Optimal Speculative Trade among Large Traders*

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Abstract

A collection of large traders hold heterogeneous prior beliefs regarding market fundamentals. This gives them a motive to engage in speculative trade with respect to market prices. Rather than assuming an exogenous set of financial instruments, we aim to characterize the financial instrument that maximizes the traders' gains from speculative trade, subject to the incentive constraints that result from the traders' ability to manipulate market prices. We show that this instrument affects price volatility without destroying ex-post efficient allocations. We also characterize the implementability of optimal speculative trade when the traders' prior beliefs are private information.

1 Introduction

When traders have heterogeneous beliefs about the future price of some commodity, they can make speculative gains by betting on the price. Such bets can be made in a forward market, where traders sign contracts that specify monetary transfers as a function of the future price. Different forms of contracts may generate different levels of speculative gains. Therefore, in order to understand the role of these financial instruments in speculative trade, it is important to have a theoretical benchmark that identifies the limits to the speculative gains that can be made using such instruments. This paper takes a first step towards this goal, adapting a theoretical framework first presented in Eliaz and Spiegler (2005).

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A major obstacle to the analysis of speculative trade in the presence of heterogeneous beliefs is that speculative gains may be unbounded. This is because risk-neutral traders with different prior beliefs would be willing to take infinite bets on the future price. We circumvent this problem in this paper by focusing on an *imperfectly competitive* market with large traders who can affect market prices. This ability to affect prices, and hence, to manipulate the outcome of bets places restrictions on its stakes.

There are a number of important commodity markets in which there are only a few participants with significant market power, who are also active in the futures market associated with that commodity. Notable examples of such commodities include electricity, natural gas, crude oil and grains (see Newbery (1984), Dong and Liu (2005), Haigh, Hranaiova and Overdahl (2005)). Indeed, there have been public concerns with regards to the possibility that large participants in these commodity markets would abuse their market power by trying to affect the price of the commodity in order to reap gains in the forward market (for concerns regarding the grain market, see Pirrong (2004), and for concerns regarding the electricity market, see Borenstein et al. (2005) and FERC (2003)).

In our focus on imperfectly competitive markets, we also follow a convention in the literature on speculative trade (most notably, Kyle (1984,1985,1989), Harris and Raviv (1993), Kyle and Wang (1997) and Odean (1998)). The main difference between these works and the present paper is that we do not impose a set of exogenous instruments for speculation. Rather, we ask the following question: if the traders themselves could design a financial instrument that maximizes their total surplus (including speculative gains due to different priors), subject to the incentive constraints that result from their ability to influence market prices, what would that instrument be?

Our contribution is modest, in the sense that we present our ideas in the context of a highly stylized market model. There are two time periods. In period 2, a commodity is traded in a market with identical sellers having a unit supply, and identical buyers having a unit demand. In addition to these traders, there is an external demand that may either be zero or high. The realization of this external demand is known to the buyers and sellers when they trade in the commodity market. Trade is carried out according to a complete-information market games adapted from Dubey (1982). In the absence of betting, Nash equilibrium (NE) induces the competitive outcome in each state.

In period 1, the traders have different prior beliefs regarding the size of external demand. We assume that neither the level of external demand nor the traders' actions in period 2 are verifiable. Therefore, the traders can give an expression to their heterogenous beliefs only by betting on the future market price. A bet is a contract that assigns transfers among traders conditional on whether trade takes place and at what price. Each bet modifies the payoffs in the market game, and therefore its NE need to be recalculated.[1] A bet is optimal if it is (constrained) interim Pareto efficient - that is, if it maximizes the sum of the traders' interim expected utilities, calculated according to their individual priors, subject to the constraint that a NE of the market game (modified by the bet) is played.

After presenting the model in Section 2, we analyze the structure of constrained interim-efficient bets in Section 3. We show that they can be interpreted as non-linear futures contracts. While in some cases the efficient bet affects market prices and introduces price volatility which would not exist in the absence of bets, it does not destroy ex-post efficiency. The transfers administered by the bet assume the role of prices and provide the incentives that restore ex-post efficiency.

In Section 4, we assume that prior beliefs are private information (drawn from some distribution) and ask whether the efficient bet can be implemented by some mechanism. To answer this question, we apply the mechanism design approach first presented in Eliaz and Spiegler (2005,2006). In these papers we focus on bilateral speculation problems, where two agents hold different priors over an unverifiable state of nature, which affects the outcome of a game they are about to play. We characterize interim-efficient bets and discuss their implementability in terms of the underlying game's payoff structure. This characterization relies on a formal analogy between the problem of implementing interim-efficient bets and the problem of efficiently dissolving a partnership, which was originally studied by Cramton, Gibbons and Klemperer (1987) - henceforth, referred to as CGK.

The main result in Section 4 establishes that the problem of implementing the interim-efficient bets in our market model is also equivalent to CGK's model. (The equivalence does not follow from Eliaz and Spiegler (2005,2006). In these papers we provide a sufficient condition for the equivalence, which is not satisfied in general by our current model.). Using this equivalence, we show that the answer depends on asymmetries between buyers and sellers in the basic market game. As the number of sellers increases, and as the gap between buyers' and seller's valuation of the traded asset diminishes, the efficient bet an be implemented for a larger set of distributions

[1] This observation was used by Allaz and Vila (1993) to derive a rationale for forward markets, in an environment *without* uncertainty. They show that producers may wish to use forward contracts in order to improve their situation in a future, imperfectly competitive spot market. In their model, producers first trade in forward contracts, and then play a Cournot game in which their payoff functions are modified by the positions they took in the forward market.

from which priors are drawn. When the numbers of buyers and sellers are identical, the efficient bet can always be implemented, using a natural, auction-like mechanism. Thus, in this case, the "derivative market" is designed as an auction for an option-like asset.

Our main point in this paper is to demonstrate, via a simple example, that meaningful statements about speculative trade (its volume and effect on prices and resource allocation) can be made in models with non-common priors, as long as one relaxes the assumption that the asset market is perfectly competitive. Imperfect competition means that traders can manipulate the outcome of bets, so that the stakes of incentive-compatible bets are bounded. Of course, bounded bets could also be generated by alternative assumptions, such as risk aversion or liquidity constraints. For a survey of some recent works on speculative trade which employ these assumptions, see Scheinkman and Xing (2003).

2 The Model

There are two time periods, 1 and 2. In period 2, the following market game is played. There are S sellers and B buyers. Each seller $s \in \{1, \ldots, S\}$ is able to supply a single unit of an indivisible good at a cost of $c \geq 0$. Sellers derive no utility from consuming the good. Each buyer $b \in \{S+1, \ldots, S+B\}$ is willing to pay 1 for a single unit, and derives no utility from consuming additional units. There is also an external demand for ν units at a price of 1. External demand behaves stochastically, depending on the state of nature, ω. There are two states of nature: $\omega = l$ (no external demand) and $\omega = h$ (high external demand), such that $\nu = 0$ in state l and $\nu = h$ (abusing notation) in state h. We assume that $h > S$.

The market agents trade according to the following simultaneous-move double-auction, adapted from Dubey (1982). Every agent (buyers and sellers alike) submits a *buy order*, consisting of a bid price and a number of demanded units, which may be any integer from 0 to S. In addition, every seller submits a *sell order*, namely an ask price for the unit he is able to produce. Both bid and ask prices must lie in $[0, 1]$. The market price is the *highest* market-clearing price, given the aggregate supply and demand curves induced by the agents' buy and sell orders. If there exists no market-clearing price, the outcome is "no trade". If there is excess demand at the market price, then agents are serviced according to their supply and demand (i.e., on the demand side, agents who submitted a higher bid get a higher priority, and on the supply side, agents who submittied a lower ask get a higher priority), and ties are broken by a symmetric lottery.

Agents have quasi-linear utilities. A buyer's payoff is $\min(1, q^{buy}) - pq^{buy}$ if he ends up buying q^{buy} units at a price p. A seller's payoff is $(p-c) \cdot q^{sell} - pq^{buy}$ if he ends up selling q^{sell} units to other agents and buying q^{buy} units from other agents at a price p (q^{buy} gets the values $0, 1, 2, ..., S$, and q^{sell} gets the values $0, 1$). Note that we assume that when a seller purchases a unit from himself, he does *not* incur the production cost c. We denote agent i's payoff function by u_i.

The realization of ν is common knowledge in period 2. Hence, in each state ω the agents play a complete information market game denoted by $G(\omega)$. This game has the following properties.

Remark 1 *The market price in any NE of $G(h)$ is 1. The market price in any NE of $G(l)$ is c if $S > B$; 1 if $S < B$; and any value in $[c, 1]$ if $S = B$.*

In period 1 agents have conflicting prior beliefs regarding the likelihood of each state. Let θ_i denote the prior probability that agent i assigns to state h. Denote $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{S+B})$. It is common knowledge that every agent i independently draws his prior belief θ_i from a continuous *cdf* F with support $[0, 1]$ and positive continuous density. Thus, F may be interpreted as the distribution of opinions regarding future external demand in the general population of traders.

A bet is a multilateral contract, which maps a set of verifiable contingencies to budget-balanced monetary transfers among the traders. We assume that neither the state of nature nor the agents' actions are verifiable. The only contingencies that can be contracted upon are whether trade occurs in the second period and at what price. For every action profile a in the market game, let $x(a) \in [0, 1] \cup D$ represent the verifiable market outcome induced by a, where $x(a) = D$ if a induces no trade, and $x(a)$ is the market price if a induces trade. Thus, a bet is a profile of functions $\mathbf{t} = (t_i(\cdot))_{i=1}^{B+S}$, where $t_i : [0, 1] \cup D \to \mathbb{R}$; $t_i(x)$ is the monetary transfer *received* by agent i when the second-period outcome is x; and $\sum_{i=1}^{B+S} t_i(x) = 0$ for all $x \in [0, 1] \cup D$.

If agents sign a bet in period 1, their second-period payoff function is modified, such that agent i's payoff from an action profile a is $u_i(a) + t_i(x(a))$. For each state ω, we let $G(\omega, \mathbf{t})$ denote the second-period market game that is played in state ω after the agents agreed on a bet $\mathbf{t}$ in the first period. Consider an agent j who signed the bet and expects the second period action profile in state $\omega \in \{h, l\}$ to be a^ω. Denote $\mathbf{a} = (a^h, a^l)$. Given the agent's first-period prior belief, his expected utility is:

$$U_i(\mathbf{a}, \mathbf{t}) \equiv \theta_i[u_i(a^h) + t_i(x(a^h))] + (1-\theta_i)[u_i(a^l) + t_i(x(a^l))]$$

We conclude the description of the model with a few comments. First, note that second-period trade takes place once and for all. If an agent purchased more units than he is able to consume, he cannot resell those extra units. Second, the model rules out short-selling: a seller cannot offer more than one unit and a buyer cannot offer any unit. Consequently, there is an asymmetry in the agents' ability to influence market outcomes. If a certain action profile induces a market price of $p < 1$, then every agent can unilaterally induce a higher price $p' \in (p, 1]$, by demanding a sufficiently large quantity at p'. In contrast, downward price manipulation is often impossible, because a comparable "dumping" strategy is unavailable. This asymmetry between upward and downward price manipulation will play an important role in the sequel. Third, the assumptions imposed on the stochastic behavior of external demand simplify the restrictions imposed on the traders' incentives to manipulate the market outcome in the second period. To see this, note that because the size of external demand in state h is higher than S, trade must take place in that state at a price of 1, regardless of the agents' actions. Hence, when agents contemplate signing a bet in period 1, they all agree that the verifiable market outcome in state h will be $x^h = 1$. Therefore, market manipulation is possible only in state l.

