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STRONGLY REPRESENTABLE ATOM STRUCTURES OF CYLINDRIC ALGEBRAS

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Abstract. A cylindric algebra atom structure is said to be *strongly representable* if all atomic cylindric algebras with that atom structure are representable. This is equivalent to saying that the full complex algebra of the atom structure is a representable cylindric algebra. We show that for any finite $n \geq 3$, the class of all strongly representable n-dimensional cylindric algebra atom structures is not closed under ultraproducts and is therefore not elementary.

Our proof is based on the following construction. From an arbitrary undirected, loop-free graph Γ, we construct an n-dimensional atom structure $\mathcal{E}(\Gamma)$, and prove, for infinite Γ, that $\mathcal{E}(\Gamma)$ is a strongly representable cylindric algebra atom structure if and only if the chromatic number of Γ is infinite. A construction of Erdős shows that there are graphs Γ_k $(k < \omega)$ with infinite chromatic number, but having a non-principal ultraproduct $\prod_D \Gamma_k$ whose chromatic number is just two. It follows that $\mathcal{E}(\Gamma_k)$ is strongly representable (each $k < \omega$) but $\prod_D \mathcal{E}(\Gamma_k)$ is not.

§1. Introduction. This paper is broadly about *algebras of α-ary relations,* for an ordinal α. An α-ary relation is a set of ordered α-tuples of elements of some base set, and an algebra of α-ary relations will consist of a set of α-ary relations, endowed with various operations. These operations include the boolean union and complement and constants denoting the empty relation and the maximum or 'unit' relation, and the algebra will be a boolean algebra under these operations. But there will also be additional operations that make use of the relational form of the elements of the algebra. Various choices of these operations can be made. The 'cylindric' approach, first taken by Alfred Tarski and his students Louise Chin and Frederick Thompson in the late 1940s, gives us *cylindric set algebras,* which have since been studied extensively, e.g., in [10, 8, 9]. These algebras include constants called diagonal elements, which are like equality, and unary functions called cylindrifications, which are like existential quantification. For finite α, the algebras are closely related to first-order logic with α variables. But relation symbols in first-order logic have finite arity, so for infinite α, the algebraic approach, which can handle relations of any arity up to α, goes further.

Roughly speaking, the class RCA_α of *'representable α-dimensional cylindric algebras'* is the closure under isomorphism of the class of all algebras of relations

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as just described. Quite a lot of work has gone into characterising RCA_α. Tarski proved in [22] that it is a variety: it can be axiomatised by equations. Explicit finite sets of equations axiomatising $\mathsf{RCA}_0, \mathsf{RCA}_1$, and RCA_2 are known — the finite set of equations defining RCA_2 is due to Henkin [9, theorem 3.2.65]. But for finite $n \geq 3$, Monk showed in [17] that RCA_n is not finitely axiomatisable, Andréka showed in [1] that any equational axiomatisation of it uses infinitely many variables, and [14] showed that RCA_n is not closed under Monk completions [18] and so, by results of Venema [24], is not axiomatisable by Sahlqvist equations. In general, characterising RCA_n for $n \geq 3$ seems to be a hard problem.

In this paper, we are concerned with the special case of *atomic* algebras in RCA_α. The boolean reduct of each $\mathcal{A} \in \mathsf{RCA}_\alpha$ is a boolean algebra. We say that $\mathcal{A}$ is *atomic* if this reduct is atomic. If $\mathcal{A}$ is atomic, then its non-boolean structure induces a dual relational structure on its set of atoms. These *'atom structures'* are the real focus of our paper. The atom structure of $\mathcal{A}$ is written $\mathrm{At}\,\mathcal{A}$; it is analogous to a Kripke frame in modal logic. This dual approach works well not just for atomic $\mathcal{A} \in \mathsf{RCA}_\alpha$ but for any algebra $\mathcal{A}$ whose boolean reduct is an atomic boolean algebra and in which the non-boolean operations preserve all boolean sums. So we restrict our attention to such $\mathcal{A}$. It turns out that each element a of such an algebra $\mathcal{A}$ can be identified with the subset of $\mathrm{At}\,\mathcal{A}$ consisting of the atoms beneath a. In this way, the entire non-boolean structure of $\mathcal{A}$ can be recovered from $\mathrm{At}\,\mathcal{A}$.

It is tempting then to work with $\mathrm{At}\,\mathcal{A}$ instead of $\mathcal{A}$, because it is smaller, and the boolean operations are absent. This does have its uses, but unfortunately, $\mathrm{At}\,\mathcal{A}$ does not always determine whether $\mathcal{A} \in \mathsf{RCA}_\alpha$ or not. For each finite $n \geq 3$, there are atomic algebras $\mathcal{A}, \mathcal{B}$ with $\mathrm{At}\,\mathcal{A} = \mathrm{At}\,\mathcal{B}$, $\mathcal{A} \in \mathsf{RCA}_n$, and $\mathcal{B} \notin \mathsf{RCA}_n$ [14]. What is going on is that $\mathcal{B}$ has more elements than $\mathcal{A}$, and these elements are incompatible with true algebras of relations.

An (abstract) *atom structure* is a relational structure of the similarity type of atom structures of atomic algebras in RCA_α. The example above suggests to define two classes of atom structure:

1. $\mathsf{At}\,\mathsf{RCA}_\alpha = \{\mathcal{S} :$ some atomic algebra $\mathcal{A}$ with atom structure $\mathcal{S}$ is in $\mathsf{RCA}_\alpha\}$,
2. $\mathsf{Str}\,\mathsf{RCA}_\alpha = \{\mathcal{S} :$ every atomic algebra $\mathcal{A}$ with atom structure $\mathcal{S}$ is in $\mathsf{RCA}_\alpha\}$.

An atom structure in $\mathsf{At}\,\mathsf{RCA}_\alpha$ will be called *weakly representable,* and one in $\mathsf{Str}\,\mathsf{RCA}_\alpha$ *strongly representable.*[1] Every atom structure is the atom structure of *some* atomic algebra, and it follows that $\mathsf{Str}\,\mathsf{RCA}_\alpha \subseteq \mathsf{At}\,\mathsf{RCA}_\alpha$. The example above shows that the inclusion is proper, for finite $\alpha \geq 3$. By a general result of Venema [23], $\mathsf{At}\,\mathsf{RCA}_\alpha$ is always elementary and effectively axiomatisable from equations defining RCA_α. For $\alpha \leq 2$, RCA_α is axiomatisable by Sahlqvist equations, and $\mathsf{Str}\,\mathsf{RCA}_\alpha$ is then the same class as $\mathsf{At}\,\mathsf{RCA}_\alpha$. It is elementary and finitely axiomatisable. See remark 7.3 for more details. However, for $\alpha \geq 3$, RCA_α is not Sahlqvist-axiomatisable and $\mathsf{Str}\,\mathsf{RCA}_\alpha$ is not so easily characterised.

In this paper, we will show (in theorem 6.1) that for finite $n \geq 3$, $\mathsf{Str}\,\mathsf{RCA}_n$ is not definable by any set of first-order sentences: it is not an elementary class. This adds to the general body of evidence that RCA_α is hard to characterise. It

[1] But 'Str' stands for 'structures for'. [7] studies these notions in a wider context.

answers [13, Problem 1] and [12, problem 14.20] for finite dimensions (admittedly, these problems were set by the authors).

We remark that RCA_α has a cousin: RRA, the class of representable relation algebras. Its members are isomorphic to algebras of binary relations, using a different choice of relational operators from RCA_α. RRA is also hard to characterise. The analogous result for RRA, that $\mathsf{Str}\,\mathsf{RRA}$ is non-elementary, was proved in [13, 12] by a similar method to the one here.

A few words about the method. Because RCA_n is a variety, an atomic algebra $\mathcal{A}$ will be in RCA_n iff all the equations defining RCA_n are valid in $\mathcal{A}$. From the point of view of $\mathrm{At}\,\mathcal{A}$, each equation corresponds to a certain universal monadic second-order statement, where the universal quantifiers are restricted to ranging over the sets of atoms that are defined by (i.e., lie underneath) elements of $\mathcal{A}$. Such a statement will fail in $\mathcal{A}$ iff $\mathrm{At}\,\mathcal{A}$ can be partitioned into finitely many $\mathcal{A}$-definable sets with certain 'bad' properties. In order to give a very rough outline of our argument, we call this a *bad partition.* This idea can be used to show that RCA_n (for $n \geq 3$) is not finitely axiomatisable, by finding a sequence of atom structures, each having some sets that form a bad partition, but with the minimal number of sets in a bad partition increasing as we go along the sequence. This can yield algebras not in RCA_n but with an ultraproduct that is in RCA_n, so reproving Monk's result that RCA_n is not finitely axiomatisable. The reader should have no trouble in using the methods of our paper to do exactly that.

Curiously, our problem here is the reverse of this. An atom structure is in $\mathsf{Str}\,\mathsf{RCA}_n$ iff it has no bad partition using any sets at all. We want to find atom structures in $\mathsf{Str}\,\mathsf{RCA}_n$ — so they have no bad partitions — with an ultraproduct that does have a bad partition. This will show that $\mathsf{Str}\,\mathsf{RCA}_n$ is not closed under ultraproducts, and so is non-elementary.

We find our source of bad partitions in graph theory. From our point of view, a bad partition of a graph is a *finite colouring:* a partition of its set of nodes into finitely many independent (edge-free) sets. Using some coding, from a graph we can create an atom structure that is strongly representable iff the graph has no finite colouring. Our problem now boils down to finding a sequence of graphs with no finite colouring, but with an ultraproduct that does have a finite colouring. In graph-theoretic language, we want graphs of infinite chromatic number, having an ultraproduct with finite chromatic number. Graphs like this can be found using a well-known theorem of Erdős [5], which shows that there exist finite graphs of arbitrarily large chromatic number and girth (length of the shortest cycle). By taking disjoint unions, we can obtain graphs of infinite chromatic number (no bad partitions) and arbitrarily large girth. A non-principal ultraproduct of these graphs has no cycles, so has chromatic number 2 (a bad partition into just two sets).

