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EXPLICIT RUNGE–KUTTA SCHEMES AND FINITE ELEMENTS WITH SYMMETRIC STABILIZATION FOR FIRST-ORDER LINEAR PDE SYSTEMS*

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Abstract. We analyze explicit Runge–Kutta schemes in time combined with stabilized finite elements in space to approximate evolution problems with a first-order linear differential operator in space of Friedrichs type. For the time discretization, we consider explicit second- and third-order Runge–Kutta schemes. We identify a general set of properties on the space stabilization, encompassing continuous and discontinuous finite elements, under which we prove stability estimates using energy arguments. Then we establish L^2-norm error estimates with quasi-optimal convergence rates for smooth solutions in space and time. These results hold under the usual CFL condition for third-order Runge–Kutta schemes and any polynomial degree in space and for second-order Runge–Kutta schemes and first-order polynomials in space. For second-order Runge–Kutta schemes and higher polynomial degrees in space, a tightened 4/3-CFL condition is required. Numerical results are presented for smooth and rough solutions. The case of finite volumes is briefly discussed.

Key words. first-order PDEs, transient problems, stabilized finite elements, discontinuous Galerkin, explicit Runge–Kutta schemes, stability, convergence

AMS subject classifications. 65M12, 65M15, 65M60, 65L06

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1. Introduction. Let Ω be an open, bounded, Lipschitz domain in $\mathbb{R}^d$, and let $T > 0$ be a finite time. We consider the following linear evolution problem:

$$\partial_t u + Au = f \qquad \text{in } \Omega \times (0,T), \tag{1.1a}$$

$$u(\cdot, t = 0) = u_0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$(M - D)u = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{1.1c}$$

Here, $u : \Omega \times (0,T) \to \mathbb{R}^m$, $m \geq 1$, is the unknown, $f : \Omega \times (0,T) \to \mathbb{R}^m$ is the source term, and A is a first-order linear differential operator in space endowed with a symmetry property specified below (the operator A can also accommodate a zero-order term). Moreover, u_0 is the initial datum, D is a symmetric-valued boundary field associated with the differential operator A by an integration by parts formula, and M is a user-dependent nonnegative boundary field used to enforce the boundary condition. The exact solution u is assumed to be smooth enough to have a trace at $t = 0$ and on the boundary $\partial\Omega$ of Ω. Typical examples include advection problems and linear wave propagation problems in electromagnetics and acoustics.

Our goal is to analyze approximations to (1.1a) using explicit Runge–Kutta (RK) schemes in time and finite elements with symmetric stabilization in space. Explicit

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RK schemes are popular methods to approximate in time systems of ordinary differential equations. In the context of space semidiscretization by discontinuous Galerkin (DG) methods, explicit RK schemes have been developed by Cockburn and Shu [11], Cockburn, Lin, and Shu [10], and Cockburn, Hou, and Shu [8] and applied to a wide range of engineering problems (see, e.g., [9] and references therein). This is in stark contrast with the case of space semidiscretization by continuous finite elements where, to our knowledge, stabilization techniques have not yet been analyzed in combination with explicit RK schemes. In particular, stabilization by consistent streamline diffusion seems not to be compatible with explicit RK schemes. In fact, the only viable explicit method with continuous approximation in space for the present evolution problems is, to our knowledge, the method of characteristics [15, 25]. Alternatively, implicit methods can be considered (see, e.g., [19, 16, 4]), i.e., based on $\mathcal{A}$-stable time discretizations or semi-implicit methods [4], resulting from the combination of an $\mathcal{A}$-stable scheme with an appropriate explicit treatment of the space stabilization operator.

The starting point of our analysis is the observation that owing to the properties of the differential operator A and the boundary condition (1.1c), there is a real number λ_0 such that (s.t.) for the exact solution,

$$(Au, u)_L \geq \tfrac{1}{2}(Mu, u)_{L,\partial\Omega} - \lambda_0 \|u\|_L^2. \tag{1.2}$$

Here, we have set $L := [L^2(\Omega)]^m$, $(\cdot,\cdot)_L$ denotes the usual scalar product in L with associated norm $\|\cdot\|_L$, and $(\cdot,\cdot)_{L,\partial\Omega}$ denotes the usual scalar product in $[L^2(\partial\Omega)]^m$. As a result, an energy can be associated with the evolution problem (1.1a). Indeed, taking the L-scalar product of (1.1a) by u, exploiting (1.2), and integrating in time, it is inferred using Gronwall's lemma that solutions to (1.1a) satisfy the energy estimate

$$\max_{t\in[0,T]} \|u\|_L^2 + \int_0^T (Mu, u)_{L,\partial\Omega} dt \leq C, \tag{1.3}$$

where the constant C depends on the initial datum u_0, the source f, the time T, and the parameter λ_0. This implies in particular that the energy, defined as the L-norm of the solution, is controlled at any time.

Following the seminal work of Levy and Tadmor [24], our analysis of explicit RK schemes with stabilized finite elements hinges on energy estimates. The crucial point is that explicit RK schemes are antidissipative (that is, they produce energy at each time step), and this energy production needs to be compensated by the dissipation of the stabilization scheme in space. In [24], a so-called coercivity condition was proposed on the discrete differential operator in space, and with this condition, the stability of the usual three-stage third-order RK (RK3) and four-stage fourth-order (RK4) schemes was proven under a CFL-type condition. The coercivity condition in [24] can for instance be satisfied if an artificial viscosity is used for space stabilization. However, artificial viscosity yields suboptimal convergence estimates in space as soon as finite elements with polynomials of degree ≥ 1 are used.

In the present paper, we improve on this point by establishing stability estimates for a wide class of high-order finite element methods (FEMs) with symmetric stabilization. High-order FEMs do not satisfy the above coercivity condition. Instead, we derive here a sharper set of conditions on the stabilization and proceed along a different path than in [24] for the stability analysis, still relying on energy arguments. Using stability arguments alone yields only suboptimal energy error estimates in space for smooth solutions. Herein, we additionally derive quasi-optimal estimates using the symmetric stabilization. We also consider fully unstructured simplicial meshes.

A salient feature is that our conditions allow for a unified analysis of several high-order stabilized FEMs encountered in the literature. Examples include in the context of continuous finite elements, e.g., interior penalty of gradient jumps (CIP) [7, 5], local projection [1, 26], subgrid viscosity [18, 19], or orthogonal subscale stabilization [12, 13], and also include discontinuous finite elements (DG methods) [23, 22, 17, 14]. Incidentally, a noteworthy point is that using the present approach, DG methods can be cast into the same unified analysis framework as stabilized FEMs, indicating that all these methods essentially share the same stability properties.

Explicit RK schemes come in various forms; see, e.g., [20]. Here, we present results for two-stage second-order RK (RK2) and RK3 schemes. These schemes are written in a specific form suitable for the present analysis, and we verify that usual RK2 and RK3 schemes encountered in the literature can be cast into this form.

Our main results can be summarized as follows:

- Under the usual CFL condition $\tau \leq \varrho(h/\sigma)$, where τ is the time step, h is the minimal mesh size, σ is a reference velocity, and ϱ is a dimensionless constant, an energy error estimate of the form $O(\tau^2 + h^{3/2})$ for the RK2 scheme and piecewise affine finite elements is shown;
- Under the tightened 4/3-CFL condition $\tau \leq \varrho'(h/\sigma)^{4/3}$, an energy error estimate of the form $O(\tau^2 + h^{p+1/2})$ for the RK2 scheme and finite elements with polynomials of total degree $\leq p$ with any $p \geq 2$ is shown. In view of this error estimate, the 4/3-CFL condition is not very restrictive since for $p = 2$, it yields an error estimate of the form $O(h^{8/3} + h^{5/2})$ so that the time and space errors are almost equilibrated, while for $p \geq 3$, a stronger restriction on the time step is needed to equilibrate the time and space errors;
- Under the usual CFL condition $\tau \leq \varrho(h/\sigma)$, an energy error estimate of the form $O(\tau^3 + h^{p+1/2})$ for the RK3 scheme and finite elements with polynomials of total degree $\leq p$ with any $p \geq 1$ is shown.

To the best of our knowledge, the above results are new for continuous FEMs. As such, they provide an attractive alternative to the method of characteristics since the present schemes are more easily extensible to higher order. For DG methods, the two above results for RK2 schemes have been obtained by Zhang and Shu for nonlinear scalar conservation laws [28] and symmetrizable systems of nonlinear conservation laws [29]. The present proofs are, however, different since they rely on the fact that DG methods can be viewed as stabilized FEMs (using suitable interelement penalties) instead of formulating them using fluxes. This approach is instrumental to the unified analysis mentioned above. Moreover, our results for the RK3-DG scheme are, to the best of our knowledge, new.

This paper is organized as follows. In section 2, we present the continuous and discrete settings and state the conditions on the stabilization of the FEM allowing for the unified analysis. In sections 3 and 4, we treat RK2 and RK3 schemes, respectively. Numerical results illustrating the theory are presented in section 5. Finally, section 6 contains some conclusions together with a brief discussion on finite volume schemes (DG methods with $p = 0$), the forward Euler method, and lines for future work.

2. The setting. This section presents the continuous and discrete settings together with some examples. We also state the conditions on the stabilization of the FEM allowing for the unified analysis.

2.1. The continuous problem. Let $\{A_i\}_{1\le i\le d}$ be fields in $[L^\infty(\Omega)]^{m,m}$ s.t.

$$A_i \text{ is symmetric a.e. in } \Omega \quad \forall i \in \{1,\dots,d\}, \tag{2.1}$$

$$\Lambda := \sum_{i=1}^{d} \partial_i A_i \in [L^\infty(\Omega)]^{m,m}. \tag{2.2}$$

The differential operator A in (1.1a) is

$$A := \sum_{i=1}^{d} A_i \partial_i. \tag{2.3}$$

For further use, we set $\sigma := \max_{1\le i\le d} \|A_i\|_{[L^\infty(\Omega)]^{m,m}}$. Assuming that the PDE system (1.1a) is written in nondimensional form for u, the components of the fields A_i scale as velocities, and the quantity σ represents a maximum wave speed. The source term f is in $C^0(0,T;L)$, the initial datum u_0 is in L, and the boundary field M used to enforce the boundary condition is in $[L^\infty(\partial\Omega)]^{m,m}$ and such that $\|M\|_{[L^\infty(\partial\Omega)]^{m,m}} \le C_M\sigma$ for some constant C_M. Let $n = (n_1,\dots,n_d)$ denote the outward unit normal to Ω. The boundary matrix field $D \in [L^\infty(\partial\Omega)]^{m,m}$ is defined s.t. a.e. on $\partial\Omega$,

$$D := \sum_{i=1}^{d} A_i n_i \tag{2.4}$$

and takes, by construction, symmetric values. Owing to the symmetry property (2.1), integration by parts yields $(Av,v)_L + \frac{1}{2}(\Lambda v, v)_L = \frac{1}{2}(Dv,v)_{L,\partial\Omega}$. Since M is nonnegative, the seminorm

$$|v|_M := (Mv,v)_{L,\partial\Omega}^{1/2} \tag{2.5}$$

is well defined. Introducing the bilinear form

$$a(v,w) = (Av,w)_L + \tfrac{1}{2}((M-D)v,w)_{L,\partial\Omega}, \tag{2.6}$$

integration by parts yields

$$a(v,v) = \tfrac{1}{2}|v|_M^2 - \tfrac{1}{2}(\Lambda v, v)_L. \tag{2.7}$$

We now give three examples of evolution problems fitting the present framework.

