follows from (4) and (6) that

$$a_k \frac{\mathrm{d}\xi_k}{\mathrm{d}t} = -\frac{a_k}{\tau_k}(\xi_k - \xi_k^o) = -\frac{1}{\tau_k} p_k^r. \tag{12}$$

Substitution of expression (12) into (9) results in the following equations for the pressure increments p_k^r

$$\frac{\mathrm{d}}{\mathrm{d}t} p_k^r = m_\rho^k \frac{\mathrm{d}\rho}{\mathrm{d}t} + m_s^k \frac{\mathrm{d}s}{\mathrm{d}t} - \frac{1}{\tau_k} p_k^r. \tag{13}$$

Coefficients m_ρ^k and m_S^k are functions of ρ, s, τ_k and ξ_k. However, since

$$m_\rho^k = -a_k \frac{\partial \xi_k^o}{\partial \rho} + \frac{\partial a_k}{\partial \rho}(\xi_k - \xi_k^o) \quad \text{and} \quad m_s^k = -a_k \frac{\partial \xi_k^o}{\partial s} + \frac{\partial a_k}{\partial s}(\xi_k - \xi_k^o), \tag{14}$$

to leading order, the dependence on ξ_k can be neglected and the coefficients m_ρ^k and m_s^k can be treated as the phenomenological functions to be determined by experiment.

The governing system can be closed if Eqs. (7) and (13) are supplemented by the equation of continuity, compressible Navier–Stokes equations, energy equation and the caloric equation of state relating the specific internal energy e, density and temperature [17,28]:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div}(\underline{v}) = 0, \tag{15}$$

$$\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial}{\partial x_i} p_\Sigma + \frac{\partial}{\partial x_k} \sigma'_{ik}, \tag{16}$$

$$\rho \frac{\mathrm{d}e}{\mathrm{d}t} = -p_\Sigma \operatorname{div}(\underline{v}) + \sigma'_{ik} \frac{\partial v_i}{\partial x_k} + \operatorname{div}(\kappa \nabla T), \tag{17}$$

$$e = e(\rho, T), \tag{18}$$

where $\underline{v} = (v_1, v_2, v_3)$ is the velocity vector, $\underline{r} = (x_1, x_2, x_3)$ is the position vector, σ'_{ik} is the viscous stress tensor defined by Eq. (19)

$$\sigma'_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik} \frac{\partial v_l}{\partial x_l}\right) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x_l}, \tag{19}$$

η is the shear viscosity, ς is the volume viscosity, κ is the coefficient of heat conductivity. The summation operation is carried out with respect to the repeating subscripts.

For complex media like biological fluids and soft tissue the structure of the caloric equation of state (18) is largely unknown to-date. For the purposes of the present work, it is convenient to use the following well-known thermodynamic relationship

$$\mathrm{d}e = \left(p - \frac{\rho T \beta c_s^2}{\gamma}\right)\frac{1}{\rho^2}\mathrm{d}\rho + c_v \mathrm{d}T, \tag{20}$$

where β is the coefficient of thermal expansion, c_s is the speed of sound at constant entropy, $\gamma = c_p/c_v$ is the specific heat ratio, c_v and c_p are the specific heats at constant volume and pressure, respectively.

No linearity assumptions are made about the quasi-equilibrium thermodynamic equation of state $p(\rho, s)$. It is taken in generic form here, since in biological fluids it is often the thermodynamic equation of state that is the main source of non-linearity.

3. Continuously distributed relaxations

To extend the above theory to continuous distribution of relaxation times, instead of p_k^r, m_ρ^k and m_s^k, it is convenient to introduce the functions:

$$\hat{p}^r(\varpi) = \varpi^{-1} p_k^r, \qquad \hat{m}_\rho(\varpi) = \varpi^{-1} m_\rho^k, \qquad \hat{m}_s(\varpi) = \varpi^{-1} m_s^k, \tag{21}$$

where $\varpi = 1/\tau_k$ is a relaxation frequency. As opposed to the discrete set of parameters $\{\tau_k\}$, we now assume that $\tau_k > 0$ is a continuous variable. Thus the variable ϖ is allowed to vary continuously also. Hence, every relaxation process is characterized by the relaxation frequency ϖ, and we shall now consider the relaxation pressure increment per unit relaxation frequency $\hat{p}^r$, instead of the relaxation pressure increment p_k^r.

Assume everywhere in what follows that the variations of the relaxation times $\tau_k(T, p)$ due to temperature and pressure changes are negligible, and ϖ can be considered as a parameter independent of the macroscopic time t and the position vector. Eq. (13) can then be rewritten as

$$\frac{\partial \hat{p}^r}{\partial t} + \underline{v} \cdot \nabla \hat{p}^r + \varpi \hat{p}^r = \hat{m}_\rho \frac{\mathrm{d}\rho}{\mathrm{d}t} + \hat{m}_s \frac{\mathrm{d}s}{\mathrm{d}t}. \tag{22}$$

The summation in Eq. (7) is substituted by the integral over the entire range of relaxation frequencies ϖ:

$$p_\Sigma = p(\rho, s) + \int_0^\infty \hat{p}^r(\varpi, t, \underline{r})\mathrm{d}\varpi. \tag{23}$$

Note that the function p_Σ does not depend on ϖ, since ϖ enters the last formula as an integration variable. Neither do any of the unknown variables $\underline{v}, \rho, e, s$ or T in the governing system (15)–(20) depend on ϖ, since the only ϖ-dependent function $\hat{p}^r$ appears in this system via the function p_Σ given by Eq. (23), that is, in the integral with respect to ϖ. Consequently, if we define functions $p^r(\tau)$, $m_\rho(\tau)$ and $m_s(\tau)$ of a new independent variable τ that has the dimension of time as the Laplace inversions of functions $\hat{p}^r(\varpi)$, $\hat{m}(\varpi)$ and $\hat{m}_s(\varpi)$ with respect to variable ϖ, that is,

$$\hat{p}^r = \int_0^\infty p^r(\tau)\mathrm{e}^{-\varpi\tau}\mathrm{d}\tau, \qquad \hat{m}_\rho = \int_0^\infty m_\rho(\tau)\mathrm{e}^{-\varpi\tau}\mathrm{d}\tau, \qquad \hat{m}_s = \int_0^\infty m_s(\tau)\mathrm{e}^{-\varpi\tau}\mathrm{d}\tau, \tag{24}$$

then the inverse Laplace transformation of the Eq. (22) for $\hat{p}^r$ with respect to the independent variable ϖ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}p^r + \frac{\partial}{\partial \tau}p^r = m_\rho \frac{\mathrm{d}\rho}{\mathrm{d}t} + m_s \frac{\mathrm{d}s}{\mathrm{d}t}, \tag{25}$$

where by definition $p^r(\tau \le 0) = 0$.

