IDENTIFICATION AND ESTIMATION OF HEDONIC MODELS

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Identification and Estimation of Hedonic Models[1]

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Abstract

This paper considers the identification and estimation of hedonic models. We establish that technology and preferences in a separable version of the hedonic model are generically identified up to affine transformations from data on demand and supply in a single hedonic market. For a very general parametric structure, preferences and technology are fully identified from demand data. Much of the confusion in the empirical literature that claims that hedonic models estimated on data from a single market are fundamentally underidentified is based on linearizations that do not use all of the information in the model. The exact economic model that justifies the linear approximations has strange properties so the approximation is doubly poor. A semiparametric estimation method is proposed, and alternative estimators are considered. Instrumental variables estimators can be applied to identify technology and preference parameters from a single market even though there are no exclusion restrictions.

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1 Introduction

Sherwin Rosen pioneered the analysis of hedonic markets in a perfectly competitive setting. He also proposed an econometric identification strategy for recovering preferences and technology from hedonic markets. His hedonic model characterizes markets for heterogeneous goods (or factors or amenities) that implicitly price out the attributes that characterize the goods (or factors or amenities).

Rosen's fundamental paper has shaped the way economists think about the pricing of heterogeneous characteristics. Yet for two reasons, the full potential of his method remains to be exploited. First, except for special cases, high dimensional hedonic models with multiple characteristics require solutions of complicated partial differential equations to fully characterize the market equilibrium. This renders difficult theoretical analyses which require computation of nonlinear implicit equations. Second, the method of identification of preferences and technology proposed by Rosen has been severely criticized in the literature. It is widely held that the preferences and technology generating hedonic models are identified only through arbitrary functional form and exclusion assumptions, especially when they are estimated from data on a single market.

This paper considers whether equilibrium in hedonic markets imposes any restrictions on estimating equations and whether it is possible to identify technology and preferences from data on a single hedonic market. We consider both parametric and nonparametric versions of these questions.

We show that the hedonic model has empirical content. For very general parametric families, the hypothesis of equilibrium imposes very tight restrictions on the data. Preferernces and technology are generically identified from data on a single hedonic market. For the nonparametric case, we establish generic identification of technology and preference parameters up to affine transformations, the standard level of identification that can be obtained from market choice equations.

We establish that commonly used linearization strategies made to simplify estimation problems produce identification problems. The hedonic model is generically nonlinear. The functional form assumptions made in the applied literature give rise to the identification problems that are widely thought to be fatal to Rosen's empirical methodology. We go on to show that the economic model for which the widely used linearization methods are exact is implausible, so the approximation is doubly poor.

Our identification analysis also applies to a broader class of empirical models of nonlinear pricing: models of the effects of taxes on behavior when taxes are set optimally (Mirrlees, 1971), and a model of monopoly pricing (Mussa and Rosen, 1978, Wilson, 1993). It also applies to the standard problem of taxes and labor supply (Heckman 1974; Hausman, 1980). For specificity, in this paper we focus on the hedonic model, briefly discussing other applications in the conclusion.

This paper proceeds in the following way. In section two, we present the hedonic model and review an important linear-quadratic special case due to Tinbergen (1956), and used by Epple (1987), that gives rise to closed form solutions. This model justifies widely used linearizations as exact solutions. In section three, we discuss the peculiar properties of this model. The influential criticism of Rosen's estimating strategy by James Brown and Harvey Rosen (1982) is based on an linear-quadratic approximation to the true model which is exact in the Tinbergen model. When the Tinbergen model is slightly perturbed, the Brown-Rosen critique no longer applies. In section four, we prove a theorem (Thm 1) that establishes that for a general class of models, the Brown-Rosen critique only applies to a special, nongeneric, case. In section three, we go on to discuss standard criticisms of instrumental variables methods applied to estimate preferences and technology in hedonic markets: (a) sorting implies that within a single market, there are no natural exclusion restrictions (Epple, 1987; Kahn and Lang, 1988) and (b) use of multimarket data identifies the hedonic model by making implicit, and implausible, assumptions about why hedonic pricing functions differ across markets.

In section four, we establish (a) the identifiability of the hedonic model within a single market for a broad class of parametric models (polynomials of any finite order or any model belonging to a finite dimensional vector space); (b) the identification of the hedonic model up to levels for a broad class of nonparametric models; and (c) that using all of the information from both sides of the hedonic market jointly adds nothing to what can be identified analyzing the supply side and demand side separately in conjunction with the hedonic pricing function. We show how extra information on levels of outcomes, rather than just pricing and demand equations, aids in identifying the missing level set information in the nonparametric case. In section five, we briefly discuss instrumental variable estimation strategies. We prove a corollary of Theorem 1 that justifies the application of IV in the general parametric case and discuss extensions of the existing literature to cover the nonparametric case. Section six presents some conclusions and suggestions for future research.

2 The Hedonic Model: General Results and An Important Special Case With A Closed Form Solution

We first present a general statement of the hedonic model. For simplicity, consider a labor market setting. The model is static. Consumers (workers) match to single worker firms. Let z be an attribute vector characterizing jobs. $P(z)$ is the earnings of workers supplying attribute vector z, which is a disamenity. Let R be unearned income. We define $U(c, z, \theta, A)$ as the preferences of workers where θ represents preference parameters that vary across persons, A represents preference parameters common across persons and c is consumption where $c = P(z) + R$. Given $P(z)$, a twice continously differentiable price function, and assuming the utility function is twice differentiable[2], we obtain the following conditions for a maximum

FOC:

$$U_c\left(c, z, \theta, A\right) P_z\left(z\right) + U_z\left(c, z, \theta, A\right) = 0 \tag{1}$$

$$SOC : \left(U_{zz'} + U_c P_{zz'} + P_z U_{cc'}\left(P_z\right)'\right) \text{ is negative definite.} \tag{2}$$

[2]For expositional convenience, we restrict our analysis to economies in which the equilibrium price function is smooth. Similar analyses can be done for economies in which the equilibrium price function is not smooth. For an example of an economy with smooth technologies and absolutely continuous distributions of consumer heterogeneity in which the equilibrium price function is piecewise twice continuously differentiable see Nesheim (2001). For other examples of sorting problems with non-smooth pricing functions see (Wilson, 1993).

Firms demand attribute z and maximize profits which are a function of output $F(z;\nu,B)$ minus production costs $P\left(z\right)$ where ν is a vector of technology parameters firms that vary across firms and B is a common technology parameter shared by all firms. We assume that the production function is twice differentiable. Profits are

$$\Pi\left(z,\nu,B,P\left(z\right)\right)=F(z;\nu,B)-P(z)$$

$$\text{FOC: } F_z\left(z,\nu,B\right)-P_z\left(z\right)=0 \tag{3}$$

$$\text{SOC:}\left(F_{zz'}-P_{zz'}\right) \text{ is negative definite.} \tag{4}$$

Throughout we assume the regular case where the second order conditions hold as strict inequalities.

Workers differ in their preference vector θ. Firms differ in their productivity vector ν. Let the density of θ be f_θ. The density of ν is f_ν. We assume that both ν and θ are absolutely continuous random variables. Analytically, it is useful to distinguish the case $\dim(\theta)\geq\dim(z)$ and $\dim(\nu)\geq\dim(z)$, where "dim" is dimension, from other possible cases. This is the case analyzed in Rosen (1974). There is no loss of generality for the purposes of this paper in setting the inequalities to strict equalities.

Assuming a local implicit function theorem applies, we can invert FOC (1) and (3) to obtain θ and ν and hence obtain the classical hedonic case analyzed by Rosen (1974). From the FOC for the firm we obtain

$$\nu=\nu\left(z,P_z,B\right).$$

From the FOC (2) for the consumer we obtain:

$$\theta = \theta\left(z, P_z, P(z) + R, A\right).$$

Using these relationships, we substitute into f_ν and f_θ to find the density of z demanded given $P(z)$ and the technology and density of heterogeneity ν and the density of z supplied given $P(z)$ and the density of worker heterogeneity, θ and the preference system of consumers.

The **Demand Density** is:

$$f_\nu\left(\nu\left(z, P_z, B\right)\right) \det\left[\frac{\partial \nu\left(z, P_z, B\right)}{\partial z}\right] d\nu.$$

This tells us the density of demand for a given price function, technology parameter and density of ν.

The **Supply Density** is:

$$f_\theta\left(\theta\left(z, P_z, P(z) + R, A\right)\right) \det\left[\frac{\partial \theta\left(z, P_z, P(z) + R, A\right)}{\partial z}\right].$$

This is the density of the amenity supplied as a function of the price function, preference parameters A and density of θ. From the second order conditions (4) and (2), respectively, the Jacobian terms are both positive.

Equilibrium in hedonic markets requires that demand and supply be equated at each point of the support of z to solve for the market clearing surface $P(z)$. Equilibrium prices must satisfy the following second order differential equation in $P(z)$

$$\begin{aligned} & f_\nu\left(\nu\left(z, P_z, B\right)\right) \det\left[\frac{\partial \nu\left(z, P_z, B\right)}{\partial z}\right] \\ = \; & f_\theta(\theta(z, P_z, P(z) + R, A)) \det\left[\frac{\partial \theta\left(z, P_z, P(z) + R, A\right)}{\partial z}\right]. \end{aligned} \tag{5}$$

The solution depends on the technology of the firms F, the utility function U of the workers, and the distributions f_ν and f_θ respectively of firms and workers in the population. We examine the empirical content of these restrictions in this paper. Economic theory implies that marginal products and marginal utilities are nonnegative in most cases. In order for agents to participate in the market, firms and workers must receive wages and profits above reservation levels. These criteria generate the boundary conditions that determine the solution of the differential equation for equilibrium prices. They also play a role in the identification analysis.

We next present a linear-quadratic model with normal heterogeneity due to Tinbergen (1956) that has a closed form expression. This is the model that justifies widely used empirical approximations as exact descriptions, and provides an intuitive introduction to the hedonic model.

2.1 A Linear-Quadratic Example

Assume preferences are quadratic in z and linear in c and that $\dim(z) = \dim(\theta)$ [3]

$$U(c, z, \theta, A) = R + P(z) + \theta' z - \frac{1}{2} z' A z.$$

The conditions determining a consumer maximum are

FOC: $\theta - Az + P_z = 0$

SOC: $(P_{zz'} - A)$ is negative definite.

On the firm side, assume the production function is quadratic in z and $\dim(z) = \dim(\nu)$.

[3] The model in this example was first analyzed by Tinbergen (1956) and has been used by Epple (1987) and Tauchen and Witte (2001) among others.

Profits are

$$\Pi\left(z,\nu,B,P\left(z\right)\right)=\nu'z-\frac{1}{2}z'Bz-P\left(z\right)$$

and the conditions determining a firm's optimum are

FOC**:** $\nu-Bz-P_z=0$

where

SOC: $-(B+P_{zz'})$ is negative definite.

The distributions of ν,θ in the population are normal. The distribution of θ is $\theta\sim N(\mu_\theta;\Sigma_\theta)$, and the distribution of ν is $\nu\sim N(\mu_\nu;\Sigma_\nu)$.

An arbitrary price function induces a density of demand and a density of supply at every location z. The equilibrium price function can be found by equating these densities at every point z and solving the differential equation (5). However, in the normal-linear-quadratic case one can guess - correctly - that the solution to the problem is quadratic in z :

$$P(z)=\pi_0+\pi_1'z+\frac{1}{2}z'\pi_2z \tag{6}$$

and then check that this guess is accurate. Assuming the price function is quadratic, the first order conditions for the firm are:

$$\text{Firm:}\qquad \nu-Bz-\pi_1-\pi_2z=0 \tag{7}$$

and for the consumer they are:

$$\text{Consumer: } \theta-Az+\pi_1+\pi_2z=0. \tag{8}$$

From the second order conditions, $B + \pi_2$ and $A - \pi_2$ are positive definite. Thus we may solve for z from (7) to obtain $z = (B + \pi_2)^{-1}(\nu - \pi_1)$ and from (8) $z = (A - \pi_2)^{-1}(\theta + \pi_1)$. Note that once we have solved for π_1 and π_2, these latter two equations define the equilibrium matching function linking the characteristics of demanders (7) to those of suppliers (8). For each z, this function is

$$(B + \pi_2)^{-1}(\nu - \pi_1) = (A - \pi_2)^{-1}(\theta + \pi_1).$$

Thus, the equilibrium relationship between ν and θ is

$$\nu = \pi_1 + (B + \pi_2)(A - \pi_2)^{-1}(\theta + \pi_1). \tag{9}$$

Equilibrium is characterized by a vector π_1, and a matrix π_2, that equate demand and supply at all z subject to all constraints.

In the normal-linear-quadratic case, we may solve for π_1 and π_2 that equate demand and supply, both of which are normally distributed. Equating normal random variables only requires equating the mean of demand with the mean of supply and the variance of demand with the variance of supply. The mean demand is obtained from (7):

$$(B + \pi_2)^{-1}E(\nu - \pi_1) = E^D(z) \qquad \text{(Average Demand)}.$$

The mean supply is obtained from (8):

$$(A - \pi_2)^{-1}E(\theta + \pi_1) = E^S(z) \qquad \text{(Average Supply)}.$$

Letting $\mu_\theta = E(\theta)$ and $\mu_\nu = E(\nu)$, $E^D(z) = E^S(z)$ implies that

$$(B+\pi_2)^{-1}(\mu_\nu - \pi_1) = (A-\pi_2)^{-1}(\mu_\theta + \pi_1) \qquad \text{(Equality of means)}.$$

Rearranging terms, we obtain an explicit expression for π_1 in terms of $A, B, \mu_\nu, \mu_\theta$ and π_2 :

$$[(A-\pi_2)^{-1} + (B+\pi_2)^{-1}]^{-1}[(B+\pi_2)^{-1}\mu_\nu - (A-\pi_2)^{-1}\mu_\theta] = \pi_1.$$

To determine π_2, compute the variances of demand and supply from (7) and (8) respectively to obtain:

$$\begin{aligned} \textstyle\sum_\nu &= (B+\pi_2)\textstyle\sum_z^D (B+\pi_2)' \\ \textstyle\sum_\theta &= (A-\pi_2)\textstyle\sum_z^S (A-\pi_2)' \end{aligned}$$

where $\sum_z^D$ is the variance of demand given the price schedule and $\sum_z^S$ is the variance of z given the supply schedule. From equality of variances of the demand and supply distributions we obtain an implicit equation for π_2 :

$$(B+\pi_2)^{-1}\sum\nolimits_\nu (B+\pi_2)^{-1} = (A-\pi_2)^{-1}\sum\nolimits_\theta (A-\pi_2)^{-1}.$$

We pin down initial conditions using the restrictions that $U \geq \bar{U}$, a reservation value, and profits are positive $(\Pi \geq 0)$. Equilibrium profits as a function of location are $\frac{1}{2}z'(B+\pi_2)z - \pi_0$. Hence nonnegativity of profits implies $-\pi_0 \geq 0$ since $(B+\pi_2)$ is positive definite by the second order conditions. A similar argument on the worker side implies $\pi_0 \geq 0$. Hence $\pi_0 = 0$.