- Advection. Let $\beta \in [L^\infty(\Omega)]^d$ with $\nabla\cdot\beta \in L^\infty(\Omega)$ and consider the PDE

  $$\partial_t u + \beta\cdot\nabla u = f. \tag{2.8}$$

  Set $m = 1$ and

  $$A_i = \beta_i \ \forall i \in \{1,\dots,d\}, \tag{2.9}$$

  yielding $\Lambda = \nabla\cdot\beta$ and $D = \beta\cdot n$. An admissible boundary condition consists of taking $M = |\beta\cdot n|$, which enforces u to zero on the inflow boundary.
- Maxwell's equations in $\mathbb{R}^3$. Let μ, ϵ be positive constants, set $c_0 = (\mu\epsilon)^{-1/2}$, and consider the PDE system

  $$\begin{cases} \mu\partial_t H + \nabla\times E = f_1, \\ \epsilon\partial_t E - \nabla\times H = f_2, \end{cases} \tag{2.10}$$

where H is the magnetic field and E the electric field. Set $m = 6$, $u = (\mu^{1/2}H, \epsilon^{1/2}E)$, and let

$$A_i = c_0 \left[\begin{array}{c:c} 0_{3,3} & R_i \\ \hdashline R_i^t & 0_{3,3} \end{array}\right] \quad \forall i \in \{1,2,3\}, \tag{2.11}$$

where $0_{3,3}$ is the null matrix in $\mathbb{R}^{3,3}$ and $(R_i)_{jk} = \epsilon_{jik}$ for $i, j, k \in \{1,2,3\}$, ϵ_{jik} being the Levi–Civita permutation tensor, so that $\Lambda = 0_{6,6}$. An admissible boundary condition is for instance to enforce a Dirichlet condition on the tangential component of the electric field. Then D and M are given by

$$D = c_0 \left[\begin{array}{c:c} 0_{3,3} & N \\ \hdashline N^t & 0_{3,3} \end{array}\right], \qquad M = c_0 \left[\begin{array}{c:c} 0_{3,3} & -N \\ \hdashline N^t & 0_{3,3} \end{array}\right], \tag{2.12}$$

where $N = \sum_{i=1}^{3} n_i R_i \in \mathbb{R}^{3,3}$ is such that $Nz = n \times z$ for all $z \in \mathbb{R}^3$.

- Acoustics equations in $\mathbb{R}^d$. Let c_0 be a positive constant and consider the PDE system

$$\begin{cases} c_0^{-2}\partial_t p + \nabla\cdot q = f_1, \\ \partial_t q + \nabla p = f_2, \end{cases} \tag{2.13}$$

where p is the pressure and q the momentum per unit volume. Set $m = d+1$, $u = (c_0^{-1}p, q)$, and let

$$A_i = c_0 \left[\begin{array}{c:c} 0 & e_i^t \\ \hdashline e_i & 0_{d,d} \end{array}\right] \quad \forall i \in \{1,\dots,d\}, \tag{2.14}$$

where $(e_1,\dots,e_d)$ denotes the Cartesian basis of $\mathbb{R}^d$, so that $\Lambda = 0_{d+1,d+1}$. An admissible boundary condition is for instance to enforce a Dirichlet condition on the normal component of the flux. Then D and M are given by

$$D = c_0 \left[\begin{array}{c:c} 0 & n^t \\ \hdashline n & 0_{d,d} \end{array}\right], \qquad M = c_0 \left[\begin{array}{c:c} 0 & -n^t \\ \hdashline n & 0_{d,d} \end{array}\right]. \tag{2.15}$$

2.2. Space semidiscretization. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of simplicial meshes of Ω where $h \le 1$ denotes the maximum diameter of elements in $\mathcal{T}_h$. For simplicity, we assume that the meshes are affine and that Ω is a polyhedron. Mesh faces are collected in the set $\mathcal{F}_h$ which is split into the set of interior faces, $\mathcal{F}_h^{\text{int}}$, and boundary faces, $\mathcal{F}_h^{\text{ext}}$. For $T \in \mathcal{T}_h$ and for $F \in \mathcal{F}_h$, $\|\cdot\|_{L,T}$ and $\|\cdot\|_{L,F}$, respectively, denote the $[L^2(T)]^m$- and $[L^2(F)]^m$-norms; moreover, we define $\|\cdot\|_{L,\mathcal{F}_h}^2 := \sum_{F\in\mathcal{F}_h} \|\cdot\|_{L,F}^2$. We assume that meshes are kept fixed in time and also that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform; see Remark 2.1 below.

Let V_h be a finite element space consisting of either continuous or discontinuous $\mathbb{R}^m$-valued piecewise polynomials of total degree $\le p$ with $p \ge 1$ (the case $p = 0$ is also possible for DG methods leading to finite volume schemes; see section 6 for a brief discussion). Let π_h denote the L-orthogonal projection onto V_h. Set $V(h) := [H^{3/2+\epsilon}(\Omega)]^m + V_h$, $\epsilon > 0$. The seminorm $|\cdot|_M$ defined above can be extended to $V(h)$. We consider a discrete version of the bilinear form a, namely, a_h, together with a stabilization bilinear form s_h. Both bilinear forms a_h and s_h are defined on $V(h) \times$

$V(h)$. It is important to distinguish a_h from s_h since s_h is instrumental in achieving quasi-optimal error estimates. In the context of DG methods, a_h corresponds to the use of centered fluxes, while adding s_h leads to upwind fluxes; see the example in section 2.4. We define the linear operators $A_h : V(h) \to V_h$ and $S_h : V(h) \to V_h$ s.t. $\forall (v, w_h) \in V(h) \times V_h$,

$$(A_h v, w_h)_L := a_h(v, w_h), \qquad (S_h v, w_h)_L := s_h(v, w_h). \tag{2.16}$$

We also define $L_h : V(h) \to V_h$ s.t.

$$L_h = A_h + S_h, \tag{2.17}$$

so that the space semidiscrete problem is for all $t \in (0, T)$, $\frac{d}{dt} u_h + L_h u_h = f_h$, where $f_h := \pi_h f$.

We now state the key design conditions on the bilinear forms a_h and s_h. The first three assumptions are the following:

(A1) for all $v_h \in V_h$, $a_h(v_h, v_h) = \frac{1}{2}|v_h|_M^2 - \frac{1}{2}(\Lambda v_h, v_h)_L$;
(A2) s_h is symmetric and nonnegative on $V(h) \times V(h)$;
(A3) the strong solution u satisfies for all $t \in (0, T)$, $\partial_t \pi_h u + A_h u = f_h$, and $S_h u = 0$.

Assumption (A3) is a (strong) consistency property; it is equivalent to the fact that for the strong solution u, for all $t \in (0, T)$ and for all $v_h \in V_h$,

$$a_h(u, v_h) = (Au, v_h)_L + \tfrac{1}{2}((M - D)u, v_h)_{L,\partial\Omega}, \tag{2.18}$$

while $s_h(u, v_h) = 0$. This consistency property can be weakened for the stabilization bilinear form s_h, in particular when analyzing local projection or orthogonal subscale stabilization. The present strong consistency assumption is sufficient to analyze interior penalty stabilization and DG methods; see section 2.4 for examples. Furthermore, owing to assumption (A2), we can define on $V(h)$ the seminorm

$$|v|_S := \left\{ s_h(v, v) + \tfrac{1}{2}|v|_M^2 \right\}^{1/2}. \tag{2.19}$$

An important consequence of (A1)–(A2) and (2.19) is the following dissipativity property of the discrete setting: For all $v_h \in V_h$,

$$(L_h v_h, v_h)_L = |v_h|_S^2 - \tfrac{1}{2}(\Lambda v_h, v_h)_L. \tag{2.20}$$

The next two assumptions are fairly standard and concern the basic stability of the discrete operators S_h and L_h, namely,

(A4) there is C_S s.t. for all $v_h \in V_h$,

$$|v_h|_S \le C_S^{1/2} \sigma^{1/2} h^{-1/2} \|v_h\|_L, \tag{2.21}$$

and there is C'_S s.t. for all $v \in [H^{p+1}(\Omega)]^m$,

$$|v - \pi_h v|_S \le C'_S \sigma^{1/2} h^{p+1/2} \|v\|_{[H^{p+1}(\Omega)]^m}; \tag{2.22}$$

(A5) there is C_L s.t. for all $z \in V(h)$,

$$\|L_h z\|_L \le \sigma \|\nabla_h z\|_{L^d} + C_L \sigma^{1/2} h^{-1/2} |z|_S, \tag{2.23}$$

where ∇_h denotes the broken gradient of z and $\|\cdot\|_{L^d}$ the usual norm in L^d (the broken gradient is needed when working with DG methods; it coincides with the usual gradient for continuous finite elements).

The last two assumptions are the most important. They can be stated as follows:

(A6) there is C_π s.t. for all $(z, v_h) \in V(h) \times V_h$,

$$|(L_h(z - \pi_h z), v_h)_L| \le C_\pi \|z - \pi_h z\|_* (|v_h|_S + \|v_h\|_L), \tag{2.24}$$

with the norm for $y \in V(h)$,

$$\|y\|_* := \sigma^{1/2} h^{1/2} \|\nabla_h y\|_{L^d} + \sigma^{1/2} h^{-1/2} \|y\|_L + \sigma^{1/2} \|y\|_{L,\mathcal{F}_h} + |y|_S; \tag{2.25}$$

(A7) in the piecewise affine case ($p = 1$), there is C'_π s.t. for all $(v_h, w_h) \in V_h \times V_h$,

$$|(L_h v_h, w_h - \pi_h^0 w_h)_L| \le C'_\pi \sigma^{1/2} h^{-1/2} (|v_h|_S + \|v_h\|_L) \|w_h - \pi_h^0 w_h\|_L, \tag{2.26}$$

where π_h^0 denotes the L-orthogonal projection onto piecewise constant functions.

Owing to the definition (2.25) of the norm $\|\cdot\|_*$, (A5) implies for all $z \in V(h)$,

$$\|L_h z\|_L \le C'_L \sigma^{1/2} h^{-1/2} \|z\|_*, \tag{2.27}$$

with $C'_L = \max(1, C_L)$. Using inverse and trace inequalities, it is inferred that there is C_* s.t. for all $v_h \in V_h$,

$$\|v_h\|_* \le C_* \sigma^{1/2} h^{-1/2} \|v_h\|_L. \tag{2.28}$$

Hence, letting $C_{L*} := C'_L C_*$, there holds for all $v_h \in V_h$,

$$\|L_h v_h\|_L \le C_{L*} \sigma h^{-1} \|v_h\|_L. \tag{2.29}$$

Furthermore, using (2.22) and usual approximation properties in finite element spaces, it is inferred that there is C'_* s.t. for all $v \in [H^{p+1}(\Omega)]^m$,

$$\|v - \pi_h v\|_* \le C'_* \sigma^{1/2} h^{p+1/2} \|v\|_{[H^{p+1}(\Omega)]^m}. \tag{2.30}$$

Assumption (A6) is essential to derive quasi-optimal error estimates. It is sharper than assumption (A5) combined with the Cauchy–Schwarz inequality, which yields only

$$|(L_h(z - \pi_h z), v_h)_L| \le C'_L \sigma^{1/2} h^{-1/2} \|z - \pi_h z\|_* \|v_h\|_L. \tag{2.31}$$

In other words, the factor $h^{-1/2}$ is removed by augmenting the norm of v_h on the right-hand side by $|v_h|_S$, and this is why controlling this seminorm in the energy estimates is crucial and allows one to compensate for the antidissipative nature of the explicit RK schemes (see also Remark 3.2 below). Without assumption (A6), the error estimates are of order h^p instead of $h^{p+1/2}$. Moreover, assumption (A7) is essential in the proof of Theorem 3.2 below to establish the convergence of the explicit RK2 scheme combined with piecewise affine finite elements under the usual CFL condition.

2.3. CFL conditions. Let τ be the time step, taken to be constant for simplicity and such that $T = N\tau$, where N is an integer. For $0 \le n \le N$, a superscript n indicates the value of a function at the discrete time $n\tau$, and for $0 \le n \le N-1$, we set $I_n = [n\tau, (n+1)\tau]$. We assume without loss of generality that $\tau \le 1$. We also assume that the following, so-called usual CFL condition holds:

$$\tau \le \varrho (h/\sigma) \tag{2.32}$$

for some positive real number ϱ. The value of ϱ will be specified below whenever relevant. Furthermore, in the case of RK2 schemes with polynomials of total degree ≥ 2, we will also need the so-called strengthened 4/3-CFL condition

$$\tau \leq \varrho'(h/\sigma)^{4/3} \tag{2.33}$$

for some positive real number ϱ'. Again, the value of ϱ' will be specified below whenever relevant. Since $\tau \leq 1$, the strengthened 4/3-CFL condition (2.33) implies the CFL condition (2.32) with $\varrho = (\varrho')^{3/4}$.