Since

$$I = \int_0^\infty \hat{p}^r(\varpi)\mathrm{d}\varpi = \int_0^\infty p^r(\tau)\int_0^\infty \mathrm{e}^{-\varpi\tau}\mathrm{d}\varpi\,\mathrm{d}\tau = \int_0^\infty p^r(\tau)\tau^{-1}\mathrm{d}\tau, \tag{26}$$

we obtain

$$p_\Sigma = p(\rho, s) + \int_0^\infty p^r(\tau)\tau^{-1}\mathrm{d}\tau, \tag{27}$$

where I is the total relaxation pressure.

As a result, an infinite set of equations for the functions p_k^r has been reduced to a single equation of Boltzmann type (25) for the function p^r.

Eqs. (15)–(19), (25), (27) constitute the final system of governing equations for media with continuously distributed relaxation times which is being presented here for the first time.

Strictly speaking, appearance of phenomenological evolutionary Eq. (25) results in the requirement that the rate of entropy production must be a positive functional of p^r on the solutions of the governing system of equations. One way to work out the structure of the entropy equation in this situation is to use the formalism of the extended irreversible thermodynamics, whose detailed description can be found in the monographs [29,30], where the relationship between the conventional quasi-equilibrium irreversible thermodynamics and the extended irreversible thermodynamics are discussed in detail. The involved analytical procedure closely follows similar derivation for viscous heat conducting fluids described in Section 7.2 of monograph [29] and is not reproduced here for this reason. It results in the following extended Gibbs equation for the specific entropy:

$$T\frac{\mathrm{d}s}{\mathrm{d}t}=\left(\frac{\mathrm{d}e}{\mathrm{d}t}+p_{\Sigma}\frac{\mathrm{d}v}{\mathrm{d}t}\right)-\frac{1}{2\rho^2}\int_0^{\infty}\left(p^r\right)^2\frac{\partial}{\partial\tau}\left(\frac{1}{\tau m_\rho}\right)\mathrm{d}\tau, \tag{28}$$

where the integral on the right-hand side of Eq. (28) must be negative. The sufficient (although not necessary) condition of this is $\partial(\tau m_\rho)/\partial\tau \geq 0$. It can be inferred from the analysis of the next section that the additional terms appearing in the entropy equation (28) have negligible higher-order effect on the properties of the acoustic field considered in this work.

System (15)–(19), (25), (27) contains phenomenological functions $m_\rho(\tau,\rho,s)$ and $m_s(\tau,\rho,s)$ which determine the absorption and dispersion properties of the medium. Our aim now is to obtain these functions in terms of the experimentally measurable absorption coefficient in the frequency domain. This will allow us to formulate a closed physical model for media with continuous distribution of relaxations.

4. Determination of the absorption functions

Assume that unsteady perturbations of the hydrodynamic parameters in the medium at rest are small, and consider a linear version of the governing equations. This can be obtained if a solution is sought in the form of the following series

$$\rho=\rho_o+\rho'+\cdots,\qquad T=T_o+T'\cdots,\qquad s=s_o+s'+\cdots, \tag{29}$$

$$p_\Sigma=p_o+p'_\Sigma+\cdots,\qquad \underline{v}=\underline{v}'+\cdots,\qquad p^r=p'_r+\cdots, \tag{30}$$

$$p=p_o+c_{so}^2\rho'+\frac{c_{po}-c_{vo}}{\beta_o c_{vo}}\rho_o s'+\cdots, \tag{31}$$

$$m_\rho=m_{\rho o}(\tau)+\cdots,\qquad m_s=m_{so}(\tau)+\cdots \tag{32}$$

where the subscript "*o*" refers to the unperturbed values of hydrodynamic parameters. All dashed variables represent the leading order corrections to the related unperturbed values.

Substitution of these expansions into the equations of motion (15)–(20), (25), (27) results in the following system

$$\frac{\partial^2\rho'}{\partial t^2}-\nabla^2 p'_\Sigma-\frac{4\eta_o/3+\varsigma_o}{\rho_o}\nabla^2\frac{\partial\rho'}{\partial t}=0, \tag{33}$$

$$\frac{\partial p'_r}{\partial t}+\frac{\partial p'_r}{\partial\tau}=m_{\rho o}\frac{\partial\rho'}{\partial t}+m_{so}\frac{\partial s'}{\partial t}, \tag{34}$$

$$p'_\Sigma=c_{so}^2\rho'+\frac{c_{po}-c_{vo}}{\beta_o c_{vo}}\rho_o s'+\int_0^{\infty}p'_r\frac{\mathrm{d}\tau}{\tau}, \tag{35}$$

$$\frac{\partial T'}{\partial t}=\frac{\beta_o T_o c_{so}^2}{c_{vo}\rho_o}\frac{\partial\rho'}{\partial t}+\frac{\kappa_o}{c_{vo}\rho_o}\nabla^2 T', \tag{36}$$

$$\frac{\partial s'}{\partial t}=\frac{\kappa_o}{\rho_o T_o}\nabla^2 T'. \tag{37}$$

In the absence of viscosity, thermal conductivity and relaxations this system leads to conventional wave equation for the function ρ'.

In the presence of absorption the shape of the linear sinusoidal sound wave changes on a length scale proportional to the reciprocal of the absorption coefficient α_{ab}. As a result, there exist two characteristic length scales in lossy media—the wavelength and the absorption length equal to $1/\alpha_{\mathrm{ab}}$. Their ratio defines the nondimensional absorption coefficient $\varepsilon=c_{so}\alpha_{\mathrm{ab}}/f$, where f is the frequency of the sound wave.

In biological media this value is, normally, small. For example in [8], the amplitude attenuation coefficient for whole human liver *in-vivo* is given by the approximate formula $\alpha_{\mathrm{ab}}\approx 0.052f$ Np/cm, where the frequency of the sound wave f is measured in MHz. This gives the attenuation length of 19.2 cm, if the reference frequency f is taken to be 1 MHz. With the speed of sound in liver being $c_{so}\approx 1580$ cm/s, the value of the wavelength corresponding to $f=1$ MHz is equal to $c_{so}/f\approx 1.58\cdot 10^{-1}$ cm. Although the absorption coefficient is not exactly equal to the attenuation coefficient (but, normally, is somewhat smaller) and the actual values of the attenuation coefficient vary between different literature sources, it is important here that the value of the nondimensional absorption coefficient is a reasonably small quantity approximately equal to $\varepsilon = 8.2 \cdot 10^{-3}$. Consequently, a typical value of the nondimensional absorption coefficient can be used as a natural small parameter ε for further asymptotic analysis of the system (33)–(37).

By virtue of the linear character of the problem, $m_{\rho o}(\tau)$ must be proportional to α_{ab}, and the coefficient $m_{\rho o}(\tau)$ can also be expected to be small.