For a separable case with $\sum_\theta$ and $\sum_\nu$ diagonal, π_2 is diagonal. Effectively, this is a scalar case. Suppose that A and B are scalars so that z is scalar. Then

$$
\begin{aligned}
\pi_1 &= \frac{\dfrac{\mu_\nu}{B+\pi_2} - \dfrac{\mu_\theta}{A-\pi_2}}{\dfrac{1}{A-\pi_2} + \dfrac{1}{B+\pi_2}} \\
&= \frac{\mu_\nu(A-\pi_2) - \mu_\theta(B+\pi_2)}{(B+\pi_2)+(A-\pi_2)} \\
&= \mu_\nu\left(\frac{A-\pi_2}{A+B}\right) - \mu_\theta\left(\frac{B+\pi_2}{A+B}\right).
\end{aligned}
$$

Recall that from the second order conditions $A - \pi_2 > 0$ and $B + \pi_2 > 0$. Equality of variances implies that $(A-\pi_2)^2\sigma_{\nu\nu} = (B+\pi_2)^2\sigma_{\theta\theta}$. Define $\sigma_\nu = (\sigma_{\nu\nu})^{1/2}$ and $\sigma_\theta = (\sigma_{\theta\theta})^{1/2}$ so

$$
\begin{gathered}
\pm\,(A-\pi_2)\sigma_\nu = (B+\pi_2)\sigma_\theta \\
\pi_2 = \frac{A\sigma_\nu - B\sigma_\theta}{\sigma_\nu + \sigma_\theta}\ .^4
\end{gathered}
$$

If $\sigma_\nu = \sigma_\theta$ and $A = B$, $\pi_2 = 0$ is a solution. This is a knife-edge result. If

$$
\begin{gathered}
\sigma_\nu = \sigma_\theta,\ A \neq B \\
\frac{(A-B)}{2} = \pi_2.
\end{gathered}
$$

If the variance $\sum_\theta = 0$ or $\sum_\nu = 0$, then there is effectively only one type of consumer or one type of firm respectively. If $\sum_\theta = 0$, $\pi_2 = A$ and $\pi_1 = \theta$, a vector of constants. If $\sum_\nu = 0$, $\pi_2 = B$ and $\pi_1 = \nu$, a vector of constants. In those cases, the hedonic price line coincides with the marginal valuations of consumers or marginal productivity of firms respectively.

[4] The other root violates second order conditions.

3 Identifying and Estimating The Model

Sherwin Rosen stressed the importance of taking theory to data. He considered the problem of recovering technology and preference parameters from data. He also framed the empirical questions about hedonic models that have occupied the attention of economists for the past 27 years.

He analyzed the problem of using data from a single market in which $P(z)$ is available and there are no missing attributes. Using the first order conditions (1) and (3) ((7) and (8) in the linear-quadratic-normal example) he proposed a two step method for estimating both preference and technology parameters. He did not consider direct estimation of production, profit or preference functions, a source of information we consider in section four. We simply note here that if there are no missing attributes, we can recover the production function directly from data on inputs and outputs using standard methods. Even if production (or profit) data are available, data on utility are not, so the problem considered by Rosen still remains for recovering the parameters of at least one side of the market.

From our discussion of the linear - quadratic - normal case, the parameters π_1 and π_2 do not directly identify either preference or technology parameters except when $\sum_\theta = 0$ or $\sum_\nu = 0$ respectively. The pricing function combines parameters in an economically uninterpretable fashion.

The most direct approach to estimating the hedonic model would be to solve equation (5) for $P(z)$ in terms of the parameters of preferences, technology and the distributions of tastes and productivity and to jointly estimate the demand functions and supply functions and distributions of preference and technology parameters exploiting all of the information in the equilibrium conditions including data on demand, supply and the pricing function.

That approach is computationally complicated and does not transparently deliver identification of the deep structural parameters.

Rosen suggested an intuitively plausible and computationally simpler two step estimation procedure that has been widely criticized. In step 1 of his procedure, the analyst estimates $P(z)$ from market data. In step 2, the analyst uses first order conditions (1) and (3) in conjunction with the marginal prices obtained from step 1 to recover preferences and technology respectively.

In the context of the linear-quadratic example, the first stage would be to estimate pricing function $P(z)$, recover π_1 and π_2, and form the marginal prices and then estimate the curvature parameters of technology, and preferences using (7) and (8) respectively. Specifically, he proposed to estimate B and the mean of ν (μ_ν) from the least squares regression

$$\hat{\pi}_1 + \hat{\pi}_2 z = \mu_\nu + Bz + \varepsilon_\nu \tag{10}$$

where $\varepsilon_\nu = \nu - \mu_\nu$, and "$\hat{\ }$" denotes estimate. A parallel proposal for preferences estimates A and the mean of $\theta(\mu_\theta)$ from the regression

$$\hat{\pi}_1 + \hat{\pi}_2 z = \mu_\theta + Az + \varepsilon_\theta \tag{11}$$

where $\varepsilon_\theta = \theta - \mu_\theta$. We assume that μ_θ and μ_ν are functions of regressors (x) and (y) respectively, $\mu_\theta(x)$ and $\mu_\nu(y)$.

In two influential papers, James Brown and Harvey Rosen (1982) and James Brown (1983) analyze the regression method based on (10) and (11). These papers contain most of the main ideas in the empirical literature on hedonics that emerged from Rosen's paper. They interpret (10) and (11) as linearized *approximations* to (1) and (3). The linear quadratic model of Section 2 is the framework for which these approximations are exact.

In this approximation interpretation, the distributions of ν and θ are kept in the background. Standard linear econometric methods are applied to identify the parameters of (10) and (11) and connections among the parameters of preferences, technology and the distributions of tastes and productivity are not made explicit. Issues of identification are confused with issues of estimation. Common to an entire genre of empirical economics, this literature focuses on finding "good instruments" and misses basic sources of identification in hedonic models.

Starting from (10) and (11), Brown (1983) and Brown and Rosen (1982) make three points which have been reiterated in the subsequent empirical literature.

Point One: Identification Can Only Be Obtained Through Arbitrary Functional Form Assumptions

Since z is on both sides of (10) and (11), by a property of least squares, a regression using the constructed price $\hat{P}_z(z) = \hat{\pi}_1 + \hat{\pi}_2 z$ as the dependent variable in (10) or (11) only identifies π_2 even if μ_ν or μ_θ are functions of regressors. In general, π_2 does not identify any technology or preference parameter. In the special cases where there is no variation in preference parameters θ or where there is no dispersion in ν, π_2 identifies preference (A) or production (B) parameters respectively.

However, if the constructed price is a nonlinear function of z, this argument no longer holds. The nonlinear variation in $\widehat{P}_z\left(z\right)$ gives an added piece of information which can help to identify technology and preference parameters.[5] This identification strategy rules out collinearity between z and $\widehat{P}_z\left(z\right)$, but such nonlinearity is widely viewed as an artificial source of identification that is thought to be "arbitrary." In Theorem 1 in section 4, we prove that this nonlinearity is generic in the hedonic model.

[5]See Fisher (1966) for an early discussion of the value of nonlinearities in identifying econometric models.

Point Two: Absence of Instruments

Even if such "arbitrary" assumptions are made, so that we can use the nonlinearity in $\widehat{P}_z(z)$ to help identify the parameters and circumvent Point One, we still face standard endogeneity problems. z is correlated with ε_ν and ε_θ in (10) and (11) respectively. Moreover, exclusion restrictions from the other side of the market cannot be justified. In the notation of this section, the equilibrium matching condition (9) of section 2 requires that

$$\varepsilon_\nu = \varepsilon_\theta + (A - B)z + \mu_\theta(x) - \mu_\nu(y) \tag{12}$$

so that conditional on z there is both functional and statistical dependence connecting ε_θ, ε_ν, z and the regressors.[6] Conditional on z, $\varepsilon_\nu, \varepsilon_\theta$, x and y become stochastically dependent even if in the underlying population initially they are mutually independent.

With data from a single market, one is forced to hunt for "clever" instruments with a questionable economic basis. Thus, even if "arbitrary" nonlinearities are invoked, standard instruments may be lacking. In sections 4 and 5 we show that the economics of the model guarantees valid instruments even though there are no exclusion restrictions.

Point Three: Use of Multimarket Data

Brown (1983), Brown and Rosen (1982), Kahn and Lang (1988), and Tauchen and Witte (2001) change Rosen's problem and consider estimation of the first order conditions using multimarket data either across regions, or across time in the same region. The motivation for this approach is that if preferences, technology, and the distributions of tastes and productivities are the same across markets but for some unspecified reason price functions are not, variation in the $P_z(z)$ across markets serves to identify preferences and technology. This source of identification is viewed as being more robust.

[6]Epple (1987), Bartik (1987) and Kahn and Lang (1988) stress this point.

The problem with this identification strategy is that it is logically inconsistent. If preferences, technology, and the distributions of tastes and productivities are the same across markets, equilibrium price functions must be as well. The strategy is apparently more robust because it is vague about the source of variation that makes price functions differ when preferences, technology, and the distributions of tastes and technology are common across markets. This approach can be used to identify the preferences or technology on one side of the market. If preferences are stable and the distributions of preferences across markets are stable, but technologies are different for exogenous reasons, then multimarket variation shifts the hedonic function against stable preferences and identifies preference parameters. Switching the roles of technology and preferences, multimarket data identifies technology and the distribution of technology parameters.

3.1 Using All Of The Economics of The Model

These criticisms are symptoms of a deeper problem: all of the economic content of the hedonic model is not being exploited. We argue that when it is exploited, the model is generically identified even within a single market without having to invoke arbitrary functional forms. We develop this point formally in the next section. Here we develop the intuition for it using the linear-quadratic model.

Consider all of the economic implications of the linear-quadratic model - not just the first order conditions (7) and (8). Any reasonable specification of the model requires that profits be non-negative, that utilities exceed threshold reservation values and that firm marginal products be non-negative while marginal utilities of consumers for disamenities be non-positive. Adopting all of these restrictions eliminates Point One within the linear-quadratic example of Section 2.

The linear-quadratic-normal model of Section 2 results in an equilibrium with a linear marginal price function. This equilibrium produces an econometric system that is not identified. (Brown-Rosen Point One). In this example, it would be arbitrary and incorrect to impose that the marginal price function is nonlinear. However, the model in Section 2 is very special. It belongs to a very small class of models that produce an equilibrium marginal price function that is linear. In the next section we prove as a special case of a more general theorem that there is an open dense set of models surrounding the linear-quadratic models of Section 2 that do not produce linear marginal price functions. In these models, it is not arbitrary to impose nonlinear marginal price functions.

The normal-linear-quadratic example has a number of peculiarities. From (7) and (8), it is evident that marginal products can become negative, and marginal disutilities of labor (z) can become positive. Nothing restricts marginal prices to be non-negative or for the demands or supplies of z to be non-negative.

To see how fragile Point One is, suppose that we perturb the scalar version of the model to have non-normal θ and ν. Profits are

$$\Pi(z) = \nu_0 + \nu_1 z - \frac{b}{2} z^2 - P'(z),$$

with first order condition

$$\nu_1 - bz - P'(z) = 0.$$

Worker preferences are

$$U(z) = \theta_0 + \theta_1 z - \frac{a}{2} z^2 + P(z)$$

with first order condition

$$\theta_1 - az + P'(z) = 0.$$

Figure 1 shows the price functions for two cases. A full specification of parameter values generating figures 1-4 is given in Table 1. The first case is for ν_1 and θ_1 normally distributed. ($\lambda = 1$; see the notes) The second case is for ν_1 and θ_1 distributed as a mixture of normals with weights $\lambda = .999$ (for the original case which produced the straight line) and $1 - \lambda = .001$. With this minor perturbation, the price function becomes highly nonlinear. The second derivative of the price function is far from zero. (Figure 2). Figure 3 and 4 show two other cases when $\lambda = .99$ and $\lambda = .90$. A small dose of nonnormality produces a highly nonlinear price function, and undercuts Brown-Rosen Point One.

These figures also reveal unattractive properties of the linear-quadratic model. Negative and positive quantities of z are demanded and supplied and marginal prices are negative for a large portion of the population. Figures 4 and 5 present a case where marginal prices are positive because we restrict $\nu_1 \geq 0$, $b > 0$, $\theta_1 \geq 0$ and $a > 0$. A full specification of parameter values is given in Table 2. We write $\ell n\ \nu_1 = \nu_{10} + \nu_{11}' x + \varepsilon_\nu$ and $\ell n\ \theta_1 = \theta_{10} + \theta_{11}' y + \varepsilon_\theta$ where (x, y, ε_ν and ε_θ) are mixtures of normals. Now marginal prices are nonlinear and positive and only positive quantities of the amenity are demanded and supplied. By imposing economically plausible restrictions, Brown-Rosen Point One is shown to be less cogent. In Section 4 we show that these examples are generic.

Even though Point One is non-generic, Point Two remains. There are apparently no valid instruments for z on the right hand sides of (10) and (11). A strategy needs to be found to deal with the endogeneity of z. In the next two sections, we discuss two such strategies and present general results for a model with a single characteristic with no arbitrary functional form restrictions or distributional assumptions and establish that the hedonic model is generically identified from data from a single market. Even though there are no exclusion restrictions, instrumental variables is a valid estimator.

4 Parametric and Nonparametric Analyses of A One Dimensional Model with Additively Separable First Order Conditions

This section analyzes a class of one dimensional models for z with additive separability in the first order conditions but with no specific functional form or distributional assumptions imposed. The one dimensional case allows us to abstract from a variety of problems that we address in our other work: (a) questions of existence of solutions to partial differential equations and (b) questions about the proper treatment of missing attributes in a multidimensional model.[7] Both types of questions are important but they distract us from the basic questions of identification and testability of the hedonic model posed in the introduction to this paper.