Remark 2.1. On shape-regular mesh families, as usual, the space scale in the CFL condition is no longer h, but the smallest element diameter in the mesh. Then the same space scale is used in the negative powers of h in assumptions (A4)–(A7). It is also possible, with additional technicalities, to consider in the CFL condition the ratio of the local meshsize to the local maximum wave speed.

2.4. Examples. In this section, we present two examples of discrete bilinear forms a_h and s_h satisfying assumptions (A1)–(A7). For $F \in \mathcal{F}_h^{\text{int}}$, there are T^-, T^+ in $\mathcal{T}_h$ such that $F = \partial T^- \cap \partial T^+$, n_F is the unit normal to F pointing from T^- to T^+, and for a smooth enough function v that is possibly double-valued at F, we define its jump and mean value at F as $[\![v]\!] := v|_{T^-} - v|_{T^+}$ and $\{\!\{v\}\!\} = \frac{1}{2}(v|_{T^-} + v|_{T^+})$, respectively. For vector-valued functions, the jump and averages are defined componentwise as above. The arbitrariness in the sign of $[\![v]\!]$ is irrelevant. Meshes can possess hanging nodes when working with discontinuous finite elements under the usual assumption that face diameters are comparable to local element diameters.

An example with continuous finite elements is that of CIP [7, 5, 2],

$$a_h^{\text{cip}}(v, w) := \sum_{T \in \mathcal{T}_h} (Av, w)_{L,T} + \sum_{F \in \mathcal{F}_h^{\text{ext}}} \tfrac{1}{2}((M - D)v, w)_{L,F}, \tag{2.34}$$

$$s_h^{\text{cip}}(v, w) := \sum_{F \in \mathcal{F}_h^{\text{ext}}} (S_F^{\text{ext}} v, w)_{L,F} + \sum_{F \in \mathcal{F}_h^{\text{int}}} h_F^2 (S_F^{\text{int}} n_F \cdot [\![\nabla v]\!], n_F \cdot [\![\nabla w]\!])_{L,F}, \tag{2.35}$$

where h_F denotes the diameter of F. An example with DG is (see, e.g., [17])

$$a_h^{\text{dg}}(v, w) := a^{\text{cip}}(v, w) - \sum_{F \in \mathcal{F}_h^{\text{int}}} (D_F [\![v]\!], \{\!\{w\}\!\})_{L,F}, \tag{2.36}$$

$$s_h^{\text{dg}}(v, w) := \sum_{F \in \mathcal{F}_h^{\text{ext}}} (S_F^{\text{ext}} v, w)_{L,F} + \sum_{F \in \mathcal{F}_h^{\text{int}}} (S_F^{\text{int}} [\![v]\!], [\![w]\!])_{L,F}, \tag{2.37}$$

where $D_F := \sum_{i=1}^d A_i n_{F,i}$ for all $F \in \mathcal{F}_h^{\text{int}}$.

The $\mathbb{R}^{m,m}$-valued fields S_F^{ext} and S_F^{int}, which are defined on boundary and interior faces, respectively, must satisfy the following design conditions:

$$\text{for the exact solution } u \text{ and } \forall t \in [0, T],\ S_F^{\text{ext}} u = 0 \text{ on } \partial\Omega, \tag{2.38}$$

$$S_F^{\text{ext}} \text{ and } S_F^{\text{int}} \text{ are symmetric and nonnegative}, \tag{2.39}$$

$$\forall F \in \mathcal{F}_h^{\text{ext}},\ S_F^{\text{ext}} \leq \alpha_1 \sigma I_m \quad \text{and} \quad \forall F \in \mathcal{F}_h^{\text{int}},\ \alpha_2 |D_F| \leq S_F^{\text{int}} \leq \alpha_3 \sigma I_m, \tag{2.40}$$

and $\forall F \in \mathcal{F}_h^{\text{ext}}$ and $\forall (y, z) \in [L^2(F)]^m \times [L^2(F)]^m$,

$$|((M - D)y, z)_{L,F}| \leq \alpha_4 \sigma^{1/2} |y|_{S,F} \|z\|_{L,F}, \tag{2.41}$$

$$|((M + D)y, z)_{L,F}| \leq \alpha_5 \sigma^{1/2} \|y\|_{L,F} |z|_{S,F}. \tag{2.42}$$

Here, $\alpha_1,\dots,\alpha_5$ are positive parameters, inequalities in (2.40) are meant on the associated quadratic forms, I_m denotes the identity matrix in $\mathbb{R}^{m,m}$, and for $F\in\mathcal{F}_h^{\text{ext}}$, $|v|_{S,F} := \{(S_F^{\text{ext}}v,v)_{L,F}+(Mv,v)_{L,F}\}^{1/2}$, which is well defined since S_F^{ext} is nonnegative. The absolute value $|D_F|$ is also well defined since D_F is, by construction, symmetric.

LEMMA 2.1. *Assume that the design conditions* (2.38)–(2.42) *hold and that for all* $T\in\mathcal{T}_h$ *and for all* $i\in\{1,\dots,d\}$, $A_i|_T\in[C^{0,1/2}(T)]^{m,m}$. *Then, assumptions* (A1)–(A7) *hold for* a_h^{cip} *and* s_h^{cip} *defined by* (2.34)–(2.35) *and for* a_h^{dg} *and* s_h^{dg} *defined by* (2.36)–(2.37).

Proof. Assumptions (A1)–(A6) can be proven as in [2, 3] for CIP and as in [17] for DG. To prove assumption (A7) for CIP, let $(v_h,w_h)\in V_h\times V_h$ with $p=1$ and set $y_h=w_h-\pi_h^0 w_h$. Since y_h may not be in V_h, we obtain

$$(L_hv_h,y_h)_L=\sum_{T\in\mathcal{T}_h}(Av_h,\pi_hy_h)_{L,T}+\sum_{F\in\mathcal{F}_h^{\text{ext}}}\tfrac{1}{2}((M-D)v_h,\pi_hy_h)_{L,F}+s_h^{\text{cip}}(v_h,\pi_hy_h).$$

For the third term, using (A2) and (A4) we obtain

$$\begin{aligned}|s_h^{\text{cip}}(v_h,\pi_hy_h)|&\le|v_h|_S|\pi_hy_h|_S\le|v_h|_SC_S^{1/2}\sigma^{1/2}h^{-1/2}\|\pi_hy_h\|_L\\&\le C_S^{1/2}\sigma^{1/2}h^{-1/2}|v_h|_S\|y_h\|_L.\end{aligned}$$

The second term is bounded similarly using (2.41) and a trace inequality to bound $\|\pi_hy_h\|_{L,F}$. For the first term, letting $\overline{A}:=\sum_{i=1}^d(\pi_h^0A_i)\partial_i$, we observe that

$$\sum_{T\in\mathcal{T}_h}(Av_h,\pi_hy_h)_{L,T}=\sum_{T\in\mathcal{T}_h}(\overline{A}v_h,\pi_hy_h)_{L,T}+\sum_{T\in\mathcal{T}_h}((A-\overline{A})v_h,\pi_hy_h)_{L,T}:=T_1+T_2.$$

Owing to the regularity of the fields A_i together with an inverse inequality, $|T_2|\le C\sigma^{1/2}h^{-1/2}\|v_h\|_L\|y_h\|_L$. Moreover, since $p=1$, $z_h:=\overline{A}v_h$ is piecewise constant so that for all $T\in\mathcal{T}_h$, $(z_h,y_h)_{L,T}=0$. Hence, $T_1=\sum_{T\in\mathcal{T}_h}(z_h,\pi_hy_h-y_h)_{L,T}$. Let $\mathcal{I}_{\text{av}}(z_h)\in V_h$ be such that its values at element vertices are the average of the values taken by z_h on the elements sharing the vertex (the high-order variant of the averaging operator $\mathcal{I}_{\text{av}}$ is used to prove (A6)). Then using the approximation properties of $\mathcal{I}_{\text{av}}$ (see [2]) and the Cauchy–Schwarz inequality yields

$$T_1=\sum_{T\in\mathcal{T}_h}(z_h-\mathcal{I}_{\text{av}}(z_h),\pi_hy_h-y_h)_{L,T}\le\left(\sum_{F\in\mathcal{F}_h^{\text{int}}}h_F\|[\![z_h]\!]\|_{L,F}^2\right)^{1/2}\|y_h\|_L.$$

Since $\|[\![z_h]\!]\|_{L,F}\le\|[\![Av_h]\!]\|_{L,F}+\|[\![(\overline{A}-A)v_h]\!]\|_{L,F}$, using (2.40), the regularity of the fields A_i, an inverse inequality, and a discrete trace inequality leads to

$$|T_1|\le C\sigma^{1/2}h^{-1/2}(|v_h|_S+\|v_h\|_L)\|y_h\|_L.$$

This proves assumption (A7) for CIP. For DG, the proof is similar, but simpler since $\pi_hy_h=y_h$ and so $T_1=0$. ∎

The above setting can be applied to the PDE systems presented in section 2.1; see [2, 17].

- Advection. Take $S_F^{\text{ext}}=0$ and $S_F^{\text{int}}=\gamma|\beta{\cdot}n_F|$ with user-defined parameter $\gamma>0$ ($\gamma=\frac{1}{2}$ amounts to so-called upwinding in the context of DG methods).

- Maxwell's equations in $\mathbb{R}^3$. Take

$$S_F^{\text{ext}} = c_0 \left[\begin{array}{c:c} 0_{3,3} & 0_{3,3} \\ \hdashline 0_{3,3} & \gamma_1 N^t N \end{array}\right], \qquad S_F^{\text{int}} = c_0 \left[\begin{array}{c:c} \gamma_2 N_F^t N_F & 0_{3,3} \\ \hdashline 0_{3,3} & \gamma_3 N_F^t N_F \end{array}\right], \tag{2.43}$$

  where γ_1, γ_2, and γ_3 are positive user-defined parameters and where N_F is defined as N by using n_F instead of n. The operator S_F^{int} amounts to penalizing on each interface the jump of the normal derivative (for CIP) or of the value (for DG) of the tangential components of the electric and magnetic fields.
- Acoustics equations in $\mathbb{R}^d$. Take

$$S_F^{\text{ext}} = c_0 \left[\begin{array}{c:c} 0 & 0_d^t \\ \hdashline 0_d & \gamma_1 n \otimes n \end{array}\right], \qquad S_F^{\text{int}} = c_0 \left[\begin{array}{c:c} \gamma_2 & 0_d^t \\ \hdashline 0_d & \gamma_3 n_F \otimes n_F \end{array}\right], \tag{2.44}$$

  where 0_d is the null vector in $\mathbb{R}^d$. The operator S_F^{int} amounts to penalizing on each interface the jump of the normal derivative (for CIP) or of the value (for DG) of the pressure and that of the normal component of the momentum per unit volume.

3. Analysis of explicit RK2 schemes. This section is devoted to the convergence analysis of explicit RK2 schemes. First, we present the specific form of the schemes on which we will work and show that usual implementations of RK2 schemes fit this form. Then we derive the error equation and establish the key energy identity. Finally, we infer quasi-optimal energy error estimates under the CFL condition (2.32) for piecewise affine finite elements and under the strengthened 4/3-CFL condition (2.33) for polynomials with total degree ≥ 2. We will keep track of the constants to derive the CFL conditions but not to state the error estimates. Henceforth, C denotes a generic constant, independent of the mesh size and the time step, but that can depend on f, u, the fields A_i and M, the constants in assumptions (A4)–(A7), and the constants ϱ and ϱ' in the CFL condition. The value of C can change at each occurrence. The inequality $a \leq Cb$, for positive real numbers a and b, is often abbreviated as $a \lesssim b$. This convention is kept for the rest of this work. Finally, we assume here $u \in C^3(0,T;L)$ and $f \in C^2(0,T;L)$.

