On zero-error communication via quantum channels in the presence of noiseless feedback^a

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We initiate the study of zero-error communication via quantum channels when the receiver and sender have at their disposal a noiseless feedback channel of unlimited quantum capacity, generalizing Shannon's zero-error communication theory with instantaneous feedback.

We first show that this capacity is a function only of the linear span of Choi-Kraus operators of the channel, which generalizes the bipartite equivocation graph of a classical channel, and which we dub "non-commutative bipartite graph". Then we go on to show that the feedback-assisted capacity is non-zero (allowing for a constant amount of activating noiseless communication) if and only if the non-commutative bipartite graph is non-trivial, and give a number of equivalent characterizations. This result involves a far-reaching extension of the "conclusive exclusion" of quantum states [Pusey/Barrett/Rudolph, *Nature Phys.* **8**(6):475-478, 2012].

We then present an upper bound on the feedback-assisted zero-error capacity, motivated by a conjecture originally made by Shannon and proved later by Ahlswede. We demonstrate this bound to have many good properties, including being additive and given by a minimax formula. We also prove a coding theorem showing that this quantity is the entanglement-assisted capacity against an adversarially chosen channel from the set of all channels with the same Choi-Kraus span, which can also be interpreted as the feedbackassisted *unambiguous* capacity. The proof relies on a generalization of the "Postselection Lemma" (de Finetti reduction) [Christandl/König/Renner, *Phys. Rev. Lett.* **102**:020504, 2009] that allows to reflect additional constraints, and which we believe to be of independent interest. This capacity is a relaxation of the feedback-assisted zero-error capacity; however, we have to leave open the question of whether they coincide in general.

We illustrate our ideas with a number of examples, including classical-quantum channels and Weyl diagonal channels, and close with an extensive discussion of open questions.

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I. ZERO-ERROR COMMUNICATION ASSISTED BY NOISELESS QUANTUM FEEDBACK

In information theory it is customary to consider not only asymptotically long messages but also asymptotically vanishing, but nonzero error probabilities, which leads to a probabilistic theory of communication characterized by entropic capacity formulas [14, 44]. It is well-known that when communicating by block codes over a discrete memoryless channel at rate below the capacity, the error probability goes to zero exponentially in the block length, and while it is one of the major open problems of information theory to characterize the tradeoff between rate and error exponent in general, we have by now a fairly good understanding of it. However, if the error probability is required to vanish faster than exponential, or equivalently is required to be zero exactly (at least in the case of finite alphabets), we enter the strange and much less understood realm of zero-error information theory [37, 45], which concerns asymptotic combinatorial problems, most of which are unsolved and are considered very difficult. There are a couple of exceptions to this rather depressing state of affairs, one having been already identified by Shannon in his founding paper [45], namely the discrete memoryless channel N(y|x) assisted by instantaneous noiseless feedback, whose capacity is given by the fractional packing number of a bipartite graph Γ representing the possible transitions N(y|x) > 0. The other one is the the recently considered assistance by no-signalling correlations [20], which is also completely solved in terms the fractional packing number of the same bipartite graph Γ .

Recent years have seen attempts to create a theory of quantum zero-error information theory [40], identifying some rather strange phenomena there such as superactivation [18, 22] or entanglement advantage for classical channels [19, 39], but resulting also in some general structural progress such as a quantum channel version of the Lovász number [23]. Motivated by the success in the above-mentioned two models, two of us in [24] (see also [25]) have developed a theory of zero-error communication over memoryless quantum channels assisted by quantum no-signalling correlations, which largely (if not completely) mirrors the classical channel case; in particular, it yielded the first capacity interpretation of the Lovász number of a graph. Some of the techniques and insights developed in [24] will play a central role also in the present paper.

In the present paper, we take as our point of departure the other successful case, Shannon's theory of zero-error communication assisted by noiseless instantaneous feedback. In detail, consider a quantum channel $\mathcal{N} : \mathcal{L}(A) \longrightarrow \mathcal{L}(B)$, i.e. a completely positive and trace preserving (cptp) linear map from the operators on A to those of B (both finite-dimensional Hilbert spaces), where $\mathcal{L}(A)$ denotes the linear operators (i.e. matrices) on A, with Choi-Kraus and Stinespring representations

$$\mathcal{N}(\rho) = \sum_{j} E_{j} \rho E_{j}^{\dagger} = \operatorname{Tr}_{C} V \rho V^{\dagger},$$

for linear operators $E_j : A \longrightarrow B$ such that $\sum_j E_j^{\dagger} E_j = 1$, and an isometry $V : A \longrightarrow B \otimes C$, respectively. The linear span of the Choi-Kraus operators is denoted by

$$K = \mathcal{K}(\mathcal{N}) := \operatorname{span}\{E_j : j\} < \mathcal{L}(A \to B),$$

where "<" means that *K* is a subspace of $\mathcal{L}(A \to B)$, the linear operators (i.e. martrices) mapping *A* to *B*. We will discuss a model of communication where Alice uses the channel *n* times in succession, allowing Bob after each round to send her back an arbitrary quantum system. They may also share an entangled state prior to the first round (if not, they can have it anyway from the second round on, since Bob could use the first feedback to create an arbitrary entangled state). Their goal is to allow Alice to send one of *M* messages down the channel uses such that Bob is able to distinguish them perfectly. More formally, the most general *quantum feedback-assisted code* consists of a state (w.l.o.g. pure) $|\phi\rangle \in X_0 \otimes Y_0$ and for each message $m = 1, \ldots, M$ isometries for encoding and feedback decoding

$$U_t^{(m)}: X_{t-1} \otimes F_{t-1} \longrightarrow A_t \otimes X_t,$$

$$W_t: Y_{t-1} \otimes B_t \longrightarrow F_t \otimes Y_t,$$
(1)

for t = 1, ..., n and appropriate local quantum systems X_t (Alice) and Y_t (Bob), as well the feedback-carrying systems F_t ; see Fig. 1. For consistency (and w.l.o.g.), $F_0 = F_n = \mathbb{C}$ are trivial. Note that Bob can use the feedback channel to create any entangled state $|\phi\rangle$ with Alice for later use before they actually send messages. We use isometries, rather than general cptp maps, to represent encoders and decoders in the feedback-assisted communication scheme, because by the Stinespring dilation [48], all local cptp maps can be "purified" to local isometries. Thus every seemingly more general protocol involving cptp maps can be purified to one of the above form. We will find this form convenient in the later analysis as it allows us to reason on the level of Hilbert space vectors.

We call this quantum feedback-assisted code a *zero-error code* if there is a measurement on Y_n that distinguishes Bob's output states $\rho^{(m)} = \sum_{\underline{j}} \rho_{\underline{j}}^{(m)}$, with certainty, where the sum is over the states

$$\rho_{\underline{j}}^{(m)} = \operatorname{Tr}_{X_n} \left(\prod_{t=n}^1 (W_t E_{j_t} U_t^{(m)}) |\phi\rangle \langle \phi| \prod_{t=1}^n (U_t^{(m)\dagger} E_{j_t}^{\dagger} W_t^{\dagger}) \right),$$
(2)

which are the output states given a specific sequence $\underline{j} = j_1 \dots j_n$ of Kraus operators. [Note that here and below, for convenience, we use $\prod_{t=n}^{1} Q_t$ to represent right-to-left multiplications



FIG. 1. Diagrammatic representation of a feedback-assisted code for messages m sent down a channel \mathcal{N} used n times, in the form of a schematic circuit diagram. All boxes are isometries (acting on suitably large input and output quantum registers), and the solid lines and arrows represent the "sending" of the respective register. Bob's final output state ρ_m after n rounds of using the channel and feedback is in register Y_n .

of operators Q_t , namely $\prod_{t=n}^{1} Q_t := Q_n \cdots Q_1$.] In other words, these states $\rho^{(m)}$ have to have mutually orthogonal supports, i.e. for all $m \neq m'$, all $\underline{j}, \underline{k}$ and all $\xi \in \mathcal{L}(X_n)$,

$$0 = \langle \phi | \prod_{t=1}^{n} (U_{t}^{(m')^{\dagger}} E_{j_{t}}^{\dagger} W_{t}^{\dagger}) \xi \prod_{t=n}^{1} (W_{t} E_{k_{t}} U_{t}^{(m)}) | \phi \rangle =: \langle \phi_{\underline{j}}^{(m')} | \xi | \phi_{\underline{k}}^{(m)} \rangle.$$

By linearity, we see that this condition depends only on the linear span of the Choi-Kraus operator space K, in fact it can evidently be expressed as the orthogonality of a tensor defined as a function of $|\phi\rangle$, the $U_t^{(m)}$ and W_t , to the subspace $(K \otimes K^{\dagger})^{\otimes n}$ – cf. similar albeit simpler characterizations of zero-error and entanglement-assisted zero-error codes in terms of the "non-commutative graph" $S = K^{\dagger}K := \operatorname{span}\{E_k^{\dagger}E_j : k, j\} < \mathcal{L}(A)$ [18, 22, 23], and of no-signalling assisted zero-error codes in terms of the "non-commutative bipartite graph" K [24]. Thus we have proved **Proposition 1** A quantum feedback-assisted code for a channel N being zero-error is a property solely of the Choi-Kraus space $K = \mathcal{K}(\mathcal{N})$. The maximum number of messages in a feedback-assisted zero-error code is denoted $M_f(n; K)$. Hence, the quantum feedback-assisted zero-error capacity of \mathcal{N} ,

$$C_{0EF}(K) := \lim_{n \to \infty} \frac{1}{n} \log M_f(n; K) = \sup_n \frac{1}{n} \log M_f(n; K),$$

is a function only of K.

In the case of a classical channel $N : \mathcal{X} \longrightarrow \mathcal{Y}$ with transition probabilities N(y|x), assisted by classical noiseless feedback, the above problem was first studied – and completely solved – by Shannon [45]. To be precise, his model has noiseless instantaneous feedback of the channel output back to the encoder; it is clear that any protocol with general actions (noisy channel acting on the output) by the receiver can be simulated by the receiver storing the output and the encoder getting a copy of the channel output, if shared randomness is available. Our model differs from this only by the additional availability of entanglement; that this does not increase further the capacity follows from [20], see our comments below.

Following Shannon, we introduce the (bipartite) *equivocation graph* $\Gamma = \Gamma(N)$ on $\mathcal{X} \times \mathcal{Y}$, which has an edge xy iff N(y|x) > 0, i.e. the adjacency matrix is $\Gamma(y|x) = \lceil N(y|x) \rceil$; furthermore the confusability graph G = G(N) on \mathcal{X} , with an edge $x \sim x'$ iff there exists a y such that N(y|x)N(y|x') > 0, i.e., iff the neighbourhoods of x and x' in Γ intersect. The *feedback-assisted zero-error capacity* $C_{0F}(N)$ of the channel N can be seen to depend only on Γ .

Note that for (the quantum realisation of) a classical channel, i.e.

$$\mathcal{N}(\rho) = \sum_{xy} N(y|x) |y\rangle \langle x|\rho|x\rangle \langle y|,$$

the corresponding subspace is given by

$$K = \operatorname{span}\{|y\rangle\!\langle x| : xy \text{ is an edge in } \Gamma\},\$$

so *K* should really be understood as the quantum generalisation of the equivocation graph (a *non-commutative bipartite graph*) [24], much as $S = K^{\dagger}K$ was advocated in [23] as a quantum generalisation of an undirected graph.

Shannon proved

$$C_{0F}(N) = C_{0F}(\Gamma) = \begin{cases} 0 & \text{if } G \text{ is a complete graph } (\text{iff } C_0(N) = 0), \\ \log \alpha^*(\Gamma) & \text{otherwise.} \end{cases}$$
(3)

Here, $\alpha^*(\Gamma)$ is the so-called *fractional packing number* of Γ , defined as a linear programme, whose dual linear programme is the *fractional covering number* [42, 45]:

$$\alpha^{*}(\Gamma) = \max \sum_{x} w_{x} \text{ s.t. } \forall x \ 0 \le w_{x}, \ \forall y \ \sum_{x} w_{x} \Gamma(y|x) \le 1,$$

$$= \min \sum_{y} v_{y} \text{ s.t. } \forall y \ 0 \le v_{y}, \ \forall x \ \sum_{y} v_{y} \Gamma(y|x) \ge 1.$$
(4)

This number appears also in other zero-error communication problems, namely as the zero-error capacity of the channel assisted by no-signalling correlations [20]. There, it is also shown to be the asymptotic simulation cost of a channel with bipartite graph Γ in the presence of shared randomness. This shows that for a classical channel with bipartite graph Γ , interpreted as a quantum channel \mathcal{N} with non-commutative bipartite graph K, $C_{0F}(\Gamma) = C_{0EF}(K)$.

The first case in eq. (3) of a complete graph *G* is easy to understand: whatever the parties do, and regardless of the use of feedback, any two inputs may lead to the same output sequence, so not a single bit can be transmitted with certainty. In either case, Shannon showed that only some arbitrarily small rate of perfect communication (actually a constant amount, dependent only on Γ) is sufficient to achieve what we might call the *activated capacity* $\overline{C}_{0F}(N)$, which is always equal to $\log \alpha^*(\Gamma)$. This was understood better in the work of Elias [28] who showed that the capacity of zero-error *list decoding* of *N* (with arbitrary but constant list size) is exactly $\log \alpha^*(\Gamma)$. Thus a coding scheme for *N* with feedback would consist of a zero-error list code with list size *L* and rate $R \ge (1 - \frac{1}{L}) \log \alpha^*(\Gamma) - O(\frac{1}{L})$ for *n* uses of the channel *N*, followed by feedback in which Bob lets Alice know the list of *L* litems in which he now knows the message falls, followed by a noiseless transmission of $\log L$ bits of Alice to resolve the remaining ambiguity. Shannon's scheme [45] is based on a similar idea, but whittles down the list by a constant factor in each round, so Bob needs to update Alice on the remaining list after each channel use. The constant noiseless communication at the end of this protocol can be transmitted using an unassisted zero-error code via the given channel *N* (at most $\log L$ uses), or via an activating noiseless channel.

The dichotomy in eq. (3) has the following quantum channel analogue (in fact, generalization):

Proposition 2 For any non-commutative bipartite graph $K = \mathcal{K}(\mathcal{N}) < \mathcal{L}(A \rightarrow B)$, the feedback-assisted zero-error capacity of K vanishes, $C_{0EF}(K) = 0$, if and only if the associated non-commutative graph is complete, i.e. $S = K^{\dagger}K = \mathcal{L}(A)$, which is equivalent to vanishing entanglement-assisted zero-error capacity, $C_{0E}(S) = 0$.

Proof Clearly $C_{0EF}(K) \ge C_{0E}(S)$ since on the right hand side we simply do not use feedback, but any code is still a feedback-assisted code. Hence, if the latter is positive then so is the former. It is well known that if $S \ne \mathcal{L}(A)$, then $C_{0E}(S) \ge 1 > 0$, in fact each channel use can transmit at least one bit [22, 23].