We analyze a class of separable preferences and technologies on the firm side. We start

[7]Existence conditions for ordinary differential equations are much easier to satisfy. See Zachmanoglou and Thoe, 1986.

with production technology $F(z, x, \varepsilon_1)$ where $(x, \varepsilon_1) = \nu$ in the notation of Section 2. We use a more symmetric notation to simplify the exposition. The first order condition is

$$F_z(z, x, \varepsilon_1) = P'(z).$$

We consider a class of models with restrictions on $F_z(z, x, \varepsilon)$ such that we can separate z from x and ε_1, and x from ε_1. For a known monotonic transformation ψ_1, we assume that

$$\psi_1(F_z(z, x, \varepsilon)) = \tau(z) + M_1(\eta_1(x) + \varepsilon_1))$$

where M_1 is monotonic in $(\eta_1(x) + \varepsilon_1)$ and $\psi_1 \in C^2$, $M_1 \in C^2$. With this restriction, we can write FOC as

$$\tau(z) + M_1(\eta_1(x) + \varepsilon_1) = \psi_1(P'(z))$$

so we can rewrite the model in the following way:

$$M_1^{-1}[\tau(z) - \psi_1(P'(z))] = \eta_1(x) + \varepsilon_1. \tag{A-1a}$$

with

$$\text{Support } \varepsilon_1 = (0, \infty). \tag{A-1b}$$

Leading cases include

1. (a) ψ_1 the identity function and $F_z(z,x,\varepsilon) = \varphi_1(z) + M_1(\eta_1(x) + \varepsilon_1)$ where $\dfrac{d\varphi_1}{dz} < 0$; M_1 can be the identity function, the exponential function or any other monotonic transformation of $\eta_1(x) + \varepsilon_1$

   or

   (b) $\psi_1(q) = \log(q)$

   $F_z(z,x,\varepsilon) = K_1(z)M_1(\eta_1(x) + \varepsilon_1)$; M_1 monotonic

   $\eta_1(x) + \varepsilon_1 = (M_1)^{-1}\left(\dfrac{P'(z)}{K_1(z)}\right)$.

For specificity, we consider one member of this class noting that we can generalize our results to the broader class at the end of this section. We specify the firm production function as

$$F(z,x,\varepsilon_1) = \Phi^1(z) + z\eta_1(x) + z\varepsilon_1 \tag{A-2}$$

where the cost of labor quality is $P(z)$, as before, and there is a unit price of output. Letting $\dfrac{\partial \Phi^1}{\partial z} = \varphi_1(z)$, the conditions for profit maximization are

$$\text{FOC: } \varphi_1(z) + \eta_1(x) + \varepsilon_1 = P'(z) \tag{13}$$

which is a special case of the transformations (A-1a) introduced above and

SOC: $\varphi_1'(z) - P''(z) < 0$.

On the worker side, we analyze preferences with a constant marginal utility of consumption of goods. For specificity, we consider a particular model:

$$U(z,y,\varepsilon_2,c) = c - \Phi^2(z) + z\eta_2(y) + z\varepsilon_2.$$

We define $\frac{\partial \Phi^2}{\partial z} = \varphi_2(z)$ and write the maximization conditions as

FOC: $P'(z) - \varphi_2(z) + \eta_2(y) + \varepsilon_2 = 0$

and

SOC: $P''(z) - \varphi_2'(z) < 0.$

Again, our analysis applies to a broader class of preferences with separable marginal utilities. We develop this point below.

We assume an equilibrium determination of prices so that equilibrium condition (5) applies. Let $q_1(x)$ be the density of x with support $\mathcal{X}$ and let $q_2(y)$ be the density of y with support $\mathcal{Y}$. Define the density of ε_1 as $g_1(\varepsilon_1)$ and the density of ε_2 as $g_2(\varepsilon_2)$. Assume x is independent of ε_1 and y is independent of ε_2. The first order conditions define mappings from (x, ε_1) to (x, z) and from (y, ε_2) to (y, z) :

$$\begin{aligned} \varepsilon_1 &= P'(z) - \varphi_1(z) - \eta_1(x) \\ x &= x \end{aligned}$$

and

$$\begin{aligned} \varepsilon_2 &= P'(z) - \varphi_2(z) - \eta_2(y) \\ y &= y. \end{aligned}$$

These expressions relate equilibrium sortings of ε_1 and ε_2 to z given x and y respectively. Such sorting is an essential feature of the hedonic equilibrium model. The associated Jacobians are $dx d\varepsilon_1 = [P''(z) - \varphi_1'(z)] dx dz$ and $dy d\varepsilon_2 = [\varphi_2'(z) - P''(z)] dy dz$, respectively. From the second order conditions, the terms in brackets are positive. Equilibrium condition

(5) implies that

$$\int_{\mathcal{X}} g_1(P'(z)\text{-}\varphi_1(z)\text{-}\eta_1(u))(P''(z)\text{-}\varphi_1'(z))q_1(u)du \tag{14}$$
$$= \int_{\mathcal{Y}} g_2(P'(z)\text{-}\varphi_2(z)\text{-}\eta_2(u))(P''(z)\text{-}\varphi_2'(z))q_2(u)du$$

where $\mathcal{X}$ and $\mathcal{Y}$ are supports of X and Y respectively. Initial conditions are provided by the requirements that $\Pi \geq 0$ and $U \geq u_0$.

We will now state a genericity result. Recall that a property $P(\theta)$, depending on a parameter $\theta \in \Theta$, is called generic if the set $\Omega \subset \Theta$ of values of the parameter for which it holds true contains a countable intersection of open dense subsets. If Θ is a complete metric space, such a set Ω will be dense in Θ, by a celebrated theorem of Baire. Moreover, the intersection of two such sets will still be dense in Θ. In other words, if a property is generic, and does not hold for a certain value $\bar{\theta}$ of the parameter, there will be in any neighbourhood of $\bar{\theta}$ some other value θ of the parameter where the property holds true. A generic property is robust in the sense that if $P_1(\theta)$ and $P_2(\theta)$ are generic, then so is their intersection $P_1(\theta) \cap P_2(\theta)$.

The "parameters" of the model which are functions are $(\varphi_1, \varphi_2), (g_1, g_2), (\eta_1, \eta_2), (q_1, q_2)$. We have the following:

Theorem 1 *Generically with respect to any of the parameter pairs, the equilibrium equations have no solution of the form* $P'(z) = a_1 + b_1\varphi_1(z)$, *nor any solution of the form* $P'(z) = a_2 + b_2\varphi_2(z)$.

The precise definitions of the parameter spaces and their respective topologies are given in Appendix A, together with the proof of the theorem. This theorem can easily be modified to prove that generically, the equilibrium equations have no solution $P'(z)$ which can be expressed as a polynomial in (φ_1, φ_2).

As a consequence of this theorem, Brown-Rosen Point One that regressions of $P'(z)$ on $\varphi_1(z)$ or $\varphi_2(z)$ simply recover the marginal price $(\hat{a}_1 = 0, \hat{b}_1 = 1; \hat{a}_2 = 0, \hat{b}_2 = 1)$ is not generically correct. The model is intrinsically nonlinear. The examples presented at the end of the section 3 are prototypical, not special. There is no arbitrariness in assuming that $P'(z)$ and $\varphi_1(z)$ do not lie in the same linear space.

Even if Point One is not generic, Point Two remains. Within a single market, there is no natural exclusion restriction. The larger question considered in this paper is whether we can identify $(g_1, g_2, \varphi_1, \varphi_2, \eta_1, \eta_2)$ from data on $P(z), z, x$, and y from a single market. We focus on identifying (g_1, φ_1, η_1) from data on $P(z), z$, and x since the analysis is symmetric for (g_2, φ_2, η_2) using data on $P(z), z$, and y. We later consider what information, if any, is available from the joint density of $(z, x, y, P(z))$.

We present two methods for recovering these functions from data in a single market. One is based on extensions of average derivative models (Powell, Stock and Stoker, 1989) and closely related transformation models (see Horowitz (1998)). We develop these methods in this section. The other is based on nonlinear instrumental variables. (Amemiya, 1975). The second method is based on a corollary of Theorem 1 which we prove in section five.

The trick in applying average derivative and transformation models to the hedonic problem is to exploit the separability of z, x and ε_1. Define

$$T_1(z) = P'(z) - \varphi_1(z).$$

This function combines price and preference data. This kind of function is called a transformation function and its nonparametric identification and estimation have received extensive theoretical attention. (See Horowitz (1998) for a survey and new results). These models extend average derivative models (Powell, Stock and Stoker, 1989) by considering nonlinear transformations of dependent variables. Observe that $T_1(z) = M_1^{-1}(\tau(z) - \psi_1(P'(z))$ as defined in (A-1a).

Let G_1 be the cumulative distribution function corresponding to g_1. Assuming X is independent of ε_1 and taking account of the first order condition (13) , we may write

$$F^1(z\,|x\,) = G_1(T_1(z) - \eta_1(x))$$

where $F^1\,(z\,|x\,)$ is the empirical cumulative density function of z conditional on x. Assuming that $\lim_{q\to\infty} T_1(q) = \infty$, which follows from the assumption that the support of $\varepsilon_1 = (0, \infty)$, and further assuming that T_1 and η_1 are twice continuously differentiable, we may write

$$F_z^1(z\,|x\,) = g_1(T_1(z) - \eta_1(x)) \cdot T_1'(z) \tag{15}$$

where $T_1'(z) > 0$ from SOC.

Moreover

$$F_{x_i}^1(z\,|x\,) = -g_1(T_1(z) - \eta_1(x)) \cdot \frac{\partial \eta_1}{\partial x_i}. \tag{16}$$

¿From (16)

$$\frac{F_{x_i}^1\,(z\,|x\,)}{F_{x_j}^1(z\,|x\,)} = \frac{\dfrac{\partial \eta_1(x)}{\partial x_i}}{\dfrac{\partial \eta_1(x)}{\partial x_j}} \quad \text{for all } i, j. \tag{17}$$

This ratio determines the level sets of $\eta_1(x)$. More generally, taking the ratio of (15) to (16) for an arbitrary argument i, we obtain

$$\left(\frac{-F_z^1(z\,|x)}{F_{x_i}^1\,(z\,|x)}\right) = \frac{T_1'(z)}{\frac{\partial \eta_1(x)}{\partial x_i}} \tag{18}$$

¿From (17), sign $(F_{x_i}^1) = -$ sign $\left(\frac{\partial \eta_i}{\partial x_i}\right)$. Assume, without loss of generality, that $\frac{\partial \eta_i}{\partial x_i} > 0$.

Then the left hand side of (18) is positive. (Recall that $T_1'(z) > 0$).

$$\frac{\partial}{\partial z} \log \left[\frac{-F_z^1(z\,|x)}{F_{x_i}^1(z\,|x)}\right] = \frac{T_1''(z)}{T_1'(z)}. \tag{19}$$

Define $h\,(z, x) = \log \left[\frac{-F_z^1(z|x)}{F_{x_i}^1(z|x)}\right]$. Since $h(z, x)$ satisfies equation (19), then $h(z, x)$ must be of the form

$$h(z, x) = h_0 + h_1(z) + h_2(x)$$

where $h_1(0) = 0$, $h_2(0) = 0$, and h_0 is a constant. h_0, $h_1(z)$ and $h_2(x)$ are known empirically. Further equation (19) can be written as

$$\frac{dh_1(z)}{dz} = \frac{T_1''(z)}{T_1'(z)}.$$

This equation has the solution

$$T_1'\,(z) = K_1 \exp\left(h_1\,(z)\right) \tag{20}$$

where K_1 is a constant of integration. Further

$$T_1(z) = C_1 + K_1 \int_0^z \exp(h_1(s))ds$$

where C_1 is another constant of integration.

This solution enables us to solve for $\eta_1(x)$. Substituting (20) into (18),

$$\begin{aligned} \frac{\partial \eta_1}{\partial x_i} \exp(h_0 + h_1(z) + h_2(x)) &= K_1 \exp\,(h_1(z)) \\ \frac{\partial \eta_1(x)}{\partial x_i} &= K_1 \exp(\,-h_0 - h_2(x)) \end{aligned} \tag{21}$$

hence

$$\eta_1(x) = R_1 + K_1 \int_0^x \exp(-h_0 - h_2(s))ds$$

where R_1 is a constant of integration and the multiple integral is taken over all the dimensions of x.

For a given K_1, we can identify $T_1(z)$ and $\eta_1(x)$ up to constants. From (15), we can identify $g_1(\varepsilon_1)$ using a normalization on ε_1 to tie down the undetermined combination of constants C_1 and R_1 which we acquire when we integrate up to the levels of $T_1(z)$ and $\eta_1(x)$).

Thus we identify

$$\tilde{\eta}_1(x) = \frac{\eta_1(x) - R_1}{K_1}$$

$$\tilde{T}_1(z) = \frac{T_1(z) - C_1}{K_1}.$$

In this notation

$$\varepsilon_1 = T_1(z) - \eta_1(x) = (C_1 - R_1) + K_1(\tilde{T}(z) - \tilde{\eta}_1(x))$$

we can identify the combination of coefficients $(C_1 - R_1)$ by assuming $E(\varepsilon_1) = 0$ or median $(\varepsilon_1) = 0$ or fixing some quantile of ε_1 to a known value. This leaves K_1 undefined (and the specific values of C_1 and R_1 that equal $C_1 - R_1$). From (15) and (16) we can identify $g_1(\varepsilon_1)$ up to scale. Specifically we define

$$\tilde{\varepsilon}_1 = (\varepsilon_1 / K_1)$$

$$g_1(\varepsilon_1) d\varepsilon_1 = K_1 g_1(\tilde{\varepsilon}_1 K_1) d\tilde{\varepsilon}_1 = \tilde{g}_1(\tilde{\varepsilon}_1).$$

Since we know $P'(z)$,we can identify $\tilde{\varphi}_1(z) = P'(z) - K_1\tilde{T}(z) - C_1$.

Using the data on $F^2(z \mid y)$, we can identify $\tilde{\eta}_2(y) = \dfrac{\eta_2(y) - R_2}{K_2}$ and $\tilde{T}_2(z) = \dfrac{T_2(z) - C_2}{K_2}$ and $\tilde{g}_2(\tilde{\varepsilon}_2) = K_2 g_2(K_2\varepsilon_2) d\varepsilon_2$ where $\tilde{\varepsilon}_2 = (\varepsilon_2 / K_2)$, and $\tilde{\varphi}_2(z) = P'(z) + K_2\tilde{T}_2(z) + C_2$ where the constants are defined in a fashion analogous to the case previously analyzed.

