Spherical symmetry and hydrostatic equilibrium in theories of gravity

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Declaration

I, Atifah Mussa, confirm that the work contained in this thesis is my own and is also thanks to some very useful discussions with my supervisor, Christian Böhmer. Parts of this thesis have been published with co-authors Christian Böhmer, Nicola Tamanini and Hâkan Andreasson in the following papers

- C. G. Böhmer and A. Mussa, "Charged perfect fluids in the presence of a cosmological constant," Gen. Rel. Grav. **43** (2011) 3033 [arXiv:1010.1367 [gr-qc]].
- H. Andreasson, C. G. Böhmer and A. Mussa, "Bounds on M/R for Charged Objects with positive Cosmological constant," Class. Quant. Grav. **29** (2012) 095012 [arXiv:1201.5725 [gr-qc]].
- C. G. Böhmer, A. Mussa and N. Tamanini, "Existence of relativistic stars in $f(T)$ gravity," Class. Quant. Grav. **28** (2011) 245020 [arXiv:1107.4455 [gr-qc]].

These papers can be found in appendices A, B, C respectively, and are referenced as [1–3] in the bibliography. All other material used to write this thesis has been cited accordingly.

Abstract

Static, spherically symmetric solutions of the Einstein-Maxwell equations in the presence of a cosmological constant are studied, and new classes of solutions are derived. Namely the charged Einstein static universe and the interior and exterior charged Nariai spacetimes, these solutions form a subclass of the RNdS solution with distinct properties. The charged Nariai solutions are then matched at a common boundary.

When constructing solutions to gravitational theories it is important that these matter distributions remain in hydrostatic equilibrium. If this equilibrium is lost, with internal gravitational forces dominating internal stresses, the solution will collapse under its gravitational field. An upper bound on the mass-radius ratio M_g/R for charged solutions in de Sitter space is derived, this bound implies hydrostatic equilibrium. The result is achieved by assuming the radial pressure $p \geq 0$ and energy density $\rho \geq 0$, plus $p + 2p_\perp \leq \rho$ where the tangential pressure $p_\perp \neq p$. The bound provides a generalisation of Buchdahl's inequality, $2M/R \leq 8/9$, valid for Schwarzschild's solution. In the limit $Q \to 0\,, \Lambda \to 0$, the bound reduces to Buchdahl's inequality.

Solutions in hydrostatic equilibrium are also considered in modified $f(T)$ gravity. It is shown that the tetrads $e^i{}_\mu$ impact the structure of the field equations, and certain tetrads impose unnecessary constraints. Two particular tetrads are studied in more detail, solutions are then found for both tetrads, and a conservation equation is obtained using an analogous method to obtaining the Tolman-Oppenheimer-Volkoff equation. Although both tetrad fields locally give rise to the spherically symmetric metric, the tetrad fields are not globally well-defined and hence cannot be described as spherically symmetric. We then derive an upper bound on M/R which also implies hydrostatic equilibrium, this yields some constraints on the form of $f(T)$ given a particular tetrad that locally gives rise to the line element $\mathrm{d}s^2 = \mathrm{e}^a \mathrm{d}t^2 - \mathrm{e}^b \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2$.

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"All of physics is either impossible or trivial. It is impossible until you understand it, and then it becomes trivial," Ernest Rutherford.

Contents

Outline

This thesis divides into two topics which share the underlying theme of spherically symmetric matter distributions. The concept of such spherically symmetric spacetimes is explored in two distinct theories of gravity, namely general relativity and $f(T)$ gravity. The latter is obtained by making modifications to an alternative theory of gravity known as teleparallelism, this theory is equivalent to general relativity and is thus often referred to as the teleparallel equivalent of general relativity or TEGR. In addition to studying such solutions, the balance of the internal gravitational force with the internal stresses in these spacetimes will be discussed in the framework of both theories.

The first topic is an investigation of particular spherically symmetric exact solutions in general relativity, for which the background is given in chapter 1. My results are presented in chapter 3 along with [1] and [2]. Two new classes of charged perfect fluid solutions in the presence of a positive cosmological constant are derived in [1], we then proceed to find an upper bound for the mass of a charged matter distribution with internal forces in equilibrium in a curved spacetime in [2]. Most of the calculations used to obtain these results are then discussed in chapter 3.

The second theme is introduced in chapter 2, this explores various modified or alternative theories of gravity, and the possibility of obtaining spherically symmetric solutions within these theories. The theories which will be considered in most detail are teleparallelism and modified teleparallelism. The latter is also referred to as $f(T)$ gravity and is discussed further with results presented in chapter 4 and in [3]. In the $f(T)$ gravity framework, I begin by discussing a recent claim that spherically symmetric solutions do not exist in this theory and discuss the complexity of achieving such solutions in this theory of gravity. I then show the role the chosen tetrad field plays in this problem. The publication inserted in [3] gives some solutions for two distinct tetrad fields, these solutions are discussed briefly in chapter 4. After considering a tetrad field which admits a wider class of $f(T)$ models for the spherically spherically symmetric metric in modified teleparallelism, that is the so-called rotated tetrad, I consider the applicability of Birkhoff's theorem given this tetrad field. Finally, an upper bound for the mass-radius ratio of a static solution in $f(T)$ gravity with the rotated tetrad field is derived. This result is obtained using an analogous method to the aforementioned bound in general relativity, and allows us to determine some constraints $f(T)$ must satisfy in order to respect this bound.

Chapter 1

Introduction to general relativity

After many years of development Einstein presented his general theory of relativity in 1915, it was then published the following year in [4]. General relativity is an extension of special relativity which includes a modification of Newton's law of gravity. It provides a relativistic description of the gravitational field exerted by a massive object and its effects on the geometric structure of the surrounding spacetime. The theory states that the gravitational interaction due to the presence of matter causes spacetime to curve hence distorting the path of a nearby object. This differs from the original foundations of Newton's laws of gravitation, where gravity is an attractive force between two massive objects which interacts instantaneously. In this description, planetary orbits are a consequence of this gravitational pull emanating from the sun, therefore in this theory the suns gravitational field interacts directly with the planet as opposed to the surrounding spacetime. However given certain circumstances Newtonian theory provides an accurate description of the gravitational interaction, this includes a weaker gravitational field. This is known as the Newtonian limit in which spacetime is asymptotically flat and the field equations can be approximated with Newton's laws of motion. General relativity is required for a more significant gravitational field, when Newtonian gravity no longer agrees with observation. For instance, the observation of the precession of the perihelion of Mercury deviated slightly from the predictions of Newton's equations, whereas solutions in general relativity describe this orbit correctly.

In general relativity, spacetime has the structure of a four-dimensional pseudo -Riemannian manifold $\mathcal{M}$, this is equipped with a metric $g_{\mu\nu}$ which can be used to determine local geometric quantities such as angles and lengths. The metric associated with a pseudo-Riemannian manifold is not positive definite, therefore it will have signature $(1,3)$ or $(3,1)$, for the purposes of this thesis I will consider a metric with signature $(-,+,+,+)$ for results in general relativity unless otherwise stated. The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are symmetric so that $g_{\mu\nu} = g_{\nu\mu}$ and $g^{\mu\nu} = g^{\nu\mu}$, where $g^{\mu\sigma}g_{\sigma\nu} = \delta^{\mu}_{\nu}$. The line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}$ is invariant under arbitrary invertible transformations known as diffeomorphisms [5].

Curvature, which is often a manifestation of the gravitational force due to the presence of a nearby matter distribution, is represented by the Riemann curvature tensor $R^{\sigma}{}_{\mu\nu\rho}$ and its contractions the Ricci tensor $R_{\mu\nu}$ and Ricci curvature scalar R. Note that the Ricci tensor equals zero in the absence of matter with a vanishing cosmological constant Λ, which will be shown later in this section, however the Riemann tensor can be non-zero in such circumstances. The Riemann tensor is defined in terms of the covariant derivative ∇_{μ}, which is the generalisation of the partial derivative ∂_{μ} for a curved spacetime. For instance, the covariant derivative of a rank-$(1,1)$ tensor $A^{\mu}{}_{\nu}$ is defined to be $\nabla_{\sigma}A^{\mu}{}_{\nu} = \partial_{\sigma}A^{\mu}{}_{\nu} + \Gamma^{\mu}_{\sigma\rho}A^{\rho}{}_{\nu} - \Gamma^{\rho}_{\sigma\nu}A^{\mu}{}_{\rho}$, note that the covariant derivative satisfies the condition $\nabla_{\sigma}g_{\mu\nu} = 0$. These derivatives do not commute when acting on vectors and tensors $\nabla_{\mu}\nabla_{\nu}v^{\sigma} \neq \nabla_{\nu}\nabla_{\mu}v^{\sigma}$ whereas partial derivatives do $\partial_{\mu}\partial_{\nu}v^{\sigma} = \partial_{\nu}\partial_{\mu}v^{\sigma}$, and $R^{\sigma}{}_{\mu\nu\rho}$ exploits this property to measure deviations from flat spacetime. The Riemann tensor on a manifold $\mathcal{M}$ is then expressed in terms of the commutator of the covariant derivative as follows

$$[\nabla_{\mu}, \nabla_{\nu}]v^{\sigma} = \nabla_{\mu}\nabla_{\nu}v^{\sigma} - \nabla_{\nu}\nabla_{\mu}v^{\sigma} = R^{\sigma}{}_{\rho\mu\nu}\, v^{\rho} - T_{\mu\nu}{}^{\rho}\nabla_{\rho}v^{\sigma}\,, \tag{1.0.1}$$

where v_{μ} is a vector field on $\mathcal{M}$ and $T^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\mu\nu}$ is the torsion tensor. Spacetime in the general relativity framework is assumed to be torsion free, this is attained by using the symmetric Levi-Civita connection $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$ associated with the metric. This connection implies the last term in equation (1.0.1) vanishes; therefore in general relativity $R^{\sigma}{}_{\rho\mu\nu}\, v^{\rho} = [\nabla_{\mu}, \nabla_{\nu}]v^{\sigma}$, and for a rank-$(1,1)$ tensor this generalises to $[\nabla_{\mu}, \nabla_{\nu}]A^{\sigma}{}_{\rho} = R^{\sigma}{}_{\lambda\mu\nu}A^{\lambda}{}_{\rho} - R^{\lambda}{}_{\rho\mu\nu}A^{\sigma}{}_{\lambda}$. The

Levi-Civita connection and the Riemann curvature tensor are defined as follows

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \Big(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \Big) ,$$
$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}_{\rho\mu} + \Gamma^{\sigma}_{\lambda\mu}\Gamma^{\lambda}_{\rho\nu} - \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\lambda}_{\rho\mu} , \tag{1.0.2}$$

where we have used the conventions in [6] for the Riemann tensor. From this the Ricci tensor $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$ and Ricci scalar $R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}$ can be constructed. Note that the Riemann tensor satisfies the following properties

$$\begin{aligned} &(\mathbf{1})\, R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu}, && (\mathbf{2})\, R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu} , \\ &(\mathbf{3})\, R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0, && (\mathbf{4})\, R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho} , \end{aligned} \tag{1.0.3}$$

where $R_{\sigma\rho\mu\nu} = g_{\sigma\lambda} R^{\lambda}{}_{\rho\mu\nu}$. The first property is clear from inspecting definition (1.0.1), and the second follows from the identity $[\nabla_\mu, \nabla_\nu] g_{\sigma\rho} = -g_{\lambda\rho} R^{\lambda}{}_{\sigma\mu\nu} - g_{\sigma\lambda} R^{\lambda}{}_{\rho\mu\nu} = 0$. The next property can be shown by using the Riemann tensor given in equation (1.0.2), all terms cancel which yields the result. The final relation follows from using (**1**)–(**3**) to write $R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} - R_{\nu\sigma\rho\mu} = R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\nu\rho\mu\sigma} + R_{\nu\mu\sigma\rho} = 0$, or equivalently $R_{\sigma\rho\mu\nu} - R_{\mu\nu\sigma\rho} = -\big(R_{\sigma\mu\nu\rho} - R_{\nu\rho\sigma\mu}\big)$. Using this we can write the following

$$R_{\sigma\rho\mu\nu} - R_{\mu\nu\sigma\rho} = -\big(R_{\sigma\mu\nu\rho} - R_{\nu\rho\sigma\mu}\big) = R_{\sigma\nu\rho\mu} - R_{\rho\mu\sigma\nu} = -\big(R_{\sigma\rho\mu\nu} - R_{\mu\nu\sigma\rho}\big) ,$$

which gives the required result.

In the field equations of the theory, matter and energy related quantities are represented by the stress-energy-momentum tensor $\mathcal{T}_{\mu\nu}$, and the geometry of the resulting curvature is described by the Einstein tensor $G_{\mu\nu}$. To derive the equations relating the effects of matter to the structure of spacetime, consider the Einstein-Hilbert action

$$S = S_{\text{gravity}} + S_{\text{matter}} = \frac{1}{16\pi} \int \big(R - 2\Lambda\big) \sqrt{-g}\ \mathrm{d}^4 x + \int L_{\text{matter}} \sqrt{-g}\ \mathrm{d}^4 x ,$$

where we have used geometric units so that the speed of light c and the gravitational constant G are set to unity, the convention $c = G = 1$ will continue to be used throughout the thesis. This is not the original form of the action since it did not initially include the cosmological constant Λ. This was later inserted by Einstein whilst developing a cosmological model of the universe, and the presence of Λ was to ensure that the model was static. Observations currently support a positive Λ [7].

There have been many reformulations of the variational principle used to obtain the field equations [8]. Here, the equation of motion is derived by varying the action with respect to the metric $g_{\mu\nu}$

$$
\begin{aligned}
G_{\mu\nu} + \Lambda g_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \mathcal{T}_{\mu\nu} \, , \\
\mathcal{T}_{\mu\nu} = M_{\mu\nu} + E_{\mu\nu} \, ,
\end{aligned}
\tag{1.0.4}
$$

where $M_{\mu\nu}$ and $E_{\mu\nu}$ represent the matter and electromagnetic stress-energy-momentum tensors respectively. Due to the symmetries of the Einstein tensor $G_{\mu\nu} = G_{\nu\mu}$ the sixteen components of the field equation (1.0.4) are reduced to ten second order differential equations which must be solved for $g_{\mu\nu}$.

Note that the Einstein tensor $G_{\mu\nu}$ satisfies $\nabla^\nu G_{\mu\nu} \equiv 0$. This follows from the combination $\nabla_\lambda R_{\sigma\rho\mu\nu} + \nabla_\mu R_{\sigma\rho\nu\lambda} + \nabla_\nu R_{\sigma\rho\lambda\mu} = 0$ which is referred to as the Bianchi identity. To show this result, we use the Jacobian identity

$$
\begin{aligned}
0 \equiv & \Big([\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + [\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] \Big) g_{\sigma\rho} v^\rho \\
= & \Big([\nabla_\lambda, R_{\sigma\rho\mu\nu}] + [\nabla_\mu, R_{\sigma\rho\nu\lambda}] + [\nabla_\nu, R_{\sigma\rho\lambda\mu}] \Big) v^\rho \\
= & \Big(\nabla_\lambda R_{\sigma\rho\mu\nu} + \nabla_\mu R_{\sigma\rho\nu\lambda} + \nabla_\nu R_{\sigma\rho\lambda\mu} \Big) v^\rho \, ,
\end{aligned}
$$

where we used the product rule to rewrite the second line $-R_{\sigma\rho\mu\nu} \nabla_\lambda v^\rho = v^\rho \nabla_\lambda R_{\sigma\rho\mu\nu} - \nabla_\lambda (R_{\sigma\rho\mu\nu} v^\rho)$. Contracting the Bianchi identity with the metric twice, and using the properties (1.0.3) yields

$$
\begin{aligned}
g^{\sigma\nu} g^{\rho\lambda} & \Big(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\lambda} + \nabla_\nu R_{\rho\sigma\lambda\mu} \Big) \\
& = \nabla^\rho R^\sigma{}_{\mu\sigma\rho} - \nabla_\mu g^{\sigma\nu} R^\lambda{}_{\sigma\lambda\nu} + \nabla^\sigma R^\lambda{}_{\sigma\lambda\mu} \, ,
\end{aligned}
$$

which can be written as $\nabla^\nu G_{\mu\nu} = \nabla^\nu R_{\mu\nu} - \frac{1}{2} \nabla_\mu R = 0$, note that the term involving the cosmological constant does not affect this result since $\nabla^\nu \big(\Lambda g_{\mu\nu} \big) \equiv 0$. Plugging this into Einstein's field equation (1.0.4) implies that the stress-energy-momentum tensor is conserved $\nabla^\nu \mathcal{T}_{\mu\nu} = 0$.

A perfect fluid with anisotropic pressure in the presence of an electromagnetic field has stress-energy-momentum tensor

$$\mathcal{T}_{\mu\nu} = \underbrace{\Big(\rho + p_\perp\Big)u_\mu u_\nu + p_\perp g_{\mu\nu} + \Big(p - p_\perp\Big)v_\mu v_\nu}_{\text{matter energy } M_{\mu\nu}} + \underbrace{\frac{1}{4\pi}\Big(F_\mu{}^\sigma F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\sigma\rho}F_{\sigma\rho}\Big)}_{\text{electromagnetic energy } E_{\mu\nu}} . \tag{1.0.5}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor, u^μ is the four-velocity so that $u^\mu u_\mu = -1$ and v^μ is a spacelike unit vector in the radial direction hence $v^\mu v_\mu = 1$. Note if one assumes the pressure is isotropic so that the radial p and tangential $p_\perp$ pressures are equal, the number of independent equations will be reduced. Moreover, imposing vanishing radial and tangential pressures $p = p_\perp = 0$ gives rise to a dust solution. Alternatively, one can assume the metric we seek exerts various symmetries, for instance a metric that is static or spherically symmetric will also simplify the form of the field equations. Putting these assumptions of a static and spherically symmetric metric together with the consideration of an isotropic perfect fluid energy-momentum tensor gives rise to a spherically symmetric isotropic fluid, and will reduce the ten field equations to two plus a conservation equation which can be then solved for $g_{\mu\nu}$. A static and spherically symmetric dust solution is determined by one field equation with a conservation equation. Many results which will be discussed in this introduction have used an isotropic pressure $p = p_\perp$ unless otherwise stated, whereas most results presented later will allow for an anisotropic pressure.

When studying spacetime in the absence of both matter and an electromagnetic field, one must consider vacuum solutions. This corresponds to a vanishing stress-energy-momentum tensor, hence the field equation becomes $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ which implies $R_{\mu\nu} - \Lambda g_{\mu\nu} = 0$. The latter is obtained by using an equivalent formulation of Einstein's equation which is often referred to as the trace-reversed field equation. Taking the trace of the Einstein tensor $G = g^{\mu\nu}G_{\mu\nu}$ yields

$$G = g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R + \Lambda g^{\mu\nu}g_{\mu\nu} = -R + 4\Lambda = 8\pi\mathcal{T} ,$$

this can then be substituted into Einstein's field equation to give the trace-reversed equation $R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi\big(\mathcal{T}_{\mu\nu} - \frac{1}{2}\mathcal{T}g_{\mu\nu}\big)$. The aforementioned result now follows from the vacuum field equations where $\mathcal{T}_{\mu\nu} = 0$.

1.1 Exact solutions

Exact solutions to Einstein's field equation have been considered since the development of the theory of general relativity, and are represented by the metric $g_{\mu\nu}$. However, after almost a century of study, there is no concise criterion or single outlined method for the construction of viable exact solutions of the theory; the derivation of a solution is an extremely broad concept because there are two rank-two tensors $\mathcal{T}_{\mu\nu}$ and $g_{\mu\nu}$ each containing up to ten independent components. This allows for many parameters when describing spacetime, hence some reasonable assumptions need to be made in order to proceed. A particular approach is to fix the form of $\mathcal{T}_{\mu\nu}$ so that the matter distribution describes for example, dust, perfect fluid or vacuum solutions. Thus the metric will usually describe parts of the spacetime only, and will need to be matched to the relevant exterior or interior solution to describe the entire spacetime. This simplification can then be used to solve the field equations to find the metric $g_{\mu\nu}$. As described above, assumptions can be imposed on the metric to reduce complexity of the field equations, this includes constructing solutions which exhibit spacetime symmetries, considering static, stationary or non-charged spacetimes, and restricting asymptotic behaviour. The conditions used to obtain solutions can include imposing both physical and mathematical assumptions, and it is important that the solutions developed are physically consistent. For example, if we consider a matter distribution in the presence of electromagnetic forces, the solution must obey the Einstein-Maxwell equations. The presence of the electromagnetic energy-momentum tensor $E_{\mu\nu}$ in the field equations (1.0.4) implies that the exterior of a charged matter distribution is not vacuum, it is instead referred to as an electro-vacuum solution ($\mathcal{T}_{\mu\nu} = E_{\mu\nu}$) whereas the neutral analogue will have a vacuum exterior ($\mathcal{T}_{\mu\nu} = 0$). It was shown that solutions can be derived from an equation of state and this is viewed as the most physical way of proceeding [9], where an equation of state relates the energy density and radial pressure with $\rho = \rho(p)$. The next few paragraphs will outline the development of some important exterior solutions in Einstein's theory, then a summary of some cosmological solutions will follow.

In 1916, shortly after the publication of general relativity, Schwarzschild discovered the first static, spherically symmetric solution which described the exterior spacetime of a massive object [10]. This solution was developed prior to the inclusion of the cosmological constant in the field equations, and before the effects of charge Q and rotation J were considered important in the construction of solutions. The absence of Λ, Q and J means it provides the simplest model of a static, spherically symmetric matter distribution. More importantly, the Schwarzschild solution gives an accurate description of Mercury's orbit. The charged analogue was also discovered in 1916 by Reissner [11], and independently by Nordström in 1918 [12], hence it is known as the Reissner-Nordström solution. This solution was formulated in Schwarzschild coordinates, and in the absence of charge it reduces to the exterior Schwarzschild solution. Several years later, in 1923, Birkhoff proved an important theorem [13]. This stated that solving the vacuum field equations for a spherically symmetric metric will yield a static spacetime, moreover the solution is unique and is the Schwarzschild solution. Birkhoff's theorem has been generalised for the Einstein-Maxwell equations, the resulting exterior is unique and is given by the Reissner-Nordström solution [14]. Birkhoff's result can be proved directly from the field equations for a spherically symmetric, time-dependent metric.

Similarly, axially symmetric solutions have been studied since the publication of general relativity, starting with Weyl [15] in 1917. A very well-known axisymmetric solution of the theory was found by Kerr in 1963 [16], this can be written in Schwarzschild-like coordinates and it reduces to the Schwarzschild solution in the absence of rotation. Naturally the charged analogue was the next generalisation, this was formulated soon after by Newman *et al* in 1965 [17] and is referred to as the Kerr-Newman solution.

All solutions mentioned so far describe the exterior of a massive object which is either a vacuum or electro-vacuum spacetime. Possible interior solutions will be looked at in more detail in section 1.4, where we will discuss in particular interior charged perfect fluid solutions in the presence of a cosmological constant and their neutral analogues.

Cosmological models

The exterior solutions discussed above are useful for describing the gravitational field outside a massive object, alternatively cosmological solutions potentially model the formation and evolution of our universe. The importance of such models in describing our universe will be discussed further in chapter 2, below we will outline some cosmological solutions in general relativity.

Shortly after Schwarzschild's solution was published, in 1917 Einstein proposed a static universe which resulted from the introduction of the cosmological constant Λ in the field equations [18]. This is known as the Einstein static universe and was intended to provide a cosmological model. The solution had a constant energy density and vanishing pressure hence described a homogeneous universe, the addition of the cosmological constant gave a solution that was static thus the universe was not expanding. It was later discovered that this does not provide a good approximation to our universe due to observations by Hubble in 1929 [19]. The observations showed that distant galaxies are moving away from the Earth and galaxies which were further away moved with an increased speed. Hubble's observation lead to the conclusion that the universe is not static but is in fact expanding, and supported some theoretical work on an expanding universe given by the de Sitter and Friedmann-Lemaître-Robertson-Walker solutions discovered in 1917 [20] and 1922 [21] respectively, these solutions will be discussed below. Additionally, the stability of the Einstein static universe has been questioned [21, 22]. Nonetheless, some generalisations of the Einstein static universe will be derived in section 3.6.

In 1917, de Sitter obtained a maximally symmetric family of vacuum cosmological solutions as an extension of the Einstein static universe. There are two solutions, they are known as the de Sitter and anti-de Sitter solutions which correspond to the presence of a cosmological constant Λ that is either positive or negative. These new solutions can be viewed as a four-dimensional hyperboloid embedded in a five-dimensional manifold, whereas de Sitter showed that Einstein's solution would be a three-sphere embedded in a four-dimensional manifold [20]. Spacetime is assumed to be homogeneous and isotropic which successfully describes an expanding universe, where a positive or negative Λ corresponds to accelerated or decelerated expansion respectively. Since observations show that the universe is experiencing accelerated expansion, a positive cosmological constant provides a more physically accurate solution. The de Sitter and anti-de Sitter solutions are often presented in non-static coordinates, however can be transformed to static form [23]. Both versions differ to the aforementioned solutions asymptotically, the previously mentioned exterior solutions tend to flat Minkowski spacetime at large distances from the matter distribution whereas the de Sitter solutions are asymptotically curved. Spacetime equipped with this metric is often studied in cosmology, and de Sitter predicted an expanding universe with this solution prior to Hubble's discovery. Additionally, the non-static form with a positive cosmological constant has been considered as a model for inflation in the early universe [24], this model uses the so-called inflationary coordinates for de Sitter space [25].

The Schwarzschild exterior in the presence of a positive cosmological constant was found in 1918 by Kottler [26] and later by Weyl in 1919 [27]. This is given by a combination of the Schwarzschild and static de Sitter solutions and is often referred to as the Schwarzschild-de Sitter or Kottler solution. In the absence of mass this solution becomes the static de Sitter universe, similarly setting the cosmological constant to zero gives rise to the Schwarzschild exterior and a negative cosmological constant yields the anti-de Sitter analogue of the Kottler solution. These solutions describe the exterior of a matter distribution in de Sitter or anti de Sitter space, and the asymptotic behaviour of the metric is dominated by the Λ contribution. Following this, such asymptotically de Sitter solutions have continued to be studied and generalised, for example the charged analogue is known as the Reissner-Nordström de Sitter (or RNdS) solution [28]. The RNdS solution will be discussed in more detain in chapter 3. Likewise the Kerr [28] and Kerr-Newman [29] solutions have been studied in the presence of a positive or negative cosmological constant Λ. The Nariai solutions, found in 1951 [30] form a particular subclass of the Kottler solution which are asymptotically distinct. Hence this spacetime requires an alternative metric to describe the exterior, this is known as the exterior Nariai solution. More details will be provided in chapter 3, and the charged analogue is derived in section 3.5 and [1].

In 1922 Friedmann discovered a time-dependent solution which paved the way for the analysis of expansion in cosmology. This solution was later rediscovered independently by Lemaître, Robertson and Walker and is therefore known as the Friedmann-Lemaître-Robertson-Walker or FLRW metric. The FLRW metric, which can be considered as a generalisation of the de Sitter solution, provides the standard model of cosmology in which it gives a description of the expanding universe. The FLRW solution was later identified as a special case (the homogeneous version) of the more general inhomogeneous Lemaître-Tolman-Bondi model [31].

There are numerous solutions to Einstein's field equations, many of these solutions have been omitted and some areas such as the axially symmetric Kerr solution, and the time-dependent FLRW model have only been mentioned very briefly above. The remainder of this chapter will concentrate on the discussion of static, spherically symmetric solutions, and more detail will be given to such solutions in the presence of charge Q which reside in de Sitter space. However, the FLRW solution will briefly enter the discussion of modified gravity in chapter 2. For a more detailed account of exact solutions of Einstein's field equations which are not mentioned here see for instance [6].

1.1.1 Causal structure of spacetime

The exterior Schwarzschild solution is given by the line element

$$\mathrm{d}s^2 = -\mathrm{e}^{a(r)}\mathrm{d}t^2 + \mathrm{e}^{b(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{1.1.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the line element of the unit two-sphere, and $\mathrm{e}^a = 1 - \frac{2M}{r} = \mathrm{e}^{-b}$. The derivation of the mass appearing in the Schwarzschild metric will be shown in section 1.2, which gives rise to this relation for e^{-b}. The solution is valid for $r \geq R$, where R is the boundary of the matter distribution. The exterior Schwarzschild solution is required to be valid at the boundary R so that it can be matched to a suitable interior, see section 1.4 for a more detailed discussion. Notice that $\mathrm{e}^{a(0)} \to \infty$ and $\mathrm{e}^{b(2M)} \to \infty$. Hence the metric would contain two singularities if the boundary lies within the region $R \leq 2M \equiv r_s$, where r_s is referred to as the Schwarzschild radius. Studying these singularities showed that the latter is a coordinate singularity as opposed to a gravitational singularity, and it can be removed by using an alternative coordinate system. However the former is a gravitational singularity, this can be seen by considering a particular contraction of the Riemann curvature tensor known as the Kretschmann invariant $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The scalar K is invariant under coordinate transformations, and for the Schwarzschild solution $K = \frac{48M^2}{r^6}$. Therefore when $r = 0$ in the Schwarzschild metric we have $K \to \infty$, thus the singularity at $r = 0$ cannot be removed by changing coordinates.

If the radius of the matter distribution reaches the Schwarzschild radius $R = r_s$, the object will collapse under its gravitational field and will develop a singularity at $r = 0$, the reason for this collapse will be discussed in section 1.3. This object is known as a black hole, the Schwarzschild radius r_s gives the location of the black hole boundary which forms a hypersurface and is referred to as the event horizon of the black hole. Beyond the event horizon lies the singularity, and if a particle passes r_s it will not return, this implies that once in the interior a particle is causally disconnected from the exterior (that is a particle will not reach the exterior in a finite amount of time). Due to this property the event horizon is often referred to as 'the point of no return'. In particular light cannot escape, which hinders our visibility of the event horizon and beyond. It is possible to construct interior Schwarzschild solutions with the boundary $R > r_s$ which do not contain a singularity, and subject to certain conditions the boundary will remain greater than r_s, this will be discussed in section 1.4. Note, when we consider solutions involving more quantities such as charge, rotation or a cosmological constant this gives rise to additional horizons. For instance, the Reissner-Nordström solution contains two horizons which are referred to as the inner r_- and outer r_+ horizons. Likewise, the Kottler solution has a cosmological horizon r_c as well as the black hole horizon r_s.

The nature of a curved spacetime can be visualised further by adopting a different coordinate system, and some alternative coordinate systems enable an analysis of the event horizon and the region beyond. This allows us to study the path of a particle and its behaviour once the event horizon has been passed, in particular its causal relation with other points in spacetime. The causal structure of spacetime can be understood further by exploring its geometric structure, and to study the geometry we study the possible paths of light. In general relativity freely falling particles travel along geodesics, which are the curved analogue of straight lines. The trajectories are governed by the geodesic equation $v^\mu \nabla_\mu v_\nu = 0$ which is obtained by finding the extrema of $\int \mathrm{d}s = \int \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\mathrm{d}\tau$, where $v^\mu = \dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ is tangent to the curve $x^\mu(\tau)$. The tangent vector v^μ can be classified as either space-like, time-like or null and this depends on whether the quantity $g_{\mu\nu}v^\mu v^\nu$ is positive, negative or zero respectively. Similarly if the quantity $g_{\mu\nu}v^\mu v^\nu$ is either positive, negative or zero at every point along the curve x^μ, the curve is described as either space-like, time-like or null. Light travels along null geodesics, where all other physical points follow a time-like trajectory and a space-like curve does not connect two events. Null curves determine the boundary of a light cone, which can be seen in figure 1.1.

In order to study null geodesics near the event horizon in the radial direction we need to employ an alternative description of coordinates. This is because radial null geodesics of the Schwarzschild metric correspond to $\mathrm{d}s^2 = 0$ with θ and ϕ constant, this gives rise to the equation $\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2M}{r}\right)^{-1}$, which is not well-behaved at the horizon. There is a collection of coordinate transformations which allow us to extend null geodesics to the event horizon and beyond this region of spacetime. More details will be provided in section 3.4 where the transformations will be given explicitly for the Reissner-Nordström spacetime.

These notions of causality can be used to define some conditions a physically reasonable matter-energy-momentum tensor must satisfy, these are known as the null, weak, strong and dominant energy conditions, see for instance [5, 38]. The null energy condition imposes that for any null vector v^μ we have $\mathcal{T}_{\mu\nu}v^\mu v^\nu \geq$

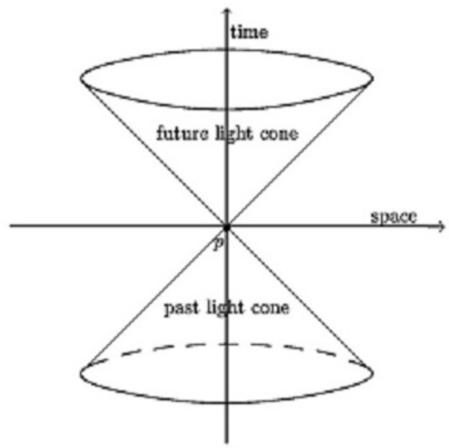

Figure 1.1: *This figure contains the light cone originating from the point p. All points inside the light cone are connected to p with a time-like curve and points along the boundary are connected to p with a null curve. Inside the light cone one can distinguish between the future and past, and each point inside the light cone is time-like separated from p, whereas each point on the light cone is null separated from p and if outside the cone they are space-like separated from p.*

0, the weak energy condition requires the same for all time-like vectors v^μ so that the energy-momentum tensor contains positive energy densities. The strong energy condition states the trace-reversed field equation satisfies $(\mathcal{T}_{\mu\nu} - \frac{1}{2}\mathcal{T} g_{\mu\nu})v^\mu v^\nu \geq 0$ for all time-like unit vectors v^μ. Finally, the dominant energy condition imposes the weak energy condition holds along with the requirement that if v^μ is time-like (or null) then $-\mathcal{T}^\mu_\nu v^\nu$ is also a time-like (or null) vector. When $\mathcal{T}_{\mu\nu}$ is a perfect fluid, these conditions can be stated explicitly in terms of the components of $\mathcal{T}_{\mu\nu}$. For an isotropic pressure $p = p_\perp$ the energy momentum tensor from equation (1.0.5) is $\mathcal{T}_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$, where u^μ is the four-velocity with $u_\mu u^\mu = -1$. Note that the quantity $g_{\mu\nu}v^\mu v^\nu = -1$ if v^μ is a time-like unit vector and vanishes if v^μ is null. Using this the null energy condition becomes $\rho + p \geq 0$, whereas the weak energy condition requires that the former holds along with $\rho \geq 0$. The strong energy condition also requires the null condition plus $\rho + 3p \geq 0$. Finally, the dominant energy condition states that $|p| \leq \rho$ holds in addition to the weak energy condition.

1.1.2 Static, spherically symmetric spacetimes

Symmetry is an important assumption when considering solutions, particularly exact solutions, it is important due to the mathematical simplifications which can be applied to the governing physics. Spherically symmetric solutions have had many successful theoretical predictions which support Einstein's theory, such as the Schwarzschild solution implying all three empirical tests of general relativity [32].