The alternative way of introducing the small parameter ε is to define it as the ratio of some typical value of the function $m_{\rho o}$ to c_{so}^2, since, in agreement with formula (10), function $m_{\rho o}$ has the same dimension as the square of the speed of sound. Thus, let

$$m_{\rho o} = \varepsilon\sigma(\tau), \qquad \eta_o = \varepsilon\eta', \qquad \zeta_o = \varepsilon\zeta', \qquad \kappa_o = \varepsilon\kappa', \tag{38}$$

where ε is a small parameter and $\sigma(\tau)$, η', ζ', κ' are assumed to be of the order of unity in ε.

Consider the problem of one-dimensional wave propagation in the direction of the positive x_1-axis. Assume that the wave field is excited by the sinusoidal motion of a piston which has started at $t = 0$. The mean position of the surface of the piston coincides with the plane $x_1 = 0$.

The solution to the system (33)–(37) can be constructed as an asymptotic expansion in ε:

$$\rho' = \rho_1(\theta, \xi) + o(1), \qquad p'_\Sigma = p_1(\theta, \xi) + o(1), \tag{39}$$

$$s' = \varepsilon s_1(\theta, \xi) + o(\varepsilon), \qquad T' = T_1(\theta, \xi) + o(1), \qquad p'_r = \varepsilon p_r(\theta, \xi, \tau) + o(\varepsilon), \tag{40}$$

where the independent variable $\theta = t - x_1/c_{so}$ describes fast changes of the phase of the wave whereas the independent variable $\xi = \varepsilon x_1$ describes slow variations of the amplitude and phase due to absorption and dispersion.

Substituting formulae (38)–(40) into the system (33)–(37) and retaining only the leading order terms in ε we obtain the following system of equations

$$2\frac{\partial p_1}{\partial \xi} = c_{so}^{-1}\frac{\partial}{\partial \theta}\int_0^\infty p_r\frac{\mathrm{d}\tau}{\tau} + \lambda\frac{\partial^2 p_1}{\partial \theta^2}, \tag{41}$$

$$\frac{\partial p_r}{\partial \theta} + \frac{\partial p_r}{\partial \tau} = \frac{\sigma(\tau)}{c_{so}^2}\frac{\partial p_1}{\partial \theta} + \mu\frac{\beta_o m_{so}(\tau)}{\rho_o}\frac{\partial^2 p_1}{\partial \theta^2}, \tag{42}$$

$$\rho_1 = c_{so}^{-2}p_1, \qquad s_1 = \mu\frac{\beta_o}{\rho_o}\frac{\partial p_1}{\partial \theta}, \qquad T_1 = \frac{\beta_o T_o}{c_{po}\rho_o}p_1, \tag{43}$$

where

$$\lambda = \frac{1}{\rho_o c_{so}^3}\left(\zeta' + \frac{4}{3}\eta' + \kappa'\left(\frac{1}{c_{vo}} - \frac{1}{c_{po}}\right)\right), \qquad \mu = \frac{\kappa'}{c_{po}\rho_o c_{so}^2}. \tag{44}$$

The initial and boundary conditions are

$$p_1(\theta \le 0) = p_r(\theta \le 0) = p_r(\tau \le 0) = 0, \tag{45}$$

$$p_1(\xi = 0) = A\sin(\omega\theta), \qquad p_1(\theta, \xi \to \infty) \to 0, \tag{46}$$

where $A > 0$ is a given amplitude of pressure perturbation at $\xi = 0$.

The solution to this problem can be obtained by means of the Laplace transformation with respect to θ, thus

$$p_1 = \frac{A\omega}{2\pi\mathrm{i}}\int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty}\varphi(u)\exp(u\theta)\mathrm{d}u, \tag{47}$$

where $\gamma' > 0$,

$$\varphi(u) = \exp(uc(u)\xi/2)/(\omega^2 + u^2), \tag{48}$$

$$c(u) = c_{so}^{-1}\int_0^\infty\int_0^\tau\left\{u\frac{\sigma(\tau')}{c_{so}^2} + \mu u^2\frac{\beta_o m_{so}(\tau')}{\rho_o}\right\}\mathrm{e}^{-u(\tau-\tau')}\mathrm{d}\tau'\frac{\mathrm{d}\tau}{\tau} + \lambda u. \tag{49}$$

At this stage of analysis the function $c(u)$ remains unknown, since our aim is to link $\sigma(\tau)$ and $m_{so}(\tau)$ to the absorption coefficient of the medium $\alpha_{\mathrm{ab}}(\omega)$.

It can be expected on physical grounds that after a sufficiently long time, the initial perturbations created when switching the piston on will have decayed, with the remaining portion of the sound field being entirely due to the forced periodic motion of the piston. For this reason, the knowledge of the transient part of the solution (47) is not needed to determine the functional relationship between functions $\sigma(\tau)$ and $m_{so}(\tau)$ and the absorption coefficient $\alpha_{\mathrm{ab}}(\omega)$. For this purpose we expand the solution (47) at long retarded time θ.

In order to obtain this asymptotic expansion, additional assumptions have to be imposed on the structure of the function $\varphi(u)$. These can be formulated on the basis of Theorem 41.1 in [31] and the discussion therein. We require that the function $\varphi(u)$ of the complex variable u (i) be analytic in the half-plane $\mathrm{Re}(u) > \gamma$; (ii) have any number of poles of any order or branching points in the left half-plane $\mathrm{Re}(u) < 0$, and (iii) have any finite number of branching points or zeros $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-2}$ on the vertical line $\mathrm{Re}(u) = 0$, but no order one poles, apart from the poles $\alpha_{n-1} = -\mathrm{i}\omega$ and $\alpha_n = \mathrm{i}\omega$ which correspond to the driving frequency of the piston, *i.e.*, in the vicinity of any point α_j, where $j = 1, 2, 3, \ldots, n-2$, the function $\varphi(u)$ can be expanded into the series

$$\varphi(u) = \sum_{\nu=0}^{\infty} c_\nu (u-\alpha_j)^{\lambda_\nu^j}, \quad -1 < \lambda_0^j < \lambda_1^j < \cdots < \infty. \tag{50}$$

According to the results formulated in [31] and also in [32], the asymptotic expansion of the inverse Laplace transform (47) as θ goes to plus infinity is then given by the term

$$p_1 = A\mathrm{e}^{-0.5\omega c_i(\omega)\xi} \sin\left(\omega\left(\theta + \frac{1}{2}c_r(\omega)\xi\right)\right) + \cdots, \tag{51}$$

plus the sum of functions that describe the decaying transient motion. Here $c_r(\omega) + \mathrm{i}c_i(\omega) = c(\mathrm{i}\omega)$,

$$c_r(\omega) = \frac{\omega}{c_{so}} \int_0^\infty \int_0^\tau \frac{\sigma(\tau')}{c_{so}^2} \sin\omega(\tau-\tau') - \mu\omega \frac{\beta_o m_{so}(\tau')}{\rho_o} \cos\omega(\tau-\tau')\mathrm{d}\tau' \frac{\mathrm{d}\tau}{\tau}, \tag{52}$$

$$c_i(\omega) = \frac{\omega}{c_{so}} \int_0^\infty \int_0^\tau \frac{\sigma(\tau')}{c_{so}^2} \cos\omega(\tau-\tau') + \mu\omega \frac{\beta_o m_{so}(\tau')}{\rho_o} \sin\omega(\tau-\tau')\mathrm{d}\tau' \frac{\mathrm{d}\tau}{\tau} + \lambda\omega, \tag{53}$$

where τ' is the integration variable.