The lack of identification of the scale of the utility function is a classical result. We do not observe utility so we can only identify level sets connected with utility. If we observe output or utility, we can determine the missing parameters by using direct analysis of the production or profit functions. Direct estimation of (A-2) entails identification of a correlated random coefficient model in a semiparametric setting.[8] Using (13) as a replacement function in the sense of Heckman and Robb (1985) or as a control function in the sense of Blundell and Powell (2001), we may solve for ε_1 and substitute in (A-2) to obtain

$$F(z,x) = \Phi^1(z) + z\eta_1(x) + z(P'(z) - \eta_1(x) - \varphi_1(z)) = \Phi^1(z) + zP'(z) - z\varphi_1(z)$$

so

$$\psi(z) = F(z,x) - zP'(z) = \int_0^z \varphi_1(t)dt - z\varphi_1(z)$$

and $\psi(z)$ is observed. We may thus estimate the derivative on the right hand side

$$\frac{\partial\psi(z)}{\partial z} = -z\varphi_1'(z).$$

Integrating up we obtain

$$C_0 + \int \left[-\frac{1}{z}\frac{\partial\psi(z)}{\partial z}\right] dz = \varphi_1(z)$$

so we determine $\varphi_1(z)$ up to an additive constant and in the context of the example for linear in parameters $\eta_1(x)$ we determine K_1.

With additional (weak) parametric structure, we can determine the scaling constants without using the output data. Thus, we can stay within the Rosen program which does not

[8]See Heckman and Vytlacil (1998) for a discussion of correlated random coefficient models.

contemplate using output data. We now assume that there is a finite-dimensional vector space E which contains both ϕ_1 and ϕ_2 and which is known *ex ante*. In other words, both ϕ_1 and ϕ_2 can be described by a finite set of parameters $(a_1, ..., a_K)$ and $(b_1, ..., b_K)$ which enter linearly: $\phi_1 = \sum a_k \bar{\phi}_k$ where the $\bar{\phi}_k$ are known functions, and similarly for ϕ_2. It will be assumed that E consists of C^1 functions, and contains the constants. For example, E could be the set of polynomials of degree less than or equal to m where m is a known integer.

Theorem 2 *Generically with respect to any of the parameter pairs in Theorem 1, no solution P of the equilibrium equation belongs to E, and φ_1, φ_2 are identified up to additive constants*

Proof. As shown above, we have:

$$\begin{aligned} \tilde{T}_1(z) &= \frac{\tilde{P}'(z)}{K_1} - \frac{C_1}{K_1} - \frac{\varphi_1(z)}{K_1} \\ \tilde{T}_2(z) &= \frac{\tilde{P}'(z)}{K_2} - \frac{C_2}{K_2} - \frac{\varphi_2(z)}{K_2} \end{aligned}$$

Arguing as in Theorem 1, we can show that generically $P \notin E$. This being the case, there must be some continuous function f such that $\int fh = 0$ for all $h \in E$, but $\int fP' \neq 0$. Applying such a function to both sides of the preceding equalities, we get:

$$\int \tilde{T}_i(z) f(z) dz = \frac{1}{K_i} \int \tilde{P}'(z) f(z) dz \quad , i = 1, 2$$

which determines $K_i, i = 1, 2$. Plugging back into the equations, we find that φ_i is determined up to an additive constant C_i ■

4.1 Is There Information In The Joint Densities?

Thus for a very general class of polynominal models, we obtain identification of the $(\varphi_{1,}\ \varphi_2)$ functions from single market data. So far we have only considered identification using data from only one side of the market. We now consider whether additional information can be extracted from the joint densities on both demand and supply sides.

Thus far we have used information on the joint densities of (x, z) and (y, z) and have shown how to identify everything except K_1, K_2. In the parametric case covered by Theorem 2, we identify K_1 and K_2.

There is one potentially powerful piece of information that we have not yet used; the joint distribution of (x, y, z). This joint distribution may have identifying power because the distribution of z conditional on x does not equal the distribution of z conditional on x and y. Where there is sorting on both sides of the market, this full joint density contains information that might be exploited.[9] We show that there is no more information available beyond what is in the marginal densities.

Recall the first-order conditions from the previous section. On the firm side we have

$$\varepsilon_1 = T_1(z) - \eta_1(x)$$

and on the worker side we have

$$\varepsilon_2 = T_2(z) - \eta_2(y).$$

These technologies are the primitives of the model. The other primitive is the joint density

[9]Epple (1987) discusses the potential importance of using the full joint density but his discussion is not complete.

for $(x, y, \varepsilon_1, \varepsilon_2)$:

$$q_1(x)\, q_2(y)\, g_1(\varepsilon_1)\, g_2(\varepsilon_2) \,.$$

By assumption $x, y, \varepsilon_1, \varepsilon_2$ are jointly independent. Note that this independence does not hold conditional on location z, but only holds across all z locations. (One can think of the hedonic equilibrium as a mapping from the joint distribution of $(x, y, \varepsilon_1, \varepsilon_2)$ to the joint distribution of $(x, y, \varepsilon_1, \varepsilon_2, z)$). This mapping does not change the marginal distribution of $(x, y, \varepsilon_1, \varepsilon_2)$. This marginal distribution is exogenous and can only change over time due to exogenous time trends, investments, exit or entry, or fundamental demographic change.[10]

Given the model primitives, we want to derive what restrictions the model places on the observable data; i.e. the joint distribution of (x, y, z). To derive these restrictions note the following. The random vector underlying the economy is $(x, y, \varepsilon_1, \varepsilon_2)$. The dimension of this random vector is $n_x + n_y + 2$ where n_x is the dimension of x, n_y is the dimension of y, and ε_1 and ε_2 are each of dimension 1. The equilibrium maps this underlying random vector into the observable random vector (x, y, z). This observable random vector is of dimension $n_x + n_y + 1$; it is of dimension one less than fundamental random vector $(x, y, \varepsilon_1, \varepsilon_2)$.

In order to derive the observed data density we first fix the functions $T_1'(z_1)$ and $T_2'(z_2)$. Imagine an economy where both firms and workers are choosing locations taking T_1' and T_2' as given, but that firms choose z_1 while workers choose z_2. For the moment, we do not impose equilibrium and allow $z_1 \neq z_2$. The following mapping generates the data from this

[10] A more complete dynamic analysis would model how this marginal distribution changes over time.

hypothetical economy:

$$
\begin{aligned}
x &= x;\ y = y \\
\varepsilon_1 &= T_1\left(z_1\right) - \eta_1\left(x\right) \\
\varepsilon_2 &= T_2\left(z_2\right) - \eta_2(y)
\end{aligned}
$$

These functions map observable and unobservable characteristics of workers and firms into the observables $\left(x, y, z_1, z_2\right)$. The Jacobian of the mapping is

$$
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\eta_1'\left(x\right) & 0 & T_1'\left(z_1\right) & 0 \\
0 & -\eta_2'(y) & 0 & T_2'\left(z_2\right)
\end{vmatrix} = T_1'\left(z_1\right) T_2'\left(z_2\right).
$$

This mapping defines a density on $\left(x, y, z_1, z_2\right)$:

$$
q_1\left(x\right) q_2\left(y\right) g_1\left(T_1\left(z_1\right) - \eta_1\left(x\right)\right) g_2\left(T_2\left(z_2\right) - \eta_2\left(y\right)\right) T_1'\left(z_1\right) T_2'\left(z_2\right).
$$

This is a well defined density for the disequilibrium economy. However, if we impose equilibrium ($z_1 = z_2 = z$), we can determine the joint density of $\left(x, y, z\right)$. It is the density of $\left(x, y, z_1, z_2\right)$ conditional on $z_1 = z_2 = z$. That is, the density of $\left(x, y, z\right)$ is

$f(x, y, z)$

$$
= \frac{q_1\left(x\right) q_2\left(y\right) g_1\left(T_1\left(z\right) - \eta_1\left(x\right)\right) g_2\left(T_2\left(z\right) - \eta_2\left(y\right)\right) T_1'\left(z\right) T_2'\left(z\right)}{\int\limits_x \int\limits_y \int\limits_z q_1\left(t_1\right) q_2\left(t_2\right) g_1\left(T_1\left(t_3\right) - \eta_1\left(t_1\right)\right) g_2\left(T_2\left(t_3\right) - \eta_2\left(t_2\right)\right) T_1'\left(t_3\right) T_2'\left(t_3\right) dt_1 dt_2 dt_3} \tag{22}
$$

In the appendix we prove that there is no more information in the joint densities than in the marginal densities.

Theorem 3 *Joint density (22) provides no more information than the marginal densities* $f(z_1, x), f(z_2, y)$.

Proof: See Appendix A. ■

4.2 The Role of Separability

The key role in identification played by separability assumption (A-1a) is demonstrated in Figures 7 and 8 which plot marginal willinging to pay and marginal products against z. The marginal pricing function is also plotted.

Separability of the first order conditions as used in this paper (see condition A-1a) gives parallel willingness to pay and marginal productivity curves. (See the two parallel curves in each figure for two values of x and y respectively). Equilibrium is at point A of each curve. As x shifts we reach a new equilibrium B. But the slope of B is the same as the slope at B' on the initial benchmark curve. Thus, with sufficient support for Z (guaranteed by the assumption (A-1) on the support of ε_1 and ε_2) we can trace out the benchmark willingness to pay ($Y = y'$) and marginal productivity curves ($X = x'$) using data on all levels of Z and X. In a nonseparable case, we cannot relate the slope at B to any particular point on the benchmark curves. The entire analysis of this section can be reproduced for any member of the class of transformations defined in (A-1).

5 Instrumental Variables

Theorem 1 supplemented with some additional regularity conditions justifies the application of instrumental variables for general parametric versions of model (13). Instrumental variables are generically valid even though there are no exclusion restrictions.

We analyze the first order condition

$$P'(z) = \varphi_1(z) + \eta_1(x) + \varepsilon_1$$

with $z \in \mathcal{Z} = (0, \infty)$, $x \in \mathcal{X} = (0, \infty)$, and $\varepsilon_1 \in E_1 = (0, \infty)$ where $(x, \varepsilon) \sim q_1(x)\, g_1(\varepsilon)$ and q_1 and g_1 are strictly positive densities, $P'(z) > 0$, $\varphi_1(z) > 0$ and $P'' - \varphi_1' > 0$. We assume $E_X(\eta_1^2(x)) < \infty$.

The literature reviewed in Section 3 establishes that in a single market setting there are no exclusion restrictions for this equation. Variables from the other side of the market are stochastically dependent on ε_1 given $Z = z$.

Although there are no natural exclusion restrictions, instruments for $\varphi_1(z)$ are still available. If $E_Z(\varphi_1(Z) \mid x)$ is not collinear with $\eta_1(x)$, then it is possible to use X as instruments for $\varphi_1(z)$ in (13). Kahn and Lang (1988) make this point by way of an example for a particular functional form. In this section we establish that generically X is a valid instrument for any arbitrary parametric functional form that satisfied the conditions required to prove a corollary to Theorem 1. This result highlights the main themes of our paper: that the hedonic model is intrinsically nonlinear, that nonlinearity is an important source of identifying information and that intuitions developed in linear econometrics when applied to a nonlinear model are misleading. We can use our result to justify the choice of parametric nonlinear IV as in Amemiya, 1975.

As a consequence of Theorem 1, instrumental variable estimation strategies for general nonparametric models are valid. In the appendix, we prove that generically the expectation of $\varphi_1(z)$ given x is not collinear with $\eta_1(x)$. This means that the X are valid instruments for $\varphi_1(z)$.

Corollary 1 of Theorem 1 *Generically with respect to any pair of the parameters in Theorem 1,* $E_Z\left(\varphi_1\left(z\right)|x\right)$ *cannot be collinear with* η_1.

Proof: See Appendix A. ■

As a consequence of this corollary, we can use X as an instrument for $\varphi_1(Z)$ using parametric nonlinear IV (Amemiya, 1975). We conjecture that this condition also justifies the application of nonparametric IV (Darolles et. al, 2001, Florens, Heckman, Meghir and Vytlacil, 2000, or Newey and Powell, 2000). However, those papers require an exclusion restruction which is not intrinsic to the model and it is necessary to extend their arguments to impose Corollary 1 as an identifying condition in the estimation. This is a task we leave for the future.

6 Summary, Conclusions and Proposed Extensions

This paper considers identification and estimation of technology and preference parameters using data on choices made in a single hedonic market. The general hedonic problem is formulated, a normal-linear-quadratic version of the model is developed and its advantages and peculiarities are exposed.

Standard criticisms directed against Sherwin Rosen's two stage estimation procedure for hedonic models are shown to be misleading. Generically, a separable nonparametric version of the model is identified up to levels. With mild functional form assumptions, the model is completely identified. Two estimation procedures are presented: (a) nonparametric transformation methods, and (b) IV in a general nonlinear but parametric setting.

The analysis developed here applies to closely related problems of estimating preferences and technology when taxes are set optimally (Mirrlees, 1971 and 1986), when monopolists price discriminate (Mussa and Rosen, 1978; Wilson, 1993) and for the standard problem of taxes and labor supply (Heckman, 1974; Hausman 1980) when tax schedules are nonlinear and continuous.

Our presentation of the hedonic model is for the vector case. Yet our basic proofs are only for the scalar case. An extension for the nonseparable vector case is underway in joint work with Rosa Matzkin. That work considers the case of identification for a nonseparable hedonic model with vector attributes when some of the attributes are missing. (Ekeland, Heckman, Matzkin and Nesheim, 2001, in preparation)

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7 Appendix: Proofs

Recall that we have denoted by $\mathcal{X}$ and $\mathcal{Y}$ the supports of q_1 and q_2, so that we may assume that $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Denote by $\mathcal{Z}$ the domain of z, so that $z \in \mathcal{Z}$; both φ_1 and φ_2 map $\mathcal{Z}$ into R. For the sake of simplicity, it will be assumed that $\mathcal{Z}$ is an interval, possibly unbounded. We denote by $C^1(\mathcal{Z})$, the space of continuously differentiable[11] functions on $\mathcal{Z}$ endowed with the following topology: $f_n \to f$ iff f_n converges to f and the derivatives f_n' converge to f', uniformly on compact subsets of $\mathcal{Z}$. It is known that this topology turns C^1 into a complete metric space.[12]

Denote by $C_1^2(R)$ the space of twice differentiable functions g on the real line, satisfying $\int g = 1$ and $g > 0$ everywhere, with g, g', g'' continuous and uniformly bounded. It is endowed with the topology of uniform convergence of g, g', g'' which turns it into a complete metric space; this is the natural space for g_1 and g_2.