3.1. RK2 schemes. We consider schemes of the form

$$w_h^n = u_h^n - \tau L_h u_h^n + \tau f_h^n, \tag{3.1}$$

$$u_h^{n+1} = \tfrac{1}{2}(u_h^n + w_h^n) - \tfrac{1}{2}\tau L_h w_h^n + \tfrac{1}{2}\tau \psi_h^n, \tag{3.2}$$

with the assumption that

$$\psi_h^n = f_h^n + \tau \partial_t f_h^n + \delta_h^n, \qquad \|\delta_h^n\|_L \lesssim \tau^2. \tag{3.3}$$

There are many ways of writing explicit RK2 schemes. Since the space differential operator is linear, they all amount in the homogeneous case ($f = 0$) to

$$u_h^{n+1} = u_h^n - \tau L_h u_h^n + \tfrac{1}{2}\tau^2 L_h^2 u_h^n. \tag{3.4}$$

Two examples of RK2 schemes that fit the present form are

- the second-order Heun method which is usually written in the form (3.1)–(3.2) with

$$\psi_h^n = f_h^{n+1}. \tag{3.5}$$

  Assumption (3.3) obviously holds.
- the second-order Runge method (also called the improved forward Euler method) which is usually written in the form

$$k_1 = -L_h u_h^n + f_h^n, \tag{3.6}$$

$$k_2 = -L_h(u_h^n + \tfrac{1}{2}\tau k_1) + f_h^{n+1/2}, \tag{3.7}$$

$$u_h^{n+1} = u_h^n + \tau k_2. \tag{3.8}$$

  Equations (3.6)–(3.8) can be rewritten in the form (3.1)–(3.2) with

$$\psi_h^n = 2f_h^{n+1/2} - f_h^n. \tag{3.9}$$

  Assumption (3.3) obviously holds.

Define

$$\xi_h^n = u_h^n - \pi_h u^n, \qquad \zeta_h^n = w_h^n - \pi_h w^n, \tag{3.10}$$

$$\xi_\pi^n = u^n - \pi_h u^n, \qquad \zeta_\pi^n = w^n - \pi_h w^n, \tag{3.11}$$

with $w = u + \tau\partial_t u$. Using these quantities, the errors can be written as

$$u^n - u_h^n = \xi_\pi^n - \xi_h^n, \qquad w^n - w_h^n = \zeta_\pi^n - \zeta_h^n. \tag{3.12}$$

The convergence analysis proceeds as follows. Since upper bounds on ξ_π^n and ζ_π^n result from standard approximation properties in finite element spaces, error upper bounds can be derived by obtaining upper bounds on ξ_h^n and ζ_h^n in terms of ξ_π^n and ζ_π^n and then using the triangle inequality. To this purpose, we first identify the error equation governing the time evolution of ξ_h^n and ζ_h^n. The form of this equation is similar to (3.1)–(3.2) with data depending on ξ_π^n, ζ_π^n, f, and u. Then we establish an energy identity associated with (3.1)–(3.2), whence the desired upper bounds on ξ_h^n and ζ_h^n are inferred using, in particular, assumption (A6).

3.2. The error equation. Our first lemma identifies the equations governing the quantities ξ_h^n and ζ_h^n.

LEMMA 3.1. *There holds*

$$\zeta_h^n = \xi_h^n - \tau L_h \xi_h^n + \tau\alpha_h^n, \tag{3.13}$$

$$\xi_h^{n+1} = \tfrac{1}{2}(\xi_h^n + \zeta_h^n) - \tfrac{1}{2}\tau L_h \zeta_h^n + \tfrac{1}{2}\tau\beta_h^n, \tag{3.14}$$

with

$$\alpha_h^n = L_h \xi_\pi^n, \qquad \beta_h^n = L_h \zeta_\pi^n - \pi_h \eta^n + \delta_h^n, \tag{3.15}$$

with $\eta^n = \tau^{-1}\int_{I_n}(t_{n+1}-t)^2\partial_{ttt}u\,dt$.

Proof. Recalling assumption (A3), namely, $\pi_h\partial_t u^n = -L_h u^n + f_h^n$, yields

$$\pi_h w^n = \pi_h u^n - \tau L_h u^n + \tau f_h^n.$$

Subtracting this equation from (3.1) yields (3.13). To derive (3.14), observe that

$$u^{n+1} = u^n + \tau\partial_t u^n + \tfrac{1}{2}\tau^2\partial_{tt}u^n + \tfrac{1}{2}\tau\eta^n,$$

and projecting yields

$$\begin{aligned}\pi_h u^{n+1} &= \pi_h w^n + \tfrac{1}{2}\tau^2\pi_h\partial_{tt}u^n + \tfrac{1}{2}\tau\pi_h\eta^n\\ &= \tfrac{1}{2}(\pi_h u^n + \pi_h w^n) - \tfrac{1}{2}\tau L_h u^n + \tfrac{1}{2}\tau f_h^n + \tfrac{1}{2}\tau^2\pi_h\partial_{tt}u^n + \tfrac{1}{2}\tau\pi_h\eta^n.\end{aligned}$$

Moreover,

$$\tau\pi_h\partial_{tt}u^n = \tau\partial_t(\pi_h\partial_t u^n) = -\tau L_h\partial_t u^n + \tau\partial_t f_h^n = -L_h(w^n - u^n) + \tau\partial_t f_h^n,$$

whence

$$\pi_h u^{n+1} = \tfrac{1}{2}(\pi_h u^n + \pi_h w^n) - \tfrac{1}{2}\tau L_h w^n + \tfrac{1}{2}\tau(\pi_h\eta^n + f_h^n + \tau\partial_t f_h^n).$$

Subtracting this equation from (3.2) yields (3.14). ∎

3.3. Energy identity and stability. Our next step is to derive an energy identity, then leading to our main stability estimate.

Lemma 3.2 (energy identity). *There holds*

$$\begin{aligned}(3.16)\quad &\|\xi_h^{n+1}\|_L^2 - \|\xi_h^n\|_L^2 + \tau|\xi_h^n|_S^2 + \tau|\zeta_h^n|_S^2\\ &= \|\xi_h^{n+1} - \zeta_h^n\|_L^2 + \tau(\alpha_h^n, \xi_h^n)_L + \tau(\beta_h^n, \zeta_h^n)_L + \tfrac{1}{2}\tau(\Lambda\xi_h^n, \xi_h^n)_L + \tfrac{1}{2}\tau(\Lambda\zeta_h^n, \zeta_h^n)_L.\end{aligned}$$

Proof. Multiply (3.13) by ξ_h^n and (3.14) by $2\zeta_h^n$, sum both equations, and use the identity $2(\xi_h^{n+1}, \zeta_h^n)_L = \|\xi_h^{n+1}\|_L^2 - \|\xi_h^{n+1} - \zeta_h^n\|_L^2 + \|\zeta_h^n\|_L^2$ together with (2.20) which yields $(L_h\xi_h^n, \xi_h^n)_L = |\xi_h^n|_S^2 - \frac{1}{2}(\Lambda\xi_h^n, \xi_h^n)_L$ and $(L_h\zeta_h^n, \zeta_h^n)_L = |\zeta_h^n|_S^2 - \frac{1}{2}(\Lambda\zeta_h^n, \zeta_h^n)_L$. ∎

Remark 3.1. The quantity $\|\xi_h^{n+1} - \zeta_h^n\|_L^2$ appearing on the right-hand side of the energy identity (3.16) is the antidissipative term associated with the explicit nature of the RK2 scheme. This term essentially amounts to a second-order time derivative. Instead, the positive terms $\tau|\xi_h^n|_S^2 + \tau|\zeta_h^n|_S^2$ on the left-hand side result from dissipation in space.

Lemma 3.3 (stability). *Under the usual CFL condition* (2.32) *for any positive real number ϱ, there holds*

$$\begin{aligned}(3.17)\quad &\|\xi_h^{n+1}\|_L^2 - \|\xi_h^n\|_L^2 + \tfrac{1}{2}\tau|\xi_h^n|_S^2 + \tfrac{1}{2}\tau|\zeta_h^n|_S^2\\ &\qquad\le \|\xi_h^{n+1} - \zeta_h^n\|_L^2 + C\tau(\tau^4 + \|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2 + \|\xi_h^n\|_L^2).\end{aligned}$$

Proof. Starting from the energy identity (3.16), we bound the last four terms on the right-hand side.

(i) We first bound α_h^n and β_h^n. Let us prove that

$$(3.18)\qquad \tau\|\alpha_h^n\|_L \lesssim \tau^{1/2}\|\xi_\pi^n\|_*, \qquad \tau\|\beta_h^n\|_L \lesssim \tau^{1/2}\|\zeta_\pi^n\|_* + \tau^3,$$

and that for all $v_h \in V_h$,

$$(3.19)\qquad \tau(\alpha_h^n, v_h)_L \lesssim \tau\|\xi_\pi^n\|_*(|v_h|_S + \|v_h\|_L),$$

$$(3.20)\qquad \tau(\beta_h^n, v_h)_L \lesssim \tau\|\zeta_\pi^n\|_*(|v_h|_S + \|v_h\|_L) + \tau^3\|v_h\|_L.$$

Using the definition of α_h^n, the bound (2.27), and the CFL condition (2.32) to eliminate the factor $h^{-1/2}$ yields

$$\tau\|\alpha_h^n\|_L = \tau\|L_h\xi_\pi^n\|_L \le \tau C_L'\sigma^{1/2}h^{-1/2}\|\xi_\pi^n\|_* \lesssim \tau^{1/2}\|\xi_\pi^n\|_*.$$

Similarly, using the definition of β_h^n, the assumption on δ_h^n, and the regularity of the strong solution u yields

$$\tau\|\beta_h^n\|_L = \tau\|L_h\zeta_\pi^n - \pi_h\eta^n + \delta_h^n\|_L \lesssim \tau\|L_h\zeta_\pi^n\|_L + \tau^3 \lesssim \tau^{1/2}\|\zeta_\pi^n\|_* + \tau^3.$$

This proves (3.18). To prove (3.19)–(3.20), we use assumption (A6) to infer

$$\tau(\alpha_h^n, v_h)_L = \tau(L_h\xi_\pi^n, v_h)_L \lesssim \tau\|\xi_\pi^n\|_*(|v_h|_S + \|v_h\|_L),$$

and similarly using assumption (A6) and the Cauchy–Schwarz inequality,

$$\tau(\beta_h^n, v_h)_L = \tau(L_h\zeta_\pi^n, v_h)_L + \tau(-\pi_h\eta^n + \delta_h^n, v_h)_L \lesssim \tau\|\zeta_\pi^n\|_*(|v_h|_S + \|v_h\|_L) + \tau^3\|v_h\|_L.$$

(ii) Owing to the bounds (3.19) and (3.20),

$$\tau(\alpha_h^n, \xi_h^n)_L + \tau(\beta_h^n, \zeta_h^n)_L \lesssim \tau\|\xi_\pi^n\|_*(|\xi_h^n|_S + \|\xi_h^n\|_L) + \tau\|\zeta_\pi^n\|_*(|\zeta_h^n|_S + \|\zeta_h^n\|_L) + \tau^3\|\zeta_h^n\|_L.$$

Moreover, it is inferred from (3.13) using the triangle inequality, the bound (2.29), and the CFL condition (2.32) that

$$\|\zeta_h^n\|_L \le \|\xi_h^n\|_L + \tau\|L_h\xi_h^n\|_L + \tau\|\alpha_h^n\|_L \lesssim \|\xi_h^n\|_L + \tau\|\alpha_h^n\|_L.$$

Hence, owing to (3.18),

$$\|\zeta_h^n\|_L \lesssim \|\xi_h^n\|_L + \tau^{1/2}\|\xi_\pi^n\|_* \le \|\xi_h^n\|_L + \|\xi_\pi^n\|_*,$$

since $\tau \le 1$. Collecting these bounds and using Young inequalities yields

$$\tau(\alpha_h^n, \xi_h^n)_L + \tau(\beta_h^n, \zeta_h^n)_L \le \tfrac{1}{2}\tau|\xi_h^n|_S^2 + \tfrac{1}{2}\tau|\zeta_h^n|_S^2 + C\tau(\tau^4 + \|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2 + \|\xi_h^n\|_L^2).$$

(iii) Finally,

$$\tfrac{1}{2}\tau(\Lambda\xi_h^n, \xi_h^n)_L + \tfrac{1}{2}\tau(\Lambda\zeta_h^n, \zeta_h^n)_L \lesssim \tau\|\xi_h^n\|_L^2 + \tau\|\zeta_h^n\|_L^2 \lesssim \tau\|\xi_h^n\|_L^2 + \tau\|\xi_\pi^n\|_*^2,$$

since $\|\zeta_h^n\|_L \lesssim \|\xi_h^n\|_L + \|\xi_\pi^n\|_*$. This concludes the proof. ∎

Remark 3.2. Assumption (A6) is crucial to obtain the bounds (3.19)–(3.20), which are sharper by a factor $\tau^{1/2}$ than the bounds resulting from (3.18) and the Cauchy–Schwarz inequality, yielding for instance $\tau(\alpha_h^n, v_h)_L \lesssim \tau^{1/2}\|\xi_\pi^n\|_*\|v_h\|_L$; that is, adding the seminorm $|v_h|_S$ on the right-hand side allows one to gain a factor $\tau^{1/2}$.