Conversely, let us assume that $C_{0E}(S) = 0$, i.e. $S = K^{\dagger}K = \mathcal{L}(A)$. We will show by induction on *t* that for any two distinct messages, w.l.o.g. b = 0, 1, Bob's output states after *t* rounds, $\rho_t^{(b)}$ on Y_t , cannot be orthogonally supported, meaning $M_f(n; K) = 1$. Here,

$$\rho_t^{(b)} = \sum_{j_1\dots j_t} \operatorname{Tr}_{X_t F_t} |\phi_{j_1\dots j_t}^{(b)}\rangle \langle \phi_{j_1\dots j_t}^{(b)}|, \text{ with}$$
$$|\phi_{j_1\dots j_t}^{(b)}\rangle = \prod_{i=t}^1 W_i E_{j_i} U_i^{(b)} |\phi\rangle \in X_t \otimes F_t \otimes Y_t.$$

This is clearly true for t = 0 since at that point Alice and Bob share only $|\phi\rangle_{X_0Y_0}$, hence $\rho_0^{(0)} = \rho_0^{(1)} = \text{Tr}_{X_0}\phi$. For t > 0, let Bob after t - 1 rounds have one of the states $\rho_{t-1}^{(b)}$; by the induction hypothesis, $\rho_{t-1}^{(0)} \neq \rho_{t-1}^{(1)}$ – by a slight abuse of notation meaning that the supports are not orthogonal, or equivalently that the operators are not orthogonal with respect to the Hilbert-Schmidt inner product. This means that there are indices $j_1 \dots j_{t-1}$ and $k_1 \dots k_{t-1}$ such that

$$\left(\phi_{t-1}^{(0)}\right)_{Y_{t-1}} := \left(\phi_{j_1\dots j_{t-1}}^{(0)}\right)_{Y_{t-1}} \not\perp \left(\phi_{k_1\dots k_{t-1}}^{(1)}\right)_{Y_{t-1}} =: \left(\phi_{t-1}^{(1)}\right)_{Y_{t-1}}.$$

This can be expressed equivalently as

$$\operatorname{Tr}_{Y_{t-1}} |\phi_{t-1}^{(0)}\rangle \langle \phi_{t-1}^{(1)} | \neq 0.$$

$$\operatorname{Tr}_{Y_{t-1}} |\psi_t^{(0)}\rangle \langle \psi_t^{(1)}| = \operatorname{Tr}_{Y_{t-1}} U_t^{(0)} |\phi_{t-1}^{(0)}\rangle \langle \phi_{t-1}^{(1)}| U_t^{(1)\dagger} \neq 0.$$
(5)

After that, the channel action consists in one of the Choi-Kraus operators $E_j : A \to B$. Let us assume, with the aim of establishing a contradiction, that Bob's states after the channel action were orthogonal, i.e. for all j and k,

$$\operatorname{Tr}_{X_t} E_j \psi_t^{(0)} E_j^{\dagger} \perp \operatorname{Tr}_{X_t} E_k \psi_t^{(1)} E_k^{\dagger}$$

In other words, for all j, k and operators ξ on X_t ,

$$0 = \langle \psi_t^{(1)} | \xi \otimes E_k^{\dagger} E_j \otimes \mathbf{1} | \psi_t^{(0)} \rangle$$

= Tr[($\xi \otimes E_k^{\dagger} E_j$) Tr_{Yt-1} | $\psi_t^{(0)} \rangle \langle \psi_t^{(1)} |$].

But since ξ is arbitrary and the $E_k^{\dagger} E_j$ span $\mathcal{L}(A)$, this would imply $\operatorname{Tr}_{Y_{t-1}} |\psi_t^{(0)}\rangle\langle\psi_t^{(1)}| = 0$, contradicting (5).

Thus, applying now also the isometry $W_t : BY_{y-1} \to F_tY_t$, we find that there exist j_t and k_t such that

$$\left(\phi_{j_1\dots j_t}^{(0)}\right)_{F_tY_t} \not\perp \left(\phi_{k_1\dots k_t}^{(1)}\right)_{F_tY_t}, \text{ hence } \left(\phi_{j_1\dots j_t}^{(0)}\right)_{Y_t} \not\perp \left(\phi_{k_1\dots k_t}^{(1)}\right)_{Y_t},$$

and so finally $\rho_t^{(0)} \not\perp \rho_t^{(1)}$, proving the induction step.

Motivated by \overline{C}_{0F} of a classical channel [45], see above, we define also feedback-assisted codes with n channel uses and up to b noiseless classical bits of forward communication. The setup is the same as in eq. (1) and Fig. 1 with n + b rounds, n of which feature the isometric dilation Vof \mathcal{N} , and b the isometry $V' : |i\rangle \mapsto |i\rangle |i\rangle$ (i = 0, 1) corresponding to the noiseless bit channel $\overline{id}_2 : \rho \mapsto \sum_{i=0}^1 |i\rangle \langle i|\rho |i\rangle \langle i|$. It is clear that the output states can be written in a way similar to eq. (2), and that the maximum number of messages in a zero-error code depends only on n, b and $K < \mathcal{L}(A \to B)$, which we denote $M_f^{+b}(n; K)$. Clearly, $M_f^{+0}(n; K) = M_f(n; K)$ and in general, $M_f^{+b+1}(n; K) \ge 2 M_f^{+b}(n; K)$. Furthermore, it can easily be verified that

$$2^{-b}M_f^{+b}(n;K) \, 2^{-c}M_f^{+c}(m;K) \le 2^{-b-c}M_f^{+b+c}(n+m;K),$$

hence we can define the activated feedback-assisted zero-error capacity

$$\overline{C}_{0EF}(K) := \sup_{b} \sup_{n} \frac{1}{n} \left(\log M_f^{+b}(n;K) - b \right)$$
$$= \sup_{b} \lim_{n \to \infty} \frac{1}{n} \log M_f^{+b}(n;K).$$

Then the above Proposition 2 can be rephrased as

$$C_{0EF}(K) = \begin{cases} \overline{C}_{0EF}(K) & \text{if } S = K^{\dagger}K \neq \mathcal{L}(A), \\ 0 & \text{if } S = \mathcal{L}(A) \text{ (iff } C_{0E}(S) = 0), \end{cases}$$
(6)

motivating our focusing on $\overline{C}_{0EF}(K)$ from now on

The rest of the present paper is organized as follows: In Section II we start with a concrete example showing the importance of measurements "conclusively excluding" hypotheses from a list of options, and go on to show several concise characterizations of nontrivial channels, i.e. those for which $\overline{C}_{0EF}(K) > 0$. In Section III we first review a characterization of the fractional packing number in terms of the Shannon capacity minimized over a set of channels, which then motivates the definition of $C_{\min E}(K)$ obtained as a minimization of the entanglement-assisted capacity over quantum channels consistent with the given non-commutative bipartite graph. $C_{\min E}(K)$ represents the best known upper bound on the feedback-assisted zero-error capacity. We illustrate the bound by showing how it allows us to determine $\overline{C}_{0EF}(K)$ for Weyl diagonal channels, i.e. K spanned by discrete Weyl unitaries. We also show that $C_{\min E}(K)$ is the ordinary (small error) capacity of the system assisted by entanglement, against an adversarial choice of the channel (proof in Appendix A, based on a novel Constrained Postselection Lemma, aka "de Finetti reduction", in Appendix B). After that, we conclude in Section IV with a discussion of open questions and future work.

II. CHARACTERIZATION OF VANISHING CAPACITY $\overline{C}_{0EF}(K)$

In this section, we will prove the following result.

Theorem 3 If the non-commutative bipartite graph $K < \mathcal{L}(A \to B)$ contains a subspace $|\beta\rangle \otimes A^{\dagger} < K$ with a state vector $|\beta\rangle \in B$, meaning that the constant channel $\mathcal{N}_0 : \rho \mapsto |\beta\rangle\langle\beta| \operatorname{Tr} \rho$ has $\mathcal{K}(\mathcal{N}_0) < K$, then $\overline{C}_{0EF}(K) = 0$; we call such K trivial.

Conversely, if K is nontrivial, then $\overline{C}_{0EF}(K) > 0$.

Proof ("trivial \Rightarrow zero capacity") We show the stronger statement $M_f^{+b}(n; K) = 2^b$ for all n and b. Indeed, as the zero-error condition is only a property of K, we may assume a concrete constant channel \mathcal{N}_0 with $\mathcal{K}(\mathcal{N}_0) = |\beta\rangle \otimes A^{\dagger} < K$. The outputs of the n copies of \mathcal{N}_0 in the feedback code do not matter at all as they are going to be $\beta^{\otimes n}$, which Bob can create himself. Hence the only information arriving at Bob's from Alice is in the b classical bits in the course of the protocol. But even assisted by entanglement and feedback, Alice can convey at most b noiseless bits in this way, due to the Quantum Reverse Shannon Theorem [5].

The opposite implication ("nontrivial \Rightarrow positive capacity") will be the subject of the remainder of this section. We will start by looking at cq-channels first – Subsection II A for pure state cqchannels, Subsection II B for a mixed state example and Subsection II C for general cq-channels –, before completing the proof for general channels in Subsection II D.

A. Pure state cq-channels

For a given orthonormal basis $\{|i\rangle\}$ of the input space *A*, and pure states $|\psi_i\rangle$ in the output space, consider the cq-channel

$$\mathcal{N}(
ho) = \sum_i |\psi_i
angle \langle i|
ho|i
angle \langle \psi_i|,$$

with Kraus subspace

$$K := \mathcal{K}(\mathcal{N}) = \operatorname{span}\{|\psi_i\rangle\langle i|\}$$

We shall demonstrate first the following result:

Proposition 4 For a pure state cq-channel, $\overline{C}_{0EF}(K)$ is always positive unless K is trivial, which is equivalent to all $|\psi_i\rangle$ being collinear, i.e. $K = |\psi\rangle \otimes A^{\dagger}$ for some pure state $|\psi\rangle$.

Proof If *K* is trivial, then the above proof of the sufficiency of triviality in Theorem 3 shows $\overline{C}_{0EF}(K) = 0$.

Conversely, if *K* is non-trivial, then there are two output vectors, denoted $|\psi_0\rangle$ and $|\psi_1\rangle$, that are not collinear, and we shall simply restrict the channel to the corresponding inputs 0 and 1. I.e., we focus only on $K' = \text{span}\{|\psi_0\rangle\langle 0|, |\psi_1\rangle\langle 1|\}$, and the corresponding channel

$$\mathcal{N}'(\rho) = |\psi_0\rangle\langle 0|\rho|0\rangle\langle\psi_0| + |\psi_1\rangle\langle 1|\rho|1\rangle\langle\psi_1|.$$

Consider using it three times, inputting only the code words 001, 010 and 100. This gives rise to output states

$$\begin{aligned} |u_a\rangle &= |\psi_0\rangle |\psi_0\rangle |\psi_1\rangle, \\ |u_b\rangle &= |\psi_0\rangle |\psi_1\rangle |\psi_0\rangle, \\ |u_c\rangle &= |\psi_1\rangle |\psi_0\rangle |\psi_0\rangle, \end{aligned}$$

which have the property that their pairwise inner products are all equal: $\langle u_x | u_y \rangle = |\langle \psi_0 | \psi_1 \rangle|^2 =: \epsilon$. By using the channel 3n times, Alice can prepare the states

$$|t_x\rangle = |u_x\rangle^{\otimes n}$$
 $(x = a, b, c)$

whose pairwise inner products are all equal and indeed ϵ^n , i.e. arbitrarily close to 0. Now, if *n* is large enough (so that $\epsilon^n \leq \frac{1}{2}$), there is a cptp map that Bob can apply to transform

$$\begin{aligned} |t_a\rangle &\longmapsto \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \\ |t_b\rangle &\longmapsto \frac{1}{\sqrt{2}}(|2\rangle + |0\rangle), \\ |t_c\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \end{aligned}$$

(This follows from well known results on pure-state transformations, see e.g. [11].) By now it may be clear where this is going: Bob measures the computational basis and overall we obtain a classical channel $P : \{a, b, c\} \rightarrow \{0, 1, 2\}$ with exactly one 0-entry in each row and column:

$$P(0|a) = P(1|b) = P(2|c) = 0,$$

which has zero-error capacity 0, but assisted by feedback and a finite number of activating noiseless bits, it is $\log \frac{3}{2}$ [45]. We conclude that $\overline{C}_{0EF}(\mathcal{N}) \geq \frac{1}{3n} \log \frac{3}{2} > 0$.

B. Mixed state cq-channel

To generalize the previous treatment to mixed states, let us first look at a specific simple example: Let $|\psi_i\rangle$ (i = 0, 1, 2) be three mutually distinct but non-orthogonal states in \mathbb{C}^3 , and define a cq-channel \mathcal{N} with three inputs i = 0, 1, 2, mapping

$$0 \longmapsto \frac{1}{2}\psi_1 + \frac{1}{2}\psi_2,$$

$$1 \longmapsto \frac{1}{2}\psi_0 + \frac{1}{2}\psi_2,$$

$$2 \longmapsto \frac{1}{2}\psi_0 + \frac{1}{2}\psi_1.$$

(7)

Thus,

$$K = \operatorname{span}\{|\psi_1\rangle\langle 0|, |\psi_2\rangle\langle 0|, |\psi_0\rangle\langle 1|, |\psi_2\rangle\langle 1|, |\psi_0\rangle\langle 2|, |\psi_1\rangle\langle 2|\}$$

and the most general channel \mathcal{N}' consistent with this *K* is a cq-channel of the form

 $i \mapsto \rho_i$, ρ_i supported on span $\{ |\psi_j\rangle : j \in \{0, 1, 2\} \setminus i \}$.

We shall show how to construct a zero-error scheme with feedback, achieving positive rate, at least for $|\psi_i\rangle$ that are sufficiently close to being orthogonal. For the zero-error properties, we may as well focus on \mathcal{N} , which is easier to reason with. For the following, it may be helpful to think of eq. (7) in a partly classical way: any input *i* is mapped to a random $|\psi_j\rangle$, subject to $j \neq i$, so that for two uses of the channel, each pair i_1i_2 is mapped randomly to one of four $|\psi_{j_1}\rangle|\psi_{j_2}\rangle$, with $j_1 \neq i_1$, $j_2 \neq i_2$. Of course, vice versa each of these nine vectors is reached from exactly four inputs.