We thank Istvan Németi and Tarek Sayed Ahmed (and others) for suggesting that we try to extend [13] to show that $\mathsf{Str}\,\mathsf{RCA}_n$ is non-elementary. We also thank the referee for helpful comments. We assume some knowledge of basic boolean notions such as atoms and ultrafilters. For those seeking more details of the topics considered here, we suggest [8, 9, 19, 21, 2], or for some parts, [12].

Layout of paper. Section 2 lays out the basic formal definitions and facts about them. In section 3 we introduce the atom structures, based on graphs, that will be used to prove our main result. In section 4 we establish some preliminary results about 'networks' and related machinery. Section 5 connects representability to chromatic number, which allows us to prove our main result in section 6. Section 7 has some remarks and problems.

§2. Representable cylindric algebras and atom structures.

This section recalls the standard definitions and facts that we will use, all well known, and some notation. We will not need to use *cylindric algebras* at all. (These are abstract versions of cylindric set algebras, defined by equations that can be found in [8].)

2.1. Representable algebras. First, we recall the formal definition of the class RCA_α. Let α be an ordinal. For a set U, ${}^\alpha U$ denotes the set of maps from α to U. We write such maps as x, y, and for $i < \alpha$ we write $x(i)$ as x_i. For finite α, we identify ${}^\alpha U$ with the cartesian product U^α, via $x \mapsto (x_0, \ldots, x_{\alpha-1})$. An *$\alpha$-ary relation* on U is a subset of ${}^\alpha U$. For $i, j < \alpha$, the *i, jth diagonal* D^U_{ij} denotes $\{x \in {}^\alpha U : x_i = x_j\}$. Given $i < \alpha$ and an α-ary relation X on U, the *ith cylindrification* $C^U_i X$ denotes the set of all elements of ${}^\alpha U$ that agree with some element of X on each coordinate except, perhaps, on the ith coordinate: $C^U_i X = \{y \in {}^\alpha U : \exists x \in X\ \forall j < \alpha (j \neq i \to y_j = x_j)\}$.

A *cylindric set algebra* of dimension α is an algebra $\mathcal{A} = (A, \cup, -, \emptyset, {}^\alpha U, D^U_{ij}, C^U_i)_{i,j<\alpha}$ consisting of a set A of α-ary relations on some non-empty base set U, equipped with the boolean constants $\emptyset, {}^\alpha U$ and boolean operations $\cup$ and $-$ (where $-X = {}^\alpha U \setminus X$), the diagonal elements D^U_{ij} $(i, j < \alpha)$, and the cylindrifications C^U_i $(i < \alpha)$. A must of course be closed under all these operations.

We wish to consider abstract algebras related to these. The signature of α-dimensional cylindric set algebras consists of a binary function symbol $+$, a unary function symbol $-$, constants $0, 1$, and d_{ij} $(i, j < \alpha)$, and unary function symbols c_i $(i < \alpha)$. (Traditionally, slightly different symbols from the 'concrete' operations $\cup$, etc., are used.) A *cylindric-type algebra* (of dimension α) is just a structure for this signature.

Our central definition is as follows.

Definition 2.1. An α-dimensional cylindric-type algebra $\mathcal{A}$ is said to be *representable* if it is isomorphic to a subalgebra of a direct product of cylindric set algebras of dimension α; such an isomorphism is called a *representation* of $\mathcal{A}$. RCA_α denotes the class of representable α-dimensional cylindric-type algebras.

[22] proves that RCA_α is a variety (an elementary class axiomatised by equations).

2.2. Notation. Until section 7, we are interested only in finite dimensions, and we fix such a dimension n, where $3 \leq n < \omega$. *Throughout, all cylindric-type algebras and atom structures will be of dimension n.* n is an ordinal, so it is $\{0, 1, \ldots, n-1\}$. Usually, i, j, k, l, etc., denote elements of n. For a set X, $\wp(X)$ denotes the set of all subsets of X, and for $m < \omega$, $[X]^m$ denotes $\{A \subseteq X : |A| = m\}$. Maps (including partial ones) are regarded formally, as sets of ordered pairs; so we may write $f \subseteq g$, etc. We write $\mathrm{dom}(f), \mathrm{rng}(f)$ for the domain and range (respectively) of a map f. We frequently identify (notationally) a structure with its domain.

2.3. Atom structures. It is well known that any algebra whose boolean reduct is an atomic boolean algebra has an atom structure, which essentially records the values of the non-boolean functions on atoms. The atom structure can be defined whatever these functions are like, but it only really comes into its own when they are completely additive, preserving all existing suprema. In that case, the atom structure determines their values on all elements of the algebra.

We would like to define a 'cylindric-type' atom structure $\mathcal{S}$ to be strongly representable if every cylindric-type algebra with atom structure $\mathcal{S}$ is representable. The problem with this is that we can always find pathological algebras with any given atom structure $\mathcal{S}$, but that cannot be representable. This can easily be done, since even if the boolean reduct of the algebra is a boolean algebra, the c_i need not be completely additive — and in any representable algebra, the c_i are completely additive. So we will restrict our consideration to cylindric-type algebras based on boolean algebras and in which the c_i are completely additive. This will yield the alternative characterisation of strong representability, in lemma 2.6 below, which is what we actually use in the proofs later.

Definition 2.2. An *(atomic) cylindric BAO* is a cylindric-type algebra $\mathcal{A}$ whose boolean reduct is an (atomic) boolean algebra, and in which $\mathsf{c}_i \sum S = \sum\{\mathsf{c}_i a : a \in S\}$ for every set S of elements of $\mathcal{A}$ with a least upper bound $\sum S$ in $\mathcal{A}$, and every $i < n$.

We are misusing 'BAO' slightly. It stands for 'boolean algebra with operators', and indeed every cylindric BAO is a boolean algebra with (normal additive) operators in the sense of [16]. But not all boolean algebra with operators are completely additive.

It can easily be verified that every algebra in RCA_n is a cylindric BAO.

Definition 2.3.

1. A *cylindric atom structure* is a structure of the form $\mathcal{S} = (H, D_{ij}, E_i : i, j < n)$, where H is a non-empty set, each D_{ij} is a subset of H, and each E_i is a binary relation on H.
2. Let $\mathcal{A}$ be an atomic cylindric BAO. The *atom structure of* $\mathcal{A}$, in symbols $\mathrm{At}\,\mathcal{A}$, is the structure $(H, D_{ij}, E_i : i, j < n)$, where H is the set of atoms of $\mathcal{A}$, $D_{ij} = \{x \in H : x \leq \mathsf{d}_{ij}\}$ for each $i, j < n$, and $x\ E_i\ y$ iff $x \leq \mathsf{c}_i y$ for each $i < n$ and $x, y \in H$.
3. The *complex algebra* $\mathcal{S}^+$ over a cylindric atom structure $\mathcal{S} = (H, D_{ij}, E_i : i, j < n)$ is the cylindric-type algebra $(\wp(H), \cup, -, \emptyset, H, D_{ij}, \mathsf{c}_i : i, j < n)$, where for $X \subseteq H$, we define $-X = H \setminus X$ and $\mathsf{c}_i X = \{x \in H : \exists x' \in X(x\ E_i\ x')\}$.

The following lemma is well known and follows from results in a slightly different setting in [16, §3]. A proof can be found in [12, proposition 2.66].

Lemma 2.4.

1. *If $\mathcal{A}$ is an atomic cylindric BAO, then $At\,\mathcal{A}$ is a cylindric atom structure. Moreover, there is an embedding $h : \mathcal{A} \to (At\,\mathcal{A})^+$ defined by $h(a) = \{x \in At\,\mathcal{A} : x \leq a\}$.*

2. *If $\mathcal{S}$ is a cylindric atom structure, then $\mathcal{S}^+$ is an atomic cylindric BAO. Moreover, $At(\mathcal{S}^+) \cong \mathcal{S}$.*

DEFINITION 2.5.

1. A cylindric atom structure $\mathcal{S}$ is said to be *strongly representable* if for every atomic cylindric BAO $\mathcal{A}$ with $\mathrm{At}\,\mathcal{A} = \mathcal{S}$, we have $\mathcal{A} \in \mathsf{RCA}_n$.
2. We write $\mathsf{Str}\,\mathsf{RCA}_n$ for the class of strongly representable cylindric atom structures.

Rather than considering every possible atomic cylindric BAO with atom structure $\mathcal{S}$, there is a more convenient way to tell whether $\mathcal{S}$ is strongly representable:

LEMMA 2.6. *A cylindric atom structure $\mathcal{S}$ is strongly representable iff $\mathcal{S}^+ \in \mathsf{RCA}_n$.*

Proof. By lemma 2.4, $\mathcal{S}^+$ is an atomic cylindric BAO and $\mathrm{At}(\mathcal{S}^+) \cong \mathcal{S}$, from which $\Rightarrow$ follows. Conversely, if $\mathcal{S}^+ \in \mathsf{RCA}_n$, then there is an embedding g from $\mathcal{S}^+$ into a product of cylindric set algebras. Let $\mathcal{A}$ be any atomic cylindric BAO with atom structure $\mathcal{S}$. By lemma 2.4, there is also an embedding $h : \mathcal{A} \to \mathcal{S}^+$. Then $g \circ h$ is a representation of $\mathcal{A}$, so $\mathcal{A} \in \mathsf{RCA}_n$. Hence, $\mathcal{S}$ is strongly representable. $\square$

§3. Atom structures from graphs. The cylindric atom structures that we will use in our theorem are made from graphs.

3.1. Graphs. In this paper, by a *graph* we will mean a pair $\Gamma = (G, E)$, where $G \neq \emptyset$ and $E \subseteq G \times G$ is an irreflexive and symmetric binary relation on G. We will often use the same notation for Γ and for its set of nodes (G above). A pair $(x, y) \in E$ will be called an *edge* of Γ. See [4] for basic information (and a lot more) about graphs.