The natural spaces for η_1 and η_2 are $C^1(\mathcal{X})$ and $C^1(\mathcal{Y})$ respectively. The natural spaces for q_1 and q_2 are $C_1^0(\mathcal{X})$ and $C_1^0(\mathcal{Y})$, where C_1^0 denotes the space of continous functions q such that $\int q = 1$ and $q > 0$ everywhere, endowed with the uniform norm. We now restate Theorem 1 more precisely:

Theorem 1 Restated *Generically with respect to any of the parameters pairs* $(\varphi_1, \varphi_2) \in C^1(\mathcal{Z}) \times C^1(\mathcal{Z})\,, (g_1, g_2) \in C_1^2(R) \times C_1^2(R)\,, (\eta_1, \eta_2) \in C^1(\mathcal{X}) \times C^1(\mathcal{Y})\,, (q_1, q_2) \in C_1^0(\mathcal{X}) \times C_1^0(\mathcal{Y})$ *the equilibrium equations have no solution of the form* $P'(z) = a_1 + b_1\varphi_1(z)$*, nor any solution of the form* $P'(z) = a_2 + b_2\varphi_2(z)\,.$

[11]If $z_0 \in \mathcal{Z}$ is the left (or right) extremity of $\mathcal{Z}$, a derivative at z_0 will be understood to mean a right (or left) derivative.

[12]And even a Banach space if $\mathcal{Z}$ is compact.

Proof of Theorem 1:

Set $(\varphi_1, \varphi_2, g_1, g_2, \eta_1, \eta_2, q_1, q_2) = \theta$ and $C^1(\mathcal{Z}) \times C^1(\mathcal{Z}) \times C_1^2(R) \times C_1^2(R) \times C^1(\mathcal{X}) \times C^1(\mathcal{Y}) \times C_1^0(\mathcal{X}) \times C_1^0(\mathcal{Y}) = \Theta$.

Define a map $\Phi : \Theta \times \mathbb{R}^4 \to C^0(\mathbb{R})$ by:

$$\begin{aligned}\Phi(\theta, a_1, b_1)(z) &= (b_1 - 1)\varphi_1'(z) \int_{\mathcal{X}} g_1(a_1 + (b_1 - 1)\varphi_1(z) - \eta_1(x)) q_1(x)\, dx \\ &\quad - (b_1\varphi_1'(z) - \varphi_2'(z)) \int_{\mathcal{Y}} g_2(a_1 + b_1\varphi_1(z) - \varphi_2(z) - \eta_2(y))\, q_2(y)\, dy\end{aligned}$$

$\Phi(\theta, a_1, b_1) = 0$, or $\Phi(\theta, a_1, b_1)(z) = 0$ for all z, means that the equilibrium equation has a solution of the form

$$P(z) = a_1 + b_1\varphi_1(z)$$

and we want to show that, generically in any of the parameter pairs, this cannot happen. To do that, fix three points z_1, z_2, z_3 in Z, all pairwise distinct, and define a map $\Psi : \Theta \times \mathbb{R}^2 \to \mathbb{R}^3$ by:

$$\Psi(\theta, a_1, b_1) = (\Phi(\theta, a_1, b_1)(z_i))_{1 \le i \le 3}$$

Lemma: The map Ψ is C^1

Proof: The Gateaux derivative $D\Psi$ of Ψ at (θ, a_1, b_1) is easily expressed. Set $\delta\theta = (\delta\varphi_1, \delta\varphi_2, \delta g_1, \delta g_2, \delta\eta_1, \delta\eta_2, \delta q_1, \delta q_2)$, where the components of $\delta\theta$ belong to the appropriate vector spaces, $\delta g_1, \delta g_2, \delta q_1, \delta q_2$ being subject to the additional requirement of integrating to zero. Similarly, set $(\delta a_1, \delta b_1) \in R^2$, and compute the first variation of Ψ:

$$
\begin{aligned}
&D\Psi(\delta\theta, \delta a_1, \delta b_1) = \qquad\qquad \text{(A-1)}\\
&[\delta\varphi_1'(z_i)(b_1-1)\int\limits_{\mathcal{X}} g_1(a_1+(b_1-1)\varphi_1(z_i)-\eta_1(x))q_1(x)dx -\\
&(b_1\delta\varphi_1'(z_i)-\delta\varphi_2'(z_i))\int\limits_{\mathcal{Y}} g_2(a_1+b_1\varphi_1(z_i)-\varphi_2(z_i)-\eta_2(y))q_2(y)dy +\\
&\delta\varphi_1(z_i)(b_1-1)^2\varphi_1'(z_i)\int\limits_{\mathcal{X}} g_1'(a_1+(b_1-1)\varphi_1(z_i)-\eta_1(x))q_1(x)dx -\\
&(b_1\delta\varphi_1(z_i)-\delta\varphi_2(z_i))\left(b_1\varphi_1'(z_i)-\varphi_2'(z_i)\right)\int\limits_{\mathcal{Y}} g_2'(a_1+b_1\varphi_1(z_i)-\varphi_2(z_i)-\eta_2(y))q_2\left(y\right)dy +\\
&(b_1-1)\,\varphi_1'\left(z_i\right)\int\limits_{\mathcal{X}} \delta g_1(a_1+(b_1-1)\,\varphi_1\left(z_i\right)-\eta_1\left(x\right))q_1\left(x\right)dx -\\
&(b_1\varphi_1'\left(z_i\right)-\varphi_2'\left(z_i\right))\int\limits_{\mathcal{Y}} \delta g_2\left(a_1+b_1\varphi_1\left(z_i\right)-\varphi_2\left(z_i\right)-\eta_2\left(y\right)\right)q_2\left(y\right)dy -\\
&(b_1-1)\,\varphi_1'\left(z_i\right)\int\limits_{\mathcal{X}} g_1'(a_1+(b_1-1)\,\varphi_1\left(z_i\right)-\eta_1\left(x\right))\delta\eta_1\left(x\right)q_1\left(x\right)dx +\\
&(b_1\varphi_1'\left(z_i\right)-\varphi_2'\left(z_i\right))\int\limits_{\mathcal{Y}} g_2'\left(a_1+b_1\varphi_1\left(z_i\right)-\varphi_2\left(z_i\right)-\eta_2\left(y\right)\right)\delta\eta_2\left(y\right)q_2\left(y\right)dy +\\
&(b_1-1)\,\varphi_1'\left(z_i\right)\int\limits_{\mathcal{X}} g_1(a_1+(b_1-1)\,\varphi_1\left(z_i\right)-\eta_1\left(x\right))\delta q_1\left(x\right)dx -\\
&(b_1\varphi_1'\left(z_i\right)-\varphi_2'\left(z_i\right))\int\limits_{\mathcal{Y}} g_2\left(a_1+b_1\varphi_1\left(z_i\right)-\varphi_2\left(z_i\right)-\eta_2\left(y\right)\right)\delta q_2\left(y\right)dy +
\end{aligned}
$$

Since the functions g_1 and g_2 are uniformly bounded, and their first derivatives also, all the integrals in these formulas are well-defined. Since the functions g_1 and g_2 are uniformly continuous, as are their first derivatives, these integrals depend continuously on (a_1, b_1) and on θ. So the function Ψ is C^1.

This ends the proof of the lemma. To prove the theorem, we have to vary each pair of parameters singly. This amounts to considering, instead of Ψ, the partial maps obtained by keeping all parameter values fixed except two, and showing that the corresponding derivative is onto. This gives four different cases.

Genericity with respect to (φ_1, φ_2) We consider the partial map $\Psi(\varphi_1, \varphi_2, a_1, b_1)$ and the derivative of the partial map $D\Psi(\delta\varphi_1, \delta\varphi_2, \delta a_1, \delta b_1)$, where it is understood that all the other parameters $g_1, g_2, \eta_1, \eta_2, q_1, q_2$ are set to fixed values. Hence the derivative of the partial map is given by $(A-1)$ with all variations other than $(\delta\varphi_1, \delta\varphi_2, \delta a_1, \delta b_1)$ set to zero.

Since the point z_i are pairwise distinct, we can choose the $(\delta\varphi_1, \delta\varphi_2)$ so that $(\delta\varphi_1(z_i), \delta\varphi_2(z_i)) = (0,0)$ for all i. Choosing in addition $(\delta a_1, \delta b_1) = (0,0)$ cancels all the terms on the right-hand side except the two first ones. Since the remaining integrals are non-zero (in fact, positive), the coefficients of $\delta\varphi_1'(z_i)$ and $\delta\varphi_2'(z_i)$ cannot vanish together. So the image by $D\Psi$ of vectors such that $(\delta\varphi_1(z_i), \delta\varphi_2(z_i)) = (0,0)$ and $(\delta a_1, \delta b_1) = (0,0)$ must be all of R^3.

Saying that $D\Psi$ is onto means that the partial map Ψ is transversal to every point in $\mathbb{R}^3$, in particular to the origin. By Thom's transversality theorem, generically in (φ_1, φ_2), the partial map

$$(a_1, b_1) \to \Psi(\varphi_1, \varphi_2, a_1, b_1)$$

is transversal to the origin. This means that whenever $\Psi(\varphi_1, \varphi_2, a_1, b_1) = 0$, the partial derivative $D_{a_1,b_1}\Psi$ must be onto; but the latter is impossible, since $D_{a_1,b_1}\Psi$ sends a two-dimensional space into a three-dimensional one. So $\Psi(\varphi_1, \varphi_2, a_1, b_1) \neq 0$ for every (a_1, b_1).

We have thus proved that, generically in (φ_1, φ_2), we must have $\Phi(\varphi_1, \varphi_2, a_1, b_1)(z_i) \neq 0$ for one i at least. This implies of course that $\Phi(\varphi_1, \varphi_2, a_1, b_1)(z)$ cannot be identically zero, and hence that the equilibrium equation does not have a solution of the form $P(z) = a_1 + b_1\varphi_1(z)$. The same argument will show that, generically in (φ_1, φ_2), the equilibrium equation does not have a solution of the form $P(z) = a_2 + b_2\varphi_2(z)$. Since the intersection of two generic properties is generic, the theorem follows for the pair (φ_1, φ_2).■

Genericity with respect to (g_1, g_2) We consider the partial map $\Psi(g_1, g_2, a_1, b_1)$, where it is understood that all the other parameters are pegged to fixed values. The partial derivative is given by

$$
\begin{aligned}
&D\Psi(\delta g_1, \delta g_2, \delta a_1, \delta b_1) = \\
&[(b_1 - 1)\varphi_1'(z_i)\int_{\mathcal{X}} \delta g_1(a_1 + (b_1 - 1)\varphi_1(z_i) - \eta_1(x))q_1(x)\,dx - \\
&(b_1\varphi_1'(z_i) - \varphi_2'(z_i))\int_{\mathcal{Y}} \delta g_2(a_1 + b_1\varphi_1(z_i) - \varphi_2(z_i) - \eta_2(y))\,q_2(y)\,dy + \\
&(\delta a_1)(b_1 - 1)\varphi_1'(z_i)\int_{\mathcal{X}} g_1'(a_1 + (b_1 - 1)\varphi_1(z_i) - \eta_1(x))q_1(x)dx - \\
&(\delta a_1)(b_1\varphi_1'(z_i) - \varphi_2'(z_i))\int_{\mathcal{Y}} g_2'(a_1 + b_1\varphi_1(z_i) - \varphi_2(z_i) - \eta_2(y))q_2(y)dy + \\
&(\delta b_1)(b_1 - 1)\varphi_1'(z_i)\varphi_1(z_i)\int_{\mathcal{X}} g_1'(a_1 + (b_1 - 1)\varphi_1(z_i) - \eta_1(x))q_1(x)dx - \\
&(\delta b_1)(b_1\varphi_1'(z_i) - \varphi_2'(z_i))\varphi_1(z_i)\int_{\mathcal{Y}} g_2'(a_1 + b_1\varphi_1(z_i) - \varphi_2(z_i) - \eta_2(y))q_2(y)]_{i=1,2,3}
\end{aligned}
$$

Introduce the distribution functions μ_1 and μ_2 of the random variables η_1 and η_2. They are probability measures on the real line, and the first two lines of the above formula can be rewritten as:

$$D\Psi\left(\delta g_1, \delta g_2\right) =$$
$$[\left(b_1-1\right)\varphi_1'\left(z_i\right)\int \delta g_1(a_1+\left(b_1-1\right)\varphi_1\left(z_i\right)+t)d\mu_1 -$$
$$\left(b_1\varphi_1'\left(z_i\right)-\varphi_2'\left(z_i\right)\right)\int \delta g_2\left(a_1+b_1\varphi_1\left(z_i\right)-\varphi_2\left(z_i\right)+t\right)d\mu_2]_{i=1,2,3}$$

Setting $c_i = a_1 + \left(b_1-1\right)\varphi_1\left(z_i\right)$ and $d_i = a_1 + b_1\varphi_1\left(z_i\right) - \varphi_2\left(z_i\right)$, and denoting by μ_1^i and μ_2^i the translates of μ_1 and μ_2 by $-c_i$ and $-d_i$ we rewrite the first two lines of the partial derivative again as:

$$D\Psi\left(\delta g_1, \delta g_2\right) = [\left(b_1-1\right)\varphi_1'\left(z_i\right)\int \delta g_1(t)d\mu_1^i - \left(b_1\varphi_1'\left(z_i\right)-\varphi_2'\left(z_i\right)\right)\int \delta g_2\left(t\right)d\mu_2^i]_{i=1,2,3}$$

We pick the z_i so that the probability measures μ_1^i and $\mu_2^i, i = 1,2,3,$ are pairwise different, and $\varphi_1'\left(z_i\right)$ and $\varphi_2'\left(z_i\right)$ do not vanish. Then the coefficients of the integrals cannot vanish simultaneously, and the right-hand side clearly spans R^3. We conclude as in the preceding case; by applying Thom's transversality theorem.