In agreement with formula (51), the absorption coefficient is given by the following expression

$$\alpha_{\mathrm{ab}}(\omega) = \varepsilon\omega c_i(\omega)/2. \tag{54}$$

After substitution of formula (54) for $c_i(\omega)$, and formula (44) for λ, Eq. (54) takes the form

$$\alpha_{\mathrm{ab}}(\omega) = \frac{\varepsilon\omega^2}{2c_{so}} \int_0^\infty \int_0^\tau \left\{ \frac{\sigma(\tau')}{c_{so}^2} \cos\omega(\tau-\tau') + \mu\omega \frac{\beta_o m_{so}(\tau')}{\rho_o} \sin\omega(\tau-\tau') \right\} \mathrm{d}\tau' \frac{\mathrm{d}\tau}{\tau} + \alpha_{\mathrm{ab}}^{\mathrm{n}}(\omega), \tag{55}$$

where

$$\alpha_{\mathrm{ab}}^{\mathrm{n}}(\omega) = \frac{\varepsilon\omega^2}{2\rho_o c_{so}^3} \left(\zeta' + \frac{4}{3}\eta' + \kappa' \left(\frac{1}{c_{vo}} - \frac{1}{c_{po}} \right) \right) \tag{56}$$

is the well known expression for the absorption coefficient in a viscous heat conducting Newtonian fluid, [2]. This is of no interest for the subsequent study, and in what follows only the relaxation part of the absorption coefficient

$$\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega) = \alpha_{\mathrm{ab}}(\omega) - \alpha_{\mathrm{ab}}^{\mathrm{n}}(\omega) \tag{57}$$

is considered.

The nondimensional combination of parameters

$$\mu\omega = \frac{\kappa'\omega}{c_{po}\rho_o c_{so}^2} \tag{58}$$

in formula (55) characterizes the influence of the thermal conductivity on the relaxation part of the absorption coefficient $\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)$. In many practical situations this value is small, and the thermal conductivity is negligible. This is certainly true for most biological applications. For example, the value of $\mu\omega$ based on the experimental data presented in [8] for the typical frequency of 1 MHz results in the assessment $\mu\omega \sim 10^{-6}$ for most soft tissues and biological fluids. For this reason, the term containing the coefficient $\mu\omega$ will be neglected.

In terms of functions $\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)$ and $m_{\rho o} = \varepsilon\sigma(\tau)$ given by formulae (38) and (57), respectively, formula (55) can then be rewritten as the following integral equation for the unknown rheological function $m_{\rho o}$:

$$\int_0^\infty \int_0^\tau m_{\rho o}(\tau') \cos\omega(\tau-\tau')\mathrm{d}\tau' \frac{\mathrm{d}\tau}{\tau} = \frac{2c_{so}^3}{\omega^2} \alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega). \tag{59}$$

4.1. *Solution of the integral equation for the absorption function $m_\rho(\tau)$*

Our aim now is to obtain an analytic formula for $m_{\rho o}(\tau)$ as a function of absorption coefficient $\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)$, in order to formulate a closed system of equations of motion for media whose absorption properties are described by the absorption coefficient $\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)$ as a function of the sound frequency $\omega \geq 0$.

If we change the order of integration in the left-hand side of formula (59), make the substitution $y = \tau - \tau'$ in the resulting inner integral and then change the order of integration to the original one, after the introduction of the function

$$\psi(y) = \int_0^\infty m_{\rho o}(\tau') \frac{\mathrm{d}\tau'}{y+\tau'} \tag{60}$$

the integral equation (59) takes the form

$$\int_0^\infty \psi(y')\cos\omega y'\,\mathrm{d}y' = f(\omega), \tag{61}$$

where $f(\omega) = 2c_{so}^3\omega^{-2}\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)$ and $0 \le \omega < +\infty$.

Multiplication of the left-hand side of Eq. (61) by $\cos\omega y\,\mathrm{d}\omega$, where $y > 0$, and integration over the interval $0 \le \omega < +\infty$ yields

$$\psi(y) = \frac{2}{\pi}\int_0^\infty f(\omega)\cos\omega y\,\mathrm{d}\omega. \tag{62}$$

Now consider analytic continuation of (60) to the right half-plane $\mathrm{Re}\,(y) > 0$. By the residue theorem, the inverse Laplace transformation of the right-hand side gives the relationship

$$\mathcal{L}_u^{-1}\{\psi(y)\} = \int_0^\infty m_{\rho o}(\tau)\mathrm{e}^{-u\tau}\mathrm{d}\tau = \hat{m}_{\rho o}(u), \tag{63}$$

i.e., the inverse Laplace transform of $\psi(y)$ is equal to the Laplace transform $\hat{m}_{\rho o}(u)$ of the function $m_{\rho o}(\tau)$.

In view of formula (62), the unknown function $m_{\rho o}(\tau)$ can be rewritten in terms of the absorption coefficient as

$$m_{\rho o}(\tau) = \frac{4c_{so}^3}{\pi}\left(\frac{1}{2\pi\mathrm{i}}\right)^2\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty}\mathrm{e}^{u\tau}\mathrm{d}u\int_{\gamma_2-\mathrm{i}\infty}^{\gamma_2+\mathrm{i}\infty}\mathrm{e}^{uy}\mathrm{d}y\int_0^\infty\frac{\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)\cos\omega y}{\omega^2}\,\mathrm{d}\omega, \tag{64}$$

where $\gamma_1, \gamma_2 \ge 0$.

Formula (64) solves the integral equation (59) and gives the required representation of the unknown phenomenological function $m_{\rho o}(\tau)$ versus the experimentally measurable value of the absorption coefficient $\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)$. Together with the relaxation equation (25) and the equation of state (27) it solves the problem of reconstruction of the equations of motion from experimental measurements.

The advantage of these equations over previous approaches to this problem that have been reported in the literature (see Introduction for more details) is that these formulae are not restricted to any specific form of the functional dependency of the absorption coefficient on the frequency ω, such as the power law absorption, for instance.

The only essential physical restrictions on applicability of formula (64) is the well-proven hypothesis of negligible thermal conductivity on the time scale of acoustic motion, and the finite value of the total relaxation pressure on the right-hand side of formula (27).

5. Examples of applications

The adiabatic version of theory ($s = \mu = 0$), proposed in the previous section, will now be applied to a number of experimental laws of sound absorption used in different applications. The following examples demonstrate the use of formula (64), in order to recover the function $m_\rho(\tau)$ which appears in the relaxation equation (25) and characterizes the absorption and dispersion properties of the medium.