DEFINITION 3.1. Let $\Gamma = (G, E)$ be a graph.

1. A set $X \subseteq G$ is said to be *independent* if $E \cap (X \times X) = \emptyset$.
2. The *chromatic number* $\chi(\Gamma)$ of Γ is the smallest $k < \omega$ such that G can be partitioned into k independent sets, and ∞ if there is no such k.
3. By a *cycle* of length k in Γ (for finite $k \geq 3$) we will mean a sequence $(x_0, \ldots, x_{k-1})$ of distinct nodes of G such that $(x_0, x_1), \ldots, (x_{k-2}, x_{k-1})$, and (x_{k-1}, x_0) are all edges of Γ.
4. An *ultrafilter on* Γ is simply an ultrafilter of the boolean algebra $(\wp(G), \cup, -, \emptyset, G)$, where $-X$ (for $X \subseteq G$) is defined to be $G \setminus X$.

LEMMA 3.2. *A graph Γ has no cycles of odd length iff $\chi(\Gamma) \leq 2$.*

Proof. See, e.g., [4, proposition 1.6.1]. The result holds for both finite and infinite graphs; the implicit assumption in [4, p. 2] that graphs are finite is not needed in the proof in [4]. In [4], reflections and cyclic permutations of a cycle $(x_0, \ldots, x_{k-1})$ are regarded as the *same* cycle. Obviously this makes no difference to the lemma. $\square$

LEMMA 3.3. *A graph Γ has infinite chromatic number iff there is an ultrafilter on Γ containing no independent sets.*

Proof. $\Leftarrow$: if Γ has a partition into finitely many independent sets, then any ultrafilter on Γ contains one of them.

$\Rightarrow$: if $\chi(\Gamma) = \infty$, let δ_0 be the set of all subsets X of (the set of nodes of) Γ such that the complement of X is the union of finitely many independent sets. It is easy to check that δ_0 has the finite intersection property. So by the boolean prime ideal theorem [3, proposition 4.1.3], it extends to an ultrafilter δ on Γ. If $X \subseteq \Gamma$ is independent, then $\Gamma \setminus X \in \delta_0$, so $X \notin \delta$. □

3.2. A cylindric atom structure. Until section 6, we fix a graph $\Gamma = (G, E)$. We write $\Gamma \times n$ for the graph

$$\big(G \times n, \{((x, i), (y, j)) : x, y \in G,\ i, j < n,\ (x, y) \in E \text{ or } i \neq j\}\big).$$

$\Gamma \times n$ can be thought of as n disjoint copies of Γ, with all possible edges between distinct copies being added. Note that $\chi(\Gamma \times n) = \chi(\Gamma) \cdot n$, where $\infty \cdot n = \infty$ of course.

DEFINITION 3.4. *For an equivalence relation $\sim$ on a set X, and $Y \subseteq X$, we write $\sim\restriction Y$ for $\sim \cap (Y \times Y)$. We write $=_X$ for the equality relation on X. For a partial map $K : n \to \Gamma \times n$ and $i, j < n$, we write $K(i) = K(j)$ to mean that either $K(i), K(j)$ are both undefined, or they are both defined and are equal.*

The following definition is a little complicated because cylindric-type algebras have diagonal elements. For diagonal-free algebras, the definition would be simpler.

DEFINITION 3.5. We define a cylindric atom structure $\mathcal{E}(\Gamma) = (H, D_{ij}, \equiv_i : i, j < n)$ as follows.

1. H is the set of all pairs $(K, \sim)$, where $K : n \to \Gamma \times n$ is a partial map and $\sim$ is an equivalence relation on n, satisfying the following conditions.
   (a) If $|n/{\sim}| = n$ (in other words, if $\sim$ is $=_n$), then $\mathrm{dom}(K) = n$ and $\mathrm{rng}(K)$ is not independent.
   (b) If $|n/{\sim}| = n - 1$, then K is defined only on the unique $\sim$-class $\{i, j\}$ (say) of size 2, and $K(i) = K(j)$.
   (c) If $|n/{\sim}| \leq n - 2$, then K is nowhere-defined (i.e., $K = \emptyset$).
2. $D_{ij} = \{(K, \sim) \in H : i \sim j\}$.
3. $(K, \sim) \equiv_i (K', \sim')$ iff $K(i) = K'(i)$ and $\sim\restriction(n \setminus \{i\}) = \sim'\restriction(n \setminus \{i\})$.

We will frequently write $\mathcal{E}(\Gamma)$ for H. It may help to think of $K(i)$ as assigning the node $K(i)$ of $\Gamma \times n$ not to i but to the set $n \setminus \{i\}$, so long as its elements are pairwise non-equivalent via $\sim$.

DEFINITION 3.6. If $\sim$ is an equivalence relation on n, and $i < n$, we say that $\sim$ is *i-distinguishing* if $\sim\restriction(n \setminus \{i\})$ is just $=_{n\setminus\{i\}}$: that is, $j \not\sim k$ for every distinct $j, k \in n \setminus \{i\}$.

The next lemma follows from definition 3.5.

LEMMA 3.7. *Let $(K, \sim) \in \mathcal{E}(\Gamma)$.*

1. *For each $i < n$, $K(i)$ is defined iff $\sim$ is i-distinguishing.*
2. *If $i \sim j$, then $K(i) = K(j)$.*
3. *If $\sim$ is $=_n$, then $\mathrm{rng}(K)$ is not an independent subset of $\Gamma \times n$.*

DEFINITION 3.8. We write $\mathcal{C}$ (or explicitly, $\mathcal{C}(\Gamma)$) for the cylindric BAO $\mathcal{E}(\Gamma)^+$. We write $\mathcal{C}_+$ for the set of all ultrafilters of (the boolean reduct of) $\mathcal{C}$. We define $\equiv_i$ on $\mathcal{C}_+$ by $\mu \equiv_i \nu$ iff $\{\mathsf{c}_i S : S \in \mu\} \subseteq \nu$.

LEMMA 3.9. *For any $\mu, \nu \in \mathcal{C}_+$ and $i < n$, the following are equivalent:*

1. $\mu \equiv_i \nu$,
2. $\{\mathsf{c}_i S : S \in \mu\} = \{\mathsf{c}_i T : T \in \nu\}$,
3. *whenever $S \in \mu$ and $T \in \nu$, there are $(X, \sim) \in S$ and $(X', \sim') \in T$ such that $(X, \sim) \equiv_i (X', \sim')$.*

Consequently, $\equiv_i$ is an equivalence relation on $\mathcal{C}_+$.

Proof.

(1) ⇒ (2): Assume (1). For any $S \in \mu$ we know that $c_i S = c_i(c_i S) \in \nu$, hence $\{c_i S : S \in \mu\} \subseteq \{c_i T : T \in \nu\}$. Conversely, if $T \in \nu$ then $-c_i T = c_i(-c_i T) \notin \nu$. Hence, by (1), $-c_i T \notin \mu$, so $c_i T \in \mu$ and $c_i T \in \{c_i S : S \in \mu\}$, proving $\{c_i T : T \in \nu\} \subseteq \{c_i S : S \in \mu\}$. This proves (2).

(2) ⇒ (3): Assume (2) and pick any $S \in \mu$ and $T \in \nu$. By (2), $c_i T \in \mu$ so $S \cap c_i T \in \mu$, hence $S \cap c_i T \neq \emptyset$. Let $(X, \sim) \in S \cap c_i T$. Since $(X, \sim) \in c_i T$ there is $(X', \sim') \in T$ with $(X', \sim') \equiv_i (X, \sim)$, establishing (3).

(3) ⇒ (1): Assume that (1) is false, so $\mu \not\equiv_i \nu$ and there is $S \in \mu$ with $c_i S \notin \nu$. Then $-c_i S \in \nu$. For any $(X, \sim) \in S$ and $(X', \sim') \in -c_i S$ we have $(X, \sim) \not\equiv_i (X', \sim')$, by definition of $-c_i S$, proving that (3) is false.

□

In fact, by defining the ijth diagonal to be $\{\mu \in \mathcal{C}_+ : D_{ij} \in \mu\}$, we can obtain a cylindric atom structure on $\mathcal{C}_+$.

§4. Networks and patch systems.

We will use networks and related machinery in the next section to study representability. Here, we lay out some necessary definitions and facts.

4.1. Projections of ultrafilters.

DEFINITION 4.1. For $i < n$, let $F_i = \{(K, \sim) \in \mathcal{E}(\Gamma) : \sim \text{ is } i\text{-distinguishing}\}$ $(\in \mathcal{C})$.

Clearly, F_i is the intersection of the sets $-D_{jk}$, taken over all distinct $j, k \in n \setminus \{i\}$. If $(K, \sim) \in \mathcal{E}(\Gamma)$, then $K(i)$ is defined iff $(K, \sim) \in F_i$.

LEMMA 4.2. *For each $i, j < n$, we have $F_i \cap D_{ij} \subseteq F_j$.*

Proof. If $(K, \sim) \in F_i \cap D_{ij}$, then $\sim$ is i-distinguishing, and $i \sim j$. It follows that $\sim$ is j-distinguishing, so that $(K, \sim) \in F_j$. □

DEFINITION 4.3. Let μ be an ultrafilter of $\mathcal{C}$, and let $i < n$. We say that μ is *i-distinguishing* if $D_{jk} \notin \mu$ for all distinct $j, k \in n \setminus \{i\}$.

Clearly, an ultrafilter of $\mathcal{C}$ is i-distinguishing iff it contains F_i.

DEFINITION 4.4. Let $i < n$.

1. For $S \subseteq F_i$, put $S(i) = \{K(i) : (K, \sim) \in S\}$.
2. For $X \subseteq \Gamma \times n$, put $X^{(i)} = \{(K, \sim) \in F_i : K(i) \in X\}$.

3. For an ultrafilter μ of $\mathcal{C}$, put $\mu(i) = \{S(i) : S \in \mu, S \subseteq F_i\}$. (This is empty if μ is not i-distinguishing.)