Symmetries are described by maps that preserve the structure of spacetime. In flat Minkowski space such maps are a part of the Lorentz group plus translations, which constitute the larger Poincaré group. For the more general, curved manifold $\mathcal{M}$ the geometric structure of spacetime is modified and so is the notion of symmetry. Given two manifolds $\mathcal{M}$ and $\mathcal{N}$ of the same dimension, the map $f : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism if the ranks of all tensor fields are preserved by f and f^{-1}. Equivalently f is a diffeomorphism if it is bijective and both f and f^{-1} are smooth, where a function is smooth (or C^∞) if all derivatives are continuous. The manifolds $\mathcal{M}$ and $\mathcal{N}$ related by the diffeomorphism f are referred to as diffeomorphic and share the same structure. The diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ is described as an isometry if it preserves the metric, and hence geometric quantities such as distances. Symmetries on a manifold are then described by a one-parameter family of isometries; such maps are generated by a vector field ξ_a which belongs to the tangent space to a given point on $\mathcal{M}$, and for each a in $\mathbb{R}$ the map $\xi_a : \mathcal{M} \to \mathcal{M}$ is an isometry. This vector field ξ_a is referred to as a Killing vector field and obeys Killings equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

Thus spacetime symmetries can simply be expressed in terms of Killing vectors and when the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is independent of a coordinate x^μ for a particular μ (in Schwarzschild coordinates $x^\mu = t, r, \theta$ or ϕ) then the unit vector in the direction of x^μ is a Killing vector of the metric. For example if the metric is independent of t then $\xi_\mu = \big(g_{tt}, 0, 0, 0\big)$ is a time-like Killing vector, where $\xi^\mu = g^{\mu\nu}\xi_\nu = \big(1, 0, 0, 0\big)$. If the metric admits a time-like Killing vector ξ_μ this implies spacetime is stationary, the spacetime can then be described as static if the Killing vector is orthogonal to a family of hypersurfaces. Additionally, an $n+1$-dimensional spherically symmetric spacetime will permit $(n-1)n/2$ rotational Killing vectors, thus there are three rotational Killing vectors in a four-dimensional spherically symmetric spacetime. An n-dimensional maximally symmetric spacetime allows $(n+1)n/2$ Killing vectors, thus in four-dimensions there are ten Killing vectors.

A four-dimensional static and spherically symmetric spacetime such as the Schwarzschild solution admits four Killing vectors. Since this solution is static it is independent of t this gives rise to one time-like $\xi^1_\mu = \left(-\mathrm{e}^a, 0, 0, 0\right)$, spherical symmetry implies that there are also three rotational Killing vectors. One of the rotational Killing vectors is obvious from the line element $\xi^2_\mu = \left(0, 0, 0, r^2 \sin^2\theta\right)$, the other two can be expressed as $\xi^3_\mu = \left(0, 0, r^2 \sin\phi, r^2 \sin^2\theta \cot\theta \cos\phi\right)$ and $\xi^4_\mu = \left(0, 0, -r^2 \cos\phi, r^2 \sin^2\theta \cot\theta \sin\phi\right)$, for a derivation of the latter two Killing vectors see [33]. On the other hand, four-dimensional de Sitter (or anti-de Sitter) space is maximally symmetric and hence permits ten killing vectors, they can similarly be obtained by solving Killings equation.

The consequences of symmetry are stressed further by an important theorem discovered by Noether [34], this states that certain symmetries of the action lead to conservation laws. Such symmetries are coordinate transformations which leave the action unchanged. Since they imply conservation of quantities such as energy and momentum they are useful when defining the mass in general relativity, in particular the Komar mass which will be discussed in the next section, 1.2. This theorem however is not relevant for calculations in this thesis, and therefore will not be considered any further.

1.2 Defining mass in general relativity

Before discussing solutions which describe the interior of a matter distribution, it is useful to understand the concept of mass in general relativity. Newtonian gravity and special relativity both have a simple definition for the mass of an object, however this single definition is lost in general relativity. Newtonian gravity assumed the equivalence of the inertial and gravitational masses, and Newton's laws of motion imply that the inertial mass of an object determines its acceleration $\vec{a}$ due to a force acting on it $\vec{F} = m\vec{a} = \frac{\mathrm{d}\vec{k}}{\mathrm{d}t}$, with $\vec{a} = \frac{\mathrm{d}\vec{u}}{\mathrm{d}t}$ where $\vec{u}$ is the velocity and $\vec{k} = m\vec{u}$ is the momentum. Alternatively, the mass in Newtonian gravity can be calculated with Gauss's law, which expresses the mass in terms of the matter density ρ, this relation is given below (1.2.1). Einstein's theory of special relativity resulted in the equivalence of the rest or proper mass m_0 and energy of an object. This equivalence arises since the total energy E

and three-momentum $\vec{k}$ of an object with three-velocity $\vec{u}$ are defined as

$$E = \sqrt{m_0^2 + \vec{k}^2}\,,$$
$$\vec{k} := m\vec{u} = m_0\vec{u}/(\sqrt{1-\vec{u}^2})\,,$$

where the momentum of a stationary object is zero, thus $E = m_0$. From the definition of $\vec{k}$ we can write down the relativistic mass (the mass of an object in motion) m and relativistic energy E as follows

$$m = \frac{m_0}{\sqrt{1-\vec{u}^2}}\,,$$
$$E = \sqrt{m_0^2 + \frac{m_0^2\vec{u}^2}{(1-\vec{u}^2)}} = \frac{m_0}{\sqrt{1-\vec{u}^2}}\,.$$

therefore when the momentum is non-zero, we also have $E = m$. From this we see the rest mass is the smallest possible mass. Note, the related four-momentum is given by the four-vector $k_\mu = (E, \vec{k}) = mu_\mu$, where u_μ is the four-velocity and E is the relativistic energy.

The equivalence of inertial and gravitational masses is not lost in general relativity due to the equivalence principle, and neither is the equivalence of energy and rest mass. However there are many possible definitions of mass in general relativity and the applicable definition is dependent on the situation. The equivalence principles imply that we can abandon the effects of gravity for particles in free fall in an inertial frame, this results in the difficulty of defining the local gravitational energy. This problem leads to a difficulty in extending the concept of total energy from special relativity for a matter distribution, and hence the related mass. In calculating the mass of a particular solution, the symmetries, asymptotic behaviour such as curvature, and presence of charge or rotation will alter the definition. In the absence of charge, the mass of a static, spherically symmetric object from its centre $r = 0$ to the radial value $0 < r \leq R$ is defined using the energy density

$$m(r) = 4\pi \int_0^r \tilde{r}^2 \rho \, \mathrm{d}\tilde{r}\,, \tag{1.2.1}$$

this definition of mass is analogous to Gauss's law which gives rise to the Newtonian mass. Since $r = R$ is the boundary of the object, where the interior and exterior solutions are matched, the metric is required to be well behaved in this region. Therefore $m(R) \equiv M$ denotes the total mass. In the Schwarzschild solution m is obtained by writing the Einstein field equations in component form for the line element (1.1.1). The (t,t) field equation can be written as $8\pi r^2\rho = \frac{\mathrm{d}}{\mathrm{d}r}\left(r - r\mathrm{e}^{-b}\right)$, then inserting this into definition (1.2.1) yields $\mathrm{e}^{-b} = 1 - \frac{2m}{r}$. The mean density of a spherically symmetric matter distribution is defined from this formulation of the mass and is given by

$$\hat{\rho}(r) = \frac{3m}{4\pi r^3}\,.$$

Now this definition of mass (1.2.1) can be extended to include the effects of the presence of charge, the stress-energy-momentum tensor in the Einstein-Maxwell field equations includes a non-zero contribution from the electromagnetic energy-momentum tensor $E_{\mu\nu}$. This was studied by Florides in 1962 [35], prior to this it was largely believed that the electromagnetic force did not contribute to the gravitational mass. However, this result was speculated prior to Florides discovery during the study of charged interior solutions, the relevant solutions will be discussed further in section 1.4.1. The non-zero component of this additional tensor quantity that contributes to the (t,t) field equation in a static and spherically symmetric spacetime is $E^t_t = -\frac{q^4}{8\pi r^4}$. Here the charge is given in terms of its charge density σ, and is defined to be

$$q(r) = 4\pi \int_0^r \mathrm{e}^{(a+b)/2}\tilde{r}^2\sigma \mathrm{d}\tilde{r}\,.$$

The governing Maxwell equations that lead to this derivation of E^t_t will be discussed further in chapter 3. Given the same line element as above, the (t,t) component of the Einstein-Maxwell equations is now given by

$$8\pi r^2\rho + \frac{q^2}{r^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r - r\mathrm{e}^{-b}\right).$$

Hence the charge contributes to the mass of the object

$$m(r) = 4\pi \int_0^r \tilde{r}^2\rho\,\mathrm{d}\tilde{r} = \frac{1}{2}\int_0^r \left\{\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\tilde{r} - \tilde{r}\mathrm{e}^{-b}\right) - \frac{q^2}{\tilde{r}^2}\right\}\mathrm{d}\tilde{r}\,, \tag{1.2.2}$$

where $q = q(r)$ cannot be integrated directly unless q is defined explicitly. Following the notation used in [1] and [2] yields $m(r) = m_i = m_g - m_q$, where m_g is the total gravitational mass and m_q is the electromagnetic contribution, these definitions will be explained further in chapter 3. The total gravitational mass at the boundary $r = R$ is given by $M_g = m_g(R) = m_i(R) + m_q(R)$, and for the Schwarzschild solution the absence of charge implies $m_g = m_i$. A similar relation can be made for the mass in the presence of a cosmological constant, for instance in the Kottler solution

$$m(r) = 4\pi \int_0^r \tilde{r}^2 \rho \mathrm{d}\tilde{r} = \frac{1}{2}\int_0^r \left\{ \frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\tilde{r} - \tilde{r}\mathrm{e}^{-b}\right) - \Lambda\tilde{r}^2 \right\} \mathrm{d}\tilde{r}\,, \tag{1.2.3}$$

thus $m_i = m_g - \frac{\Lambda r^3}{6}$. However this distinction is not as important as in the presence of charge, this is because $\Lambda =$ constant this can be directly integrated unlike $q(r)$. Therefore in the Kottler solution m_i is typically used instead of m_g, where $\mathrm{e}^{-b} = 1 - \frac{2M_i}{r} - \frac{\Lambda r^2}{3} = 1 - \frac{2M_g}{r}$. When we consider the Reissner-Nordström de Sitter solution in chapter 3 we will not absorb the cosmological constant contribution into m_g.

Alternatively, the Komar mass describes the mass in any stationary and asymptotically flat spacetime [36]. The only requirements are that spacetime admits a time-like killing vector field and the exterior solution is vacuum. Then Noether's theorem states that this symmetry defined by the killing vector η^μ leads to a conserved current which is in turn used to define the Komar mass. The Komar mass can be shown to be equal to the total mass $M = m(R)$ given in equation (1.2.1) for the spherically symmetric and static Schwarzschild solution, see [5]. There has been much more work on this area of general relativity, which resulted in many alternative definitions dependent on the situation, for instance [37]. However, these other definitions of mass are not relevant for the work in this thesis, for the Reissner-Nordström de Sitter solution the definition (1.2.2) in the presence of a cosmological constant will be used.

1.3 Compact stars and black holes

The theoretical idea of the existence of an object so massive that no particle, including light, can escape its gravitational field was first introduced in 1798 by Laplace [38], but this concept did not receive much attention due to its unusual properties. Over a century later, Einstein's theory of relativity predicted the existence of such objects, now known as black holes. However, at this time, these exotic objects still proved to be a controversial area with much doubt on the possibility of their existence [39]. This prediction had followed

Schwarzschild's solution, where the singular behaviour at the Schwarzschild radius initiated further study alongside debate. It remained a controversial area until approximately half a century later in 1958, when research [40–43] showed that the black hole solution is an unavoidable consequence of general relativity. This cleared up some previously unaccepted properties of black holes such as their associated 'singularities'. The Schwarzschild solution now gives the first black hole solution [44] and the Schwarzschild radius $r_s = 2M$ is known not to be a singularity [40, 41, 45] but the black hole event horizon.

During this development of the understanding of black holes, the possibility of another exotic object emerged. In 1932 it was speculated that a supernova explosion is due to the gravitational collapse of a stellar object, and this explosion transforms the star into a neutron star [46, 47]. A neutron star is an extremely dense but relatively small object, and is classified as a compact object due to its compressed density. The first observation to support this theory was more than three decades later [48] in 1968. Other compact objects include white dwarfs. Since these objects are so dense, it is possible that they will attract and collect more mass. If enough mass is collected, the compact object may experience gravitational collapse again and become a black hole. These two developments led to widespread interest in compact objects and black holes in general relativity. The next section will discuss the life cycle of a star and the possibility of gravitational collapse.

1.3.1 Star formation and gravitational collapse

From observation and theory, we now have an idea of the life cycle of a star, from birth when it is only visible via an infra-red telescope, up to the gravitational collapse which forms a compact object.

A star eventually emerges from a nebula, which is an interstellar cloud of cosmic dust and gases such as helium and hydrogen. Formation begins when regions of increased gravity cause the gas and dust to condense, as these regions become more massive they then collapse under the strengthened gravitational field, causing an increase in temperature and thus forming a protostar. The star remains in this stage for many thousands of years, until the core temperature and density reach the necessary levels for nuclear fusion to occur and support its gravitational field; this conversion of hydrogen into helium will produce enough energy to stop the internal gravitational force causing collapse. At this stage the star is often referred to as a main sequence star, although there are many subclassifications of stars.

In order to avoid gravitational collapse or explosion, a massive object such as a main sequence star must remain in hydrostatic equilibrium. That is the gravitational force pushing inwards and internal stresses (due to nuclear fusion etc) pushing outwards need to be balanced. If the gravitational force exceeds that of the internal stresses, the object will collapse under the dominating gravitational field and will eventually be reduced to a compact object such as a white dwarf or neutron star. The compact object also needs to remain in hydrostatic equilibrium to avoid collecting more mass, and consequently experiencing gravitational collapse again which will result in a black hole. Hoyle and Narlikar [49] stated that given Einstein's field equations, a large enough mass in the process of gravitational collapse will continue to collapse until a singularity is developed. It is possible that a charged interior matter distribution subject to certain constraints can avoid becoming massive enough to develop such a singularity. For instance, some work which will be discussed in section 1.4 shows that the balance of the gravitational attraction with the electric repulsion gives rise to a charged object in hydrostatic equilibrium, however there are charged interior spacetimes which do contain a singularity. This can also be achieved for neutral solutions, the corresponding bounds on M/R required to maintain this equilibrium will then be discussed in section 1.5. The possible fate of a star after gravitational collapse and hence the classification of compact objects, depends on the configuration of the matter distribution; this includes the mass, density, internal stresses and forces involved. This configuration is equally important in the initial stages when denser regions of begin to form in the nebula and throughout the life of the star. The gravitational collapse of a stellar object indicates the radial value has reduced hence the collected mass is concentrated into this smaller region of spacetime, causing it to become more dense. If the radius coincides with the event horizon the object is reduced to a black hole.

Maintaining hydrostatic equilibrium, and circumstances which allow this are of importance for a solution to be viable. In order to determine these circumstances, the radial value at which hydrostatic equilibrium is lost and gravitational collapse occurs needs to be studied further. There have been many results that show the Schwarzschild solution subject to particular conditions, may get rather close but will not reach r_s. Therefore it will not experience gravitational collapse provided that the restrictions are satisfied. Section 1.5 will discuss some of these results in more detail, but first we will discuss an important result on maintaining hydrostatic equilibrium.

1.3.2 Maintaining hydrostatic equilibrium

A star will remain in hydrostatic equilibrium provided that its gravitational force which points inwards balances against its outward pointing internal pressures. Given a particular solution, the Tolman-Oppenheimer-Volkoff equation [50] states a relationship between the mass, density and pressure which will allow the star to remain in hydrostatic equilibrium. In this section, we will derive this result, which is valid for a static, spherically symmetric solution describing for instance a neutron star in Schwarzschild coordinates (1.1.1). With the perfect fluid energy-momentum tensor for an isotropic pressure, in the absence of charge $\mathcal{T}_{\mu\nu} = \big(\rho + p\big)u_\mu u_\nu + p g_{\mu\nu} = \mathrm{diag}\big(-\mathrm{e}^a \rho, \mathrm{e}^b p, r^2 p, r^2 \sin^2\theta\, p\big)$. Computing the field equations yields

$$8\pi\rho = \frac{b'\mathrm{e}^{-b}}{r} + \frac{1}{r^2}\Big(1 - \mathrm{e}^{-b}\Big)\,, \tag{1.3.1}$$

$$8\pi p = \frac{a'\mathrm{e}^{-b}}{r} - \frac{1}{r^2}\Big(1 - \mathrm{e}^{-b}\Big)\,, \tag{1.3.2}$$

$$8\pi p_\perp = \frac{\mathrm{e}^{-b}}{2}\left(a'' + \Big(\frac{a'}{2} + \frac{1}{r}\Big)\big(a' - b'\big)\right)\,, \tag{1.3.3}$$

for this calculation the pressure is assumed to be isotropic, so that $p = p_\perp$ and the last two equations are required to be equal. This gives rise to the isotropy condition $\frac{\mathrm{e}^{-b}}{2}\Big(a'' + \frac{a'}{2}\big(a' - b'\big) - \frac{1}{r}\big(a' + b'\big)\Big) + \frac{1}{r^2}\big(1 - \mathrm{e}^{-b}\big) = 0$, which means only two independent field equations remain. Taking the derivative of the expression for p given by (1.3.2) with respect to r gives

$$8\pi p' = \frac{\mathrm{e}^{-b}}{r}\left(a'' - a'b' - \frac{1}{r}(a' + b')\right) + \frac{2}{r^3}\big(1 - \mathrm{e}^{-b}\big)\,, \tag{1.3.4}$$

while the field equations can be manipulated to construct the following equations

$$\begin{aligned} 0 =& \frac{\mathrm{e}^{-b}}{r}\left(a'' + \frac{a'}{2}\big(a' - b'\big) - \frac{1}{r}\big(a' + b'\big)\right) + \frac{2}{r^3}\big(1 - \mathrm{e}^{-b}\big)\,, \\ 4\pi a'(\rho + p) =& \frac{a'\mathrm{e}^{-b}}{2r}(a' + b') \\ =& -\frac{\mathrm{e}^{-b}}{r}\left(a'' - a'b' - \frac{1}{r}\big(a' + b'\big)\right) - \frac{2}{r^3}\big(1 - \mathrm{e}^{-b}\big)\,, \end{aligned}$$

the former is obtained by subtracting (1.3.2) from (1.3.3), this combination is the isotropy condition which is then multiplied by $2/r$. The latter is given by adding (1.3.1) to (1.3.2) and multiplying by $a'/2$, then the former expression is utilised to rewrite the right-hand side of the latter. The expression for $\rho + p$ can now be substituted into equation (1.3.4) which yields

$$p' = -\frac{a'}{2}\left(\rho + p\right) , \tag{1.3.5}$$

this is a conservation equation for the pressure. Note that the general conservation equation $\nabla^{\nu}\big(G_{\mu\nu} - 8\pi\mathcal{T}_{\mu\nu}\big) = 0$ for this static solution reduces to $\nabla^{r}\big(G_{rr} - 8\pi\mathcal{T}_{rr}\big) = \mathrm{e}^{-b}\partial_r\Big(\frac{a'}{r} - \frac{1}{r^2}\big(\mathrm{e}^{b} - 1\big) - 8\pi\mathrm{e}^{b}p\Big) = 0$. This coincides with equation (1.3.4) when expanded out, then following the same steps as above leads to equation (1.3.5). We now require an expression for a' in the interior, which can be obtained by firstly rewriting the (t,t) field equation

$$8\pi r^2\rho = \frac{\mathrm{d}}{\mathrm{d}r}\Big(r - r\mathrm{e}^{-b}\Big) ,$$

and integrating with respect to r, this then leaves an equation which can be solved for b in the interior

$$\mathrm{e}^{-b} = 1 - \frac{1}{r}\int_0^r 8\pi\tilde{r}^2\rho\,\mathrm{d}\tilde{r} = 1 - \frac{2m}{r} , \tag{1.3.6}$$

where we have used the definition of the mass given in equation (1.2.1), which states

$$m(r) = \int_0^r 4\pi\tilde{r}^2\rho\,\mathrm{d}\tilde{r} ,$$

for $0 < r \leq R$ and the total mass M is given by the integral evaluated at the boundary $r = R$, hence $m(R) = M$. Now this can be inserted into the (r,r) field equation, (1.3.2), to obtain an expression for a'

$$a' = \frac{2m + 8\pi r^3 p}{r^2 - 2mr} , \tag{1.3.7}$$

which is then substituted into the equation for p' given by equation (1.3.5), this yields

$$p' = -\frac{m + 4\pi r^3 p}{r^2\big(1 - \frac{2m}{r}\big)}\Big(\rho + p\Big)\,. \tag{1.3.8}$$

Equation (1.3.8) is the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium for a static, spherically symmetric solution (1.3.3). Note in the Newtonian limit, when $m \ll r\,, p \ll \rho$, the equation of hydrostatic equilibrium is given by $p' = -\frac{m}{r^2}\rho$. The TOV equation follows directly from the field equations and can be generalised further for an anisotropic pressure, so that $p \neq p_\perp$. We will also see that anisotropy of the energy-momentum tensor affects the mass-radius ratio, which is obviously due to the additional independent (θ, θ) field equation involving $p_\perp$. This in turn gives rise to additional values appearing in the conservation equation, the modified equation for hydrostatic equilibrium was generalised for the anisotropic pressure in 1973 [51]. This result will be generalised further for the Reissner-Nordström de Sitter solution with anisotropic pressure in chapter 3. A similar method will be utilised to derive a conservation equation in the modified gravity framework in chapter 4 and [3].

1.4 Interior solutions

So far, we have seen numerous exterior solutions with a wide range of properties, including the Schwarzschild, Kottler and Nariai solutions along with various charged analogues. In studying such objects, we are also interested in modelling the interior spacetime. The interiors need to be matched at the boundary to their exterior counterparts using the correct matching conditions [52–55], the boundary $r = R$ forms a hypersurface which is referred to as the matching hypersurface. At this matching hypersurface the metrics must take the same value, also both the interior and exterior metrics are required to be at least C^1. A metric is C^k if the first k derivatives are continuous in the required region. Before the solutions can be matched we require that the interior is regular and hence singularity free, in particular the metric should be finite at the centre $r = 0$ with finite energy density and pressure throughout the interior. In addition to the first fundamental forms which are given by the metric coefficients, the second fundamental forms must also agree at the hypersurface [52–54], this will be discussed in more detail in section 3.5.2. Alternatively, both the interior and exterior metrics can be rescaled into Gauss coordinates, so that the coefficient of the transformed radial coordinate in the metric is set to unity, then it must be shown that the metrics are continuous and differentiable at the boundary or hypersurface [55]. See section 3.5.2 for a comparison of matching methods with the interior and exterior charged Nariai solutions, or [1] for the matching with the latter method in more detail. Some solutions assume the boundary to be the vanishing pressure hypersurface $p(R) = 0$ in order to match [55], which we will see more of in the discussion below.

In 1916, Schwarzschild developed the first interior solution which matched to the exterior vacuum Schwarzschild solution at the boundary, where both metrics were C^1 [10]. This was an incompressible fluid with the feature of a constant energy density ρ. For a further investigation on Schwarzschild's interior as an incompressible fluid see [56], the solution was later redeveloped in isotropic coordinates [57]. Further analysis of the Schwarzschild interior was carried out in 1975, it was shown that this interior is conformally flat and is the only static and spherically symmetric perfect fluid solution of Einstein's field equations to have this property [58]. A spacetime manifold $\mathcal{M}$ is conformally flat if for every neighbourhood about a point on $\mathcal{M}$ there is a conformal mapping to Minkowski spacetime, that is the spacetime metric of $\mathcal{M}$ can be expressed as a multiple of the Minkowski metric $g_{\mu\nu} = \alpha\eta_{\mu\nu}$ for some scalar α, such a conformal map preserves angles. It was also shown that the homogeneous limit of the Schwarzschild interior yields the Einstein static or de Sitter universe [58], in the homogeneous limit we restrict the pressure $p(r) =$ constant.

In 1939, Tolman developed several interior solutions, where hydrostatic equilibrium was imposed with various constraints, these solutions were matched to the relevant exterior [50]. Among the new solutions developed in [50], a particular model was shown to give rise to either the Einstein universe, Schwarzschild interior or Kottler solutions under the correct constraints. The remaining solutions and their properties have been studied further and generalised in for instance [59]. The Whittaker solutions [60] were developed in 1968, this abandoned the concept of constant energy density solutions and instead imposed $\rho + 3p =$ constant, which allowed for a variable ρ and p.

The interior solutions discussed in this section so far have an isotropic pressure, this means it is assumed that the radial and tangential pressures are equal $p = p_\perp$. It was realised in the 1970's [51] that solutions with anisotropic pressure, $p \neq p_\perp$ provide more realistic stellar models, which generated interest [61, 62]. In [63], some known interior Schwarzschild solutions were generalised to accommodate $p_\perp$, among the anisotropic pressure solutions was a Schwarzschild-like model and various Tolman-like solutions. These solutions are derived from the TOV-like conservation equation which is generalised for an anisotropic pressure. In 1973, Florides introduced a new class of Schwarzschild interior solutions with a regular centre and anisotropic pressure [64]. This solution was matched to the exterior Schwarzschild solution, in order to do this, Florides assumed the existence of a vanishing pressure hypersurface at the boundary. In fact, this solution had the distinguishing property of vanishing radial pressure throughout the interior, although the tangential pressure and energy density were allowed to vary. Koftini also introduced a Florides style interior solution [65]. In this solution, the matching conditions required at the boundary did not constrain the solutions since the radial value was allowed to be arbitrarily close to the Schwarzschild radius r_s. A generalisation of the Florides interior solution, in the presence of a cosmological constant [66], was obtained in 1986. The Florides solution, and its generalisations are of particular importance for some of the work carried out in this thesis, in particular a similar method is applied in [1] to obtain the interior charged Nariai solution, see chapter 3 for more details.

Interior solutions which can be embedded within cosmological models are also of importance. Analogously to the Kottler solution, McVittie combined the Schwarzschild solution in isotropic coordinates with the FLRW solution in 1933, this is known as the McVittie solution and describes an interior of a matter distribution embedded in an expanding universe [67], for a more detailed analysis of this solution see for instance [6,68]. However the interior Kottler solution was derived much later [69, 70]. This interior solution was among various constant and non-constant density solutions in the presence of a cosmological constant derived in [70] and [71] respectively. Also accompanying the eleven new solutions in [70] was the interior Nariai solution, which was matched to its exterior [30] thus describing the entire Nariai spacetime.

1.4.1 Charged interior solutions

It is clear from the field equations of general relativity that the exterior of a charged matter distribution will not be vacuum, this is due to the presence of electromagnetic energy. In addition to this, the presence of electromagnetic energy affects the interior since the gravitational and electromagnetic forces are required to be in equilibrium in this region. The Reissner-Nordström exterior provides some information on the nature of the relationship between M, Q and the effects on hydrostatic equilibrium. However either the mass or charge can be set to zero without affecting equilibrium in the exterior solution, but this is not always possible in the interior [72]. Constructing an interior with the internal gravitational and electromagnetic forces in equilibrium will tell us more about the required relationship between the parameters. The search for interior charged solutions began to gather more interest from the 1960's, after a charged solution with an infinite mass singularity was obtained by Papapetrou in 1947 [73]. For several decades many interior solutions, with various defining properties and symmetries, were matched to the Reissner-Nordström exterior at the boundary where both metrics are at least C^1. This section will be devoted to a brief overview of the timeline and induced properties of such solutions, and their de Sitter analogues.

Bonnor [72] derived some interior, spherically symmetric charged solutions in 1960, including a charged dust sphere with vanishing pressure throughout the interior. It was shown with a particular model, that a solution with vanishing mass $m = 0$ is only possible if the matter density ρ is negative, and concluded that the electromagnetic energy made some small contributions to the overall gravitational mass [72]. Another regular, spherically symmetric charged dust interior solution was obtained by Bonnor several years later. This remained in equilibrium subject to the assumption that the matter density was equal to the charge density $\sigma = |\rho|$ [74], in both cases the solutions were C^1 at the boundary where they were matched to the exterior Reissner-Nordström spacetime and it was shown that for very large masses the electric repulsion can balance the gravitational pull and prevent gravitational collapse. Effinger, in 1965, constructed a charged interior solution, however this solution contained a singularity at the origin $r = 0$ [75]. This singularity was removed in 1967 by Kyle's constant density solution [76], and separately by Wilson [77]. Although it was only shifted to another point in spacetime in both solutions, the required restrictions for the spacetime metric to avoid these singularities were outlined. The following year, Cohen [78] developed a static, spherically symmetric charged interior solution that satisfied the conditions to be matched to the Reissner-Nordström exterior. Similarly to Bonnor's conclusion [72] Cohen deduced that in this solution, the charge density generated mass, which can be seen in equation (1.2.2). Bailyn subsequently obtained an interior solution with the gravitational and electromagnetic forces in equilibrium, which again like Bonner's solution resulted negative matter density when Q was much larger than M to attain this equilibrium [79]. It was concluded that this implied that matter cannot exist without charge since the overall energy density $\mathrm{d}m_g/\mathrm{d}r|_{r=R}$, with the electric contribution from (1.2.2), is positive similarly to [78]. Krori and Barura, in 1975, obtained a singularity free, static and spherically symmetric spacetime with non-zero charge, this was matched to the Reissner-Nordström exterior [80]. In [81], several charged interiors which match to the exterior Reissner-Nordström solution were obtained, and a distinction was made between this analysis and the solutions in [75–77,80]. Buchdahl analysed the previous paper in [82], and stated that these solutions were derived without imposing any physically reasonable assumptions which leads to certain problems. Physical properties in the context of charged solutions includes an equation of state along with equations relating the matter and charge densities, Buchdahl considered the latter in more detail to derive interior solutions in [82].

In 1977, Cooperstock highlighted that most Reissner-Nordström interior solutions were developed mathematically without considering physical interpretations or restrictions [83], and developed various alternative solutions. Amongst the solutions was a charged analogue of the interior Schwarzschild incompressible fluid and a solution that assumed $\rho + q^2/r^2 = \text{constant}$ (where $q = q(r)$ is the charge in the interior), a similar method will be used for the interior charged Nariai solution derived in chapter 3 and [1]. Florides derived the charged analogue of his original interior solution [84] also in 1977, where a charged dust solution with vanishing pressure is provided as a special case. Following this, Florides obtained several charged solutions with various features [85], in particular the method used in [83] was adapted.

Still in the pursuit of a physically consistent interior charged solution, in 1979, Mehra obtained a singularity free spherically symmetric spacetime with constant energy density [86]. The following year, in 1980, Mehra [87] then stated that a problem with some previous solutions [72, 75–77, 80] arises since the matter density is non-zero at the boundary, and derived a regular interior which satisfies this with $\sigma = \text{constant}$. Then, in 1992, Mehra developed another static, spherically symmetric class of interior solutions with maximum matter density in the centre, which decreased as the radius increased and eventually vanished at the matching surface [88]. In this solution, the charge density σ was not restricted to be constant. The charged interiors considered above are not conformally flat, Shi-Chang considered the results in [58] and derived singularity free, conformally flat solutions for various mass densities in [89]. Following this Tiwari *et al* derived an interior metric with the condition $g_{tt}g_{rr} = -\mathrm{e}^{a+b} = -1$, the mass in this solution was shown to have electromagnetic contributions only [90]. A regular interior solution with finite pressure that matched to the exterior Reissner-Nordström solution was found by Xingxiang [91], the solution was not completely fixed and under the correct conditions reduced to either [83, 87, 89]. Recently, a charged analogue of the Whittaker solution was obtained and was matched to the Reissner-Nordström exterior at a vanishing pressure hypersurface, this solution again imposed $\rho + 3p =$ constant [92] like its neutral counterpart. However the energy density increases as r moves away from the origin towards the boundary contrary to the results in [87, 88], which is largely considered an non-physical solution.

All the solutions [72–81, 83–89, 91, 92] exhibit spherical symmetry and are static, various solutions which did not impose such constraints were also constructed during this period. A collection of static charged interiors were examined by Das in 1962, with no spatial symmetries assumed. It was concluded that the solutions only remain static if the electromagnetic and energy densities are equal in magnitude [93]. Non-static, charged interiors were then constructed almost a decade later by Bekenstein [94], this analysis showed that the non-static interior must be matched to the static Reissner-Nordström exterior. In 1968, Raychaudhuri derived charged dust distributions with various charge and mass densities which remain in hydrostatic equilibrium, a result showed that the equilibrium of the solutions may be lost if spherical symmetry is assumed. In developing the solutions, various relationships were imposed between the configuration of matter and charge in order to maintain hydrostatic equilibrium, this construction lead to the derivation singularity free spacetimes [95].

Equilibrium of the gravitational and electromagnetic forces for the charged counterparts have been an issue, initiated in 1962 when Dirac contemplated the stability of the electron. In 1964, Bonnor [74] discussed hydrostatic equilibrium of a charged solution, results showed that in particular models the electric repulsion can counteract the gravitational field exerted by large masses and stop gravitational collapse. On the other hand, Bekenstein [94] also considered hydrostatic equilibrium and gravitational collapse in 1971, and concluded that the generalised charged analogue of the TOV conservation equation may not apply to neutron stars. In [94] it was also stated that the conservation equation does not imply that the gravitational attraction and electric repulsion will remain in equilibrium, which opposes the results in [74].

Whilst many contributions were being made to modelling the interior of a charged matter distribution, which are C^1 at the boundary and can be matched to the exterior Reissner-Nordström solution, some results embedded such solutions in an expanding universe. Bertotti [96] and Robinson [97] considered a similar model in 1959, this solution described a uniform magnetic field in the presence of a cosmological constant, however it was stated by Bertotti that the particular model may not have cosmological applications [96]. In 1966, Vaidya and Shah obtained a charged particle in an FLRW expanding universe [98], which can be considered as the charged generalisation of the McVittie solution. A decade later, Bohra and Mehra introduced a cosmological solution in the form of charged dust and radiation, they found that matter has a positive effect on curvature and charge has a negative effect [99]. However, these solutions do not describe the interior of the Reissner-Nordström de Sitter solution, and to the best of my knowledge no such solutions have been found other than the solutions we derive in [1]. A discussion of these solutions will be provided in chapter 3.

1.5 Bounds on M/R

There have been many results devoted to the study of solutions which remain in hydrostatic equilibrium, particularly since the discovery of neutron stars and the concept of black holes became more accepted in general relativity [40, 41, 46, 47]. An area of particular interest is determining what prevents an already extremely dense compact object, such as a neutron star, collecting more mass and eventually becoming a black hole. A neutron star also requires that the gravitational force and internal pressures remain in hydrostatic equilibrium, and as we have seen in equation (1.3.8) this can be expressed as a relationship between the mass, radius and internal pressures for particular solutions modelling a neutron star. Equivalently, given a particular solution, it is possible to develop some conditions which imply the solution will not reach its event horizon, this is shown by deriving a lower bound on the total radius R. This lower bound will be larger than the event horizon, and shows that R will not reduce to the event horizon provided that the conditions are satisfied. For example it was shown that the Schwarzschild interior in hydrostatic equilibrium, which satisfies particular conditions such as constant energy density and isotropic pressure, is bounded by $R \geq \frac{9}{8} r_s$ [10]. This result was initially derived by Schwarzschild which was only valid for a solution that is an incompressible fluid. Note that the bound $R \geq \frac{9}{8} r_s$ is often rewritten as an upper limit on M/R. In 1959 the result was derived based on different conditions in Buchdahl's theorem [100], which extended the result to hold for a compressible fluid. The conditions imposed and the derivation of the bound will be discussed further in the next section 1.5.1. It has also been shown that in isotropic coordinates, the upper bound on M/R is smaller [57] than that found by Schwarzschild and Buchdahl.