Finally, note that our model is formally indistinguishable from a model in which every agent i assigns probability $\frac{1}{2}$ to each state, and his utility function is multiplied by a state-dependent constant (θ_i in one state and $1 - \theta_i$ in the other state). The motivation for signing side contracts under this re-interpretation is risk sharing rather than speculative trade. Note that this re-interpretation requires us to assume that the utility from money is state-*dependent*, whereas the trade-off between money and consumption is state-*independent*. We find it hard to imagine other motivations than non-common priors for such preferences.

Constrained interim-efficient bets

We are now able to define the limits to potential gains from speculative bets, imposed by the agents' ability to manipulate market prices. Consider the following constrained optimization problem. For every profile of priors $\boldsymbol{\theta}$, choose a bet $\mathbf{t}(\boldsymbol{\theta})$ and a state-contingent action profile $\mathbf{a}(\boldsymbol{\theta})$ so as to maximize

$$\sum_{i=1}^{B+S} U_i[\mathbf{a}(\boldsymbol{\theta}), \mathbf{t}(\boldsymbol{\theta})] \tag{1}$$

subject to constraint that for every state $\omega \in \{h, l\}$, the outcome $a^{\omega}(\boldsymbol{\theta})$ is a NE in the modified market game in which agent i's payoff function is $u_i(a^{\omega}) + t_i(x(a^{\omega}))$. We refer to this constraint as "second-period incentive compatibility" (**SPIC**). In order to be sustainable, a bet must satisfy the SPIC constraints - that is, it must provide the agents with incentives not to manipulate the market price.

A solution $(\mathbf{a}(\boldsymbol{\theta}), \mathbf{t}(\boldsymbol{\theta}))$ to the constrained optimization problem is referred to as *constrained interim efficient* (or **CIE** for short). In other words, for any pair $(\mathbf{a}, \mathbf{t})$ which is not a solution, the agents can find a bet $\mathbf{t}'$ and a state-contingent action profile $\mathbf{a}'$, such that $(\mathbf{a}', \mathbf{t}')$ satisfies the SPIC constraints and every agent prefers $(\mathbf{a}', \mathbf{t}')$ to $(\mathbf{a}, \mathbf{t})$, given his prior. We refer to the optimal value of (1) as the "CIE surplus". Occasionally, we refer to $\mathbf{t}(\boldsymbol{\theta})$ as a "CIE bet". We shall say that a pair $(\mathbf{a}, \mathbf{t})$ is a *candidate solution* if the action profiles, $\mathbf{a}^h$ and $\mathbf{a}^l$, are NE of the $\mathbf{t}$-modified game in states h and l, respectively.

The following pair of examples illustrates how the SPIC constraints affect the sustainability of bets. In both examples, $S = B = 1$. Our first example describes a bet which cannot be sustained, once SPIC constraints are taken into account. Suppose that b and s sign a bet requiring s to pay b the amount A if trade occurs in period 2, and receive A from b if trade does not occur. Thus, $t_s(D) = -t_b(D) = A$, and $t_s(x) = -t_b(x) = -A$ for every $x \in [0, 1]$. Occurrence of trade in state h is assured, regardless of the players' actions. Suppose that there is an action profile a^l such that $x(a^l) = D$. Then, the agents' first-period interim expected utilities are:

$$\begin{aligned} U_s(\mathbf{a}, \mathbf{t}) &\equiv \theta_s \cdot [1 - c - A] + (1 - \theta_s) \cdot A \\ U_b(\mathbf{a}, \mathbf{t}) &\equiv \theta_b \cdot [1 - 1 + A] - (1 - \theta_b) \cdot A \end{aligned}$$

However, the buyer can impose trade in state l by demanding one unit at $p = 1$. Both before and after this deviation, his bare-game payoff is zero, but the deviation tilts the outcome of the bet in his favor. Therefore, as long as $A > 0$, there is no action profile that satisfies the SPIC constraints.

Now suppose that b and s sign an alternative bet requiring s to pay $p - c$ if there is trade at a price of $p > c$, and zero if there is no trade, or if there is trade at a price of $p \leq c$. This contract resembles a call option which is settled in cash, giving the buyer the right to purchase a unit of the good for a price of c in period 2. In state h trade occurs at $p = 1$, regardless of the agents' actions. Suppose that in state l, s offers one unit at a price of c, and b demands one unit at this price. Let us show that this action profile constitutes a NE in the market game modified by the bet. The only way a seller can manipulate the outcome of the bet is by raising the ask price to $p > c$. However, his bare-game payoff will remain zero and in addition he will have to pay $p - c$ to the buyer. The buyer can manipulate the outcome by raising his bid price also to p. The increase in the side payment that the buyer receives as a result of this deviation is exactly offset by the decrease in his bare-game payoff. Therefore, none of the agents wish to manipulate the bet's outcome. It follows that the bet and the constructed action profile satisfy the SPIC constraints.

3 Characterization of interim-efficient bets

A priori, it is not clear whether traders would prefer to sacrifice efficiency in the bare market game in order to increase their speculative gains. But as our first result shows, interim-efficient bets do not compromise ex-post efficiency.

Proposition 1 *For every profile of priors $\boldsymbol{\theta}$, there exists a CIE solution, which is also ex-post efficient.*

There are two reasons why this result is not self-evident. First, in principle, the traders could sustain a no-trade outcome in state l. This could be achieved with a pair $(\mathbf{a}, \mathbf{t})$ in which all sellers submit an ask of 1, all buyers submit a bid of 0, and $\mathbf{t}$ is defined such that it is not profitable for a seller or for a buyer to unilaterally induce trade. However, as we show in the proof of the proposition, the SPIC constraint that prevents trade from occuring in l implies, by budget-balanceness, that $\mathbf{t}(D|\boldsymbol{\theta}) = \mathbf{t}(1|\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$. This means that by enforcing no trade in state l, not only do traders earn zero speculative gains, they also lose the bare-game surplus that is available in that state. Hence, it cannot be interim-efficient not to trade in state l.

Second, the agents could also use the bets' transfers to sustain inefficient trade where some agents either do not trade, or purchase useless units. However, as we show in the Appendix (see Lemma 1), the total surplus attained by a bet that induces inefficient trade cannot exceed the surplus that is generated by a bet that induces efficient trade.

In light of Proposition 1, we proceed to explore the properties of those CIE solutions that are ex-post efficient. More specifically, we ask whether there exist such solutions in which the agents' behavior in each state is similar to their behavior in the NE of the bare game in the following sense: the agents' actions are independent of their prior beliefs, all buyers submit the same buy order, all sellers submit the same sell order and the outcome is individually rational (i.e., sellers do not purchase any units, buyers purchase at most one unit and the price is in $[c, 1]$). We refer to efficient CIE solutions with these properties as *natural* CIE solutions.

Definition 1 *The CIE surplus is attained by natural solutions if there exists a bet* $\mathbf{t}(\boldsymbol{\theta})$ *and a pair of action profiles,* a^h *and* a^l*, which are efficient and individually rational in states* h *and* l *respectively, such that for all* $\boldsymbol{\theta}$*, the tuple* $((a^h, a^l), \mathbf{t}(\boldsymbol{\theta}))$ *is CIE.*

We begin by characterizing the bets that are used in any natural CIE solution. Since there is no speculation when $p^l = p^h = 1$, we focus on the case in which a natural CIE solution induces $p^l < 1$. For our characterization we shall need the following notation. Let p^ω denote a NE market price in state ω. Let $i^*(\boldsymbol{\theta}) = \min_i \theta_i$. Given that F is continuous, we ignore the case in which several agents share the same prior. To simplify the exposition, we shall refer to i^* as the *l-optimistic agent.*

Proposition 2 *Suppose the CIE surplus is attained by a natural CIE solution that induces* $p^l < 1$*. Then a bet* $\mathbf{t}(\boldsymbol{\theta})$ *attains this surplus if and only if for every* s*,* b *and* $p \in (p^l, 1]$,

$$t_s(p|\boldsymbol{\theta}) - t_s(p^l|\boldsymbol{\theta}) \leq \min\{1, \frac{B}{S}\} \cdot (p^l - c) + p(S-1) \tag{2}$$

$$t_b(p|\boldsymbol{\theta}) - t_b(p^l|\boldsymbol{\theta}) \leq \min\{1, \frac{S}{B}\} \cdot (1 - p^l) + pS - 1 \tag{3}$$

and if at $p = 1$*, the above conditions hold with equality for every* $s \neq i^*(\boldsymbol{\theta})$ *and* $b \neq i^*(\boldsymbol{\theta})$*.*

Since in equilibrium, the market prices in states h and l will be 1 and p^l respectively, we may interpret $\left|t_j(1) - t_j(p^l)\right|$ as the *stakes* of the "bilateral bet" between i^* and $j \neq i^*$. In other words, this is the *volume of the speculative trade* between these agents. it is constrained by j's gain from following his equilibrium strategy, relative to manipulating the market price in state l from p^l to 1 (the fact that only *upward* price manipulation is relevant can be traced to our assumption of no short-selling). The loss from manipulating the market price is equal to the net cost of buying S units at a price of 1. For a buyer, who consumes the first unit he buys, this cost is equal to his expenditure, S, minus his utility from the first unit. For a seller, who can purchase the first unit from himself, this cost is equal to his expenditure on the remaining $S-1$

units. This buyer-seller difference in manipulation costs will play in important role in the next section. Finally, note that the CIE bet characterized in Proposition 2 has the property that the stakes of the bet between i^* and every $j \neq i^*$ are *independent* of $\boldsymbol{\theta}$.

Proposition 2 is proven by actually constructing a natural CIE solution in which the *l-optimistic agent* essentially bets on a low price ($p^l < 1$) against each of his opponents, where this price p^l is independent of $\boldsymbol{\theta}$. The following result is an immediate corollary of this construction in the proof.