LEMMA 4.5. *For i, S, X as above, $X^{(i)}(i) = X$ and $S(i)^{(i)} \supseteq S$.*

Proof. Well, if $(K, \sim) \in S$, then $(K, \sim) \in F_i$ and $K(i) \in S(i)$, so $(K, \sim) \in S(i)^{(i)}$. Also, $X^{(i)}(i) = \{K(i) : (K, \sim) \in X^{(i)}\} = \{K(i) : (K, \sim) \in F_i,\ K(i) \in X\} \subseteq X$. Now fix arbitrary $x \in X$. Pick any $j \neq i$, let $\sim$ be the unique i-distinguishing equivalence relation on n with $i \sim j$, and define K by $K(i) = K(j) = x$, while $K(k)$ is undefined for $k \neq i, j$. Then $(K, \sim) \in \mathcal{E}(\Gamma)$. We have $(K, \sim) \in X^{(i)}$ and $x = K(i) \in X^{(i)}(i)$. As x was arbitrary, $X \subseteq X^{(i)}(i)$. □

LEMMA 4.6. *Let μ be an i-distinguishing ultrafilter of $\mathcal{C}$. Then*

1. *$\mu(i)$ is an ultrafilter on $\Gamma \times n$.*
2. *If $j < n$ and $D_{ij} \in \mu$, then μ is also j-distinguishing and $\mu(j) = \mu(i)$.*
3. *For any ultrafilter ν of $\mathcal{C}$, we have $\mu \equiv_i \nu$ iff ν is i-distinguishing and $\mu(i) = \nu(i)$.*

Proof. We will use lemma 4.5, and obvious facts such as $X \subseteq Y \subseteq \Gamma \times n \Rightarrow X^{(i)} \subseteq Y^{(i)}$ and $S \subseteq T \subseteq F_i \Rightarrow S(i) \subseteq T(i)$, without explicit mention.

1. Take an arbitrary element of $\mu(i)$: it is of the form $S(i)$, where $S \in \mu$ and $S \subseteq F_i$. Suppose that $S(i) \subseteq X \subseteq \Gamma \times n$. Then $S \subseteq S(i)^{(i)} \subseteq X^{(i)}$ and so $X^{(i)} \in \mu$. Also, $X^{(i)} \subseteq F_i$. So $X = X^{(i)}(i) \in \mu(i)$. Hence, $\mu(i)$ is closed under supersets.

   Take arbitrary elements $S(i), T(i) \in \mu(i)$, where $S, T \in \mu$ and $S, T \subseteq F_i$. Then $S \cap T \in \mu$ and $S(i) \cap T(i) \supseteq (S \cap T)(i) \in \mu(i)$. So by the first part, $\mu(i)$ is closed under intersection.

   Let $X \subseteq \Gamma \times n$ be arbitrary, and write $-X$ for $(\Gamma \times n) \setminus X$. Then $X^{(i)} \cup (-X)^{(i)} = F_i \in \mu$, so one of $X^{(i)}, (-X)^{(i)}$ is in μ, and one of $X, -X$ is in $\mu(i)$. Note that $\mu(i)$ is a proper filter, because there is no $S \in \mu$ with $S \subseteq F_i$ and $S(i) = \emptyset$. So it is an ultrafilter.
2. This is trivial if $i = j$, so suppose not. Suppose also that $D_{ij} \in \mu$. Then $F_i \cap D_{ij} \in \mu$, so by lemma 4.2, $F_j \in \mu$, and μ is j-distinguishing. Now take an arbitrary element $S(i)$ of $\mu(i)$, where $S \in \mu$ and $S \subseteq F_i$. Put $T = S \cap D_{ij}$. Then $T \in \mu$ too, $T \subseteq F_j$ by lemma 4.2, and clearly $S(i) \supseteq T(i) = T(j) \in \mu(j)$. Hence, $S(i) \in \mu(j)$. So $\mu(i) \subseteq \mu(j)$, and as they are both ultrafilters on $\Gamma \times n$, they are equal.
3. Assume that $\mu \equiv_i \nu$. Then $\mathsf{c}_i F_i \in \nu$. But $\mathsf{c}_i F_i = F_i$, as is easy to check. So $F_i \in \nu$, and it follows that ν is i-distinguishing. Moreover, if $S \in \mu$ and $S \subseteq F_i$, then $\mathsf{c}_i S \in \nu$ and $\mathsf{c}_i S \subseteq F_i$, so $S(i) = (\mathsf{c}_i S)(i) \in \nu(i)$. It follows that $\mu(i) \subseteq \nu(i)$, and since both are ultrafilters on $\Gamma \times n$, they must be equal.

   Conversely, suppose that ν is also i-distinguishing, and $\mu(i) = \nu(i)$. Take arbitrary $S \in \mu$ and $T \in \nu$; by lemma 3.9, it is enough if we find $(X, \sim) \in S$ and $(X', \sim') \in T$ with $(X, \sim) \equiv_i (X', \sim')$. We can assume that $S, T \subseteq F_i$. Then $S(i) \in \mu(i) = \nu(i) \ni T(i)$, so $S(i) \cap T(i) \neq \emptyset$. Hence, there are $(X, \sim) \in S$ and $(X', \sim') \in T$ with $X(i) = X'(i)$. But $(X, \sim), (X', \sim') \in F_i$, so $\sim\restriction(n \setminus \{i\})$ and $\sim'\restriction(n \setminus \{i\})$ are both equality on $n \setminus \{i\}$, so are equal. So $(X, \sim) \equiv_i (X', \sim')$ as required.

□

4.2. Ultrafilter networks. These are approximations of representations.

Definition 4.7. Let X be a set.

1. An *n-tuple of elements of X* is an element of X^n. We write $\bar{a}, \bar{b}, \dots$ for n-tuples, and implicitly $\bar{a} = (a_0, \dots, a_{n-1})$, etc.
2. For n-tuples $\bar{a}, \bar{b} \in X^n$, and $i < n$, we write $\bar{a} \equiv_i \bar{b}$ if $a_j = b_j$ for all $j < n$, $j \neq i$.
3. For a tuple $\bar{a}$ and $j < n$, we let $\bar{a}[i/j]$ denote the tuple $\bar{b}$ defined by $\bar{b} \equiv_i \bar{a}$ and $b_i = a_j$.
4. We say that $\bar{a}$ is *i-distinguishing* if $a_j \neq a_k$ for all distinct $j, k \in n \setminus \{i\}$.

Definition 4.8. A *partial ultrafilter network over $\mathcal{C}$* is a pair $N = (N_1, N_2)$, where N_1 is a set (of 'nodes'), and $N_2 : N_1^n \to \mathcal{C}_+$ is a partial map, such that the following hold, for all $\bar{a}, \bar{b}$ on which N_2 is defined.

1. for $i, j < n$, we have $D_{ij} \in N_2(\bar{a})$ iff $a_i = a_j$,
2. for $i < n$, if $\bar{a} \equiv_i \bar{b}$ then $N_2(\bar{a}) \equiv_i N_2(\bar{b})$.

For partial ultrafilter networks $N = (N_1, N_2)$ and $M = (M_1, M_2)$, we write $N \subseteq M$ if $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$. We say that N is *total* if $N_2 : N_1^n \to \mathcal{C}_+$ is a total map; in this case, we call N an *ultrafilter network over $\mathcal{C}$*.

In case of need, we write Nodes(N) for N_1, but generally we write N for any of N, N_1, N_2.

4.3. Patch systems. These help us to examine the way projections of ultrafilters in a network interact. To give a very rough idea of how they arise, imagine that $\bar{a} = (a_0, \dots, a_{n-1})$ is a tuple of distinct nodes of an ultrafilter network N. For each $i < n$, it turns out that the projection $N(\bar{a})(i)$ depends only on the set $\{a_j : j < n,\ j \neq i\}$, and not on the order of entries in $\bar{a}$ or the omitted element a_i. Thus, N yields an assignment, which we call a *patch system,* of ultrafilters on $\Gamma \times n$ to subsets of N of size $n-1$. This represents much of the information in N in a simpler way. The ultrafilters $N(\bar{a})(i)$ $(i < n)$ will be mutually 'coherent', and any coherent allotment of ultrafilters to the sets in $[N]^{n-1}$ is induced by an ultrafilter network. So we can build ultrafilter networks by building patch systems, which is easier.

We now formalise this in a sharper way.

Definition 4.9.

1. A *patch system* (for Γ) is a pair $P = (P_1, P_2)$, where P_1 is a set, and P_2 assigns an ultrafilter $P_2(A)$ on $\Gamma \times n$ to every subset A of P_1 of size $n-1$. (We think of the As as 'patches'. If $|P_1| < n-1$ then $P_2 = \emptyset$.)
2. Let $P = (P_1, P_2)$ be a patch system. A set $A = \{a_0, \dots, a_{n-1}\} \subseteq P_1$ of size n is said to be *P-coherent* if whenever $X_i \in P_2(A \setminus \{a_i\})$ (for each $i < n$), there are $x_i \in X_i$ $(i < n)$ such that $\{x_0, \dots, x_{n-1}\}$ is not an independent subset of $\Gamma \times n$.
3. A patch system $P = (P_1, P_2)$ is said to be *coherent* if every $A \subseteq P_1$ of size n is P-coherent.

As with ultrafilter networks, we will often write P for any of P, P_1, P_2. We write simply 'coherent' when P is clear from the context.

Lemma 4.10. *Let $P = (P_1, P_2)$ be a patch system and let $A = \{a_0, \ldots, a_{n-1}\} \in [P_1]^n$. For each $i < n$, let $A_i = A \setminus \{a_i\}$. Then A is P-coherent iff there exists an ultrafilter μ of $\mathcal{C}$ that is i-distinguishing for all $i < n$ and with $\mu(i) = P_2(A_i)$ for every $i < n$.*

Proof. Write $\bar{a}$ for the tuple $(a_0, \ldots, a_{n-1})$. Suppose first that μ exists as stated. Let sets $X_i \in P_2(A_i) = \mu(i)$ be given, for each $i < n$. For each i, choose $S_i \in \mu$ with $S_i \subseteq F_i$ and $S_i(i) = X_i$. Put $S = \bigcap_{i<n} S_i$. Then $S \in \mu$ and $S \subseteq F_i$ for all i. Take any $(K, \sim) \in S$. Then $K(i)$ is defined for all i, and $K(i) \in X_i$. By definition of $\mathcal{E}(\Gamma)$, $\text{rng}(K)$ is not independent. This shows that A is coherent.