Genericity with respect to (q_1, q_2) We consider the partial map $\Psi\left(g_1, g_2, a_1, b_1\right)$, where it is understood that all the other parameters are pegged to fixed values. The partial derivative is given by:

$$D\Psi(\delta q_1, \delta q_2, \delta a_1, \delta b_1) =$$
$$[\left(b_1-1\right)\varphi_1'\left(z_i\right)\int_{\mathcal{X}} g_1(a_1+\left(b_1-1\right)\varphi_1\left(z_i\right)-\eta_1\left(x\right))\delta q_1\left(x\right)dx-$$
$$\left(b_1\varphi_1'\left(z_i\right)-\varphi_2'\left(z_i\right)\right)\int_{\mathcal{Y}} g_2\left(a_1+b_1\varphi_1\left(z_i\right)-\varphi_2\left(z_i\right)-\eta_2\left(y\right)\right)\delta q_2\left(y\right)dy+$$

$$(\delta a_1)(b_1-1)\varphi_1'(z_i)\int\limits_{\mathcal{X}} g_1'(a_1+(b_1-1)\varphi_1(z_i)-\eta_1(x))q_1(x)dx -$$

$$(\delta a_1)(b_1\varphi_1'(z_i)-\varphi_2'(z_i))\int\limits_{\mathcal{Y}} g_2'(a_1+b_1\varphi_1(z_i)-\varphi_2(z_i)-\eta_2(y))q_2(y)dy$$

$$+(\delta b_1)(b_1-1)\varphi_1'(z_i)\varphi_1(z_i)\int\limits_{\mathcal{X}} g_1'(a_1+(b_1-1)\varphi_1(z_i)-\eta_1(x))q_1(x)dx$$

$$-(\delta b_1)(b_1\varphi_1'(z_i)-\varphi_2'(z_i))\varphi_1(z_i)\int\limits_{\mathcal{Y}} g_2'(a_1+b_1\varphi_1(z_i)-\varphi_2(z_i)-\eta_2(y))q_2(y)]_{i=1,2,3}$$

We claim that the partial map obtained by setting $(\delta a_1, \delta b_1) = 0$ is onto. We get

$$D\Psi(\delta q_1, \delta q_2) =$$

$$[\quad (b_1-1)\,\varphi_1'(z_i)\int\limits_{\mathcal{X}} g_1(a_1+(b_1-1)\,\varphi_1(z_i)-\eta_1(x))\delta q_1(x)\,dx -$$

$$(b_1\varphi_1'(z_i)-\varphi_2'(z_i))\int\limits_{\mathcal{Y}} g_2\left(a_1+b_1\varphi_1(z_i)-\varphi_2(z_i)-\eta_2(y)\right)\delta q_2(y)\,dy]_{i=1,2,3}$$

We choose the z_i so that the $\varphi_1'(z_i)$ and the $\varphi_2'(z_i)$ do not vanish, and so that the functions $g_1(a_1+(b_1-1)\,\varphi_1(z_i)-\eta_1(x))$ and $g_2\left(a_1+b_1\varphi_1(z_i)-\varphi_2(z_i)-\eta_2(y)\right)$ are pairwise different on a set of positive measure. The claim then follows, and genericity obtains as in the preceding cases.

Genericity with respect to (η_1, η_2) We consider

$$D\Psi(\delta\eta_1, \delta\eta_2) = (b_1-1)\,\varphi_1'(z_i)\int\limits_{\mathcal{X}} g_1'(a_1+(b_1-1)\,\varphi_1(z_i)-\eta_1(x))\delta\eta_1(x)\,q_1(x)\,dx +$$

$$(b_1\varphi_1'(z_i)-\varphi_2'(z_i))\int\limits_{\mathcal{Y}} g_2'\left(a_1+b_1\varphi_1(z_i)-\varphi_2(z_i)-\eta_2(y)\right)\delta\eta_2(y)\,q_2(y)\,dy$$

and we argue as in the preceding case.

Proof of Theorem 3:

The strategy of the proof is to determine whether taking the objects determined from the marginal densities as demonstrated in the previous subsection and plugging them into (22) provides any more information about the parameters that are not identified.

Write the conditional distribution of z given x and y as

$$F(z|x,y) = \frac{\int\limits_{-\infty}^{z} g_1(T_1(s) - \eta_1(x))\, g_2(T_2(s) - \eta_2(x))\, T_1'(s)\, T_2'(s)\, ds}{Q}$$

where

$$Q = \int\limits_{x}\int\limits_{y}\int\limits_{z} q_1(t_1)\, q_2(t_2)\, g_1(T_1(s) - \eta_1(x))\, g_2(T_2(s) - \eta_2(x))\, T_1'(s)\, T_2'(s)\, dt_1 dt_2 ds.$$

Using the information secured from the marginals, we obtain

$$\begin{aligned}
T_1(z) &= K_1\tilde{T}_1(z) + C_1;\ T_2(z) = K_2\tilde{T}_2(z) + C_2 \\
\tilde{g}_1(\tilde{\varepsilon}_1) &= g_1(K_1\tilde{\varepsilon}_1)\, K_1;\quad \tilde{g}_2(\tilde{\varepsilon}_2) = g_2(K_2\tilde{\varepsilon}_2)\, K_2 \\
\eta_1(x) &= K_1\tilde{\eta}_1(x) + R_1;\ \eta_2(x) = K_2\tilde{\eta}_2(x) + R_2
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\varepsilon}_1(z,x) &= \tilde{T}_1(z) - \tilde{\eta}_1(x) + \left(\frac{C_1 - R_1.}{K_1}\right) \\
\tilde{\varepsilon}_2(z,x) &= \tilde{T}_2(z) - \tilde{\eta}_2(x) + \left(\frac{C_2 - R_2}{K_2}\right)
\end{aligned}$$

and "$\sim$" denotes that this information is known from the marginals.

Substituting into $F(z|x,y)$, using the appropriate Jacobians of transformation, we obtain

$$F(z|x,y) = \frac{1}{Q}\int_{-\infty}^{z} \tilde{T}_1'(s)\, K_1 g_1\left(K_1 \tilde{\varepsilon}_1(s,x)\right) \tilde{T}_2'(s) K_2 g_2\left(K_2 \tilde{\varepsilon}_2(s,y)\right) ds.$$

where $\tilde{\varepsilon}_1(s,x)$ and $\tilde{\varepsilon}_2(s,y)$ are written as explicit functions of (s,x) and (s,y) respectively where s is an argument of integration. From the analysis of the marginals, we do not know g_1, g_2 but rather $\tilde{g}_1, \tilde{g}_2$ which are functions of $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ respectively. Thus the unidentified constants K_1, K_2 drop out and equation (22) provides no additional information on them:

$$F(z|x,y) = \frac{1}{Q}\int_{-\infty}^{z} \tilde{T}_1'(s)\, \hat{g}_1\left(\hat{\varepsilon}_1(s,x)\right) \tilde{T}_2'(s)\, \tilde{g}_2\left(\tilde{\varepsilon}_2(s,y)\right) ds.$$

Proof of Corollary 1 of Theorem 1:

By definition,

$$P'(z) - \varphi_1(z) = \eta_1(x) + \varepsilon_1.$$

Because of the second-order condition $P''(z) - \varphi_1'(z) > 0$ so that the left side can be inverted uniquely (globally) to obtain

$$z = \Lambda\left(\eta_1(x) + \varepsilon_1\right)$$

where by the implicit function theorem $\Lambda'(q) = \left[P''(\Lambda(q)) - \varphi_1'(\Lambda(q))\right]^{-1}$. Define the mapping

$$\begin{aligned} h &= \varphi_1\left(\Lambda\left(\eta_1(x) + \varepsilon_1\right)\right) \\ x &= x, \end{aligned}$$

where

$$E_z\left(\varphi_1(z)\,|x\right) = E_z\left(h\,|x\right) = \int\limits_E \varphi_1\left(\Lambda\left(\eta_1(x)+\varepsilon_1\right)\right) g_1(\varepsilon_1)\, d\varepsilon_1.$$

This conditional expectation is a functional of η_1.Assume it is linear with respect to η_1. Pick a direction $\delta\eta_1$, and define a function $f(t)$ on the real line by:

$$f(t) = \int\limits_E [\varphi_1\left(\Lambda\left(\eta_1(x) + t\;\delta\eta_1(x) + \varepsilon_1\right)\right) - \varphi_1\left(\Lambda\left(\eta_1(x)+\varepsilon_1\right)\right)] g_1(\varepsilon_1)\, d\varepsilon_1$$

Then $f(t)$ is linear, so that $f''(0) = 0$. Performing the computations, we get the equation:

$$[\delta\eta_1(x)]^2 \int \lambda\left(\eta_1(x)+\varepsilon_1\right) g_1(\varepsilon_1)\, d\varepsilon_1 = 0$$

where $\lambda = \left[\left(\varphi_1''\left(P''-\varphi_1'\right) - \varphi_1'\left(P'''-\varphi_1''\right)\right)\left(P''-\varphi_1'\right)^{-3}\right] \circ \Lambda$. This reduces to:

$$\int \lambda\left(\eta_1(x)+\varepsilon_1\right) g_1(\varepsilon_1)\, d\varepsilon_1 = 0 \;\text{ a.e.}x$$

and the function λ has the property that every translate of λ by any amount $\eta_1(x)$ integrates to 0 against g_1, which is a fixed probability density. If the support of g_1 is unbounded, it follows that $\lambda = 0$ a.e. so $\left[\left(\varphi_1'' - \varphi_1'\left(P'''-\varphi_1''\right)\right)\left(P''-\varphi_1'\right)^{-2}\right] \circ \Lambda$ vanishes, meaning that $\left(\varphi_1'' - \varphi_1'\left(P'''-\varphi_1''\right)\right)\left(P''-\varphi_1'\right)^{-2}$ vanishes on the range of Λ, which is precisely the domain $\mathcal{Z}$ of z. This proves that

$$\left(\varphi_1''\left(P''-\varphi_1'\right) - \varphi_1'\left(P'''-\varphi_1''\right)\right) = 0 \;\text{ on }\; \mathcal{Z}$$

so that

$$\frac{\varphi_1''}{\varphi_1'} = \frac{P'''}{P''}$$

and $P'' = a\varphi_1'$, ending with $P' = a\varphi_1 + b$. From Theorem 1, genreically this cannot happen and the proof is conluded.

Table 1

Model 1 – Linear quadratic technologies – Unrestricted z

Figures (1) through (4) display the slope and curvature of the equilibrium price function in the unrestricted linear-quadratic hedonic economy in four cases. All four cases use the parameters $a = 2.0$ and $b = 1.0$. In addition, each case assumes that both ν_1 and θ_1 are distributed as mixtures of two normals. The parameters describing the components of the two normal distributions are listed in the table below.

	Components	
	1	2
μ_{ν_1}	-1.0	0.0
μ_{θ_1}	1.0	0.0
$\sigma^2_{\nu_1}$	0.3	0.3
$\sigma^2_{\theta_1}$	0.5	0.3

Each of the four cases is distinguished by the weights on the two components. The benchmark case assumes that λ, the weight on component one of the mixture is 1.0. Hence, case 1 is the benchmark linear-quadratic-normal model. The other three cases use different values of λ to show how the slope and curvature of the price function vary with λ.

Figures (1) and (2) were generated using $\lambda = 1.00$ and $\lambda = 0.999$. Figures (3) and (4) were generated using $\lambda = 0.99$ and $\lambda = 0.90$.

Table 2
Model 2
Linear Quadratic Technologies
Non-Negative z

Firms	$\Pi(z) = \nu_0 + \nu_1 z - \frac{1}{2} b z^2 - p(z)$
	$\nu_1, b \geq 0$
	$\ln \nu_1 = \nu_{10} + \nu_{11}' x + \varepsilon_1$
	x and ε_1 are both distributed as a mixture of normals
	(the mixtures could have only one component).
FOC	$\nu_1 - bz - p'(z) = 0$
SOC	$-b - p''(z) < 0$
Workers	$V(z) = \theta_0 + \theta_1 z - \frac{1}{2} a z^2 + p(z)$
	$\theta_1, a \geq 0$
	$\ln \theta_1 = \theta_{10} + \theta_{11}' y + \varepsilon_2$
	y and ε_2 are both distributed as mixtures of normals

Equilibrium

Figures (5) and (6) display the equilibrium slope and curvature of the price function in the linear-quadratic hedonic model with the following restrictions $\nu_1 \geq 0$, $\theta_1 \geq 0$, and $z \geq 0$.

The parameter values $a = 2.0$ and $b = 1.0$ were used. In addition, ν_1 and θ_1 were assumed to be distributed as mixtures of two normals. The parameters of the two normals are listed in the following table.

	Components	
	1	2
μ_ν	1.0	2.0
μ_θ	-0.5	0.5
σ_ν^2	0.21	0.21
σ_θ^2	0.61	0.61
λ_ν	0.5	0.5
λ_θ	0.5	0.5