Corollary 1 *The CIE surplus is attained by natural solutions.*

It is interesting to note that the bets in natural CIE solutions can have a simple form that resembles some real-life financial instruments. In particular, such bets can take the form of a contract that specifies constant transfers as long as trade occurs at a price $p \leq p^l$ or does not occur at all, but if $p > p^l$, agent i^* pays an additional amount which is linear in p. This contract may be interpreted as a non-linear option.

Proposition 2 allows us to analyze the effect of speculative trade - at the CIE solution - on prices. Our assumption on the size of external demand in state h implies that the NE in that state must be 1, regardless of the form of bets signed in the first period. We therefore focus on the effect that CIE bets have on the price in state l.

Proposition 3 *CIE solutions have the following implications for market prices.*

(i) When $S > B$, there exists a CIE solution with the property that p^l is perfectly competitive.

(ii) When $B > S > 1$, the CIE surplus is attained without bets and with perfectly competitive prices.

(iii) When $B > S = 1$ and the l-optimistic agent is a buyer (i.e., $i^* > 1$), the CIE surplus is attained with non-trivial bets, sustaining any $p^l \in [c, 1)$. When $B > S = 1$ and the l-optimistic agent is a seller (i.e., $i^* = 1$), the CIE surplus is attained without bets and with perfectly competitive prices, as well as with bets and any $p^l \in [c, 1)$.

(iv) When $B = S$, there exists a CIE solution with the property that p^l is perfectly competitive. However, there exists no CIE solution that induces $p^l = 1$.

This result highlights some important features of the market price in the presence of CIE bets. First, although the CIE bet is a function of the priors, the market price does not depend on them. Second, when $B > S > 1$, there is *no* speculation. In this case, the competitive forces push the price in state l all the way up to 1, regardless of the bet. But this means that the prices in both states are equal, hence traders cannot bet. Competition pushes the price to its competitive level in the case of $B < S$, too. In this case, however, the price is c, hence there are non-trivial CIE bets. These bets are "purely speculative" in the sense that they have no effect on the second period market outcome. That is, the NE of the modified game is exactly the same as the NE of the bare game: the outcome is efficient and the price is perfectly competitive. When $B > S = 1$, CIE bets lead to indeterminacy, since p^l can get any value in $[c, 1)$. In this case, bets assume the role otherwise played by prices, providing the incentives needed for an efficient allocation. Finally, when $S = B$, $p^l = 1$ is not a NE price in the modified game, although it is a NE of the bare game.

Comment: the upper bound on bid and ask prices

In our market model, bid and ask prices are bounded in $[0, 1]$. What is the economic justification for this assumption? Recall that the agents' valuations are common knowledge in the model. Therefore, it is also common knowledge that if an agent submits a bid price above 1, he must be exploiting his market power to tilt the outcome of a previously signed bet. An external regulatory agency may respond to such a transparent attempt to manipulate the price by shutting down the market, or by punishing the manipulator.

Suppose that we relax this assumption, and allow agents to submit any non-negative bid and ask price. When $S = B = 1$, this perturbation does not alter our analysis. The reason is that every agent can unilaterally impose no trade whenever the market price is strictly between 0 and 1. The SPIC constraints that follow are sufficiently strong to render the bounds on bid and ask prices irrelevant.

When there are more than two agents, removing all bounds on prices implies that CIE bets do not exist, because the agents can sustain bets with arbitrarily high stakes. The trick is to set p^l between c and 1, and let p^h be arbitrarily high (the external demand thus becomes irrelevant). In this way, the cost of manipulating the price from p^l to p^h is also arbitrarily large, which allows agents to raise the stakes of their bet without limit.

Actually constructing a second-period NE that will sustain an arbitrarily high p^h is not trivial. The reason is that such a price exceeds the buyers' willingness to pay, hence at most one buyer purchases the good at this high price (otherwise, buyers could profitably deviate by demanding zero units, thereby cutting their loss without affecting the market price). The equilibrium construction takes this into account: only one buyer b^* purchases the good in h. The bet is designed such that when $p = p^h$, a second agent i gives b^* a transfer that compensates him for purchasing the good at $p^h > 1$. The reason i is willing to incur this cost is that he bets with a third agent that the second-period price will be p^h, and the speculative gains in this bilateral bet are sufficient to cover the compensatory transfer to b^*.

To conclude, while the bounds on market prices are irrelevant in the $S = B = 1$ case, they are crucial for our results when there are more than two agents.

4 Mechanism design

Corollary 1 in the previous section established that given any profile of priors $\boldsymbol{\theta}$, there exists a natural CIE solution in which the market outcome in each state is efficient, individually rational and independent of the agents' priors. However, the CIE bet in these solutions does depend on $\boldsymbol{\theta}$. This raises the question: would agents be able to sign such a bet when their prior beliefs are private information?

Our approach to addressing this question is borrowed from the mechanism-design literature. We ask, is there a mechanism - i.e., a game played in the first period whose outcome is a bet - such that for every $\boldsymbol{\theta}$, there exists a perfect Bayesian Nash Equilibrium (PBNE) in which the first-period outcome is a bet $\mathbf{t}(\boldsymbol{\theta})$ and the second-period outcome is a^h in state h and a^l in state l, such that $((a^h, a^l), \mathbf{t}(\boldsymbol{\theta}))$ is a natural CIE solution? Whenever the answer to this question is positive, we say that the mechanism implements a natural CIE solution.[2]

We require the mechanism to satisfy a participation constraint. Every agent can veto the mechanism, in which case the agents play a NE of the bare market game in period 2. Therefore, the interim expected utility that any agent earns in the PBNE of the two-stage game induced by the mechanism cannot be lower than his interim expected utility in the NE of the bare game. Note that when $S = B$, there are multiple equilibria in the bare game. In this case, the participation constraint is non-standard, in the sense that the agents' reservation utility is determined in equilibrium, rather than being exogenous.

We consider implementation via a direct mechanism. This means that the agents play a two-period game, denoted $\Gamma(\mathbf{t})$ In the first period, every agent submits a report $\hat{\theta}_j \in [0, 1]$ or chooses to veto the mechanism. If all agents choose to participate,

[2]Our focus on this class of CIE solutions may entail some loss of generality in that there may be cases where a symmetric, natural CIE solution is not implementable for some distribution F, but another CIE solution is. However, we have not been able to obtain necessary or sufficient conditions for implementing *some* CIE solution.

their profile of reports $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, ..., \hat{\theta}_{S+B})$ is assigned a profile of transfer functions $\mathbf{t}(x|\hat{\boldsymbol{\theta}}) = (t_1(x|\hat{\boldsymbol{\theta}}), ..., t_{S+B}(x|\hat{\boldsymbol{\theta}}))$. In period 2, the state of Nature is realized and the agents play the market game whose payoffs are modified by $\mathbf{t}(x|\hat{\boldsymbol{\theta}})$. We identify the direct mechanism with $\mathbf{t}(x|\hat{\boldsymbol{\theta}})$.

In order to formally define our notion of implementation, we require the following notation. Given an action profile in state ω, a^ω, we denote $x^\omega \equiv x(a^\omega)$. For each state ω, define $T_i^\omega(\theta_i') \equiv E_{\theta_{-i}} t_i(x^\omega(\theta_i', \theta_{-i})|\theta_i', \theta_{-i})$. That is, if agent i reports a prior θ_i', while all other agents are truthful, then $T_i^\omega(\theta_i')$ is agent i's expected transfer in state ω under the mechanism $\mathbf{t}(x|\hat{\boldsymbol{\theta}})$.

Definition 2 *A direct mechanism* $\mathbf{t}(x \mid \hat{\boldsymbol{\theta}})$ *implements a natural CIE solution for a given distribution* F *if there exist two pairs of action profiles,* $\mathbf{a} = (a^h, a^l)$ *and* $\mathbf{d} = (d^h, d^l)$ *such that for every* $\boldsymbol{\theta}$ *the following conditions hold:*

(CIE) $[\mathbf{a}, \mathbf{t}(x|\boldsymbol{\theta})]$ *is a natural CIE solution,*

and there exists a PBNE in $\Gamma(\mathbf{t})$ *satisfying:*

(veto-SPIC) In each state ω*, the action profile* d^ω *is a pure-strategy NE in* $G(\omega)$

(IC) Conditional on participating, agent i *weakly prefers to report his true prior in period* 1*. That is, for every* i *and every* θ_i, θ_i'*:*

$$\theta_i T_i^h(\theta_i) + (1-\theta_i) T_i^l(\theta_i) \geq \theta_i T_i^h(\theta_i') + (1-\theta_i) T_i^l(\theta_i')$$

(IR) Each agent chooses to participate in period 1*. That is, for every* i *and every* θ_i*:*

$$\theta_i [T_i^h(\theta_i) + u_i(a^h)] + (1-\theta_i)[T_i^l(\theta_i) + u_i(a^l)] \geq \theta_i u_i(d^h) + (1-\theta_i) u_i(d^l)$$

Condition (CIE) and (veto-SPIC) imply that in the second stage of $\Gamma(\mathbf{t})$, the agents play a NE of the market game. If at least one of the agents vetoed the mechanism, they play a NE of the *bare* game. If all agents opted to participate, then they coordinate on a NE of the $\mathbf{t}$-*modified* game, independently of their first-stage announcements. The IC and IR constraints refer to the agents' first-period decisions. Bare-game payoffs are supressed in the IC constraint because the agents' second-period actions are independent of the reported priors.

Let us begin with the case of $S \neq B$, in which the bare game admits a unique equilibrium price in each state. We ignore the case of $B > S > 1$, since as we already know from Proposition 3, , there is no speculation in this case, and therefore no implementation problem.

Proposition 4 *When $B > S = 1$, the CIE surplus is not implementable for any F. When $S > B$, there exists a distribution F for which the CIE surplus is implementable.*

To develop an intuition for the above result, define $i^*(\hat{\boldsymbol{\theta}})$ to be the l-optimistic agent according to the profile of reports $\hat{\boldsymbol{\theta}}$. Consider a mechanism that satisfies the following condition:

$$t_i(1 \mid \hat{\boldsymbol{\theta}}) - t_i(p^l \mid \hat{\boldsymbol{\theta}}) = \begin{cases} \min\{1, \frac{B}{S}\} \cdot (p^l - c) + S - 1 & if \quad i = s \text{ and } s \neq i_*(\hat{\boldsymbol{\theta}}) \\ \min\{1, \frac{S}{B}\} \cdot (1 - p^l) + S - 1 & if \quad i = b \text{ and } b \neq i_*(\hat{\boldsymbol{\theta}}) \\ -\sum_{j \neq j^*}[t_j(1 \mid \hat{\boldsymbol{\theta}}) - t_j(p^l \mid \hat{\boldsymbol{\theta}})] & if \qquad i = i_*(\hat{\boldsymbol{\theta}}) \end{cases} \quad (4)$$

Note that when $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, equation (4) is consistent with the condition given by Proposition 2. From the proof of Proposition 2, it follows that if a mechanism satisfies (4), then regardless of the first-period outcome, the following are NE in the second period (modified) market game. In state h, regardless of whether $S > B$ or $B > S$,each seller submits an ask of 1 and each buyer demands one unit and bids 1. If $S > B$, then in state l, each seller submits an ask of c, and each buyer demands one unit and bids c. If, however, $B > S = 1$, then in state l, the seller submits an ask of $p^l \in [c, 1)$, and each buyer demands one unit and bids p^l. Thus, if $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, the bet assigned by the above mechanism is CIE. The problem is to design such a mechanism $\mathbf{t}(x \mid \hat{\boldsymbol{\theta}})$, which also ensures that the parties participate and report their true priors.