For the converse, assume that A is coherent. Write $f_i = P_2(A_i)$, for each $i < n$. This is an ultrafilter on $\Gamma \times n$. Consider

$$\Theta = \{X_i^{(i)} : i < n,\ X_i \in f_i\}.$$

(recall from definition 4.4 that $X^{(i)} = \{(K, \sim) \in F_i : K(i) \in X\}$). Θ has the finite intersection property. To see this, it is enough to show that for any $X_i \in f_i$ (for each $i < n$), there is $(K, \sim) \in \bigcap_{i<n} F_i$ with $K(i) \in X_i$ for each i. But by coherence, there are $x_i \in X_i$ $(i < n)$ such that $\{x_i : i < n\}$ is not independent. Define K by $K(i) = x_i$ (each i). Then $(K, =_n) \in \mathcal{E}(\Gamma)$ is as required.

We define μ to be any ultrafilter of $\mathcal{C}$ extending Θ. (Existence uses the boolean prime ideal theorem.) Clearly, for each i, $F_i = (\Gamma \times n)^{(i)} \in \Theta$, so μ is i-distinguishing. Let $X \in f_i$ be arbitrary. Then $X^{(i)} \in \Theta \subseteq \mu$, so $X = X^{(i)}(i) \in \mu(i)$. As X was arbitrary, $f_i \subseteq \mu(i)$. As both are ultrafilters, $\mu(i) = f_i$. ☐

Definition 4.11. For any ultrafilter network $N = (N_1, N_2)$, define ∂N to be the patch system (N_1, P_2), where P_2 is a function from subsets of N_1 of size $n-1$ to ultrafilters on $\Gamma \times n$, defined by

$$P_2(\{a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}\}) = N_2(\bar{a})(i), \tag{1}$$

for each $i < n$ and each i-distinguishing $\bar{a} \in N_1^n$.

The following lemma shows that ∂N is well defined.

Lemma 4.12. *Let $N = (N_1, N_2)$ be a partial ultrafilter network.*

1. *For each $\bar{a} \in \text{dom}(N_2)$ and $i < n$, $N_2(\bar{a})$ is i-distinguishing iff $\bar{a}$ is i-distinguishing.*
2. *If N is total, then ∂N is a well defined and coherent patch system.*
3. *Suppose that $P = (N_1, P_2)$ is a coherent patch system and that the above condition (1) holds for any i-distinguishing $\bar{a} \in \text{dom}(N_2)$. Then there is a (total) ultrafilter network $N^+ = (N_1, N_2^+)$ with $N^+ \supseteq N$ and $\partial N^+ = P$.*

Proof. 1. Easy; left to the reader.

2. If $\bar{a} \in N_1^n$ is i-distinguishing, then by the first part, $N_2(\bar{a})$ is also i-distinguishing, so by lemma 4.6, $N_2(\bar{a})(i)$ is a well-defined ultrafilter on $\Gamma \times n$. We have to show that it depends only on $\{a_k : k < n,\ k \neq i\}$.

   Claim. Let $\bar{a}$ be i-distinguishing and $\bar{b}$ be j-distinguishing tuples in N_1^n, and suppose that $\{a_k : k < n,\ k \neq i\} = \{b_k : k < n,\ k \neq j\}$. Then $N_2(\bar{a})(i) = N_2(\bar{b})(j)$.

   Proof of claim. We first establish a useful fact. Take any i-distinguishing $\bar{a}$ and $j < n$, and let $\bar{b} = \bar{a}[i/j]$; it is also i-distinguishing. Now $\bar{a} \equiv_i \bar{b}$, so as N is a network, $N_2(\bar{a}) \equiv_i N_2(\bar{b})$. By lemma 4.6, $N_2(\bar{a})(i) = N_2(\bar{b})(i)$. Also, $D_{ij} \in N_2(\bar{b})$. So by the lemma again, $N_2(\bar{b})$ is j-distinguishing, and $N_2(\bar{b})(i) = N_2(\bar{b})(j)$. We conclude that if $\bar{a} \in N_2^n$ is i-distinguishing, then $\bar{a}[i/j]$ is j-distinguishing and $N_2(\bar{a})(i) = N_2(\bar{a}[i/j])(j)$.

Now take $\bar{a}, i, \bar{b}, j$ as in the claim. By replacing $\bar{a}$ by $\bar{a}[i/0]$ and $\bar{b}$ by $\bar{b}[j/0]$, we can assume that $i = j = 0$. We now prove the claim by induction on $d(\bar{a}, \bar{b}) = \max\{k < n : a_k \neq b_k\}$. If this is 0 or undefined, then $\bar{a} \equiv_0 \bar{b}$, so $N_2(\bar{a}) \equiv_0 N_2(\bar{b})$; by lemma 4.6(3), $N_2(\bar{a})(0) = N_2(\bar{b})(0)$ as required. Otherwise, let $i > 0$ be greatest such that $a_i \neq b_i$. Then $\{a_k : k \neq 0\} = \{b_k : k \neq 0\}$, so $b_i = a_j$ for some $j \neq i$, $j > 0$. If $j > i$, then $b_i = a_j = b_j$, contradicting that $\bar{b}$ is 0-distinguishing. So $j < i$. Put $\bar{c} = \bar{a}[0/i][i/j][j/0]$. That is,

$$\bar{c} = (a_i, a_1, \ldots, a_{j-1}, a_i, a_{j+1}, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_{n-1}).$$

By the above, $N_2(\bar{a})(0) = N_2(\bar{c})(0)$. Also, $c_i = a_j = b_i$, and $c_k = a_k = b_k$ for all $k > i$. Hence, $\bar{c}$ is 0-distinguishing, $\{c_k : k \neq 0\} = \{b_k : k \neq 0\}$, and $d(\bar{c}, \bar{b}) < d(\bar{a}, \bar{b})$. So inductively, $N_2(\bar{c})(0) = N_2(\bar{b})(0)$. The claim follows.

By the claim, P_2 is well defined. By lemma 4.10, every $A \in [N_1]^n$ is coherent, and hence so is P.

3. We must put $N_2^+(\bar{a}) = N_2(\bar{a})$ for $\bar{a} \in \mathrm{dom}(N_2)$. We need to define $N_2^+(\bar{a})$ for every $\bar{a} \in N_1^n \setminus \mathrm{dom}(N_2)$. Fix such an $\bar{a}$.
   - If $|\mathrm{rng}(\bar{a})| \leq n - 2$, define $N_2^+(\bar{a})$ to be the principal ultrafilter of $\mathcal{C}$ generated by $\{(\emptyset, \sim_{\bar{a}})\}$, where $\sim_{\bar{a}}$ is defined by $i \sim_{\bar{a}} j$ iff $a_i = a_j$.
   - If $|\mathrm{rng}(\bar{a})| = n - 1$, there are unique $i < j < n$ with $a_i = a_j$. Write f for $P_2(\mathrm{rng}(\bar{a}))$ — an ultrafilter on $\Gamma \times n$. Let $\Delta = F_i \cap F_j \cap D_{ij}$, and define

     $$N_2^+(\bar{a}) = \{S \in \mathcal{C} : (S \cap \Delta)(i) \in f\}.$$

     As can be verified, this is an ultrafilter of $\mathcal{C}$. Clearly, $\Delta(i) = \Gamma \times n \in f$. So $\Delta \in N_2^+(\bar{a})$, and hence $D_{kl} \in N_2^+(\bar{a})$ iff $a_k = a_l$, for all $k, l < n$. Also, if $S \in N_2^+(\bar{a})$ and $S \subseteq F_i$, then $S(i) \supseteq (S \cap \Delta)(i) \in f$. Hence, $N_2^+(\bar{a})(i) \subseteq f$, so as both are ultrafilters, $N_2^+(\bar{a})(i) = f$. By lemma 4.6, $N_2^+(\bar{a})(j) = f$ as well.
   - If $|\mathrm{rng}(\bar{a})| = n$, then by lemma 4.10, there is an ultrafilter μ that is i-distinguishing for all i, and with $\mu(i) = P(\{a_j : j < n,\ j \neq i\})$ for every $i < n$. We define $N_2^+(\bar{a}) = \mu$.

   We now check that $N^+ = (N_1, N_2^+)$ is an ultrafilter network. By construction and because $N = (N_1, N_2)$ is already a partial ultrafilter network,

   $$D_{ij} \in N_2^+(\bar{a}) \iff a_i = a_j, \quad \text{for any } \bar{a} \in N_1^n \text{ and } i, j < n. \tag{2}$$

   It follows that for each $i < n$, $\bar{a}$ is i-distinguishing iff $N_2^+(\bar{a})$ is i-distinguishing, and

   $$N_2^+(\bar{a})(i) = P_2(\{a_j : j < n,\ j \neq i\}) \text{ for any } i\text{-distinguishing } \bar{a} \in N_1^n. \tag{3}$$

   This was assumed to hold already for any i-distinguishing $\bar{a} \in \mathrm{dom}(N_2)$, and by construction it holds for all remaining tuples in N_1^n.

Suppose that $\bar{a} \equiv_i \bar{b}$. We check that $N_2^+(\bar{a}) \equiv_i N_2^+(\bar{b})$. Assume first that $\bar{a}$ is i-distinguishing. Then by (3), $N_2^+(\bar{a})(i) = P_2(\{a_j : j \neq i\})$. Also, $\bar{b}$ is clearly i-distinguishing too, so $N_2^+(\bar{b})(i) = P_2(\{b_j : j \neq i\})$. These sets are the same, so $N_2^+(\bar{a})(i) = N_2^+(\bar{b})(i)$. By lemma 4.6(3), $N_2^+(\bar{a}) \equiv_i N_2^+(\bar{b})$.