3.4. Error estimates. Starting from the stability estimate (3.17), there are two ways to bound the positive term $\|\xi_h^{n+1} - \zeta_h^n\|_L^2$ appearing on the right-hand side. In the general case $p \ge 2$, the strengthened 4/3-CFL condition (2.33) is needed. By proceeding differently and using assumption (A7) for $p = 1$, it is possible to control this term using only the usual CFL condition (2.32).

3.4.1. General case: 4/3-CFL condition. The next theorem provides a quasi-optimal energy error estimate in the general case $p \geq 2$ under the strengthened 4/3-CFL condition.

THEOREM 3.1. *Assume that* $u \in C^3(0,T;L) \cap C^s(0,T;[H^{p+1-s}(\Omega)]^m)$ *for* $s \in \{0,1\}$. *Then under the strengthened 4/3-CFL condition* (2.33) *for any positive real number* ϱ', *there holds*

$$\|u^N - u_h^N\|_L + \left(\sum_{n=0}^{N-1} \tfrac{1}{2}\tau |u_h^n|_S^2 + \tfrac{1}{2}\tau |w_h^n|_S^2 \right)^{1/2} \lesssim \tau^2 + h^{p+1/2}. \tag{3.21}$$

Proof. The proof is decomposed into three steps.

(i) Bound on $\|\xi_h^{n+1} - \zeta_h^n\|_L^2$. Combining (3.13) and (3.14) yields

$$\begin{aligned} \xi_h^{n+1} - \zeta_h^n &= -\tfrac{1}{2}\tau L_h(\zeta_h^n - \xi_h^n) + \tfrac{1}{2}\tau(\beta_h^n - \alpha_h^n) \\ &= \tfrac{1}{2}\tau^2 L_h^2 \xi_h^n + \tfrac{1}{2}\tau(\beta_h^n - \alpha_h^n - \tau L_h \alpha_h^n). \end{aligned}$$

Set $R_h^n = \frac{1}{2}\tau(\beta_h^n - \alpha_h^n - \tau L_h \alpha_h^n)$. Using the triangle inequality, the bound (2.29), and the CFL condition (2.32) yields $\|R_h^n\|_L \lesssim \tau \|\beta_h^n\|_L + \tau \|\alpha_h^n\|_L$, so that (3.18) implies

$$\|R_h^n\|_L \lesssim \tau^3 + \tau^{1/2}\|\zeta_\pi^n\|_* + \tau^{1/2}\|\xi_\pi^n\|_*.$$

As a result,

$$\|\xi_h^{n+1} - \zeta_h^n\|_L^2 \lesssim \tau^4 \|L_h^2 \xi_h^n\|_L^2 + \tau(\tau^5 + \|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2).$$

(ii) Using the above bound together with the stability estimate (3.17) leads to

$$\|\xi_h^{n+1}\|_L^2 - \|\xi_h^n\|_L^2 + \tfrac{1}{2}\tau|\xi_h^n|_S^2 + \tfrac{1}{2}\tau|\zeta_h^n|_S^2 \lesssim \tau^4\|L_h^2\xi_h^n\|_L^2 + \tau(\tau^4 + \|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2 + \|\xi_h^n\|_L^2),$$

since $\tau^5 \leq \tau^4$. The strengthened 4/3-CFL condition together with (2.29) imply

$$\tau^4 \|L_h^2 \xi_h^n\|_L^2 \lesssim \tau \|\xi_h^n\|_L^2.$$

Hence,

$$\|\xi_h^{n+1}\|_L^2 - \|\xi_h^n\|_L^2 + \tfrac{1}{2}\tau|\xi_h^n|_S^2 + \tfrac{1}{2}\tau|\zeta_h^n|_S^2 \lesssim \tau\|\xi_h^n\|_L^2 + \tau(\tau^4 + \|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2).$$

(iii) Summing over n in the above estimate and using Gronwall's lemma, it is inferred that

$$\|\xi_h^N\|_L^2 + \sum_{n=0}^{N-1} \tfrac{1}{2}\tau(|\xi_h^n|_S^2 + |\zeta_h^n|_S^2) \lesssim \tau^4 + \sum_{n=0}^{N-1} \tau(\|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2),$$

and using the approximation property (2.30) yields

$$\|\xi_h^N\|_L^2 + \sum_{n=0}^{N-1} \tfrac{1}{2}\tau(|\xi_h^n|_S^2 + |\zeta_h^n|_S^2) \lesssim \tau^4 + h^{2p+1},$$

whence (3.21) readily follows using the triangle inequality and the fact that $|u - u_h^n|_S = |u_h^n|_S$ and $|w - w_h^n|_S = |w_h^n|_S$. ∎

Remark 3.3. Although there is no specific limit to the value of the constant ϱ' in the 4/3-CFL condition for the convergence result of Theorem 3.1 to hold, the constant in the error estimate depends exponentially on ϱ'. Hence, in practice, a value small enough should be considered for ϱ'.

3.4.2. Piecewise affine finite elements: Usual CFL condition. The next theorem provides a quasi-optimal energy error estimate for $p = 1$ under the usual CFL condition.

THEOREM 3.2. *Assume that piecewise affine finite elements are used and that* $u \in C^3(0,T;L) \cap C^s(0,T;[H^{2-s}(\Omega)]^m)$ *for* $s \in \{0,1\}$. *Then under the usual CFL condition* (2.32) *with*

$$\varrho \le \min\left\{\tfrac{1}{8}C_L^{-2}, \tfrac{1}{2}(C_i')^{-2/3}\right\}, \tag{3.22}$$

with $C_i' = C_i C_\pi'$ *and where* C_i *is the constant in the inverse inequality* $\|\nabla_h v_h\|_{L^d} \le C_i h^{-1}\|v_h - \pi_h^0 v_h\|_L$ *valid for all* $v_h \in V_h$, *there holds*

$$\|u^N - u_h^N\|_L + \left(\sum_{n=0}^{N-1} \tfrac{1}{8}\tau|u_h^n|_S^2 + \tfrac{1}{8}\tau|w_h^n|_S^2\right)^{1/2} \lesssim \tau^2 + h^{3/2}. \tag{3.23}$$

Proof. We bound $\|\xi_h^{n+1} - \zeta_h^n\|_L^2$ differently from the proof of Theorem 3.1. Set $x_h^n := \xi_h^n - \zeta_h^n$, so that (3.13)–(3.14) yield

$$\xi_h^{n+1} - \zeta_h^n = \tfrac{1}{2}\tau L_h x_h^n + \tfrac{1}{2}\tau(\beta_h^n - \alpha_h^n).$$

Hence, using the triangle inequality and assumption (A5) yields

$$\begin{aligned}\|\xi_h^{n+1} - \zeta_h^n\|_L \le \tfrac{1}{2}\sigma\tau\|\nabla_h x_h^n\|_{L^d} &+ \tfrac{1}{2}C_L\sigma^{1/2}h^{-1/2}\tau|x_h^n|_S \\ &+ \tfrac{1}{2}\tau\|\beta_h^n - \alpha_h^n\|_L.\end{aligned} \tag{3.24}$$

The first step is to control $\|\nabla_h x_h^n\|_{L^d}$ using the inverse inequality $\|\nabla_h x_h^n\|_{L^d} \le C_i h^{-1}\|y_h^n\|_L$, where $y_h^n = x_h^n - \pi_h^0 x_h^n$. The key idea is to exploit the fact that, up to a nonessential perturbation, x_h^n is in the range of the discrete operator L_h since $x_h^n = \tau L_h \xi_h^n - \tau\alpha_h^n$. Thus,

$$\|y_h^n\|_L^2 = (x_h^n, y_h^n)_L = \tau(L_h\xi_h^n, y_h^n)_L - \tau(\alpha_h^n, y_h^n)_L.$$

To bound the first term on the right-hand side, we use assumption (A7) to infer

$$\tau|(L_h\xi_h^n, y_h^n)_L| \le C_\pi'\sigma^{1/2}h^{-1/2}\tau(|\xi_h^n|_S + \|\xi_h^n\|_L)\|y_h^n\|_L.$$

Furthermore, bounding the second term by the Cauchy–Schwarz inequality, using the CFL condition, and simplifying by $\|y_h^n\|_L$ leads to

$$\|y_h^n\|_L \le C_\pi'\sigma^{1/2}h^{-1/2}\tau|\xi_h^n|_S + \tau\|\alpha_h^n\|_L + C\tau^{1/2}\|\xi_h^n\|_L.$$

Thus, owing to the above inverse inequality on $\|\nabla_h x_h^n\|_{L^d}$,

$$\|\nabla_h x_h^n\|_{L^d} \le C_i h^{-1}\|y_h^n\|_L \le C_i'\sigma^{1/2}h^{-3/2}\tau|\xi_h^n|_S + Ch^{-1}(\tau\|\alpha_h^n\|_L + \tau^{1/2}\|\xi_h^n\|_L),$$

with $C_i' = C_i C_\pi'$. Hence, substituting back into (3.24) yields

$$\begin{aligned}\|\xi_h^{n+1} - \zeta_h^n\|_L \le\ & \tfrac{1}{2}C_i'\sigma^{3/2}h^{-3/2}\tau^2|\xi_h^n|_S \\ &+ \tfrac{1}{2}C_L\sigma^{1/2}h^{-1/2}\tau|\xi_h^n - \zeta_h^n|_S + C\tau^{1/2}(\tau^{5/2} + \|\xi_\pi^n\|_* + \|\zeta_\pi^n\|_* + \|\xi_h^n\|_L),\end{aligned}$$

where the contributions of α_h^n and β_h^n have been bounded using (3.18) and the (generic) CFL condition (2.32). Owing to the condition (3.22), it is now inferred that

$$\begin{aligned}\|\xi_h^{n+1} - \zeta_h^n\|_L &\leq 2^{-5/2}\tau^{1/2}(|\xi_h^n|_S + |\zeta_h^n - \xi_h^n|_S) + C\tau^{1/2}(\tau^{5/2} + \|\xi_\pi^n\|_* + \|\zeta_\pi^n\|_* + \|\xi_h^n\|_L) \\ &\leq 2^{-3/2}\tau^{1/2}(|\xi_h^n|_S + |\zeta_h^n|_S) + C\tau^{1/2}(\tau^{5/2} + \|\xi_\pi^n\|_* + \|\zeta_\pi^n\|_* + \|\xi_h^n\|_L).\end{aligned}$$

Squaring yields

$$\|\xi_h^{n+1} - \zeta_h^n\|_L^2 \leq \tfrac{3}{8}\tau|\xi_h^n|_S^2 + \tfrac{3}{8}\tau|\zeta_h^n|_S^2 + C\tau(\tau^5 + \|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2 + \|\xi_h^n\|_L^2).$$

The conclusion readily follows by proceeding as in the proof of Theorem 3.1. ∎

4. Analysis of explicit RK3 schemes. This section is devoted to the convergence analysis of explicit RK3 schemes. We proceed similarly to section 3. The main difference is that a sharper stability estimate can be derived for RK3 schemes, so that the strengthened 4/3-CFL condition (2.33) is no longer needed. Furthermore, we assume here $u \in C^4(0,T;L)$ and $f \in C^3(0,T;L)$.