Our approach to analyzing this problem involves reinterpreting it as a problem of allocating an asset to the person who values it the most. Suppose that in period 1, all agents agree to participate and report their true priors. Consider the decision problem faced by a single agent, say a buyer, in the second period. What is his gain from following the action profile a^l, relative to unilaterally moving the price up to 1? By definition, the gain is zero in state h since the market price in that state is already 1. However, in state l the gain is

$$\min\{1, \frac{S}{B}\} \cdot (1 - p^l) + S - 1 - [t_b(1 \mid \boldsymbol{\theta}) - t_b(p^l \mid \boldsymbol{\theta})] \quad (5)$$

By our construction of $\mathbf{t}(x \mid \hat{\boldsymbol{\theta}})$, and the assumption that $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, the value of this expression depends on the relation between S and B and whether $b = i^*$. If $b \neq i^*$, then the gain given by (5) is zero. If $b = i^*$, then (5) is equal to $S(S-1) + B(S-c)$ when $S > B$, and equal to $1 - c$ when $B > S = 1$.

Thus, the agent's gain may be interpreted as a right to receive a prize whenever the second period market price is p^l or lower. The size of the prize is $S(S-1) + B(S-c)$ if $S > B$, and $1 - c$ if $B > S = 1$. We may therefore describe the right to receive the above prize as an asset, whose first-period valuation by each party i is $(1-\theta_i)[S(S-1) + B(S-c)]$ if $S > B$, and $(1-\theta_i)(1-c)$ if $B > S = 1$. Note that the buyer receives this asset if and only if $(1-\theta_b) > (1-\theta_i)$ for all i. This is analogous to allocating the asset to the party who values it the most.

When no bet is signed in period 1, the agents play the bare market game. Note that when $S \neq B$, this game has a unique NE in state l: each seller offers one unit and each buyer demands one unit at a price p^l, where $p^l = c$ if $S > B$ and $p^l = 1$ if $S < B$. It follows that the buyer's gain from following his equilibrium action, relative to pushing the price up to 1 is again zero in state h. But in state l, this gain is $S - c$ if $S > B$ and 0 if $B > S = 1$. Thus, when $S > B$, it is as if the buyer initially holds a share of $S - c$ in the asset described above. Similarly, when $B > S = 1$, it is as if the buyer initially holds *zero* shares in the asset. His first-period valuation of this asset is $(1-\theta_b)(S-c)$ in the former case and 0 in the latter case.

These observations suggest that the problem of implementing the CIE surplus is analogous to the problem of dissolving a partnership efficiently. In this problem, $S + B$ agents jointly hold an asset. If $S > B$, then the asset is of size $S(S-1) + B(S-c)$ and the agents' shares in the asset are $\frac{S-c}{S(S-1)+B(S-c)}$ for B of the agents and $\frac{S-1}{S(S-1)+B(S-c)}$ for S of the agents. If $B > S = 1$, then the asset is of size $1 - c$ and the agents' shares in the asset are 1 for one of the $B + 1$ agents and 0 for all other agents. Each agent privately and independently draws a valuation of the asset. The problem is to design a mechanism that allocates the entire asset to the agent with the highest valuation, subject to the constraint that all agents agree to participate in this mechanism.

CGK showed that implementing this objective depends on the initial ownership structure. If $(S-c)/(S-1)$ is close to 1 - that is, if some of the agents enter the negotiation mostly as "sellers" of the asset - the same forces that underlie the Myerson-Satterthwaite theorem make it hard to allocate the asset efficiently. As the gap between $S - 1$ and $S - c$ shrinks, each agent enters the negotiation both as a seller and a buyer, and thus he has "countervailing incentives" when reporting his valuation. Translated into the language of our model, this result means that implementing the CIE bet becomes easier when the equilibrium payoffs in the bare game become more equal across traders. Put differently, when the value of not speculating is more or less the same for all traders, it becomes easier to implement the CIE bet.

The technical basis for this result is a formal analogy to the partnership dissolution model of CGK. In that model, an asset is jointly owned by a collection of agents, who are characterized by their initial ownership share and their valuation of the asset, which is independently drawn from a commonly known distribution. A partnership is dissolved efficiently if full ownership of the asset is assigned to the agent with the highest valuation. CGK show that implementability of efficient partnership dissolution diminishes as the initial ownership structure becomes more asymmetric.

At the other extreme, when $B > S = 1$, the net gain of a buyer from following his equilibrium strategy in the bare game, relative to pushing the price up to 1, is always zero. This is because the price of the good is 1 in each state. In contrast, the seller's net gain is $1 - c$, which is precisely the entire size of the asset in the analogous partnership problem. This extreme buyer-seller asymmetry leads to a Myerson-Satterthwaite impossibility result.

Let us turn to comparative statics.

Proposition 5 *Fix F, and suppose that the CIE surplus is implementable for some S, B, c, $S > B$. Then:*
(i) The CIE surplus is also implementable for $c' \in (c, 1)$.
(ii) The CIE surplus is also implementable for S', B' satisfying $S' > S, B'$.

As c approaches 1, the NE payoffs in the bare game become similar for buyers and sellers. Similarly, when the number of sellers becomes larger, the difference between buyers' and sellers' valuation of a single unit becomes negligible relative to the number of units that need to be purchased in order to drive the price up. Therefore, these changes in market fundamentals facilitate implementability of the CIE bet.

The case of $S = B$ turns out to be special because of the multiplicity of prices in the NE of the bare game. Consider the following indirect mechanism. In period 1, every agent exercises a veto option, or submits a bid for a lottery ticket which entitles its owner to a prize of $Z = 2B \cdot (B - \frac{1+c}{2})$ conditional on the occurrence of trade at a price $p < 1$. If at least one agent exercises his veto option, the agents play the bare market game in the second period. Otherwise, the lottery ticket is assigned to the highest-bidding agent, who then pays his bid. After the ticket is allocated to the winner, the agents play the market game in the second period. Both the revenues from the winner's bid and the cost of paying the prize are distributed equally among all agents.

The first-period auctioning of the lottery ticket modifies second-period payoffs as follows (the bids are sunk at that stage, and therefore we can ignore them). If the market price is below 1, then in addition to the bare-game payoff, the auction winner receives a net payment of $(2B-1)\cdot(B-\frac{1+c}{2})$. The other agents' net payoff is their bare-game payoff minus $B-\frac{1+c}{2}$. If the market price is equal to 1, or if there is no trade, then the agents' net payoff is equal to their bare-game payoff. Let Γ denote the two-stage game induced by the betting auction.

Proposition 6 *Let $S=B$. Then, Γ implements the CIE surplus for all distributions F. Moreover, in the PBNE that implements the CIE, $p^h=1$ and $p^l=\frac{1+c}{2}$ in period 2, after every history.*

As we explained above, implementability of the CIE bet is easier when all traders face the same reservation value from not speculating. This value is equal to their NE payoffs in the bare game. When $S=B$ any price in $[c,1]$ can be sustained in the NE of $G(l)$. We may therefore select a NE in which buyers and sellers have the same payoffs. This can be achieved by choosing a NE in which $p^l=\frac{1+c}{2}$, such that $1-p^l=p^l-c$. Our implementation problem then becomes formally equivalent to the problem of implementing efficient dissolution of an *equal-share* partnership, which CGK show to be possible under any distribution of valuations. Moreover, CGK show that such a partnership can be efficiently dissolved using a simple indirect mechanism.

Proposition 6 highlights two important features of the CIE bet. First, this bet may be interpreted as a future contract (which is essentially a step function of the market price if we ignore the possibility of no trade), competed for in a market which is designed as a first-price auction. Thus, the indirect mechanism described above may serve as a theoretical benchmark for the design of market institutions for speculative trade in derivatives.

Second, the PBNE that implements the CIE surplus has the property that market prices are history-*independent*. In other words, the bets induced by the mechanism are "purely speculative", in the sense that they do not affect the outcome in the second-period market.

5 Conclusion

In this paper we characterized "efficient speculative trade", where the object of speculation is the future price in a simple, imperfectly competitive market model, populated by the betting parties themselves and external, "noise" traders. The main features of "optimal speculation" can be summarized as follows. First, optimal speculation may involve price volatility which does not exist in the absence of bets. However, it does not compromise the ex-post efficiency of resource allocation. Second, a *non-linear* contract of the future market price, which may be contrasted with the linearity in prices of standard options, is an efficient instrument for speculation (the CIE bet). Finally, optimal speculation depends on the profile of prior beliefs only in-so-far as it relies on the identity of the agent with the lowest assessment of external demand.

Applying the mechanism design approach of Eliaz and Spiegler (2005,2006), we discussed the implementability of optimal speculation when the agents' prior beliefs are private information. When there is a single sellers, it is impossible to implement optimal speculation. When there are more sellers than buyers, implementation becomes possible for a larger set of distributions of priors as the number of sellers increases and as the gap between buyers' and sellers' valuations decreases. When the numbers of buyers and sellers are the same, optimal speculation is implementable for any distribution of priors. Furthermore, it is implementable via an auction-like mechanism.

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Appendix: Proofs

The following Lemma will be instrumental in proving Propositions 1-3.

Lemma 1 *Let $((a^h, a^l), \mathbf{t})$ be any candidate solution with the property that a^l induces trade in state l. There exist a bet $\hat{\mathbf{t}}$ and an ex-post efficient action profile $\hat{a}^l$ such that $((a^h, \hat{a}^l), \hat{\mathbf{t}})$ is a candidate solution with a total interim-expected surplus, which is higher or equal to the surplus generated by $((a^h, a^l), \mathbf{t})$. Moreover, $\hat{a}^l$ can be chosen so that every seller submits the same ask $\hat{p} \in [c, 1]$, and every buyer demands one unit and bids $\hat{p}$.*

Proof. Let $\hat{a}^l$ be the following action profile: every buyer submits a bid of $\hat{p}$ and demands exactly one unit and every seller submits an ask of $\hat{p}$. Under this action profile, each seller sells a unit with probability $\min\{\frac{B}{S}, 1\}$ and each buyer buys a unit with probability $\min\{\frac{S}{B}, 1\}$. Let p be the market price induced by a^l. If $p \geq c$, then $\hat{p} = p$. Otherwise, $\hat{p} = c$. Let $u_i(a^l)$ and $u_i(\hat{a}^l)$ denote the bare game payoffs induced by a^l and $\hat{a}^l$ respectively. Note that $u_b(\hat{a}^l) = \min\{\frac{S}{B}, 1\} \cdot (1 - \hat{p})$ and $u_s(\hat{a}^l) = \min\{\frac{B}{S}, 1\} \cdot (\hat{p} - c)$.