Now, assuming that the pairwise inner products of the $|\psi_i\rangle$ are small enough, i.e.

$$|\langle \psi_0 | \psi_1 \rangle|, \ |\langle \psi_0 | \psi_2 \rangle|, \ |\langle \psi_1 | \psi_2 \rangle| \le \epsilon,$$

to guarantee that there is a deterministic pure state transformation (by cptp map) $|\psi_{j_1}\rangle|\psi_{j_2}\rangle \mapsto |\varphi_{j_1j_2}\rangle$ [11], where

$$|\varphi_{j_1j_2}\rangle = \frac{1}{\sqrt{8}} \sum_{\substack{j_1j_2 \in I \subset \{0,1,2\}^2 \\ |I|=2}} |I\rangle \in \mathbb{C}^{36}.$$

On these states, Bob performs a measurement in the computational basis of the $|I\rangle$, and we get an effective classical channel mapping $i_1i_2 \in \{0,1,2\}^2$ randomly to some $\{j_1j_2,k_1k_2\} = I \subset \{0,1,2\}^2$, subject to the constraint

$$(j_1 \neq i_1 \& j_2 \neq i_2)$$
 or $(k_1 \neq i_1 \& k_2 \neq i_2)$,

which means that each I is reached from at most eight out of the nine pairs i_1i_2 . In fact, the observation of $I = \{j_1j_2, k_1k_2\}$ excludes at least two out of nine input symbols, namely j_1k_2 and k_1j_2 , meaning that this classical channel has zero-error capacity (plus feedback plus a finite number of noiseless bits) of $\geq \log \frac{9}{7}$. In conclusion, we achieve for N, and hence for any N' with $\mathcal{K}(\mathcal{N}') < K$, a rate of $\geq \frac{1}{2} \log \frac{9}{7} > 0$.

C. General cq-channels

The above examples rely on measuring the output states ρ_i of the cq-channel \mathcal{N} by a POVM (M_j) such that the resulting classical(!) channel $N : i \to j$ with $N(j|i) = \text{Tr } \rho_i M_j$ has an equivocation graph Γ with $\alpha^*(\Gamma) > 1$, because then $\overline{C}_{0EF}(K) \ge \overline{C}_{0F}(\Gamma) = \log \alpha^*(\Gamma) > 0$. For this, cf. eq. (4), it is necessary and sufficient that each outcome j excludes at least one input i, i.e. $N(j|i) = \text{Tr } \rho_i M_j = 0$, or equivalently $\rho_i \perp M_j$. A POVM (M_j) with this property is said to "conclusively exclude" the set $\{\rho_i\}$ of states [4, 41]. It is clearly only a property of the support projections P_i of ρ_i , and w.l.o.g. the POVM is indexed by the same i's, i.e. (R_i) such that $P_iR_i = 0$ for all i, as well as $R_i \ge 0$ and $\sum_i R_i = 1$.

Our approach in the following will be to characterize when a set $\{\rho_i\}$ of states, or one of its tensor powers $\{\rho_i\}^{\otimes n} = \{\rho_{\underline{i}} = \rho_{i_1} \otimes \cdots \otimes \rho_{i_n}\}$, can be conclusively excluded. For instance, Pusey, Barrett and Rudolph [41] showed that for any two linearly independent pure states $|\psi_0\rangle$ and $|\psi_1\rangle$, it is always possible to find an integer n and a 2^n -outcome POVM $(R_i : \underline{i} \in \{0, 1\}^n)$ such that

$$\operatorname{Tr} R_{\underline{i}} |\psi_{\underline{i}}\rangle \langle \psi_{\underline{i}}| = 0, \quad |\psi_{\underline{i}}\rangle = |\psi_{i_1}\rangle \otimes |\psi_{i_2}\rangle \otimes \cdots \otimes |\psi_{i_n}\rangle.$$

I.e. we can design a quantum measurement that can conclusively exclude the *n*-fold states $|\psi_{\underline{i}}\rangle$ with *n*-bit strings $\underline{i} = i_1 \dots i_n$ as outcomes, even when $|\psi_0\rangle$ and $|\psi_1\rangle$ are not orthogonal.

We will employ the powerful techniques developed in the proof of [24, Prop. 14], allowing us to show a far-reaching generalization of the Pusey/Barrett/Rudolph result [41]. The version we need can be stated as follows; it is adapted to a cq-channel with *a*-dimensional input space *A* and output states ρ_i (i = 1, ..., a), whose support projectors are P_i and supports K_i , so that the non-commutative graph is

$$K = \sum_{i} K_{i} \otimes \langle i | := \operatorname{span}\{ |\psi_{i}\rangle\langle i | : |\psi_{i}\rangle \in K_{i}, i = 1, \cdots, a \}.$$

Proposition 5 Let $(P_i)_{i=1}^a$ be projectors on a Hilbert space B, with a transitive group action by unitary conjugation on the P_i , i.e. we have a finite group G acting transitively on the labels i, and a unitary representation U^g such that $P_{ig} = (U^g)^{\dagger} P_i U^g$ for $g \in G$.

Consider the isotypical decomposition of U^g ,

$$B = \bigoplus_{\lambda} \mathcal{Q}_{\lambda} \otimes \mathcal{R}_{\lambda}$$

into irreps Q_{λ} of U^{g} , with multiplicity spaces \mathcal{R}_{λ} (cf. [31], see also [12, 32]). Denote the number of terms λ by L, and the largest occurring multiplicity by $M = \max_{\lambda} |\mathcal{R}_{\lambda}|$. If now

$$\frac{a}{\left\|\sum_{i} P_{i}\right\|_{\infty}} > 16L^{6}M^{9},$$

then there exists a POVM (R_i) with $P_iR_i = 0$ for all *i*. In other words, any set $\{\rho_i\}$ with supp $\rho_i < K_i$ can be conclusively excluded.

Before we prove it, we use it to derive the following general result. To state it, we need some notation: For a set $\mathcal{E} = \{\rho_i\}_{i=1}^a$ of states, let

$$\mathcal{E}^{\otimes n} = \left\{ \rho_{\underline{i}} = \rho_{i_1} \otimes \cdots \otimes \rho_{i_n} : \underline{i} = i_1 \dots i_n \in [a]^n \right\}.$$

The strings $\underline{i} = i_1 \dots i_n$ are classified according to $type \tau$ [17], which is the empirical distribution of the letters $i_t, t = 1, \dots, n$. There are only $\binom{n+a-1}{a-1} \leq (n+1)^a$ many different types. The subset of $\mathcal{E}^{\otimes n}$ corresponding to type τ is denoted

$$\mathcal{E}_{\tau}^{(n)} = \left\{ \rho_{\underline{i}} = \rho_{i_1} \otimes \cdots \otimes \rho_{i_n} : \underline{i} = i_1 \dots i_n \text{ has type } \tau \right\}$$

We also recall the definition of the *semidefinite packing number* [24] of a non-commutative bipartite graph *K* with support projection P_{AB} onto the Choi-Jamiołkowski range $(\mathbb{1} \otimes K) |\Phi\rangle$, where $|\Phi\rangle = \frac{1}{\sqrt{|A|}} \sum_{i=1}^{|A|} |i\rangle |i\rangle$ is the maximally entangled state:

$$A(K) = \max \operatorname{Tr} S_A \quad \text{s.t.} \quad 0 \le S_A, \ \operatorname{Tr}_A P_{AB}(S_A \otimes \mathbb{1}_B) \le \mathbb{1}_B$$

= min Tr T_B \quad \text{s.t.} \quad 0 \le T_B, \ \operatorname{Tr}_B P_{AB}(\mathbb{1}_A \otimes T_B) \ge \mathbb{1}_A.
(8)

For the cq-channel case, $P_{AB} = \sum_i |i\rangle \langle i|^A \otimes P_i^B$, this simplifies to

$$\mathsf{A}(K) := \max \sum_{i} s_i \text{ s.t. } 0 \le s_i, \sum_{i} s_i P_i \le \mathbb{1}.$$
(9)

In particular, for the cq-graph K induced by projections $\{P_i\}$ in Proposition 5, we have

$$A(K) = \frac{a}{\left\|\sum_{i} P_i\right\|_{\infty}}.$$

Theorem 6 Let $\mathcal{E} = \{\rho_i\}_{i=1}^a$ be a finite set of quantum states with supports $K_i = \text{supp } \rho_i$, and let K be the associated non-commutative bipartite graph $\sum_i K_i \otimes \langle i |$. Then the following are equivalent:

- *i.* $\overline{C}_{0EF}(K) > 0$;
- *ii. K is nontrivial;*
- *iii.* $\bigcap_i K_i = 0$;
- *iv.* $\|\sum_{i} P_{i}\|_{\infty} < a;$
- *v.* A(K) > 1;
- vi. For sufficiently large n and a suitable type τ , the set $\mathcal{E}_{\tau}^{(n)}$ can be conclusively excluded.
- **Proof** *i*. \Rightarrow *ii*. has been shown in the first part (necessity) of Theorem 3, at the start of this section. *ii*. \Leftrightarrow *iii*. $|\beta\rangle \otimes A^{\dagger} < K = \sum_{i} K_{i} \otimes \langle i|$ if and only if $|\beta\rangle \in \bigcap_{i} K_{i}$.
- *iii.* \Leftrightarrow *iv.* $\|\sum_i P_i\|_{\infty} \leq \sum_i \|P_i\|_{\infty} = a$ with equality if and only if there is a common eigenvector $|\beta\rangle$ with eigenvalue 1 for all of the P_i , i.e. $|\beta\rangle \in \bigcap_i K_i$.
- $|\beta\rangle$ with eigenvalue 1 for all of the P_i , i.e. $|\beta\rangle \in \bigcap_i K_i$. *iv.* \Rightarrow *v*. We check that $s_i = \frac{1}{\|\sum_i P_i\|_{\infty}}$ is feasible for A(K); indeed,

$$\sum_{i} s_i P_i = \frac{1}{\left\|\sum_{i} P_i\right\|_{\infty}} \sum_{i} P_i \le \mathbb{1},$$

thus $\mathsf{A}(K) \ge \frac{a}{\left\|\sum_{i} P_{i}\right\|_{\infty}} > 1.$

 $v. \Rightarrow vi.$ Note that the non-commutative bipartite graph corresponding to $\mathcal{E}^{\otimes n}$ is $K^{\otimes n}$. Let's denote the graph of $\mathcal{E}_{\tau}^{(n)}$ by $K_{\tau}^{(n)}$. In [24] it is shown that A(K) is multiplicative, $A(K^{\otimes n}) = A(K)^n$; indeed, for an optimal assignment of weights s_i feasible for A(K), $s_{\underline{i}} = s_{i_1} \cdots s_{i_n}$ is feasible (and optimal) for $A(K^{\otimes n})$. Hence, there exists a type τ such that

$$\mathsf{A}(K_{\tau}^{(n)}) \ge \sum_{\underline{i} \in \tau} s_{\underline{i}} \ge \frac{1}{\operatorname{poly}(n)} \mathsf{A}(K)^{n}.$$
(10)

On the other hand, the symmetric group S_n acts transitively by permutation on the strings of type τ , and equivalently by permutation of the *n* tensor factors of B^n . This representation is well known to have only $L \leq \text{poly}(n)$ irreps, each of which has multiplicity $M \leq \text{poly}(n)$. Thus, from eq. (10), we deduce that for sufficiently large n, $A(K_{\tau}^{(n)}) \geq 16L^6M^9$, which by Proposition 5 implies that the set $\mathcal{E}_{\tau}^{(n)}$ can be conclusively excluded.

 $vi. \Rightarrow i.$ By sending signals $\underline{i} = i_1 \dots i_n \in \tau$ and measuring the output states $\rho_{\underline{i}}$ with a conclusively excluding POVM $(M_{\underline{i}} : \underline{i} \in \tau)$, we simulate a classical channel whose bipartite equivocation graph Γ has $\alpha^*(\Gamma) > 1$, hence $\overline{C}_{0EF}(K) \ge \frac{1}{n}\overline{C}_{0F}(\Gamma) > 0$.

Proof (of Proposition 5) Assume that we have a feasible $s_i = s^*$ (i = 1, ..., a) for A(K) such that $A(K) \ge \sum_i s_i = s^* a \ge 16L^6 M^9$. Concretely, this means that $\sum_i s_i P_i = s^* \sum_i P_i \le 1$.

We will show that a desired POVM (R_i) can be found, such that $R_{i^g} = (U^g)^{\dagger} R_i U^g$ for all *i* and *g*. The problem of finding the POVM (R_i) then becomes equivalent to finding $0 \le R_0 \le 1 - P_0$ such that

$$\frac{1}{a}\sum_{i=1}^{a}R_{i} = \frac{1}{|G|}\sum_{g\in G} (U^{g})^{\dagger}R_{0}U^{g} = \frac{1}{a}\mathbb{1}.$$
(11)

Schur's Lemma [31] tells us

$$\frac{1}{|G|}\sum_{g} (U^g)^{\dagger} R_0 U^g = \frac{1}{a} \sum_{\lambda} Q_{\lambda} \otimes \zeta_{\lambda},$$

where Q_{λ} is the projection onto the irrep Q_{λ} , ζ_{λ} is a semidefinite operator on \mathcal{R}_{λ} . The equality constraints (11) on R_0 are equivalent to $\zeta_{\lambda} = \prod_{\lambda}$, the projection onto \mathcal{R}_{λ} , for all λ .

Now, for each λ choose an orthogonal basis $\{Z_{\mu}^{(\lambda)}\}$ of Hermitians over \mathcal{R}_{λ} , with $Z_{0}^{(\lambda)} = \frac{1}{\operatorname{Tr}\Pi_{\lambda}}\Pi_{\lambda}$ and $\|Z_{\mu}^{(\lambda)}\|_{2} = 1$ for $\mu \neq 0$. Then the operators $\frac{1}{\operatorname{Tr}Q_{\lambda}}Q_{\lambda} \otimes Z_{\mu}^{(\lambda)}$ form a basis of the U^{g} -invariant operators, hence our constraints on R_{0} can be rephrased as

$$0 \le R_0 \le 1 - P_0, \quad \operatorname{Tr} R_0 \left(\frac{Q_\lambda}{\operatorname{Tr} Q_\lambda} \otimes Z_\mu^{(\lambda)} \right) = \frac{1}{a} \delta_{\mu 0} \,\forall \,\lambda,\mu.$$
(12)

Notice that here, the semidefinite constraints on R_0 leave quite some room, whereas we have "only" LM^2 linear conditions to satisfy. Given s^* satisfying the constraint of A(K), our strategy now will be to show that we can construct a $0 \le R_0 \le \frac{2}{a}(1 - P_0)$ such that Eqs. (12) hold.