Deriving such ratios will enable us develop an accurate model of a neutron star, this can be used to categorise such compact objects in terms of their mass. It was stated in [101] that this will allow us to detect black hole candidates, since if the mass of an observed dense object exceeds that of a neutron star it can be distinguished from a compact object and hence classified as a black hole, see [101]. Additionally, it has been shown that upper and lower bounds on M/R can be translated into bounds on the gravitational redshift z resulting from the star [100–102].

In Newtonian gravity, the Chandrasekhar limit provides an upper bound for the mass of a stable white dwarf star subject to necessary conditions in [103]. If any of the restrictions are violated and the mass exceeds this bound, gravitational collapse will occur and the result will be a denser compact object. It was the following year, in 1932, that Baade and Zwicky [46] and Landau [47] suggested the existence of a neutron star which is categorised as a compact object. The TOV equation found in 1939 [50], can be used to derive a bound on M/R for such relativistic neutron stars. This bound is analogous to the Chandrasekhar limit for a white dwarf, since it provides the necessary conditions for a neutron star to remain stable. The TOV and Chandrasekhar limits are given in terms of solar masses.

From 1965, it was discovered that nuclear forces need to also be considered when checking hydrostatic equilibrium [104] and abandoning these forces will alter the value for which equilibrium is maintained or therefore lost. However, neglecting these forces in a calculation can still provide useful information on maintaining hydrostatic equilibrium and avoiding gravitational collapse as outlined in [38]. In order to derive a bound that implies hydrostatic equilibrium, an appropriate model of a stellar object is required, whilst we can neglect nuclear forces, we need to have physically viable models. For example, models of compact objects with a constant energy density or those assumed to have most of its density concentrated in the core describe physically allowed solutions, whereas a solution whose density increases as the boundary is approached is largely considered non-physical. Following one of the former assumptions, in [105] an upper bound on the mass-radius ratio was derived, but constrained to the mass and radius of spherical regions of the stars, and in particular this region was restricted to be the core of the star.

An important quality in deriving these bounds is ensuring that they are optimal, or sharp, so that at least a particular solution will saturate the inequality. For example in the Schwarzschild solution, an appropriate bound could be $R \geq r_s$ which was shown in [106, 107]. But there are no solutions which saturate this inequality, other than solutions with the radius coinciding with its event horizon and hence describe black holes. Therefore the inequality can be made more accurate such as the bound provided above by Schwarzschild and Buchdahl [10, 100].

When extra parameters are considered such as charge, or the cosmological constant, there is an additional event horizon, hence the mass-radius bounds and equations for hydrostatic equilibrium will need to be redeveloped to include more information. Therefore, constraints on M/R have been studied for other solutions, for instance a bound was developed for the Krori-Barua solution which is an interior of the Reissner-Nordström solution [108]. Extensions of the Buchdahl limit will be considered in the presence of the cosmological constant or charge, and will be discussed further at the end of this chapter 1.5.2.

The majority of research devoted to finding bounds on M/R is restricted to providing upper bounds for compact objects in hydrostatic equilibrium, where exceeding the limit means equilibrium is broken by the internal gravitational force overcoming its internal pressures. The Eddington limit or Eddington luminosity provides a bound which if exceeded means nuclear fusion has broken hydrostatic equilibrium, as opposed to gravity overcoming the internal stresses [109]. Additionally, minimum bounds on M/R have been considered [110–113] and mass-radius bounds have been studied on non-compact objects in [114].

As we can see, limits on M/R apply to a stellar object at different stages in its life cycle in order for its internal forces to remain in equilibrium and avoid becoming compact object, and such limits are useful for studying other aspects of the object. In this thesis, we will concentrate on upper bounds on M/R, this will be studied in various contexts, including in the RNdS solution which will be discussed in chapter 3 and [1, 2], and also will be considered in an alternative theory of gravity in chapter 4 . In the next two sections, we will show some previous and pioneering results beginning with a result from Buchdahl.

1.5.1 Buchdahl's theorem

In 1959, Buchdahl generalised an existing result on maintaining hydrostatic equilibrium in [100], where he derived an upper bound in which the mass must satisfy for its internal forces to remain in equilibrium, $M/R \leq 4/9$. This had previously only applied to the Schwarzschild interior solution [10]. Buchdahl's theorem states that a neutral spherically symmetric perfect fluid solution of Einstein's field equation will also satisfy the bound $M/R \leq 4/9$. This holds for any solution with energy density $\rho \geq 0$ and the isotropic pressure $p_\perp = p \geq 0$. It is also required that ρ is non-increasing outwards which means it is a decreasing function of r, in terms of its derivative $\rho_r \leq 0$ in the interior. A solution satisfying these conditions is considered to be in hydrostatic equilibrium, and provided the restrictions are not broken, the radius R will remain larger than the Schwarzschild radius $r_s = 2M$ or more precisely $R \geq \frac{9}{8} r_s$. Moreover, this result somewhat restricts the equation of state which relates ρ and p.

In order to derive Buchdahl's result we will follow the relabelling of variables outlined in [100] whilst bearing in mind the results obtained in section 1.3.2. Consider the following relabelling of variables $x = r^2$, $y = \mathrm{e}^{-b/2}$, $z = \mathrm{e}^{a/2}$ and $\omega = \frac{4\pi}{3}\hat{\rho} \equiv m/r^3$, using this we can write the combination $\mathrm{e}^{-b} = 1 - 2m/r$ as $y^2 = 1 - 2x\omega$ provided that the interior is regular. Next we rewrite the (r, r) field equation given by equation (1.3.2) and ρ by taking the derivative of equation (1.2.1) in new variables

$$
\begin{aligned}
4\pi p &= \frac{2z_x}{z}\Big(1 - 2x\omega\Big) - \omega \, , \\
4\pi\rho &= \frac{m_r}{r^2} = 2x\omega_x + 3\omega \, ,
\end{aligned}
$$

where the latter is obtained using the definition of the mass. Now equation (1.3.5) can be written in terms of new variables

$$
4\pi p_x = -\frac{2z_x}{z}\left(x\omega_x + \omega + \Big(1 - 2x\omega\Big)\frac{z_x}{z}\right) ,
$$

additionally the expression for p in new variables can be used to eliminate p_x, first we take its derivative with respect to x

$$\begin{aligned}4\pi p_x &= \frac{2}{z}\left(y^2 z_{xx} + 2yy_x z_x - \frac{y^2 z_x^2}{z}\right) - \omega_x \\ &= \frac{2}{z}\left(\left(1-2x\omega\right)z_{xx} - 2\left(x\omega_x + \omega\right)z_x - \left(1-2x\omega\right)\frac{z_x^2}{z}\right) - \omega_x\,.\end{aligned}$$

Putting the two expressions for p_x together and cancelling the relevant terms yields

$$\left(1-2x\omega\right)z_{xx} - \left(x\omega_x + \omega\right)z_x - \frac{z\omega_x}{2} = 0\,. \tag{1.5.1}$$

This equation can be simplified further by considering the function α defined by the relation

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} = \frac{1}{\sqrt{1-2x\omega}} = \frac{1}{y}\,,$$

now derivatives with respect to x can be written as $z_x = \alpha_x z_\alpha$ thus $z_{\alpha\alpha} = y^2 z_{xx} - yy_x z_x$. With this equation (1.5.1) becomes

$$z_{\alpha\alpha} = \frac{z}{2}\omega_x\,. \tag{1.5.2}$$

Since we have assumed that ρ decreases with r so that $\rho_r \leq 0$, this implies the mean density $\hat{\rho}$ is such that $\hat{\rho}_r \leq 0$. Therefore $\omega_r \leq 0$ from the definition of ω, and thus ω also decreases with r. Hence the value of ω at the origin $r = 0$ will be larger than the value at the boundary $r = R$, then for $0 \leq r \leq R$ we have $\omega(0) \geq \omega(r) \geq \omega(R)$ and thus $\frac{w(R)}{w(r)} \leq 1$. This implies the following relationship

$$y^2 = 1 - 2x\omega(r) \leq 1 - 2x\omega(R)\,. \tag{1.5.3}$$

The restriction $w_r \leq 0$ also yields a relationship between p and ω, namely if $\rho \geq 3\gamma^{-1}p$ for some constant γ, this can be written as $r\omega_r + 3\omega \geq 3\gamma^{-1}p$. Thus the constraint $w_r \leq 0$ implies

$$\gamma\omega \geq p\,.$$

Additionally, due to the positivity of $z = \mathrm{e}^{a(r)/2}$ and negativity of ω_r, equation (1.5.2) implies $z_{\alpha\alpha} \leq 0$. This means z_α decreases as a function of α, and again the value of z_α is larger at the origin $r = 0$ than at the boundary $r = R$ hence

$$\left.\frac{\mathrm{d}z}{\mathrm{d}\alpha}\right|_{r=0} \geq \frac{\mathrm{d}z}{\mathrm{d}\alpha} \geq \left.\frac{\mathrm{d}z}{\mathrm{d}\alpha}\right|_{r=R} ,$$

where $\frac{\mathrm{d}z}{\mathrm{d}\alpha} = \frac{yz_r}{2r}$. At the boundary R and for all $r \geq R$ the solution becomes the Schwarzschild exterior so that $z = \mathrm{e}^{a/2} = \sqrt{1 - \frac{2M}{r}}$, where $M = m(R)$ is the total mass. Then in particular at the boundary $\frac{\mathrm{d}z}{\mathrm{d}\alpha}\big|_R = \frac{M}{2R^3} = \frac{\omega(R)}{2}$, therefore

$$\int_0^R \mathrm{d}z \geq \frac{1}{2}\int_0^R \omega(R)\mathrm{d}\alpha = w(R)\int_0^R \frac{r\mathrm{d}r}{y} ,$$

where the left hand side of this inequality is simply $z(R) - z(0) = \sqrt{1 - \frac{2M}{R}} - z(0) \leq z(R)$. Using equation (1.5.3) the right hand side becomes

$$\omega(R)\int_0^R \frac{r\mathrm{d}r}{y} \geq \frac{\omega(R)}{y(R)}\int_0^R r\mathrm{d}r = \frac{1 - y(R)^2}{2y(R)} \geq \frac{1 - y(R)^2}{2(1 + y(R))} = \frac{1}{2}\big(1 - y(R)\big) .$$

Putting this together, the inequality reduces to

$$\sqrt{1 - \frac{2M}{R}} = y(R) \geq \frac{1}{2}\big(1 - y(R)\big) , \tag{1.5.4}$$

or equivalently $y(R) \geq \frac{1}{3}$ which yields the result $1 - \frac{2M}{R} \geq \frac{1}{9}$. For a full discussion of the proof and result see [100].

1.5.2 Beyond Buchdahl's theorem

In response to Buchdahl's work, Bondi provided an extension in 1964 [115], in which he studied some circumstances which allow the radius to go below $\frac{9}{4}M$ as it decreases so that it approaches and eventually reaches r_s, although it was shown that in general Buchdahl's bound is satisfied.

It was later pointed out that the conditions imposed in Buchdahl's theorem are rather restrictive due to the assumptions on the energy density ρ and since the tangential pressure $p_\perp$ is not allowed to differ from the radial stresses. The assumptions were then relaxed to allow for $p_\perp \leq p$ and $p_\perp > p$ separately, the former gives a result similar to that of Buchdahl whereas the latter differs more significantly whilst remaining above the Schwarzschild radius [116].

Since these restrictions have been highlighted, the mass radius ratio has been generalised many times under various assumptions for various solutions. From 2000, the charged [117] and de Sitter [118] analogues of Buchdahl's theorem and hence the gravitational redshift limits were obtained, each corresponding to the Reissner-Nordström and Schwarzschild de Sitter interiors respectively. Several years later, it was generalised further for anisotropic objects in de Sitter space [119]. This bound explicitly relied on the difference between the tangential and radial stresses $p_\perp - p$ at the surface $r = R$. During the same period an alternative method for obtaining the Buchdahl-like upper bound on M/R was used for isotropic, constant density [70] and non-constant density [71] perfect fluids in the presence of a cosmological constant. It was later shown that the limit obtained, which utilized the central pressures approach, can be used to bound the cosmological constant [120].

Buchdahl's restrictions were then relaxed further in 2007 to obtain a sharp bound on M/R, the new conditions only require that the energy density and radial pressure are positive with $p \neq p_\perp$ and satisfy $p + 2p_\perp \leq \Omega\rho$ for some constant Ω. Consequently, the resulting inequality differs to that of Buchdahl's, however $\Omega = 1$ yields that particular result [121].

Following this revival of Buchdahl type inequalities and their importance in general relativity, using a different method some sharp bounds were obtained for M/R given various conditions on the energy density ρ and pressures p and $p_\perp$ [122]. This analysis covered results presented in [100, 121], a specific result outlined in this paper is of importance for various calculations in this thesis and the method will be outlined below.

Consider the relabelling of variables $x = 1 - \mathrm{e}^{-b}$ and $y = 8\pi r^2 p$ then the definition of the mass yields $x = 2m/r$ and the conditions on ρ and p imply that the new variables belong to the set

$$\mathcal{U} = \Big\{(x, y) : \, 0 \leq x < 1, y \geq 0\Big\} \,.$$

The first two field equations (1.3.1) and (1.3.2) can now be written in new coordinates

$$\begin{aligned} 8\pi r^2 \rho =& 2\dot{x} + x \,, \\ 8\pi r^2 p =& y = 2\dot{a}\big(1 - x\big) - x \,, \end{aligned}$$

where $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\beta}$ and $\beta = 2\ln r$, so that for instance $2\dot{x} = rx'$. The (r, r) field equation defined to be equal to y can be used to obtain $\dot{a}$

$$\dot{a} = \frac{x+y}{2(1-x)}\,,$$

taking the derivative of this with respect to β yields a first order expression for $\ddot{a}$. Using this, the third field equation (1.3.3) can also be transformed into a first order equation

$$\begin{aligned} 8\pi r^2 p_\perp =& (1-x)\left(2\ddot{a} + \dot{a}\left(\dot{a} - \frac{\dot{x}}{1-x}\right) - \frac{\dot{x}}{1-x}\right) \\ =& \frac{\dot{x}\big(x+y\big)}{2\big(1-x\big)} + \dot{y} + \frac{\big(x+y\big)^2}{4\big(1-x\big)}\,. \end{aligned}$$

Then writing the inequality $p + 2p_\perp \leq \rho$ in new variables leaves

$$\begin{aligned} \dot{x}\Big(3x + y - 2\Big) + 2\dot{y}\big(1-x\big) \leq& -\frac{1}{2}\Big(3x^2 + y^2 - 2\big(x-y\big)\Big) \\ =& -\frac{1}{2}u(x,y)\,. \end{aligned}$$

The next step is to find a function $w(x,y)$ of the form $w = \gamma^2/(1-x)$ such that

$$\begin{aligned} \dot{w} =& \frac{\gamma}{(1-x)^2}\left\{\dot{x}\Big(2\gamma_x(1-x) + \gamma\Big) + 2\dot{y}\gamma_y(1-x)\right\} \\ =& \frac{\gamma}{(1-x)^2}\left\{\dot{x}\Big(3x + y - 2\Big) + 2\dot{y}(1-x)\right\}, \end{aligned}$$

where $\gamma = \gamma(x,y)$ can be found by comparing coefficients and solving the resulting differential equations. The function that satisfies this is

$$w = \frac{(4-3x+y)^2}{1-x}\,, \tag{1.5.5}$$

which is a specific case of the function derived in [122]. By deriving the optimal value of w, this can be rearranged and translated into an upper bound on $2M/R$, therefore we refer to w as the optimisation function. To proceed we put the expressions for $\dot{w}$ and $p + 2p_\perp \leq \rho$ together which yields

$$\dot{w}(x,y) \leq -\frac{4-3x+y}{2(1-x)^2}\,u(x,y)\,, \tag{1.5.6}$$

the sign of u is fixed based on the sign of its prefactor $4 - 3x + y$, which is positive since $0 \le x < 1$ and $y \ge 0$. The function w is required to be increasing, and we assume w has a maximum value so that $w \le \sup_{\mathcal{U}} w$. Therefore, since the aforementioned combination is positive we must maximise w given the constraints on x and y plus $u \le 0$. Here $u \le 0$ implies

$$0 \ge 3x^2 + y^2 - 2(x - y) = (3x + 1)(x - 1) + (y + 1)^2 \,,$$

this condition can be used to write down the following two conditions

$$\begin{aligned} (y+1)^2 &\le (3x+1)(1-x) \,, \\ 2y \le (3x+1)(1-x) - y^2 - 1 &\le (3x+1)(1-x) - 1 \,. \end{aligned} \tag{1.5.7}$$

The first equation in (1.5.7) can now be used to write w as

$$\begin{aligned} w = \frac{(4 - 3x + y)^2}{1 - x} &= \frac{(y+1)^2}{1-x} + 6(y+1) + 9(1-x) \\ &\le 3x + 1 + 6(y+1) + 9(1-x) = 16 - 6(x - y) \,, \end{aligned}$$

using the second relation in (1.5.7) to replace $6y$, this then reduces to $w \le 16 - 9x^2 \le 16$. The maximum value is therefore $\sup_{\mathcal{U}} w = 16$ where this value is attained at $x = y = 0$, from this the bound on M/R can be derived. Rearranging the combination $w(x, y) \le 16$ by collecting the terms involving x leads to

$$\left(2 - 3\sqrt{1-x}\right)^2 \le 1 \implies 2 - 3\sqrt{1-x} \le 1 \,,$$

or equivalently $\sqrt{1-x} \ge \frac{1}{3}$. Using $x = 2M/R$ at the boundary $r = R$, then rearranging gives the desired inequality $2M/R \le 8/9$.

This particular method has been extended and includes the Schwarzschild de Sitter [123] and Reissner-Nordström solutions [124] and relies on the condition $p + 2p_\perp \le \rho$ with the energy density ρ and the anisotropic pressures p, $p_\perp$ positive. These conditions imply that the dominant energy condition holds. The bound on M/R is then obtained as above, by representing this combination of the field equations as an optimisation problem for w and solving for the maximum.

The optimisation method outlined above [122–124] will be generalised to give an upper bound on M/R in the Reissner-Nordström de Sitter class of solutions in chapter 3 and [1]. Whilst transforming the corresponding field equations into an optimisation problem, it became clear that a particular subclass of solutions were excluded from this result due to the coordinate system used. These solutions are known as the charged Nariai solutions, and occur when the inner, outer and cosmological horizons are degenerate. The degenerate horizons manifest when the coefficient of the RNdS solution $\mathrm{e}^a = 0$, hence the a more suitable coordinate system is required. I developed an interest for this peculiar class of solutions and began to study its neutral counterpart, the Nariai solutions which are a subclass of the Kottler solutions. However the interior Nairiai does not match to the exterior Kottler spacetime. In deriving the charged analogues we utilised the central pressures approach used in [70, 71] to derive the same bound, but in this case it was valid for the charged Nariai solutions also, see chapter 3 and [2].

Chapter 2

Introduction to non-classical gravity

General relativity is often referred to as classical gravity to distinguish it from alternative or modified theories of gravity. There has been much research dedicated to developing an alternative theory that delivers the possibility of unifying gravity with the remaining fundamental forces, such as electromagnetism. Similarly, paralleling the development of cosmological theories, modifications have been applied to general relativity to incorporate our new understanding of the universe. This chapter will discuss both ideas, concentrating the former to the developments in the first few decades following Einstein's theory of relativity, therefore omitting alternatives to classical gravity which require quantum mechanics.

2.1 Alternative theories of gravity

Although alternative theories of gravity have received an increased amount of interest over recent decades, physicists and mathematicians have worked on such theories by making modifications to general relativity since its birth in 1915. Initially, such modifications were made in order to find a theory which unified the two fundamental forces known at that time, namely the gravitational and electromagnetic forces. The remaining two fundamental forces, known as the weak and strong nuclear forces, were not discovered until the revelation that the atom is made up of smaller particles [125] which inspired work by

Fermi in 1934 [126] and others thereafter. Although most of these attempts were unsuccessful at unifying the two forces, we will see that most endeavours at early unification introduced important concepts which remain prominent in parts of physics and mathematics we study today.

There was much disagreement on how to incorporate the electromagnetic field as an amendment to general relativity. In 1917, there were some ideas regarding an asymmetric metric [127], this change intended to describe electromagnetism alongside gravity, but this approach was quickly abandoned. Weyl was the first to introduce a method of unification that captured attention in 1918 [128]. The connection in this theory differed from the Levi-Civita connection since it was defined from the metric plus an additional vector quantity, which represent the gravitational and electromagnetic fields respectively. However, Einstein found some problems with the physical interpretation [127]. Despite this, the theory introduced gauge transformations which motivated the notion of gauge invariance and these concepts now provide the foundations for gauge theory.

Kaluza, in 1921 [129], introduced a five-dimensional theory in which the extra spatial dimension adhered to the cylindrical condition [127]. The cylindrical condition imposes that physical phenomena do not interact with the fifth dimension. That is if the indices X, Y denote five-dimensional coordinates with $X, Y = 0, 1, 2, 3, 4$ and 4 is the coordinate relating to the fifth dimension, the cylindrical condition becomes $\partial_4 g_{XY} = 0$. However, Einstein proved that spherically symmetric exact solutions did not exist in Kaluza's theory [130]. This five-dimensional theory is now known as Kaluza-Klein theory [131], where the additional dimension is compact as opposed to obeying the cylindrical condition [132] and the field equations separate into Einstein's and Maxwell's field equations. The extra dimension implies the metric now has fifteen components, of which four are required for the electromagnetic four-potential and ten for gravity, the remaining component corresponds to a scalar field. Kaluza-Klein theory remains an attempt to unify gravity with electromagnetism which continues to be studied today [133], and has set the foundations for higher dimensional theories which intend to unify the four fundamental forces, such as string theory.

During the same year Eddington developed an affine theory [134], which treated the metric and connection as independent variables in the construction of the field equations governing spacetime. Einstein initially favoured this method, but eventually realised the theory lacked physical applications and shifted to the opposing view held by Weyl and Pauli [135]. The approach of treating the connection and metric independently did not lead to a unified theory, but it is still utilized in the Palatini variational principle. This particular method is vital for the Palatini formalism in both classical and modified gravity.

In 1928, Einstein proposed a unification based on the notion of absolute or distant parallelism [136], although unsuccessful as a unified theory, this idea later regained popularity. The theory is now known as the teleparallel equivalent of general relativity (TEGR), this abandons the aim of unification and surprisingly gives an alternative yet equivalent formulation of general relativity. TEGR and the developments the theory has experienced will be discussed further in the next section. After teleparallelism failed to unify gravity and electromagnetism, Einstein began work on a theory that assumed non-zero torsion and curvature which was initiated by Cartan in 1922. This theory is now referred to as Einstein-Cartan theory [137], however this was not an attempt at unification and remains a theory of gravitation alone.

Currently, electromagnetic phenomena in the presence of a gravitational field are usually described by the Einstein-Maxwell field equations of general relativity, see chapter 3 for more details. Unification attempts no longer aim to modify classical gravity to include the effects of electromagnetism, but now intend to unify all four forces. For instance, areas such as quantum gravity and string theory concentrate on unifying classical gravity with quantum mechanics, where the latter successfully describes the electromagnetic and nuclear forces. Black hole thermodynamics provides an example of quantum field theory and classical gravity being applicable at the same scale [138]. The no hair conjecture states that a black hole solution to Einstein's field equations is characterised by up to three parameters, this includes its mass M, electric charge Q and angular momentum J. It was observed that this implies the laws of thermodynamics do not apply to black holes [139] since there is no temperature or visible entropy present from these variables. Following this the laws were reformulated to apply to black holes where temperature was represented by the surface gravity and entropy by the horizon area, the relationships between the parameters were merely an analogy. These modified laws are referred to as the laws of black hole thermodynamics and were originally obtained purely using differential geometry and concepts in general relativity [139]. It later became clear that quantum field theory can provide some useful contributions to the laws of black hole thermodynamics, studying these quantum effects gave rise to Hawking radiation which confirmed a physical relationship between temperature and surface gravity [140].

2.2 Teleparallelism

In 1928, Einstein attempted to unify gravity and electromagnetism based on the notion of teleparallelism, also referred to as 'distant parallelism', 'absolute parallelism' or 'fernparallelismus' [136]. This formulation differs from general relativity since teleparallelism assumes vanishing curvature and non-zero torsion. Hence the gravitational force exerted by a massive object causes a change in torsion as opposed to resulting in the curvature of spacetime, which in turn leads to the distortion of geodesics.

Moreover, teleparallelism utilizes objects which belong to the tangent space to a given point on a manifold, the tangent space is Minkowski space with the metric $\eta_{ij} = \text{diag}(1, -1, -1, -1)$. In the teleparallel formalism we consider spacetime metrics with signature $(+, -, -, -)$ which differs from our convention in general relativity. Latin indices $i, j, k\ldots$ denote tangent space coordinates and such indices are raised and lowered with η_{ij}, whereas Greek indices $\mu, \nu, \rho\ldots$ denote spacetime coordinates which are raised and lowered using $g_{\mu\nu}$. Given a tangent space to a point p on the manifold, there exists a field of orthonormal unit vectors forming a basis of the local tangent space. This is referred to as the tetrad, vierbein or n-bein field $e^i{}_\mu$, whose inverse is such that $e^i{}_\mu e_i{}^\nu = \delta^\nu_\mu$ and $e^i{}_\mu e_j{}^\mu = \delta^i_j$. Note that all tensors defined on the spacetime can be expressed in terms of the tetrad field, for example given a vector v^μ we have the relationship $e^i{}_\mu v^\mu = v^i$. The metric tensor, and hence other geometric quantities are determined by the tetrad field with the relation $g_{\mu\nu} = \eta_{ij} e^i{}_\mu e^j{}_\nu$, where $e^i{}_\mu \mathrm{d}x^\mu = e^i$ thus the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ij} e^i e^j$. In addition to this, $e^i{}_\mu$ enables us to assign 'direction' to the line element $\mathrm{d}s^2$ between two points which are a finite distance apart and hence provides the notion of parallelism, whereas the metric alone does not allow for this distinction to be made in general relativity. The tetrad field associated with the metric is not unique, since arbitrary Lorentz transformations applied to $e^i{}_\mu$ will leave the metric unchanged (see section 4.2), therefore $e^i{}_\mu$ is not fully determined by the metric. The tetrad field has sixteen degrees of freedom, whereas the metric tensor is symmetric ($g_{\mu\nu} = g_{\nu\mu}$) and thus only has ten degrees of freedom. Since the description of gravity only requires ten degrees of freedom this leaves six extra variables to fix. Einstein's attempt at unification aimed to utilize the six additional degrees of freedom to include a description of electromagnetism alongside gravity, it seemed promising since the electric and magnetic fields of Maxwell's theory require precisely six degrees of freedom.

Einstein proposed the tetrad field since it enables us to define the notion of parallelism which can be expressed as $\nabla_\nu v^\sigma = 0$ or equivalently $\partial_\nu v^\sigma = -\Gamma^\sigma_{\mu\nu} v^\mu$ where the latter is obtained by using the definition of the covariant derivative ∇_ν for a vector v^σ. This gives rise to the connection that is asymmetric in its lower indices and invariant under rotations $\Gamma^\sigma_{\mu\nu} = -e^i{}_\mu \partial_\nu e_i{}^\sigma$. It was later discovered that this connection had already been derived by assuming that the Riemann tensor vanishes $R^\sigma{}_{\mu\nu\rho} = 0$ by Weitzenböck [141] in 1921, hence this is known as the Weitzenböck connection

$$\Gamma^\sigma_{\mu\nu} = e_i{}^\sigma \partial_\nu e^i{}_\mu = -e^i{}_\mu \partial_\nu e_i{}^\sigma \,.$$

The defined symmetry of the Levi-Civita connection results in non-zero curvature and vanishing torsion, on the other hand the Weitzenböck connection results in vanishing curvature and non-zero torsion, this can be seen by expressing the Riemann curvature tensor in terms of the new connection, where $R^\sigma{}_{\mu\nu\rho} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\rho \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\lambda_{\mu\rho} - \Gamma^\sigma_{\lambda\rho}\Gamma^\lambda_{\mu\nu}$. Writing this explicitly in terms of the tetrad field gives the expression

$$\begin{aligned} R^\sigma{}_{\mu\nu\rho} = \partial_\nu \Big(e_i{}^\sigma \partial_\rho e^i{}_\mu \Big) - \partial_\rho \Big(e_i{}^\sigma \partial_\nu e^i{}_\mu \Big) & \\ + e_i{}^\sigma \Big(\partial_\nu e^i{}_\lambda \Big) e_j{}^\lambda \Big(\partial_\rho e^j{}_\mu \Big) - e_i{}^\sigma \Big(\partial_\rho e^i{}_\lambda \Big) e_j{}^\lambda \Big(\partial_\nu e^j{}_\mu \Big) \,, & \end{aligned}$$

then applying the product rule to expand the first two terms, and utilising the defining property of the Weitzenböck connection to rewrite the last two terms yields

$$\begin{aligned} R^\sigma{}_{\mu\nu\rho} = \Big(\partial_\nu e_i{}^\sigma\Big)\Big(\partial_\rho e^i{}_\mu\Big) - \Big(\partial_\rho e_i{}^\sigma\Big)\Big(\partial_\nu e^i{}_\mu\Big) + e_i{}^\sigma \Big(\partial_\nu \partial_\rho e^i{}_\mu - \partial_\rho \partial_\nu e^i{}_\mu\Big) \\ - e^i{}_\lambda \Big(\partial_\nu e_i{}^\sigma\Big) e_j{}^\lambda \Big(\partial_\rho e^j{}_\mu\Big) + e^i{}_\lambda \Big(\partial_\rho e_i{}^\sigma\Big) e_j{}^\lambda \Big(\partial_\nu e^j{}_\mu\Big) \equiv 0 \,, \end{aligned}$$

where $e^i_\lambda e^\lambda_j = \delta^i_j$, using this identity the last two terms can be written as $-\delta^i_j(\partial_\nu e_i{}^\sigma)(\partial_\rho e^j{}_\mu) + \delta^i_j(\partial_\rho e_i{}^\sigma)(\partial_\nu e^j{}_\mu) = -(\partial_\nu e_i{}^\sigma)(\partial_\rho e^i{}_\mu) + (\partial_\rho e_i{}^\sigma)(\partial_\nu e^i{}_\mu)$ which cancels the first two terms. The remaining two terms vanish since partial derivatives commute.

Although the connection is not a tensor, Einstein found that the skew symmetric part of $\Gamma^\sigma_{\mu\nu}$ gives rise to what is now known as the torsion tensor

$$T^\sigma{}_{\mu\nu} = 2\Gamma^\sigma_{[\nu\mu]} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu} \,,$$

Einstein constructed another tensor from the connection which is now referred to as the contortion tensor $K^{\sigma}{}_{\mu\nu}$. The contortion tensor is defined as the difference between the Weitzenböck connection $\Gamma^{\sigma}_{\mu\nu}$ used in teleparallelism and the Levi-Civita connection $\hat{\Gamma}^{\sigma}_{\mu\nu}$ used in general relativity, $K^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu} - \hat{\Gamma}^{\sigma}_{\mu\nu}$. Alternatively, this can be written in terms of the torsion tensor

$$K^{\mu\nu}{}_{\sigma} = -\frac{1}{2}\Big(T^{\mu\nu}{}_{\sigma} - T^{\nu\mu}{}_{\sigma} - T_{\sigma}{}^{\mu\nu}\Big) .$$

Additionally, in order to write down the field equations, Einstein introduced a tensor which provides a useful combination of the quantities $K^{\sigma}{}_{\mu\nu}$ and $T^{\sigma}{}_{\mu\nu}$, this is given by the tensor

$$S_{\sigma}{}^{\mu\nu} =\frac{1}{2}\Big(K^{\mu\nu}{}_{\sigma} + \delta^{\mu}_{\sigma}T_{\rho}{}^{\rho\nu} - \delta^{\nu}_{\sigma}T_{\rho}{}^{\rho\mu}\Big) ,$$

$S_{\sigma}{}^{\mu\nu}$ is antisymmetric so that $S_{\sigma}{}^{\mu\nu} = -S_{\sigma}{}^{\nu\mu}$ and is sometimes referred to as the superpotential. This quantity is useful since it enables us to define the torsion scalar T in the following way

$$T =S_{\sigma}{}^{\mu\nu}T^{\sigma}{}_{\mu\nu} , \tag{2.2.1}$$

In teleparallelism, the Lagrangian density appearing in the action contains a contraction of the torsion tensor as opposed to the Riemann curvature tensor, Einstein outlined several possibilities for the Lagrangian, the second was a generalised version of the others and is given by the combination

$$\begin{aligned} L_{\text{gravity}} &= e\Big(c_1 T^{\sigma}{}_{\mu\nu}T_{\sigma}{}^{\mu\nu} + c_2 T^{\sigma}{}_{\mu\nu}T^{\nu\mu}{}_{\sigma} + c_3 T_{\sigma\mu}{}^{\sigma}T^{\nu\mu}{}_{\nu}\Big) \\ &= e\Big(c_1 L_1 + c_2 L_2 + c_3 L_3\Big) , \end{aligned} \tag{2.2.2}$$

where $e = \det\big(e^{i}{}_{\mu}\big) = \sqrt{-g}$, note that this combination is in the absence of matter hence any resulting field equations will be vacuum. The components of the Lagrangian density L_1, L_2, L_3 are invariants of the torsion tensor and c_1, c_2, c_3 are constants. The quantities L_1, L_2 and L_3 had also been outlined by Weitzenböck in [142]. The constants c_1, c_2 and c_3 mean teleparallelism has three free parameters, hence it is often referred to as a three parameter theory of gravity.

The first Lagrangian Einstein tried was that given in equation (2.2.2) with $c_2 = 1$ and $c_1 = c_3 = 0$, which was later shown to be equivalent to the action with $c_1 = 1$ and $c_2 = c_3 = 0$. These actions gave rise to two sets of equations which could be interpreted as the vacuum Einstein and Maxwell equations, however this separation was viewed as synthetic. Following this many other combinations were considered, including a particular combination given by the Lagrangian with $c_1 = 1/2$, $c_2 = 1/4$ and $c_3 = -1$ which gives rise to a set of vacuum field equations equivalent to those of general relativity. Einstein experienced a lengthy search for the field equations, many of the systems of equations obtained contained additional components. This meant the sixteen degrees of freedom offered by the tetrad were described by more than sixteen equations and the system of equations was over-determined, these were then fixed with added constraints. Since the action with $c_1 = 1/2$, $c_2 = 1/4$ and $c_3 = -1$ implies general relativity, Einstein altered the Lagrangian to $\mathcal{L} = \frac{1+\alpha_1}{2}L_1 + \frac{1-4\alpha_2}{4}L_2 - \frac{4+\alpha_1}{4}L_3$ and again obtained the equations of general relativity plus electromagnetism provided the condition $\frac{\alpha_2}{\alpha_1} \to 0$ was satisfied.