For $i = b, s$, let q_i denote the expected number of units that agent i buys in the profile a^l. Define $\delta_q \equiv \min\{S, B\} - \sum_b \min\{q_b, 1\}$. Note that $\delta_q \geq 0$. Similarly, for every seller s let α_s denote the probability that s sells his unit to another agent under the profile a^l. Define $\delta_\alpha \equiv \min\{S, B\} - \sum_s \alpha_s$. Note that $\delta_\alpha \geq 0$. Note also that because there may be a seller s with $q_s > 0$, it follows that $\sum_b q_b \leq \sum_s \alpha_s$.

To define the bet $\hat{\mathbf{t}}$, we distinguish between two cases.

Case $1 : p \geq c$

In this case, $\hat{p} = p$. We construct $\hat{\mathbf{t}}$ as follows. First, for every agent i and for every $x \in (p, 1]$, let $\hat{t}_i(x) = t_i(x)$. Second, for every $x \in [0, p] \cup D$ and for every $s > 1$, let

$$\hat{t}_s(x) = t_s(x) + (\alpha_s - \min\{\frac{B}{S}, 1\})(p - c)$$

and for $s = 1$, let

$$\hat{t}_1(x) = t_1(x) + (\alpha_s - \min\{\frac{B}{S}, 1\} + \delta_\alpha)(p - c)$$

Finally, for $x \in [0, p] \cup D$ and for $b > S + 1$, let

$$\hat{t}_b(x) = t_b(x) + (\min\{q_b, 1\} - \min\{\frac{S}{B}, 1\})(1 - p)$$

and for $b = S + 1$, let

$$\hat{t}_b(x) = t_b(x) + (\min\{q_b, 1\} - \min\{\frac{S}{B}, 1\} + \delta_q)(1 - p)$$

Note that by construction, the bet $\hat{t}$ is budget balanced ($\sum_i \hat{t}_i(x) = 0$), and has the property that for every agent i,

$$u_i(\hat{a}^l) + \hat{t}_i[x(\hat{a}^l)] \geq u_i(a^l) + t_i[x(a^l)] \tag{6}$$

This inequality is strict if $q_b > 1$ and $q_s > 0$ for some buyer b and seller s. It follows that $\sum_i U_i(\hat{a}, \hat{t}) \geq \sum_i U(a, t)$.

It remains to show that $\hat{a}^l$ is a NE of the $\hat{\mathbf{t}}$-modified market game in state l. To show this, note first that the action profile $\hat{a}^l$ has the property that no agent i can manipulate the price downwards. Second, note that by our assumption that (a, t) is CIE, no agent has any incentive in state l to push the price up to some $p' > p$. This implies that for every buyer b and seller s,

$$\begin{aligned} u_b(a^l) + t_b[x(a^l)] &\geq 1 - Sp' + t_b(p') \\ u_s(a^l) + t_s[x(a^l)] &\geq p' - c - Sp' + t_s(p') \end{aligned}$$

In addition, if any agent i can unilaterally impose no trade, he has no incentive to do so:

$$u_i(a^l) + t_i[x(a^l)] \geq t_i(D)$$

But by (6), no agent has any incentive to either push the price up or to impose no trade in the profile $\hat{a}^l$.

Case $2 : p < c$

Noting that in this case $\hat{p} = c$, we construct $\hat{\mathbf{t}}$ as follows. First, for every agent i and for all $x \in (c, 1]$, let $\hat{t}_i(x) = t_i(x)$. Second, for $x \in [0, c] \cup D$, let

$$\begin{aligned} \hat{t}_s(c) &= t_s(p) + \min\{\frac{B}{S}, 1\}(p - c) + (\alpha_s - \min\{\frac{B}{S}, 1\})(p - c) + q_s(1 - p) && (7) \\ \hat{t}_{b<S+B}(c) &= t_b(p) + \min\{\frac{S}{B}, 1\}(c - p) + (q_b - \min\{\frac{S}{B}, 1\})(1 - p) && (8) \\ \hat{t}_{S+B}(c) &= t_{S+B}(p) + \min\{\frac{S}{B}, 1\}(c - p) + (q_{S+B} - \min\{\frac{S}{B}, 1\})(1 - p) && (9) \\ &\quad -(\sum_{i=1}^{S+B} q_i - \min\{S, B\})(1 - c) \end{aligned}$$

Note first that for every outcome $x \in [0,1] \cup D$, we have $\sum_i \hat{t}_i(x) = 0$. This is obvious for $x \in (c,1]$. To see that this also holds for $x \in [0,c] \cup D$, note that since the number of units sold must equal the number of units bought, $\sum_{i=1}^{S+B} q_i = \sum_s \alpha_s$.

The bet $\hat{\mathbf{t}}$ also has the property that (6) holds. To see this, note that for every buyer b:

$$u_b(a^l) + \hat{t}_b[x(a^l)] = \min\{q_b, 1\} - q_b p + t_b(p)$$

Compare this with $u_b(\hat{a}^l) + \hat{t}_b[x(\hat{a}^l)]$, which for every $b < S + B$ is given by

$$\min\{\frac{S}{B}, 1\} \cdot (1-c) + \hat{t}_b(c)$$

Substituting $\hat{t}_b(c)$ with the expression in (8) yields

$$\begin{aligned} & \min\{\frac{S}{B}, 1\} \cdot (1-c) + t_b(p) + \min\{\frac{S}{B}, 1\} \cdot (c-p) + (q_b - \min\{\frac{S}{B}, 1\})(1-p) \\ = \; & q_b(1-p) + t_b(p) \\ \geq \; & \min\{q_b, 1\} - q_b p + t_b(p) \end{aligned}$$

Similarly, for $b = S + B$,

$$\begin{aligned} u_{S+B}(a^l) + \hat{t}_{S+B}[x(a^l)] &= q_b(1-p) + t_b(p) - (\sum_{i=1}^{S+B} q_i - \min\{S, B\})(1-c) \\ &\geq \min\{q_b, 1\} - q_b p + t_b(p) \end{aligned}$$

where the last inequality follows from the fact that $\sum_{i=1}^{S+B} q_i \leq \min\{S, B\}$.

With regards to the sellers,

$$u_s(a^l) + \hat{t}_s[x(a^l)] = \alpha_s(p-c) + t_s(p) - q_s p$$

while

$$\begin{aligned} u_s(\hat{a}^l) + \hat{t}_s[x(\hat{a}^l)] &= \hat{t}_s(c) \\ &= \alpha_s(p-c) + t_s(p) + q_s(1-p) \\ &\geq \alpha_s(p-c) + t_s(p) - q_s p \end{aligned}$$

By essentially the same argument given in Case 1 above, it follows that $\hat{a}^l$ is a NE of the $\hat{\mathbf{t}}$-modified market game in state l. ■

Proof of Proposition 1

The proof proceeds in several steps.

Step 1. There exists a CIE solution, $(\mathbf{a}(\boldsymbol{\theta}), \mathbf{t}(\boldsymbol{\theta}))$.

Because traders are risk-neutral and have quasi-linear utilities, their payoffs are linear in the market prices and in the transfers. Because bids and asks must lies in $[0, 1]$, market prices must also lie in this interval. By the SPIC constraints, the differences, $t_j(1) - t_j(p^l)$, are bounded for every agent j. Hence, the constrained optimization problem that defines the CIE surplus must have a solution.

Step 2. The outcome in state h is ex-post efficient.

In state h, the market price is $p^h = 1$, regardless of the agents' actions. Therefore, in equilibrium they will act as price takers: each seller will offer one unit and demand zero units, while each buyer is indifferent between demanding one unit and demanding zero units. Therefore, $\mathbf{a}(\boldsymbol{\theta})$ must induce an efficient outcome regardless of the buyers' strategies.

Step 3. There is trade in state l.

Assume the contrary. Each agent can manipulate the outcome and impose trade at $p = 1$, by demanding a single unit at a price of 1. Moreover, each seller can impose this at no cost by simultaneously submitting a bid of 1 and and an ask of 0, in which case he would buy the good from himself (it must be the case that all the other sellers quote a strictly positive ask price - otherwise, trade would occur). It follows that the SPIC constraints in the no-trade state must include the following inequalities:

$$\begin{aligned} t_s(D) &\geq 1 - 1 + t_s(1) \\ t_b(D) &\geq 1 - 1 + t_b(1) \end{aligned}$$

By budget balancedness, $\Sigma_i t_i(D) = \Sigma_i t_i(1) = 0$. Hence, $t_i(D) = t_i(1)$ for all i, such that total surplus is equal to the bare-game surplus, given the agents' behavior. But since the bare-game outcome is ex-post inefficient in state l, it obviously does not maximize total surplus.

Step 4. There exists an ex-post efficient CIE solution.

By Steps 1-3, there exists a CIE solution $[(a^h(\boldsymbol{\theta}), a^l(\boldsymbol{\theta})), \mathbf{t}]$with the following properties: (i) the outcome in state h is ex-post efficient, and (ii) there is trade in state l. By Lemma 1, there must exist a CIE solution $[(a^h(\boldsymbol{\theta}), \hat{a}^l(\boldsymbol{\theta})), \hat{\mathbf{t}}]$ where $\hat{a}^l(\boldsymbol{\theta})$ is ex-post efficient. ■

Proof of Proposition 2

Let a^h be the profile of actions in which every seller submits an ask of 1 and every buyer bids 1 and demands a single unit. Let a^l be the profile of actions in which every seller submits the same ask $p^l \in [c, 1]$, and every buyer demands one unit and bids p^l. By Proposition 1, there exists an ex-post efficient CIE solution, hence, by Lemma 1, for every profile of priors $\boldsymbol{\theta}$, there exists a bet $\mathbf{t}(\boldsymbol{\theta})$ such that $((a^h, a^l), \mathbf{t}(\boldsymbol{\theta}))$ is CIE. We show that we can let $\mathbf{t}(\boldsymbol{\theta})$ be a bet that satisfies (2) and (3).