In detail, introduce a new variable $X \ge 0$, with

$$R_0 = \frac{1}{a}(1 - P_0)X(1 - P_0)$$

which makes sure that R_0 is automatically supported on the complement of P_0 . Now rewrite the conditions (12) in terms of X, introducing the notation

$$C_{\lambda\mu} = \frac{1}{\operatorname{Tr} Q_{\lambda}} Q_{\lambda} \otimes Z_{\mu}^{(\lambda)}, \quad D_{\lambda\mu} = (\mathbb{1} - P_0) C_{\lambda\mu} (\mathbb{1} - P_0).$$

This gives the new form of the constraints as

$$\operatorname{Tr} X D_{\lambda\mu} = \delta_{\mu0}.\tag{13}$$

Our goal will be to find a "nice" dual set $\{\widehat{D}_{\lambda\mu}\}$ to the $\{D_{\lambda\mu}\}$, i.e. Tr $D_{\lambda\mu}\widehat{D}_{\lambda'\mu'} = \delta_{\lambda\lambda'}\delta_{\mu\mu'}$, with which we can write a solution $X = \sum_{\lambda\mu} \delta_{\mu0}\widehat{D}_{\lambda\mu} = \sum_{\lambda} \widehat{D}_{\lambda0}$. To this end, we construct first the dual set $\widehat{C}_{\lambda\mu}$ of the $\{C_{\lambda\mu}\}$, which is easy:

$$\widehat{C}_{\lambda\mu} = Q_{\lambda} \otimes \widehat{Z}_{\mu}^{(\lambda)} = \begin{cases} Q_{\lambda} \otimes \Pi_{\lambda} & \text{ for } \mu = 0, \\ Q_{\lambda} \otimes Z_{\mu}^{(\lambda)} & \text{ for } \mu \neq 0, \end{cases}$$

so that indeed $\operatorname{Tr} C_{\lambda\mu} \widehat{C}_{\lambda'\mu'} = \delta_{\lambda\lambda'} \delta_{\mu\mu'}$. Now, consider the $LM^2 \times LM^2$ -matrix T,

$$T_{\lambda\mu,\lambda'\mu'} = \operatorname{Tr} D_{\lambda\mu} C_{\lambda'\mu'}$$

= $\operatorname{Tr}(\mathbf{1} - P_0) C_{\lambda\mu} (\mathbf{1} - P_0) \widehat{C}_{\lambda'\mu'}$
= $\delta_{\lambda\lambda'} \delta_{\mu\mu'} - \Delta_{\lambda\mu,\lambda'\mu'},$

with the deviation

$$\Delta_{\lambda\mu,\lambda'\mu'} = \operatorname{Tr} P_0 C_{\lambda\mu} (\mathbb{1} - P_0) \widehat{C}_{\lambda'\mu'} + \operatorname{Tr} C_{\lambda\mu} P_0 \widehat{C}_{\lambda'\mu'}.$$

Here,

$$\begin{aligned} |\Delta_{\lambda\mu,\lambda'\mu'}| &\leq 2 \|P_0 C_{\lambda\mu}\|_1 \|\widehat{C}_{\lambda'\mu'}\|_{\infty} \\ &\leq 2 \|P_0 C_{\lambda\mu}\|_1 = 2 \|P_0 |C_{\lambda\mu}|\|_1 \\ &\leq 2 \sqrt{\operatorname{Tr} P_0 |C_{\lambda\mu}|} \sqrt{\|C_{\lambda\mu}\|_1}, \end{aligned}$$

using $\|\widehat{C}_{\lambda'\mu'}\|_{\infty} \leq 1$, the unitary invariance of the trace norm, and Lemma 7 stated below. Since $|C_{\lambda\mu}| = \frac{1}{\operatorname{Tr} Q_{\lambda}} Q_{\lambda} \otimes |Z_{\mu}^{(\lambda)}|$ is invariant under the action of U^g , we have $\operatorname{Tr} P_0|C_{\lambda\mu}| = \operatorname{Tr} P_i|C_{\lambda\mu}|$ for all *i*, and using $\sum_i s^* P_i \leq 1$ we get

$$|\Delta_{\lambda\mu,\lambda'\mu'}| \le 2\sqrt{\frac{1}{s^*a}} \|C_{\lambda\mu}\|_1^2 \le 2\sqrt{M}(s^*a)^{-1/2}.$$
(14)

With this and introducing a new parameter β we get that

$$\|T - \mathbf{1}\|_{\infty} \le \|T - \mathbf{1}\|_{2} = \sqrt{\sum_{\lambda \mu \lambda' \mu'} |\Delta_{\lambda \mu, \lambda' \mu'}|^{2}} \le \sqrt{L^{2} M^{4} 4 M(s^{*}a)^{-1}} \le \frac{1}{\beta},$$
(15)

where $s^*a \ge 4\beta^2 L^2 M^5$. Assuming $\beta \ge 2$ (which will be the case with our later choice), we thus know that *T* is invertible; in fact, we have $T = 1 - \Delta$ with $\|\Delta\|_{\infty} \le \frac{1}{\beta} \le \frac{1}{2}$, hence $T^{-1} = \sum_{k=0}^{\infty} \Delta^k$ and so

$$\left\|T^{-1} - \mathbf{1}\right\|_{\infty} = \left\|\sum_{k=1}^{\infty} \Delta^k\right\|_{\infty} \le \sum_{k=1}^{\infty} \|\Delta\|_{\infty}^k \le \frac{1}{\beta - 1} \le \frac{2}{\beta}.$$

I.e., writing $T^{-1} = 1 + \widetilde{\Delta}_{\lambda\mu,\lambda'\mu'}$ we get

$$|\widetilde{\Delta}_{\lambda\mu,\lambda'\mu'}| \le \|\widetilde{\Delta}\|_{\infty} \le \frac{2}{\beta}.$$
(16)

The invertibility of *T* implies that there is a dual set to $\{D_{\lambda\mu}\}$ in span $\{\widehat{C}_{\lambda\mu}\}$. Indeed, from the definition of $T_{\lambda\mu,\lambda'\mu'}$ and the dual sets,

$$\widehat{C}_{\lambda'\mu'} = \sum_{\lambda\mu} T_{\lambda\mu,\lambda'\mu'} \widehat{D}_{\lambda\mu}, \text{ which can be rewritten as}$$
$$\widehat{D}_{\lambda\mu} = \sum_{\lambda'\mu'} (T^{-1})_{\lambda'\mu',\lambda\mu} \widehat{C}_{\lambda'\mu'}.$$

Now we can finally write down our candidate solution to Eq. (13):

$$\begin{split} X &= \sum_{\lambda\mu} \delta_{\mu0} \widehat{D}_{\lambda\mu} \\ &= \sum_{\lambda\mu} \delta_{\mu0} \sum_{\lambda'\mu'} (T^{-1})_{\lambda'\mu',\lambda\mu} \widehat{C}_{\lambda'\mu'} \\ &= \sum_{\lambda} \widehat{C}_{\lambda0} + \sum_{\lambda\lambda'\mu'} \widetilde{\Delta}_{\lambda'\mu',\lambda0} \widehat{C}_{\lambda'\mu'} \\ &= 1 + \text{Rest.} \end{split}$$

The rest term can be bounded as follows:

$$\|\operatorname{Rest}\|_{\infty} \leq \sum_{\lambda\lambda'\mu'} \frac{2}{\beta} = \frac{2}{\beta}L^2M^2$$

using Eq. (16). Thus we find $\|\text{Rest}\|_{\infty} \leq 1$ if $\beta \geq 2L^2M^2$ and $s^*a \geq 4\beta^2L^2M^5 \geq 16L^6M^9$. In this case, we will have $0 \leq X \leq 2$ and $R_0 := \frac{1}{a}(\mathbb{1} - P_0)X(\mathbb{1} - P_0)$ satisfies

$$0 \le R_0 \le \frac{2}{a}(\mathbf{1} - P_0) \le \mathbf{1} - P_0,$$

as well as

$$\frac{1}{|G|} \sum_{g \in G} (U^g)^{\dagger} R_0 U^g = \frac{1}{a} \mathbb{1}.$$

Thus we get the desired POVM $\left(R_i = \frac{a}{|G|} \sum_{g \text{ s.t. } 0^g = i} (U^g)^{\dagger} R_0 U^g\right)$ such that

$$\sum_{i} R_i = \mathbb{1}, \ R_i \ge 0, \ \operatorname{Tr} P_i R_i = 0,$$

and we are done.

Lemma 7 (Lemma 15 in [24]) Let ρ be a state and P a projection in a Hilbert space H. Then,

$$\operatorname{Tr} \rho P \le \|\rho P\|_1 \le \sqrt{\operatorname{Tr} \rho P}$$

More generally, for $X \ge 0$ *and a POVM element* $0 \le E \le 1$ *,*

$$\operatorname{Tr} XE \le \|XE\|_1 \le \sqrt{\operatorname{Tr} X}\sqrt{\operatorname{Tr} XE}.$$

We even recover the Pusey/Barrett/Rudolph result [41] as a corollary: There, $\mathcal{E} = \{|\psi_0\rangle, |\psi_1\rangle\}$ with (w.l.o.g.) $|\psi_{0,1}\rangle = \alpha|0\rangle \pm \beta|1\rangle$ qubit states, $1 > \alpha \ge \beta > 0$. We have the unitary phase action of $\mathbb{Z}_2 = \{\mathbb{1}, Z\}, Z|\psi_{0,1}\rangle = |\psi_{1,0}\rangle$, and hence on $\mathcal{E}^{\otimes n}$ we have a transitive action of $G = \mathbb{Z}_2^n \rtimes S_n$ (the semidirect product), the symmetric group S_n acting by permutation of the tensor factors and \mathbb{Z}_2^n as $\bigotimes_{t=1}^n Z^{b_t}$. It has L = n and $M \le n + 1$ [31], whereas

$$\frac{a}{\left\|\sum_{\underline{i}} P_{\underline{i}}\right\|_{\infty}} = \frac{2^n}{(2\alpha^2)^n} = \frac{1}{\alpha^{2n}}.$$

Hence, for large enough n, we have that the latter exceeds $16L^6M^9 = \text{poly}(n)$, and then Proposition 5 above implies that $\mathcal{E}^{\otimes n}$ can be conclusively excluded.

D. General case

We shall reduce the case of a general channel to that of a cq-channel. Indeed, recall that we allow Alice and Bob to share entanglement, so Alice can encode information into the Bell states

$$|\Phi_{uv}\rangle = (\mathbb{1} \otimes Z^{v} X^{u})|\Phi\rangle = (X^{u} Z^{v} \otimes \mathbb{1})|\Phi\rangle,$$

with the maximally entangled state $|\Phi\rangle = \frac{1}{\sqrt{|A|}} \sum_{i=1}^{|A|} |i\rangle |i\rangle$ and the discrete Weyl operators X and Z (basis and phase shift). This effectively constructs a cq-channel (with a = |A|)

$$\mathcal{M}: [a]^2 \ni uv \mapsto (\mathrm{id} \otimes \mathcal{N}) |\Phi_{uv}\rangle \langle \Phi_{uv}| = (X^u Z^v \otimes \mathbb{1}) \rho_{00}(Z^{-v} X^{-u} \otimes \mathbb{1}) \in \mathcal{S}(AB),$$
(17)

with the Choi-Jamiołkowski state $\rho_{00} = (id \otimes \mathcal{N}) |\Phi\rangle\langle\Phi|$. Applying Theorem 6 to this channel is the key to obtain the following result, which in turn directly implies the reverse direction ("nontrivial \Rightarrow positive capacity") in Theorem 3, concluding its proof.

Proposition 8 A non-commutative bipartite graph K with support projection P_{AB} onto the Choi-

Jamiołkowski range $(\mathbb{1} \otimes K) |\Phi\rangle$ *has positive activated feedback assisted zero-error capacity,* $\overline{C}_{0EF}(K) > 0$, *if and only if one of the following equivalent conditions hold:*

- *i. K* is non-trivial, i.e. there is no constant channel \mathcal{N}_0 with $\mathcal{K}(\mathcal{N}_0) < K$;
- *ii.* There is no state $|\beta\rangle \in B$ with $|\beta\rangle \otimes A^{\dagger} < K$;
- *iii.* $||P_B||_{\infty} < |A|$;
- iv. $\operatorname{Tr}_A(\mathbb{1} P_{AB})$ has full rank;
- *v.* A(K) > 1.

Proof $\overline{C}_{0EF}(K) > 0 \Rightarrow i$. has been shown in the first part (necessity) of Theorem 3, at the start of this section, likewise *i*. \Leftrightarrow *ii*..

ii. \Leftrightarrow *iii.* $P_{AB} \leq \mathbb{1}_A \otimes \mathbb{1}_B$, hence $P_B \leq |A|\mathbb{1}_B$, i.e. $||P_B||_{\infty} \leq |A|$. Equality is attained if and only if there exists an eigenvector $|\beta\rangle$ of P_B with eigenvalue |A|, which is equivalent to $|A| = \text{Tr} |\beta\rangle\langle\beta|P_B = \text{Tr}(\mathbb{1}_A \otimes |\beta\rangle\langle\beta|)P_{AB}$. But since $\mathbb{1}_A \otimes |\beta\rangle\langle\beta|$ has trace |A| and P_{AB} is a projector, this is equivalent to $\mathbb{1}_A \otimes |\beta\rangle\langle\beta| \leq P_{AB}$, or again equivalently $|\beta\rangle \otimes A^{\dagger} < K$.

 $\begin{aligned} &iii. \Leftrightarrow iv. \|P_B\|_{\infty} < |A| \text{ if and only if } P_B = \operatorname{Tr}_A P_{AB} < |A| \mathbb{1}_B, \text{ if and only if } \operatorname{Tr}_A(\mathbb{1} - P_{AB}) > 0. \\ &iii. \Rightarrow v. \text{ Simply observe that } S = \frac{1}{\|P_B\|_{\infty}} \mathbb{1}_A \text{ is feasible for } \mathsf{A}(K), \text{ since } \operatorname{Tr}_A(S \otimes \mathbb{1})P_AB = \frac{1}{\|P_B\|_{\infty}} P_B \leq \mathbb{1}, \text{ hence } \mathsf{A}(K) \geq \operatorname{Tr} S = \frac{|A|}{\|P_B\|_{\infty}} > 1. \end{aligned}$

 $v. \Rightarrow ii.$ We show the contrapositive: If $|\beta\rangle \otimes A^{\dagger} < K$, then $\mathbb{1} \otimes |\beta\rangle\langle\beta| \leq P_{AB}$. Now, if S is feasible for A(K), we have $\mathbb{1}_B \geq \text{Tr}_A(S \otimes \mathbb{1})P_{AB} \geq \text{Tr}_A(S \otimes \mathbb{1})(\mathbb{1} \otimes |\beta\rangle\langle\beta|) = (\text{Tr} S)|\beta\rangle\langle\beta|$, hence $\text{Tr} S \leq 1$, and so A(K) = 1.

iii. $\Rightarrow \overline{C}_{0EF}(K) > 0$. Consider the cq-channel \mathcal{M} in eq. (17). It has output state support projectors

$$P_{uv} = (X^u Z^v \otimes \mathbb{1}) P_{AB}(Z^{-v} X^{-u} \otimes \mathbb{1}), \quad u, v = 1, \dots, a,$$

and we can verify directly that $\sum_{uv} P_{uv} = |A| \mathbf{1}_A \otimes P_B$, so its norm satisfies

$$\left\| \sum_{u,v} P_{uv} \right\|_{\infty} = |A| \, \|P_B\|_{\infty} < |A|^2.$$

In other words, it satisfies the requirements of item iv) in Theorem 6, hence $\overline{C}_{0EF}(K) \geq \overline{C}_{0EF}(\mathcal{M}) > 0$.

III. SHANNON THEORETIC UPPER BOUND ON $\overline{\mathbf{C}}_{0\mathbf{EF}}(\mathbf{K})$

In this section we will develop an upper bound on the feedback-assisted zero-error capacity via information theoretic ideas. For this purpose we first review the classical case, due to Shannon.