Now assume that $\bar{a}$ is not i-distinguishing. Let $\Delta = \bigcap\{D_{jk} : j, k \neq i, a_j = a_k\} \cap \bigcap\{-D_{jk} : j, k \neq i, a_j \neq a_k\}$. By (2), $\Delta \in N_2^+(\bar{a})$ and (since $\bar{b} \equiv_i \bar{a}$) $\Delta \in N_2^+(\bar{b})$. Take any $S \in N_2^+(\bar{a})$ and $S' \in N_2^+(\bar{b})$. By lemma 3.9, it suffices to find some $(X, \sim) \in S$ and $(X', \sim') \in S'$ with $(X, \sim) \equiv_i (X', \sim')$. We simply take any $(X, \sim) \in S \cap \Delta$ and $(X', \sim') \in S' \cap \Delta$. There are distinct $j, k \neq i$ with $a_j = a_k$, so $(X, \sim), (X', \sim') \in D_{jk}$ and hence $X(i), X'(i)$ are undefined. Clearly, $\sim\restriction(n \setminus \{i\}) = \sim'\restriction(n \setminus \{i\})$. Hence, $(X, \sim) \equiv_i (X', \sim')$ as required.

So N^+ is an ultrafilter network. Certainly, $N \subseteq N^+$, and it is immediate from (3) that $\partial N^+ = P$.

□

§5. Representations. Recall that Γ is a fixed graph, and $\mathcal{C} = \mathcal{C}(\Gamma)$. We are going to show that if Γ is infinite, $\mathcal{C}$ is representable iff $\chi(\Gamma) = \infty$. We will need the following lemma. Recall that an algebra $\mathcal{A}$ is *simple* if for any algebra $\mathcal{B}$ of the same signature, any homomorphism $h : \mathcal{A} \to \mathcal{B}$ is either trivial (i.e., $h(x) = h(y)$ for all $x, y \in \mathcal{A}$) or one-one.[2]

LEMMA 5.1. *$\mathcal{C}$ is simple, as is any subalgebra of $\mathcal{C}$.*

Proof. Let $(K, \sim) \in \mathcal{E}(\Gamma)$, and let i with $1 \leq i < n$ be arbitrary. Define K_i to be the partial function from n to $\Gamma \times n$ given by $K_i(0) = K_i(i) = K(i)$ (this may be undefined), $K_i(j)$ being undefined for $j \in n \setminus \{0, i\}$. Also define $\sim_i$ to be the unique equivalence relation on n satisfying $\sim_i\restriction(n \setminus \{i\}) = \sim\restriction(n \setminus \{i\})$ and $i \sim_i 0$ ($\sim_i$ is the reflexive transitive closure of the binary relation just defined). Then $(K_i, \sim_i) \in \mathcal{E}(\Gamma)$ and $(K, \sim) \equiv_i (K_i, \sim_i)$. So, writing K_{ij} for $(K_i)_j$, etc., we have

$$(K, \sim) \equiv_1 (K_1, \sim_1) \equiv_2 (K_{12}, \sim_{12}) \cdots \equiv_{n-1} (K_{12\ldots n-1}, \sim_{12\ldots n-1}) = (L, \approx), \text{ say.}$$

So $(L, \approx) \in \mathsf{c}_{n-1} \ldots \mathsf{c}_1\{(K, \sim)\}$, and $(K, \sim) \in \mathsf{c}_1 \ldots \mathsf{c}_{n-1}\{(L, \approx)\}$.

Recall that $n \geq 3$. Now 2 is not in the domain of K_1. Therefore, K_{12} has empty domain, and hence $K_{12} = \cdots = K_{12\ldots n-1} = L = \emptyset$. Also, it is clear that $\approx = n \times n$. We conclude that $(L, \approx)$ has a fixed value, independent of $(K, \sim)$. So for any $(K, \sim) \in \mathcal{E}(\Gamma)$,

$$(K', \sim') \in \mathsf{c}_1 \ldots \mathsf{c}_{n-1}\mathsf{c}_{n-1} \ldots \mathsf{c}_1\{(K, \sim)\} \tag{4}$$

for every $(K', \sim') \in \mathcal{E}(\Gamma)$. Thus, the right-hand side of (4) is 1. Since every non-zero element of $\mathcal{C}$ lies above some $(K, \sim)$, and the c_i are additive,

$$\mathsf{c}_1 \ldots \mathsf{c}_{n-1}\mathsf{c}_{n-1} \ldots \mathsf{c}_1 S = 1 \quad \text{for every } S \in \mathcal{C} \setminus \{0\}. \tag{5}$$

Now let h be a homomorphism defined on some subalgebra $\mathcal{D}$ of $\mathcal{C}$. Notice that if $S \in \mathcal{D}$ and $h(S) = 0$, then

$$h(\mathsf{c}_i S) = \mathsf{c}_i h(S) = \mathsf{c}_i 0 = \mathsf{c}_i h(0) = h(\mathsf{c}_i 0) = h(0) = 0 \text{ for every } i < n. \tag{6}$$

[2]Some definitions require also that $|\mathcal{A}| > 1$. This does not affect the next lemma.

Assume that h is not one-one. We need to show that $h(0) = h(1)$. As h preserves the boolean operations, there is non-zero $S \in \mathcal{D}$ such that $h(S) = 0$. Now, by (5) and repeated application of (6) we obtain $h(1) = h(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}\mathsf{c}_{n-1} \ldots \mathsf{c}_1 S) = 0 = h(0)$ as required. ☐

PROPOSITION 5.2. *Suppose that $\chi(\Gamma) = \infty$. Then $\mathcal{C}$ is representable.*

Proof. We use the following game played by players $\forall$, $\exists$. The game constructs a chain $N_0 \subseteq N_1 \subseteq \cdots$ of (total) ultrafilter networks over $\mathcal{C}$. The game starts with the unique one-point network N_0. There are ω rounds, numbered $0, 1, \ldots$. In each round t, where the current network is N_t, $\forall$ chooses an n-tuple $\bar{a} \in N_t^n$, an $i < n$, and an element $S \in \mathcal{C}$ such that $\mathsf{c}_i S \in N_t(\bar{a})$. $\exists$ must respond with an ultrafilter network $N_{t+1} \supseteq N_t$ such that there is $\bar{b} \in N_{t+1}^n$ with $\bar{b} \equiv_i \bar{a}$ and $S \in N_{t+1}(\bar{b})$. $\exists$ wins if she succeeds in moving according to the rules in each round.

LEMMA 5.3. *If $\exists$ has a winning strategy in the game, then $\mathcal{C}$ is representable.*

Proof. Using the downward Löwenheim–Skolem–Tarski theorem [3, theorem 3.1.6], choose a countable elementary subalgebra $\mathcal{C}_0$ of $\mathcal{C}$. Let $N_0 \subseteq N_1 \subseteq \cdots$ be a play of the game in which $\forall$ plays every possible move $(\bar{a}, i, S)$ for $S \in \mathcal{C}_0$ at some stage, and $\exists$ uses her winning strategy. We can define an ultrafilter network $N = \bigcup_{t<\omega} N_t$ over $\mathcal{C}$ in the obvious way. N can be checked to induce a homomorphism h of $\mathcal{C}_0$ into a cylindric set algebra as follows:

$$h : \mathcal{C}_0 \to \big(\wp(N^n), \cup, -, \emptyset, N^n, D_{ij}^N, C_i^N\big)_{i,j<n}$$
$$h : S \mapsto \{\bar{a} \in N^n : S \in N(\bar{a})\}.$$

Clearly, $h(1) = N^n \neq h(0) = \emptyset$. By lemma 5.1, $\mathcal{C}_0$ is simple, so h is an embedding and $\mathcal{C}_0$ is representable. As RCA_n is an elementary class, $\mathcal{C}$ is representable too. ☐

The converse of the lemma also holds, but we will not need it.

So it is enough to show that $\exists$ has a winning strategy in this game. To this end, suppose that we are in round t, and the current network is N_t. Let $\forall$ choose $\bar{a}, i, S$ as per the rules: so $\mathsf{c}_i S \in N_t(\bar{a})$. If there is $\bar{c} \in N_t^n$ with $\bar{c} \equiv_i \bar{a}$ and $S \in N_t(\bar{c})$, then $\exists$ may play $N_{t+1} = N_t$. So assume not.

$\exists$ needs to define $N_{t+1} \supseteq N_t$. She first defines $\mathrm{Nodes}(N_{t+1})$ to be $\mathrm{Nodes}(N_t) \cup \{z\}$, where $z \notin N_t$ is a new node. Now she has to assign ultrafilters to n-tuples from N_{t+1}. For n-tuples from N_t, this is done already by N_t itself. Let $\bar{b}$ denote the n-tuple given by $\bar{b} \equiv_i \bar{a}$, $b_i = z$. $\exists$'s next task is to choose an ultrafilter for $\bar{b}$.

CLAIM. $\mathsf{c}_i(S \cap \bigcap_{j \neq i} -D_{ij}) \in N_t(\bar{a})$.

Proof of claim. Write Δ for $\bigcap_{j \neq i} -D_{ij}$. Plainly, $\Delta \cup \bigcup_{j \neq i} D_{ij} = \mathcal{E}(\Gamma)$. So it is easily seen that $\mathsf{c}_i S = \mathsf{c}_i(S \cap \Delta) \cup \bigcup_{j \neq i} \mathsf{c}_i(S \cap D_{ij})$. Assume for contradiction that the claim fails. So $\mathsf{c}_i(S \cap D_{ij}) \in N_t(\bar{a})$ for some $j \neq i$. Let $\bar{c} = \bar{a}[i/j] \in N_t^n$. Then $\bar{a} \equiv_i \bar{c}$. Because N_t is a network, $N_t(\bar{a}) \equiv_i N_t(\bar{c})$. By lemma 3.9, $\mathsf{c}_i(S \cap D_{ij}) \in N_t(\bar{c})$ as well. Now $D_{ij} \in N_t(\bar{c})$. So $D_{ij} \cap \mathsf{c}_i(S \cap D_{ij}) \in N_t(\bar{c})$. But it is easily checked that $D_{ij} \cap \mathsf{c}_i(S \cap D_{ij}) = D_{ij} \cap S$. So $S \in N_t(\bar{c})$, contradicting our assumption above. This proves the claim.