We begin by characterizing the SPIC constraints that the solution $((a^h, a^l), \mathbf{t}(\boldsymbol{\theta}))$ must satisfy. Note that if $p^l = 1$, there is no speculation and the CIE surplus is attained without any transfers by simply playing in each state a NE of the bare game. We, therefore, focus on the case in which $p^l < 1$. By definition, the action profiles (a^h, a^l) have the property that no player can unilaterally impose a price *below* p^l. In addition, a^h has the property that no player can impose a higher price. Therefore, the only relevant SPIC constraints are those that a^l and $\mathbf{t}(\boldsymbol{\theta})$ must satisfy in order to prevent a single agent from unilaterally imposing a price $p > p^l$. These constraints are given by the following inequalities:

$$\min\{1, \frac{B}{S}\} \cdot (p^l - c) + t_s(p^l) \quad \geq \quad p - Sp + t_s(p) \tag{10}$$

$$\min\{1, \frac{S}{B}\} \cdot (1 - p^l) + t_b(p^l) \quad \geq \quad 1 - Sp + t_b(p) \tag{11}$$

Note these are precisely the constraints given by (2) and (3).

If $\min\{S, B\} = 1$, then there is at least one agent who can unilaterally impose no trade in state l. Hence there are additional SPIC constraints that are needed to prevent such a deviation. We can minimize these constraints by having all agents quote the same price. Hence, if $S = 1$ and $B > 1$, or if $B = 1$ and $S > 1$, only a single agent - either the single buyer or the single seller - can impose no trade. To prevent him from doing so, we can impose an infinite fine on him whenever there is no trade. That is, if $S = 1$ and $B > 1$ we set $t_s(D) = -\infty$, and if $B = 1$ and $S > 1$, we set $t_b(D) = -\infty$.

If $S = B = 1$, then each agent can unilaterally impose no trade in every state. This means that we need to satisfy additional SPIC constraints:

$$p^l - c + t_s(p^l) \quad \geq \quad t_s(D) \tag{12}$$

$$1 - p^l + t_b(p^l) \quad \geq \quad t_b(D) \tag{13}$$

Note that when $S = B = 1$, the SPIC constraints (10) and (11) with respect to $p = 1$

become

$$p^l - c + t_s(p^l) \;\geq\; t_s(1) \tag{14}$$
$$1 - p^l + t_b(p^l) \;\geq\; t_b(1) \tag{15}$$

Hence, by setting $t_s(D) = t_s(1)$ and $t_b(D) = t_b(1)$, we make the constraints (12) and (13) equivalent to (14) and (15). It follows that the additional constraints required to prevent no trade when $\min\{S, B\} = 1$ can be satisfied without imposing further restrictions on $t_j(p)$, beyond those implied by (10) and (11). Hence, inequalilties (2) and (3) are necessary for attaining the CIE surplus with a natural solution.

The candidate solution $((a^h, a^l), t(\boldsymbol{\theta}))$ generates a total interim-expected surplus equal to

$$\sum_s \{\theta_s[(1-c) + t_s(1|\boldsymbol{\theta})] + (1-\theta_s)[\min\{\frac{B}{S}, 1\} \cdot (p^l - c) + t_s(p^l|\boldsymbol{\theta})]\} \tag{16}$$
$$+ \sum_b \{\theta_b t_b(1|\boldsymbol{\theta}) + (1-\theta_B)[\min\{\frac{S}{B}, 1\} \cdot (1 - p^l) + t_b(p^l|\boldsymbol{\theta})]\}$$

where $t_i(x|\boldsymbol{\theta})$ denotes i's transfer when the outcome is x, given that the profile of priors is $\boldsymbol{\theta}$.

To simplify the exposition, let $\alpha_b^l = \min\{\frac{S}{B}, 1\}$ and $\alpha_s^l = \min\{\frac{B}{S}, 1\}$. For each $p > p^l$, define

$$z_s(p; \boldsymbol{\theta}) \;\equiv\; \alpha_s^l p + t_s(p; \boldsymbol{\theta}) - \alpha_s^l p^l - t_s(p^l; \boldsymbol{\theta}) \tag{17}$$
$$z_b(p; \boldsymbol{\theta}) \;\equiv\; -\alpha_b^l p + t_b(p; \boldsymbol{\theta}) + \alpha_b^l p^l - t_b(p^l; \boldsymbol{\theta}) \tag{18}$$

where we Note that by budget-balanceness, $\sum_{i=1}^{S+B} z_i(p; \boldsymbol{\theta}) = 0$ for all p. The total interim-expected surplus may then be written more compactly as follows:

$$\sum_i \theta_i z_i(1; \boldsymbol{\theta}) + (1-c) \sum_s \theta_s (1 - \alpha_s^l) + (1-c) \min\{S, B\} \tag{19}$$

Notice that we have no freedom in choosing the values of the second and third terms in the above expression for the surplus. These are uniquely determined by the realized vector of priors and by the values of S and B. However, we can affect the first term in (19) through the bet we choose. Thus, the problem of achieving the CIE surplus can be reduced to the problem of maximizing $\sum_i \theta_i z_i(1; \boldsymbol{\theta})$, subject to the SPIC constraints. These constraints impose an upper bound on $\sum_i \theta_i z_i(1; \boldsymbol{\theta})$, and hence, on the total surplus.

To derive this bound, we rewrite the SPIC constraints, given by (10) and (11), as follows:

$$
\begin{aligned}
z_{s\neq i^*}(p;\boldsymbol{\theta}) &\leq (S+\alpha^l_{s\neq i^*}-1)p-\alpha^l_{s\neq i^*}c && (20)\\
z_{b\neq i^*}(p;\boldsymbol{\theta}) &\leq (S-\alpha^l_{b\neq i^*})p+\alpha^l_{b\neq i^*}-1 && (21)
\end{aligned}
$$

These inequalities imply an upper bound on (19), and hence, on the total interim-expected surplus. To compute this bound, it is useful to rewrite (19) as follows (we use here the property that $\sum_{i=1}^{S+B} z_i(p;\boldsymbol{\theta})=0$ for all p):

$$
\sum_{i\neq i^*(\boldsymbol{\theta})}(\theta_i-\theta_{i^*(\boldsymbol{\theta})})\cdot z_{i\neq i^*(\boldsymbol{\theta})}(1;\boldsymbol{\theta})+(1-c)\sum_s \theta_s(1-\alpha^l_s)+(1-c)\min\{S,B\} \qquad (22)
$$

By (20) and (21), the upper bound on total surplus is obtained by substituting

$$
\begin{aligned}
z_{s\neq i^*(\boldsymbol{\theta})}(1;\boldsymbol{\theta}) &= S-1+(1-c)\min\{\frac{B}{S},1\} && (23)\\
z_{b\neq i^*(\boldsymbol{\theta})}(1;\boldsymbol{\theta}) &= S-1 && (24)
\end{aligned}
$$

into (22). Note that these equations are obtained by requiring the constraints (2) and (3) to be binding at $p=1$ for every $s\neq i^*(\boldsymbol{\theta})$ and $b\neq i^*(\boldsymbol{\theta})$. Hence, this added requirement is also necessary for attaining the CIE surplus. Since the bet we constructed attains the CIE surplus while satisfying (2) and (3), these properties are also sufficient for attaining the CIE surplus with a natural solution. It follows that the properties described in the statement of the proposition are both necessary and sufficient. ■

Proof of Proposition 3

By Lemma 1 and Propositions 1-2 there exists a CIE solution with the following properties: (i) in state l every seller submits the same ask $p^l\in[c,1]$, and every buyer demands one unit and bids p^l, and (ii) if $p^l<1$, then the CIE bet satisfies (2)-(3). Property (i) guarantees that no agent can unilaterally impose a price lower than p^l. Property (ii) ensures that no agent has any incentive to impose no trade or a price above p^l.

Suppose $S>B$. The competitive price level in this case is c. If $p^l=c$, then no agent can deviate and increase his bare game payoff without affecting the market price. But since no agent can lower the market price, and none has any incentive to raise it, we conclude that the CIE surplus can be achieved with $p^l=c$.

Suppose instead that $B>S>1$. The competitive price level in this case is 1. If

$p^l < 1$, then any buyer who deviates to a bid of 1 raises the probability that he trades without affecting the price, a contradiction. But if $p^l = 1$, then agents cannot bet and the CIE surplus is attained without transfers.

Suppose next that $B > S = 1$. The competitive price level in this case is still 1. Let us distinguish between two cases. First, assume that $i^* = 1$ - i.e., the l-optimistic agent is the seller. In this case, according to expressions (23)-(24), $z_i(1; \boldsymbol{\theta}) \leq 0$ for every $i \neq i^*$. Therefore, total interim-expected surplus cannot exceed $1 - c$. One way to attaing this surplus is without bets, so that $p^l = 1$. Another way is to set $p^l \in [c, 1)$, and set t so that inequalities (20)-(21) are binding for every $p \in (p^l, 1]$.

Second, assume that $i^* > 1$ - i.e., the l-optimistic agent is a buyer. In this case, according to expressions (23)-(24), $z_s(1; \boldsymbol{\theta}) \leq 1 - c$ for the seller s and $z_b(1; \boldsymbol{\theta}) \leq 0$ for every buyer $b \neq i^*$. Therefore, total interim-expected surplus cannot exceed $(1 - c) \cdot (1 + \theta_s - \theta_{i^*})$, and it is attained with a bet that sustains any $p^l \in [c, 1)$, by setting t so that inequalities (20)-(21) are binding for every $p \in (p^l, 1]$.

Finally, suppose that $S = B$. Then any price level in $[c, 1]$ is competitive. If $p^l \in [c, 1]$, then since $S = B$, any buyer who deviates to a bid higher than p^l would still trade with probability one and would not affect the price. Hence, any $p^l \in [c, 1]$ can be sustained in the NE of the modified market game. However, since the total interim-expected surplus generated by any $p^l \in [c, 1)$ equals (22), where $z_{i \neq i^*(\boldsymbol{\theta})}(1; \boldsymbol{\theta})$ satisfy (23) and (24), there cannot be a CIE solution with $p^l = 1$. ■

Proof of Proposition 4

The proof relies on a formal relation between the problem of implementing the CIE surplus in our model and the problem of efficiently dissolving a partnership. This latter problem is defined as follows. A partnership with $S + B$ members is a tuple $\langle r_1, \ldots, r_{S+B}, F \rangle$, where $r_i \geq 0$ is partner i's initial share in the jointly owned asset and F is the continuous distribution on $[0, 1]$ from which all partners independently (but privately) draw their valuations of the asset. The partners are assumed to be risk neutral with quasi-linear preferences, where $1 - \theta_i$ denotes partner i's value for a unit of the asset. A partnership is dissolved efficiently if the entire asset $\sum_i r_i$ is allocated to the partner with the highest valuation.