The stronger stability properties of explicit RK3 schemes are reflected by the fact that these schemes are stable under the usual CFL condition (2.32) without any stabilization ($s_h = 0$); see [27, Theorem 2]. This is, however, not sufficient to derive quasi-optimal error estimates of the form $O(\tau^3 + h^{p+1/2})$. Indeed, in the absence of stabilization, assumption (A6) cannot be invoked leading to suboptimal error estimates of the form $O(\tau^3 + h^p)$. Therefore, it is important to revisit the stability analysis of explicit RK3 schemes in the presence of stabilization so as to ensure that the corresponding energy identity provides enough dissipation in space.

4.1. RK3 schemes. We consider schemes of the form

$$w_h^n = u_h^n - \tau L_h u_h^n + \tau f_h^n, \tag{4.1}$$

$$y_h^n = \tfrac{1}{2}(u_h^n + w_h^n) - \tfrac{1}{2}\tau L_h w_h^n + \tfrac{1}{2}\tau(f_h^n + \tau\partial_t f_h^n), \tag{4.2}$$

$$u_h^{n+1} = \tfrac{1}{3}(u_h^n + w_h^n + y_h^n) - \tfrac{1}{3}\tau L_h y_h^n + \tfrac{1}{3}\tau\psi_h^n, \tag{4.3}$$

with the assumption that

$$\psi_h^n = f_h^n + \tau\partial_t f_h^n + \tfrac{1}{2}\tau^2\partial_{tt} f_h^n + \delta_h^n, \qquad \|\delta_h^n\|_L \lesssim \tau^3. \tag{4.4}$$

There are many ways of writing explicit RK3 schemes. Since the space differential operator is linear, they all amount in the homogeneous case ($f = 0$) to

$$u_h^{n+1} = u_h^n - \tau L_h u_h^n + \tfrac{1}{2}\tau^2 L_h^2 u_h^n - \tfrac{1}{6}\tau^3 L_h^3 u_h^n. \tag{4.5}$$

One example that fits the above form is the third-order Heun method, which is usually written as follows:

$$k_1 = -L_h u_h^n + f_h^n, \tag{4.6}$$

$$k_2 = -L_h(u_h^n + \tfrac{1}{3}\tau k_1) + f_h^{n+1/3}, \tag{4.7}$$

$$k_3 = -L_h(u_h^n + \tfrac{2}{3}\tau k_2) + f_h^{n+2/3}, \tag{4.8}$$

$$u_h^{n+1} = u_h^n + \tfrac{1}{4}\tau(k_1 + 3k_3). \tag{4.9}$$

Straightforward algebra yields

$$\psi_h^n = -\tfrac{5}{4}f_h^n + \tfrac{9}{4}f_h^{n+2/3} - \tfrac{1}{2}\tau\partial_t f_h^n - \tfrac{3}{2}\tau L_h(f_h^{n+1/3} - f_h^n - \tfrac{1}{3}\tau\partial_t f_h^n). \tag{4.10}$$

Proposition 4.1. *Assume that $f \in C^2(0,T;[H^1(\Omega)]^m)$ and that $s_h = s_h^{\mathrm{cip}}$ as defined by* (2.35) *or that $s_h = s_h^{\mathrm{dg}}$ as defined by* (2.37). *Then* (4.4) *holds.*

Proof. We need to prove that $\|\psi_h^n - (f_h^n + \tau\partial_t f_h^n + \frac{1}{2}\tau^2\partial_{tt} f_h^n)\|_L \lesssim \tau^3$. Set $\psi_h^n = A + B$ with $A = -\frac{5}{4} f_h^n + \frac{9}{4} f_h^{n+2/3} - \frac{1}{2}\tau\partial_t f_h^n$ and $B = -\frac{3}{2}\tau L_h(f_h^{n+1/3} - f_h^n - \frac{1}{3}\tau\partial_t f_h^n)$. Using Taylor expansions yields

$$\|A - (f_h^n + \tau\partial_t f_h^n + \tfrac{1}{2}\tau^2\partial_{tt} f_h^n)\|_L \lesssim \tau^3 \|f\|_{C^3(0,T;L)}.$$

Consider now the term B and set $z^n := f^{n+1/3} - f^n - \frac{1}{3}\tau\partial_t f^n$ so that $B = -\frac{3}{2}\tau L_h(\pi_h z^n)$. Assumption (A5) yields $\|L_h(\pi_h z^n)\|_L \lesssim \|\nabla z^n\|_{L^d} + h^{-1/2}|\pi_h z^n|_S$, where we have used the H^1-stability of π_h in writing $\|\nabla z^n\|_{L^d}$. When $s_h = s_h^{\mathrm{dg}}$, observe that $|z^n|_S = 0$ since $f \in C^2(0,T;[H^1(\Omega)]^m)$ so that

$$h^{-1/2}|\pi_h z^n|_S = h^{-1/2}|z^n - \pi_h z^n|_S \lesssim \|\nabla z^n\|_{L^d}.$$

When $s_h = s_h^{\mathrm{cip}}$, the boundary contribution is bounded as above, while the interior contribution is bounded by a trace inequality and the H^1-stability of π_h, yielding again $h^{-1/2}|\pi_h z^n|_S \lesssim \|\nabla z^n\|_{L^d}$. As a result, using Taylor expansions yields

$$\|B\|_L \lesssim \tau\|\nabla(f^{n+1/3} - f^n - \tfrac{1}{3}\tau\partial_t f^n)\|_{L^d} \lesssim \tau^3\|f\|_{C^2(0,T;[H^1(\Omega)]^m)},$$

completing the proof. ∎

4.2. The error equation. Along with definitions (3.10)–(3.11), let $\theta_h^n = y_h^n - \pi_h y^n$ and $\theta_\pi^n = y^n - \pi_h y^n$, with $y = u + \tau\partial_t u + \frac{1}{2}\tau^2\partial_{tt} u$.

Lemma 4.1. *There holds*

$$\zeta_h^n = \xi_h^n - \tau L_h \xi_h^n + \tau\alpha_h^n, \tag{4.11}$$

$$\theta_h^n = \tfrac{1}{2}(\xi_h^n + \zeta_h^n) - \tfrac{1}{2}\tau L_h\zeta_h^n + \tfrac{1}{2}\tau\beta_h^n, \tag{4.12}$$

$$\xi_h^{n+1} = \tfrac{1}{3}(\xi_h^n + \zeta_h^n + \theta_h^n) - \tfrac{1}{3}\tau L_h\theta_h^n + \tfrac{1}{3}\tau\gamma_h^n, \tag{4.13}$$

with

$$\alpha_h^n = L_h\xi_\pi^n, \qquad \beta_h^n = L_h\zeta_\pi^n, \qquad \gamma_h^n = L_h\theta_\pi^n - \pi_h\eta^n + \delta_h^n, \tag{4.14}$$

where $\eta^n = \tau^{-1}\int_{I_n} \frac{1}{2}(t_{n+1} - t)^3\partial_{tttt} u\, dt$.

Proof. Equations (4.11) and (4.12) are obtained as in Lemma 3.1. To derive (4.13), observe that $u^{n+1} = u^n + \tau\partial_t u^n + \frac{1}{2}\tau^2\partial_{tt}u^n + \frac{1}{6}\tau^3\partial_{ttt}u^n + \frac{1}{3}\tau\eta^n$, and proceed again as in Lemma 3.1. ∎

4.3. Energy identity and stability. Our goal is now to derive an energy identity, then leading to our main stability estimate.

Lemma 4.2 (energy identity). *There holds*

$$\begin{aligned}\tfrac{1}{2}\|\xi_h^{n+1}\|_L^2 - \tfrac{1}{2}\|\xi_h^n\|_L^2 + \tfrac{1}{2}\tau|\xi_h^n|_S^2 + \tfrac{1}{6}\tau|\zeta_h^n|_S^2 + \tfrac{1}{3}\tau|\theta_h^n|_S^2 + \tfrac{1}{6}\|\theta_h^n - \zeta_h^n\|_L^2 \\ = \tfrac{1}{6}\tau|\zeta_h^n - \xi_h^n|_S^2 + \tfrac{1}{2}\|\xi_h^{n+1} - \theta_h^n\|_L^2 + \Lambda_h^n \\ + \tfrac{1}{3}\tau(\gamma_h^n, \theta_h^n)_L + \tfrac{1}{6}\tau(\beta_h^n, \xi_h^n)_L + \tfrac{1}{3}\tau(\alpha_h^n, \xi_h^n + \tfrac{1}{2}\zeta_h^n)_L,\end{aligned} \tag{4.15}$$

where $\Lambda_h^n := \frac{1}{6}\tau(\Lambda\xi_h^n, \xi_h^n)_L - \frac{1}{6}\tau(\Lambda\zeta_h^n, \xi_h^n)_L + \frac{1}{6}\tau(\Lambda\theta_h^n, \theta_h^n)_L$.

Remark 4.1. The quantities $\frac{1}{6}\tau|\zeta_h^n - \xi_h^n|_S^2$ and $\frac{1}{2}\|\xi_h^{n+1} - \theta_h^n\|_L^2$ appearing on the right-hand side of the energy identity (4.15) are the antidissipative terms associated with the explicit nature of the RK3 scheme. The term involving the $|\cdot|_S$-seminorm on the right-hand side indicates in particular that the coupling between the explicit RK3 scheme and space stabilization deserves a careful analysis. A further important fact is that, contrary to the RK2 scheme, there is now a new positive term on the left-hand side, namely, $\frac{1}{6}\|\theta_h^n - \zeta_h^n\|_L^2$, which significantly improves the stability properties of the RK3 scheme, in particular circumventing the need for the strengthened 4/3-CFL condition for high-order polynomials.

Proof. We first observe that combining (4.11)–(4.13) yields

$$\theta_h^n = \zeta_h^n - \tfrac{1}{2}\tau L_h(\zeta_h^n - \xi_h^n) + \tfrac{1}{2}\tau(\beta_h^n - \alpha_h^n), \tag{4.16}$$

$$\xi_h^{n+1} = \theta_h^n - \tfrac{1}{3}\tau L_h(\theta_h^n - \zeta_h^n) + \tfrac{1}{3}\tau(\gamma_h^n - \beta_h^n). \tag{4.17}$$