A knowledge of the rheological function $m_\rho(\tau)$ is important because it gives a closed set of equations which describes the wave propagation process. This new set of equations consists of the standard hydrodynamic equations plus an additional equation for the new state variable p^r which appears in the thermodynamic equation of state, and introduces coupling with the internal structure of the material. Further theoretical discussion of this approach can be found in [24] and references therein.

A full nonlinear version of the resulting governing system is given by (15)–(20), (25), (27). Its linear counterpart for uniform media is the system (33)–(37).

In many practical situations the non-dimensionalized values of the acoustic velocity $\underline{v}$, the density derivative $\mathrm{d}\rho/\mathrm{d}t$ and the absorption function m_ρ are small. Since the relaxation pressure p^r is of the order of the product $m_\rho\mathrm{d}\rho/\mathrm{d}t$, the convective term $\underline{v}\cdot\nabla\hat{p}^r$ in (22) is $o\left(\underline{v}^2\right)$ and can often be neglected. This is certainly true for a quadratically nonlinear theory. In this case the linear version of the Eq. (22) can be used. Its solution is given by the following expression

$$\hat{p}^r = \int_0^t \frac{\mathrm{d}\rho}{\mathrm{d}t'}\hat{m}_\rho(\varpi)\exp(-\varpi(t-t'))\mathrm{d}t', \tag{65}$$

which after the substitution into the first equality (26) gives the following formula for the total relaxation pressure

$$I = \int_0^t \frac{\mathrm{d}\rho}{\mathrm{d}t'}J_i(t-t')\mathrm{d}t', \tag{66}$$

where $i = 1, 2$ and

$$J_1(y) = \int_0^\infty \frac{m_\rho(\tau)\mathrm{d}\tau}{\tau+y} \quad \text{or} \quad J_2(y) = \frac{4}{\pi}c_s^3\int_0^\infty\frac{\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)\cos\omega y}{\omega^2}\mathrm{d}\omega, \tag{67}$$

if use of formula (64) is made. Since $p_\Sigma = p(\rho, s) + I$, the nonlinear momentum equation then takes the following compact form:

$$\frac{\mathrm{d}\underline{v}}{\mathrm{d}t} + \frac{1}{\rho}\nabla p(\rho, s) - \frac{1}{\rho}\nabla \int_0^t t\{\rho\nabla \cdot \underline{v}\}(t')J(t-t')\mathrm{d}t' = 0, \tag{68}$$

where $J(y) = J_1$ or $J(y) = J_2$. It must be supplemented with the continuity Eq. (15) and the equation of state

$$p(\rho, s) = \mathrm{const} + c_{so}^2(\rho - \rho_o) + \frac{B}{2}(\rho - \rho_o)^2 + \cdots, \tag{69}$$

where B is the nonlinearity parameter of the medium, and $\rho_o(\underline{r})$ is the distribution of density in the medium at rest.

A number of examples of how the wave equations in different lossy media can be generated with the help of this system is given in this section. The methods for their numerical solution comprise a separate topic which goes beyond the scope of this paper and will be addressed elsewhere.

5.1. *Example 1: finite number of relaxations*

This example demonstrates how the case of a medium with a finite number of relaxation frequencies can be derived from the present theory with continuously distributed relaxations proposed in the previous sections. A finite number of relaxations is typical, say, for atmospheric absorption, where the absorption coefficient is the sum of two relaxation terms associated with the vibration of nitrogen and oxygen molecules plus the Newtonian term (56), [3]:

$$\alpha_{\mathrm{ab}}(\omega) = \alpha_{\mathrm{ab}}^{\mathrm{n}}(\omega) + \sum_n \frac{\varepsilon}{2}\frac{m_n}{\varpi_n}\frac{\omega^2}{1 + \omega^2\varpi_n^{-2}}, \tag{70}$$

where m_n are constant, $\varpi_n > 0$ are the discrete relaxation frequencies of the medium, and n is an integer.

If the expression for $\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega) = \alpha_{\mathrm{ab}}(\omega) - \alpha_{\mathrm{ab}}^{\mathrm{n}}(\omega)$, where $\alpha_{\mathrm{ab}}(\omega)$ is defined by (70), is substituted into the second formula (67), it can be shown after some algebra that

$$J(y) = \varepsilon \sum_n m_n \mathrm{e}^{-y\varpi_n}. \tag{71}$$

Hence, the final expression for the total relaxation pressure is:

$$I = \varepsilon c_{so}^3 \sum_n m_n \int_0^t \frac{\mathrm{d}\rho}{\mathrm{d}t'}\mathrm{e}^{-\varpi_n(t-t')}\mathrm{d}t' \tag{72}$$

which coincides with the well known formula for multiple discrete relaxations, [2].

5.2. *Example 2: absorption of sound in soft tissue and biological fluids*

For many types of soft tissue and biological fluids the attenuation coefficient is given by the empirical formula, [8]:

$$\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega) = \varepsilon\alpha_o\omega^{1+r}, \tag{73}$$

known as a power law of absorption. Here $\alpha_o > 0$ and $0 < r < 1$ are experimental constants.

According to the second formula (67), the power law of absorption (73) implies that

$$J(y) = \frac{4}{\pi}c_{so}^3\varepsilon\alpha_o\Gamma(r)\cos(\pi r/2)y^{-r}. \tag{74}$$

Consequently, the total relaxation pressure (66) is the fractional derivative of the density

$$I = \frac{4\cos(\pi r/2)\Gamma(r)}{\pi}\varepsilon\alpha_o c_{so}^3 \int_0^t \frac{\mathrm{d}\rho}{\mathrm{d}t'}\frac{\mathrm{d}t'}{(t-t')^r}. \tag{75}$$

If $r = 0$ (*i.e.*, when $\alpha_{\mathrm{ab}}(\omega)$ is linear in ω), then $\Gamma(r) = \infty$ and the right-hand is unbounded.

Calculation of the absorption function results in the expression

$$m_{\rho o}(\tau) = \frac{4\cos(\pi r/2)\sin(\pi r)\Gamma(r)}{\pi^2}\begin{cases}0, & \text{if } \tau \leq 0\\ \varepsilon\alpha_o c_{so}^3\tau^{-r}, & \text{if } \tau > 0.\end{cases} \tag{76}$$

The unidirectional 1D linear problem (41)–(46) for the pressure field p_1 then reduces to an integro-differential equation of the Volterra type:

$$\partial_\xi p_1 = \frac{2\cos(\pi r/2)\Gamma(r)}{\pi}\alpha_o\partial_\theta \int_0^\theta \frac{\partial p_1(\theta')}{\partial\theta'}\frac{\mathrm{d}\theta'}{(\theta-\theta')^r} \tag{77}$$

with the related initial and boundary conditions. Here, as elsewhere in the paper, and for consistency with other literature sources mentioned in the Introduction, we discard the Newtonian absorption term and set $\lambda = 0$.