Figure 1: Slope of Price Function - Model 1a

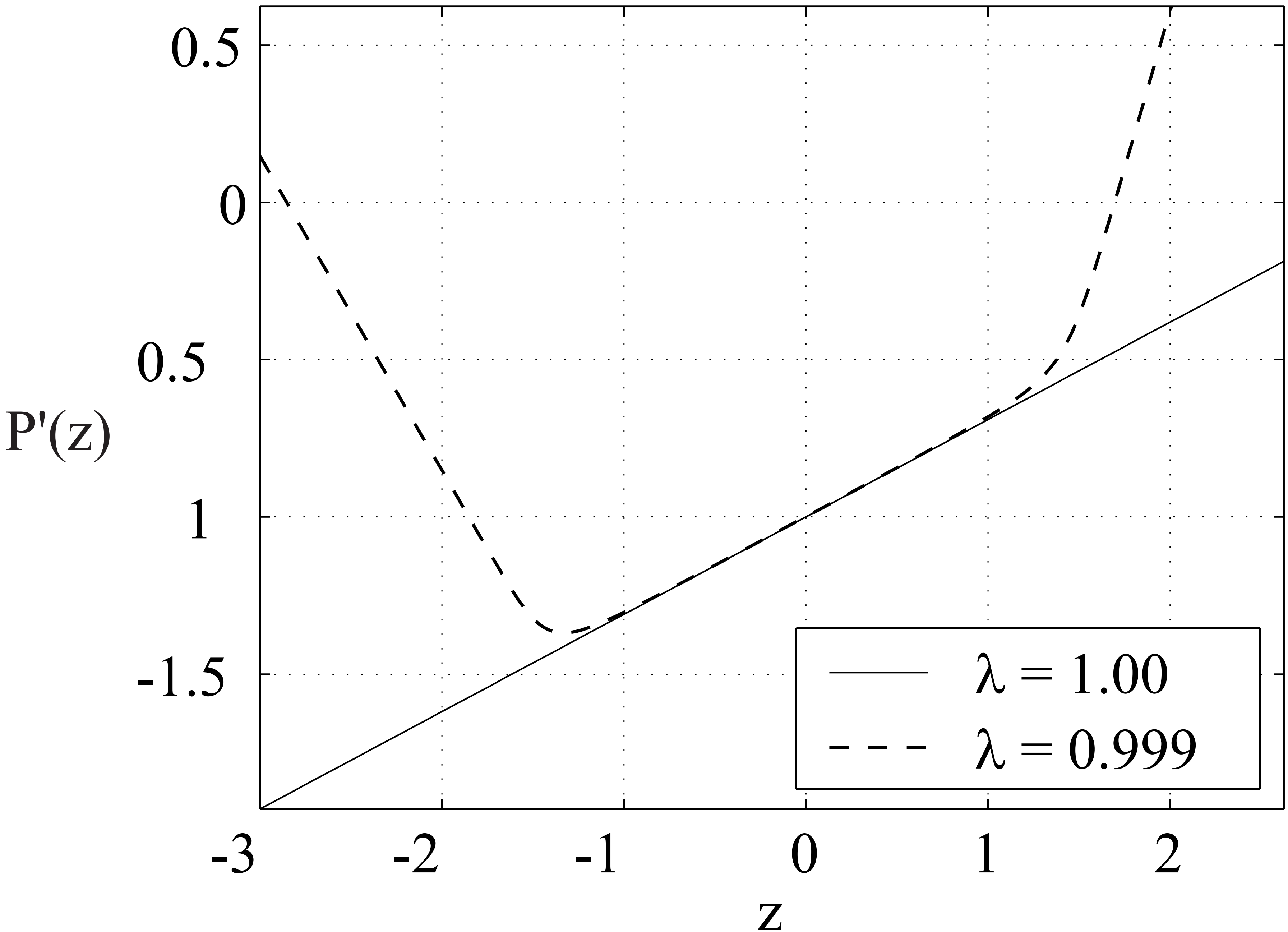


Figure 2: Curvature of Price Function - Model 1a

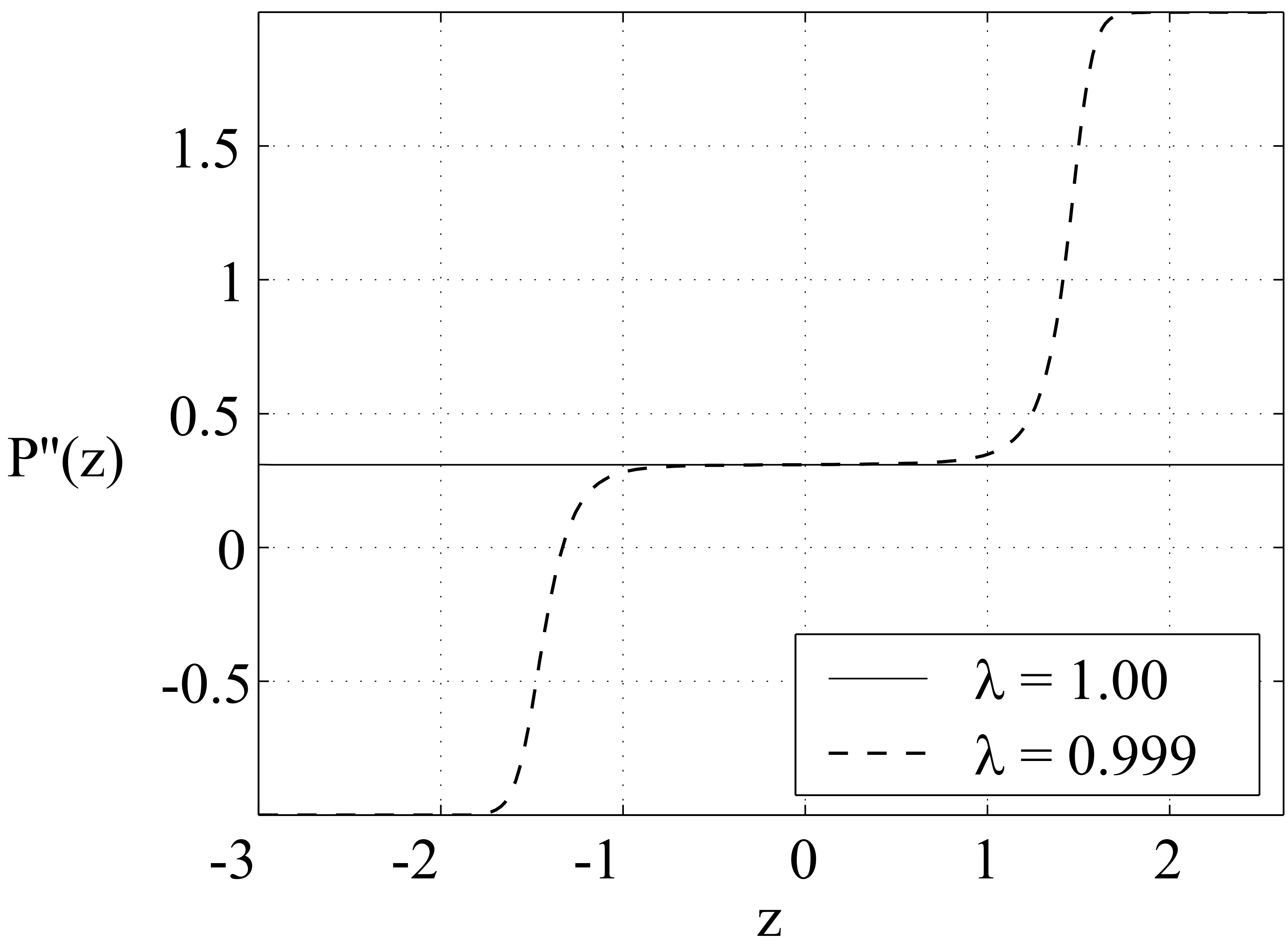


Figure 3: Slope of Price Function - Model 1b

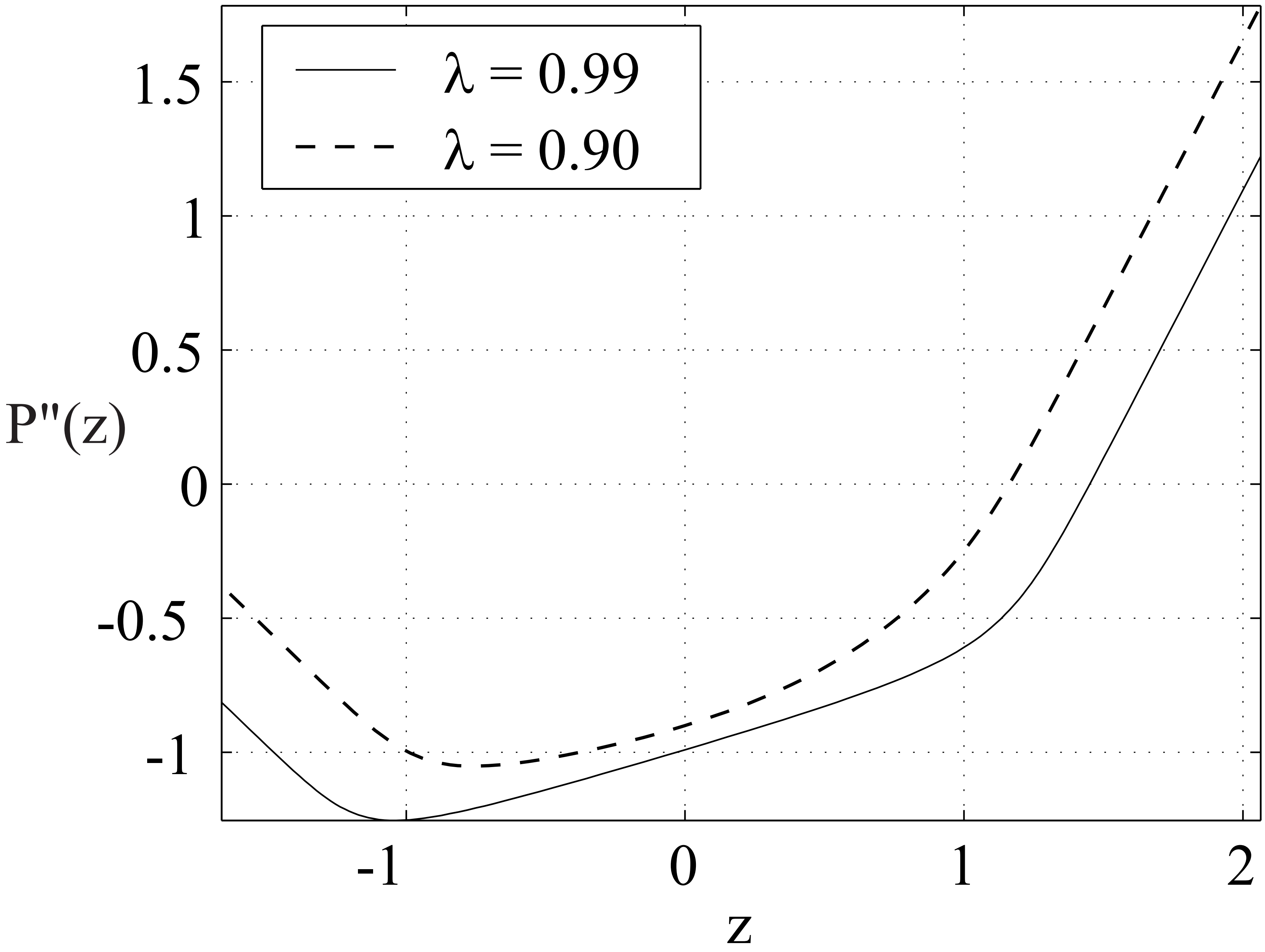


Figure 4: Curvature of Price Function - Model 1b

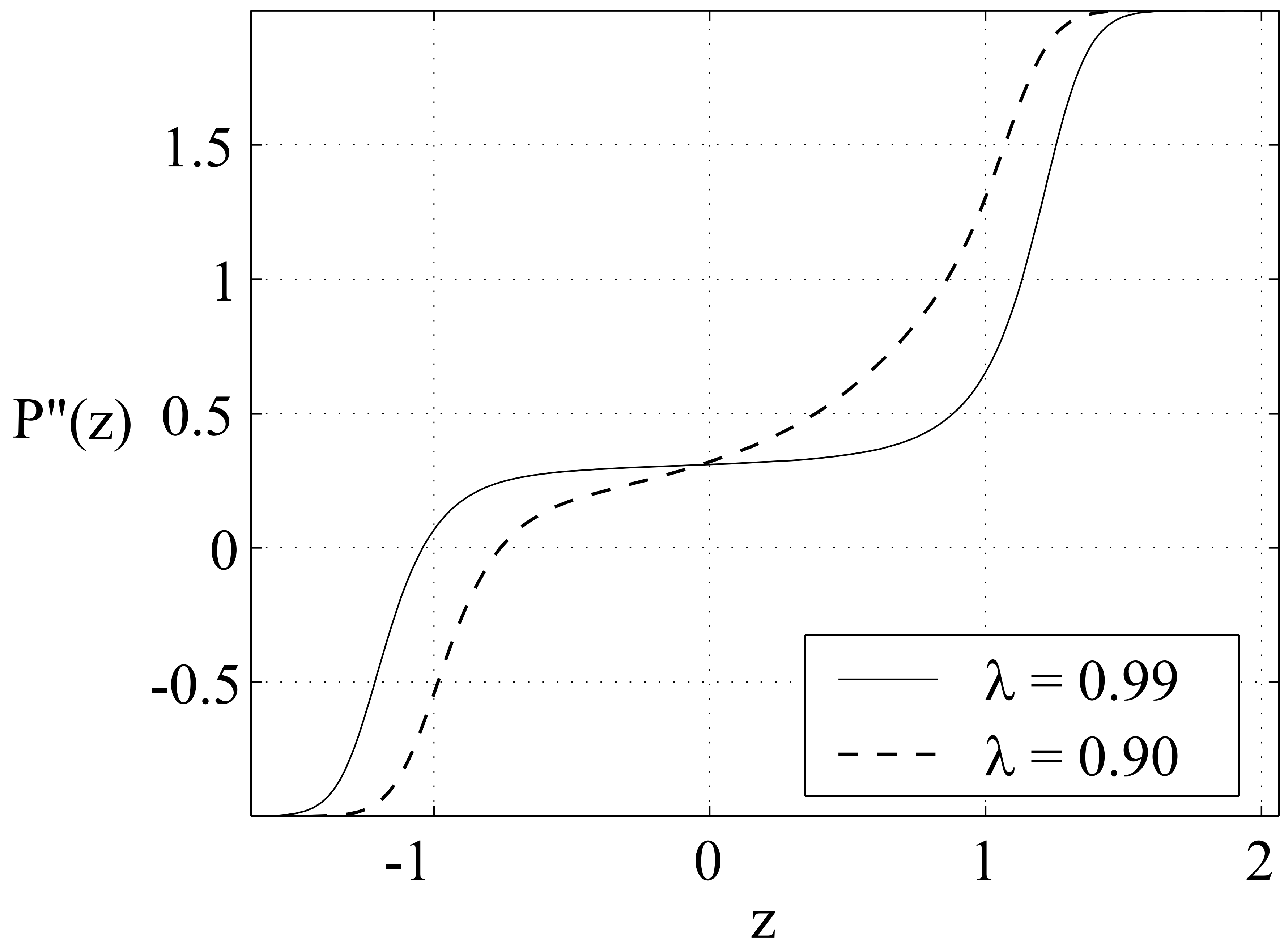


Figure 5: Slope of Pricing Function - Model 2