However, it soon became clear that the electromagnetic field must be treated separately from the gravitational field because of their vast differences in strength and range of spacetime affected. The gravitational field is always attractive, the collected gravitational field of a relatively small massive object is weak and does not have much of an effect on its surroundings, whereas the accumulated gravitational field of a sufficiently large massive object will have more significant effects on the surrounding spacetime. On the other hand, charged particles are either attractive or repulsive, and the electromagnetic field produced by a charged particle is much stronger than the gravitational force exerted by a massive particle. But it is believed that the overall charge of the universe is neutral due to the balance of attractive and repulsive charges. Therefore gravity dominates on the large scales and electromagnetic forces dominate smaller scales, although both fields have infinite ranges unlike the weak and strong nuclear forces. It was realised that this attempt at unification failed in 1930, when the search for solutions was unsuccessful since there was no Schwarzschild solution and the field equations allowed for an non-physical solution [143]. Additionally, McVittie discovered that his axially symmetric solution from general relativity was not a solution of teleparallelism [144]. Following this, Einstein discontinued further research on teleparallelism and worked on Kaluza-Klein and Einstein-Cartan theory.

2.2.1 New general relativity and TEGR

Many years after the abandonment of teleparallelism, in 1961, Møller reconsidered Einstein's teleparallel theory, but as an alternative theory of gravity [145] as opposed to a unification of gravity and electromagnetism. This was formulated on the Weitzenböck spacetime and utilized the tetrad field approach, similarly to Einstein's original theory, however the six additional degrees of freedom due to the tetrad were fixed with an extra field equation $\psi_{\mu\nu} = 0$ containing six components. This method gives rise to the Schwarzschild solution unlike Einstein's unification approach, thus this gravitational theory also validates the three empirical tests of general relativity. Møller's idea quickly developed interest as the following year, Pellegrini and Plebanski reformulated Møller's approach to teleparallelism, and attempted to obtain the field equations from a modified version of the Einstein-Hilbert action [146]. They first considered a general form of the Lagrangian density containing invariants of the torsion tensor as above, this choice can then be narrowed down by checking which combination reduces to Newtonian gravity under the correct conditions, the Schwarzschild solution is again recovered. However both these theories were only invariant with respect to constant tetrad rotations, and were based on the assumption that all sixteen degrees of freedom from the tetrad are of physical significance.

Hayashi and Nakano, in 1967, imposed that the classical gravitational Lagrangian is invariant under translations and hence developed a gauge theory for the translation group [147], and later associated this formulation to the geometric structure of teleparallelism on the Weitzenböck spacetime [148]. In this paper, the phrase 'new general relativity' was coined as a synonym for the teleparallel theory of gravity, it was also shown that this gauge theory does not lead to general relativity only new general relativity. Hayashi's approach to teleparallelism differed from Einstein's since the action was allowed to be invariant under global gauge transformations instead of local [149] and it differed from the reformulation in [145, 146] since an alternative combination of contractions of the torsion tensor were used for the Lagrangian. More importantly, new general relativity differs from the teleparallel theory of gravity that is studied in this thesis (TEGR), this is again is due to the differences in the Lagrangian where new general relativity uses the Lagrangian from equation (2.2.2) plus an additional quantity see [149]. In 1979 Hayashi and Shirafuji considered a particular Lagrangian density which fixed two of the three free parameters of new general relativity thus leaving a one-parameter teleparallel theory of gravity, similar to the theory considered by Møller. There was much debate on the validity of the revised theory, in 1982, Kopczyński stated that the Lagrangian used at the time led to unpredictable behaviour of torsion since the field equations did not fully determine the torsion tensor [150]. To resolve this, an alternative formulation of the Lagrangian was suggested.

As mentioned in section 2.2 a particular combination of the three free parameters in the action, $c_1 = 1/2$, $c_2 = 1/4$ and $c_3 = -1$, gives rise to the field equations of general relativity. This theory is now known as the teleparallel equivalent to general relativity or TEGR. It was shown in [151] that the gravitational Lagrangian with this combination of parameters can be written as $L_{\text{gravity}} = S_\sigma{}^{\mu\nu} T^\sigma{}_{\mu\nu} = T$. This combination continues to constitute the action of TEGR and its modifications today, since this particular theory is utilised in this thesis, the words TEGR and teleparallelism will be used as synonyms for the remainder of the thesis. General relativity and teleparallelism are found to be equivalent theories which give rise to the same solutions, despite some major differences in their description of gravity and its effects on spacetime.

The quantities $T_\sigma{}^{\mu\nu}$, $K_\sigma{}^{\mu\nu}$ and $S_\sigma{}^{\mu\nu}$ are required to construct the field equations, which are obtained in an analogous way to general relativity, with the torsion scalar $T = S_\sigma{}^{\mu\nu} T^\sigma{}_{\mu\nu}$ replacing the Ricci curvature scalar R in the Einstein-Hilbert action

$$S = S_{\text{gravity}} + S_{\text{matter}} = \frac{1}{16\pi} \int e\, T \,\mathrm{d}^4x + \int e\, L_{\text{matter}} \,\mathrm{d}^4x \,.$$

The field equations are then obtained by varying the action with respect to the tetrad field $e^i{}_\mu$

$$e^{-1}\partial_\mu \big(e S_i{}^{\mu\nu}\big) - T^\sigma{}_{\mu i} S_\sigma{}^{\nu\mu} - \frac{1}{4} e_i{}^\nu T = -4\pi \mathcal{T}_i^\nu \,,$$

where $\mathcal{T}_i^\nu = e_i{}^\mu \mathcal{T}_\mu^\nu$ denotes the energy-momentum tensor. For a perfect fluid we can ensure conservation due to the antisymmetry of $S_\sigma{}^{\mu\nu} = -S_\sigma{}^{\nu\mu}$

$$4\pi \partial_\nu \Big(e\big(j_i{}^\nu + \mathcal{T}_i{}^\nu\big)\Big) = 0 \,, \tag{2.2.3}$$

where j_i^ν denotes the gauge current and represents the energy-momentum of the gravitational field

$$j_i{}^\nu = \frac{1}{4\pi}\Big(e_i{}^\sigma T^\rho{}_{\mu\sigma} S_\rho{}^{\nu\mu} - \frac{1}{4} e_i{}^\nu T\Big) \,.$$

Note that equation (2.2.3) is the teleparallel analogue of the conservation equation from general relativity, where the latter is given by $\nabla^{\nu}\big(G_{\mu\nu} + 8\pi\mathcal{T}_{\mu\nu}\big)$ (the energy-momentum tensor has the opposite sign to chapter 1 since we have adopted the opposite sign convention in teleparallelism).

2.3 Modified gravity

Shortly after Einstein's theory of gravity emerged, observational data and theoretical predictions began to revolutionise our view of the universe. Prior to the observational facts, there was disagreement on particular properties of our universe predicted by theoretical models. The FLRW solution predicted that the universe originated from a singularity or a 'big bang' from which it continues to expand [21], this theory contradicted Einstein's idea of a static universe with no beginning [18]. In fact many opposed the big bang theory and agreed the universe had no beginning or end, [152, 153]. For instance, the main rival to the big bang model was the steady state theory.

The first major observation is owed to Hubble in 1929 [19]. This confirmed speculation that the universe is expanding, where the idea of expansion was already modelled by some cosmological solutions to Einstein's field equations several years prior to the observations [20, 21]. The steady state theory was then modified by Hoyle to incorporate this discovery, this now stated while the universe is expanding matter is continuously being created thus we observe a homogeneous and isotropic universe [153]. Many years later, in 1965, the big bang theory had finally become accepted as the mainstream cosmological model. This was due to various observations which led to the discovery of cosmic microwave background radiation (CMB) [154], these observations followed the preceding development in radio astronomy. The discovery of CMB supported another prediction from the big bang model outlined by the FLRW solution, and contradicted the steady state model.

However, over the years, the big bang theory has also seen modifications based on observations of some issues with the model. This began with the flatness problem, which is the realisation that only a particular set of initial conditions are admissible after the big bang to create our universe, and any small changes would yield significant differences. Secondly, the assumption of homogeneity and the measured CMB radiation implies a uniform background temperature and energy density across the entirety of our vast universe, this led to what is known as the horizon problem. These problems initiated the notion of early time inflation as a possible solution [155], this imposes a very short period of exponential inflation after the big bang. It was previously believed that the universe expanded at a uniform rate, now it is known to have undergone certain phases of expansion [156], and the current phase is not uniform but expanding at an accelerated rate [7, 157]. The accelerated phase of expansion is often attributed to dark energy, and this exotic energy is believed to constitute a large part of our universe. Including the big bang and short inflationary period the universe is viewed to have undergone four main stages of expansion, the remaining three are known as the radiation, matter and dark energy dominated eras, in each stage the evolution of the universe is governed by either radiation, matter or dark energy. There are various cosmological models which describe these phases of the universe for example, the ΛCDM model (Λ cold dark matter) [158]. Also, quintessence and kinetic quintessence aim to provide an alternative description of accelerated expansion compared with dark energy [159].

Modified theories of gravity have developed a considerable amount of interest, primarily due to their ability to provide an alternative framework for understanding the effects which are normally attributed to dark energy [160]. Additionally, these theories are viewed to have the potential to unify the stages of early time inflation and late time accelerated expansion. Similarly to the teleparallel equivalent of general relativity, such theories of gravity are typically obtained by modifying the Einstein-Hilbert action accordingly. But this does not necessarily require defining additional objects or changing concepts of the underlying theory of gravity as in TEGR. We will limit our discussion to modified general relativity or teleparallelism, for instance such alterations to Newtonian gravity will not be considered here.

Following the adjustments to general relativity discussed previously, alternative modifications to the Lagrangian have been considered further. In 1934 Born and Infeld [161] investigated transformations of the Lagrangian which removed divergences from the theory, but the effects of gravity were abandoned in this particular calculation and only applied it to electromagnetic phenomena. Many years later, in 1962, actions which included higher order invariants of the Ricci curvature were considered. Such modifications were viewed as necessary in order to quantise general relativity [162]. Almost a decade later, in 1971 Lovelock considered generalisations to Einstein's field equations by considering the Lagrangian in higher dimensions, this can be applied to theories of gravity which require more than four dimensions of spacetime [163].

In 1961, general relativity was modified to incorporate Mach's principle, which states that there is a relationship between local and distant motion [5], this entailed allowing Newton's gravitational constant to evolve in spacetime $G \rightarrow G(\phi)$ where ϕ is a scalar field. The idea of the fundamental constants of nature being allowed to evolve with time had already been raised by Dirac in 1937 [164], which in particular encompassed the gravitational constant becoming time dependent. This modification to general relativity is now referred to as Jordan-Brans-Dicke theory [165,166] or more often just Brans-Dicke theory, and provides an alternative description to the classical model of gravity outlined by general relativity. Brans-Dicke theory is in fact a particular model of a wider class of modifications known as scalar tensor theory which was first investigated by Jordan [165]. The foundations of this theory came from the Kaluza-Klein model, and is also general relativity coupled to a scalar field, see [167] for more information.

Another approach to modifying general relativity is given by $f(R)$ gravity, this modification was initiated by Buchdahl in 1970 [168] and has captured much interest over the years [169]. Prior to this arbitrary modification, the widely accepted generalisation to the Einstein Hilbert action was the ΛCDM model given by $f(R) = R - 2\Lambda$. There are various formalisms in $f(R)$ gravity, which depend on the variational principle used to obtain the field equations from the action. These variational approaches are analogues to the methods utilised in general relativity [8] and give rise to the metric, Palatini, and metric-affine formalisms in $f(R)$ gravity. It has been shown that these theories are equivalent to Brans-Dicke theory, see [169,170] for a thorough review. Each of these $f(R)$ theories are described by fourth order field equations, whereas the fourth order terms vanish in general relativity and hence gives rise to second order field equations [171]. Note that some view the Palatini $f(R)$ field equations as second order, however the Ricci scalar $\mathcal{R}(\Gamma)$ contains higher derivatives of the function f making it a fourth order theory. It is believed that these higher order corrections to general relativity will provide a description of the accelerated expansion of our universe without the need for dark energy. However, these modified equations prove to be extremely complex, even for static solutions in the absence of rotation, and because of this results presented are often in the absence of matter. During this advancement in modified $f(R)$ gravity, the Born-Infeld approach was applied to the gravitational action. This was first considered in 1998 [172], it has subsequently been shown that this modification also successfully describes accelerated expansion without the need for dark energy and also provides a description of dark matter [173].

The modification in $f(R)$ gravity only modifies R in the classic Einstein-Hilbert action of general relativity, therefore only the gravitational action is changed. It was recently proposed that modifications should also be applied to the matter action, and that this will allow for a non-minimal coupling between curvature and matter. This particular modification is often referred to as $f(R, L_m)$ gravity, where $L_m = L_{\text{matter}}$ is the matter Lagrangian. In 2011 the concept of modifying our description of matter was extended and modified further in $f(R, \mathcal{T})$ gravity, here the action is a function of the Ricci scalar R and the trace of the energy-momentum tensor $\mathcal{T}$ [174]. However it was shown that most models of $f(R, \mathcal{T})$ gravity violate the first law of black hole thermodynamics [175]. Note that $f(R, T)$ is sometimes interpreted as a function of the Ricci scalar R and torsion tensor T.

For a more in depth review of the various modifications to general relativity described in, or omitted from, this section see [176].

2.3.1 Modified Teleparallelism

Modifications to the teleparallel equivalent of general relativity have been constructed more recently. These theories are obtained in an analogous manner to $f(R)$ gravity, by modifying the gravitational Lagrangian to be an arbitrary function of its original argument T. In 2007, the Born and Infeld approach to modifying the Lagrangian was applied to TEGR, this gave rise to a theory that successfully described early time inflation [177]. This particular modification also contained black hole solutions and cosmological models without a singularity [178]. The theory is now known as $f(T)$ gravity [179, 180], although the modifications are more arbitrary than in [177, 178]. The field equations of the theory are second order and hence provide a simpler approach compared with the fourth order field equations of $f(R)$ gravity.

It was later shown that modifications to teleparallel gravity give rise to an alternative framework for dark energy by describing accelerated expansion [179]-[186] and this was unified with a description of the radiation and matter dominated stages of the universe [179, 182]. Additionally, $f(T)$ gravity offers a unification of early time inflation with the current accelerated expansion phase [183]. However some oppose that $f(T)$ gravity successfully provides a description of the accelerated expansion epoch, for example it was shown that certain $f(T)$

models which describe this phase differ greatly from the ΛCDM model and hence requires too many restrictions to avoid this [187]. It was also shown that various solutions do not allow for the crossing of the phantom divide [188] whereas others do [189–191], the phantom divide refers to the equation of state of dark energy ($p/\rho = -1$) in the description of the accelerated expansion epoch [192].

There have been many results comparing modified teleparallel gravity with observational results, and the constraints this data will impose on the allowed models of $f(T)$ have been presented [179, 188, 191, 193]. In addition to an alternative to dark energy, it was shown that modifications to teleparallelism can provide an alternative to dark matter [194]. The study of cosmological models was extended to inhomogeneous and anisotropic universes in order to derive additional early time inflation models such as the de Sitter solution [195]. It was found that the de Sitter solution in an anisotropic universe will not exist unless an anisotropic pressure is imposed.

The models which imply accelerated expansion in $f(T)$ gravity have been compared to alternative theories of gravity which have a successful description of this phase, the alternative theories considered were $f(R)$ gravity [196], kinetic quintessence models [197] and ΛCDM models [187, 198, 199]. Various models in $f(T)$ gravity provide results which differ to that of the ΛCDM model [187,198,199]. In addition to this, Dirac's idea of a time-dependent gravitational constant was applied to modified teleparallel gravity, and the consequential constraints on $f(T)$ were examined [200]. Certain black hole solutions in modified teleparallel gravity supposedly violate the first law of black hole thermodynamics due to the absence of local Lorentz invariance [201]. However, the second law is satisfied from early time inflation up to the present late time acceleration phase, though it has been stated it can be violated in later phases for various models [202]. It was then shown that the first law is satisfied for most models and that the second law of black hole thermodynamics is always satisfied unless the law is misinterpreted [203, 204]. Comparisons have also been made with general relativity with the conclusion that $f(T)$ may provide a simpler modification to general relativity than $f(R)$ gravity [179, 180].

2.3.2 $f(T)$ field equations

Modified teleparallelism is constructed on the Weitzenböck spacetime, hence the underlying theory of teleparallelism and the tensors constructed from the Weitzenböck connection ${T_\sigma}^{\mu\nu}$, ${K_\sigma}^{\mu\nu}$, ${S_\sigma}^{\mu\nu}$ remain unchanged. The modification is made to the action

$$S = S_{\rm gravity} + S_{\rm matter} = \frac{1}{16\pi}\int e\, f(T)\, {\rm d}^4x + \int e\, L_{\rm matter}\, {\rm d}^4x\,. \qquad (2.3.1)$$

Note that some authors make a slightly different modification to the action [199], where the gravitational Lagrangian in the action is instead of the form $L_{\rm gravity} = T + f(T)$, and it was shown that when $f(T) =$ constant for such an action this theory reduces to general relativity [198]. In this thesis, we will only consider the action (2.3.1) for calculations in modified teleparallelism, however in chapter 4 we will make a similar relabelling $f(T) = T + h(T)$ in the field equations. It has been observed that both versions of the action do not obey local Lorentz symmetries, only global [187, 205, 206], it was then deduced that this causes problems with the dynamics of the field equations [187] and gives rise to additional degrees of freedom [206]. Recently, the impact of the extra degrees of freedom were discussed further in [207], it was concluded that this can cause problems including the possibility of particles travelling faster than the speed of light (superluminal propagation) unless $f_{TT} = 0$. In [208], the Noether symmetries of the Lagrangian were discussed, this analysis was then used to derive a time dependent solution.

Varying the action (2.3.1) with respect to the tetrad field $e^i{}_\mu$ gives rise to the $f(T)$ field equations

$$S_i{}^{\mu\nu} f_{TT}\partial_\mu T + \Big(e^{-1}\partial_\mu\big(eS_i{}^{\mu\nu}\big) - T^\sigma{}_{\mu i}S_\sigma{}^{\nu\mu}\Big)f_T - \frac{e_i{}^\nu f}{4} = -4\pi \mathcal{T}_i{}^\nu\,, \qquad (2.3.2)$$

where $S_i{}^{\mu\nu} = e_i{}^\sigma S_\sigma{}^{\mu\nu}$. In [3] the conservation equation (2.2.3) for a perfect fluid energy-momentum tensor in teleparallel gravity was generalised for $f(T)$ gravity, in order to proceed we rewrite the field equations as

$$e\delta^\sigma_\rho S_\sigma{}^{\mu\nu} f_{TT}\partial_\mu T + \Big(e^i{}_\rho\partial_\mu\big(eS_i{}^{\mu\nu}\big) - e\delta^\sigma_\rho T^\lambda{}_{\mu\sigma}S_\lambda{}^{\nu\mu}\Big)f_T - \frac{e}{4}\delta^\nu_\rho f = -4\pi e\delta^\sigma_\rho \mathcal{T}_\sigma{}^\nu\,,$$

where we have just multiplied the equations of motion by $ee^i{}_\rho$. This yields the conservation equation

$$4\pi\partial_\nu\Big(e\big(j_i{}^\nu + \mathcal{T}_i{}^\nu\big)\Big) = 0\,, \qquad (2.3.3)$$

which is similar to the conservation equation in TEGR, see equation (2.2.3), however the energy-momentum for the gravitational field $j_i{}^\nu$ has been modified to

$$j_i{}^\nu = -\frac{1}{4\pi}\left(e_i{}^\sigma T^\rho{}_{\mu\sigma} S_\rho{}^{\nu\mu} f_T + \frac{1}{4} e_i{}^\nu f\right). \tag{2.3.4}$$

This is a possible reformulation of the gauge current $j_i{}^\nu$, for example, one could also consider the alternative which gives rise to the same conservation equation

$$j_i{}^\nu = -\frac{1}{4\pi}\left(e_i{}^\sigma S_\sigma{}^{\mu\nu} f_{TT} \partial_\mu T - e_i{}^\sigma T^\rho{}_{\mu\sigma} S_\rho{}^{\nu\mu} f_T + \frac{1}{4} e_i{}^\nu f\right).$$

Spherically symmetric solutions, without restricting to cosmological models as described above, have been analysed in $f(T)$ gravity more recently [209–214]. There was a recent claim that spherically symmetric solutions cannot be attained in modified teleparallelism [209]. Whilst investigating this claim, we found that the usual diagonal tetrad field chosen gives rise to an additional off-diagonal field equation. This extra equation imposes a severe constraint on the torsion scalar $T' = 0$ which in turn restricts $f(T)$. In addition to this, the diagonal tetrad field is not a global solution since it contains singularities, this will be discussed further in section 4.2.1. By rotating the diagonal tetrad we removed these restrictions on T, however the rotated tetrad still contains the aforementioned singularities, this tetrad field will be discussed in section 4.2.2, and possible alternative tetrads will also be discussed in section 4.2.3. These results show that the modified teleparallel field equations and hence constraints on $f(T)$ depend on the chosen tetrad field. In [210], static and spherically symmetric solutions were analysed further, and constraints on $f(T)$ were presented which allow for the existence of various solutions. This included the the Schwarzschild and Reissner-Nordström solutions in the presence of a cosmological constant, the sign of Λ was not restricted.

Following this, static solutions with anisotropic pressure have been considered for the diagonal tetrad [211]. This analysis gives rise to wormhole and black hole solutions, such models were analysed further for the off-diagonal tetrad and it was shown that the anisotropic case violates the null energy condition [212]. However in [213, 214] wormhole and black hole solutions with the same setting of anisotropic pressure and off-diagonal tetrad field were derived, in which the null energy condition is satisfied. Recently, solutions exhibiting circular [215] and planar symmetry [216] were derived, the latter aided the construction of various solutions including the Møller, Kottler and Whittaker solutions.

Recall that for a given metric $g_{\mu\nu}$ the associated tetrad field $e_i{}^\mu$ is not unique since one can perform arbitrary Lorentz transformations on the tetrad field without changing the metric, whereas a given tetrad field corresponds to a unique metric. So far, the literature discussed above uses the diagonal and rotated tetrad fields, where the latter is simply a rotated version of the former. However, the rotated tetrad is not the most general form and applying a Lorentz boost to the diagonal tetrad field will result in a more generalised version [217]. It was shown that this gives rise to the Schwarzschild solution. In [218, 219] further transformations of the tetrad fields were derived, where tetrads which give rise to the FLRW solution (for both positive and negative curvature scenarios) were examined in [218]. Additionally, some 'good' tetrad fields were described in the context of $f(T)$ gravity, where it was shown that the so called rotated or off-diagonal tetrad field is a special case of a more general rotated tetrad, and the Schwarzschild de Sitter vacuum was derived [219].

Birkhoff's theorem was shown to hold for the diagonal tetrad field [220], but the off-diagonal field equation and hence the restriction on T had been overlooked. I considered Birkhoff's theorem for the rotated tetrad, so these restrictions no longer apply in chapter 4. Additionally, I derived a bound on M/R in $f(T)$ gravity with the rotated tetrad field, using the usual method of constrained optimization, which can also be found in chapter 4. This has given some insight on the required constraints on f given the existence of a solution with this mass radius bound.

Chapter 3

Charged solutions in de Sitter space

In this chapter we will first derive the Reissner-Nordström de Sitter spacetime from the Einstein-Maxwell field equations with a non-zero cosmological constant in section 3.1 and the TOV conservation equation will be generalised to include the effects of charge in the presence of a cosmological constant in section 3.2. Hydrostatic equilibrium will be explored further in section 3.3, where we will derive an upper limit on M_g/R. Following this, we will discuss the causal structure of the RNdS spacetime and present the corresponding Carter-Penrose diagrams, an outline of the derivation for the Reissner-Nordström solution using a method that can be extended for the RNdS solution.

A peculiar subclass of the Reissner-Nordström de Sitter spacetime will be derived in section 3.5.1, these are known as the charged Nariai solutions, this follows a discussion of the neutral analogue in section 3.5. Finally, we will use results from this chapter to derive generalisations of the Einstein static universe in the presence of an electromagnetic field in section 3.6.

3.1 Reissner-Nordström de Sitter spacetime

The Reissner-Nordström de Sitter metric is a known exterior, electro-vacuum solution of the Einstein-Maxwell equations. Einstein's field equation was given in equation (1.0.4) and can be written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\Big(M_{\mu\nu} + E_{\mu\nu}\Big)\,. \tag{3.1.1}$$

$M_{\mu\nu}$ and $E_{\mu\nu}$ are the matter and electromagnetic energy-momentum tensors respectively. The electromagnetic energy-momentum tensor $E_{\mu\nu}$ is given in terms of the electromagnetic field strength tensor $F_{\mu\nu}$, which is often referred to as the Faraday tensor

$$E_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu}{}^{\sigma}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\sigma\rho}F_{\sigma\rho}\right), \tag{3.1.2}$$

where $F_{\mu\nu}$ is an anti-symmetric rank-two tensor so that $F_{\mu\nu} = -F_{\nu\mu}$. This tensor is defined in terms of the spatial three-vectors $\vec{E}$ and $\vec{B}$ which represent the electric and magnetic fields respectively. These vectors can be expressed using the electromagnetic four-potential $A^{\mu} = (\psi, \vec{A^c})$ with $\vec{A^c} = \big(\tilde{A}_x, \tilde{A}_y, \tilde{A}_z\big)$, where $\vec{E} = -\vec{\nabla}\psi - \partial_t\vec{A^c} = \Big(-\partial_x\psi - \partial_t A_x\,, -\partial_y\psi - \partial_t A_y\,, -\partial_z\psi - \partial_t A_z\Big)\,, \vec{B} = \vec{\nabla}\times\vec{A^c} = \Big(\partial_y A_z - \partial_z A_y\,, \partial_z A_x - \partial_x A_z\,, \partial_x A_y - \partial_y A_x\Big)$. The Faraday tensor is then given by the relation $\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$, thus in Cartesian coordinates this tensor becomes

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{3.1.3}$$

To obtain this tensor in spherical coordinates, consider the transformation $r = \sqrt{x^2+y^2+z^2}\,, \cos\theta = z/r\,, \tan\phi = y/x$ with inverse $x = r\cos\phi\sin\theta\,, y = r\sin\phi\sin\theta\,, z = r\cos\theta$. Using this we can now rewrite the electric and magnetic fields in new coordinates

$$\vec{E} = \begin{pmatrix} \sin\theta\Big(\cos\phi E_x + \sin\phi E_y\Big) + \cos\theta E_z \\ \cos\theta\Big(\cos\phi E_x + \sin\phi E_y\Big) - \sin\theta E_z \\ \cos\phi E_y - \sin\phi E_x \end{pmatrix} = \begin{pmatrix} E_r \\ E_\theta \\ E_\phi \end{pmatrix},$$

$$\vec{B} = \begin{pmatrix} \sin\theta\Big(\cos\phi B_x + \sin\phi B_y\Big) + \cos\theta B_z \\ \cos\theta\Big(\cos\phi B_x + \sin\phi B_y\Big) - \sin\theta B_z \\ \cos\phi B_y - \sin\phi B_x \end{pmatrix} = \begin{pmatrix} B_r \\ B_\theta \\ B_\phi \end{pmatrix}, \tag{3.1.4}$$

these can be written in terms of the four-potential A^μ with the relation given above, but this is not necessary for this calculation.

Now the electromagnetic field strength tensor can be given in spherical coordinates, using the tensor transformation law $F^{\mu\nu} = \frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma}\tilde{F}^{\rho\sigma}$ we can convert each component of (3.1.3) and make use of the relations in (3.1.4) to simplify the resulting expressions. This yields for instance, $F^{tr} = \sin\theta\big(\cos\phi E_x + \sin\phi E_y\big) + \cos\theta E_z = E_r = -F^{rt}$ and $F^{t\theta} = \frac{\cos\theta}{r}\big(\cos\phi E_x + \sin\phi E_y\big) - \frac{\sin\theta E_z}{r} = \frac{E_\theta}{r} = -F^{\theta t}$. Computing the remaining components yields the Faraday tensor in spherical coordinates

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_r & \frac{E_\theta}{r} & \frac{E_\phi}{r\sin\theta} \\ -E_r & 0 & \frac{B_\phi}{r} & -\frac{B_\theta}{r\sin\theta} \\ -\frac{E_\theta}{r} & -\frac{B_\phi}{r} & 0 & \frac{B_r}{r^2\sin\theta} \\ -\frac{E_\phi}{r\sin\theta} & \frac{B_\theta}{r\sin\theta} & -\frac{B_r}{r^2\sin\theta} & 0 \end{pmatrix}, \qquad (3.1.5)$$

then indices can be lowered with $F_{\mu\nu} = g_{\sigma\mu}F^{\sigma\rho}g_{\rho\nu}$, using the spherically symmetric metric $g_{\mu\nu} = \big(-\mathrm{e}^a, \mathrm{e}^b, r^2, r^2\sin^2\theta\big)$. This gives for example $F_{tr} = -\mathrm{e}^{a+b}E_r = -F_{rt}$, and the remaining components can be calculated similarly.

The electromagnetic field strength tensor $F_{\mu\nu}$ must satisfy Maxwell's equations, which can be written as

$$\partial_\mu\Big(\sqrt{-g}F^{\mu\nu}\Big) = \sqrt{-g}J^\nu\,, \qquad (3.1.6)$$

$$\partial_{[\mu}F_{\nu\sigma]} = 0\,, \qquad (3.1.7)$$

$J^\mu = 4\pi\sigma u^\mu$ is the four-current density and is a product of the proper charge density σ and the four-velocity $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ which satisfies $u^\mu u_\mu = -1$. The definition of the electromagnetic field strength tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, implies that Maxwell's equation (3.1.7) is automatically satisfied.

Since we are working in a static and spherically symmetric spacetime with line element of the form

$$\mathrm{d}s^2 = -\mathrm{e}^a\mathrm{d}t^2 + \mathrm{e}^b\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \qquad (3.1.8)$$

where $\mathrm{d}\Omega^2 = \mathrm{d}r^2 + \sin^2\theta\mathrm{d}\phi^2$ is the line element for the two-sphere with unit radius, this implies J^μ is independent of the temporal and angular coordinates so that the only contribution is the radial component J^r. Thus equation (3.1.6) reduces the components of the field strength tensor to just the $F^{\mu r}$ entries in (3.1.5). Since we consider solutions in the absence of a magnetic field $\vec{B}$, this means the only non-zero components of $F^{\mu\nu}$ in a static and spherically symmetric spacetime are $F^{tr} = -F^{rt} = E_r$. Using Maxwell's equation (3.1.6) we obtain $\mathrm{e}^{(a+b)/2} r^2 \sin\theta J^r = \partial_t\Big(\mathrm{e}^{(a+b)/2} r^2 \sin\theta F^{tr}\Big)$, where $J^r = 4\pi\sigma u^r = 4\pi\sigma \frac{\mathrm{d}r}{\mathrm{d}t}$ therefore

$$\begin{aligned} F^{tr} &= 4\pi \frac{\mathrm{e}^{-(a+b)/2}}{r^2} \int_0^r \left(\mathrm{e}^{(a+b)/2} \tilde{r}^2 \sigma \frac{\mathrm{d}\tilde{r}}{\mathrm{d}t} \right) \mathrm{d}t \\ &= \frac{\mathrm{e}^{-(a+b)/2}}{r^2} q(r)\,, \end{aligned}$$

or equivalently $F_{rt} = \mathrm{e}^{(a+b)/2} q/r^2 = -F_{tr}$. Here $q = q(r)$ is the total charge in the region $[0, r]$ and is defined in terms of the charge density σ. Thus $q(r)$ is given by

$$q(r) = 4\pi \int_0^r \mathrm{e}^{(a+b)/2} \tilde{r}^2 \sigma \mathrm{d}\tilde{r}\,. \tag{3.1.9}$$

Using this we can compute the components of the electromagnetic energy-momentum tensor (3.1.2), the non-zero entries are the four diagonal elements $E_t{}^t = E_r{}^r = -E_\theta{}^\theta = -E_\phi{}^\phi = -\frac{q^2}{8\pi r^4}$. The matter energy-momentum tensor for a perfect fluid is given in section 1.3.2, with an anisotropic pressure this becomes $M_\mu{}^\nu = \mathrm{diag}\Big(-\rho, p, p_\perp, p_\perp\Big)$. This can be put together to yield the components of the total energy momentum tensor

$$\mathcal{T}_\mu{}^\nu = \mathrm{diag}\Big(-\rho - \frac{q^2}{8\pi r^4}\,,\ p - \frac{q^2}{8\pi r^4}\,,\ p_\perp + \frac{q^2}{8\pi r^4}\,,\ p_\perp + \frac{q^2}{8\pi r^4}\Big)\,.$$

Note that for non-zero charge q and pressures $p\,, p_\perp$, the condition of isotropy in the total energy momentum tensor $\mathcal{T}_r{}^r - \mathcal{T}_\theta{}^\theta = 0$ becomes $p - p_\perp = \frac{q^2}{4\pi r^2}$. Now the field equations can be written in component form

$$8\pi\rho + \frac{q^2}{r^4} = \frac{b' \mathrm{e}^{-b}}{r} + \frac{1}{r^2}\Big(1 - \mathrm{e}^{-b}\Big) - \Lambda\,, \tag{3.1.10}$$

$$8\pi p - \frac{q^2}{r^4} = \frac{a' \mathrm{e}^{-b}}{r} - \frac{1}{r^2}\Big(1 - \mathrm{e}^{-b}\Big) + \Lambda\,, \tag{3.1.11}$$

$$8\pi p_\perp + \frac{q^2}{r^4} = \frac{\mathrm{e}^{-b}}{2}\left(a'' + \Big(\frac{a'}{2} + \frac{1}{r}\Big)\Big(a' - b'\Big) \right) + \Lambda\,. \tag{3.1.12}$$

The (t, t) field equation, given by equation (3.1.10), can be used to obtain the mass, similarly to the method used in section 1.2 in the absence of charge. The total gravitational mass is defined to be

$$m_g(r) := \underbrace{4\pi \int_0^r \tilde{r}^2 \rho \, \mathrm{d}\tilde{r}}_{m_i(r)} + \underbrace{4\pi \int_0^r \mathrm{e}^{(a+b)/2} \tilde{r} \sigma q \, \mathrm{d}\tilde{r}}_{m_q(r)} \,. \tag{3.1.13}$$

In the presence of a cosmological constant and charge, m_i can be obtained by inserting (3.1.10) in place of the energy density ρ. This yields

$$\begin{aligned} m_i =& \frac{1}{2} \int_0^r \left\{ \frac{\mathrm{d}}{\mathrm{d}\tilde{r}} \left(\tilde{r} - \tilde{r}\mathrm{e}^{-b} \right) - \frac{q^2}{\tilde{r}^2} - \Lambda \tilde{r}^2 \right\} \mathrm{d}\tilde{r} \\ =& \frac{r(1-\mathrm{e}^{-b})}{2} - \frac{\Lambda}{6} r^3 - \int_0^r \frac{q^2}{2\tilde{r}^2} \mathrm{d}\tilde{r} \,. \end{aligned} \tag{3.1.14}$$

Next, in order to obtain the exterior Reissner-Nordström de Sitter line element, the electromagnetic mass m_q must be written as

$$m_q = \int_0^r \frac{q^2}{2\tilde{r}^2} \mathrm{d}\tilde{r} + \frac{q^2}{2r} \,, \tag{3.1.15}$$

see for instance [35, 82] for further discussion of the electromagnetic mass and a derivation of (3.1.15). Substituting m_i and m_q into the definition of m_g in equation (3.1.13) and rearranging gives $\mathrm{e}^{-b} = 1 - \frac{2m_g}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3} r^2$. In the Reissner-Nordström de Sitter exterior, we have $r \geq R$, where R is the boundary of the matter distribution and is often referred to as the total radius. This implies the metric coefficient

$$\mathrm{e}^{-b} = 1 - \frac{2M_g}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2 \,, \tag{3.1.16}$$

where $M_g = m_g(R)$ and $Q = q(R)$ represent the total gravitational mass and charge respectively.