A direct mechanism for dissolving a partnership is a pair of functions $(q(\hat{\boldsymbol{\theta}}), m(\hat{\boldsymbol{\theta}}))$ that assign, for each profile of reported values $\hat{\boldsymbol{\theta}}$, an allocation of shares, $q_1(\hat{\boldsymbol{\theta}}), \ldots, q_{S+B}(\hat{\boldsymbol{\theta}})$, and a profile of monetary transfers, $m_1(\hat{\boldsymbol{\theta}}), \ldots, m_{S+B}(\hat{\boldsymbol{\theta}})$, such that for all $\hat{\boldsymbol{\theta}}$, $q_i(\hat{\boldsymbol{\theta}}) \geq 0$, $\sum_i q_i(\hat{\boldsymbol{\theta}}) = \sum_i r_i$ and $\sum_i m_i(\hat{\boldsymbol{\theta}}) = 0$.

Definition 3 *A mechanism* $(q(\hat{\boldsymbol{\theta}}), m(\hat{\boldsymbol{\theta}}))$ *efficiently dissolves a partnership* $\langle r_1, \ldots, r_{S+B}, F\rangle$ *if it satisfies the following properties for* $i = 1, \ldots, S+B$*:*

(EFF) Whenever $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$*,*

$$q_i(\boldsymbol{\theta}) = \begin{cases} \sum_i r_i & if \quad \theta_i \leq \theta_j \text{ for all } j \\ 0 & if \quad \theta_j < \theta_i \text{ for some } j \end{cases}$$

(IC) There is a Bayesian NE in which every partner reports his true value. That is, for every* i *and every* θ_i, θ_i'*:*

$$(1-\theta_i)Q_i(\theta_i) + M_i(\theta_i) \geq (1-\theta_i)Q_i(\theta_i') + M_i(\theta_i')$$

where $Q_i(\hat{\theta}_i) \equiv E_{\theta-i}q_i(\hat{\theta}_i, \theta_{-i})$ *and* $M_i(\hat{\theta}_i) \equiv E_{\theta-i}m_i(\hat{\theta}_i, \theta_{-i})$*.*

(IR) Each partner's interim-expected payoff in the truth-telling Bayesian NE is at least as high as the value he assigns to his initial share. That is, for every* i *and every* θ_i :

$$(1-\theta_i)Q_i(\theta_i) + M_i(\theta_i) \geq (1-\theta_i)r_i$$

We say that a partnership can be dissolved efficiently if there exists a direct mechanism that implements its efficient dissolution.

Define $i_*(\hat{\boldsymbol{\theta}})$ to be the lowest indexed agent among those agents with the lowest reported prior on h. Consider a direct mechanism $\mathbf{t}(x|\hat{\boldsymbol{\theta}})$ that satisfies (2) and (3) with $\boldsymbol{\theta}$ replaced by $\hat{\boldsymbol{\theta}}$ and i^* replaced by $i_*(\hat{\boldsymbol{\theta}})$ (note that there are many such mechanisms). We distinguish between two cases: $S > B$ and $B > S = 1$.

Case 1: $B > S = 1$

We begin by constructing second-period continuation strategies, which are necessary for implementation of a natural CIE . If at least one agent refuses to participate in the first-period mechanism, the seller submits an ask of 1 and each buyer demands one unit at a price of 1. This action profile is the NE of both $G(h)$ and $G(l)$, hence it is also a NE in the corresponding second-period subgame. Now suppose that all agents agreed to participate in the first-period mechanism and submitted a profile of reports $\hat{\boldsymbol{\theta}}$. In state h, the seller submits an ask of 1, and each buyer demands one unit at this price. Denote this action profile by a^h. It is independent of $\hat{\boldsymbol{\theta}}$. In state l, the seller submits an ask of $p^l(\hat{\boldsymbol{\theta}}) \in [c, 1)$, and every buyer demands one unit and bids $p^l(\hat{\boldsymbol{\theta}})$. Denote this action profile by $a^l(\hat{\boldsymbol{\theta}})$. By Propositions 2 and 3, $a^l(\hat{\boldsymbol{\theta}})$ constitutes a NE of $G(l, \mathbf{t})$ for any $\mathbf{t}$ that satisfies (2) and (3).

Our objective is to examine whether there exist distributions F for which agreeing to participate in the mechanism and reporting one's true prior, together with the second-period continuation strategies described above, constitute a PBNE. Our approach is to show that if this is the case, then there is a corresponding partnership $\langle r_1, \ldots, r_{B+1}\rangle$ that can be efficiently dissolved for some F whenever $(a^h, a^l(\boldsymbol{\theta}), \mathbf{t}(x|\boldsymbol{\theta}))$ is implementable for that F. Using Proposition 2 of CGK, we obtain a contradiction.

Assume $(a^h, a^l(\boldsymbol{\theta}), \mathbf{t}(x|\boldsymbol{\theta}))$ is implementable for F. Consider the partnership $\langle r_1, \ldots, r_{B+1}, F\rangle$ where $r_1 = 1 - c$ and $r_b = 0$ for every $b > 1$. Let $(q(\hat{\boldsymbol{\theta}}), w(\hat{\boldsymbol{\theta}}))$ be the following mechanism: for each agent i and for every pair of reports $\hat{\boldsymbol{\theta}}$,

$$\begin{aligned} q_1(\hat{\boldsymbol{\theta}}) &\equiv p^l(\hat{\boldsymbol{\theta}}) - c - [t_1(1|\hat{\boldsymbol{\theta}}) - t_1(p^l(\hat{\boldsymbol{\theta}})|\hat{\boldsymbol{\theta}})] \\ q_b(\hat{\boldsymbol{\theta}}) &\equiv \frac{1 - p^l(\hat{\boldsymbol{\theta}})}{B} - [t_b(1|\hat{\boldsymbol{\theta}}) - t_b(p^l(\hat{\boldsymbol{\theta}})|\hat{\boldsymbol{\theta}})] \\ m_1(\hat{\boldsymbol{\theta}}) &= t_1(1|\hat{\boldsymbol{\theta}}) \\ m_b(\hat{\boldsymbol{\theta}}) &= t_b(1|\hat{\boldsymbol{\theta}}) \end{aligned}$$

Because $t_1(1|\hat{\boldsymbol{\theta}})$ and $t_b(1|\hat{\boldsymbol{\theta}})$ satisfy (CIE), (veto-SPIC), (IC) and (IR) it follows that the mechanism $(q(\hat{\boldsymbol{\theta}}), m(\hat{\boldsymbol{\theta}}))$ has the following properties. First, by (CIE), whenever $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$,

$$q_i(\hat{\boldsymbol{\theta}}) = \begin{cases} 1 - c & if \quad i = i^*(\hat{\boldsymbol{\theta}}) \\ 0 & if \quad i \neq i^*(\hat{\boldsymbol{\theta}}) \end{cases}$$

Hence, $q(\hat{\boldsymbol{\theta}})$ satisfies (EFF).

Second, by (IC), we have that for every $\theta_i' \in [0, 1]$,

$$(1 - \theta_1)[Q_1(\theta_1) - (p^l(\hat{\boldsymbol{\theta}}) - c)] + M_1(\theta_1) \geq (1 - \theta_1)[Q_1(\theta_1') - (p^l(\hat{\boldsymbol{\theta}}) - c)] + M_1(\theta_1')$$

and for every $b \geq 2$,

$$(1 - \theta_b)[Q_b(\theta_1) - \frac{1 - p^l(\hat{\boldsymbol{\theta}})}{B}] + M_b(\theta_b) \geq (1 - \theta_b)[Q_b(\theta_b') - \frac{1 - p^l(\hat{\boldsymbol{\theta}})}{B}] + M_b(\theta_b')$$

By (IR),

$$(1 - \theta_1)[Q_1(\theta_1) - (p^l(\hat{\boldsymbol{\theta}}) - c)] + M_1(\theta_1) \geq (1 - \theta_1)[(1 - c) - (p^l(\hat{\boldsymbol{\theta}}) - c)]$$

and for every $b \geq m+1$,

$$(1-\theta_b)[Q_b(\theta_b) - \frac{1-p^l(\hat{\boldsymbol{\theta}})}{B}] + M_b(\theta_b) \geq (1-\theta_b)[-\frac{1-p^l(\hat{\boldsymbol{\theta}})}{B}]$$

These two inequalities imply that $(q(\hat{\boldsymbol{\theta}}), m(\hat{\boldsymbol{\theta}}))$ satisfies (IC*) and (IR*).

It follows that given F, the mechanism $(q(\hat{\boldsymbol{\theta}}), m(\hat{\boldsymbol{\theta}}))$ efficiently dissolves the partnership $\langle 1-c, 0, \ldots, 0\rangle$. But this contradicts Proposition 2 of CGK, which states that there exists no F for which one can efficiently dissolve a partnership whose entire assets are owned by a single partner. This implies that our initial assertion - that $(a^h, a^l(\boldsymbol{\theta}), \mathbf{t}(x|\boldsymbol{\theta}))$ is implementable for F - is false.

Case 2: $B < S$

We begin by constructing second-period continuation strategies, which are necessary for implementation of a natural CIE. Regardless of the agents' actions in the first period, if state h is realized in the second period, the agents coordinate on the action profile a^h described in the previous case. If state l is realized, then regardless of the agents' actions in the first period, the agents play the action profile a^l from the previous case, only with $p^l = c$ for any profile of reported priors. Since a^h and a^l are NE of $G(h)$ and $G(l)$ respectively, they are also NE of the second-period subgame when at least one agent vetoes the mechanism. In what follows, we shall construct a mapping from announced priors to bets such that a^h and a^l will constitute NE of the modified second-period market game.

We now show that there exists a partnership $\langle r_1', \ldots, r_{S+B}', F\rangle$ that can be efficiently dissolved *if and only if* $(a^h, a^l, \mathbf{t}(x|\boldsymbol{\theta}))$ is implementable for F. We then apply Proposition 3 of CGK to obtain that there exists a distribution F for which this partership can be efficiently dissolved, which implies that $(a^h, a^l, \mathbf{t}(x|\boldsymbol{\theta}))$ for that F.