Set $A = \frac{1}{2}\|\xi_h^{n+1}\|_L^2 - \frac{1}{2}\|\xi_h^{n+1} - \theta_h^n\|_L^2 - \frac{1}{2}\|\xi_h^n\|_L^2$. Then

$$\begin{aligned} A &= (\xi_h^{n+1} - \tfrac{1}{2}\theta_h^n, \theta_h^n)_L - \tfrac{1}{2}\|\xi_h^n\|_L^2 \\ &= \tfrac{1}{2}\|\theta_h^n\|_L^2 + (\xi_h^{n+1} - \theta_h^n, \theta_h^n)_L - \tfrac{1}{2}\|\xi_h^n\|_L^2 \\ &= \tfrac{1}{2}\|\theta_h^n\|_L^2 - \tfrac{1}{2}\|\xi_h^n\|_L^2 - \tfrac{1}{3}\tau(L_h(\theta_h^n - \zeta_h^n), \theta_h^n)_L + \tfrac{1}{3}\tau(\gamma_h^n - \beta_h^n, \theta_h^n)_L, \end{aligned}$$

where (4.17) has been used. Set $D_1 := \frac{1}{3}\tau(\gamma_h^n - \beta_h^n, \theta_h^n)_L$. The term $\frac{1}{2}\|\theta_h^n\|_L^2 - \frac{1}{2}\|\xi_h^n\|_L^2$ can be evaluated using the energy identity (3.16) for the RK2 scheme applied to (4.11)–(4.12). Setting

$$D_2 := D_1 + \tfrac{1}{2}\tau(\alpha_h^n, \xi_h^n)_L + \tfrac{1}{2}\tau(\beta_h^n, \zeta_h^n)_L + \tfrac{1}{4}\tau(\Lambda\xi_h^n, \xi_h^n)_L + \tfrac{1}{4}\tau(\Lambda\zeta_h^n, \zeta_h^n)_L,$$

yields

$$\begin{aligned} A &= -\tfrac{1}{2}\tau|\xi_h^n|_S^2 - \tfrac{1}{2}\tau|\zeta_h^n|_S^2 + \tfrac{1}{2}\|\theta_h^n - \zeta_h^n\|_L^2 - \tfrac{1}{3}\tau(L_h(\theta_h^n - \zeta_h^n), \theta_h^n)_L + D_2 \\ &= -\tfrac{1}{2}\tau|\xi_h^n|_S^2 - \tfrac{1}{2}\tau|\zeta_h^n|_S^2 - \tfrac{1}{3}\tau|\theta_h^n|_S^2 + \tfrac{1}{2}\|\theta_h^n - \zeta_h^n\|_L^2 + \tfrac{1}{3}\tau(L_h\zeta_h^n, \theta_h^n)_L + D_2' \\ &= -\tfrac{1}{2}\tau|\xi_h^n|_S^2 - \tfrac{1}{6}\tau|\zeta_h^n|_S^2 - \tfrac{1}{3}\tau|\theta_h^n|_S^2 + \tfrac{1}{2}\|\theta_h^n - \zeta_h^n\|_L^2 + \tfrac{1}{3}\tau(L_h\zeta_h^n, \theta_h^n - \zeta_h^n)_L + D_2'' \\ &= -\tfrac{1}{2}\tau|\xi_h^n|_S^2 - \tfrac{1}{6}\tau|\zeta_h^n|_S^2 - \tfrac{1}{3}\tau|\theta_h^n|_S^2 + \tfrac{1}{2}(\theta_h^n - \zeta_h^n, \theta_h^n - \zeta_h^n + \tfrac{2}{3}\tau L_h\zeta_h^n)_L + D_2'', \end{aligned}$$

with $D_2' := D_2 + \frac{1}{6}\tau(\Lambda\theta_h^n, \theta_h^n)_L$ and $D_2'' := D_2' - \frac{1}{6}\tau(\Lambda\zeta_h^n, \zeta_h^n)_L$. Consider the fourth term on the right-hand side, say, B, and observe that owing to (4.11) and (4.16),

$$\begin{aligned} B &= \tfrac{1}{2}(\theta_h^n - \zeta_h^n, \theta_h^n - \zeta_h^n + \tfrac{2}{3}\tau L_h\zeta_h^n)_L \\ &= \tfrac{1}{2}(\theta_h^n - \zeta_h^n, -\tfrac{1}{3}(\theta_h^n - \zeta_h^n) + \tfrac{2}{3}\tau L_h\xi_h^n + \tfrac{2}{3}\tau(\beta_h^n - \alpha_h^n))_L \\ &= -\tfrac{1}{6}\|\theta_h^n - \zeta_h^n\|_L^2 + \tfrac{1}{3}(\theta_h^n - \zeta_h^n, \xi_h^n - \zeta_h^n)_L + \tfrac{1}{3}\tau(\beta_h^n, \theta_h^n - \zeta_h^n)_L. \end{aligned}$$

Set $G = -\frac{1}{2}\tau|\xi_h^n|_S^2 - \frac{1}{6}\tau|\zeta_h^n|_S^2 - \frac{1}{3}\tau|\theta_h^n|_S^2 - \frac{1}{6}\|\theta_h^n - \zeta_h^n\|_L^2$ and $D_3 = D_2'' + \frac{1}{3}\tau(\beta_h^n, \theta_h^n - \zeta_h^n)_L$, so that

$$A = G + \tfrac{1}{3}(\theta_h^n - \zeta_h^n, \xi_h^n - \zeta_h^n)_L + D_3.$$

Then using (4.16) again leads to

$$\begin{aligned} A &= G + \tfrac{1}{3}(-\tfrac{1}{2}\tau L_h(\zeta_h^n - \xi_h^n) + \tfrac{1}{2}\tau(\beta_h^n - \alpha_h^n), \xi_h^n - \zeta_h^n)_L + D_3 \\ &= G + \tfrac{1}{6}\tau|\zeta_h^n - \xi_h^n|_S^2 + D_4, \end{aligned}$$

with $D_4 = D_3 + \frac{1}{6}\tau(\beta_h^n - \alpha_h^n, \xi_h^n - \zeta_h^n)_L - \frac{1}{12}\tau(\Lambda(\zeta_h^n - \xi_h^n), \zeta_h^n - \xi_h^n)_L$. Using the expressions for D_1, D_2'', and D_3 in D_4 yields (4.15). ∎

LEMMA 4.3 (stability). *Under the usual CFL condition* (2.32) *with*

$$\varrho \le \min\left(\tfrac{5}{154}C_S^{-1}, (\tfrac{3}{4})^{1/2}C_{L*}^{-1}\right), \tag{4.18}$$

there holds

$$\|\xi_h^{n+1}\|_L^2 - \|\xi_h^n\|_L^2 + \tfrac{1}{48}\tau|\xi_h^n|_S^2 + \tfrac{1}{12}\tau|\zeta_h^n|_S^2 + \tfrac{1}{48}\tau|\theta_h^n|_S^2 \\ \le C\tau(\tau^6 + \|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2 + \|\theta_\pi^n\|_*^2 + \|\xi_h^n\|_L^2). \tag{4.19}$$

Proof. We bound the terms on the right-hand side of the energy identity (4.15).

(i) Bound on $\frac{1}{6}\tau|\zeta_h^n - \xi_h^n|_S^2$. Let ϵ and $\hat{\epsilon}$ be positive real numbers to be chosen later. Observe that

$$\begin{aligned}|\zeta_h^n - \xi_h^n|_S^2 &\le (1+\epsilon)|\theta_h^n - \xi_h^n|_S^2 + (1+\epsilon^{-1})|\theta_h^n - \zeta_h^n|_S^2 \\ &\le (1+\epsilon)(1+\hat{\epsilon})|\theta_h^n|_S^2 + (1+\epsilon)(1+\hat{\epsilon}^{-1})|\xi_h^n|_S^2 + (1+\epsilon^{-1})|\theta_h^n - \zeta_h^n|_S^2 \\ &\le (1+\epsilon)(1+\hat{\epsilon})|\theta_h^n|_S^2 + (1+\epsilon)(1+\hat{\epsilon}^{-1})|\xi_h^n|_S^2 \\ &\quad + (1+\epsilon^{-1})C_S\sigma h^{-1}\|\theta_h^n - \zeta_h^n\|_L^2,\end{aligned}$$

where assumption (A4) has been used. Then taking $\epsilon = \frac{5}{72}$ and $\hat{\epsilon} = \frac{7}{11}$ and observing that $\frac{1}{6}(1+\epsilon)(1+\hat{\epsilon}) = \frac{7}{24}$, $\frac{1}{6}(1+\epsilon)(1+\hat{\epsilon}^{-1}) = \frac{11}{24}$, and that

$$\tfrac{1}{6}\tau(1+\epsilon^{-1})C_S\sigma h^{-1} \le \tfrac{1}{12},$$

owing to the choice (4.18) for the CFL condition, it is inferred that

$$\tfrac{1}{6}\tau|\zeta_h^n - \xi_h^n|_S^2 \le \tfrac{7}{24}\tau|\theta_h^n|_S^2 + \tfrac{11}{24}\tau|\xi_h^n|_S^2 + \tfrac{1}{12}\|\theta_h^n - \zeta_h^n\|_L^2.$$

(ii) Bound on $\frac{1}{2}\|\xi_h^{n+1} - \theta_h^n\|_L^2$. Using (4.17) and the bound (2.29) yields

$$\begin{aligned}\tfrac{1}{2}\|\xi_h^{n+1} - \theta_h^n\|_L^2 &\le \tfrac{1}{9}\tau^2\|L_h(\theta_h^n - \zeta_h^n)\|_L^2 + \tfrac{1}{9}\tau^2\|\gamma_h^n - \beta_h^n\|_L^2 \\ &\le \tfrac{1}{9}(\tau C_{L*}\sigma h^{-1})^2\|\theta_h^n - \zeta_h^n\|_L^2 + \tfrac{1}{9}\tau^2\|\gamma_h^n - \beta_h^n\|_L^2 \\ &\le \tfrac{1}{12}\|\theta_h^n - \zeta_h^n\|_L^2 + \tfrac{1}{9}\tau^2\|\gamma_h^n - \beta_h^n\|_L^2,\end{aligned}$$

owing to the choice (4.18) for the CFL condition.

(iii) Inserting the bounds delivered in steps (i) and (ii) into (4.15) yields

$$\begin{aligned}&\tfrac{1}{2}\|\xi_h^{n+1}\|_L^2 - \tfrac{1}{2}\|\xi_h^n\|_L^2 + \tfrac{1}{24}\tau|\xi_h^n|_S^2 + \tfrac{1}{6}\tau|\zeta_h^n|_S^2 + \tfrac{1}{24}\tau|\theta_h^n|_S^2 \\ &\quad \le \tfrac{1}{9}\tau^2\|\gamma_h^n - \beta_h^n\|_L^2 + \tfrac{1}{3}\tau(\gamma_h^n, \theta_h^n)_L + \tfrac{1}{6}\tau(\beta_h^n, \zeta_h^n)_L + \tfrac{1}{3}\tau(\alpha_h^n, \xi_h^n + \tfrac{1}{2}\zeta_h^n)_L + \Lambda_h^n.\end{aligned}$$

(iv) It remains to bound the five terms on the right-hand side, say, T_1–T_5. To this purpose, we first bound the quantities α_h^n, β_h^n, and γ_h^n by proceeding as for the RK2 scheme. It is readily inferred that

$$\tau\|\alpha_h^n\|_L \lesssim \tau^{1/2}\|\xi_\pi^n\|_*, \quad \tau\|\beta_h^n\|_L \lesssim \tau^{1/2}\|\zeta_\pi^n\|_*, \quad \tau\|\gamma_h^n\|_L \lesssim \tau^{1/2}\|\theta_\pi^n\|_* + \tau^4,$$

and that for all $v_h \in V_h$,

$$\begin{aligned}\tau(\alpha_h^n, v_h)_L &\lesssim \tau\|\xi_\pi^n\|_*(|v_h|_S + \|v_h\|_L), \\ \tau(\beta_h^n, v_h)_L &\lesssim \tau\|\zeta_\pi^n\|_*(|v_h|_S + \|v_h\|_L), \\ \tau(\gamma_h^n, v_h)_L &\lesssim \tau\|\theta_h^n\|_*(|v_h|_S + \|v_h\|_L) + \tau^4\|v_h\|_L.\end{aligned}$$

Moreover, still proceeding as for the RK2 scheme,

$$\|\zeta_h^n\|_L \lesssim \|\xi_h^n\|_L + \|\xi_\pi\|_*, \qquad \|\theta_h^n\|_L \lesssim \|\xi_h^n\|_L + \|\xi_\pi\|_* + \|\zeta_\pi^n\|_*.$$

Using these estimates yields $T_1 \lesssim \tau(\tau^7 + \|\zeta_\pi^n\|_*^2 + \|\theta_\pi^n\|_*^2)$. Furthermore, using Young inequalities leads to

$$T_2 + T_3 + T_4 \leq \tfrac{1}{48}\tau|\xi_h^n|_S^2 + \tfrac{1}{12}\tau|\zeta_h^n|_S^2 + \tfrac{1}{48}\tau|\theta_h^n|_S^2 + C\tau(\tau^6 + \|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2 + \|\theta_\pi^n\|_*^2 + \|\xi_h^n\|_L^2).$$