A. Shannon theoretic characterization of the fractional packing number: Shannon's Conjecture

The following characterization of the feedback-assisted zero-error capacity of a classical channel was conjectured by Shannon at the end of his seminal paper [45], and to our knowledge proved first by Ahlswede [2], in the context of his treatment of the capacity of arbitrarily varying (classical) channels with instantaneous feedback, and using his very general results in that theory. Our proof seems more direct, but then it is specially geared towards the zero-error setting.

Proposition 9 For a bipartite graph Γ on $\mathcal{X} \times \mathcal{Y}$ such that every $x \in \mathcal{X}$ is adjacent to at least one $y \in \mathcal{Y}$,

$$\log \alpha^*(\Gamma) = C_{\min}(\Gamma) := \min\{C(N) : \Gamma(N) \subseteq \Gamma\}$$

where C(N) is the usual Shannon capacity of a noisy classical channel [44].

Proof The left hand side is the zero-error capacity of Γ , assisted by feedback (plus some finite amount of communication), $\overline{C}_{0F}(\Gamma)$ [45]. From this, and the fact that feedback does not increase the Shannon capacity of a channel [45] (which may also be proved invoking the Reverse Shannon Theorem [6]), it follows that $C(N) \ge \log \alpha^*(\Gamma)$ for any eligible N, hence $C_{\min}(\Gamma) \ge \log \alpha^*(\Gamma)$.

There is also a direct proof of this that avoids operational arguments, relying instead only on elementary combinatorial notions. It goes via showing that for every eligible channel N and input probability distribution p,

$$V(p) := \log \min_{y} \frac{1}{\sum_{x} \Gamma(y|x) p_x} \le I(X:Y), \tag{18}$$

which is enough because $\max_p V(p) = \log \alpha^*(\Gamma)$, while of course the maximum of I(X : Y) equals C(N). Now, eq. (18) is easily seen to be true for uniform distribution $p_x = \frac{1}{|\mathcal{X}|}$. Namely, with the equivocation sets $\mathcal{E}_y = \{x : \Gamma(y|x) = 1\}$ and the output probability distribution $q_y = \sum_x p_x N(y|x)$:

$$V(p) = \log |\mathcal{X}| - \max_{y} \log |\mathcal{E}_{y}|$$

$$\leq \log |\mathcal{X}| - \sum_{y} q_{y} \log |\mathcal{E}_{y}|$$

$$\leq \log |\mathcal{X}| - \sum_{y} q_{y} H(X|Y = y)$$

$$= H(X) - H(X|Y) = I(X : Y)$$

where we have used the fact that $P_{X|Y=y}$ is supported on \mathcal{E}_y , and the uniformity of the distribution of *X*. For non-uniform *p*, we use the method of types [17] to reduce to the uniform case. In detail, consider the product distribution $p^{\otimes n}$ and $X^n \sim p^{\otimes n}$ as input to the i.i.d. channel $N^{\otimes n}$. Introducing the type $T = T(X^n)$ of the string X^n , we have:

$$nI(X:Y) = I(X^n:Y^n) = I(TX^n:Y^n) = I(T:Y^n) + I(X^n:Y^n|T).$$

On the other hand, for every type τ ,

$$2^{-nV(p)} = 2^{-V(p^{\otimes n})}$$

= $\max_{y^n} \sum_{x^n} \Gamma(y^n | x^n) p_{x^n}$
 $\geq \max_{y^n} \sum_{x^n \in \tau} \Gamma(y^n | x^n) p_{x^n}$
= $p^{\otimes n}(\tau) \max_{y^n} \sum_{x^n \in \tau} \frac{1}{|\tau|} \Gamma(y^n | x^n),$

since conditioned on $T(X^n) = \tau$, $X^n \sim u_{\tau}$ is uniformly distributed. Hence, using the uniform case of the inequality (18),

$$nV(P) \le \log \frac{1}{p^{\otimes n}(\tau)} + V(u_{\tau}) \le \log \frac{1}{p^{\otimes n}(\tau)} + I(X^n : Y^n | T = \tau),$$

and averaging over the different types this gives

$$nV(P) \le H(T) + I(X^n : Y^n | T) \le O(\log n) + nI(X : Y),$$

because there are only poly(n) many different types, and letting $n \to \infty$ we are done.

So it remains only to show the opposite inequality. The proof uses the primal and dual linear programming [15] (LP) characterisations of $\alpha^*(\Gamma)$ to construct an optimal channel N(y|x), and in fact also an optimal input distribution p_x , such that $C(N) = I(X : Y) = \log \alpha^*(\Gamma)$.

Recall the fractional packing number, eq. (4), and choose optimal primal and dual solutions. Define an input distribution $p_x := \frac{w_x}{\alpha^*(\Gamma)}$. This is the one that appears in Shannon's [45, Thm. 7], and his $\frac{1}{P_0}$ is the same as $\alpha^*(\Gamma)$. Now, by complementary slackness [15], if $M_x := \sum_y \Gamma(y|x)v_y > 1$, then $w_x = p_x = 0$; per contrapositive, if $p_x > 0$, then $M_x = \sum_y \Gamma(y|x)v_y = 1$. Hence, we can define, for these latter x,

$$N(y|x) := \Gamma(y|x)v_y,$$

and in general for all x,

$$N(y|x) := \frac{1}{M_x} \Gamma(y|x) v_y.$$

This is our candidate channel, and we have to convince ourselves that indeed $C(N) = \log \alpha^*(\Gamma)$. First of all, let's confirm that with the above distribution p, the mutual information I(X : Y) equals $\log \alpha^*(\Gamma)$. Let $D(p||q) = \sum_x p(x) \log \frac{p(x)}{q(x)}$ be the relative entropy between two probability distributions $\{p_x\}$ and $\{q_x\}$, cf. [14]. Recall $I(X : Y) = \sum_x p_x D(N(\cdot|x)||q)$, with the output distribution

$$q_y = \sum_x p_x N(y|x) = \sum_x p_x \Gamma(y|x) v_y = \frac{v_y}{\alpha^*(\Gamma)},$$

using once more complementary slackness: the equality is trivial if $v_y = 0$, and if $v_y > 0$ then $\sum_x \Gamma(y|x)w_x = 1$. In the present case, we calculate for all x,

$$D(N(\cdot|x)||q) = \sum_{y} \frac{\Gamma(y|x)v_y}{M_x} \log \frac{\Gamma(y|x)\frac{v_y}{M_x}}{\frac{v_y}{\alpha^*(\Gamma)}} = \log \frac{\alpha^*(\Gamma)}{M_x},$$

which is $\log \alpha^*(\Gamma)$ for all $p_x > 0$ as then $M_x = 1$. So indeed $I(X : Y) = \log \alpha^*(\Gamma)$. But we see even more: While all the relative entropies $D(N(\cdot|x)||q)$ with $p_x > 0$ are equal to $\log \alpha^*(\Gamma)$, for $p_x = 0$ instead,

$$D(N(\cdot|x)||q) = \log \frac{\alpha^*(\Gamma)}{M_x} \le \log \alpha^*(\Gamma),$$

because $M_x \ge 1$. These two conditions (for $p_x > 0$ and $p_x = 0$) are well known, classic characterizations of the Shannon capacity (cf. [16, 43]); they characterize an optimal input distribution for given channel N, so indeed we prove $C(N) = \log \alpha^*(\Gamma)$.

Remark. Note that neither is C_{\min} altered by allowing the use of entanglement as well as feedback [6], nor C_{0F} by allowing the use of entanglement and other no-signalling correlations [20].

B. Quantum generalization of the Shannon bound

Recall that for a channel $\mathcal{N} : \mathcal{S}(A) \longrightarrow \mathcal{S}(B)$, the entanglement-assisted classical capacity [6], i.e. the maximum rate of asymptotically error-free communication via many uses of the channel assisted by a suitable pre-shared entangled state, is given by

$$C_E(\mathcal{N}) = \max_{\rho} I(A:B)_{\sigma} = \max_{\rho} \left\{ S(\rho) + S(\mathcal{N}(\rho)) - S((\mathrm{id} \otimes \mathcal{N})\phi) \right\}$$

where $\sigma_{AB} = (id \otimes \mathcal{N})\phi_{AA'}$ is the joint input-output state, $\phi_{AA'}$ is a purification of ρ , and $I(A : B) = S(\sigma_A) + S(\sigma_B) - S(\sigma_{AB})$ is the quantum mutual information. In the particular case above, we also write it $I(\rho; \mathcal{N}) = S(\rho) + S(\mathcal{N}(\rho)) - S((id \otimes \mathcal{N})\phi)$.

Using this, we define for a non-commutative bipartite graph $K < \mathcal{L}(A \rightarrow B)$ such that $\mathbb{1} \in K^{\dagger}K$ (these are precisely the possible Kraus subspaces of channels):

$$C_{\min E}(K) := \min\{C_E(\mathcal{N}) : \mathcal{K}(\mathcal{N}) < K\}.$$

That this is indeed a minimum follows from continuity of C_E and the fact that the eligible channels form a compact convex set. This definition is of course motivated by Proposition 9, suggesting $2^{C_{\min E}(K)}$ as a possible quantum generalisation of the fractional packing number. For one thing, for the quantum realisation K of a classical equivocation graph Γ , it is easy to see that indeed $C_{\min E}(K) = C_{\min}(\Gamma) = \log \alpha^*(\Gamma)$, see the remark at the end of the preceeding Subsection III A.

At least, this quantity is related to the feedback-assisted zero-error capacity: Indeed, the result of Bowen [10] (alternatively the Quantum Reverse Shannon Theorem [5, 7]) tells us that $C_E(\mathcal{N})$ is not increased even by allowing feedback, so that $C_{0EF}(K)$ (and actually even $\overline{C}_{0EF}(K)$) is upper bounded by the entanglement-assisted capacity $C_E(\mathcal{N})$ for any channel \mathcal{N} such that $\mathcal{K}(\mathcal{N}) < K$, hence

Theorem 10 $\overline{C}_{0EF}(K) \leq C_{\min E}(K)$ for any non-commutative bipartite graph $K < \mathcal{L}(A \rightarrow B)$. \Box

 $C_{\min E}(K)$ shares many properties with $C_{\min}(\Gamma)$, to which it reduces for classical channels. First, $C_{\min E}(K)$ is given by a minimax formula (min over channels and max over quantum mutual information – see below) to which the minimax theorem applies, so it is also given by a maximin (Lemma 11 below). Second, using this characterisation and properties of the von Neumann entropy, it can be shown that $C_{\min E}$ is additive (Lemma 12 below). Third, thanks to the operational definition of C_E , it can be easily seen to be monotonic under pre- and post-processing (Lemma 13 below).

We shall need some well-known mathematical properties of the quantum mutual information. The first is that $I(\rho; \mathcal{N})$ is concave in ρ and convex in \mathcal{N} , just like its classical counterpart [1, 6]. The convexity in \mathcal{N} follows from strong subadditivity: Let

$$\sigma^{AB} = \left(\mathrm{id} \otimes (\lambda \mathcal{N}^{(1)} + (1 - \lambda) \mathcal{N}^{(2)}) \right) \phi_{\rho}$$
$$= \lambda \sigma_{AB}^{(1)} + (1 - \lambda) \sigma_{AB}^{(2)}$$
$$= \mathrm{Tr}_{B'} \widetilde{\sigma}_{ABB'},$$

with $\widetilde{\sigma}_{ABB'} = \lambda \sigma_{AB}^{(1)} \otimes |1\rangle \langle 1|_{B'} + (1-\lambda) \sigma_{AB}^{(2)} \otimes |2\rangle \langle 2|_{B'}$. Then,

$$I(\rho; \lambda \mathcal{N}^{(1)} + (1-\lambda)\mathcal{N}^{(2)}) = I(A:B)_{\sigma}$$

$$\leq I(A:BB')_{\widetilde{\sigma}}$$

$$= \lambda I(A:B)_{\sigma^{(1)}} + (1-\lambda)I(A:B)_{\sigma^{(2)}}$$

$$= \lambda I(\rho; \mathcal{N}^{(1)}) + (1-\lambda)I(\rho; \mathcal{N}^{(2)}).$$

The concavity in ρ can be seen as follows, using strong subadditivity again: For states $\rho^{(1)}$, $\rho^{(2)}$ with purifications $\phi^{(1)}$, $\phi^{(2)}$, respectively, and $0 \le \lambda \le 1$, we construct a purification of the mixture $\lambda \rho^{(1)} + (1 - \lambda)\rho^{(2)}$, as follows:

$$|\phi\rangle = \sqrt{\lambda} |\phi^{(1)}\rangle |11\rangle_{A'A''} + \sqrt{1-\lambda} |\phi^{(2)}\rangle |22\rangle_{A'A''}.$$

With $\sigma_{AA'A''B} = (\mathrm{id}_{AA'A''} \otimes \mathcal{N})\phi$, we have

$$I(\lambda \rho^{(1)} + (1 - \lambda)\rho^{(2)}; \mathcal{N}) = I(AA'A'' : B)_{\sigma}$$

$$\geq I(AA' : B)_{\sigma}$$

$$\geq I(A : B|A')_{\sigma}$$

$$= \lambda I(\rho^{(1)}; \mathcal{N}) + (1 - \lambda)I(\rho^{(2)}; \mathcal{N})$$

Lemma 11 For any non-commutative bipartite graph $K < \mathcal{L}(A \rightarrow B)$,

$$C_{\min E}(K) = \min_{\substack{\mathcal{N} \ s.t.\\ \mathcal{K}(\mathcal{N}) < K}} \max_{\rho} I(\rho; \mathcal{N})$$
$$= \max_{\substack{\rho \\ \mathcal{K}(\mathcal{N}) < K}} \min_{\substack{\mathcal{N} \ s.t.\\ \mathcal{K}(\mathcal{N}) < K}} I(\rho; \mathcal{N}).$$

Proof The first equation is the definition of $C_{\min E}(K)$, with the formula for $C_E(\mathcal{N})$ inserted. Above we saw that the argument $I(\rho; \mathcal{N})$ is concave in the first and convex in the second argument. Hence von Neumann's minimax theorem, or rather its generalisation due to Sion [47] applies, allowing us to interchange the order of min and max.

Lemma 12 For non-commutative bipartite graphs $K_1 < \mathcal{L}(A_1 \rightarrow B_1)$ and $K_2 < \mathcal{L}(A_2 \rightarrow B_2)$,

$$C_{\min E}(K_1 \otimes K_2) = C_{\min E}(K_1) + C_{\min E}(K_2).$$

Proof We show this by separately demonstrating " \leq " and " \geq " in the above relation, using the two expressions for $C_{\min E}$ from Lemma 11. In the following, choose optimal states ρ_1 , ρ_2 and channels \mathcal{N}_1 , \mathcal{N}_2 for K_1 , K_2 , respectively.

" \leq ": By the first expression in Lemma 11,

$$C_{\min E}(K_1 \otimes K_2) \le \max_{\rho} I(\rho; \mathcal{N}_1 \otimes \mathcal{N}_2)$$

= $C_E(\mathcal{N}_1 \otimes \mathcal{N}_2)$
= $C_E(\mathcal{N}_1) + C_E(\mathcal{N}_2)$
= $C_{\min E}(K_1) + C_{\min E}(K_2),$

using the fact that the entanglement-assisted capacity is additive, proved by Adami and Cerf in [1]. Note that $\mathcal{K}(\mathcal{N}_1 \otimes \mathcal{N}_2) = \mathcal{K}(\mathcal{N}_1) \otimes \mathcal{K}(\mathcal{N}_2) < K_1 \otimes K_2$.