This equation is also known in the literature as the equation of fractional dynamics, [33]. An in depth discussion of the fundamental issues of derivation of such equations can be found in [34]. In recent years this topic has received a great deal of attention in several acoustics publications, such as [9,35,11], in which the absorption mechanism was modelled in terms of fractional derivatives. In this section we have demonstrated how this approach can be derived as a special case of the theory with a continuous distribution of relaxation processes.

The solution to the unidirectional equation (77) is

$$p_1 = A\omega\mathcal{L}_u^{-1}\left\{(\omega^2+u^2)^{-1}\exp\left(u\theta + \frac{\alpha_o}{\sin\pi r/2}u^{r+1}\xi\right)\right\}. \tag{78}$$

The leading term of the asymptotic expansion of integrals of this type when θ goes to infinity is determined by the poles of the integrand, [32], that is,

$$p_1 \approx A\exp(-\alpha_o\omega^{r+1}\xi)\sin(\omega\theta + \cot(\pi r/2)\alpha_o\omega^{r+1}\xi). \tag{79}$$

As expected, this last formula gives the same absorption coefficient $\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)$ as that quoted at the beginning of this subsection. The increase of the speed of sound over the speed of sound in the medium at rest given by this formula is

$$\Delta c_s = \cot(\pi r/2)\varepsilon\alpha_o\omega^{r+1}. \tag{80}$$

This result is consistent with the related dispersion relation obtained in [36] from formula (73) using the causality argument based on the regularized Kramers–Kronig relation.

The related non-unidirectional linear equation of fractional dynamics considered in [12] is readily recovered on linearization of the system (15), (68), (69), with the function J being defined by formula (74).

5.3. *Example 3: linear law of absorption*

Consider now the linear law of absorption

$$\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega) = \varepsilon\alpha_o\omega. \tag{81}$$

Formula (81) gives a reasonably good fit to experimental data in the MHz frequency range for soft tissue and some biological fluids, [8], and, owing to its simple structure, plays an important role in theoretical studies, [37].

Formula (81) is the special case of the power law of absorption (73) with $r = 0$. In this case the system of governing equations becomes singular, since $\Gamma\ (r = +0) = +\infty$ in (74).

Formula (80) also shows that in the limit of linear absorption the solution obtained in the previous subsection for the power law of absorption develops a singularity. Indeed, Δc_s blows up as r goes to zero.

In order to overcome this difficulty, the authors of [36] proposed a form of regularized Kramers–Kronig relation specifically tailored for the case of the linear absorption law (81). In the limit $r = +0$ their dispersion relation (*cf.* formula (88)) results in the logarithmic dependence of Δc_s on ω, rather than expression (80). However, the exact structure of the governing equation of motion in physical time-space has remained unclear.

The approach we propose in this paper is particularly suitable for answering this question. It also sheds extra light on the mathematical and physical nature of the singularity.

In order to illustrate the essence of the problem and to recover the system of governing equations which corresponds to the regularized Kramers–Kronig relation used in [36] to treat the case of linear absorption law, consider the function $m_{\rho o}(\tau) = \varepsilon\sigma(\tau)$, where $\sigma(\tau)$ is taken in the form

$$\sigma(\tau) = \alpha_o c_{so}^3\begin{cases}0, & \text{if } \tau < 0\\ \exp(-\Delta\tau), & \text{if } \tau > 0\end{cases} \tag{82}$$

where $\Delta > 0$ is a constant. Then, the first formula (67) results in the expression

$$J(y) = \frac{4}{\pi}c_{so}^3\varepsilon\alpha_o\mathrm{e}^{\Delta y}\int_{\Delta y}^{\infty}\frac{\mathrm{e}^{-z}\mathrm{d}z}{z}, \tag{83}$$

with the related nonlinear governing system given by the Eqs. (15), (68), (69). Its linear unidirectional 1D version (41), (46) reduces to the following integro-differential equation:

$$\partial_\xi p_1 = 2\alpha_o\int_0^{\theta}\frac{\partial}{\partial\theta'}p_1(\theta')\int_0^{\infty}\frac{\mathrm{e}^{-\Delta z}}{z+\theta-\theta'}\mathrm{d}z\mathrm{d}\theta'. \tag{84}$$

Solution of Eq. (84) by means of the Laplace transformation in θ and subsequent evaluation of the asymptotic behaviour of the function p_1 for large θ results in the following formula

$$p_1(\theta\to+\infty) = \exp(-\alpha_{\mathrm{ab}}^{\mathrm{r}}(\omega)\xi)\sin\left(\omega\theta - \frac{0.5\alpha_o\omega^2}{\Delta^2+\omega^2}\xi\left(\omega\ln\left(\frac{\Delta}{\omega}\right) - \Delta\frac{\pi}{2}\right)\right), \tag{85}$$

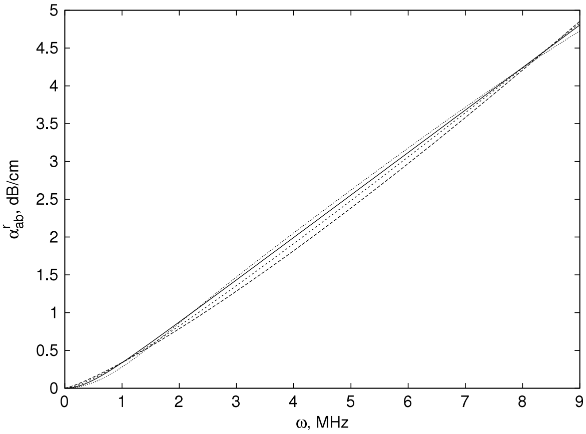

Fig. 1. Absorption coefficient $\alpha^{\mathrm{r}}_{\mathrm{ab}}$ versus frequency ω: solid line – combined formula (89) obtained experimentally in [38], dashed line – power law (73), short dashed line – formula (86), dotted line – two discrete relaxations (70). In agreement with formulae (70), (73), (86), (89), in all cases $\alpha^{\mathrm{r}}_{\mathrm{ab}}(\omega = 0) = 0$.

where the absorption coefficient is

$$\alpha^{\mathrm{r}}_{\mathrm{ab}}(\omega) = \frac{0.5\varepsilon\alpha_o\omega^2}{\Delta^2+\omega^2}\left(\omega\frac{\pi}{2} - \Delta\ln\left(\frac{\omega}{\Delta}\right)\right). \tag{86}$$

For low frequencies ($\omega \to 0$) this gives the slightly higher than quadratic growth of absorption with frequency observed in Newtonian fluids.