It is now easily seen that

$$\Sigma = \{S\} \cup \{-D_{ij} : j < n,\ j \neq i\} \cup \{-\mathsf{c}_i - T : T \in N_t(\bar{a})\}$$

has the finite intersection property. $\exists$ chooses an ultrafilter μ of $\mathcal{C}$ containing Σ, and defines $N_{t+1}(\bar{b}) = \mu$. By construction, $\mu \equiv_i N_t(\bar{a})$. Moreover, for all $j, k \neq i$, we have

$$\begin{array}{lllllll} b_i \neq b_j & \text{and} & -D_{ij} \in \Sigma, & & & & \\ b_j = b_k & \Rightarrow & a_j = a_k & \Rightarrow & D_{jk} \in N_t(\bar{a}) & \Rightarrow & D_{jk} = -\mathsf{c}_i - D_{jk} \in \Sigma, \\ b_j \neq b_k & \Rightarrow & a_j \neq a_k & \Rightarrow & -D_{jk} \in N_t(\bar{a}) & \Rightarrow & -D_{jk} = -\mathsf{c}_i D_{jk} \in \Sigma. \end{array}$$

So

$$D_{jk} \in \mu \iff b_j = b_k, \quad \text{for every } j, k < n. \tag{7}$$

Therefore, we can define a partial ultrafilter network $N' \supseteq N_t$, whose set of nodes is $\text{Nodes}(N_{t+1})$, and with $N'(\bar{b}) = \mu$. $N'(\bar{c})$ is defined iff $\bar{c} \in N_t^n$ or $\bar{c} = \bar{b}$.

To help her assign ultrafilters to the remaining tuples, $\exists$ now defines a patch system $P = (\text{Nodes}(N_{t+1}), P_2)$ as follows.

1. For any $A \in [\text{Nodes}(N_t)]^{n-1}$, she defines $P_2(A) = \partial N_t(A)$.
2. For each $j < n$ put $B_j = \{b_k : k < n,\ k \neq j\}$. $\exists$ has to define $P_2(B_j)$ for each j such that $|B_j| = n-1$. If $|B_i| = n-1$, then $P_2(B_i)$ was defined above, since $B_i \subseteq \text{Nodes}(N_t)$. (Note that $P_2(B_i) = \mu(i)$ in this case.) Consider each $j \neq i$ with $|B_j| = n-1$. As $\bar{b}$ is j-distinguishing, μ is j-distinguishing by (7), so $\mu(j)$ is well defined. $\exists$ defines $P_2(B_j) = \mu(j)$. This is well defined. For suppose that $j, k < n$, $j, k \neq i$, $|B_j| = |B_k| = n-1$, and $B_j = B_k$. Then $b_j = b_k$, so by (7), $D_{jk} \in \mu$. By lemma 4.6, $\mu(j) = \mu(k)$.
3. Now $\Gamma \times n$ is partitioned by the sets $\Gamma \times \{l\}$ for $l < n$. Each $\mu(j)$ (for each $j \neq i$ such that μ is j-distinguishing) contains exactly one set $\Gamma \times \{l\}$. There are n ls and at most $n-1$ js. So there is $l < n$ such that $\Gamma \times \{l\} \notin \mu(j)$ for each such j. Since $\chi(\Gamma) = \infty$, it can easily be seen by lemma 3.3 that there is an ultrafilter δ on $\Gamma \times n$ containing $\Gamma \times \{l\}$ and not containing any independent sets. $\exists$ defines $P_2(A) = \delta$ for all remaining $A \in [\text{Nodes}(N_{t+1})]^{n-1}$. (These are the A that contain z and are not contained in $\text{rng}(\bar{b})$.)

It is plain that P satisfies condition (1) of definition 4.11 for the partial ultrafilter network N' introduced above: that is, $N'(\bar{c})(j) = P_2(\{c_k : k < n,\ k \neq j\})$ for each $j < n$ and each j-distinguishing $\bar{c} \in N_t^n \cup \{\bar{b}\}$.

We now show that P is coherent. Let $C = \{c_0, \ldots, c_{n-1}\} \in [\text{Nodes}(N_{t+1})]^n$ be given. We check that C is P-coherent. Write C_j for $C \setminus \{c_j\}$, for each $j < n$.

- If $z \notin C$, then $C \subseteq N_t$ and C is P-coherent because (by lemma 4.12) ∂N_t is coherent.
- If $C = \text{rng}(\bar{b})$, coherence follows from lemma 4.10.
- If $z \in C$ and $|C \cap \text{rng}(\bar{b})| = n-1$,[3] let $j, k < n$ be such that $C_j = C \cap \text{rng}(\bar{b})$ and $C_k \not\subseteq N_t$, $C_k \not\subseteq \text{rng}(\bar{b})$. Then $\Gamma \times \{l\} \in P_2(C_k)$ (l as above). Note that $z \in C_j$. So by choice of l, there is $m \neq l$ with $\Gamma \times \{m\} \in P_2(C_j)$. Now, if $X_s \in P_2(C_s)$ are given, for each $s < n$, we choose $x_s \in X_s$ for each s, with

[3] This case is only needed if $n = 3$. For $n \geq 4$, it is subsumed by the next one. Moreover, for $n \geq 4$, $\Gamma \times n$ can be replaced by Γ throughout the proof.

$x_j \in X_j \cap (\Gamma \times \{m\})$ and $x_k \in X_k \cap (\Gamma \times \{l\})$. Since $l \neq m$, (x_j, x_k) is an edge of $\Gamma \times n$. So $\{x_0, \ldots, x_{n-1}\}$ is not independent.

- If $z \in C$ and $|C \cap \operatorname{rng}(\bar{b})| < n-1$, there are distinct $j, k < n-1$ such that neither C_j nor C_k are contained in N_t or in $\operatorname{rng}(\bar{b})$. So $P_2(C_j) = P_2(C_k) = \delta$. Suppose that we are given $X_s \in P_2(C_s)$ for each s. Then $X_j, X_k \in \delta$, so $X_j \cap X_k \in \delta$ and hence this set is not independent. Choose an edge (x_j, x_k) of $\Gamma \times n$, with $x_j, x_k \in X_j \cap X_k$. For each $s \neq j, k$, choose any $x_s \in X_s$. Then $x_s \in X_s$ for all s, and $\{x_0, \ldots, x_{n-1}\}$ is not independent.

So P is coherent. By lemma 4.12(3) applied to P and N', there is a (total) ultrafilter network $N_{t+1} \supseteq N'$ with $\partial(N_{t+1}) = P$. Hence, $N_{t+1} \supseteq N_t$, $N_{t+1}(\bar{b}) = \mu$, and $S \in N_{t+1}(\bar{b})$. $\exists$ plays such an N_{t+1} as her response to $\forall$'s move. We have described a winning strategy for $\exists$. This proves proposition 5.2. □

We now show that when Γ is infinite, the converse of proposition 5.2 holds.

PROPOSITION 5.4. *Suppose that Γ is infinite and $\chi(\Gamma) < \infty$. Then $\mathcal{C}$ is not representable.*

Proof. Suppose otherwise. Then there is an embedding $h : \mathcal{C} \to \prod_{q \in Q} \mathcal{A}_q$, where for each $q \in Q$, $\mathcal{A}_q = (A_q, \cup, -, \emptyset, U_q^n, D_{ij}^{U_q}, C_i^{U_q})_{ij<n}$ is a cylindric set algebra with non-empty base set U_q. Because h is one-one and $|\mathcal{C}| > 1$, $Q \neq \emptyset$. Choose any $q \in Q$, and let π denote the projection of $\prod_{q \in Q} \mathcal{A}_q$ onto $\mathcal{A}_q$. Then $g = \pi \circ h$ is a homomorphism defined on $\mathcal{C}$. Since $U_q \neq \emptyset$, we have $g(1) = U_q^n \neq \emptyset = g(0)$. By lemma 5.1, g is one-one.

We can view $\mathcal{A}_q$ as an ultrafilter network M, via $M(\bar{a}) = \{S \in \mathcal{C} : \bar{a} \in g(S)\}$, for each $\bar{a} \in U_q^n$. This can be checked to be a bona fide ultrafilter network. By lemma 4.12, ∂M is well defined and is a coherent patch system.

As $\chi(\Gamma) < \infty$, also $\chi(\Gamma \times n) < \infty$, and we can choose a finite partition of $\Gamma \times n$ into independent sets $I_0, \ldots, I_{k-1}$. Now Γ is infinite, and hence so is $\mathcal{C}$. Because g is one-one, U_q must be infinite as well. Choose distinct elements $a_0, a_1, \ldots$ of U_q, and define $f : [\omega]^{n-1} \to k$ by letting $f(\{i_1, \ldots, i_{n-1}\})$ be the unique $j < k$ such that $I_j \in \partial M(\{a_{i_1}, \ldots, a_{i_{n-1}}\})$. By Ramsey's theorem [20], we may assume that the value of f is constant — say, c. Let $A = \{a_0, \ldots, a_{n-1}\}$. Then $I_c \in \partial M(A \setminus \{a_i\})$ for each $i < n$. By coherence, there are $x_i \in I_c$ (for each $i < n$) such that $\{x_0, \ldots, x_{n-1}\}$ is not independent. This is impossible, as $\{x_0, \ldots, x_{n-1}\} \subseteq I_c$ which is independent. □

§6. The main result.