" \geq ": By the second expression in Lemma 11,

$$C_{\min E}(K_1 \otimes K_2) \ge \min_{\substack{\mathcal{N} \text{ s.t.}\\\mathcal{K}(\mathcal{N}) < K_1 \otimes K_2}} I(\rho_1 \otimes \rho_2; \mathcal{N}),$$

and we need only to show that the minimum is attained at a product channel $\mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2$ with $\mathcal{K}(\mathcal{N}_i) < K_i$. For this purpose, consider the state

$$\sigma_{A_1A_2B_1B_2} = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{A_2} \otimes \mathcal{N})(\phi_1 \otimes \phi_2),$$

for the purifications ϕ_i of ρ_i (i = 1, 2). Now observe that with respect to σ ,

$$\begin{split} I(A_1A_2:B_1B_2) - I(A_1:B_1) - I(A_2:B_2) &= S(A_1A_2) + S(B_1B_2) - S(A_1A_2B_1B_2) \\ &\quad - S(A_1) - S(B_1) + S(A_1B_1) \\ &\quad - S(A_2) - S(B_2) + S(A_2B_2) \\ &= I(A_1B_1:A_2B_2) - I(B_1:B_2) - I(A_1:A_2) \ge 0, \end{split}$$

because $I(A_1 : A_2) = 0$ and by strong subadditivity. In other words,

$$I(A_1A_2: B_1B_2)_{\sigma} \ge I(A_1: B_1)_{\sigma_1} + I(A_2: B_2)_{\sigma_2}$$

= $I(A_1A_2: B_1B_2)_{\sigma_1 \otimes \sigma_2},$

with the reduced states

$$\sigma_1 = \sigma_{A_1B_1} = \operatorname{Tr}_{A_2B_2} \sigma = (\operatorname{id}_{A_1} \otimes (\operatorname{Tr}_{B_2} \circ \mathcal{N}))(\phi_1 \otimes \rho_2),$$

$$\sigma_2 = \sigma_{A_2B_2} = \operatorname{Tr}_{A_1B_1} \sigma = (\operatorname{id}_{A_2} \otimes (\operatorname{Tr}_{B_1} \circ \mathcal{N}))(\rho_1 \otimes \phi_2).$$

I.e.,

$$I(\rho_1 \otimes \rho_2; \mathcal{N}) \ge I(\rho_1; \operatorname{Tr}_{B_2} \circ \mathcal{N}(\cdot \otimes \rho_2)) + I(\rho_2; \operatorname{Tr}_{B_1} \circ \mathcal{N}(\rho_1 \otimes \cdot)).$$

Finally, $\operatorname{Tr}_{B_2} \circ \mathcal{N}(\cdot \otimes \rho_2)$ is eligible: If \mathcal{N} has Kraus operators $E_i \in K_1 \otimes K_2 < \mathcal{L}(A_1A_2 \rightarrow B_1B_2)$, and choosing an eigenbasis of ρ_2 and an arbitrary basis of B_2 ,

$$\mathcal{K}(\mathrm{Tr}_{B_2} \circ \mathcal{N}(\cdot \otimes \rho_2)) = \mathrm{span}\left\{ \langle j |_{B_2} E_i | k \rangle_{A_2} : i, j, k \right\} < K_1.$$

 $\mathcal{K}(\operatorname{Tr}_{B_1} \circ \mathcal{N}(\rho_1 \otimes \cdot)) < K_2$ is analogous, and we are done.

Lemma 13 All of C_{0EF} , \overline{C}_{0EF} and $C_{\min E}$ are monotonic under pre- and post-processing of the channel: For non-commutative bipartite graphs $K < \mathcal{L}(A \rightarrow B)$ and $K_A < \mathcal{L}(U \rightarrow A)$, $K_B < \mathcal{L}(B \rightarrow V)$, the matrix-multiplied space $K_B K K_A < \mathcal{L}(U \rightarrow V)$ is a non-commutative bipartite graph, and

$$C_{0EF}(K) \ge C_{0EF}(K_BKK_A),$$

$$\overline{C}_{0EF}(K) \ge \overline{C}_{0EF}(K_BKK_A),$$

$$C_{\min E}(K) \ge C_{\min E}(K_BKK_A).$$

Proof For C_{0EF} and \overline{C}_{0EF} this follows directly from the operational definition: the pre- and postprocessings may be absorbed into the input modulation and feedback-decoding, respectively, showing that a zero-error code for $K_B K K_A$ yields one for K.

For $C_{\min E}$, the argument is similar using the fact that $C_E(\mathcal{N})$ is the operational entanglementassisted capacity of the channel \mathcal{N} [6].

We can now give yet another characterization of the feasibility of $\overline{C}_{0EF}(K) > 0$, adding to the list of Theorem 3 and Proposition 8.

Theorem 14 For any non-commutative bipartite graph K, $\overline{C}_{0EF}(K) > 0$ if and only if $C_{\min E}(K) > 0$.

Proof The only way in which $C_{\min E}(K)$ can be 0 is that there is a channel \mathcal{N} with $\mathcal{K}(\mathcal{N}) < K$ and $C_E(\mathcal{N}) = 0$, i.e. \mathcal{N} has to be constant. We have seen that this is equivalent to $|\beta\rangle \otimes A < K$ for a state vector $|\beta\rangle \in B$. But by Theorem 3 this is precisely the characterization of $\overline{C}_{0EF}(K)$ being 0.

To illustrate the bound of Theorem 10, we consider the example of Weyl diagonal channels and the dependence on the output state geometry for cq-channels.

Weyl diagonal channels. Denoting by *X* and *Z* the discrete translation and phase shift (which generate a subgroup of the unitary group of cardinality d^3 , thanks to the commutation relation $XZ = \omega ZX$, $\omega = e^{2\pi i/d}$), consider the channel

$$\mathcal{N}(\rho) = \sum_{a,b=0}^{d-1} p_{ab} X^a Z^b \rho Z^{-b} X^{-a},$$

with probabilities $p_{ab} \ge 0$ summing to 1. Clearly,

$$\mathcal{K}(\mathcal{N}) = \operatorname{span}\{W_{ab} := X^a Z^b : p_{ab} > 0\}$$

i.e. this *K* is characterised by a subset $S \subset \mathbb{Z}_d \times \mathbb{Z}_d$. It supports precisely those Weyl diagonal channels \mathcal{N} with $p_{ab} = 0$ for $ab \notin S$ – and of course many channels that are not Weyl diagonal.

First, note that \mathcal{N} above is Weyl-covariant:

$$\mathcal{N}(W_{ab}\rho W_{ab}^{\dagger}) = W_{ab}\mathcal{N}(\rho)W_{ab}^{\dagger}$$

for all *ab*. From this, and the irreducibility of the action of the Weyl operators on \mathbb{C}^d , it follows that

$$C_E(\mathcal{N}) = I\left(\frac{1}{d}\mathbf{1}; \mathcal{N}\right) = 2\log d - H(\vec{p}),$$

where $\vec{p} = (p_{ab} : a, b = 0, \dots, d-1)$ is the probability vector. This means that for a *k*-element $S \subset \mathbb{Z}_d \times \mathbb{Z}_d$ and $K = \operatorname{span}\{W_{ab} : ab \in S\}$,

$$\min_{\substack{\mathcal{N} \text{ Weyl-diag.} \\ \mathcal{K}(\mathcal{N}) < K}} C_E(\mathcal{N}) = 2\log d - \log k,\tag{19}$$

the minimum being attained at the uniform distribution on S: $p_{ab} = \frac{1}{k}$ for $ab \in S$, and 0 otherwise.

We will now show that $2 \log d - \log k$ is an achievable rate of zero-error communication via this channel when assisted by feedback (plus a constant activating amount of noiseless communication). The key is the observation that if we use

$$\mathcal{N}_0(\rho) = \frac{1}{k} \sum_{ab \in \mathcal{S}} W_{ab} \rho W_{ab}^{\dagger}$$

with dense coding, i.e. with a maximally entangled state $|\Phi_d\rangle$ and sender modulation by the very Weyl operators W_{ab} , the receiver making a Bell measurement in the basis $(W_{ab} \otimes \mathbb{1})|\Phi_d\rangle$, we obtain a generalised typewriter channel

$$T: \mathbb{Z}_d \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_d \times \mathbb{Z}_d,$$
$$T(ab|cd) = \begin{cases} \frac{1}{k} & \text{if } (a-c, b-d) \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

(And choosing a different \mathcal{N} supported by K changes only the non-zero transition probabilities.) T is easily seen to have fractional packing number d^2/k , so its activated feedback-assisted zeroerror capacity is $2 \log d - \log k$. Hence $\overline{C}_{0EF}(K) \ge 2 \log d - \log k$, and together with eq. (19), we conclude

$$\overline{C}_{0EF}(K) = C_{\min E}(K) = \overline{C}_{0F}(T) = 2\log d - \log k.$$

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Finally, this is also the minimal zero-error communication cost to simulate a channel supported by *K* (using entanglement and shared randomness), making use of an idea in [6]: By the results of [20], one can simulate *T* with free shared randomness at communication rate $2 \log d - \log k$. Now, if in the teleportation protocol using a maximally entangled state and the Weyl unitaries W_{ab} , we replace the noiseless channel of d^2 messages by this *T*, one simulates exactly \mathcal{N}_0 .

Nontrivial dependence of \overline{C}_{0EF} on the channel geometry. Consider a non-commutative bipartite graph corresponding to a pure state cq-channel, $K = \text{span}\{|\psi_i\rangle\langle i|\}$. We can see that $\overline{C}_{0EF}(K)$ depends nontrivially on the geometry of the vector arrangement of the $|\psi_i\rangle$, even if they are all pairwise non-orthogonal: Indeed, when they are close to parallel, $\overline{C}_{0EF}(K)$ is arbitrarily close to 0, but when they are sufficiently close to being mutually orthogonal, $\overline{C}_{0EF}(K)$ is arbitrarily close to $\log |A|$.

Clearly, the closer to being parallel the $|\psi_i\rangle$ are, the larger the required *n* in the argument in Subsection II A becomes, so the lower bound moves closer to 0. On the other hand, this is really necessary, since

$$C_{\min E}(K) = \max_{(p_i)} S\left(\sum_i p_i |\psi_i\rangle\!\langle\psi_i|\right)$$

converges to 0 as the $|\psi_i\rangle$ get closer to being collinear.

In the other extreme, to show that $C_{0EF}(K) \rightarrow \log |A|$ when $C_{\min E}(K) \rightarrow \log |A|$, i.e. when the ψ_i become closer and closer to being orthogonal, we use once more the ideas from Subsection II A: Assume that for all $i \neq j$, $|\langle \psi_i | \psi_j \rangle| \leq \epsilon$, which is a more convenient expression for $C_{\min E}(K) \geq \log |A| - \delta$.

We claim that if ϵ is small enough, we can use K to simulate a "random superset channel" (cf. [20]): for integers t < a = |A| define the classical channel $S_{1,t}^a : [a] \to {[a] \choose t}$ such that

$$S_{1,t}^a: [a] \ni i \longmapsto J \in {[a] \choose t}$$
 randomly with $i \in J$,

where $\binom{[a]}{t} = \{J : J \subseteq [a], |J| = t\}$, the collection of all subsets of [a] with t elements. Note that the transition probability matrix of $S_{1,t}^a$ is given by $\{p(J|i)\}$ such that

$$p(J|i) = {\binom{a-1}{t-1}}^{-1}, \ i \in [a], \ J \in {\binom{[a]}{t}}.$$

Indeed, we use the characterization of [11], which will show that there is a deterministic transformation of the set $\{|\psi_i\rangle\}$ to the set $\{|\varphi_i\rangle\}$, with

$$|\varphi_i\rangle = \frac{1}{\sqrt{\binom{a-1}{t-1}}} \sum_{i \in J \in \binom{[a]}{t}} |J\rangle \in \mathbb{C}^{\binom{a}{t}}.$$

Once this is achieved, Bob measures the states $|\varphi_i\rangle$ in the computational basis, resulting in an output of the channel $S_{1,t}^a$. To see this in detail, let us focus on the smallest possible case t = 2, for which we see that for $i \neq j$, $\langle \varphi_i | \varphi_j \rangle = \frac{1}{a-1}$. The necessary and sufficient condition required in [11] for the existence of a cptp map transforming $\{|\psi_i\rangle\}$ into $\{|\varphi_i\rangle\}$ is that there exists a positive semidefinite $a \times a$ -matrix M such that $\Psi = \Phi \circ M$, where $\Psi = [\langle \psi_i | \psi_j \rangle]$ and $\Phi = [\langle \varphi_i | \varphi_j \rangle]$ are the Gram matrices of the two input/output state sets, and \circ denotes the elementwise (Hadamard/Schur) product. In other words,

$$M = \Psi \circ \Phi^{\circ -1} \ge 0$$
, i.e. $(a-1)\Psi \ge (a-2)\mathbb{1}$.

However, all eigenvalues of Ψ are lower bounded by $1 - (a - 1)\epsilon$, which is $\geq \frac{a-2}{a-1}$ as soon as $\epsilon \leq \frac{1}{(a-1)^2}$. In this case, we find $\overline{C}_{0EF}(K) \geq \overline{C}_{0F}(S_{1,2}^a) = \log a - 1$. Applying the same to multiple copies of the channel, this reasoning shows that if $\epsilon \leq (|A| - 1)^{-2n}$, then $\overline{C}_{0EF}(K) \geq \log a - \frac{1}{n}$. \Box

We do not know whether in general \overline{C}_{0EF} equals $C_{\min E}$ or not. However, we can show that the latter is a genuine capacity, as per the following theorem, whose proof however we relegate to Appendix A because it would detract from our principal, zero-error argument.

Theorem 15 For any non-commutative bipartite graph K, the adversarial entanglement-assisted classical capacity of K is given by $C_{*E}(K) = C_{\min E}(K)$.