To fix the remaining metric coefficient e^a, consider the (r, r) component of the electro-vacuum field equations, that is when $r \geq R$. Rearranging for a' yields

$$a' = \frac{2\mathrm{e}^b}{r} \left(\frac{M_g}{r} - \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2 \right) .$$

Note, taking the derivative of equation (3.1.16) and comparing with the equation above yields the relation $b' = -\frac{2\mathrm{e}^b}{r}\left(\frac{M_g}{r} - \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right) = -a'$, this implies $\mathrm{e}^a = \mathrm{e}^{-b}$. Therefore the remaining metric coefficient for the exterior solution can be written explicitly as

$$\mathrm{e}^a = 1 - \frac{2M_g}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2 \,. \tag{3.1.17}$$

In the interior, the metric coefficients can be written as

$$\mathrm{e}^a = 1 - \frac{2m_g}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3}r^2 = \mathrm{e}^{-b} \,. \tag{3.1.18}$$

The solution (3.1.8) with metric coefficients (3.1.16) and (3.1.17) gives the exterior Reissner-Nordström de Sitter solution. This reduces to the Kottler solution when $Q = 0$ or the Reissner-Nordström solution when $\Lambda = 0$.

3.2 Hydrostatic equilibrium

In this section, we will construct a TOV-like conservation equation which governs the hydrostatic equilibrium of a charged spherically symmetric solution in de Sitter space. Given the RNdS spacetime with anisotropic pressure $p \neq p_\perp$, first we differentiate p with respect to r using equation (3.1.11), this becomes

$$8\pi p' = \frac{\mathrm{e}^{-b}}{r}\left(a'' - a'b' - \frac{1}{r}\left(a' + b'\right)\right) + \frac{2}{r^3}\left(1 - \mathrm{e}^{-b}\right) - \frac{4q^2}{r^5} + \frac{2qq'}{r^4} \,, \tag{3.2.1}$$

similarly to section 1.3.2, we combine the first two field equations (3.1.10) and (3.1.11), then the second two field equations (3.1.11) and (3.1.12)

$$8\pi\left(\rho + p\right) = \frac{\mathrm{e}^{-b}}{r}\left(a' + b'\right) \,, \tag{3.2.2}$$

$$8\pi\left(p - p_\perp\right) = -\frac{\mathrm{e}^{-b}}{2}\left(a'' + \frac{a'}{2}\left(a' - b'\right) - \frac{1}{r}\left(a' + b'\right)\right) - \frac{1}{r^2}\left(1 - \mathrm{e}^{-b}\right) + \frac{2q^2}{r^4} \,, \tag{3.2.3}$$

multiplying equation (3.2.2) by $a'/2$ and equation (3.2.3) by $2/r$

$$\begin{aligned}4\pi a'\big(\rho+p\big) &= \frac{a'\mathrm{e}^{-b}}{2r}\Big(a'+b'\Big)\,,\\ \frac{16\pi}{r}\big(p-p_\perp\big) &= -\,\frac{\mathrm{e}^{-b}}{r}\left(a''+\frac{a'}{2}\big(a'-b'\big)-\frac{1}{r}\big(a'+b'\big)\right)-\frac{2}{r^3}\Big(1-\mathrm{e}^{-b}\Big)+\frac{4q^2}{r^5}\,,\end{aligned}$$

combining these equations gives

$$\begin{aligned}4\pi a'\big(\rho+p\big)+\frac{16\pi}{r}\big(p-p_\perp\big) &= \frac{4q^2}{r^5}\\ &\quad -\frac{\mathrm{e}^{-b}}{r}\left(a''-a'b'-\frac{1}{r}\big(a'+b'\big)\right)-\frac{2}{r^3}\Big(1-\mathrm{e}^{-b}\Big)\,. \end{aligned} \tag{3.2.4}$$

Now we can use this equation to eliminate some terms in equation (3.2.1), which yields the conservation equation

$$p'+\frac{a'}{2}\big(\rho+p\big)+\frac{2}{r}\big(p-p_\perp\big)-\frac{qq'}{4\pi r^4}=0\,. \tag{3.2.5}$$

In order to proceed we make use of the expression for e^{-b} in the interior, given by (3.1.18), and insert this into the (r,r) field equation. This can be rearranged to give the following expression for a'

$$\frac{a'}{2}=\frac{4\pi rp+\frac{m_g}{r^2}-\frac{q^2}{r^3}-\frac{\Lambda}{3}r}{1-\frac{2m_g}{r}+\frac{q^2}{r^2}-\frac{\Lambda}{3}r^2}\,, \tag{3.2.6}$$

putting this in place of a' in equation (3.2.4) yields

$$p'+\frac{4\pi rp+\frac{m_g}{r^2}-\frac{q^2}{r^3}-\frac{\Lambda}{3}r}{1-\frac{2m_g}{r}+\frac{q^2}{r^2}-\frac{\Lambda}{3}r^2}\Big(\rho+p\Big)+\frac{2}{r}\big(p-p_\perp\big)-\frac{qq'}{4\pi r^2}=0\,, \tag{3.2.7}$$

this is the TOV-like conservation equation for a charged massive object in de Sitter space.

Now we have a set of four independent equations, this includes the three independent field equations (3.1.10)–(3.1.12) plus the conservation equation (3.2.7). Initially, there were six unknown functions, namely the energy density, pressures, charge and metric coefficients $\big\{\rho\,,p\,,p_\perp\,,q\,,a\,,b\big\}$, in this case the system of equations was under-determined. Since we have fixed the metric coefficients $\mathrm{e}^a\,,\mathrm{e}^b$ with the relation in equation (3.1.18), four unknowns remain therefore the system of equations is closed.

3.3 Bounds on M_g/R in RNdS spacetime

In this section, the results obtained in [2] will be summarised. Given the field equations (3.1.10)–(3.1.12), we can relabel the variables in the following way

$$\begin{aligned} x &= \frac{2m_g}{r} - \frac{q^2}{r^2} + \frac{\Lambda r^2}{3} = 1 - \mathrm{e}^{-b}\,, \\ y &= 8\pi r^2 p\,, \\ z_1 &= \frac{q^2}{r^2}\,, \\ z_2 &= \Lambda r^2\,. \end{aligned}$$

Based on the ranges of the variables $b\,, p\,, q$ and Λ, the new variables are valid over the set

$$\mathcal{U} = \Big\{ \big(x, y, z_1, z_2\big) : 0 \leq x < 1\,, y \geq 0\,, z_1 \geq 0\,, z_2 \geq 0, z_1 + z_2 \leq 1 \Big\}\,, \tag{3.3.1}$$

where we also impose $z_1 + z_2 \leq 1$. We can now reconstruct the field equations in a similar manner to section 1.5.2 in the absence of q and Λ. First we need the following quantities in new variables

$$\begin{aligned} \mathrm{e}^{-b} &= 1 - x\,, \\ \dot{b} &= \frac{r x'}{2(1-x)} = \frac{\dot{x}}{1-x}\,, \\ 2\dot{a} &= \frac{x + y - z_1 - z_2}{1-x}\,, \\ 2\ddot{a} &= \frac{\dot{x}\big(y - z_1 - z_2 + 1\big)}{(1-x)^2} + \frac{\dot{y} - \dot{z}_1 - \dot{z}_2}{1-x} \\ &= \frac{r^2 a''}{2} + \dot{a}\,, \end{aligned}$$

where $\beta = 2\log r$ so that $\dot{r} = \mathrm{d}r/\mathrm{d}\beta = r/2$, hence $\dot{b} = rb'/2$. Note that the (r, r) component of the field equations was used to obtain $\dot{a}$. Putting this together gives three first order field equations in new variables

$$8\pi r^2\rho = 2\dot{x} + x - z_1 - z_2\,, \tag{3.3.2}$$

$$8\pi r^2 p = y\,, \tag{3.3.3}$$

$$8\pi r^2 p_\perp = \frac{\dot{x}(x+y-z_1-z_2)}{2(1-x)} + \dot{y} - \dot{z}_1 - z_1 + \frac{(x+y-z_1-z_2)^2}{4(1-x)}\,. \tag{3.3.4}$$

Using this the combination $p + 2p_\perp \leq \rho$ becomes

$$\begin{aligned}
&\dot{x}\Big(3x + y - z_1 - z_2 - 2\Big) + 2\big(1-x\big)\big(\dot{y} - \dot{z}_1 - z_2\big)\\
&\leq -\frac{1}{2}\Big(3x^2 + \big(y - z_1 - z_2\big)^2 - 2\big(x-y\big) - 2\big(z_2(4x-3) + z_1\big)\Big)\\
&\leq -\frac{1}{2}u\big(x, y, z_1, z_2\big)\,,
\end{aligned}$$

where we have defined the right hand side of the inequality to be u. The next section will give some insight on the derivation of the optimisation function w which is required for the calculation in the aforementioned paper [2].

3.3.1 Finding w

In order to represent this combination of the field equations as a problem that can be optimised, we require a function $w(x, y, z_1, z_2)$ such that the derivative $\dot{w}$ is equal to the left hand side of the inequality $p + 2p_\perp \leq \rho$ with some undetermined pre-factor, this pre-factor will also be a function of x, y, z_1 and z_2. In the interest of generalising the result from [122], we deduce that this function should be of the form $w = \gamma^2/(1-x)$ for some function $\gamma(x, y, z_1, z_2)$, and in this section we will check whether this hypothesis is valid.

Calculating the one-form or exterior derivative $\mathrm{d}w = \dot{w}\mathrm{d}\beta$

$$\begin{aligned}
\mathrm{d}w =& \frac{\gamma}{(1-x)^2}\Big\{\mathrm{d}x\Big(3x + y - z_1 - z_2 - 2\Big) + 2\big(1-x\big)\big(\mathrm{d}y - \mathrm{d}z_1 - \mathrm{d}z_2\big)\Big\}\\
=& \frac{\gamma}{(1-x)^2}\Bigg\{\mathrm{d}x\Big(\gamma + 2\gamma_x(1-x)\Big) + 2\big(1-x\big)\big(\gamma_y\mathrm{d}y + \gamma_{z_1}\mathrm{d}z_1 + \gamma_{z_2}\mathrm{d}z_2\big)\Bigg\}\\
=& w_x\mathrm{d}x + w_y\mathrm{d}y + w_{z_1}\mathrm{d}z_1 + w_{z_2}\mathrm{d}z_2\,,
\end{aligned} \tag{3.3.5}$$

where $\gamma_x = \partial_x\gamma$. The first line gives the form of the function we are searching for and the last two lines are implied by the chain rule, comparing the two we can see the restrictions $\gamma_y = 1$ and $\gamma_{z_1} = \gamma_{z_2} = -1$.

Note that in the three-dimensional case $\vec{w} = (w_1, w_2, w_3)$ the following is always true $\vec{\nabla}.\big(\vec{\nabla} \times \vec{w}\big) = \partial_x\big(\partial_y w_3 - \partial_z w_2\big) + \partial_y\big(\partial_z w_1 - \partial_x w_3\big) + \partial_z\big(\partial_x w_2 - \partial_y w_1\big) = 0$, in four-dimensions this becomes $\mathrm{d}^2 w = 0$. Thus if the w we require exists then the exterior derivative must satisfy $\mathrm{d}^2 w = 0$, where

$$\begin{aligned} \mathrm{d}^2 w = (w_{yx} - w_{xy})\mathrm{d}x \wedge \mathrm{d}y + (w_{z_1 x} - w_{x z_1})\mathrm{d}x \wedge \mathrm{d}z_1 + (w_{z_2 x} - w_{x z_2})\mathrm{d}x \wedge \mathrm{d}z_2 \\ + (w_{z_1 y} - w_{y z_1})\mathrm{d}y \wedge \mathrm{d}z_1 + (w_{z_2 y} - w_{y z_2})\mathrm{d}y \wedge \mathrm{d}z_2 + (w_{z_2 z_1} - w_{z_1 z_2})\mathrm{d}z_1 \wedge dz_2 \,. \end{aligned}$$

Writing this out explicitly gives rise to the following equations

$$\begin{aligned} w_{yx} - w_{xy} &= -\frac{\gamma_y(3x + y - z_1 - z_2 - 2)}{(1-x)^2} + \frac{2\gamma_x}{1-x} + \frac{\gamma}{(1-x)^2} = 0\,, \\ w_{z_1 y} - w_{y z_1} &= -\frac{2(\gamma_y + \gamma_{z_1})}{1-x} = 0\,, \\ w_{z_2 y} - w_{y z_2} &= -\frac{2\,(\gamma_y + \gamma_{z_2})}{1-x} = 0\,, \\ w_{z_1 x} - w_{x z_1} &= -\frac{\gamma_{z_1}(3x + y - z_1 - z_2 - 2)}{(1-x)^2} - \frac{2\gamma_x}{1-x} - \frac{\gamma}{(1-x)^2} = 0\,, \\ w_{z_2 x} - w_{x z_2} &= -\frac{\gamma_{z_2}(3x + y - z_1 - z_2)}{(1-x)^2} - \frac{2\gamma_x}{1-x} - \frac{\gamma}{(1-x)^2} = 0\,, \\ w_{z_1 z_2} - w_{z_2 z_1} &= \frac{2(\gamma_{z_1} - \gamma_{z_2})}{1-x} = 0\,. \end{aligned}$$

Inserting the condition $\gamma_y = -\gamma_{z_1} = -\gamma_{z_2} = 1$ we found from (3.3.5) reduces three of the six equations to zero as we require. The remaining three are reduced to one non trivial equation which will be used to solve for γ

$$\begin{aligned} w_{yx} - w_{xy} = -\big(w_{z_1 x} - w_{x z_1}\big) = -\big(w_{z_2 x} - w_{x z_2}\big) \\ = -\frac{3x + y - z_1 - z_2 - 2}{(1-x)^2} + \frac{2\gamma_x}{1-x} + \frac{\gamma}{(1-x)^2} = 0\,, \end{aligned}$$

hence we have

$$2\gamma_x(1-x) + \gamma - 3x - y + z_1 + z_2 + 2 = 0\,,$$

this is a first order differential equation for γ, and is exactly what we would obtain by comparing the coefficients of $\mathrm{d}x$ in equation (3.3.5). Solving this first order differential equation yields

$$\gamma = 4 - 3x + y - z_1 - z_2 + \sqrt{1-x}\,\Gamma(y, z_1, z_2)\,,$$

where Γ is a function that is constant when integrating with respect to x, this function may have dependence on the variables y, z_1, z_2. However, the contributions from $y\,, z_1$ and z_2 are consistent with the observation that $\gamma_y = -\gamma_{z_1} = -\gamma_{z_2} = 1$ which implies $\Gamma =$ constant. Since w must coincide with equation (1.5.5) in the limit $z_1 \to 0$ and $z_2 \to 0$ this implies $\Gamma = 0$. Thus the function we seek is

$$w = \frac{\big(4 - 3x + y - z_1 - z_2\big)^2}{1-x}\,. \tag{3.3.6}$$

This function will be maximised to give the required ratio on m_g/r in the next section.

3.3.2 Maximising w

The derivative of w with respect to β is

$$\begin{aligned} \dot{w} &= \frac{\big(4 - 3x + y - z_1 - z_2\big)}{\big(1-x\big)^2}\Big(\dot{x}\big(3x + y - z_1 - z_2 - 2\big) + 2\big(\dot{y} - \dot{z_1} - z_2\big)\big(1-x\big)\Big) \\ &\leq -\frac{1}{2}\frac{\big(4 - 3x + y - z_1 - z_2\big)}{\big(1-x\big)^2}u(x, y, z_1, z_2)\,. \end{aligned} \tag{3.3.7}$$

Given the constraints on x, y, z_1 and z_2 in (3.3.1) it is straightforward to see that $4 - 3x + y - z_1 - z_2 \geq 0$, therefore if $u(x, y, z_1, z_2) \leq 0$ then $w(x, y, z_1, z_2)$ is increasing. Hence we firstly require that $u \leq 0$, which implies

$$\begin{aligned} 0 &\geq 3x^2 + \big(y - z_1 - z_2\big)^2 - 2\big(x - y\big) - 2\Big(z_2\big(4x - 3\big) + z_1\Big) \\ &= 3x\big(x-1\big) + x - 8xz_2 + \big(y - z_1 - z_2\big)^2 + 2y - 2z_1 + 6z_2 \\ &= \big(3x - 8z_2 + 1\big)\big(x-1\big) + \big(y - z_1 - z_2 + 1\big)^2\,, \end{aligned}$$

or equivalently

$$\big(y - z_1 - z_2 + 1\big)^2 \leq \big(3x - 8z_2 + 1\big)\big(1-x\big)\,.$$

This condition can be rearranged to give the relation

$$\begin{aligned} 2\big(y - z_1 - z_2\big) &\leq \big(3x - 8z_2 + 1\big)\big(1 - x\big) - 1 - \big(y - z_1 - z_2\big)^2 \\ &\leq \big(3x - 8z_2 + 1\big)\big(1 - x\big) - 1 \,. \end{aligned} \tag{3.3.8}$$

Secondly, we deduce that w has a maximum $w \leq \sup_{\mathcal{U}} w(x, y, z_1, z_2)$. Next, we will determine the maximum of w and the values of x, y, z_1 and z_2 for which this is attained. Note that

$$\begin{aligned} w &= \frac{(4 - 3x + y - z_1 - z_2)^2}{1 - x} \\ &= \frac{(1 + y - z_1 - z_2)^2}{1 - x} + 6(1 + y - z_1 - z_2) + 9(1 - x) \\ &\leq 3x - 8z_2 + 1 + 6(1 + y - z_1 - z_2) + 9(1 - x) \\ &= 16 - 6x + 6y - 6z_1 - 14z_2 = 16 - 6x + 6(y - z_1 - z_2) - 8z_2 \,, \end{aligned} \tag{3.3.9}$$

Using equation (3.3.8) gives

$$\begin{aligned} w \leq& 16 - 6x + 3(3x - 8z_2 + 1)(1 - x) - 3 - 8z_2 \\ =& 16 - 9x^2 - 24z_2(1 - x) - 8z_2 \leq 16 \,, \end{aligned} \tag{3.3.10}$$

thus $\sup_{\mathcal{U}} w = 16$, for $0 \leq x < 1$ and $z_1 \,, z_2 \geq 0$. From equation (3.3.6) we can see the maximum value of 16 is achieved at $x = y = z_1 = z_2 = 0$.

3.3.3 Deriving the bound

Thus, when the variables $x \,, y \,, z_1 \,, z_2$ satisfy the conditions outlined by $\mathcal{U}$ and $u \leq 0$ the function w has an upper bound

$$w = \frac{\big(4 - 3x + y - z_1 - z_2\big)^2}{1 - x} \leq 16 \,, \tag{3.3.11}$$

this can be translated into an upper bound on M_g/R. Rearranging equation (3.3.11) gives

$$4 - 3x - z_1 - z_2 \leq 4\sqrt{1 - x} \,,$$

collecting the terms involving $1-x$ and grouping them on the left hand side of the inequality yields

$$\left(2-3\sqrt{1-x}\right)^2 \leq 1+3z_1+3z_2 \implies 2-3\sqrt{1-x} \leq \sqrt{1+3z_1+3z_2}\,,$$

which reduces to

$$\begin{aligned}\sqrt{1-x} &\leq \frac{2}{3}-\frac{1}{3}\sqrt{1+3z_1+3z_2}\,,\\ 1-x &\leq \frac{4}{9}+\frac{1}{9}\left(1+3z_1+3z_2\right)-\frac{4}{9}\sqrt{1+3z_1+3z_2}\,.\end{aligned}$$

Therefore we have found the following upper bound for x

$$x \leq \frac{4}{9}-\frac{z_1}{3}-\frac{z_2}{3}+\frac{4}{9}\sqrt{1+3z_1+3z_2}\,, \tag{3.3.12}$$

this gives rise to the required result since $x=\frac{2m_g}{r}-\frac{q^2}{r^2}+\frac{\Lambda r^2}{3}$, $z_1=\frac{q^2}{r^2}$ and $z_2=\Lambda r^2$. Reverting to original variables yields

$$\frac{m_g}{r} \leq \frac{2}{9}+\frac{q^2}{3r^2}-\frac{\Lambda r^2}{3}+\frac{2}{9}\sqrt{1+\frac{3q^2}{r^2}+3\Lambda r^2}\,.$$

Then at the boundary when $R=r$ we have the total mass and total charge given by $m_g(R)=M$ and $q(R)=Q$ respectively, therefore

$$\frac{M_g}{R} \leq \frac{2}{9}+\frac{Q^2}{3R^2}-\frac{\Lambda R^2}{3}+\frac{2}{9}\sqrt{1+\frac{3Q^2}{R^2}+3\Lambda R^2}\,, \tag{3.3.13}$$

when $\Lambda=Q=0$, this equation reduces to $2M/R \leq 8/9$ which agrees with the bound derived in section 1.5.2. It was shown in [2] that the inequality (3.3.13) is sharp.

3.4 Maximally extended RNdS spacetime

So far we have studied some conditions for maintaining hydrostatic equilibrium in the Reissner-Nordström de Sitter spacetime. In this section we will consider the effects of the radius not respecting the bound (3.3.13). In this case, the radius of the matter distribution R will reduce to the event horizon and develop a singularity at $r=0$.

As mentioned in section 1.1.1 there are various coordinate systems which allow us to see the full causal structure of a solution, and explore sections of spacetime that might be hidden behind an event horizon. The RNdS solution contains three horizons, namely the cosmological r_c inner r_+ and outer r_- horizons. Due to the complicated nature of these horizons, in this section we will set $\Lambda = 0$ and show the derivation of the maximally extended Reissner-Nordström solution. This method can be generalised to obtain the maximally extended RNdS spacetime, but will not be shown in this thesis.

3.4.1 Extension of the Reissner-Nordström solution

The Reissner-Nordström solution is given by the line element

$$\mathrm{d}s^2 = -\Big(1 - \frac{2M_g}{r} + \frac{Q^2}{r^2}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2M_g}{r} + \frac{Q^2}{r^2}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,,$$

and the event horizons correspond to the radial values that satisfy the condition $1 - \frac{2M_g}{r} + \frac{Q^2}{r^2} = 0$ which generates the outer r_+ and inner r_- horizons

$$r_\pm = M_g \pm \sqrt{M_g^2 - Q^2}\,.$$

This solution has four possibilities depending on the relationship between Q and M_g, and only one of these cases gives rise to a spacetime that will contain two event horizons if gravitational collapse occurs. The first case $Q = 0$ yields the trivial result $r_+ = r_s\,, r_- = 0$, secondly when $Q^2 > M_g^2$ this gives rise to an non-physical solution that contains a naked singularity with no event horizons. The remaining two cases correspond to $Q^2 = M_g^2$ and $Q^2 < M_g^2$, the former implies the black hole will have the minimal possible mass with degenerate horizons $r_+ = r_-$ and this is referred to as an extremal black hole, whereas the latter yields a solution with $r_+ > r_-$ which is divided into three regions with two event horizons. In the construction of the spacetime diagram, we will consider the case $r_+ > r_-$ thus $Q^2 < M_g^2$.

If we use the coordinate system (t, r, θ, ϕ) to construct a spacetime diagram, it will not be possible to extend beyond the coordinate singularity located at r_+. This can be seen by studying null geodesics in the radial direction for constant θ and ϕ, then examining the behaviour as r_+ is approached. Such geodesics travel along the boundary of the light cone and can be used to measure the slope of the boundary (or tilting of the light cone). The geodesics are obtained by calculating $\frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} = 0$ where τ denotes the proper time, this yields

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \left(1 - \frac{2M_g}{r} + \frac{Q^2}{r^2}\right)^{-1} = \frac{r^2}{(r - r_+)(r - r_-)}\,.$$

As the horizon r_+ is approached $\frac{\mathrm{d}t}{\mathrm{d}r} \to \infty$ and the light cones become smaller then eventually are reduced to a line at $r = r_+$, therefore null geodesics cannot be extended beyond the outer horizon.

We can now introduce the so-called tortoise coordinate r^*, by integrating the null geodesic $r^* = \int \left(1 - \frac{2M_g}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}\right)^{-1} \mathrm{d}\tilde{r}$ which yields

$$r^* = r + \frac{1}{2\sqrt{M_g^2 - Q^2}}\Big(r_+^2 \ln(r - r_+) - r_-^2 \ln(r - r_-)\Big)\,. \tag{3.4.1}$$

Therefore r is defined implicitly in terms of r^* by the relation in (3.4.1). The new line element is then given by

$$\mathrm{d}s^2 = \left(1 - \frac{2M_g}{r} + \frac{Q^2}{r^2}\right)\left(\mathrm{d}r^{*2} - \mathrm{d}t^2\right) + r^2 \mathrm{d}\Omega^2\,,$$

where $r = r(r^*)$ is viewed as a function of r^*. We can see the (t, t) and (r^*, r^*) metric coefficients now vanish at r_+ as opposed to becoming singular, however when $r = r_+$, $r^* \to -\infty$ therefore r^* is not well-defined beyond the horizon. Thus we need another change of coordinates to describe this region of spacetime.

To achieve this, we extend the tortoise transformation to eliminate t and r^* from the metric by adopting the Eddington-Finkelstein coordinates [40, 41] as follows $u = t - r^*$ and $v = t + r^*$, these are often referred to as the ingoing and outgoing null coordinates respectively. Note that $v - u = 2r^*$, and setting $u =$ constant or $v =$ constant gives rise to ingoing or outgoing null geodesics respectively. Eddington-Finkelstein coordinates enable a better understanding of spacetime, particularly in the region of the horizon r_+. In this region we can write down two metrics corresponding to (u, r) and (v, r), each metric describes versions of the spacetime geometry relating to ingoing and outgoing null geodesics respectively. In each spacetime, geodesics can be extended beyond the horizon therefore the light cones do not tilt to the extent that they close up as the horizon is approached. Alternatively we can write the metric in double null coordinates (u, v) where r is given in terms of u and v implicitly by the relations $v - u = 2r^*$ and equation (3.4.1). Using $uv = t^2 - r^{*2}$ the first two terms of the line element reduce to

$$\mathrm{d}s^2 = -\left(1 - \frac{2M_g}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}u\mathrm{d}v = -\frac{\left(r - r_-\right)}{r^2}^{\left(1+\frac{r_-^2}{r_+^2}\right)} \mathrm{e}^{\frac{\sqrt{M_g^2-Q^2}}{r_+^2}\left(v-u-2r\right)}\mathrm{d}u\mathrm{d}v\,, \tag{3.4.2}$$

where we used $r^* = \frac{1}{2}(v-u)$ to rewrite the metric coefficient. Alternatively, the metric can be written as follows

$$\mathrm{d}s^2 = -\left(1 - \frac{2M_g}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}u\mathrm{d}v = -\frac{\left(r - r_+\right)}{r^2}^{\left(1+\frac{r_+^2}{r_-^2}\right)} \mathrm{e}^{\frac{-\sqrt{M_g^2-Q^2}}{r_-^2}\left(v-u-2r\right)}\mathrm{d}u\mathrm{d}v\,. \tag{3.4.3}$$

The line elements in double null Eddington-Finkelstein coordinates (3.4.2) and (3.4.3) need to be transformed further in order to be extended beyond the horizons $r_\pm$. The new coordinates required are referred to as the Kruskal-Szekeres representation, which can be used to construct the Kruskal-Szekeres extension and from this the Carter-Penrose diagrams. The latter allows a visualisation of both the ingoing and outgoing null geodesics in one spacetime diagram. We will show a derivation of the metrics used to obtain this diagram, but will not explicitly show the Kruskal-Szekeres spacetime diagram. First we relabel the coordinates as

$$U_+ = -\mathrm{e}^{-\frac{\sqrt{M_g^2-Q^2}}{r_+^2}u}\,,\; V_+ = \mathrm{e}^{\frac{\sqrt{M_g^2-Q^2}}{r_+^2}v}\,,$$
$$U_- = \mathrm{e}^{\frac{\sqrt{M_g^2-Q^2}}{r_-^2}u}\,,\; V_- = -\mathrm{e}^{-\frac{\sqrt{M_g^2-Q^2}}{r_-^2}v}\,,$$

therefore $r(U_+, V_+)$ and $r(U_-, V_-)$ are both defined implicitly by the relation given in equation (3.4.1) and

$$U_+V_+ = -\,\mathrm{e}^{\frac{\sqrt{M_g^2-Q^2}}{r_+^2}(v-u)} = -\mathrm{e}^{\frac{2\sqrt{M_g^2-Q^2}}{r_+^2}r^*}\,,$$
$$U_-V_- = -\,\mathrm{e}^{-\frac{\sqrt{M_g^2-Q^2}}{r_-^2}(v-u)} = -\mathrm{e}^{-\frac{2\sqrt{M_g^2-Q^2}}{r_-^2}r^*}\,.$$

This can be used to transform equations (3.4.2) and (3.4.3) to the two-dimensional (U_+, V_+) and (U_-, V_-) line elements respectively, where the line element in equation (3.4.2) becomes

$$\mathrm{d}s^2 = -\frac{r_+^4\left(r-r_-\right)^{\left(1+\frac{r_-^2}{r_+^2}\right)}\mathrm{e}^{-\frac{2r\sqrt{M_g^2-Q^2}}{r_+^2}}}{r^2(M_g^2-Q^2)}\mathrm{d}U_+\mathrm{d}V_+\,,$$

inspecting the metric reveals that the singularity at $r = r_+$ has been removed, but at $r = r_-$ the coefficient vanishes. Using (U_-, V_-) coordinates yields an alternative metric that is well-defined at $r = r_-$ but vanishes at $r = r_+$, this can be written as

$$\mathrm{d}s^2 = -\frac{r_-^4\left(r-r_+\right)^{\left(1+\frac{r_+^2}{r_-^2}\right)}\mathrm{e}^{\frac{2r\sqrt{M_g^2-Q^2}}{r_-^2}}}{r^2(M_g^2-Q^2)}\mathrm{d}U_-\mathrm{d}V_-\,.$$

Now we can transform the metric into Kruskal-Szekeres coordinates $(T_\pm, R_\pm)$ defined by $T_+ - R_+ = U_+\,, T_+ + R_+ = V_+\,,$ so that $-\mathrm{d}T_+^2 + \mathrm{d}R_+^2 = -\mathrm{d}U_+\mathrm{d}V_+$ to obtain the metric

$$\mathrm{d}s^2 = \frac{r_+^4\left(r-r_-\right)^{\left(1+\frac{r_-^2}{r_+^2}\right)}\mathrm{e}^{-\frac{2r\sqrt{M_g^2-Q^2}}{r_+^2}}}{r^2(M_g^2-Q^2)}\left(-\mathrm{d}T_+^2+\mathrm{d}R_+^2\right)+r^2\mathrm{d}\Omega^2\,.$$

A similar relabelling can be made for the (U_-, V_-) coordinates with $T_- - R_- = U_-$ and $T_- + R_- = V_-$. The Carter-Penrose diagrams can be obtained from the Kruskal-Szekeres extension in either $(U_\pm, V_\pm)$ or $(T_\pm, R_\pm)$ coordinates, and will be shown in the next section.

3.4.2 Carter-Penrose diagrams

Carter-Penrose diagrams are often referred to as conformal diagrams, they enable us to visualise the entire spacetime in a two-dimensional diagram. This is achieved by suppressing the two spatial dimensions θ, ϕ and considering the two-dimensional metric with coordinates $(t, r) \to (U_\pm, V_\pm)$. Then each point on the diagram represents a two-sphere with radius r and null geodesics are depicted by straight lines.

Applying a conformal transformation of the form $U'_\pm = \arctan U_\pm +\,, V'_\pm =$

arctan $V_\pm$ gives rise to the rescaled metrics

$$\mathrm{d}s^2 = \frac{r_+^4\left(r-r_-\right)^{\left(1+\frac{r_-^2}{r_+^2}\right)} \mathrm{e}^{-\frac{2r\sqrt{M_g^2-Q^2}}{r_+^2}}}{r^2\cos^2 U'_+ \cos^2 V'_+ (M_g^2-Q^2)} \mathrm{d}U'_+ \mathrm{d}V'_+ \,,$$

$$\mathrm{d}s^2 = \frac{r_-^4\left(r-r_+\right)^{\left(1+\frac{r_+^2}{r_-^2}\right)} \mathrm{e}^{\frac{2r\sqrt{M_g^2-Q^2}}{r_-^2}}}{r^2\cos^2 U'_- \cos^2 V'_- (M_g^2-Q^2)} \mathrm{d}U'_- \mathrm{d}V'_- \,.$$

This conformal transformation allows the diagram to include infinity in a finite diagram. There are five distinct infinities appearing on the diagram including the future and past null $\mathcal{J}^\pm$, future and past time-like $i^\pm$ infinities and spatial infinity i^0. Here $\mathcal{J}^-$ and $\mathcal{J}^+$ correspond to $V'_+ = \pi/2$ or $U'_+ = \pi/2$ respectively, then i^- and i^+ satisfy $V'_+ = -U'_+ = \pi/2$ or $U'_+ = -V'_+ = \pi/2$ respectively, finally spatial infinity i^0 is located at $U'_+ = V'_+ = \pi/2$. The region $r > r_+$ can be described by either the Kruskal-Szekeres extension or the exterior Reissner-Nordstrom solution, however extending geodesics into the region $r_- < r \leq r_+$ will require the use of Kruskal-Szekeres coordinates (U_+, V_+) and likewise the region $r \leq r_-$ will need the alternative description given by (U_-, V_-).

In the construction of the Carter-Penrose diagram for the Schwarzschild solution, passing the horizon r_s results in the time-like coordinate t becoming space-like and thus r becomes time-like, the singularity at $r = 0$ is space-like. However, the singularity in the Reissner-Nordström solution is time-like.

Below are some Carter-Penrose diagrams for the Reissner-Nordström de Sitter spacetime. In particular we can see how the path of a particle is affected as it approaches the horizon, and its fate once the horizon is crossed.