In the limit of small Δ the solution is a pressure wave with the absorption coefficient linearly dependent on the frequency:

$$p_1(\theta \to \infty) = \mathrm{e}^{-\frac{\pi}{4}\alpha_o\omega\xi}\sin\left(\omega\theta + \frac{\alpha_o\omega}{2}\ln\left(\frac{\omega}{\Delta}\right)\xi\right) + \cdots. \tag{87}$$

This formula gives exactly the same dispersion relation as the dispersion relation given as Eq. (13) in [36] derived from the regularized Kramers–Kronig relation for the case of linear absorption law (81):

$$\frac{1}{c_{\mathrm{rel}}(\omega)} - \frac{1}{c_{so}} = -\frac{\alpha_o\omega}{2}\ln\left(\frac{\omega}{\Delta}\right), \tag{88}$$

where $c_{\mathrm{rel}}(\omega)$ is the phase speed at frequency ω in the medium with relaxations. It is essential that the parameter Δ is identical to the reference frequency ω_o of [36]. Consequently, we see that this reference frequency is not arbitrary, but has a precise physical meaning. It determines the rate of decay of the rheological function (82) at infinity and the slope of the absorption coefficient (86) at low frequencies in the law of absorption (86) which correspond to the regularized dispersion relation (88).

5.4. *Combined absorption function for biological tissues*

Although the power law of absorption (73) fits well with experimental data in the MHz frequency range, at low and very high frequencies it may not be appropriate, [8].

An interesting two parameter absorption model

$$\alpha^{\mathrm{r}}_{\mathrm{ab}}(\omega) = \varepsilon\alpha_*\omega\arctan\beta_*\omega, \qquad \alpha_*, \beta_* = \mathrm{const} \tag{89}$$

which combines the classical quadratic law in the lower frequency limit with the linear law of absorption in the MHz frequency range was proposed in [38] for biological tissues. According to [38], this model is almost an order of magnitude more accurate than the power law of absorption. The major improvement of formula (89) over the linear formula (81) is in the lower frequency range.

In agreement with the second formula (67), the function $J(y)$ for the absorption law (89) is given by the formula

$$J(y) = 2c_{so}^3\varepsilon\alpha_o\int_{y/\beta}^{\infty}\frac{\mathrm{e}^{-z}\mathrm{d}z}{z}, \tag{90}$$

with the related governing equations being (15), (68), (69).

Fig. 1 presents the comparison of absorption models for the function $\alpha^{\mathrm{r}}_{\mathrm{ab}}$ based on the formulae (70), (73), (86), (89) for the case of homogenized bovine liver studied experimentally in [38].

Following [38], we take $\varepsilon\alpha_* = 2\pi \times 0.057$ dB/(cm MHz), $\beta_* = 2\pi \times 0.22$ 1/MHz in formula (89) and $\varepsilon\alpha_o = 0.34$ dB/cm, $r = 0.21$ in the power law formula (73), respectively. According to Jongen et al. [38], the minimal least square

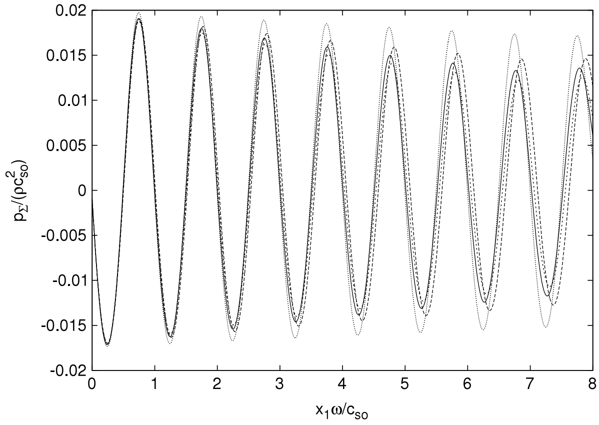

Fig. 2. Solution of the linear system of equations. Nondimensional sound pressure $p_\Sigma/(c_{so}^2\rho_o)$ versus nondimensional distance $x_1\omega/c_{so}$ at time $t = 50.26/\omega$, $\omega = 1$ MHz: solid line – combined formula (89), dashed line – power law (73), short dashed line – formula (86), dotted line – two discrete relaxations (70).

deviations from the power law (73), and the combined absorption function (89) from the experimental data obtained by these authors in the range between 1 and 9 MHz are equal to 0.047 and 0.008, respectively. Jongen et al. report that the agreement between their measurements made, for samples obtained from 7 livers, is within 5%. Further details concerning the accuracy of the above experimental data and comparisons with the fits provided by formulae (73) and (89) can be found in [38].

Unfortunately, the absence of tabulated experimental data in [38] precludes making similar fits and related mean square error assessments for the absorption laws (70) and (86). For this reason, the unknown parameters in the absorption laws (70) and (86) were chosen so as to insure the best agreement with the experimental formula (89) in the frequency range $0 < \omega < 9$ MHz, for which the experiments were conducted in [38]. These are as follows: $\varepsilon\alpha_o = 0.8$ dB/cm, $\Delta = 0.82$ MHz for the formula (86) and $n = 1, 2$, $\varepsilon m_1 = 1.25$ 1/MHz, $\varepsilon m_2 = 1.5$ 1/MHz, $\varpi_1 = 2.5$ MHz, $\varpi_2 = 11.5$ MHz for formula (70), respectively. It can be seen from Fig. 1 that the overall quality of approximations based on the formulae (70) and (86) is at least as good as that provided by the experimentally obtained power law formula (73).

Our aim now is to compare the implications for the solutions of the lossy wave Eqs. (15), (68), (69) which follow from the four absorption laws shown in Fig. 1. This can be done by numerical solution of the linear and nonlinear versions of this system of equations, with the functions $J(y)$ being given by formulae (71), (74), (83) and (90), respectively. Since the objective of this work is to study the properties of the absorption operators, only the one-dimensional version of the governing equations is considered, in order to exclude spatial effects. For the sake of simplicity, in our numerical study we neglect all nonlinear terms except the term $0.5B(\rho - \rho_o)^2$ in the equation of state (69). Normally, in soft biological tissues this term dominates all other nonlinearities, owing to the large value of the coefficient of nonlinearity B. The following parameters of the medium are used in the computations, [8]: $\rho_o = 1070$ kg/m^3, $c_{so} = 1600$ m/s, $B = 7.5c_{so}^2/\rho_o$.

The medium occupies the interval $0 < x_1 < x_{\max}$ and is set into motion by the sinusoidal perturbation of the velocity on its left boundary $x_1 = 0$ with frequency $\omega = 1$ MHz. The initial and boundary conditions are: $v_1 = 0$, $\rho = \rho_o$ at $t = 0$, $v_1(x_1 = 0) = A\sin(\omega t)$ and $v_1(x_1 = x_{\max}) = 0$. The position of the solid wall $x_1 = x_{\max}$ which bounds the computational domain on the right is taken far enough to ensure the absence of the reflected wave pattern in the bulk of the computational domain for sufficiently large times. To obtain an unbiased comparison between different lossy operators, no flux or slope limiters are implemented in the nonlinear numerical scheme in order to ensure computational stability on the shock fronts. Instead, this is achieved solely through the increase of the number of mesh points till the shock structure can be properly resolved.