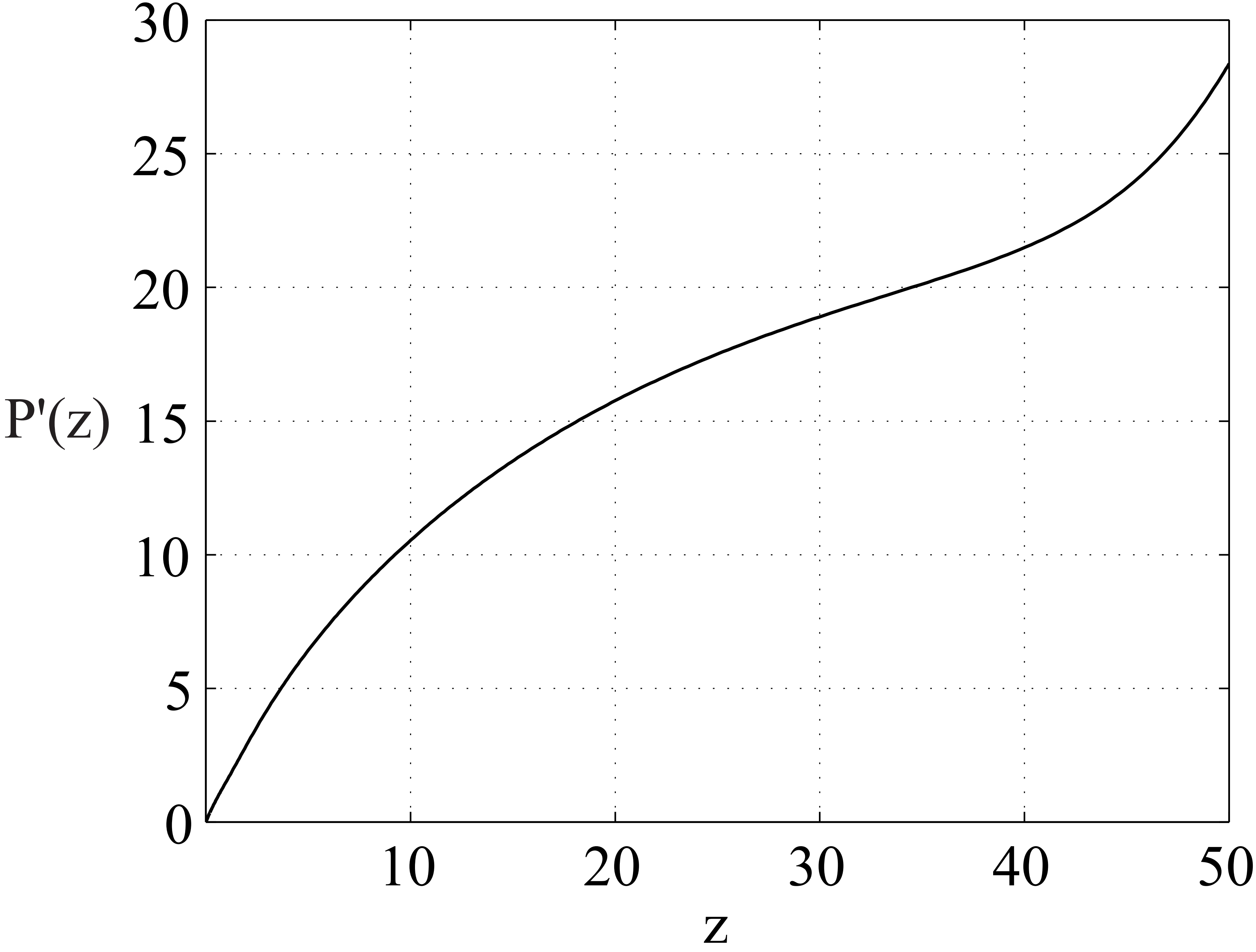


Figure 6: Curvature of Price Function - Model 2

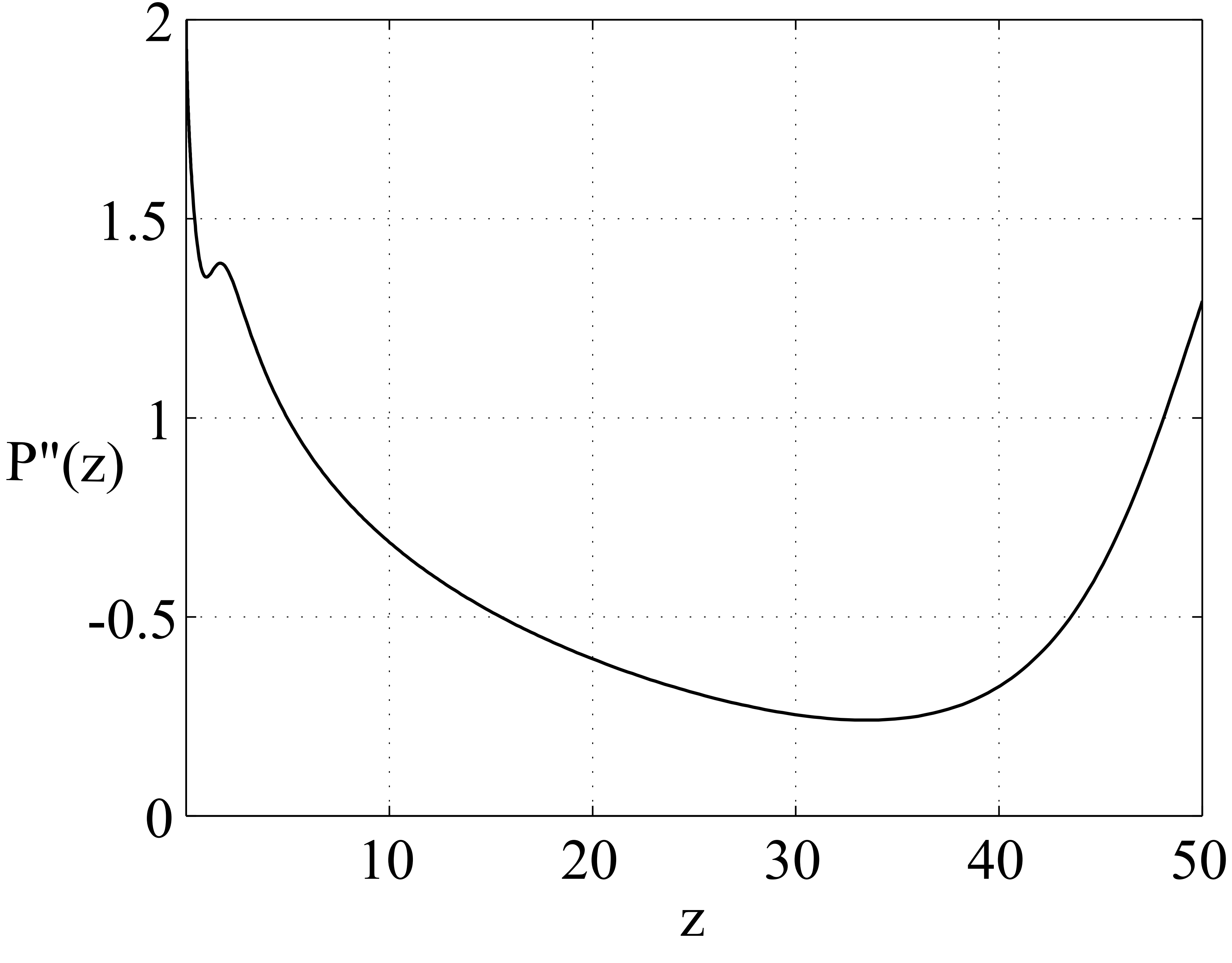
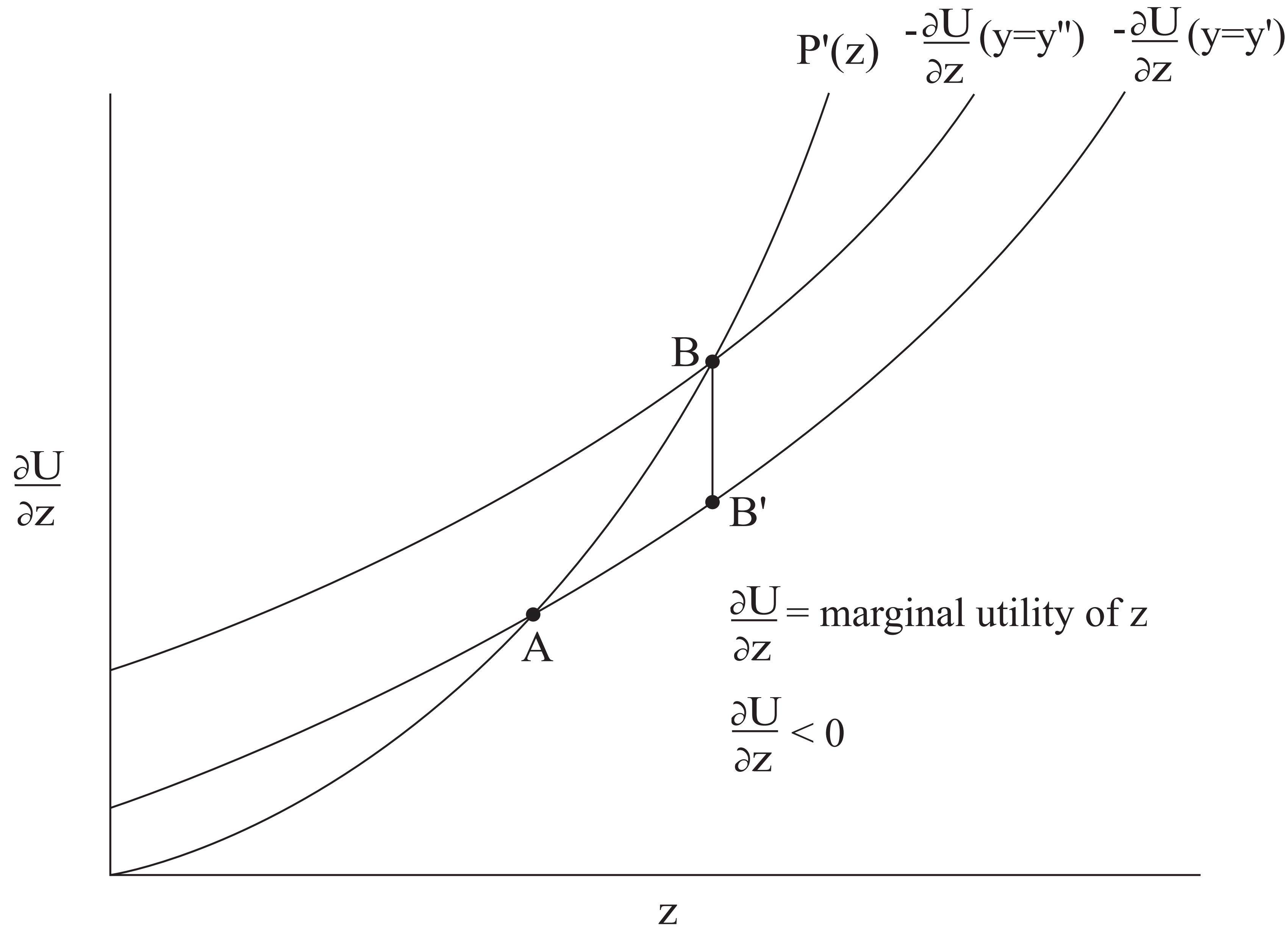


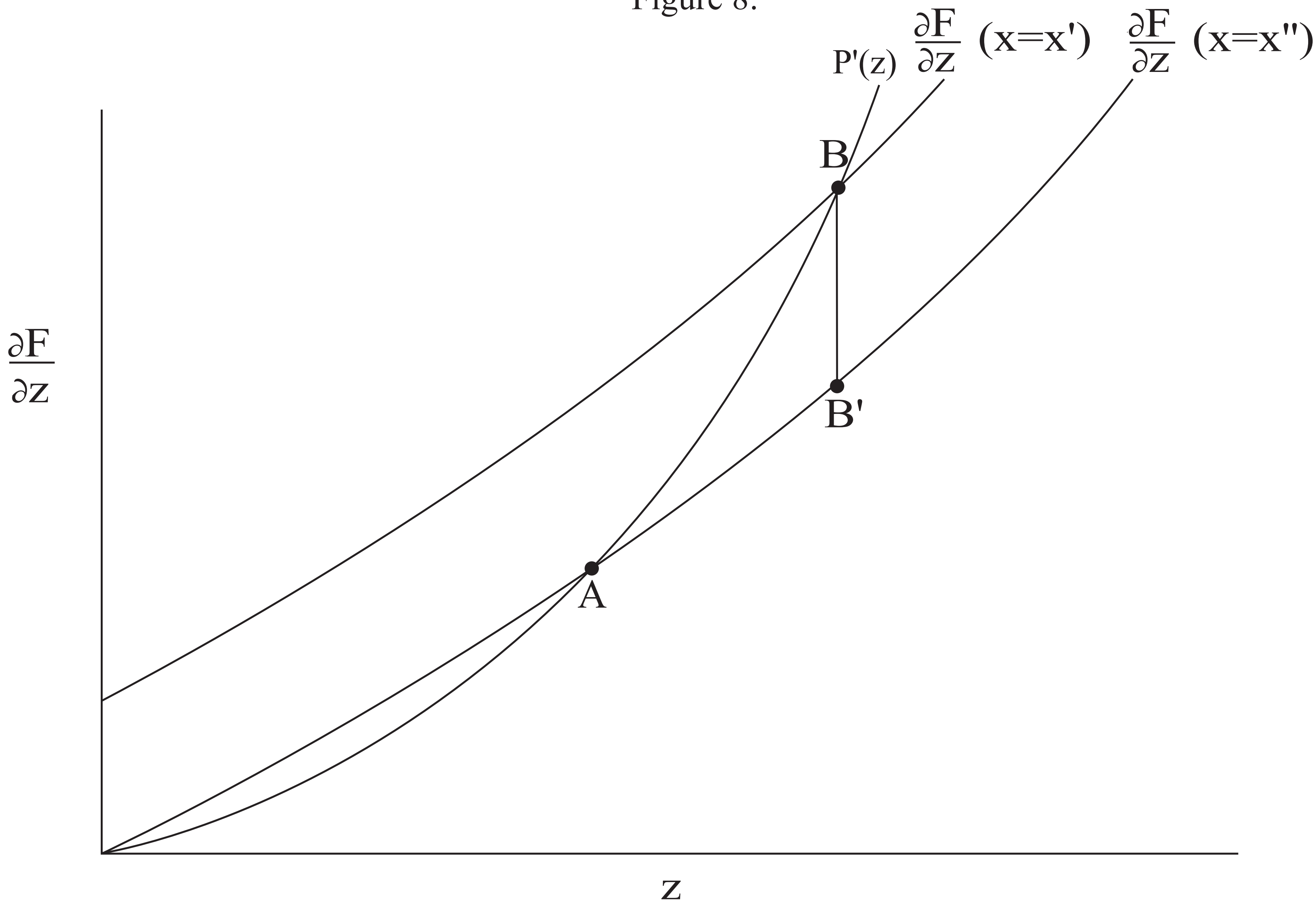
Figure 7:



Identifying Information From Separability of Marginal

Willingness to Pay in Terms of y (or ε_1)

Figure 8:



Identifying Information From Separability of Marginal Products in Terms of x (or ε_2)