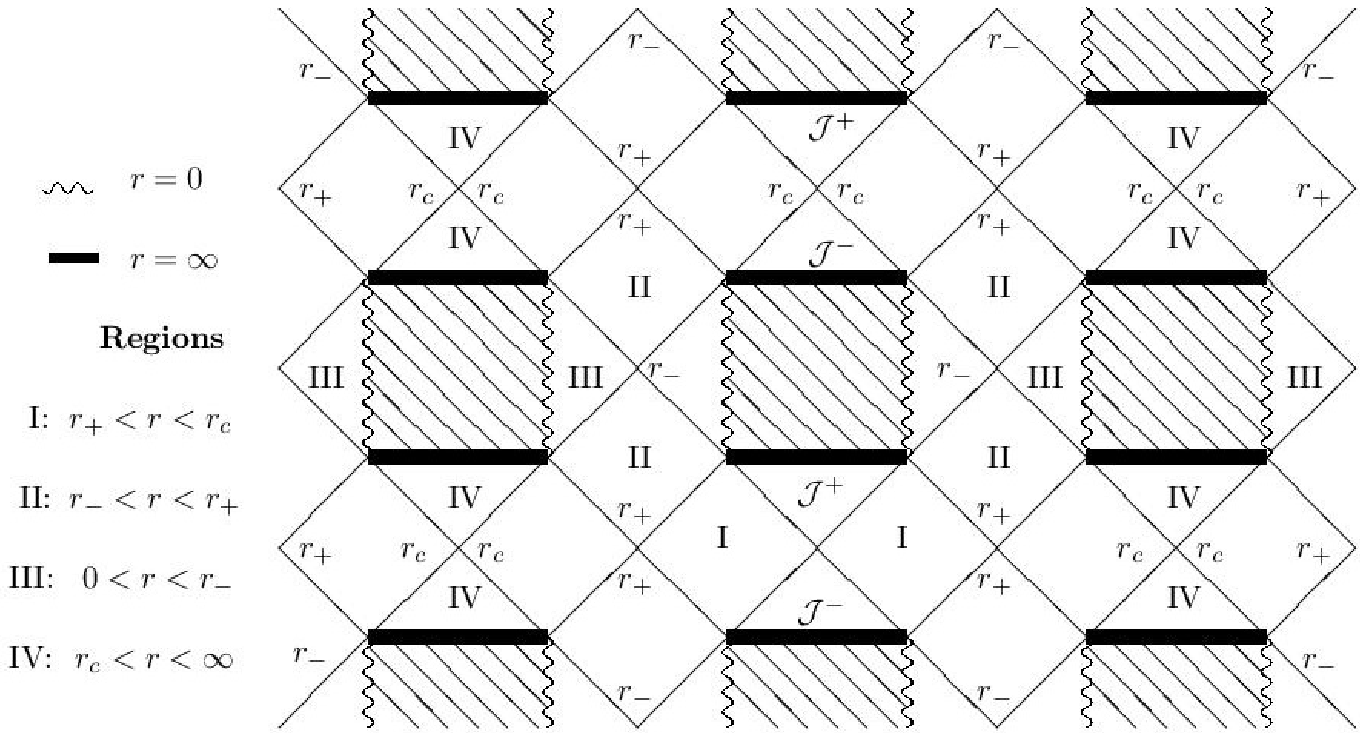

Figure 3.1: *Each point corresponds to a two-sphere with horizontal and vertical lines representing space and time respectively. Therefore the singularities seen in the RNdS solution are timelike. We denote the inner, outer and cosmological horizons r_-, r_+ and r_c. Now we are able deduce the causal structure of spacetime. Notice that the horizons divide spacetime into several repeated regions, for which we can determine the fate of an observer.*

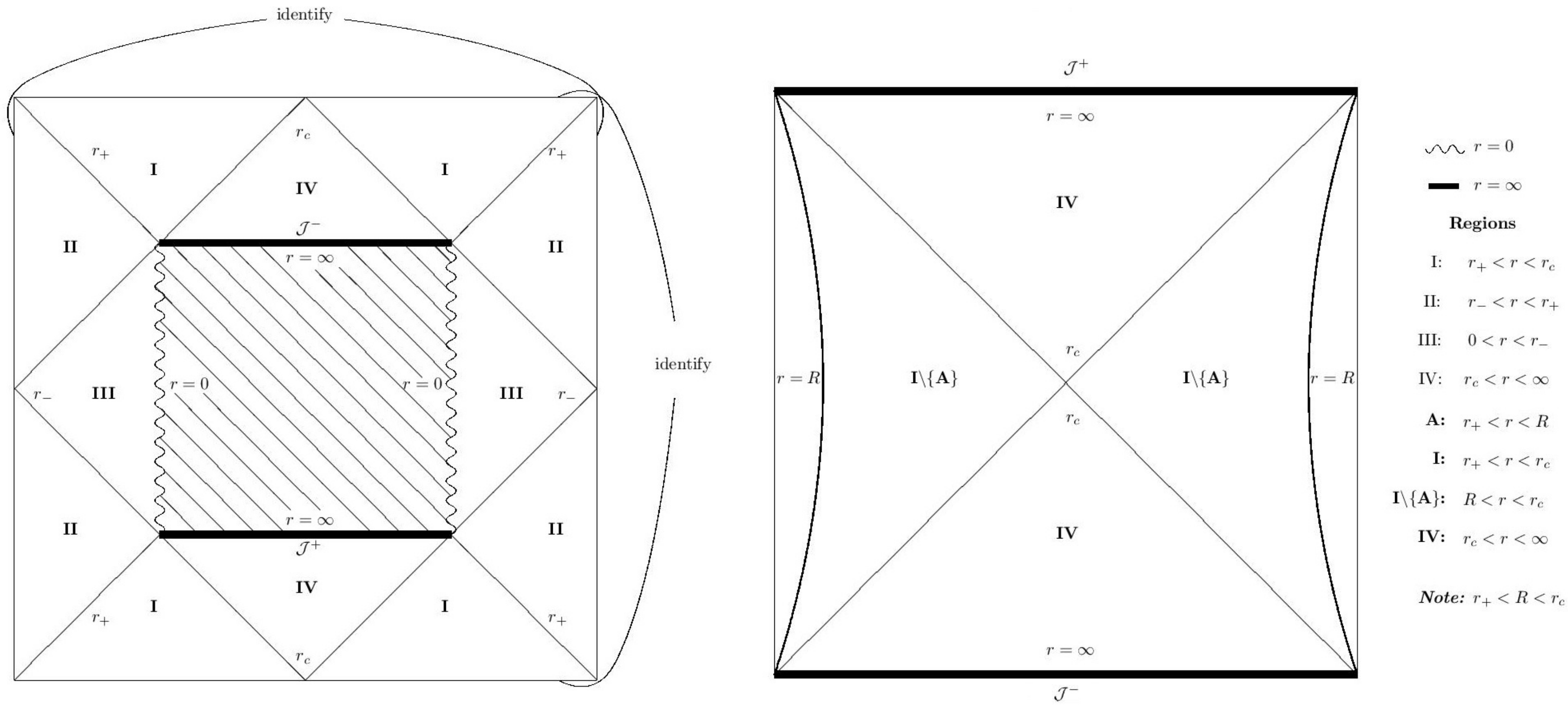

Figure 3.2: *On the left is a segment of the maximally extended RNdS solution, the edges can be identified and 'glued' together to obtain the maximal extension.*

The right includes part of the same diagram with the boundary of a matter distribution inserted in the region $r_+ < R < r_c$. *In this case the singularity is not a part of the spacetime diagram, provided that hydrostatic equilibrium is maintained.*

3.5 The Nariai solutions

The Kottler solution has line element $\mathrm{d}s^2 = \mathrm{e}^a\mathrm{d}t^2 + \mathrm{e}^b\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$ with $\mathrm{e}^a = \mathrm{e}^{-b} = 1 - \frac{2M_i}{r} - \frac{\Lambda}{3}r^2$. This solution contains two horizons, namely the event r_s and cosmological r_c horizons. Let

$$\mathcal{H}(r) = 3r\mathrm{e}^{-b} = -\Lambda r^3 + 3r - 6M_i\,,$$

then the cubic $\mathcal{H} = 0$ will have three distinct roots provided that the discriminant $\mathcal{D}_3 > 0$. For the function $\mathcal{H}$ the discriminant $\mathcal{D}_3$ only contains two terms since it has no quadratic term, where $\mathcal{D}_3 = -4\alpha\beta^3 - 27\alpha^2\gamma^2$ with $\alpha = -\Lambda\,, \beta = 3$ and $\gamma = -6M_i$. The presence of a quadratic term would make $\mathcal{D}_3$ considerably longer. Then $\mathcal{D}_3 > 0$ yields the condition $9\Lambda M_i^2 < 1$. The function $\mathcal{H}$ contains three roots, and two of these roots give the locations of the horizons r_c and r_s, whereas the third root is negative and does not have any physical significance since the radial value must be positive.

The Kottler spacetime has a subset of solutions with maximal mass, this is achieved when the horizons r_c and r_s are degenerate or equivalent. The repeated root implies the discriminant of $\mathcal{H} = 0$ satisfies $\mathcal{D}_3 = 0$ therefore $9\Lambda M_i^2 = 1$. In the region of the repeated horizons the metric coefficients in Schwarzschild-like coordinates become $\mathrm{e}^a = \mathrm{e}^{-b} = 1 - \frac{2M_i}{r} - \frac{\Lambda}{3}r^2 = 0$, therefore the (t, r) coordinates in the Kottler solution are not sufficient to describe this region of spacetime and a more suitable description is required. This is given by the Nariai solution, where the exterior Nariai solution has been derived using various methods [30, 221, 222]. In [221], it was shown that the Nariai spacetime can be viewed as a four-dimensional submanifold of a flat six-dimensional Lorentzian manifold with inner product structure. This is described by the line element

$$\mathrm{d}s^2 = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2 + \mathrm{d}x_5^2\,, \tag{3.5.1}$$

subject to the constraints $-x_0^2 + x_1^2 + x_2^2 = \frac{1}{\Lambda}$ and $x_3^2 + x_4^2 + x_5^2 = \frac{1}{\Lambda}$, these conditions state that the line element (3.5.1) is a product of a three-sphere and hyperbolic three-space. Using a coordinate transformation we obtain the exterior Nariai metric

$$\mathrm{d}s^2_{\mathrm{ext}} = \frac{1}{\Lambda}\Big(-(\alpha\sin\chi + \beta\cos\chi)^2\mathrm{d}t^2 + \mathrm{d}\chi^2 + \mathrm{d}\Omega^2\Big)\,, \tag{3.5.2}$$

this exterior solution was obtained by Nariai in 1951, [30]. The coordinate transformation will be described in more detail in the presence of charge in the next section 3.5.1, where setting $Q = 0$ will yield the above metric. Note that α and β are constants which will be fixed when matching to the interior metric. The Nariai metric is more commonly written with the constants already fixed to $\alpha = 0$ and $\beta = 1$.

An alternative method was used to derive the exterior Nariai solution in [222], where (3.5.2) was obtained by considering a slight deviation from the maximal mass Nariai solutions. This was achieved with $9\Lambda M_i^2 = 1 - \epsilon^2$ for $0 \leq \epsilon \ll 1$, then letting $\epsilon \to 0$ implies $r_c \to r_s$ and yields the Kottler spacetime with degenerate horizons. Then the relevant coordinate transformation, which is outlined in [222], will give rise to a spacetime which deviates from the Nariai spacetime slightly, and letting $\epsilon \to 0$ gives the Nariai metric (3.5.2) with $\alpha = 0$ and $\beta = 1$.

The interior metric can be obtained directly from the TOV-like equation (3.2.7) with isotropic pressure, constant density $4\pi\rho = \Lambda$ and in the absence of charge, similarly to [70]. The interior Nariai metric is given by

$$\mathrm{d}s^2_{\mathrm{int}} = -\left(\frac{\rho + p_c}{\rho}\right)^2 \left(1 - \frac{p_c \cos\chi}{p_c + \rho}\right)^2 \mathrm{d}t^2 + \frac{1}{\Lambda}\left(\mathrm{d}\chi^2 + \mathrm{d}\Omega^2\right), \tag{3.5.3}$$

this metric was obtained much later than the exterior Nariai solution, following it by over half a century in [70]. The derivation will be shown in the next section in the presence of charge, and by setting $q = 0$ and assuming a constant density $4\pi\rho = \Lambda$ will yield the interior Nariai solution (3.5.3).

3.5.1 Charged Nariai counterpart

The charged Nariai spacetime forms a subclass of the Reissner-Nordström de Sitter solution and contains three degenerate horizons referred to as the inner, outer and cosmological horizons. In the region of the three horizons, the RNdS metric coefficient $\mathrm{e}^{-b} \to 0$, therefore two of the RNdS coordinates become unsuitable in this spacetime region, namely the (t, r) coordinates. Hence an alternative description of spacetime is required in this region, analogously to the Nariai spacetime as a subset of the Kottler solution. In what follows we will derive the exterior and interior metric and match these metrics at the boundary.

The method utilised in [221] can also be applied to the charged Nariai spacetime, which is embedded in a six-dimensional flat Lorentzian manifold with line element (3.5.1). This line element is now subject to the constraints

$-x_0^2 + x_1^2 + x_2^2 = \frac{1}{A}$ and $x_3^2 + x_4^2 + x_5^2 = \frac{1}{B}$, where $A \neq B$ are constants with $2\Lambda = A + B$ and $2Q^2 = B - A$ therefore $A = \Lambda - Q^2$ and $B = \Lambda + Q^2$. The metric is reformulated in the following way

$$
\begin{aligned}
x_0 =& \frac{\sinh\Big(t\sqrt{\alpha^2+\beta^2}\Big)}{\sqrt{A(\alpha^2+\beta^2)}}\Big(\alpha\sin\chi + \beta\cos\chi\Big)\,,\\
x_1 =& \frac{\cosh\Big(t\sqrt{\alpha^2+\beta^2}\Big)}{\sqrt{A(\alpha^2+\beta^2)}}\Big(\alpha\sin\chi + \beta\cos\chi\Big)\,,\\
x_2 =& \frac{1}{\sqrt{A(\alpha^2+\beta^2)}}\Big(\beta\sin\chi - \alpha\cos\chi\Big)\,,
\end{aligned}
$$

which parametrises hyperbolic three-space. The remaining three components describe the three-sphere with radius $1/\sqrt{B}$ and are given by

$$
x_3 = \frac{1}{\sqrt{B}}\sin\theta\cos\phi\,, \quad x_4 = \frac{1}{\sqrt{B}}\sin\theta\sin\phi\,, \quad x_5 = \frac{1}{\sqrt{B}}\cos\theta\,.
$$

This coordinate system yields the exterior charged Nariai metric

$$
\mathrm{d}s^2_{\mathrm{ext}} = \frac{1}{A}\Big(-\big(\alpha\sin\chi_{\mathrm{ext}} + \beta\cos\chi_{\mathrm{ext}}\big)^2\mathrm{d}t^2 + \mathrm{d}\chi^2_{\mathrm{ext}}\Big) + \frac{1}{B}\mathrm{d}\Omega^2\,, \tag{3.5.4}
$$

where the subscript 'ext' is used to distinguish between this and the interior charged Nariai spacetime. In the absence of charge, we have $A = B = \Lambda$ which yields the Nariai metric (3.5.2).

The interior, first derived in [1], is also obtained from the conservation equation (3.2.5) with isotropic energy-momentum tensor, which implies $p - p_\perp = \frac{q^2}{4\pi r^4}$. In order to proceed, we write

$$
8\pi\mu := 8\pi\rho + \frac{q^2}{r^4} := \text{constant}\,,
$$

with this assumption we can use the (t,t) field equation (3.1.10) to write the metric coefficient e^{-b} as

$$
\frac{\mathrm{d}}{\mathrm{d}r}\Big(r - r\mathrm{e}^{-b}\Big) = 8\pi r^2\rho + \frac{q^2}{r^2} + \Lambda r^2\,. \tag{3.5.5}
$$

Integrating both sides gives

$$
\begin{aligned}
r - r\mathrm{e}^{-b} &= \int_0^r \Big(8\pi\tilde{r}^2\rho + \frac{q^2}{\tilde{r}^2} + \Lambda\tilde{r}^2\Big)\mathrm{d}\tilde{r} \\
&= \int_0^r \Big(8\pi\tilde{r}^2\mu + \Lambda\tilde{r}^2\Big)\mathrm{d}\tilde{r} = \frac{8\pi r^3\mu}{3} + \frac{\Lambda r^3}{3}\,. \qquad (3.5.6)
\end{aligned}
$$

Thus the metric coefficient e^{-b} becomes

$$
\begin{aligned}
\mathrm{e}^{-b} &= 1 - \frac{8\pi r^2\mu}{3} - \frac{\Lambda r^2}{3} := 1 - \frac{r^2}{R^2}\,, \\
b' &= \frac{2r}{R^2 - r^2}\,, \qquad (3.5.7)
\end{aligned}
$$

where we have defined $\frac{3}{R^2} := 8\pi\mu + \Lambda = \text{constant}$. Using equation (3.2.2) we can write $a' = 8\pi r\mathrm{e}^b\big(\rho + p\big) - b'$. Now let $z = \rho + p$, then the conservation equation (3.2.5) becomes

$$
z' + \frac{4\pi r R^2}{R^2 - r^2}\,z^2 - \frac{r}{R^2 - r^2}\,z - \frac{\mathrm{d}}{\mathrm{d}r}\left(\rho + \frac{q^2}{8\pi r^4}\right) = 0\,, \qquad (3.5.8)
$$

we have used the relations in equation (3.5.7) in place of e^{-b} and b'. Since the quantity $\rho + \frac{q^2}{8\pi r^4} = \text{constant}$, this can be integrated to obtain z, which yields

$$
z = \rho + p = \frac{1}{4\pi R^2}\left(1 + C\sqrt{1 - \frac{r^2}{R^2}}\right)^{-1}\,, \qquad (3.5.9)
$$

where C is an integration constant. From equation (3.2.2), we can obtain an expression for a', thus fixing the remaining metric component $\mathrm{e}^{a/2}$

$$
\begin{aligned}
a' &= \frac{2r}{R^2 - r^2}\left\{\left(1 + C\sqrt{1 - \frac{r^2}{R^2}}\right) - 1\right\}, \\
\mathrm{e}^{a/2} &= B\left(1 + C\sqrt{1 - \frac{r^2}{R^2}}\right) = \frac{1 + C\sqrt{1 - \frac{r^2}{R^2}}}{1 + C}\,, \qquad (3.5.10)
\end{aligned}
$$

the constant of integration B is fixed by setting $\mathrm{e}^{a(0)} = 1 = B(1 + C)$. The remaining constant of integration C is now fixed by considering the values of the energy density and pressure at the origin $r = 0$, given by $\rho(0) = \rho_c$ and $p(0) = p_c$, these are referred to as the central density and pressure respectively. Evaluating the central value of z gives $z_c^{-1} = \big(\rho_c + p_c\big)^{-1} = 4\pi R^2\big(1 + C\big)$ hence

$C = \frac{1}{4\pi R^2\left(\rho_c + p_c\right)} - 1.$

The current coordinate system has a singularity as $r \to R$, since $\mathrm{e}^b = \frac{R^2}{R^2 - r^2}$. The spatial part of the interior solution is a three-sphere, where the $\mathrm{d}\Omega^2$ component of the line element describes the two-sphere in terms of the two Euler angles θ and ϕ, we can thus introduce the third Euler angle χ such that $r = R\sin\chi$. Using this coordinate we have $1 - \frac{r^2}{R^2} = \cos^2\chi$. This gives rise to the interior metric

$$\mathrm{d}s^2_{\mathrm{int}} = -\left(\frac{1 + C\cos\chi_{\mathrm{int}}}{1 + C}\right)^2 \mathrm{d}t^2 + R^2\left(\mathrm{d}\chi^2_{\mathrm{int}} + \sin^2\chi_{\mathrm{int}}\,\mathrm{d}\Omega^2\right). \tag{3.5.11}$$

The charge parameter q does not appear explicitly in the metric since it is contained in the constant R with the relation $\frac{3}{R^2} = 8\pi\rho + \frac{q^2}{r^4} + \Lambda$, this quantity is constant since $8\pi\mu = 8\pi\rho + \frac{q^2}{r^4} = \text{constant}$. The constant of integration C also contains R since $C = -1 + 1/4\pi R^2(\rho_c + p_c)$, therefore q enters the metric through both R and C. Using these expressions the metric can be written explicitly in terms of q and Λ as

$$\begin{aligned}\mathrm{d}s^2 = \left(\frac{12\pi(\rho_c + p_c)}{8\pi\rho + \frac{q^2}{r^4} + \Lambda}\right)^2 &\left(1 + \cos\chi_{\mathrm{int}} - \frac{8\pi\rho + \frac{q^2}{r^4} + \Lambda}{12\pi(\rho_c + p_c)}\cos\chi_{\mathrm{int}}\right)^2 \mathrm{d}t^2 \\ &+ \frac{3}{8\pi\rho + \frac{q^2}{r^4} + \Lambda}\left(\mathrm{d}\chi^2_{\mathrm{int}} + \sin^2\chi_{\mathrm{int}}\mathrm{d}\Omega^2\right).\end{aligned} \tag{3.5.12}$$

The same method as above can be used in the derivation of the Nariai interior (3.5.3) in the absence of charge, with $q = 0$ and a constant energy density $4\pi\rho = \Lambda$. This implies that $\frac{1}{R^2} = 4\pi\rho$, following the same steps with these changes and writing the integration constant C in terms of central values yields the metric (3.5.3).

3.5.2 Matching the charged Nariai spacetimes

Let $q = er^2 = eR^2\sin^2\chi$, then for consistency the constant e must have dimension of charge/length^2. This implies $8\pi\mu = 8\pi\rho + e^2 = \text{constant}$, since $e = \text{constant}$ this yields a constant energy density ρ.

The interior metrics (3.5.11) and (3.5.12) are singularity free, and in particular the metric coefficients are finite and non-zero at the centre. Thus the interior is regular and can be matched to the exterior charged Nariai metric at the boundary χ_b. The boundary and matching conditions will be discussed further in this section.

Matching hypersurface

The boundary shared by the two metrics is referred to as the matching hypersurface Σ_b where $\chi_{\rm int} = \chi_b$, the pressure is assumed to vanish here in order to match to the electro-vacuum exterior without any discontinuities in the field equations. The pressure can be written as $p = z - \rho$, which using equation (3.1.11) in new coordinates $r \to \chi_{\rm int}$ yields

$$p(\chi_{\rm int}) = \frac{\big(\rho_c + p_c\big)\big(4\pi\rho_c - \Lambda - e^2\big) - \rho_c\Big(4\pi\big(\rho_c + 3p_c\big) - \Lambda - e^2\Big)\cos\chi_{\rm int}}{\Big(4\pi\big(\rho_c + 3p_c\big) - \Lambda - e^2\Big)\cos\chi_{\rm int} - 12\pi\big(\rho_c + p_c\big)}\,, \tag{3.5.13}$$

this equation (3.5.13) and the equations below which follow from (3.5.13) are valid for all solutions to the field equations (3.1.10)–(3.1.12) which satisfy equation (3.5.7) with $8\pi\mu = \text{constant}$.

Since the pressure vanishes at the matching hypersurface when $\chi_{\rm int} = \chi_b$, the condition $p(\chi_b) = 0$ can be used to give an expression for χ_b

$$\cos\chi_b = \frac{\big(\rho_c + p_c\big)\big(4\pi - \Lambda - e^2\big)}{\rho_c\Big(4\pi\big(\rho_c + 3p_c\big) - \Lambda - e^2\Big)}\,, \tag{3.5.14}$$

this can be rearranged to obtain an expression for the central pressure

$$p_c = \frac{\rho_c\big(1 - \cos\chi_b\big)\big(4\pi\rho_c - \Lambda - e^2\big)}{4\pi\rho_c\big(3\cos\chi_b - 1\big) + \Lambda + e^2}\,.$$

Since p_c is required to be positive and finite, this implies the denominator is positive $4\pi\rho_c\big(3\cos\chi_b - 1\big) + \Lambda + e^2 > 0$, which yields

$$\cos\chi_b > \frac{1}{3} - \frac{\Lambda + e^2}{12\pi\rho_c}\,. \tag{3.5.15}$$

This inequality can be rearranged to provide an upper bound on M/r_b which is valid for the charged Nariai spacetime. Here $r = r_b$ denotes the radial value at the boundary, or the total radius. The previous bound derived (3.3.13) does not include the charged Nariai solutions; this is due to the condition $x = 1 - \mathrm{e}^{-b} <$

1 which excludes the degenerate horizon RNdS solution. Reverting back to original coordinates $\chi_b = \arcsin(r_b/R)$, then using the trigonometric identity $\cos\bigl(\arcsin(r_b/R)\bigr) = \sqrt{1 - r_b^2/R^2}$, this implies

$$\cos\chi_b = \cos\Bigl(\arcsin\frac{r_b}{R}\Bigr) = \sqrt{1 - \frac{r_b^2}{R^2}} = \mathrm{e}^{-b/2},$$
$$\mathrm{e}^{-b/2} = \sqrt{1 - \frac{2M_g}{r_b} - \frac{\Lambda + e^2}{3} r_b^2}\,.$$

Therefore the inequality becomes

$$\sqrt{1 - \frac{2M_g}{r_b} - \frac{\Lambda + e^2}{3} r_b^2} > \frac{1}{3} - \frac{\Lambda + e^2}{12\pi\rho_c}\,, \tag{3.5.16}$$

setting $\Lambda = e^2 = 0$ and rearranging slightly yields Buchdahl's inequality $\frac{2M_g}{r_b} < \frac{8}{9}$.

Matching conditions

There are two methods for matching solutions at a common hypersuface. Firstly one can transform the metrics to Gaussian coordinates, which requires the relabelling $\psi_{\text{ext}} = \chi_{\text{ext}}/\sqrt{A}$ and $\psi_{\text{int}} = R\chi_{\text{int}}$ in the exterior and interior respectively. Then we must show that the metrics and their derivative take the same value at the matching hypersurface. Alternatively, one can use the Darmois-Israel method and show that both the first and second fundamental forms agree at the matching hypersurface [53, 54]. We used the former approach in [1] to match the charged Nariai solutions, therefore here we will follow the latter to illustrate the equivalence of the two methods.

The metrics now need to be matched at Σ_b where $\chi_{\text{int}} = \pi/2$. Using the method in [55], the metrics (3.5.4) and (3.5.11) can be written of the form

$$\mathrm{d}s^2 = -\mathrm{e}^{f(\chi)}\mathrm{d}t^2 + \frac{1}{g(\chi)^2}\mathrm{d}\chi^2 + h(\chi)^2\mathrm{d}\Omega^2\,.$$

In the exterior charged Nariai line element (3.5.4) $g(\chi) = \frac{1}{\sqrt{A}}$ and $h(\chi) = \frac{1}{\sqrt{B}}$ are constants.

The value of χ_{ext} at the matching hypersurface Σ_b is yet to be determined, this can be obtained using the coordinate transformation between the χ coordinate of the two metrics (3.5.4) and (3.5.11). This is given by

$$\frac{1}{\sqrt{A}}\mathrm{d}\chi_{\text{ext}} \to R\mathrm{d}\chi_{\text{int}}\,.$$

Which yields $\chi_{\text{ext}} \to R\sqrt{A}\,\chi_{\text{int}}$, this transfers between χ_{ext} and χ_{int} coordinates on the matching hypersurface Σ_b. Thus when $\chi_{\text{int}} = \pi/2$ we have $\chi_{\text{ext}} = R\sqrt{A}\,\pi/2$.

The first fundamental form is simply given by the metric components and hence the coefficients of the line elements $\mathrm{d}s^2_{\text{int}}$ and $\mathrm{d}s^2_{\text{ext}}$ are required to be equal at Σ_b. This yields the conditions

$$\begin{aligned} R^2 &= \frac{1}{B}\,, \\ \frac{1}{1+C} &= \frac{1}{\sqrt{A}}\left(\alpha\sin\left(R\sqrt{A}\frac{\pi}{2}\right) + \beta\cos\left(R\sqrt{A}\frac{\pi}{2}\right)\right) \\ &= \frac{1}{\sqrt{A}}\left(\alpha\sin\left(\sqrt{\frac{A}{B}}\frac{\pi}{2}\right) + \beta\cos\left(\sqrt{\frac{A}{B}}\frac{\pi}{2}\right)\right). \end{aligned} \tag{3.5.17}$$

The second fundamental form is represented by the extrinsic curvature $\mathcal{K}_{\mu\nu}$ of the matching hypersurface, the extrinsic curvature is $\mathcal{K}_{\mu\nu} = h_\mu{}^\rho\nabla_\rho n_\nu$, see [5,55]. Here $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is a three-dimensional (t,θ,ϕ) projection of the four-dimensional (t,r,θ,ϕ) metric $g_{\mu\nu}$, and n_μ is the outward pointing unit normal vector to Σ_b which has constant χ in both the interior and exterior spacetimes. The unit normal vectors are $n^\mu_{\text{ext}} = (0,\sqrt{A},0,0)$ and $n^\mu_{\text{int}} = (0,\frac{1}{R},0,0)$. The projection metric and extrinsic curvature are given by the following

$$\begin{aligned} h_{\mu\nu} &= \mathrm{diag}\left(-\mathrm{e}^f\,, h^2\,, h^2\sin^2\theta\right), \\ \mathcal{K}_{\mu\nu} &= \mathrm{diag}\left(-\frac{g\mathrm{e}^f}{2}\frac{\mathrm{d}f}{\mathrm{d}\chi}\,, \frac{gh}{2}\frac{\mathrm{d}h}{\mathrm{d}\chi}\,, \frac{gh}{2}\frac{\mathrm{d}h}{\mathrm{d}\chi}\sin^2\theta\right). \end{aligned}$$

In the exterior and interior charged Nariai spacetimes the projection metrics and second fundamental forms are given by

$$
\begin{aligned}
h_{\mu\nu}^{\text{ext}} &= \text{diag}\left(-\frac{\big(\alpha\sin\chi_{\text{ext}} + \beta\cos\chi_{\text{ext}}\big)^2}{A}\,, \frac{1}{B}\,, \frac{\sin^2\theta}{B} \right), \\
\mathcal{K}_{\mu\nu}^{\text{ext}} &= \text{diag}\left(-\frac{\big(\alpha\sin\chi_{\text{ext}} + \beta\cos\chi_{\text{ext}}\big)\big(\alpha\cos\chi_{\text{ext}} - \beta\sin\chi_{\text{ext}}\big)}{\sqrt{A}}\,, 0\,, 0 \right), \\
h_{\mu\nu}^{\text{int}} &= \text{diag}\left(-\Big(\frac{1 + C\cos\chi_{\text{int}}}{1+C}\Big)^2\,, R^2\sin^2\chi_{\text{int}}\,, R^2\sin^2\chi_{\text{int}}\sin^2\theta \right), \\
\mathcal{K}_{\mu\nu}^{\text{int}} &= \text{diag}\left(\frac{2C\sin\chi_{\text{int}}\big(1 + C\cos\chi_{\text{int}}\big)}{R\big(1+C\big)^2}\,, \right. \\
&\qquad\qquad \left. \frac{R^3}{2}\sin\chi_{\text{int}}\cos\chi_{\text{int}}\,, \frac{R^3}{2}\sin\chi_{\text{int}}\cos\chi_{\text{int}}\sin^2\theta \right).
\end{aligned}
$$

The remaining matching conditions are obtained by imposing $h_{\mu\nu}^{\text{ext}} = h_{\mu\nu}^{\text{int}}$ and $\mathcal{K}_{\mu\nu}^{\text{ext}} = \mathcal{K}_{\mu\nu}^{\text{int}}$ at Σ_b, that is when $\chi_{\text{int}} = \pi/2$ and $\chi_{\text{ext}} = R\sqrt{A}\,\pi/2 := \xi$. The first condition $h_{\mu\nu}^{\text{ext}} = h_{\mu\nu}^{\text{int}}$ confirms the matching conditions (3.5.17) obtained by matching the first fundamental forms, thus we only need to check $\mathcal{K}_{\mu\nu}^{\text{ext}} = \mathcal{K}_{\mu\nu}^{\text{int}}$. When $\chi_{\text{int}} = \pi/2$ the conditions $\mathcal{K}_{\theta\theta}^{\text{ext}} = \mathcal{K}_{\theta\theta}^{\text{int}}$ and $\mathcal{K}_{\phi\phi}^{\text{ext}} = \mathcal{K}_{\phi\phi}^{\text{int}}$ are satisfied trivially, therefore one condition remains. Namely $\mathcal{K}_{tt}^{\text{ext}} = \mathcal{K}_{tt}^{\text{int}}$, which yields

$$
-\frac{2C}{R(1+C)^2} = \frac{1}{\sqrt{A}}\Big(\alpha\sin\xi + \beta\cos\xi\Big)\Big(\alpha\cos\xi - \beta\sin\xi\Big)\,, \tag{3.5.18}
$$

the matching conditions given in equations (3.5.17) and (3.5.18) coincide with those derived in [1], thus illustrating the equivalence of the aforementioned matching methods. In [1], the matching conditions are put together and written as

$$
\begin{pmatrix} \sin\xi & \cos\xi \\ \cos\xi & -\sin\xi \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sqrt{A}/(1+C) \\ -2C/R \end{pmatrix}, \tag{3.5.19}
$$

where the matching conditions also imposes a relationship between χ_{int} and χ_{ext} which can be expressed as $\xi = R\sqrt{A}\,\pi/2 = \sqrt{A/B}\,\pi/2$.

3.6 Generalised charged Einstein static universes

In this section we will utilise some results derived in this chapter to obtain generalisations of the Einstein static universe, and amongst these models will be the usual Einstein static universe in the presence of charge.

The RNdS metric is given by equations (3.1.8) with metric coefficients (3.1.16)–(3.1.17), the centre of this solution lies at $r = 0$. In new coordinates $r = \sin\chi$, thus $r = 0$ gives rise to two regular centres at $\chi_{c_1} = 0$ and $\chi_{c_2} = \pi$. Note that the Nariai universe does not contain a second centre since the interior is valid for the range $0 \le \chi \le \pi/2$, whereas this solution does.

We assume a constant energy density $\rho \equiv \rho_c$, similarly to the neutral Einstein static universe. We also require that the pressure is strictly positive and finite for all values of χ including χ_{c_1} and χ_{c_2}, where the latter implies $0 < p(\pi) < \infty$. Using the pressure equation (3.5.13) evaluated at $\chi = \pi$, this condition is then expressed as

$$0 < \frac{(\Lambda + e^2)(2\rho_c + p_c) - 8\pi\rho_c(\rho_c + 2p_c)}{8\pi(2\rho_c + 3p_c) - \Lambda - e^2} < \infty\,,$$

which is finite and strictly positive provided that both the numerator and denominator are strictly positive, this yields the respective constraints

$$\begin{aligned}\Lambda + e^2 >&\ \frac{8\pi\rho_c(\rho_c + 2p_c)}{2\rho_c + p_c}\,,\\ \Lambda + e^2 <&\ 8\pi(2\rho_c + 3p_c)\,.\end{aligned}$$

Thus in the generalised Einstein static universe the combination $\Lambda + e^2$ is restricted to the range

$$\frac{8\pi\rho_c\big(\rho_c + 2p_c\big)}{2\rho_c + p_c} < \Lambda + e^2 < 8\pi\big(2\rho_c + 3p_c\big)\,.$$

By examining the pressure equation (3.5.13) it is straightforward to see that this becomes constant if we let $\Lambda + e^2 = 4\pi(\rho_c + 3p_c)$, since the terms involving the parameter χ vanish. With this constraint the pressure becomes $p = \frac{p_c(\rho_c + p_c)}{\rho_c + p_c}$, and the radius $\frac{3}{R^2} = 8\pi\rho_c + \Lambda + e^2$ reduces to $R^2 = \frac{1}{4\pi(\rho_c + p_c)}$. This gives rise to the charged analogue of the usual Einstein static universe, with constant energy density and pressure.