Finally, since Λ is symmetric,

$$T_5 \lesssim \tau(\|\xi_h^n\|_L^2 + \|\zeta_h^n\|_L^2 + \|\theta_h^n\|_L^2) \lesssim \tau(\|\xi_\pi^n\|_*^2 + \|\zeta_\pi^n\|_*^2 + \|\xi_h^n\|_L^2).$$

Collecting the above bounds and since $\tau^7 \leq \tau^6$ concludes the proof. ∎

4.4. Error estimate. The next theorem provides a quasi-optimal energy error estimate under the usual CFL condition.

THEOREM 4.1. *Assume that* $u \in C^4(0,T;L) \cap C^s(0,T;[H^{p+1-s}(\Omega)]^m)$ *for* $s \in \{0,1,2\}$. *Then under the usual CFL condition* (2.32) *with the choice* (4.18) *for* ϱ, *there holds*

$$\|u^N - u_h^N\|_L + \left(\sum_{n=0}^{N-1} \tfrac{1}{48}\tau|u_h^n|_S^2 + \tfrac{1}{12}\tau|w_h^n|_S^2 + \tfrac{1}{48}\tau|y_h^n|_S^2\right)^{1/2} \lesssim \tau^3 + h^{p+1/2}. \tag{4.20}$$

Proof. Starting with estimate (4.19), sum over n, apply Gronwall's lemma, and use the approximation property (2.30). ∎

5. Numerical results. In this section we investigate numerically explicit RK2 and RK3 schemes using, respectively, their implementations (3.6)–(3.8) and (4.6)–(4.9). We discretize in space using the continuous (CIP stabilized) and DG FEMs discussed in section 2.4, using piecewise affine ($p = 1$) and quadratic ($p = 2$) polynomials. We consider the advection equation (2.8) with $\beta = (y, -x)^{\mathrm{t}}$, $f = 0$, and $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. We first illustrate the convergence properties of the various schemes on a test case with smooth initial datum. Then we investigate on a test case with rough initial datum the capability of the present schemes to control global spurious oscillations. The numerical computations have been carried out using FreeFem++ [21].

5.1. Convergence rates for smooth solutions. The initial datum is the Gaussian function $u_0(x, y) = e^{-\rho(x,y)}$ with $\rho(x, y) = 10[(x - 0.3)^2 + (y - 0.3)^2]$. The stabilization parameter γ in S_F^{int} is set to 0.5 for DG (i.e., upwinding) and to 0.005 and 0.001 for CIP with $p = 1$ and $p = 2$, respectively (improvements by further tuning of these parameters goes beyond the present scope; see, e.g., [6] for such an investigation in the steady case). In Table 5.1, we report the energy errors $\|u^N - u_h^N\|_L$ at the final time $T = 2\pi$, i.e., after a complete rotation of the initial datum. The scaling τ/h is chosen to satisfy the appropriate CFL condition and, according to the error estimates (3.21), (3.23), or (4.20), in such a way that the error in time dominates the error in space. The RK2 schemes with $p = 1$ exhibit second-order accuracy in time, as stated in Theorem 3.2. On a fixed mesh, the DG formulation yields more accurate results and uses a larger number of degrees of freedom. The RK2 schemes with

Table 5.1
Energy error at final time for smooth initial datum.

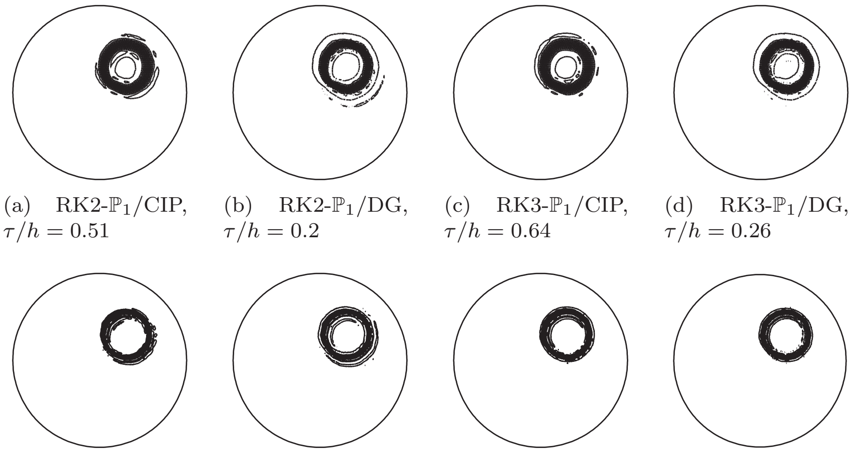
$p = 1$					
RK2 ($\tau = 0.2h$)			RK3 (see text)		
τ	CIP	DG	τ	CIP	DG
9.82e−3	4.31e−3	1.79e−3	1.05e−2	8.45e−2	3.92e−2
4.91e−3	7.20e−4	3.91e−4	5.24e−3	4.30e−3	1.72e−3
2.45e−3	1.58e−4	8.86e−5	2.62e−3	1.55e−4	8.36e−5
$p = 2$					
RK2 ($\tau = 0.14h^{4/3}$)			RK3 ($\tau = 0.08h$)		
τ	CIP	DG	τ	CIP	DG
3.98e−2	5.60e−2	4.27e−2	3.14e−2	5.57e−2	4.13e−2
1.58e−2	5.54e−3	3.74e−3	1.57e−2	5.16e−3	3.44e−3
6.28e−3	6.08e−4	4.38e−4	7.85e−3	5.47e−4	3.90e−4

$p = 2$ yield the $O(\tau^2)$ accuracy predicted by Theorem 3.1. We also observed that the strengthened 4/3-CFL condition seems to be numerically sharp. For the RK3 schemes with $p = 1$, τ scales as $h^{1/2}$. This choice is made in order to reduce the spatial error which scales as $h^{3/2}$ only; thus, the usual CFL condition is satisfied with an increasing parameter ϱ, up to 0.21. The achieved convergence rate is slightly higher than the theoretical $O(\tau^3)$ stated in Theorem 4.1. A possible explanation is that contributions of the spatial error can be dominant on coarser meshes. Finally, the RK3 schemes with $p = 2$ yield the $O(\tau^3)$ accuracy stated in Theorem 4.1; slight perturbations, due to the $O(h^{5/2})$ spatial contribution of error, can appear on the finer meshes.

5.2. Controlling oscillations in rough solutions. We now consider the initial datum $u_0(x, y) = \frac{1}{2}(\tanh(10^3(e^{-\rho(x,y)} - 0.5)) + 1)$. Owing to the sharp layer (with thickness of order 0.001), spurious oscillations are obtained when using unstabilized continuous finite elements (or, equivalently, centered fluxes) on meshes that are too coarse to resolve the internal layer. We consider a fixed uniform mesh with 256 elements along the boundary of Ω ($h \approx 0.025$). The sharp layer is thus underresolved. Approximate solutions are presented in Figure 1, confirming that the present methods are able to avoid the global spreading of spurious oscillations, so that the discrete solution after a complete rotation still exhibits a sharp layer. In all cases, the largest possible time step has been used; the corresponding value of ϱ in the CFL condition is reported in Figure 1.

6. Concluding remarks. In this work, we have analyzed several approximation methods to the evolution problem (1.1a)–(1.1c), combining explicit RK schemes in time and stabilized finite elements in space. In our opinion, salient features of this work are the following: (i) the fact that continuous and discontinuous finite element approximation in space can be cast into a unified analysis framework, (ii) the possibility to use high-order approximation methods in space and time (at least up to third-order in time, but an extension to RK4 schemes proceeding along the same lines should be feasible) in an explicit framework, (iii) the intertwined stability effects coupling stabilization in space and antidissipativity in time. This last point is important in the analysis of RK2 schemes with piecewise affine finite elements, but it also plays a role in bounding the α_h^n, β_h^n, and γ_h^n terms in the energy identities owing to assumption (A6).

The techniques presented herein can also be used to analyze finite volume schemes with upwinding by interpreting them as DG methods with polynomial degree $p = 0$.



(a) RK2-$\mathbb{P}_1$/CIP, $\tau/h = 0.51$ (b) RK2-$\mathbb{P}_1$/DG, $\tau/h = 0.2$ (c) RK3-$\mathbb{P}_1$/CIP, $\tau/h = 0.64$ (d) RK3-$\mathbb{P}_1$/DG, $\tau/h = 0.26$

(e) RK2-$\mathbb{P}_2$/CIP, $\tau/h = 0.2$ (f) RK2-$\mathbb{P}_2$/DG, $\tau/h = 0.12$ (g) RK3-$\mathbb{P}_2$/CIP, $\tau/h = 0.26$ (h) RK3-$\mathbb{P}_2$/DG, $\tau/h = 0.15$

FIG. 1. *Contour-lines of discrete solution at final time for rough initial datum.*

The forward Euler scheme takes the form $u_h^{n+1} = u_h^n - \tau L_h u_h^n + \tau f_h^n$ so that the error equation becomes $\xi_h^{n+1} = \xi_h^n - \tau L_h \xi_h^n + \tau \alpha_h^n$, where $\alpha_h^n = L_h \xi_\pi^n - \pi_h \eta^n$ with $\eta^n = \tau^{-1} \int_{I_n} (t_{n+1} - t) \partial_{tt} u \, dt$. Multiplying by ξ_h^n yields the energy identity

$$\tfrac{1}{2}\|\xi_h^{n+1}\|_L^2 - \tfrac{1}{2}\|\xi_h^n\|_L^2 + \tau|\xi_h^n|_S^2 = \tfrac{1}{2}\|\xi_h^{n+1} - \xi_h^n\|_L^2 + \tau(\alpha_h^n, \xi_h^n)_L + \tfrac{1}{2}\tau(\Lambda\xi_h^n, \xi_h^n)_L.$$

In the finite volume case, a remarkable property is that there is C_L^0 s.t. for all $v_h \in V_h$,

$$\|L_h v_h\|_L \leq C_L^0 \sigma^{1/2} h^{-1/2} |v_h|_S. \tag{6.1}$$

(This property is equivalent to the coercivity condition considered in [24].) As a result, under the usual CFL condition (2.32) with $\varrho = (2C_L^0)^{-2}$, the first term on the right-hand side of the energy identity, which results from the antidissipative nature of the forward Euler scheme, can be bounded as

$$\tfrac{1}{2}\|\xi_h^{n+1} - \xi_h^n\|_L^2 \leq \tfrac{1}{2}\tau|\xi_h^n|_S^2 + \tau^2\|\alpha_h^n\|_L^2,$$

whence proceeding as above, an energy error estimate of the form $O(\tau + h^{1/2})$ is inferred. In the absence of upwinding (that is, if centered fluxes are used), the above argument breaks down since $|\cdot|_S = 0$, and it is necessary to invoke a stronger 2-CFL condition of the form $\tau \leq \varrho'(h/\sigma)^2$. Using upwinding with the usual CFL condition or centered fluxes with the 2-CFL condition both yield a consistent approximation method in the sense that (A3) holds true. Furthermore, the stability argument also breaks down for high-order DG methods ($p \geq 1$) since (6.1) no longer holds true, thereby requiring again a 2-CFL condition. Alternatively, an inconsistent approximation with the usual CFL condition can be derived using continuous, piecewise affine finite elements and first-order streamline diffusion for which $s(v, w) = h(Av, Aw)_L$. Finally, we observe that explicit RK2 schemes can also be used in conjunction with finite volume schemes. Indeed, the proof of Theorem 3.2 is still valid for $p = 0$ under the usual CFL condition (2.32) with $\varrho = \frac{1}{8}C_L^{-2}$.

Extensions of this work can explore various directions. Firstly, a more general form for the evolution problem (1.1a) can be considered, namely, $A_0\partial_t u + Au = f$ on $\Omega \times (0,T)$. When A_0 is smooth, the above analysis carries over with minor modifications. When A_0 is not smooth (e.g., piecewise constant on the mesh), the analysis must be modified; DG methods appear to be more appropriate to handle this case. A further extension of the present analysis is to tackle error estimates in the graph norm. Finally, ongoing work focuses on combining explicit and implicit schemes for the approximation of evolution problems with stiff terms resulting from diffusion.

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