Post Nonlinear Independent Subspace Analysis

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Cocktail party problem:

- independent groups of people / music bands,
- nonlinear (post nonlinear) mixing.

PNL ISA Equations

- The PNL ISA model:

$$\mathbf{x}(t) = \mathbf{f}[\mathbf{A}\mathbf{s}(t)]. \tag{1}$$

- Assumptions ($\mathbf{s} = [\mathbf{s}_1; \ldots; \mathbf{s}_M] \in \mathbb{R}^{Md} = \mathbb{R}^D$):
  - source $\mathbf{s}$ is *d-independent*: $I(\mathbf{s}_1, \ldots, \mathbf{s}_M) = 0$,
  - $\mathbf{s}(t) \in \mathbb{R}^D$ is i.i.d. in time t,
  - $\mathbf{A} \in \mathbb{R}^{D \times D}$ invertible and 'mixing', that is: $\mathbf{A} = [\mathbf{A}_{ij} \in \mathbb{R}^{d \times d}]$, $\forall\, i \Rightarrow \exists (j,k) : \mathbf{A}_{ij}$ and $\mathbf{A}_{ik}$ are invertible.
  - $\mathbf{f} : \mathbb{R}^D \to \mathbb{R}^D$ invertible, acts component-wise.
- Goal: $\hat{\mathbf{s}}$.

Ambiguites of PNL ISA

- PNL mixing structure $\Rightarrow$ mirror demixing [$\hat{\mathbf{s}} = \mathbf{W}\mathbf{g}(\mathbf{x})$].
- Question: d-independence of $\hat{\mathbf{s}}$ $\Rightarrow$ true $\mathbf{s}$ has been found?

$$\Downarrow$$

- Yes: PNL ISA separability theorem.

Separability; PNL-ISA Ambiguities with Locally-Constant Nonzero C^2 Densities

Theorem

Supposing that:

- $\mathbf{A}, \mathbf{W}$: *invertible and 'mixing' matrices,*
- $\mathbf{s}$: *(i) existing covariance matrix, (ii) somewhere locally constant, C^2 density function,*
- $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ *is a component-wise bijection, with analytical coordinate functions.*

In this case, if $\mathbf{e} := [\mathbf{e}_1; \ldots; \mathbf{e}_M] = \mathbf{Wh}(\mathbf{As})$ *is d-independent with somewhere locally constant density function, then:*

- $\mathbf{e}$ *recovers the hidden source (up to ISA ambiguities + constant translation within subspaces).*

Separability $\Rightarrow$ PNL ISA algorithm

Sketch:

1. Estimate $\mathbf{g} = \hat{\mathbf{f}}^{-1}$:

   d-dependent Central Limit Theorem

   $\Downarrow$

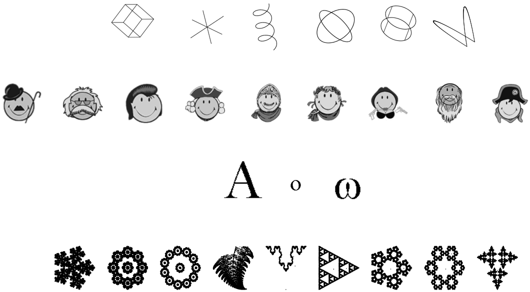
   As is asymptotically Gaussian ($D \to \infty$)

   $\Downarrow$

   g: 'gaussianization' transformation
2. Estimate **W**: linear ISA to $\mathbf{g}(\mathbf{x})$.

Test databases (s)

- I.i.d. tests:
  1. *3D-geom* ($d = 3, M = 6$),
  2. *celebrities* ($d = 2, M = 10$),
  3. *letters* ($d = 2, M \leq 50$).
- Non-i.i.d. test: *IFS* (self-similar structures; $d = 2, M = 9$).