Next, allowing a non-constant pressure $p(\chi)$ and evaluating the pressure equation (3.5.13) at the two centres $\chi_{c_1} = 0$ and $\chi_{c_2} = \pi$ yields the central pressures p_{c_1} and p_{c_2}

$$p_{c_1} := p(\chi = 0) = p_c \,,$$
$$p_{c_2} := p(\chi = \pi) = \frac{8\pi\rho_c(\rho_c + 2p_{c_1}) - (\Lambda + e^2)(2\rho_c + p_{c_1})}{\Lambda + e^2 - 8\pi(2\rho_c + 3p_{c_1})} \,.$$

The expression for p_{c_2} can be rewritten to give an equation for $\Lambda + e^2$ in terms of the central energy density ρ_c and central pressures $p_{c_1}\,, p_{c_2}$

$$\Lambda + e^2 = \frac{8\pi\rho_c(\rho_c + 2p_{c_1}) + 8\pi p_{c_2}(2\rho_c + 3p_{c_1})}{2\rho_c + p_{c_1} + p_{c_2}}$$

Now this quantity can be eliminated from the pressure equation (3.5.13), and it can therefore be written as a function of the central values $\rho_c\,, p_{c_1}\,, p_{c_2}$ and χ

$$p(\chi, \rho_c, p_{c_1}, p_{c_2}) = \frac{\rho_c(p_{c_1} + p_{c_2}) + 2p_{c_1}p_{c_2} + \rho_c(p_{c_1} - p_{c_2})\cos\chi}{2\rho_c + p_{c_1} + p_{c_2} - (p_{c_1} - p_{c_2})\cos\chi} \,. \qquad (3.6.1)$$

Additionally, the vanishing pressure hypersurface χ_b given by equation (3.5.14) can be expressed in terms of central values $\rho_c\,, p_{c_1}$ and p_{c_2},

$$\cos\chi_b = \frac{\big(\rho_c - p_{c_1}\big)\big(p_{c_2}(2p_{c_1} - 1) - p_{c_1}\big)}{p_{c_1}(\rho_c + p_{c_1} + p_{c_2}(2\rho_c p_{c_1} - \rho_c + p_{c_1})} \,. \qquad (3.6.2)$$

Chapter 4

Modified $f(T)$ gravity

The tetrad field plays an extremely important role in determining the field equations of modified $f(T)$ gravity, this will be illustrated with an example in section 4.1. Given a metric $g_{\mu\nu}$ the associated tetrad field $e^i{}_\mu$ not unique, hence an arbitrary Lorentz transformation applied to the tetrad field will leave the corresponding metric unchanged, this statement will be shown explicitly in section 4.2, and various tetrads will be summarised. The remaining results will utilise the so-called rotated tetrad, first we will explore Birkhoff's theorem in the context of $f(T)$ gravity in section 4.3. Next we will consider solutions in hydrostatic equilibrium in section 4.4, we will derive an upper bound on M/R in $f(T)$ gravity using an analogous method from chapter 3. This bound gives rise to some constraints on the form of $f(T)$ given a static solution in hydrostatic equilibrium.

4.1 Absence of relativistic stars

This section is dedicated to review the results obtained in [209], which aimed to show that spherically symmetric solutions are not attainable in modified teleparallelism. This result was obtained by deriving the analogue of the TOV-like conservation equation (1.3.5) in $f(T)$ gravity for the static, spherically symmetric line element

$$\mathrm{d}s^2 = \mathrm{e}^a \mathrm{d}t^2 - \mathrm{e}^b \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2 \,, \tag{4.1.1}$$

which is generated locally by the tetrad field $e^i{}_\mu = \mathrm{diag}\big(\mathrm{e}^{a/2}, \mathrm{e}^{b/2}, r, r\sin\theta\big)$ whose determinant is the product of its diagonal elements $e = \mathrm{e}^{(a+b)/2} r^2 \sin\theta = \sqrt{-g}$. We refer to this as the diagonal tetrad field to distinguish from other tetrads used later, note that the opposite sign convention was used in [209].

Although the diagonal tetrad field locally gives rise to the spherically symmetric line element (4.1.1), it should be noted that the tetrad field contains singularities at the north ($\theta = 0$) and south ($\theta = \pi$) poles[2], therefore it is not well-defined globally and hence cannot be described as spherically symmetric.

The torsion scalar in a Weitzenböck spacetime (2.2.1) and the field equations (2.3.2) can be written explicitly in terms of the metric, and in component form

$$T(r) = \frac{2\mathrm{e}^{-b}}{r}\left(a' + \frac{1}{r}\right), \tag{4.1.2}$$

$$4\pi\rho = -\frac{\mathrm{e}^{-b}}{r} T' f_{TT} - \left(T - \frac{1}{r^2} - \frac{\mathrm{e}^{-b}}{r}\left(a' - b'\right)\right)\frac{f_T}{2} + \frac{f}{4}, \tag{4.1.3}$$

$$4\pi p = \left(T - \frac{1}{r^2}\right)\frac{f_T}{2} - \frac{f}{4}, \tag{4.1.4}$$

$$\begin{aligned} 4\pi p_\perp = {} & \frac{\mathrm{e}^{-b}}{2}\left(\frac{a'}{2} + \frac{1}{r}\right) T' f_{TT} \\ & + \left(\frac{T}{2} + \frac{\mathrm{e}^{-b}}{2}\left(a'' + \left(\frac{a'}{2} + \frac{1}{r}\right)\left(a' - b'\right)\right)\right)\frac{f_T}{2} - \frac{f}{4}. \end{aligned} \tag{4.1.5}$$

These are the field equations derived for the diagonal tetrads in [209], where an isotropic pressure was used so $p_\perp = p$. This particular tetrad field is the simplest possible orthonormal basis of $T\mathcal{M}$ that locally gives rise to $g_{\mu\nu}$, however using the diagonal tetrad field results in an additional off-diagonal field equation

$$\frac{\mathrm{e}^{-b/2}\cot\theta}{2r^2} T' f_{TT} = 0, \tag{4.1.6}$$

this equation was overlooked in [209] and [220]. Note that when using the diagonal tetrad field in TEGR, $f_{TT} = 0$ therefore this additional off-diagonal equation is not among the field equations.

However in the absence of this equation, the remaining field equations still do not give rise to the aforementioned result. From the three field equations above with isotropic pressure, without the restriction (4.1.6), we can derive a

[2] I am very grateful to Dmitri Vassiliev for bringing this to my attention.

conservation equation. This is obtained in an analogous way to general relativity; add equation (4.1.3) to equation (4.1.4) to give $\rho + p$ then multiply this by $a'/2$, subtract (4.1.4) from (4.1.5) and multiply the result by $2/r$, the latter is then used to simplify the former. The resulting equation is substituted into the derivative of (4.1.3) which yields

$$4\pi p' = -\frac{T'}{2r^2} f_{TT} - 2\pi a'(\rho + p) \,. \tag{4.1.7}$$

This is the same equation derived in [209], it was deduced that since this conservation equation does not agree with the conservation equation (1.3.5) from general relativity, the additional term $\frac{T'}{2r^2} f_{TT}$ must vanish. But modified teleparallelism is not required to be equivalent to general relativity. The limit $f(T) = T$ which is teleparallelism is required to be equivalent to general relativity, and the additional term in the the conservation equation vanishes in this limit. Therefore (4.1.7) would be an acceptable conservation equation if the field equations were correct, and the aforementioned result does not follow from the field equations used in [209] for the diagonal tetrad field.

However, the additional field equation (4.1.6) implies that either $T' = 0$ or $f_{TT} = 0$, the latter would yield $f(T) = \alpha T + \beta$ (where α and β are constants of integration) which would restrict our results to general relativity thus we assume the former condition. This condition $T' = 0$ means $T = \text{constant} = T_0$ therefore $f(T_0) = \text{constant}$, which causes too much of a restriction on the allowed $f(T)$ models and the possible solutions which satisfy all four field equations. But this does not imply that spherically symmetric solutions do not exist in modified teleparallelism, it just means an alternative tetrad field would be more suitable. The singularities of the diagonal tetrad field also imply alternative tetrads are required. For instance, in [3] we rotated the diagonal tetrad field, this change removes the restrictions on T however the singularities of the tetrads remain. Therefore we conclude that an alternative coordinate system for the metric (4.1.1) will produce spherically symmetric solutions, possible alternatives include Gullstrand-Painlevé coordinates [210] or isotropic coordinates [217]. For instance, the Schwarzschild solution was obtained using Gullstrand-Painlevé coordinates in [210] and boosted isotropic coordinates in [217]. Some of the results in [3] will be summarised in the next section.

4.2 Various tetrad fields

Different tetrad fields corresponding the the same metric $g_{\mu\nu}$ do not give rise to the same field equations, and as we have seen above choosing the simplest form of the tetrad will not necessarily yield meaningful field equations. This section will make use of results derived in [3, 217–219], and will provide a derivation of some of the results obtained in [3]. The results in this section will assume an isotropic pressure $p = p_\perp$.

4.2.1 Diagonal tetrad field

Recall the diagonal tetrad field used above, it is also used in some literature on teleparallelism and its modification including [149, 209, 220]. Here we will state the slightly generalised version which we utilised in [3]

$$e^i{}_\mu = \mathrm{diag}\Big(\mathrm{e}^{a(r)/2}, \mathrm{e}^{b(r)/2}, R(r), R(r)\sin\theta\Big)\,, \tag{4.2.1}$$

the corresponding torsion scalar and its derivative is given below

$$\begin{aligned} T(r) &= 2\mathrm{e}^{-b}\frac{R'}{R}\left(a' + \frac{R'}{R}\right)\,, \\ T'(r) &= -2\mathrm{e}^{-b}\left(-a''\frac{R'}{R} - a'\frac{R''}{R} - \frac{2R'R''}{R^2} + \frac{R'^3}{R^3}\right) - T\left(b' + \frac{R'}{R}\right)\,. \end{aligned} \tag{4.2.2}$$

This can be used to write the tensors $T_\sigma{}^{\mu\nu}$ and $S_\sigma{}^{\mu\nu}$ more explicitly, with this the field equations become

$$(t,t) : 4\pi\rho = -\frac{\mathrm{e}^{-b}R'}{R}T' f_{TT} + \left(\frac{1}{R^2} - T + \mathrm{e}^{-b}\left((a'-b')\frac{R'}{R} - \frac{2R''}{R}\right)\right)\frac{f_T}{2} + \frac{f}{4}\,, \tag{4.2.3}$$

$$(r,r) : 4\pi p = \left(T - \frac{1}{R^2}\right)\frac{f_T}{2} - \frac{f}{4}\,, \tag{4.2.4}$$

$$\begin{aligned} (\theta,\theta) : 4\pi p &= \frac{\mathrm{e}^{-b}}{2}\left(\frac{a'}{2} + \frac{R'}{R}\right)T' f_{TT} \\ &\quad + \left(\frac{T}{2} + \mathrm{e}^{-b}\left(\frac{R''}{R} + \frac{a''}{2} + \left(\frac{a'}{4} + \frac{R'}{2R}\right)(a'-b')\right)\right)\frac{f_T}{2} - \frac{f}{4}\,, \end{aligned} \tag{4.2.5}$$

$$(r,\theta) : \frac{\mathrm{e}^{-b/2}\cot\theta}{2R^2}T' f_{TT} = 0\,. \tag{4.2.6}$$

Using the same method outlined in the previous section, 4.1, the conservation equations becomes

$$4\pi p' + 2\pi a'(\rho + p) = -\frac{T'}{2R^2} f_{TT} \,. \tag{4.2.7}$$

Note, this can be obtained directly from the general conservation equation (2.3.3) which states $4\pi\partial_\nu\big(e(j_i{}^\nu + \mathcal{T}_i{}^\nu)\big) = 0$. Writing this explicitly in terms of the diagonal tetrad yields

$$\frac{\mathrm{e}^{a/2}\sin\theta T' f_{TT}}{2} + 4\pi\,\mathrm{e}^{a/2} R^2 \sin\theta \left(p' + \frac{a'p}{2} + \frac{2p}{R} + \frac{a'\rho}{2} - \frac{2p}{R}\right) = 0\,,$$

which simplifies to $4\pi p' + 2\pi a'(\rho + p) + \frac{T' f_{TT}}{2R^2} = 0$. Since the right hand side vanishes due to the field equation (4.2.6), the conservation equation becomes $p' + \frac{a'}{2}\big(\rho + p\big) = 0$, which unexpectedly coincides with the conservation equation of general relativity (1.3.5).

The field equations and conservation equation (4.2.3)–(4.2.7) provide five equations for the six unknowns $\big\{\rho(r)\,, p(r)\,, a(r)\,, b(r)\,, R(r)\,, f(T)\big\}$. Therefore the system of equations is under-determined, and we need to make some reasonable assumptions in order to close the set of equations and find solutions. In what follows, various assumptions will be made in order to obtain a solution.

Solutions with diagonal tetrad field

As we have seen, this tetrad field gives rise to a constraint $T' = 0$, (4.2.6), which will in turn restrict the allowed solutions and the form of $f(T)$. Two of the three solutions which we derived in [3] are given below.

Solutions with $T = 0$:

Assuming a vanishing torsion scalar T will give rise to the simplest solution that satisfies $T' = 0$. Inserting $T = 0$ into the torsion scalar (4.2.2) gives rise to an equation that can be solved for a'

$$a' = -\frac{R'}{R}\,, \quad a(r) = \ln\big(c_1/R(r)\big)\,, \tag{4.2.8}$$

where c_1 is an integration constant. The constraint $T = 0$ implies $f(0), f_T(0)$ and $f_{TT}(0)$ are constants. Inserting this into the field equations (4.2.4)–(4.2.5) yields

$$4\pi\rho = \left(\frac{1}{2R^2} - \frac{\mathrm{e}^{-b}}{2}\left(\frac{R'}{R}\left(\frac{R'}{R} + b'\right) + \frac{2R''}{R}\right)\right) f_T(0) + \frac{f(0)}{4}\,, \tag{4.2.9}$$

$$4\pi p = -\frac{f_T(0)}{2R^2} - \frac{f(0)}{4}\,, \tag{4.2.10}$$

$$4\pi p = \frac{\mathrm{e}^{-b}}{4}\left(\frac{R''}{R} + \frac{R'^2}{2R^2} - \frac{R'b'}{2R}\right) f_T(0) - \frac{f(0)}{4}\,. \tag{4.2.11}$$

Inserting $a' = -\frac{R'}{R}$ into the conservation equation (4.2.7) with $T' = 0$ leaves

$$4\pi p' = \frac{2\pi R'}{R}(\rho + p)\,.$$

Since we have assumed an isotropic pressure, the following must be satisfied

$$\frac{\mathrm{e}^{-b}}{2}\left(\frac{R''}{R} + \frac{R'^2}{2R^2} - \frac{R'b'}{2R}\right) + \frac{1}{R^2} = 0\,.$$

This can be solved for b, which yields

$$b = -\ln\left(\frac{c_2 - 4R}{RR'^2}\right)\,, \tag{4.2.12}$$

where c_2 is a constant of integration. Therefore, the metric coefficients are given by

$$\mathrm{e}^a = \frac{c_1}{R}\,, \qquad \mathrm{e}^b = \frac{RR'^2}{c_2 - 4R}\,,$$

thus the line element becomes

$$\mathrm{d}s^2 = \frac{c_1}{R}\mathrm{d}t^2 - \frac{RR'^2}{c_2 - 4R}\mathrm{d}r^2 - R^2\mathrm{d}\Omega^2\,. \tag{4.2.13}$$

Notice that the metric (4.2.13) becomes singular as $R \to 0$ and $R \to c_2/4$, we expect the latter to be a coordinate singularity, however the true nature of these singularities requires a more detailed analysis.

Now we can use the field equations to find ρ and p, let $f(T = 0) = f_0 =$ constant and also let $f_T(T = 0) = 0$, where f and f_T are both non-singular at

$T = 0$. This yields a constant energy density and pressure $\rho_0\,, p_0$ respectively

$$\rho_0 = -p_0 = -\frac{f_0 a}{16\pi}\,,$$

this closes our set of equations, and all unknowns are fixed. Note that ρ_0 and p_0 obey the dark energy equation of state which in its general form is given by $p/\rho = -1$.

Triviality of the Einstein static universe

For this solution, we will assume a constant energy density $\rho = \rho_0$. Consider the case $R(r) = r$ and the metric (4.1.1) with $\mathrm{e}^{a(r)}$ and $\mathrm{e}^{b(r)}$ fixed such that

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{1}{1-kr^2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{4.2.14}$$

so that the corresponding diagonal tetrad field is given by

$$e^i{}_\mu = \mathrm{diag}\left(1, \frac{1}{\sqrt{1-kr^2}}, r, r\sin\theta\right)\,. \tag{4.2.15}$$

Since we have now chosen three functions, we have three remaining functions and the system of equations is closed. For this choice the torsion scalar reads

$$T = -\frac{2(1-kr^2)}{r^2}\,, \quad T' = \frac{4}{r^3}\,, \tag{4.2.16}$$

and the field equations (4.2.3)–(4.2.5) become

$$4\pi\rho_0 = -\frac{4(1-kr^2)}{r^4}f_{TT} - \left(\frac{1}{2r^2} - k\right)f_T - \frac{f}{4}\,, \tag{4.2.17}$$

$$4\pi p = \left(\frac{1}{2r^2} - k\right)f_T + \frac{f}{4}\,, \tag{4.2.18}$$

$$4\pi p = \frac{2(1-kr^2)}{r^4}f_{TT} + \left(\frac{1}{2r^2} - k\right)f_T + \frac{f}{4}\,, \tag{4.2.19}$$

$$0 = \frac{\mathrm{e}^{-b/2}\cot\theta}{2r^2}T' f_{TT}\,. \tag{4.2.20}$$

The last field equation, and the isotropy of the pressure respectively imply

$$T' f_{TT} = 0\,, \qquad -\frac{2(1-kr^2)}{r^4}f_{TT} = 0\,. \tag{4.2.21}$$

Since $1 - kr^2$ cannot be zero, this can only be satisfied if $f_{TT} = 0$ which takes us back to TEGR. Note that we cannot achieve $T' = 0$ due to (4.2.16).

4.2.2 Rotated tetrad field

Since $e^i{}_\mu$ belongs to the local tangent space, which is Minkowski space equipped with the metric η_{ij}, a local Lorentz transformation will leave the metric unchanged. A Lorentz transformation is of the form

$$\tilde{e}^i{}_\mu = \Lambda^i{}_j e^j{}_\mu \tag{4.2.22}$$

where $\Lambda^i{}_j$ is a four-dimensional Lorentz transformation matrix which belongs to the Lorentz group

$$\mathcal{L} = \left\{ \Lambda^i{}_j \in \mathrm{GL}(4, \mathbb{R}) : \eta_{ij} \Lambda^i{}_k \Lambda^j{}_l = \eta_{kl} \right\}, \tag{4.2.23}$$

$\mathrm{GL}(4, \mathbb{R})$ is referred to as the general linear group, this group contains all 4×4 real valued matrices with non-zero determinant, the latter condition implies the inverse is well-defined for all matrices in $\mathrm{GL}(4, \mathbb{R})$. The condition $\eta_{ij} \Lambda^i{}_k \Lambda^j{}_l = \eta_{kl}$ is the defining relation of the Lorentz group. The Lorentz group $\mathcal{L}$ has the following four connected components

$$\mathcal{L}_+^\uparrow = \left\{ \Lambda^i{}_j \in \mathcal{L} : \det\Lambda = 1, \Lambda^0{}_0 > 0 \right\}, \mathcal{L}_-^\uparrow = \left\{ \Lambda^i{}_j \in \mathcal{L} : \det\Lambda = -1, \Lambda^0{}_0 > 0 \right\},$$
$$\mathcal{L}_+^\downarrow = \left\{ \Lambda^i{}_j \in \mathcal{L} : \det\Lambda = 1, \Lambda^0{}_0 < 0 \right\}, \mathcal{L}_-^\downarrow = \left\{ \Lambda^i{}_j \in \mathcal{L} : \det\Lambda = -1, \Lambda^0{}_0 < 0 \right\},$$

where the groups $\mathcal{L}_+^\uparrow$ and $\mathcal{L}_-^\uparrow$ are referred to as orthochronous groups since they preserve the direction of time ($\Lambda^0{}_0 > 0$), and $\mathcal{L}_+^\uparrow$, $\mathcal{L}_+^\downarrow$ are referred to proper groups since they preserve orientation ($\det\Lambda = 1$). Therefore, $\mathcal{L}_+^\uparrow$ is called the proper orthochronous Lorentz group, or sometimes the restricted Lorentz group.

A Lorentz transformation applied to $e^i{}_\mu$ leaves the metric unchanged. This can be seen by writing the metric explicitly in terms of the transformed tetrad field $\tilde{e}^i{}_\mu$

$$\begin{aligned} g_{\mu\nu} &= \eta_{ij} \tilde{e}^i{}_\mu \tilde{e}^j{}_\nu = \eta_{ij} \Lambda^i{}_k \Lambda^j{}_l e^k{}_\mu e^l{}_\nu \\ &= \eta_{kl} e^k{}_\mu e^l{}_\nu \,, \end{aligned}$$

where the defining relation of the Lorentz group (4.2.23) was used to obtain the last line. The group $\mathcal{L}$ contains all spatial rotations and boosts, we begin by considering a rotation that belongs to $\mathcal{L}$ and applying it to the diagonal tetrad. A tetrad field previously used in teleparallelism to obtain the Schwarzschild solution is obtained by applying the following spatial rotation

$$\Lambda^i{}_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix} ,$$

this is a spatial rotation of the elements of the diagonal tetrad by the angles θ and ϕ. Since $\Lambda^0{}_0 > 0$ and $\det\Lambda = 1$ this belongs to the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$. Applying this rotation to the diagonal tetrads yields the tetrad field

$$\tilde{e}^i{}_\mu = \begin{pmatrix} \mathrm{e}^{a/2} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{b/2}\sin\theta\cos\phi & R\cos\theta\cos\phi & -R\sin\theta\sin\phi \\ 0 & \mathrm{e}^{b/2}\sin\theta\sin\phi & R\cos\theta\sin\phi & R\sin\theta\cos\phi \\ 0 & \mathrm{e}^{b/2}\cos\theta & -R\sin\theta & 0 \end{pmatrix} , \qquad (4.2.24)$$

we often refer to this as the off-diagonal or rotated tetrad, and the field equations are significantly different to those corresponding to the diagonal tetrad. However, the tetrad field is still not globally defined due to the singularities at the poles, thus we can obtain the spherically symmetric line element (4.1.1) using (4.2.24) but this tetrad field is not spherically symmetric. The field equations and conservation equation, along with some solutions are derived for the rotated tetrad in [3], and will be summarised in this section. The field equations for the rotated tetrad field are given by

$$
\begin{aligned}
4\pi\rho = & -\frac{\mathrm{e}^{-b/2}}{R}\Big(R'\mathrm{e}^{-b/2}-1\Big)T'f_{TT}+\left(\frac{1}{2R^2}-\frac{T}{4}\right)f_T \\
& -\frac{\mathrm{e}^{-b}}{2R^2}\left(2RR''-RR'b'+R'^2\right)f_T+\frac{f}{4}\,, \qquad (4.2.25)\\
4\pi p = & \left(\frac{T}{4}-\frac{1}{2R^2}+\frac{\mathrm{e}^{-b}}{2R^2}R'\big(R'+Ra'\big)\right)f_T-\frac{f}{4}\,, \qquad (4.2.26)\\
4\pi p = & \frac{\mathrm{e}^{-b}}{2}\left(\frac{a'}{2}+\frac{R'}{R}-\frac{\mathrm{e}^{b/2}}{R}\right)T'f_{TT} \\
& +\left(\frac{T}{4}+\frac{\mathrm{e}^{-b}}{2R}\left(R''+\left(\frac{R'}{2}+\frac{Ra'}{4}\right)(a'-b')+\frac{Ra''}{2}\right)\right)f_T-\frac{f}{4}\,, \qquad (4.2.27)
\end{aligned}
$$

where the torsion scalar is given by

$$
\begin{aligned}
T(r) = & \frac{2\mathrm{e}^{-b}}{R^2}\big(\mathrm{e}^{b/2}-R'\big)\big(\mathrm{e}^{b/2}-Ra'-R'\big)\,,\\
T'(r) = & \frac{2\mathrm{e}^{-b}}{R^2}\big(\mathrm{e}^{b/2}-R'\big)\big(\mathrm{e}^{b/2}b'-Ra''-R'a'-2R''\big)\\
& +\frac{\mathrm{e}^{-b}a'}{R}\big(b'\mathrm{e}^{b/2}-2R''\big)-\Big(b'+\frac{2R'}{R}\Big)T
\end{aligned}
$$

Therefore the tetrad (4.2.24) does not yield the additional off-diagonal field equation which appears in the field equations corresponding to (4.2.1). Thus we no longer are restricted to solutions $g_{\mu\nu}$ with $T' = 0$, also as a result this admits a wider class of $f(T)$ models. In [3], and what follows some solutions which satisfy the field equations (4.2.25)–(4.2.27) will be derived.

Conservation equation

The conservation equation will now be obtained for the rotated tetrad. Taking the derivative of (4.2.26) gives

$$
\begin{aligned}
4\pi p'(r) = & -\frac{\mathrm{e}^{-b}}{2R^2}\left(2R'\Big(\mathrm{e}^{b/2}-Ra'-R'\Big)+Ra'\mathrm{e}^{b/2}\right)T'f_{TT}+\frac{R'}{R^3}f_T\\
& +\frac{\mathrm{e}^{-b}}{2R}\left(\frac{R'}{R}\Big(R'\big(a'+b'\big)-2R''\Big)+\Big(a'\big(R'b'-R''\big)-a''R'\Big)+\frac{2R'^3}{R^2}\right)f_T\,.
\end{aligned}
\qquad (4.2.28)
$$

Next, we take a combination of the field equations (4.2.25) and (4.2.26) to obtain

$$4\pi(\rho+p) = -\frac{\mathrm{e}^{-b/2}}{R}\left(R'\mathrm{e}^{-b/2}-1\right)T' f_{TT} + \frac{R'\mathrm{e}^{-b}}{2R^2}\left(R'+Ra'\right)f_T - \frac{\mathrm{e}^{-b}}{2R^2}\left(2RR''-RR'b'+R'^2\right)f_T\,. \quad (4.2.29)$$

The isotropy of pressure allows us to write

$$-\frac{\mathrm{e}^{-b}}{2}\left(\frac{a'}{2}+\frac{R'}{R}-\frac{\mathrm{e}^{b/2}}{R}\right)T' f_{TT} + \frac{R'\mathrm{e}^{-b}}{2R^2}(R'+Ra')f_T - \frac{\mathrm{e}^{-b}}{2R}\left(R''+\left(\frac{R'}{2}+\frac{Ra'}{4}\right)(a'-b')+\frac{Ra''}{2}\right)f_T - \frac{1}{2R^2}f_T = 0\,. \quad (4.2.30)$$

We now multiply (4.2.29) by $a'/2$ and (4.2.30) by $2R'/R$ and subtract the two resulting equations

$$2\pi a'(\rho+p) = \frac{\mathrm{e}^{-b}}{2R^2}\left(2R'\left(\mathrm{e}^{b/2}-Ra'-R'\right)+Ra'\mathrm{e}^{b/2}\right)T' f_{TT} - \frac{R'}{R^3}f_T - \frac{\mathrm{e}^{-b}}{2R}\left(\frac{R'}{R}\left(R'(a'+b')-2R''\right)+\left(a'\left(R'b'-R''\right)-a''R'\right)+\frac{2R'^3}{R^2}\right)f_T\,, \quad (4.2.31)$$

comparing this with equation (4.2.28), we obtain the conservation equation for the off diagonal tetrad field (4.2.24)

$$p' + \frac{a'}{2}(\rho+p) = 0\,. \quad (4.2.32)$$

Thus the conservation equation for the rotated tetrad coincides with the general relativistic conservation equation, surprisingly this is true without any restrictions on T or f.

Below is an outline of one of the five solutions found in [3].

Solutions with $b = 0$

Consider our metric (4.1.1) and tetrad field (4.2.24) with $b(r) = 0$ and $R(r) = r$, then for all values of a and r the torsion scalar is given by $T(r) = 0$. Again, since we have a vanishing torsion scalar, f and its derivatives are constant. The field equations simplify to

$$4\pi\rho = \frac{f(0)}{4}\,, \tag{4.2.33}$$

$$4\pi p = \frac{a'}{2r}f_T(0) - \frac{f(0)}{4}\,, \tag{4.2.34}$$

$$4\pi p = \left(\frac{a''}{4} + \frac{a'}{4}\left(\frac{a'}{2} + \frac{1}{r}\right)\right)f_T(0) - \frac{f(0)}{4}\,. \tag{4.2.35}$$

Isotropy of the pressure yields $a'' + \frac{a'^2}{2} - \frac{a'}{r} = 0$, which we can solve for a and find $a = 2\ln(r^2 + c_1) + \ln c_2$, where c_1 and c_2 are constants of integration. Thus, with $b = 0$, we arrive at the metric

$$\mathrm{d}s^2 = c_2(r^2 + c_1)^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,. \tag{4.2.36}$$

Since $f(0)$ and $f_T(0)$ are constants we can label $f(0) = f_1$ and $f_T(0) = f_2$. Using this and the metric coefficient a, we can write the field equations more explicitly

$$4\pi\rho = \frac{f_1}{4}\,, \tag{4.2.37}$$

$$4\pi p = \frac{2}{r^2 + c_1}f_2 - \frac{f_1}{4}\,, \tag{4.2.38}$$

$$4\pi p = \frac{2}{r^2 + c_1}f_2 - \frac{f_1}{4}\,. \tag{4.2.39}$$

One can immediately see that ρ and p are given by

$$\rho_0 = \frac{f_1}{16\pi}\,, \qquad p = \frac{f_2}{2\pi(r^2 + c_1)} - \frac{f_1}{16\pi}\,.$$

Notice that the pressure is regular everywhere provided that $c_1 > 0$.

The rotated tetrad field will be utilised further in the remainder of this thesis to derive some more results in the $f(T)$ framework. In section 4.3, we will consider Birkhoff's theorem for the non-static rotated tetrad field, and in section 4.4 we will derive an upper bound on M/R for a static and spherically symmetric metric locally generated by the rotated tetrad.

4.2.3 Alternative tetrad fields

It was later shown, in [219], that the rotated tetrad field (4.2.24) is a special case of a more general rotated tetrad. In this paper, the Schwarzschild, Kottler and FLRW solutions were obtained for this more general tetrad field. Similarly, we can perform a Lorentz boost to the diagonal tetrad, a particular boost was used in [218] to obtain the Schwarzschild solution in isotropic coordinates in the $f(T)$ gravity framework. A more general tetrad field used in teleparallelism is given in [223], this is a composition of various rotations and boosts, and under the correct restriction it reduces to either the diagonal or rotated tetrad field (4.2.24) or the tetrads used in [218,219]. This however has not been applied to the modified teleparallel framework in its full generality, due to the complexity of the resulting field equations. The field equations corresponding to this tetrad field are under-determined due to the number of variables appearing in the tetrads, therefore assumptions are required to simplify the tetrad field in order to derive any results.

As mentioned above, the diagonal (4.2.1) and rotated (4.2.24) tetrad fields contain singularities. This means that these particular tetrad fields are not global solutions of the field equations, and cannot be described as spherically symmetric. However, the diagonal and rotated tetrads do give rise to the spherically symmetric metric (4.1.1). This problem can be eliminated by considering the metric in Cartesian coordinates then using the appropriate tetrad field for this metric, for instance isotropic or Gullstrand-Painlevé coordinates have been used to give the Schwarzschild solution in [210, 217] respectively.

As we can see, there are many tetrad fields for a given metric. Certain tetrads will give rise to unnecessary constraints, whereas others will yield under-determined field equations which will then require additional assumptions in order to derive solutions. The remainder of calculations in this thesis in the $f(T)$ framework will utilise the rotated tetrad.

4.3 Birkhoff's theorem for the rotated tetrad

Consider the time dependent, spherically symmetric metric

$$\mathrm{d}s^2 = \mathrm{e}^{a(t,r)}\mathrm{d}t^2 - \mathrm{e}^{b(t,r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{4.3.1}$$

where e^a and e^b now depend on time and we let $R(r) = r$. With these changes (4.3.1) is locally generated by the following rotated tetrad field

$$e^i{}_\mu = \begin{pmatrix} e^{a(t,r)/2} & 0 & 0 & 0 \\ 0 & e^{b(t,r)/2}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & e^{b(t,r)/2}\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & e^{b(t,r)/2}\cos\theta & -r\sin\theta & 0 \end{pmatrix} . \quad (4.3.2)$$

The torsion scalar is also time-dependent due to the non-static metric components

$$T(t,r) = \frac{2e^{-b}}{r^2}\left(e^{b/2}-1\right)\left(e^{b/2}-ra_r-1\right) , \quad (4.3.3)$$

and the derivatives of the torsion scalar can be written as

$$\begin{aligned} T_t(t,r) &= \frac{2e^{-b}}{r^2}\left(e^{b/2}-1\right)\left(e^{b/2}b_t - ra_{rt}\right) - \frac{e^{-b/2}}{r}a_r b_t - T b_t \, , \\ T_r(t,r) &= \frac{2e^{-b}}{r^2}\left(e^{b/2}-1\right)\left(e^{b/2}b_r - a_r - ra_{rr}\right) - \frac{e^{-b/2}}{r}a_r b_r - \left(b_r + \frac{2}{r}\right)T \, . \end{aligned}$$

The time-dependence of the tetrads means the of the sixteen components of the field equations, we have the following five independent components

$$(t,t): 4\pi\rho = \frac{e^{-b/2}-e^{-b}}{r}T_r f_{TT} + \left(\frac{1-e^{-b}}{r^2} - \frac{T}{2} + \frac{b_r e^{-b}}{r}\right)\frac{f_T}{2} + \frac{f}{4}, \quad (4.3.4)$$

$$(r,r): 4\pi p = \left(\frac{T}{2} - \frac{1-e^{-b}}{r^2} + \frac{a_r e^{-b}}{r}\right)\frac{f_T}{2} - \frac{f}{4} , \quad (4.3.5)$$

$$\begin{aligned} (\theta,\theta): 4\pi p = &\left(\left(\frac{a_r}{2} + \frac{1-e^{b/2}}{r}\right)e^{-b}T_r - \frac{e^{-a}b_t T_t}{2}\right)\frac{f_{TT}}{2} - \frac{f}{4} + \left(e^{-b}a_{rr}\right. \\ &\left. - e^{-a}b_{tt} + T + \frac{e^{-a}b_t}{2}\left(a_t - b_t\right) + e^{-b}(a_r - b_r)\left(\frac{1}{r} + \frac{a_r}{2}\right)\right)\frac{f_T}{4} , \quad (4.3.6) \end{aligned}$$

$$(t,r): \left(1-e^{-b/2}\right)\frac{e^{b/2}T_t f_{TT}}{r} + \frac{b_t f_T}{2r} = 0 , \quad (4.3.7)$$

$$(r,t): \frac{b_t f_T}{2r} = 0 . \quad (4.3.8)$$

The remaining components of the field equations are related to the three field equations (4.3.5), (4.3.6) and (4.3.8) as follows

$$
\begin{aligned}
&(r,r) = (\theta,r) = (\phi,r)\,,\\
&(\theta,\theta) = (r,\theta) = (\phi,\theta) = (\phi,\phi) = (r,\phi) = (\theta,\phi)\,,\\
&(r,t) = (\theta,t) = (\phi,t)\,,\\
&(t,\theta) = (t,\phi) = 0\,,
\end{aligned}
$$

therefore the (t,t) and (t,r) components of the field equation are found to be independent of the other components.