The definition of this capacity is as follows: An entanglement-assisted *n*-block code consists of an entangled state (w.l.o.g. pure) $|\phi\rangle^{A_0B_0}$, *N* modulation cptp maps $\mathcal{E}_i : \mathcal{L}(A_0) \to \mathcal{L}(A^n)$ (m = 1, ..., N), and a POVM $(D_i)_{i=1}^N$ on B_0B^n . The code is said to have error ϵ for $K^{\otimes n}$ if the (average) error probability,

$$P_{\rm err}\big(\mathcal{N}^{(n)}\big) = \frac{1}{N} \sum_{i=1}^{N} \Big(1 - {\rm Tr}\big((\mathcal{N}^{(n)} \circ \mathcal{E}_i \otimes {\rm id})\phi\big)D_i\Big),$$

is $\leq \epsilon$ for every channel $\mathcal{N}^{(n)}$ with $\mathcal{K}(\mathcal{N}^{(n)}) < K^{\otimes n}$. In this case, we call the collection $(\phi; \mathcal{E}_i, D_i)$ an (n, ϵ) -code for $K^{\otimes n}$. Denoting the largest number N of messages of an (n, ϵ) -code as $N(n, \epsilon; K)$, the *adversarial entanglement-assisted classical capacity* is defined as

$$C_{*E}(K) := \inf_{\epsilon > 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, \epsilon; K).$$

In Appendix A we shall actually show that

$$\lim_{n \to \infty} \frac{1}{n} \log N(n, \epsilon; K) = C_{\min E}(K)$$

for every $0 < \epsilon < 1$ (this is known as a strong converse). There we will see that even allowing entanglement and arbitrary feedback in the communication protocol does not increase the capacity $C_{*E}(K)$ beyond $C_{\min E}(K)$, hence we may also address it as *feedback-assisted adversarial capacity* $C_{*EF}(K)$.

IV. CONCLUSION

We have introduced the problem of determining the zero-error capacity of a quantum channel assisted by noiseless feedback. We showed that the capacity only depends on the "noncommutative bipartite graph" *K* of the channel, and that every nontrivial *K* has positive capacity.

Motivated by Shannon's treatment of the classical case, we considered the minimisation of entanglement-assisted classical capacities over all channels with the same non-commutative bipartite graph and proved several properties of this definition: it is an upper bound on the activated feedback-assisted zero-error capacity, it is given by a minimax/maximin formula, and is additive. It is also shown to be equal to the adversarial entanglement-assisted capacity.

Note that when restricting all statements above to classical channels, which are given by a bipartite equivocation graph Γ , all of these quantities boil down to the fractional packing number:

$$2^{C_{\min E}(K)} = 2^{C_{\min}(\Gamma)} = 2^{\overline{C}_{0F}(\Gamma)} = \alpha^*(\Gamma),$$

which furthermore quantifies the zero-error capacity and simulation cost of Γ when assisted by general no-signalling correlations [20], $2^{C_{0,NS}(\Gamma)} = 2^{S_{0,NS}(\Gamma)} = \alpha^*(\Gamma)$. However, for quantum channels and non-commutative bipartite graphs these notions start diverging, so none of them can be considered as a preferred "quantum fractional packing number": In [24], no-signalling assisted zero-error capacity and simulation cost were determined for cq-channels, $C_{0,NS}(K) = \log A(K)$ and $S_{0,NS}(K) = \log \Sigma(K)$, with the semidefinite packing number A(K) and another SDP $\Sigma(K)$, and while in general (for cq-channels)

$$\log \mathsf{A}(K) \le C_{\min E}(K) \le \log \Sigma(K),$$

both inequalities can be strict [24]. It remains an open question how $\overline{C}_{0EF}(K)$ fits into this picture, and in particular whether it is equal to or sometimes strictly smaller than $C_{\min E}(K)$. We believe that pure state cq-channels offer a good testing ground for ideas; we might take encouragement from [49], where it was shown that the *unambiguous capacity* of a pure state cq-graph K equals $C_{\min E}(K)$. Other interesting K are those that admit only one channel \mathcal{N} , for instance channels extremal in the set of cptp maps, cf. [24], an example of which is the amplitude damping channel; in this case, $C_{\min E}(K) = C_E(\mathcal{N})$.

Next, motivated by the fact that both A(K) and $\Sigma(K)$ are SDPs (at least for cq-graphs), we ask if there is a manifestly semidefinite programming (or even just convex optimisation) characterisation of $2^{C_{\min E}(K)}$? To make progress, we need at least to understand some properties of an optimal \mathcal{N} for given K, and potentially also an optimal input state.

To offer a concrete approach to the question whether $C_{\min E}(K)$ is an achievable rate for pure state cq-graph K, we suggest to look at the possible use of conclusive exclusion to implement a list-decoding protocol, by excluding more than one state by each outcome – cf. [4].

List-decoding from approximate decoding? Given state vectors $|\psi_1\rangle, \ldots, |\psi_N\rangle \in B$ (w.l.o.g. |B| = N) that are sufficiently orthogonal in the sense that there exists an orthonormal basis $\{|v_1\rangle, \ldots, |v_N\rangle\}$ of B such that

$$\forall i \quad |\langle v_i | \psi_i \rangle|^2 \ge 1 - \epsilon. \left(\text{For instance, this holds if for each } i, \sum_{j \neq i} |\langle \psi_i | \psi_j \rangle|^2 \le \epsilon, \text{ by [33].} \right)$$

Then, does there exist a subset of $N' \ge \Omega(N^{1-\delta})$ of these states, $\{|\psi_{i_j}\rangle : j = 1, \dots, N'\}$, $L \le O(N^{\delta}) \ (\delta \to 0 \text{ with } \epsilon \to 0 \text{ uniformly}) \text{ and a POVM } \left(M_S : S \in \binom{[N']}{L}\right)$, such that $\{j : \langle \psi_{i_j} | M_S | \psi_{i_j} \rangle \ne 0\} \subset S \text{ for all } S \in \binom{[N']}{L}$?

Note that a positive answer would imply that by preparing ψ_{i_j} and measuring the POVM elements M_S , we construct a classical channel/hypergraph Γ with $\alpha^*(\Gamma) \geq \frac{N'}{L}$. To see this, observe that each output S is reached from at most L inputs j, namely those $j \in S$, so the weight distribution $w_j = \frac{1}{L}$ for all i is admissible in the definition of $\alpha^*(\Gamma)$. Thus we would obtain

$$\overline{C}_{0EF}(K) \ge \overline{C}_{0F}(\Gamma) \ge \log \frac{N'}{L} \ge (1 - 2\delta) \log N - O(1),$$

which is at least consistent with C(N) being of the order $(1 - \epsilon) \log N - O(1)$, by the existence of the basis $\{|v_1\rangle, \ldots, |v_N\rangle\}$ and Fano's inequality.

By Hausladen et al. [33] this would imply that we can asymptotically achieve the rate $C(\mathcal{N}) = C_{\min E}(K)$ as activated feedback-assisted zero-error capacity, where $K = \text{span}\{|\psi_i\rangle\langle i| : i = 1, ..., N\}$. It would also imply a new proof of the result of [49], since we could use the Shannon scheme [45] to get arbitrarily close to the rate $\log \alpha^*(\Gamma)$ by a deterministic list-decoding with

constant list size, and then constant activating communication, which we clearly can realize in an unambiguous fashion with constant overhead.

Finally, there is another generalization of the instantaneous feedback considered by Shannon, which was dubbed "coherent feedback" in [5], and which consist in the channel environment C from the Stinespring isometry $V : A \hookrightarrow B \otimes C$ to be handed back to Alice. More like Shannon's model, it is completely passive as it doesn't involve any action of Bob's. The resulting zero-error capacity, $C_{0|F\rangle}(V)$ is not even obviously a function of K only, nor is it clear whether additional free entanglement or free active feedback from Bob to Alice will increase it, though it is clear from the Quantum Reverse Shannon Theorem that all of $C_{0|F\rangle}(V)$ and its variants are upper bounded by $C_E(\mathcal{N})$.

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Appendix A: $C_{\min E}(K)$ equals the adversarial entanglement-assisted capacity

Here we give a complete proof of the following theorem from Section III.

Theorem 15 For any non-commutative bipartite graph K, the *adversarial entanglement-assisted classical capacity* of K is given by $C_{*E}(K) = C_{\min E}(K)$.

Before proving it, we show a simpler statement on so-called compound channels, which will be pivotal for the general proof, however. For a non-commutative bipartite graph $K < \mathcal{L}(A \rightarrow B)$, and a pure state $|\phi\rangle \in AA'$ such that $\phi^A = \phi^{A'} = \rho$, define $X = (\mathbb{1} \otimes K) |\phi\rangle < A \otimes B$ and the sets of states,

$$\mathcal{S}_{K,\rho} := \{ (\mathrm{id} \otimes \mathcal{N})\phi : \mathcal{K}(\mathcal{N}) < K \} = \{ \sigma \in \mathcal{S}(AB) : \mathrm{supp}\,\sigma < X, \ \sigma^A = \rho \},\$$

as well as, for $\epsilon > 0$,

$$\mathcal{S}_{K,\rho}^{(\epsilon)} = \big\{ \sigma \in \mathcal{S}(AB) : \exists \sigma' \in \mathcal{S}_{K,\rho} \text{ s.t. } \|\sigma - \sigma'\|_1 \le \epsilon \big\}.$$

Proposition 16 For any non-commutative bipartite graph $K < \mathcal{L}(A \to B)$, a test state ρ on A, and parameters $\epsilon > 0$ and an integer k, consider the family of cq-channels $[W^{\sigma} : S_k \to \mathcal{S}(A^k \otimes B^k) : \sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}]$, with

$$W^{\sigma}: \pi \longmapsto (\mathbb{1} \otimes U_{\pi}) \sigma^{\otimes k} (\mathbb{1} \otimes U_{\pi})^{\dagger}.$$

Then, for sufficiently large ℓ , there is an ℓ -block code of $N = 2^{nR}$ messages $(n = k\ell)$ and decoding POVM $(D_i)_{i=1}^N$, with rate

$$R \ge \min_{\sigma \in \mathcal{S}_{K,\rho}} I(A:B)_{\sigma} - 2\delta,$$

and uniformly bounded error probability

$$P_{err}((W^{\sigma})^{\otimes \ell}) = \frac{1}{N} \sum_{i=1}^{N} \left(1 - \operatorname{Tr} \left(W^{\sigma}_{\pi_{1}(i)} \otimes \cdots \otimes W^{\sigma}_{\pi_{\ell}(i)} \right) D_{i} \right) \leq c^{\ell}$$

for all $\sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}$. Here, c < 1 and $\delta = 2\epsilon \log(|A||B|) + \frac{3}{k} + 2|B|^2 \frac{\log(k+|B|)}{k}$.

Proof The family of cq-channels $[W^{\sigma} : S_k \to \mathcal{S}(A^k \otimes B^k) : \sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}]$ generates a compound channel, meaning that on block length ℓ , the communicating parties face one of the i.i.d. channels $(W^{\sigma})^{\otimes \ell}, \sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}$, but they do not know beforehand which one, so they need to use a code that is good for all of them.

For this we invoke the general result of Bjelakovic and Boche [8], which states that there are such codes with rate

$$\min_{\sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}} \chi\left(\left\{p_{\pi} = \frac{1}{k!}, W_{\pi}^{\sigma} = (\mathbb{1} \otimes U_{\pi})\sigma^{\otimes k} (\mathbb{1} \otimes U_{\pi})^{\dagger}\right\}\right) - k\delta$$

for any $\delta > 0$ and with error probability uniformly bounded by c^{ℓ} , $c = c(\delta) < 1$.

By Lemma 17 below,

$$\chi\left(\left\{p_{\pi}=\frac{1}{k!}, W_{\pi}^{\sigma}=(\mathbb{1}\otimes U_{\pi})\sigma^{\otimes k}(\mathbb{1}\otimes U_{\pi})^{\dagger}\right\}\right)\geq k\,I(A:B)_{\sigma}-2|B|^{2}\log(k+|B|),$$

and because there is $\sigma' \in S_{K,\rho}$ with $\|\sigma - \sigma'\|_1 \leq \epsilon$, Fannes' inequality [29] shows that the rate (over $n = k\ell$) is

$$\geq \min_{\sigma \in \mathcal{S}_{K,\rho}} I(A:B)_{\sigma} - 2\epsilon \log(|A||B|) - \frac{3}{k} - 2|B|^2 \frac{\log(k+|B|)}{k} - \delta,$$

and we are done, choosing δ as advertised.

We end this proof pointing out a rather nice feature of the code: each message is encoded as an ℓ -tuple of permutations from S_k , $i \mapsto \underline{\pi}(i) = \pi_1(i) \dots \pi_\ell(i)$, which we may view naturally as an element of $S_k \times \dots \times S_k \subset S_n$, acting on B^n by permuting the tensor factors, each $\pi_j(i)$ on its own block of k, hence message i is mapped to the state $W_{\underline{\pi}(i)}^{\sigma} = (\mathbb{1} \otimes U_{\underline{\pi}(i)}) \sigma^{\otimes n} (\mathbb{1} \otimes U_{\underline{\pi}(i)})^{\dagger}$ on $A^n B^n$. \Box

Lemma 17 (Cf. Shor [46]) For any channel $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ and a state ρ on A with purification $|\phi\rangle \in AA'$, and let $\sigma^{AB} = (\mathrm{id} \otimes \mathcal{N})\phi$. Then, for any integer k,

$$\chi\left(\left\{p_{\pi}=\frac{1}{k!}, W_{\pi}^{\sigma}=(\mathbb{1}\otimes U_{\pi})\sigma^{\otimes k}(\mathbb{1}\otimes U_{\pi})^{\dagger}\right\}\right)\geq k\,I(A:B)_{\sigma}-2|B|^{2}\log(k+|B|),$$

where π ranges over the symmetric group S_k , acting on B^k by permuting the tensor factors.

Proof With the average state

$$\Omega^{A^k B^k} = \frac{1}{k!} \sum_{\pi \in S_k} (\mathbb{1} \otimes U_\pi) \sigma^{\otimes k} (\mathbb{1} \otimes U_\pi)^{\dagger},$$

we have

$$\chi\left(\left\{\frac{1}{k!}, (\mathbb{1} \otimes U_{\pi})\sigma^{\otimes k}(\mathbb{1} \otimes U_{\pi})^{\dagger}\right\}\right) = S(\Omega^{A^{k}B^{k}}) - S(\sigma^{\otimes k})$$
$$= S(\Omega^{A^{k}}) + S(\Omega^{B^{k}}) - I(A^{k}:B^{k})_{\Omega} - S(\sigma^{\otimes k})$$
$$= kI(A:B)_{\sigma} - I(A^{k}:B^{k})_{\Omega},$$

where we have used that all ensemble members are just unitary transformed versions of $\sigma^{\otimes k}$ (first line), the definition of the mutual information (second line), the fact that $\Omega^{A^k} = (\sigma^A)^{\otimes k}$ and $\Omega^{B^k} = (\sigma^B)^{\otimes k}$ as well as additivity of the von Neumann entropy (third line).