Fig. 2 illustrates the behaviour of the sound pressure versus the propagation distance obtained from the solution of the linearized system of governing equations for the above four absorption laws. It can be inferred from Fig. 2 that the solutions based on the combined formula (89), formula (86) and the power law (73) are in a reasonably good agreement with each other, although the power law is in a somewhat worse agreement with the first two of these solutions. The solution based on the formula with two discrete relaxations (70) works well only within the distance equal to one wave period from the source, but does not give good agreement with the other three absorption models elsewhere.

Fig. 3 demonstrates the behaviour of the nonlinear solutions corresponding to the above four absorption laws. There is good agreement between pressure amplitude predictions based on the combined formula (89) and formula (86). The power law (73) and the formula with two discrete relaxations (70) result in higher pressure amplitudes and feature shock wave formation which do not occur in the solutions corresponding to formulae (89) and (86). Compared to the combined formula (89), at least a twelvefold increase in the number of mesh points per wavelength was needed to suppress numerical oscillations near the shock fronts, if formula (70) with two discrete relaxations was used. For the power law formula (73) an equivalent fivefold increase was required. However, the power law solution features large phase differences relative to the combined formula and the formula with two discrete relaxations which are in phase with each other.

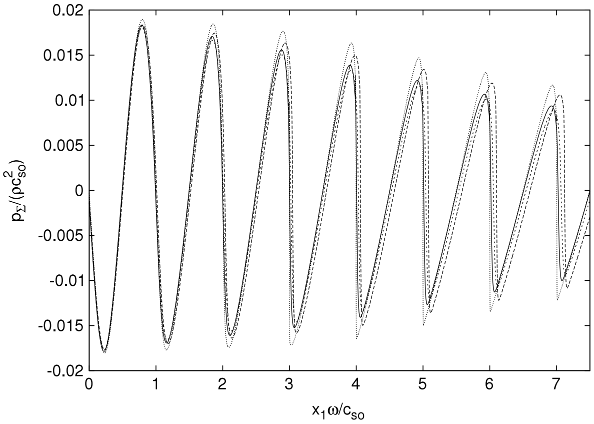

Fig. 3. Solution of the nonlinear system of equations. Nondimensional sound pressure $p_\Sigma/(c_{so}^2\rho_o)$ versus nondimensional distance $x_1\omega/c_{so}$ at time $t = 50.26/\omega$, $\omega = 1$ MHz: solid line – combined formula (89), dashed line – power law (73), short dashed line – formula (86), dotted line – two discrete relaxations (70).

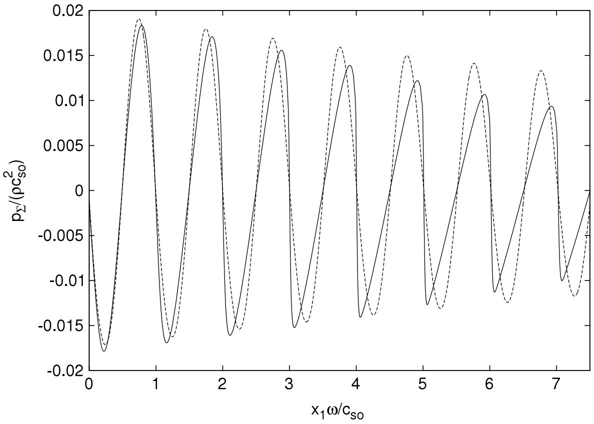

Fig. 4. Comparison of linear and nonlinear solutions based on the combined formula (89). Nondimensional sound pressure $p_\Sigma/(c_{so}^2\rho_o)$ versus nondimensional distance $x_1\omega/c_{so}$ at time $t = 50.26/\omega$, $\omega = 1$ MHz: solid line – nonlinear solution, dashed line – linear solution.

Fig. 4 features the effects of nonlinearity on the solution. It depicts the sound pressure plots presented in Figs. 2 and 3 for the case of the combined absorption law (89). Nonlinear steepening of the wave profiles can be clearly observed. Also visible is the effect of strong nonlinear damping, as would be expected *a priori* from general theoretical considerations [2].

Overall, we see that the proposed methodology makes it possible to derive a complete set of equations for modelling the mechanism of absorption in tissue. In terms of numerical modelling, and in the context of HIFU applications, it makes sense to also employ models (86) and (89) described in Sections 5.3 and 5.4, along with the power law of absorption and the corresponding fractional wave equation. This is especially true bearing in mind that, over a wide range of frequencies, formula (89) is excellent agreement with the experimental data, [38]. Our experience indicates that the combined formula (89) can be computationally more favourable for the numerical treatment of acoustic problems with strong nonlinearity.

6. Conclusions

Accurate theoretical description of acoustic waves in fluids with complex internal structure is not normally possible within the framework of Newtonian fluid dynamics because this does not provide an adequate model of the absorption mechanism. Since the pioneering work of [16], it has been recognized that an extension of classical hydrodynamics incorporating the interaction between the macroscopic motion of the fluid and its internal structure is required in order to overcome this difficulty.

This can often be done by the introduction of a finite number of additional parameters characterizing the internal state of the liquid. This must be accompanied by the related modification of the governing hydrodynamic equations, in order to accommodate the additional set of dynamic equations for these new variables in a self-consistent way, [24].

However, in media such as biological fluids the situation is more complex. Since 1970, it has been realized that a finite number of internal variables is not sufficient to account for the properties of absorption in biological media, [6]. Although the concept of a continuous distribution of relaxation degrees of freedom was put forward, over the next forty years no relaxation theory with continuous distribution of relaxation frequencies had been formulated in acoustics.

The main result of this work is that it bridges this theoretical gap. More specifically, the coupled system of governing equations (15)–(20), (25), (27), and their linear counterpart (33)–(37), which link acoustic motion with the dynamic reaction of the internal structure of the fluid to this motion has been formulated in Section 3.

In the important case of adiabatic motion, when this system contains only one unknown rheological function $m_\rho(\tau, \rho, s)$, explicit expressions for this function in terms of the experimentally measurable absorption coefficient as a function of the wave frequency have been derived in Section 4. As a result, a closed system of equations of motion can be formulated directly from absorption measurement data on the basis of the proposed theory.

As demonstrated in Section 5, the theory presented here incorporates, as its special cases, the classical theory of discrete relaxations and the fractional wave equations recently proposed for biological tissues, [9,11–13].

The work presented clarifies the physical nature of the singularity which develops in the fractional wave equations in the case of linear absorption and provides a method for reconstruction of the wave equation corresponding to the regularization procedure described in the literature, [36], in order to remove this singularity.

One of the major methodological differences between the present theory and the fractal ladder approach of [13], is that the use of a discrete set of internal parameters in [13] results in the fact that their theory does not allow for continuous variation of the exponent (2) in the power law absorption. The present theory is free from this restriction. It also provides a much wider range of absorption laws than a power law. This feature of the present theory gives a much needed flexibility in the numerical treatment of the problem as has been illustrated by the examples presented in Section 5.

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