In addition to the five field equations (4.3.4)–(4.3.8), we must write down the conservation equation (2.3.3) explicitly. Since we are interested in the vacuum equations, the conservation equation becomes

$$\partial_\nu\left(e\Big(e_i{}^\sigma T^\rho{}_{\mu\sigma}S_\rho{}^{\nu\mu}f_T + \frac{1}{4}e_i{}^\nu f\Big)\right) = 0\,, \tag{4.3.9}$$

before computing this, we will examine the field equations more closely. Equation 4.3.8 is satisfied if $f_T \equiv 0$ or $b_t = 0$, the former would again restrict us to models with $f(T) =$ constant, so we consider the latter. This implies b is independent of time, so

$$b(t,r) \equiv b(r)\,.$$

We can now insert this into the four remaining independent field equations. Firstly, the three diagonal components become

$$4\pi\rho = \frac{\mathrm{e}^{-b/2}}{r}\Big(1-\mathrm{e}^{-b/2}\Big)T_r f_{TT} + \left(\frac{1}{r^2} - \frac{T}{2} + \frac{\mathrm{e}^{-b}}{r^2}(rb_r - 1)\right)\frac{f_T}{2} + \frac{f}{4}\,, \tag{4.3.10}$$

$$4\pi p = \left(\frac{T}{2} - \frac{1}{r^2} + \frac{\mathrm{e}^{-b}}{r^2}(ra_r + 1)\right)\frac{f_T}{2} - \frac{f}{4}\,, \tag{4.3.11}$$

$$
\begin{aligned}
4\pi p =& \frac{\mathrm{e}^{-b}}{2}\left(\frac{a_r}{2} + \frac{1-\mathrm{e}^{b/2}}{r}\right)T_r f_{TT}\\
&+ \left(T + \mathrm{e}^{-b}\left(a_{rr} + \left(\frac{1}{r} + \frac{a_r}{2}\right)(a_r - b_r)\right)\right)\frac{f_T}{4} - \frac{f}{4}\,,
\end{aligned} \tag{4.3.12}
$$

the diagonal components with $b = b(r)$ now coincide with the field equations (4.2.25)–(4.2.27) with $R = r$, however a (and thus T) still have time dependence.

One independent field equation still remains, this is the (t,r) component given by equation (4.3.7). The condition $b_t = 0$ implies $T_t = \frac{2a_{rt}}{r}\left(1-\mathrm{e}^{b/2}\right)$ and the (t,r) field equation becomes

$$\left(1-\mathrm{e}^{-b/2}\right)\frac{\mathrm{e}^{b/2}T_t f_{TT}}{r} = -\left(1-\mathrm{e}^{-b/2}\right)^2\frac{a_{rt}f_{TT}}{r^2} = 0\,. \tag{4.3.13}$$

Returning back to the conservation equation (4.3.9), this can now be written more explicitly as follows

$$\begin{aligned}\partial_\nu\left(ej_i{}^\nu\right) =& TT_r f_{TT} + \left(\frac{3T_r}{2} + \left(\frac{a_r}{2} + \frac{2}{r} - \frac{e^{b/2}}{r^3}\right)T - \frac{a_r}{r^2}(1-e^{-b/2})\right)f_T \\ &+ \frac{e^{(b-a)/2}T_t}{\sin\theta(\sin\phi+\cos\phi)+\cos\theta}f_T + \left(\frac{a_r}{4} + \frac{1-e^{b/2}}{r}\right)f = 0\,,\end{aligned} \tag{4.3.14}$$

where we have utilised the conditions $b \equiv b(r)$ and (4.3.13) to simplify the expressions.

Now, equation (4.3.13) gives rise to four possibilities. Firstly it is satisfied if $f_{TT} \equiv 0$ so that $f(T) = c_1 T + c_2$ for c_1 and c_2 constants. This condition leads to TEGR where Birkhoff's theorem is already valid, hence to avoid such limitations we will look at the remaining three cases in more detail

1. $b = 0$: from equation (4.3.3), we see that this condition leads to a vanishing torsion scalar. Such solutions encompass $f(T)$ models with $f(0)$, $f_T(0)$ and $f_{TT}(0)$ constants. Consider the vacuum field equations with $T = b = f(0) = 0$, then equations (4.3.11) and (4.3.12) impose the following

$$a_r f_T = 0\,, \qquad \left(a_{rr} + \frac{a_r}{r} + \frac{a_r^2}{2}\right)f_T = 0\,, \tag{4.3.15}$$

both of these are satisfied if either $a(t,r) =$ constant, $a(t,r) = a(t)$ or $f_T(0) = 0$. The latter will again put stringent restrictions on the $f(T)$ models we can work with, and $a = a(t)$ will be considered as the next possible solution to (4.3.13). We are now left with $a =$ constant $= a_0$ and hence obtain the static exterior metric

$$\mathrm{d}s^2 = \mathrm{e}^{a_0}\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,,$$

this solution is Minkowski space which can be seen by rescaling the temporal coordinate $\mathrm{e}^{a_0/2}\mathrm{d}t \to \mathrm{d}\tau$.

2. $a(t,r) = a(t)$: this ansatz leads to the metric

$$\mathrm{d}s^2 = \mathrm{e}^{a(t)}\mathrm{d}t^2 - \mathrm{e}^{b(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,. \tag{4.3.16}$$

   The temporal coordinate can be rescaled so that $\mathrm{e}^{a(t)/2}\mathrm{d}t \to \mathrm{d}\tau$, then the metric (4.3.16) becomes

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{e}^{b(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,,$$

   which is independent of time.

3. $a(t,r) = a(r)$: this constraint on a gives rise to the static, spherically symmetric exterior metric

$$\mathrm{d}s^2 = \mathrm{e}^{a(r)}\mathrm{d}t^2 - \mathrm{e}^{b(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,,$$

   therefore Birkhoff's theorem is automatically satisfied.

Hence given the non-static rotated tetrad (4.3.2), the second off-diagonal field equation (4.3.13) and the conservation equation (4.3.14) will be satisfied if the line element is static and spherically symmetric, therefore yields the result required by Birkhoff's theorem for this tetrad. Notice that, provided $b(r,t) = b(r) \neq 0$, this avoids additional constraints on f or T.

4.4 Bounds on M/R in $f(T)$ gravity

Here we will apply the method used in sections 1.5.2, 3.3 and [2] to the $f(T)$ framework, recall that this method was adapted from [122]. We will obtain a bound on M/R for a solution with the rotated tetrad, this bound ensures the solution will remain hydrostatic equilibrium subject to some constraints on $f(T)$ and $f_T(T)$.

This result will make use of the rotated tetrad (4.2.24) and the corresponding field equations (4.2.25)–(4.2.27) with $R(r) = r$.

4.4.1 New variables

Let

$$f(T) = \frac{h(\zeta^2 T)}{\zeta^2} + T\,, \tag{4.4.1}$$

where ζ^2 is a constant of dimension length^2 and h is dimensionless, this ensures that the dimension of the first term is consistent with that of T. Then $f_T = h'+1$ and $f_{TT} = \zeta^2 h''$, where a prime always denotes differentiation with respect to the argument. Additionally, let $\beta = 2\log r$, we denote differentiation with respect to β as $\dot{r} = \mathrm{d}r/\mathrm{d}\beta = r/2$, so that for example $2\dot{T} = rT'$. With this relabelling of $f(T)$, the field equations (4.2.25)–(4.2.27) become

$$\begin{aligned}
8\pi r^2\rho =& -4\zeta^2 \mathrm{e}^{-b/2}\big(\mathrm{e}^{-b/2}-1\big)\dot{T}h'' + \left(1+\mathrm{e}^{-b}\Big(2\dot{b}-1\Big)\right)\big(h'+1\big) \\
&+ \frac{r^2}{2}\left(\frac{h}{\zeta^2} - Th'\right), & (4.4.2) \\
8\pi r^2 p =& \left(\mathrm{e}^{-b}\Big(2\dot{a}+1\Big)-1\right)\big(h'+1\big) - \frac{r^2}{2}\left(\frac{h}{\zeta^2} - Th'\right), & (4.4.3) \\
8\pi r^2 p_\perp =& \, 2\zeta^2 \mathrm{e}^{-b}\Big(\dot{a}+1-\mathrm{e}^{b/2}\Big)\dot{T}h'' \\
&+ \frac{\mathrm{e}^{-b}}{2}\Big(r^2 a'' + 2\big(\dot{a}+1\big)\big(\dot{a}-\dot{b}\big)\Big)\big(h'+1\big) - \frac{r^2}{2}\left(\frac{h}{\zeta^2} - Th'\right). & (4.4.4)
\end{aligned}$$

In order to simplify the field equations, we change variables in the following way

$$\begin{aligned}
x &= 1 - \mathrm{e}^{-b}\,, \\
y &= 8\pi r^2 p\,, \\
z_1 &= h'\,, \\
z_2 &= r^2\left(\frac{h}{\zeta^2} - Th'\right) = r^2\left(\frac{h}{\zeta^2} - Tz_1\right).
\end{aligned}$$

The new variables x, y, z_1, z_2 are all dimensionless. In order to rewrite the field equations in these variables, we first convert the following expressions

$$\begin{aligned}
\mathrm{e}^b &= \frac{1}{1-x}\,, \\
r^2 a'' &= 4\ddot{a} - 2\dot{a}\,, \\
\dot{b} &= \frac{rb'}{2} = \frac{\dot{x}}{1-x}\,, \\
h' &= z_1\,, \\
\frac{r\zeta^2 T' h''}{2} &= \zeta^2 \dot{T} h'' = \dot{z}_1\,, \\
\frac{r^2}{2}\left(\frac{h}{\zeta^2} - Th'\right) &= \frac{z_2}{2}\,.
\end{aligned}$$

Inserting this into the field equations gives

$$8\pi r^2 \rho = (2\dot{x} + x)(z_1 + 1) - 4\dot{z}_1(1 - x - \sqrt{1-x}) + \frac{z_2}{2}\,, \tag{4.4.5}$$

$$8\pi r^2 p = y = \big((1-x)(2\dot{a}+1) - 1\big)\big(z_1 + 1\big) - \frac{z_2}{2}\,, \tag{4.4.6}$$

$$\begin{aligned}
8\pi r^2 p_\perp = {}& 2\dot{z}_1(1 - x - \sqrt{1-x}) - \frac{z_2}{2} \\
&+ (1-x)(z_1+1)\left(2\ddot{a} + \dot{a}\left(\dot{a} - \frac{\dot{x}}{1-x} + \frac{2\dot{z}_1}{1+z_1}\right) - \frac{\dot{x}}{1-x}\right)\,,
\end{aligned} \tag{4.4.7}$$

now we can rearrange (4.4.6) to obtain $\dot{a}$, and hence deduce $\ddot{a}$, in new variables. This equation for $\ddot{a}$ will only involve first order derivatives, therefore inserting into (4.4.7) will then leave us with three first order field equations. The functions $\dot{a}$ and $\ddot{a}$ are now

$$\dot{a} = \frac{x + y + xz_1 + \frac{z_2}{2}}{2(1-x)(z_1+1)}\,,$$

$$\begin{aligned}
2\ddot{a} = {}& \frac{\dot{x}(y+1) + \dot{y}(1-x)}{(1-x)^2(z_1+1)} + \frac{2z_1\dot{x} + \dot{z}_2}{2(1-x)(z_1+1)} \\
&+ \frac{\dot{x}\big(2xz_1 + z_2\big)}{2(1-x)^2(z_1+1)} - \frac{\dot{z}_1\big(2y + z_2\big)}{2(1-x)(z_1+1)^2}\,.
\end{aligned}$$

Inserting this into equation (4.4.7) eliminates $\ddot{a}$ hence $p_\perp$ is now a first order equation

$$8\pi r^2 p_\perp = \frac{\dot{x}(x+y)}{2(1-x)} + \dot{y} + \frac{\big(x+y\big)^2}{4(1-x)(z_1+1)} - \frac{z_2}{2} + 2\dot{z}_1(1-x-\sqrt{1-x}) + x\dot{z}_1 + \frac{\dot{z}_2}{2} + \frac{2xz_1+z_2}{4(1-x)}\left(\frac{x+y}{z_1+1} + \frac{2xz_1+z_2}{4(z_1+1)} + \dot{x}\right) .$$

To summarize, using the new variables $x,\ y,\ z_1$ and z_2 we can reduce the field equations to three first order equations

$$8\pi r^2 \rho = (2\dot{x}+x)(z_1+1) - 4\dot{z}_1(1-x-\sqrt{1-x}) + \frac{z_2}{2}\,, \tag{4.4.8}$$

$$8\pi r^2 p = y\,, \tag{4.4.9}$$

$$8\pi r^2 p_\perp = \frac{\dot{x}(x+y)}{2(1-x)} + \dot{y} + \frac{\big(x+y\big)^2}{4(1-x)(z_1+1)} - \frac{z_2}{2} + 2\dot{z}_1(1-x-\sqrt{1-x}) + x\dot{z}_1 + \frac{\dot{z}_2}{2} + \frac{2xz_1+z_2}{4(1-x)}\left(\frac{x+y}{z_1+1} + \frac{2xz_1+z_2}{4(z_1+1)} + \dot{x}\right) . \tag{4.4.10}$$

4.4.2 Main result

Consider the combination

$$p + 2p_\perp \le \rho\,,$$

inserting the field equations in new variables (4.4.8)–(4.4.10), this expression becomes

$$\begin{aligned} & y + \frac{\dot{x}(x+y)}{(1-x)} + 2\dot{y} + \frac{\big(x+y\big)^2}{2(1-x)(z_1+1)} - z_2 + 2x\dot{z}_1 + \dot{z}_2 \\ & + 4\dot{z}_1(1-x-\sqrt{1-x}) + \frac{2xz_1+z_2}{2(1-x)}\left(\frac{x+y}{z_1+1} + \frac{2xz_1+z_2}{4(z_1+1)} + \dot{x}\right) \\ \le\ & (2\dot{x}+x)(z_1+1) - 4\dot{z}_1(1-x_1-\sqrt{1-x}) + \frac{z_2}{2}\,. \end{aligned}$$

Next, we rearrange by grouping the derivatives on the left hand side of the inequality

$$\frac{\dot{x}(x+y)}{(1-x)} - 2\dot{x}(z_1+1) + \frac{\dot{x}(2xz_1+z_2)}{2(1-x)} + 2\dot{y} + 8\dot{z}_1(1-x-\sqrt{1-x}) + 2x\dot{z}_1 + \dot{z}_2$$
$$\leq x(z_1+1) - \frac{\big(x+y\big)^2}{2(1-x)(z_1+1)} - y + \frac{3z_2}{2} - \frac{2xz_1+z_2}{2(1-x)}\left(\frac{x+y}{z_1+1} + \frac{2xz_1+z_2}{4(z_1+1)}\right),$$

multiplying this expression by $(1-x)$ yields

$$\begin{aligned}
&\dot{x}\left(\big(3x-2\big)\big(z_1+1\big) + y + \frac{z_2}{2}\right) \\
&+ 2\dot{z}_1\left(4-3x-4\sqrt{1-x}\right)\big(1-x\big) + \big(2\dot{y}+\dot{z}_2\big)\big(1-x\big) \\
\leq &-\frac{1}{2}\left(\frac{3x^2+y^2-2(x-y)}{z_1+1} - \big(1-x\big)\left(\frac{2xz_1(z_1+2)}{z_1+1} + 3z_2\right)\right) \\
&- \frac{2xz_1+z_2}{2(z_1+1)}\left(x+y+\frac{2xz_1+z_2}{4}\right) \\
=: &-\frac{1}{2}u(x,y,z_1,z_2)\,.
\end{aligned}$$

We can see that setting $z_1 = z_2 = 0$ reduces this expression to the combination $p+2p_\perp \leq \rho$ written explicitly with the field equations of general relativity which was derived in section 1.5.2.

4.4.3 Finding the optimisation function

Following the results obtained in section 3.3.1, we now derive the optimisation function $w(x,y,z_1,z_2) \equiv w$ which we again want to be of the form

$$w = \frac{\gamma^2}{1-x}\,, \tag{4.4.11}$$

such that

$$\begin{aligned}\dot{w} =&\frac{\gamma}{\left(1-x\right)^2}\Bigg\{\dot{x}\Big(\big(3x-2\big)\big(z_1+1\big)+y+\frac{z_2}{2}\Big)\\&+2\dot{z}_1\big(1-x\big)\Big(4-3x-4\sqrt{1-x}\Big)+\big(2\dot{y}+\dot{z}_2\big)\big(1-x\big)\Bigg\}\\=&\frac{\gamma}{\left(1-x\right)^2}\Bigg\{\dot{x}\Big(2\gamma_x\big(1-x\big)+\gamma\Big)+2\big(1-x\big)\Big(\dot{y}\gamma_y+\dot{z}_1\gamma_{z_1}+\dot{z}_2\gamma_{z_2}\Big)\Bigg\}\,.\end{aligned}\tag{4.4.12}$$

where the second line was obtained using the chain rule, note that $\gamma_x = \partial_x\gamma$. Using this we can write down $\mathrm{d}w = \frac{\mathrm{d}w}{\mathrm{d}\beta}\mathrm{d}\beta$, which yields

$$\begin{aligned}\mathrm{d}w =&\frac{\gamma}{\left(1-x\right)^2}\Bigg\{\mathrm{d}x\Big(\big(3x-2\big)\big(z_1+1\big)+y+\frac{z_2}{2}\Big)+\big(2\mathrm{d}y+\mathrm{d}z_2\big)\big(1-x\big)\\&+2\mathrm{d}z_1\big(1-x\big)\Big(4-3x+\sqrt{1-x}\Big)\Bigg\}\\=&w_x\mathrm{d}x+w_y\mathrm{d}y+w_{z_1}\mathrm{d}z_1+w_{z_2}\mathrm{d}z_2\,,\end{aligned}\tag{4.4.13}$$

To ensure such a w exists, we must check that $\mathrm{d}^2w = 0$, analogously to section 3.3.1 this becomes

$$\begin{aligned}\mathrm{d}^2w =&\Big(w_{yx}-w_{xy}\Big)\mathrm{d}x\wedge\mathrm{d}y+\Big(w_{z_1y}-w_{yz_1}\Big)\mathrm{d}y\wedge\mathrm{d}z_1\\&+\Big(w_{z_2y}-w_{yz_2}\Big)\mathrm{d}y\wedge dz_2+\Big(w_{xz_1}-w_{z_1x}\Big)\mathrm{d}z_1\wedge\mathrm{d}x\\&+\Big(w_{xz_2}-w_{z_2x}\Big)\mathrm{d}z_2\wedge\mathrm{d}x+\Big(w_{z_2z_1}-w_{z_1z_2}\Big)\mathrm{d}z_1\wedge\mathrm{d}z_2=0\,,\end{aligned}$$

where $w_{xy} = \partial_x\partial_y w$. The constraints resulting from $\mathrm{d}^2w = 0$ can be used to solve for γ, alternatively comparing the two lines in equation (4.4.12) will also give the required function. In what follows, we will use the latter and check that the resulting γ satisfies $\mathrm{d}^2w = 0$.

To find an expression for γ, first compare the coefficients of the two lines in equation (4.4.12). Comparing the pre-factors of $\dot{x}$ in the two equations gives a differential equation involving γ_x which can be partially solved for γ

$$2\gamma_x(1-x)+\gamma=\big(3x-2\big)\big(z_1+1\big)+y+\frac{z_2}{2}\,.$$

Solving this first order differential equation yields

$$\gamma = \big(4-3x\big)\big(z_1+1\big) + y + \frac{z_2}{2} + \sqrt{1-x}\,\Gamma_1(y,z_1,z_2)\,, \tag{4.4.14}$$

where $\Gamma_1(y,z_1,z_2)$ is a function that is constant when integrating with respect to x, and Γ_1 may depend on the other variables y, z_1, z_2. Inspecting equation (4.4.12) further, and comparing the $\dot{y}, \dot{z}_1$ and $\dot{z}_2$ coefficients gives the following three equations

$$\begin{aligned} \gamma_y = 1 \quad &\text{or} \quad \gamma = y + \Gamma_2(x,z_1,z_2)\,, \\ \gamma_{z_1} = 4-3x-4\sqrt{1-x} \quad &\text{or} \quad \gamma = \Big(4-3x-\sqrt{1-x}\Big)z_1 + \Gamma_3(x,y,z_2)\,, \\ \gamma_{z_2} = \frac{1}{2} \quad &\text{or} \quad \gamma = \frac{z_2}{2} + \Gamma_4(x,y,z_1)\,, \end{aligned} \tag{4.4.15}$$

where $\Gamma_2\,, \Gamma_3\,, \Gamma_4$ are functions which are constant when integrating with respect to y, z_1 and z_2 respectively, thus act as constants of integration. Comparing with equation (4.4.14), we see that Γ_1 is independent of y and z_2 since there are no terms involving these variables with the pre-factor $\sqrt{1-x}$. The only term that does involve this pre-factor is $-4z_1\sqrt{1-x}$, thus $\Gamma_1 = -4z_1 + \text{constant}$. The remaining terms in the set of equations (4.4.15) are already included in (4.4.14), and since this equation is already consistent with equation (1.5.5) from general relativity in the limit $z_1 \to 0$ and $z_2 \to 0$ we deduce that the functions Γ_i do not contribute any constants to γ (for $i = 1,2,3,4$). Thus $\Gamma_1 = -4z_1\,, \Gamma_2 = \Gamma_3 = \Gamma_4 = 0$, putting this together w becomes

$$w = \frac{\big(4-3x+y+z_1(4-3x-4\sqrt{1-x}) + \frac{z_2}{2}\big)^2}{1-x}\,, \tag{4.4.16}$$

which yields the required result outlined in equation (4.4.12). It is straightforward to check that $\mathrm{d}^2 w = 0$ is satisfied by this expression for w.

4.4.4 Maximising w

Determining the maximum of w will enable us to deduce a bound on M/R. First we recall the derivative of w with respect to β satisfies the following inequality

$$\begin{aligned} \dot{w} &\leq -\frac{\left(4-3x+y+z_1\left(4-3x-4\sqrt{1-x}\,\right)+\frac{z_2}{2}\right)}{2\left(1-x\right)^2}u(x,y,z_1,z_2) \\ &=: -\frac{\gamma(x,y,z_1,z_2)}{2\left(1-x\right)^2}\,u(x,y,z_1,z_2)\,. \end{aligned} \tag{4.4.17}$$

Note that $w(x,y,z_1,z_2)$ is decreasing if the right hand side of the above inequality is negative. Alternatively, given certain conditions on x, y, z_1 and z_2 which impose $\dot{w} \geq 0$ and if the right hand side is positive then w is increasing. This will also put constraints on the sign of u and γ. In general relativity the requirement that w is increasing imposes the conditions $\gamma \geq 0$ and $u \leq 0$, this was shown in [122] where the calculation was valid for solutions with $0 \leq x = 1 - \mathrm{e}^{-b} < 1$, this is further stressed by our result in section 3.3. Notice that if we instead assumed $\gamma \leq 0$ and $u \geq 0$ in general relativity, the former would imply $y \leq 4 - 3x$ thus since $0 \leq x < 1$ this would restrict the pressure y. The definition of u implies that $h' = z_1 \neq -1$, which just means this calculation does not include models with $f(T) = \text{constant}$. Additionally the pressure p must be positive so $y \geq 0$, and in general relativity w attained the maximum value of 16 at $x = y = 0$. Following these constraints; $u \leq 0$ and $\gamma \geq 0$ with $0 \leq x < 1\,,\ y \geq 0$ and $z_1 \neq -1$, we can vary the ranges of z_1 and z_2 to determine the behaviour of w. Using Mathematica to numerically maximise this function subject to these constraints, we find that $w \leq 16$ when $-1 < z_1 \leq 0$ and $z_2 \leq 0$. The maximum value is achieved at $x = y = z_1 = z_2 = 0$. This can now be translated back to original variables so that we can determine which functions $f(T)$ satisfy the constraints on z_1 and z_2. The two conditions $-1 < z_1 \leq 0$ and $z_2 = r^2\left(\frac{h}{\zeta^2} - Th'\right) \leq 0$, translate to

$$\begin{aligned} -1 < h' &\leq 0\,, \\ h &\leq \zeta^2 T h'\,, \end{aligned} \tag{4.4.18}$$

It is possible to find reasonable functions $f(T) = h/\zeta^2 + T$ which satisfy both constraints on z_1 and z_2 given by the relations in equation (4.4.18). However, these functions will not be explored in this thesis, the aim of this calculation is derive the bound on M/R which will be discussed in the next section.

4.4.5 Mass-radius ratio

In this section the bound $w \leq 16$ will be used to derive an upper bound for $2M/R$

$$\Big(4 - 3x + y + z_1\big(4 - 3x - 4\sqrt{1-x}\big) + z_2/2\Big)^2 \leq 16\big(1-x\big)\,, \tag{4.4.19}$$

since $y \geq 0$, the left hand side of (4.4.19) will still satisfy the bound if y is omitted from the equation. Then taking the square root and rearranging slightly gives

$$3(1-x)(z_1+1) + z_1 + 1 + z_2/2 \leq 4\sqrt{1-x}\,\big(z_1+1\big)\,. \tag{4.4.20}$$

Collecting the terms involving x leaves the expression

$$\Big(2 - 3\sqrt{1-x}\Big)^2 \leq 1 - \frac{3z_2}{2(z_1+1)} \implies 2 - 3\sqrt{1-x} \leq \sqrt{1 - \frac{3z_2}{2(z_1+1)}}\,,$$

or equivalently

$$\sqrt{1-x} \geq \frac{2}{3} - \frac{1}{3}\sqrt{1 - \frac{3z_2}{2(z_1+1)}}\,.$$

This can now be rearranged to give an upper bound for x

$$x \leq \frac{4}{9} + \frac{3z_2}{18(z_1+1)} + \frac{4}{9}\sqrt{1 - \frac{3z_2}{2(z_1+1)}}\,. \tag{4.4.21}$$

In order to translate this bound to an upper bound on the mass, we need to define the mass in $f(T)$ gravity and find a valid relation between x and m. One possibility is to recall that in general relativity for the Schwarzschild solution we know $x = \frac{2m}{r}$ (section 1.5.2) then in the presence of charge and a cosmological constant this becomes $x = \frac{2m_g}{r} - \frac{q^2}{r^2} + \frac{\Lambda r^2}{3}$ (section 3.3), therefore we postulate $x = \frac{2m_f}{r} = \frac{2m_g}{r} + g(f, f', f'')$. Here m_f has possible dependence on $f(T)$ and its derivatives via the unknown function g, for instance this is equivalent to rewriting $x = \frac{2m_g}{r} - \frac{q^2}{r^2} + \frac{\Lambda r^2}{3} = \frac{2m_{\Lambda,q}}{r}$ in section 3.3. At the boundary $r = R$

we have $m_f(R) = M_f$ and $x = \frac{2M_f}{R}$, using this estimate the bound becomes

$$\sqrt{1 - \frac{2M_f}{R}} \geq \frac{2}{3} - \frac{1}{3}\sqrt{1 - \frac{3R^2\left(h/\zeta^2 - Th'\right)}{2(h'+1)}}\,,$$

which can be expanded out and written in terms of $f(T)$ and f_T using $f(T) = \frac{h}{\zeta^2} + T$ and $f_T = h' + 1$. The bound then becomes

$$\frac{2M_f}{R} \leq \frac{4}{9} - \frac{R^2\big(f - Tf_T\big)}{6f_T} + \frac{4}{9}\sqrt{1 - \frac{3R^2\big(f - Tf_T\big)}{2f_T}}\,.$$

This bound is an estimate, and can depend on f and other quantities which may be contained in M_f. The general relativity limit is achieved when $f(T) = T$ and the inequality then reduces to Buchdahl's inequality $2M/R \leq 8/9$, provided that $2M_f/R \to 2M/R$ in this limit. Notice writing the quantity $L = \frac{f - Tf_T}{2f_T}$ puts the inequality in the form

$$\frac{2M_f}{R} \leq \frac{4}{9} - \frac{LR^2}{3} + \frac{4}{9}\sqrt{1 - 3LR^2}\,,$$

which is similar to the mass-radius ratio for the Kottler solution given by equation (3.3.13) with $Q = 0$, that is $\frac{2M}{R} \leq \frac{4}{9} - \frac{2\Lambda R^2}{3} + \frac{4}{9}\sqrt{1 - 3\Lambda R^2}$. Note that once the function $f(T)$ is specified, the upper bound on M_f can be translated into an upper bound on M_i using the (t,t) field equation (4.4.2).

In what follows, we will use the (t,t) field equation, that is equation (4.2.25) with $R(r) = r$, and the constraints on the variables x, z_1, z_2 to provide an alternative estimate for the mass, this estimate will be referred to as $\tilde{m}$. Again, $\tilde{m}$ is related to the mass m_i and may have additional dependence on $f(T)$ and its derivatives. Using the definition of mass from section 1.2 gives rise to the integral

$$\begin{aligned}
&2\tilde{m}(R) = 8\pi\int_0^R \rho r^2 \mathrm{d}r \\
&= \int_0^R \left\{2\zeta^2 r\left(\mathrm{e}^{-b/2} - \mathrm{e}^{-b}\right)T'h'' + \frac{r^2\big(h/\zeta^2 - Th'\big)}{2} + (h'+1)\frac{\mathrm{d}}{\mathrm{d}r}\left(r - r\mathrm{e}^{-b}\right)\right\}\mathrm{d}r \\
&= R\big(h'+1\big)\big(1 - \mathrm{e}^{-b}\big) + \int_0^R \left\{-\zeta^2 r\left(1 - \mathrm{e}^{-b/2}\right)^2 T'h'' + \frac{r^2\big(h/\zeta^2 - Th'\big)}{2}\right\}\mathrm{d}r\,,
\end{aligned}$$

where we have used $(h'+1)(1-\mathrm{e}^{-b}+rb'\mathrm{e}^{-b}) = (h'+1)\frac{\mathrm{d}}{\mathrm{d}r}\big(r - r\mathrm{e}^{-b}\big)$ to rewrite part of the (t,t) field equation (4.2.25), then integrated by parts to evaluate this term. Since $0 \le x < 1$, this implies $(1-\mathrm{e}^{-b/2})^2 = \big(1-\sqrt{1-x}\big)^2 < 1$, then provided that $\zeta^2 T' h'' = \frac{\mathrm{d}h'}{\mathrm{d}r} \le 0$ the following inequality holds and we can integrate by parts

$$
\begin{aligned}
2\tilde{M} &= R\big(h'+1\big)\big(1-\mathrm{e}^{-b}\big) + \int_0^R \left\{ -\zeta^2 r \Big(1-\mathrm{e}^{-b/2}\Big)^2 T' h'' + \frac{r^2\big(h/\zeta^2 - Th'\big)}{2} \right\} \mathrm{d}r \\
&\le R\big(h'+1\big)\big(1-\mathrm{e}^{-b}\big) + \int_0^R \left\{ -r\frac{\mathrm{d}h'}{\mathrm{d}r} + \frac{r^2\big(h/\zeta^2 - Th'\big)}{2} \right\} \mathrm{d}r \\
&= R\big(h'+1\big)\big(1-\mathrm{e}^{-b}\big) - rh' + \int_0^R \left\{ h' + \frac{r^2\big(h/\zeta^2 - Th'\big)}{2} \right\} \mathrm{d}r \,.
\end{aligned}
$$

In new variables, this becomes

$$
2\tilde{M} \le Rx\big(z_1+1\big) - Rz_1 + \int_0^R \left\{ z_1 + \frac{z_2}{2} \right\} \mathrm{d}r \,,
$$

where $\tilde{m}(R) = \tilde{M}$. Given the restrictions on the variables $x\,, z_1$ we see $x(z_1+1) - z_1 \ge 0$, and since both $z_1\,, z_2$ are negative we can omit the last term so that $2\tilde{M}/R \le x\big(z_1+1\big) - z_1$, this can be rearranged slightly to give

$$
\frac{2\tilde{M}/R + z_1}{z_1+1} \le x \,. \tag{4.4.22}
$$

Putting the inequalities (4.4.21) and (4.4.22) together leaves

$$
2\tilde{M}/R \le \frac{4}{9}\Big(z_1+1\Big) + \frac{z_2 - 6z_1}{6} + \frac{4}{9}\Big(z_1+1\Big)\sqrt{1 - \frac{3z_2}{2(z_1+1)}} \,, \tag{4.4.23}
$$

or equivalently

$$
\begin{aligned}
2\tilde{M}/R \le \frac{4}{9}\Big(h'+1\Big) + \frac{R^2}{6}\Big(h/\zeta^2 - (T + 6/R^2)h'\Big) \\
+ \frac{4}{9}\Big(h'+1\Big)\sqrt{1 - \frac{3R^2(h/\zeta^2 - Th')}{2(h'+1)}} \,,
\end{aligned}
$$

In the general relativity framework when $z_1 = z_2 = 0$ we require $\tilde{M} = M$, the inequality again reduces to $2M/R \le 8/9$. As mentioned earlier, this is an estimate based on the condition $\zeta^2 T' h'' \leq 0$ and can thus be made more precise by directly evaluating the integral $2\tilde{m}(R) = 8\pi \int_0^R \rho r^2 \mathrm{d}r$, and plugging this into the inequality (4.4.21). This would be more straightforward to compute given a particular function $f(T)$ that satisfies (4.4.18).

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Appendix A

Charged perfect fluids in the presence of a cosmological constant

Authors:

Atifah Mussa and Christian Böhmer.

Journal:

Published in General Relativity and Gravitation (2011). Volume **43** issue 11, pages 3033-3046 *(14 pages).*

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`http://link.springer.com/article/10.1007%2Fs10714-011-1223-5`

Appendix B

Bounds on M/R for charged objects with a positive cosmological constant

Authors:

Atifah Mussa, Christian Böhmer and Håkan Andreasson.

Journal:

Published in Classical and Quantum Gravity (2012). Volume **29** issue 9, paper 095012 *(10 pages)*.

This paper is available at:
`http://iopscience.iop.org/0264-9381/29/9/095012/`

Appendix C

Existence of relativistic stars in $f(T)$ gravity

Authors

Atifah Mussa, Christian Böhmer and Nicola Tamanini.

Journal:

Published in Classical and Quantum Gravity (2011). Volume **28** issue 24, paper 245020 *(16 pages)*.

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