Now we use the representation theory of S_k acting on B^k to bound the mutual information remaining: From Schur-Weyl duality [31] it is known that

$$B^k = \bigoplus_{\lambda} Q^b_{\lambda} \otimes P_{\lambda},$$

where λ are Young diagrams with at most b = |B| rows, P_{λ} are the corresponding irreps of S_k and Q_{λ}^b is the multiplicity space, which is an irrep of the commutant representation, SU(*b*). With the maximally mixed state τ_{λ} on P_{λ} , Schur's Lemma implies that

$$\Omega^{A^k B^k} = \bigoplus_{\lambda} q_{\lambda} \omega_{\lambda}^{A^k Q_{\lambda}^b} \otimes \tau_{\lambda}^{P_{\lambda}}.$$

Now observe that $\Omega^{A^kB^k}$ can by local operations $B^k \leftrightarrow D := \bigoplus_{\lambda} Q_{\lambda}^b$ be reversibly transformed into

$$\widetilde{\Omega}^{A^k D} = \bigoplus_{\lambda} q_{\lambda} \omega_{\lambda}^{A^k Q_{\lambda}^b}$$

hence

$$I(A^k : B^k)_{\Omega} = I(A^k : D)_{\widetilde{\Omega}} \le 2\log|D| \le 2b^2\log(k+b).$$

The latter because it is known that there are only $L \leq (k+1)^b$ Young diagrams and each SU(b) irrep has dimension $|Q_{\lambda}^b| \leq M = (k+b)^{\frac{1}{2}b^2}$, hence $|D| \leq LM = (k+1)^b(k+b)^{\frac{1}{2}b^2} \leq (k+b)^{b^2}$, as we only need to consider the case $b \geq 2$.

Proof (of Theorem 15) First we show the upper bound, to be precise the strong converse. Because among the eligible channels is $\mathcal{N}^{\otimes n}$ with $\mathcal{K}(\mathcal{N}) < K$ attaining the minimum in $C_{\min E}(K)$, we see immediately that $C_{*E}(K) \leq C_E(\mathcal{N}) = C_{\min E}(K)$. In fact, the Quantum Reverse Shannon Theorem for $\mathcal{N}^{\otimes n}$ [5, 7] implies the strong converse as well, i.e. for all $\epsilon < 1$,

$$\limsup_{n \to \infty} \frac{1}{n} N(n, \epsilon; K) \le C_E(\mathcal{N}) = C_{\min E}(K).$$

A direct proof of this can be found in [26] (see also [27]). Furthermore, Bowen [10] (alternatively again the Quantum Reverse Shannon Theorem) showed that feedback does not increase the entanglement-assisted capacity.

It remains to show achievability of $C_{\min E}(K)$; for this it will be enough to show that for any test state ρ on A, $C_{*E}(K) \ge \min_{\mathcal{K}(\mathcal{N}) < K} I(\rho; \mathcal{N})$, by exhibiting a sequence of codes with this rate and error probability going to 0, exponentially in n. Choose a purification $|\phi\rangle^{AA'}$ of ρ and let Alice and Bob share $\phi^{\otimes n}$ as well as a maximally entangled state of Schmidt rank n!, which is

measured by both parties in the computational basis to obtain a shared random permutation $\tau \in S_n$. Alice's encoding will be to subject her *n* input *A*'-systems to a permutation $\underline{\pi}(i)$ for each message $i = 1, \ldots, N$, then apply τ and send the resulting state through the channel $\mathcal{N}^{(n)}$; Bob will apply the permutation τ^{-1} to his *n* output *B*-systems. The state this prepares for Bob is

$$\omega(i)^{A^{n}B^{n}} = \frac{1}{n!} \sum_{\tau \in S_{n}} (\mathbf{1} \otimes U_{\tau})^{\dagger} \Big[(\mathrm{id} \otimes \mathcal{N}^{(n)}) \big((\mathbf{1} \otimes U_{\tau} U_{\underline{\pi}(i)}) \phi^{\otimes n} (\mathbf{1} \otimes U_{\tau} U_{\underline{\pi}(i)})^{\dagger} \big) \Big] (\mathbf{1} \otimes U_{\tau})$$
$$= (\mathbf{1} \otimes U_{\underline{\pi}(i)}) \Big[(\mathrm{id} \otimes \overline{\mathcal{N}}^{(n)}) \phi^{\otimes n} \Big] (\mathbf{1} \otimes U_{\underline{\pi}(i)})^{\dagger}$$
$$=: (\mathbf{1} \otimes U_{\pi(i)}) \sigma^{(n)} (\mathbf{1} \otimes U_{\pi(i)})^{\dagger},$$

with the permutation-symmetrized channel

$$\overline{\mathcal{N}}^{(n)}(\rho) = \frac{1}{n!} \sum_{\tau \in S_n} U_{\tau}^{\dagger} \mathcal{N}^{(n)} \big(U_{\tau} \rho U_{\tau}^{\dagger} \big) U_{\tau}$$

Note that as $\mathcal{K}(\mathcal{N}^{(n)}) < K^{\otimes n}$, the same holds for $\overline{\mathcal{N}}^{(n)}$. The permutations $\underline{\pi}(i)$ form a code for the compound channel

$$\left[W_{\pi}^{\sigma} = (\mathbb{1} \otimes U_{\pi})\sigma^{\otimes k}(\mathbb{1} \otimes U_{\pi})^{\dagger} : \sigma^{AB} \in \mathcal{S}_{K,\rho}^{(\epsilon)}\right]$$

according to Proposition 16 and its proof; here, $n = k\ell$, and we will determine k and ϵ later. Bob will use the very decoding POVM (D_i) from the same proposition.

To analyze the performance of this strategy, we apply the Constrained Postselection Lemma 18 to the permutation-symmetric state $\sigma^{(n)} = (\mathrm{id} \otimes \overline{\mathcal{N}}^{(n)})\phi^{\otimes n}$, $X = (\mathbb{1} \otimes K)|\phi\rangle < A \otimes B$ and $\mathcal{R} = \mathrm{Tr}_B$:

$$\sigma^{(n)} \leq (n+1)^{3|A|^2|B|^2} \int \mathrm{d}\sigma \, \sigma^{\otimes n} \, F(\sigma^A, \rho^A)^{2n},$$

where the integral is over states σ^{AB} supported on X < AB. We split the integral into two parts, a first where $F(\sigma^A, \rho^A) < 1 - \alpha$ and a second one where $F(\sigma^A, \rho^A) \ge 1 - \alpha$. Choosing α small enough ensures that those σ^{AB} are in $S_{K,\rho}^{(\epsilon)}$. Thus,

$$\sigma^{(n)} \le (n+1)^{3|A|^2|B|^2} (1-\alpha)^{2n} \sigma_0 + (n+1)^{3|A|^2|B|^2} \int_{F(\sigma^A, \rho^A) \ge 1-\alpha} \mathrm{d}\sigma \, \sigma^{\otimes n},$$

with some state σ_0 . At this point we can evaluate the error probability:

$$P_{\text{err}} = \frac{1}{N} \sum_{i=1}^{N} \operatorname{Tr} \left((\mathbb{1} \otimes U_{\underline{\pi}(i)}) \sigma^{(n)} (\mathbb{1} \otimes U_{\underline{\pi}(i)})^{\dagger} (\mathbb{1} - D_{i}) \right)$$

$$\leq \operatorname{poly}(n) \left[(1 - \alpha)^{2n} + \max_{\sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}} \operatorname{Tr} \left((\mathbb{1} \otimes U_{\underline{\pi}(i)}) \sigma^{(n)} (\mathbb{1} \otimes U_{\underline{\pi}(i)})^{\dagger} (\mathbb{1} - D_{i}) \right) \right]$$

$$\leq \operatorname{poly}(n) \left((1 - \alpha)^{2n} + c^{n/k} \right),$$

showing that for every *n* and ϵ the error probability goes to zero exponentially – in fact, at the same rate as the corresponding compound channel, except for the additional term $(1 - \alpha)^{2n}$.

The rate, according to Proposition 16 is $\geq \min_{\mathcal{K}(\mathcal{N}) < K} I(\rho; \mathcal{N}) - 2\delta$, where $\delta = 2\epsilon \log(|A||B|) + \frac{3}{k} + 2|B|^2 \frac{\log(k+|B|)}{k}$ can be made arbitrarily small by choosing ϵ small enough and k large enough.

by reducing it to a compound cq-channel [8], cf. also [36]. Observe however that what we treated here is not an "arbitarily varying quantum channel" in any sense previously considered [3, 9], going beyond the model in [36], too.

Appendix B: A Constrained Post-Selection Lemma

Here we show the following extension of the main technical result of [13] (albeit with a worse polynomial prefactor).

Lemma 18 For given Hilbert space X with dimension d, denote by $d\sigma$ the measure on the quantum states S(X) obtained by drawing a pure state from $X \otimes X'$ uniformly at random (i.e., from the unitarily invariant probability measure) and tracing out X'.

Then, for any S_n -invariant state $\rho^{(n)}$ on X^n ,

$$\rho^{(n)} \le (n+1)^{3d^2} \int \mathrm{d}\sigma \, \sigma^{\otimes n} \, F(\rho^{(n)}, \sigma^{\otimes n})^2.$$

The measure $d\sigma$ is universal in the sense that it depends only on the space *X*.

Furthermore, let $\mathcal{R} : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a cptp map, $\eta \in \mathcal{S}(Y)$ a state. Then, for every S_n -invariant state $\rho^{(n)}$ on X^n with $\mathcal{R}^{\otimes n}(\rho^{(n)}) = \eta^{\otimes n}$,

$$\rho^{(n)} \le (n+1)^{3d^2} \int \mathrm{d}\sigma \, \sigma^{\otimes n} \, F\big(\mathcal{R}(\sigma),\eta\big)^{2n}.$$

Note that the right hand side depends only on X, \mathcal{R} , η and n.

Here, $F(\xi, \eta) = \|\sqrt{\xi}\sqrt{\eta}\|_1$ is the fidelity between (mixed) states $\xi, \eta \in \mathcal{S}(X)$ [30, 35, 50].

Remark Note that in $\Omega^{(n)}$, the contribution of states σ with $F(\mathcal{R}(\sigma), \eta) < 1 - \epsilon$ is exponentially small in *n*. I.e., for a symmetric state with an additional constraint, expressed by \mathcal{R} and η , the universal de Finetti state from [13] may be chosen in such a way that almost all its contributions also approximately obey the constraint.

Proof Denoting the uniform (i.e. unitarily invariant) probability measure over *pure* states $\zeta = |\zeta\rangle\langle\zeta|$ on $X \otimes X'$ by $d\zeta$, it is well known that

$$\int \mathrm{d}\zeta \,\zeta^{\otimes n} = \frac{1}{\binom{n+d^2-1}{d^2-1}} \Pi_{\mathrm{Sym}^n(X\otimes X')},$$

with $\Pi_{\text{Sym}^n(X \otimes X')}$ denoting the projector onto the (Bose) symmetric subspace of $(X \otimes X')^{\otimes n}$. The reason is that the latter is an irrep of the $U^{\otimes n}$ -representation for $U \in \text{SU}(d^2)$, so Schur's Lemma applies. Now we apply Caratheodory's Theorem, which says that $d\zeta$ can be convex-decomposed into measures with finite support, more precisely ensembles $\{q_i, \zeta_i\}_{i=1}^{D^2}$, with $D = \binom{n+d^2-1}{d^2-1} \leq (n+1)^{d^2}$, the dimension of the Bose symmetric subspace of $(X \otimes X')^{\otimes n}$, and

$$\sum_{i} q_i \zeta_i^{\otimes n} = \frac{1}{D} \Pi_{\operatorname{Sym}^n(X \otimes X')}.$$

For the moment we shall focus on one of these measures/ensembles.

It is also well known that one can purify $\rho^{(n)}$ in a Bose symmetric way, i.e. $\rho^{(n)} = \operatorname{Tr}_{X^{(n)}} \varphi^{(n)}$, with $\varphi^{(n)} = |\varphi^{(n)}\rangle\langle\varphi^{(n)}|$ a pure state supported on the Bose symmetric subspace. Thus, with the

$$\begin{split} \varphi^{(n)} &= \Pi_{\operatorname{Sym}^{n}(X\otimes X')} \varphi^{(n)} \Pi_{\operatorname{Sym}^{n}(X\otimes X')} \\ &= D^{2} \sum_{ij} q_{i} q_{j} \zeta_{i}^{\otimes n} \varphi^{(n)} \zeta_{j}^{\otimes n} \\ &= D^{2} A \left(\sum_{ij} q_{i} q_{j} |i\rangle \langle j| \langle \zeta_{i}|^{\otimes n} \varphi^{(n)} |\zeta_{j}\rangle^{\otimes n} \right) A^{\dagger} \\ &\leq D^{4} A \left(\sum_{i} q_{i}^{2} |i\rangle \langle i| \langle \zeta_{i}|^{\otimes n} \varphi^{(n)} |\zeta_{i}\rangle^{\otimes n} \right) A^{\dagger} \\ &\leq D^{3} A \left(\sum_{i} q_{i} |i\rangle \langle i| F \left(\zeta_{i}^{\otimes n}, \varphi^{(n)} \right)^{2} \right) A^{\dagger} \\ &\leq D^{3} \sum_{i} q_{i} \zeta_{i}^{\otimes n} F \left((\operatorname{Tr}_{X'} \zeta_{i})^{\otimes n}, \rho^{(n)} \right)^{2}, \end{split}$$

where in the fourth line we have used Hayashi's pinching inequality [34], and in the fifth $q_i \leq \frac{1}{D}$; in line six we have invoked the monotonicity of the fidelity under cptp maps, here the partial trace, as well as $\operatorname{Tr}_{X'^n} \varphi^{(n)} = \rho^{(n)}$.

Now we remember that $\{q_i, \zeta_i\}$ was just one of the Caratheodory components of the uniform measure $d\zeta$, so by convex combination,

$$\varphi^{(n)} \leq D^3 \int \mathrm{d}\zeta \,\zeta^{\otimes n} F\left((\mathrm{Tr}_{X'}\zeta)^{\otimes n}, \rho^{(n)}\right)^2,$$

hence by partial trace over X'^n , and recalling the definition of $d\sigma$, we arrive at

$$\rho^{(n)} \le D^3 \int \mathrm{d}\sigma \, \sigma^{\otimes n} \, F\big(\sigma^{\otimes n}, \rho^{(n)}\big)^2.$$

To obtain the second bound, we apply the map $\mathcal{R}^{\otimes n}$ to the states inside the above fidelity; by monotonicity of the fidelity once more,

$$F(\sigma^{\otimes n}, \rho^{(n)}) \leq F\left(\mathcal{R}^{\otimes n}(\sigma^{\otimes n}), \mathcal{R}^{\otimes n}(\rho^{(n)})\right)$$
$$= F\left(\left(\mathcal{R}(\sigma)\right)^{\otimes n}, \eta^{\otimes n}\right)$$
$$= F\left(\mathcal{R}(\sigma), \eta)^{n},$$

as desired.

Remark It is the trick to sandwich the Bose-symmetric state $\varphi^{(n)}$ between symmetric subspace projectors – rather than bounding it directly by that projector –, which allows the introduction of fidelities between the state and "test" product states.

Here we have used this to enforce a linear constraint valid for $\rho^{(n)}$ on the components of the de Finetti state on the right hand side. It turns out, perhaps unsurprisingly, that also other convex constraints (with a "good" behaviour linking n = 1 with the general case) are amenable to

operator $A := \sum_{i} |\zeta_i\rangle^{\otimes n} \langle i|$,

the same treatment, for instance membership in the convex set of separable states for a multipartite space $X = X_1 \otimes \cdots \otimes X_k$, and other similar sets, or even non-convex constraints. Such generalizations and their applications are discussed in [38].

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