THEOREM 6.1. *For each finite $n \geq 3$ the class* $\mathsf{Str}\,\mathsf{RCA}_n$ *of strongly representable n-dimensional cylindric atom structures is not closed under ultraproducts, and so is non-elementary.*

Proof. By Erdős's famous 1959 theorem [5], for each finite k there is a finite graph G_k with $\chi(G_k) > k$ and with no cycles of length $< k$. Let Γ_k be the disjoint union of the G_l for $l > k$. Clearly, $\chi(\Gamma_k) = \infty$. So by proposition 5.2, $\mathcal{C}(\Gamma_k) = \mathcal{E}(\Gamma_k)^+$ is representable. By lemma 2.6, $\mathcal{E}(\Gamma_k) \in \mathsf{Str}\,\mathsf{RCA}_n$ for each finite k.

Now let Γ be a non-principal ultraproduct $\prod_D \Gamma_k$ for the Γ_k. It is certainly infinite. For $k < \omega$, let σ_k be a first-order sentence of the signature of graphs, stating that there are no cycles of length less than k. Then $\Gamma_l \models \sigma_k$ for all $l \geq k$. By Łoś's theorem [3, theorem 4.1.9], $\Gamma \models \sigma_k$ for all k. So Γ has no cycles, and hence by lemma 3.2, $\chi(\Gamma) \leq 2$. By proposition 5.4, $\mathcal{C}(\Gamma)$ is not representable. So $\mathcal{E}(\Gamma) \notin \mathsf{Str}\,\mathsf{RCA}_n$.

Now it is easily seen (e.g., because $\mathcal{E}(\Gamma)$ is first-order interpretable in Γ, for any Γ) that

$$\prod_D \mathcal{E}(\Gamma_k) \cong \mathcal{E}(\prod_D \Gamma_k).$$

So $\mathsf{Str}\,\mathsf{RCA}_n$ is not closed under ultraproducts, and, by [3, theorem 4.1.12], is non-elementary. □

§7. Conclusion. We end with some remarks and problems.

Remark 7.1. By [6, theorem 3.8.4], $\mathsf{Str}\,\mathsf{RCA}_n$ is elementary iff it is closed under elementary equivalence, iff it is closed under ultrapowers, iff it is closed under ultraproducts. Hence, for finite $n \geq 3$, $\mathsf{Str}\,\mathsf{RCA}_n$ has none of these closure properties. However, its complement is closed under ultrapowers and so $\mathsf{Str}\,\mathsf{RCA}_n$ is closed under ultraroots [6, theorem 3.8.1(1)].

Problem 7.2. For finite $n \geq 3$, is $\mathsf{Str}\,\mathsf{RCA}_n$ closed under $L_{\infty\omega}$-equivalence?

Remark 7.3. For $n \leq 2$, RCA_n is known to be axiomatisable by a finite set of Sahlqvist equations (see, e.g., [9, 3.2.56, 3.2.65], or [12, §5.3]). Hence (e.g., by [23, page 2 and theorem 1.3] or [12, proposition 2.91]), $\mathsf{Str}\,\mathsf{RCA}_n$ is the same as $\mathsf{At}\,\mathsf{RCA}_\alpha$. It is elementary and finitely axiomatisable by an explicit set of first-order sentences: the 'Sahlqvist correspondents' of the Sahlqvist equations defining RCA_n.

Problem 7.4. For infinite α, is $\mathsf{Str}\,\mathsf{RCA}_\alpha$ elementary?

Remark 7.5. Strongly representable atom structures are connected to 'completions'. Let $3 \leq n < \omega$. As we mentioned in the introduction, it follows from a general result in [23] that $\mathrm{At}\,\mathsf{RCA}_n$ is elementary. Since clearly, $\mathsf{Str}\,\mathsf{RCA}_n \subseteq \mathrm{At}\,\mathsf{RCA}_n$, by theorem 6.1 the inclusion is strict. Now take any $\mathcal{S} \in \mathrm{At}\,\mathsf{RCA}_n \setminus \mathsf{Str}\,\mathsf{RCA}_n$ (e.g., the $\mathcal{E}(\Gamma)$ of theorem 6.1). Then $\mathcal{C} = \mathcal{S}^+$ is a non-representable atomic n-dimensional cylindric BAO that has a representable subalgebra, say $\mathcal{A}$, with the same atoms as $\mathcal{C}$. It is well known that the *completion* of $\mathcal{A}$ (in the sense of [18]) is $\mathcal{C}$. Hence, RCA_n is not closed under completions. (This is known and was proved in [14].)

Strongly representable atom structures are also connected to 'complete representations'. A *complete representation* of a cylindric-type algebra $\mathcal{A}$ is a representation that respects all existing (possibly infinitary) sums and products in $\mathcal{A}$. If $\mathcal{A}$ has a complete representation then $\mathcal{A}$ is atomic, and every atomic cylindric BAO with atom structure $\mathrm{At}\,\mathcal{A}$ has a complete representation ([12, corollary 2.22] can be used to prove both statements). By lemma 2.6, which holds for any dimension, $\mathrm{At}\,\mathcal{A}$ is strongly representable.

So for any ordinal α, we may define the class 'CRAS_α' of atom structures of α-dimensional cylindric-type algebras with a complete representation. By the above, $\mathsf{CRAS}_\alpha \subseteq \mathsf{Str}\,\mathsf{RCA}_\alpha$. For $3 \leq \alpha < \omega$, the inclusion is strict because

CRAS_α is pseudo-elementary and so closed under ultraproducts (see [3, exercise 4.1.17]), while by theorem 6.1, $\mathsf{Str}\,\mathsf{RCA}_\alpha$ is not.

We should mention that for $\alpha \geq 3$, the class of α-dimensional cylindric-type algebras that have a complete representation is not closed under elementary equivalence, and so is non-elementary [11, theorem 34]. Since the atom structure of an atomic cylindric BAO is first-order interpretable in the algebra, it follows that CRAS_α is also non-elementary.

[2] uses these and related notions to show that the omitting types theorem fails for n-variable first-order logic.

Remark 7.6. Although (for finite $n \geq 3$) RCA_n is a canonical variety, we believe that the ideas of the current paper and [15] can be combined to show that RCA_n is only *barely canonical,* meaning that every first-order axiomatisation of it has infinitely many non-canonical formulas. We hope to do this in a future publication.

References

[1] H. Andréka, *Complexity of equations valid in algebras of relations, Part I: Strong non-finitizability*, ***Annals of Pure and Applied Logic***, vol. 89 (1997), pp. 149–209.

[2] H. Andréka, I. Németi, and T. Sayed Ahmed, *Omitting types for finite variable fragments and complete representations of algebras*, ***J. Symbolic Logic***, vol. 73 (2008), pp. 65–89.

[3] C. C. Chang and H. J. Keisler, ***Model theory***, 3rd ed., North-Holland, Amsterdam, 1990.

[4] R. Diestel, ***Graph theory***, Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 1997.

[5] P. Erdős, *Graph theory and probability*, ***Canad. J. Math.***, vol. 11 (1959), pp. 34–38.

[6] R. Goldblatt, *Varieties of complex algebras*, ***Ann. Pure. Appl. Logic***, vol. 44 (1989), pp. 173–242.

[7] ———, *Elementary generation and canonicity for varieties of boolean algebras with operators*, ***Algebra Universalis***, vol. 34 (1995), pp. 551–607.

[8] L. Henkin, J. D. Monk, and A. Tarski, ***Cylindric algebras part I***, North-Holland, 1971.

[9] ———, ***Cylindric algebras part II***, North-Holland, 1985.

[10] L. Henkin, J. D. Monk, A. Tarski, H. Andréka, and I. Németi, ***Cylindric set algebras***, Lecture Notes in Math., vol. 883, Springer, Berlin, 1981.

[11] R. Hirsch and I. Hodkinson, *Complete representations in algebraic logic*, ***J. Symbolic Logic***, vol. 62 (1997), no. 3, pp. 816–847.

[12] ———, ***Relation algebras by games***, Studies in Logic and the Foundations of Mathematics, vol. 147, North-Holland, Amsterdam, 2002.

[13] ———, *Strongly representable atom structures of relation algebras*, ***Proc. Amer. Math. Soc.***, vol. 130 (2002), pp. 1819–1831.

[14] I. Hodkinson, *Atom structures of cylindric algebras and relation algebras*, ***Annals of Pure and Applied Logic***, vol. 89 (1997), pp. 117–148.

[15] I. Hodkinson and Y. Venema, *Canonical varieties with no canonical axiomatisation*, ***Trans. Amer. Math. Soc.***, vol. 357 (2005), pp. 4579–4605.

[16] B. Jónsson and A. Tarski, *Boolean algebras with operators I*, ***American Journal of Mathematics***, vol. 73 (1951), pp. 891–939.

[17] J. D. Monk, *Nonfinitizability of classes of representable cylindric algebras*, ***J. Symbolic Logic***, vol. 34 (1969), pp. 331–343.

[18] ———, *Completions of boolean algebras with operators*, ***Mathematische Nachrichten***, vol. 46 (1970), pp. 47–55.

[19] ———, *An introduction to cylindric set algebras*, ***Logic J. IGPL***, vol. 8 (2000), no. 4, pp. 451–496.

[20] F. P. Ramsey, *On a problem of formal logic*, ***Proc. London Math. Soc.***, vol. 30 (1930), pp. 264–286.

[21] T. Sayed Ahmed, *Martin's axiom, omitting types, and complete representations in algebraic logic*, ***Studia Logica***, vol. 72 (2002), no. 2, pp. 285–309.

[22] A. Tarski, *Contributions to the theory of models, III*, ***Koninkl. Nederl. Akad. Wetensch Proc.***, vol. 58 (= Indag. Math. 17) (1955), pp. 56–64.

[23] Y. Venema, *Atom structures*, ***Advances in modal logic '96*** (M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors), CSLI Publications, Stanford, 1997, pp. 291–305.

[24] ———, *Atom structures and Sahlqvist equations*, ***Algebra Universalis***, vol. 38 (1997), pp. 185–199.

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