Simulations

- Performance index: Amari-index ($r \in [0,1]$) to measure the *block-permutation matrix* property of the linear approximation of

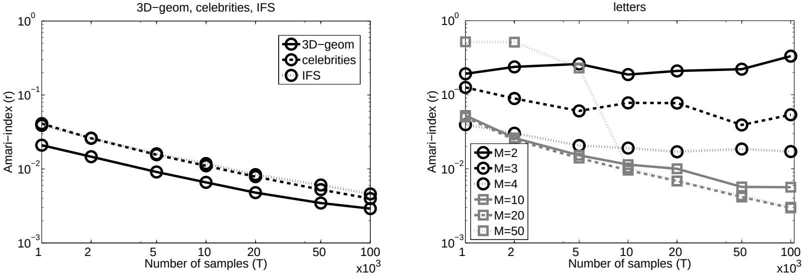
$$\mathbf{s} - E[\mathbf{s}] \in \mathbb{R}^D \mapsto \hat{\mathbf{s}} := \mathbf{Wg}[\mathbf{f}(\mathbf{As})] - E[\hat{\mathbf{s}}] \in \mathbb{R}^D.$$

- Components of the PNL ISA algorithm:
  - gaussianization based on ranks of samples,
  - ISA by joint f-decorrelation (JFD).
- Simulation parameters:
  - goodness: average of 50 random ($\mathbf{A}, \mathbf{s}, \mathbf{f}$) runs,
  - mixing matrix $\mathbf{A}$: random orthogonal,
  - coordinate-wise distortions: $f_i(z) = c_i[a_i z + \tanh(b_i z)] + d_i$.

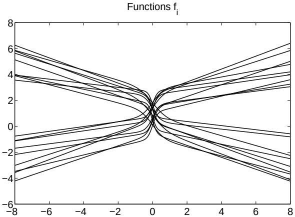
Illustrations-1: $r(T)$

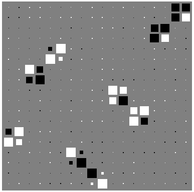
Amari-index as a function of

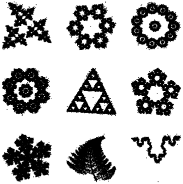
- the sample number (T): *3D-geom*, *celebrites*, *IFS*.
- dimensionality of the problem ($\leftrightarrow M$): *letters*.



Power-law decline: $r(T) \propto T^{-c}$ $(c > 0)$, as $D\nearrow$.

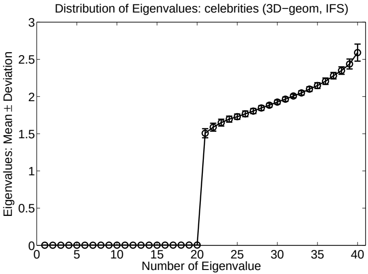

Functions f_i
8
6
4
2
0
−2
−4
−6
−8
−6
−4
−2
0
2
4
6
8

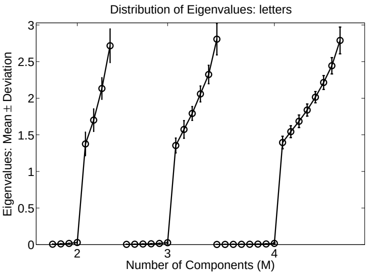




Illustrations-2: $\hat{D}$ $(T = 10,000)$

- Estimation of $D = dim(\mathbf{s})$ - in the background: $D_x = 2D$.
- Gaussianization $\rightarrow$ ordered eigenvalues of $cov[\mathbf{g}(\mathbf{x})]$.
- Results: average over 50 random runs $(\mathbf{A}, \mathbf{f})$.





Summary

- PNL ISA problem
- Separability of PNL ISA

$\Downarrow$ ($\leftarrow$ d-dependent Central Limit Theorem)

$$\boxed{\text{PNL ISA = gaussianization + ISA}}$$

- Simulations:
  - Estimation error vs. sample number:
    - power-law decline, as $D\nearrow$.
  - Possibility to estimate the dimension of the hidden source.
    - The dimensions of the hidden sources: can also be estimated using the ISA Separation Theorem [Szabó et al., JMLR 8 (2007), 1063-1095] . . .

Thank you for the attention!