Define:

$$\begin{aligned} r'_{i\leq S} &= S - 1 + \frac{B}{S}(1-c) \\ r'_{i\geq S+1} &= S-1 \\ q_i(\hat{\boldsymbol{\theta}}) &= t_i(c|\hat{\boldsymbol{\theta}}) - t_i(1|\hat{\boldsymbol{\theta}}) + r'_i \\ m_i(\hat{\boldsymbol{\theta}}) &= t_i(1|\hat{\boldsymbol{\theta}}) \\ t_i(x|\hat{\boldsymbol{\theta}}) &= \begin{cases} t_i(c|\hat{\boldsymbol{\theta}}) & if \quad x \neq 1 \\ t_i(1|\hat{\boldsymbol{\theta}}) & if \quad x = 1 \end{cases} \end{aligned}$$

Note that (EFF) holds if and only if $t_i(1|\hat{\boldsymbol{\theta}}) - t_i(c|\hat{\boldsymbol{\theta}}) = z_i(1;\boldsymbol{\theta})$ for every $i \neq i^*(\boldsymbol{\theta})$, where $z_{i\neq i^*(\boldsymbol{\theta})}(1;\boldsymbol{\theta})$ satisfies (23) and (24). Since $t_i(x|\hat{\boldsymbol{\theta}})$ is constant for all $x \neq 1$, it follows from Proposition 2 that (EFF) holds if and only if (CIE) holds. As we have noted above, our construction of the second-period action profiles satisfies (veto-SPIC). By simple algebra, we obtain that (IR) and (IC) hold if and only if (IR*) and (IC*) hold, respectively. Therefore, the mechanism $\mathbf{t}(x|\hat{\boldsymbol{\theta}})$ implements a natural CIE solution if and only if the mechanism $(q(\hat{\boldsymbol{\theta}}), m(\hat{\boldsymbol{\theta}}))$ efficiently dissolves the partnership $\left\langle r'_1, \ldots, r'_{S+B}, F \right\rangle$. Now, we can apply Proposition 1-3 in CGK to obtain the desired result. ■

Proof of Proposition 5

Let $(a^h, a^l, \mathbf{t}(x;\boldsymbol{\theta}))$ be a natural CIE solution, and consider an $(S+B)$-member partnership where S members own $S-1$ shares of the asset and B members own $S-c$ shares. By Proposition 4, this partnership can be efficiently dissolved if and only if $(a^h, a^l, \mathbf{t}(x;\boldsymbol{\theta}))$ is PBNE-implementable. By Proposition 1 of CGK, as the ratio $r_{i\leq S}/r_{i\geq S+1}$ become closer to one, the set of distributions for which efficient dissolution is possible weakly expands. This implies that the set of distributions for which $(a^h, a^l, \mathbf{t}(x;\boldsymbol{\theta}))$ is implementable, weakly as S increases and as c increases and becomes closer to one. ■

Proof of Proposition 6

We proceed in four steps.

Step 1 : Construction of second-period action profiles.

Regardless of the agents' actions in the first period, they coordinate on the following action profiles in the second period. In state h each seller submits an ask of 1 and each buyer demands one unit and bids 1. In state l, each seller submits an ask of $\frac{1+c}{2}$ and each buyer demands one unit and bids $\frac{1+c}{2}$. Denote these action profiles by a^h and a^l respectively. It is straightforward to verify that a^h is a NE in $G(h)$ and a^l is a NE in $G(l)$.

Step 2 : The action profiles a^h and a^l are NE in the games that are induced by $G(h)$ and $G(l)$ and the proposed auction.

Let $Z \equiv 2B(B - \frac{1+c}{2})$ denote the prize, awarded to the highest bidder in the auction. In state h, no agent can unilaterally alter the outcome. In state l, an agent can alter the outcome (in the sense of preventing the prize) only by demanding B units at a price of 1. A seller has no incentive to deviate in this manner if he won the auction and

$$\frac{1+c}{2} - c + (1 - \frac{1}{2B})Z \geq 1 - B$$

or if he lost and

$$\frac{1+c}{2} - c - \frac{Z}{2B} \geq 1 - B$$

Similarly, a buyer has no incentive to demand B units and bid 1 if he won the auction and

$$1 - \frac{1+c}{2} + (1 - \frac{1}{2B})Z \geq 1 - B$$

or if he lost and

$$1 - \frac{1+c}{2} - \frac{Z}{2B} \geq 1 - B$$

(note that the first-period bids are sunk, hence they are left out of these constraints). It is easy to verify that since $B \geq 1$, the above inequalities must hold.

When $S = B = 1$, we need to consider the agents' ability to impose no trade in state l. If the seller won the auction, the additional SPIC constraints are:

$$\begin{aligned} 1 - \frac{1+c}{2} - (\frac{1-c}{2}) &\geq 0 \\ \frac{1+c}{2} - c + (\frac{1-c}{2}) &\geq 0 \end{aligned}$$

and if the buyer won the auction, the constraints are:

$$\begin{aligned} 1 - \frac{1+c}{2} + (\frac{1-c}{2}) &\geq 0 \\ \frac{1+c}{2} - c - (\frac{1-c}{2}) &\geq 0 \end{aligned}$$

It is easy to verify that these additional constraints are also satisfied.

Step 3 : There exists a PBNE in which (i) all agents participate in the auction, (ii) the agent with the lowest prior on h wins the auction, and (iii) the agents play a^h and a^l in the second period.

The previous step already established that a^h and a^l are NE in the second period modified games. It remains to show that if agents expect to play these action profiles in the second period, then the first-period auction has a BNE in which all agents participate and the winning agent is the one who assigns the highest prior to state l.

Let $\mathbf{t}$ denote the proposed auction mechanism. Let j^* denote the agent who wins the auction. Let $\boldsymbol{\beta} \equiv (\beta_i(\theta_i))_{i=1}^{S+B}$ denote a profile of bidding strategies. Let $\lambda_i(\theta_i, \boldsymbol{\beta})$ denote the probability that agent i wins the auction, given that i's prior on state h is θ_i and that the agents play the strategy profile $\boldsymbol{\beta}$. The interim expected payoff of agent i, given $\boldsymbol{\beta}$, is equal to:

$$\lambda_i(\theta_i, \boldsymbol{\beta}) U_{i=j^*}(a^h, a^l, \mathbf{t}) + [1 - \lambda_i(\theta_i, \boldsymbol{\beta})] \cdot E_{\theta_{-i}} U_{i \neq j^*}(a^h, a^l, \mathbf{t}) \tag{25}$$

where:

$$\begin{aligned}
U_{s\neq j^*}(a^h, a^l, \mathbf{t}) &= (1-\theta_s)[\frac{1+c}{2} - c - \frac{Z}{2B}] + \theta_s(1-c) + \frac{\beta_{j^*}}{2B} \\
U_{s=j^*}(a^h, a^l, \mathbf{t}) &= (1-\theta_{j^*})[\frac{1+c}{2} - c + (1-\frac{1}{2B})Z] + \theta_{j^*}(1-c) - (1-\frac{1}{2B})\beta_{j^*} \\
U_{b\neq j^*}(a^h, a^l, \mathbf{t}) &= (1-\theta_b)[1 - \frac{1+c}{2} - \frac{Z}{2B}] + \theta_s(1-1) + \frac{\beta_{j^*}}{2B} \\
U_{b=j^*}(a^h, a^l, \mathbf{t}) &= (1-\theta_{j^*})[1 - \frac{1+c}{2} + (1-\frac{1}{2B})Z] + \theta_{j^*}(1-1) - (1-\frac{1}{2B})\beta_{j^*}
\end{aligned}$$

If at least one agent vetoes the auction, then the seller's interim expected payoff is

$$U_s(a^h, a^l) = \theta_s(1-c) + (1-\theta_s)(\frac{1+c}{2} - c) = -(1-\theta_s)(\frac{1-c}{2}) + (1-c)$$

while the buyer's interim expected payoff is

$$U_b(a^h, a^l) = (1-\theta_b)[1 - \frac{1+c}{2}] + \theta_b(1-1) = (1-\theta_b)(\frac{1-c}{2})$$

We wish to show that that there exists a symmetric BNE in which all agents use the same bid function $\beta(\theta_i)$, which is monotonically *decreasing* in θ_i. To show this, we again rely on a formal relation between the problem of implementing a CIE solution and the problem of efficiently dissolving a partnership.

Consider the problem of efficiently dissolving an equal-share partnership with $2B$ members, who each owns $\frac{1}{2B}$ of an asset of size $Z = 2B(B - \frac{1+c}{2})$. Agent i's valuation of the partnership is $1 - \theta_i$. Suppose the members of this partnership could participate in a first-price, sealed-bid auction in which the highest bidder wins all shares of the asset, and the auction's revenues are equally shared among all partners.

Suppose agent i won this auction. Then his payoff would be

$$(1-\theta_i)Z - (1 - \frac{1}{2B})\hat{\beta}_i$$

where $\hat{\beta}_i$ denotes his bid. If $j \neq i$ won the auction, then i's payoff would be

$$\frac{1}{2B}\hat{\beta}_j$$

If the auction is not conducted, then i's payoff is

$$(1-\theta_i)\frac{Z}{2B}$$

By Proposition 5 of CGK, this auction has a symmetric BNE in which every player uses the same bidding function $\hat{\beta}(\theta_i)$, which is a monotonically decreasing function of θ_i. Moreover, the expected payoff of each bidder in this equilibrium is at least as high as his payoff when the auction is not conducted.

Note that the payoff of player $i \leq S = B$ in the above partnership-dissolution auction may be obtained by subtracting $(1-\theta_i)(B-c)-(1-c)$ from his payoff in our proposed auction. Similarly, the payoff of player $i \geq B+1$ in the partnership-dissolution auction may be obtained by subtracting $(1-\theta_i)(B-1)$ from his payoff in our proposed auction. Therefore, if $(\hat{\beta}(\theta_1),\ldots,\hat{\beta}(\theta_{2B}))$ is a symmetric BNE in the partnership-dissolution auction, then it must also be a BNE in our proposed auction. Moreover, the expected payoff of each agent in the proposed auction must be at least as high as his payoff when at least one agent vetoes the mechanism.

Step 4 : The proposed auction attains the CIE surplus in the PBNE described in Step 3.

Step 3 described a PBNE in which all agents participate in the proposed auction, the agent with the lowest prior on h wins, and in the second period the agents play a^h in state h and a^l in state l. By Proposition 2, the CIE surplus is achieved by a bet with the following properties: (i) it induces a market game where a^h is a NE in state h and a^l is a NE in state l, and (ii) the difference between the transfer in state h and the transfer in state l for an agent $i \neq i^*(\boldsymbol{\theta})$ (i.e., an agent, who is not the most optimistic about state l) is

$$(\frac{1+c}{2}-c)+(B-1)=B-\frac{1+c}{2}$$

for a seller and

$$(1-\frac{1+c}{2})+(B-1)=B-\frac{1+c}{2}$$

for a buyer (where the difference in these transfers for the most pessimistic agent is minus the sum of these differences across all the other agents). Hence, to establish that our proposed auction attains the CIE surplus in the PBNE of Step 3, it suffices to show that the difference between the equilibrium transfer that any agent $i \neq i^*$ receives in this auction in state h and his transfer in state l is $B-\frac{1+c}{2}$ (recall that we have shown that i^* wins the auction).

To show this, consider any agent who loses in the auction. In state h the price is above $\frac{1+c}{2}$, hence, his payoff in the auction is only his share in the revenue, $\frac{1}{2B}\hat{\beta}(\theta_j)$, where $\hat{\beta}(\theta_j)$ is the equilibrium bid of the winner. In state l, the price is at or below $\frac{1+c}{2}$, hence, the agent has to pay $B - \frac{1+c}{2}$ to the winner, in addition to his share in the revenue. Therefore, the difference in transfers is precisely $B - \frac{1+c}{2}$. ■