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Developing and evaluating alternative technological infrastructures for learning mathematics

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Foreword

We would like to dedicate this paper to our friend and colleague, Jim Kaput, who was tragically killed in a road accident shortly before the conference. His work on representational infrastructures pointed to a crucial challenge for mathematics education. By situating current representational systems in their historical context, Jim showed the possibilities for designing alternatives. For Jim, and for us, a key concern was to open up, democratise and make more learnable the complex ideas of mathematics, ideas whose complexity often owes as much to the way they are represented, as to the ideas themselves. This paper is the worse for the lack of Jim's critical comments.

1. Introduction

The challenge that this chapter addresses is to explore how computers can make it possible for students to engage with mathematics that they either might have failed to engage with in a traditional school setting or which they might not have encountered. We will consider infrastructures for mathematical expression, that is to say systems of representations of mathematical ideas and objects, and the means to manage them to enhance engagement with mathematical ideas.

One focus of our chapter is algebra. Paper/pencil algebraic infrastructures made it necessary for individuals to pay considerable attention to manipulation, and key mathematical topics were only amenable to those who had already been inducted into fluent algebraic representations and calculations. This meant that many never engaged with the mathematical topic at hand and the learning of algebraic notation became a thing-in-itself, rather than a means to an end – learning to play scales without ever playing the music. We will demonstrate that digital technologies can radically change this scenario.

Cultural demands on curricula have encouraged (not always altogether thoughtfully) the use of technology, and stressed its utility for experimentation and exploration. The availability of computers in mathematical research and in the classroom has suggested the development of curricula that urge students and teachers to replicate computer-supported experimental methods used by mathematical and scientific researchers. With dynamic media, mathematics can become (and has already become, in parts of the academic field of mathematics) an experimental science, one in which the activities of experiment and observation is as important as logic and proof. So an important question arises as to the kind of assistance technological tools can bring to student experimental activity with dynamic mathematical representations, and under what conditions.

A variety of technological tools, especially Computer symbolic systems (CAS), have been presented as a means to overcome students' difficulties in paper/pencil manipulations, offering them opportunities to develop exploratory approaches inspired from research. These tools stand as candidates for new expressive infrastructures, while maintaining more or less intact the usual representations - including algebraic notation - and using the power of the computer to perform actions on these representations to obtaining diverse graphs, tables and transformations of expressions. Although promising, this approach has not been, in our view, sufficiently discussed from an epistemological point of view, and its 'viability', that is to say the conditions in which it could be effective in actual classrooms, remains problematic. We will have more to say on this below, particularly in section 2.

At the same time, there is a need for new and alternative *representations* for algebra. While the need to think creatively about representational forms arose less obviously in settings where things were mechanical and much more visible (i.e. objects had gears, levers, pulleys etc.), the devolution of processing power to the computer has generated the need for individuals *to represent for themselves* models of how things work, what makes systems fail, and what would be needed to correct them (see Noss, 1998 for an elaboration of this point; see also Hoyles, Morgan and Woodhouse, 1999). In terms of the didactical implications of this trend, perhaps the best known, at least theoretically, has been the *constructionist* proposition (see Harel and Papert 1991) that has emphasised how building and constructing physical and virtual models of situations is an effective means to construct corresponding mental notions.

The authors of this chapter are involved in two distinct projects on which they will draw to exemplify the potential exploitation of technology designed for a more learnable mathematics. These two projects converge in their goal – the design of expressive infrastructures to enhance learnability - yet they adopt different orientations. The first project is the *Casyopée* project (Lagrange 2005b). It is designed to encourage students to use existing mathematical representations with the support of the computer. The second is *WebLabs* (see, for example, Noss & Hoyles, 2006), based on the idea of building new representations for mathematical models. We present an overview of the theoretical approach of both projects, along with some findings and illustrative extracts. Reflecting on the two projects inspired by their different orientations leads us to consider a basis for the evaluation by drawing on a *plurality of dimensions* put forward by Lagrange et al. (2003). Since the classroom is a complex reality, we argue that observation and intervention is needed from a wide range of perspectives, and studies should adopt approaches that span a range of dimensions or themes.

The structure of the chapter will be as follows. First a description and exemplification of the Casyopée project, which is followed by a section about WebLabs. Finally the discussion will compare and contrast the approaches, and point to the different roles that systems like Casyopée and WebLabs can respectively play in the future.

2. Casyopée: Making algebraic notation more learnable through problem explorations

We identified above two main orientations in which computers can make it possible for young students who have little prior acquaintance or proficiency in paper/pencil algebraic representation to express rich mathematics. The first is to give students an easier and more motivating access to existing algebraic representations; the second is to search for new representations that are easier and more motivating in themselves. In this first orientation, the purpose of the Casyopée project is to offer upper secondary students an open environment for problem solving about functions, with capabilities of formal calculation, and graphic and numerical exploration, encouraging the use of algebraic representations.

Kieran (2006) recalls that difficulties with the algebraic notation has been for many years a major question for mathematics education:

> *"while arithmetic and algebra share many of the same signs and symbols, such as the equal sign, addition and subtraction signs, even the use of letters, many conceptual adjustments are required of the beginning algebra student as these signs and symbols shift in meaning from those commonly held in arithmetic".*

Teaching generally does not deal with these difficulties: algebra is very often taught as procedures disconnected from meaning and purpose. Beyond problem solving "in a narrow sense", authors promoted experimental approaches (or exploration) of problems as a way to reconnect the algebraic activity to meaning and purpose.

> *For introducing and developing algebra (...) the essential mathematics activity is that of exploring problems in an open way, extending and developing them in the search for more results and more general ones. Hence [all algebraic learning] is based on problem explorations. This is the broad sense of the term.* (Bell, 1996, p.167)

The development of computer technology supported this shift towards experimental approaches. For instance, the capacity to carry out many calculations rapidly was thought to assist the transition from an examination of single cases towards the resolution of groups of cases. Graphical and tabular representations or even the possibility of having a spreadsheet recalculate a series of expressions as a particular cell is varied similarly supported this view.

The ambition of the Casyopée project is to contribute to a change towards experimental approaches in classrooms in order to access a meaningful use of algebraic representation. Educational research has stressed the potential of such technology-aided approaches, but this does not mean that actual classroom implementation is straightforward. Considering this objective, the Casyopée team[1] identified three concerns:

1. Students' experimental activity. For a long time, authors and curricula advocated the advantages of classroom problem exploration, often by referring to professional mathematicians' activity and recently to the use of technology. Nonetheless, obstacles persist that cannot be simply attributed to teachers' unfamiliarity with this approach.

2. Students' algebraic activity and the influence of technological tools on this activity. Algebraic activity is multifaceted and involves a plurality of concepts. Technology also offers varied possibilities. A careful examination is necessary to identify the support it might bring to the transition to using algebraic notation.

3. The design of an algebraic software application that can actually be used in classrooms. Many excellent ideas have underpinned the creation of new tools for

[1] The team includes two teachers of the Institute for Research in Mathematical Education (IREM) of Rennes and the author. The project is supported by the French National Institute for Pedagogical Research (INRP).

> teaching and learning mathematics, yet it is not so clear that these ideas match the needs and constraints of 'real' classrooms and 'real' teaching.

Any mathematical education research bases its analysis on a theoretical framework. According to Mewborne (2005, p.3 & 4), using a framework brings a researcher two main benefits: "it serves as a sort of binocular that allows one to narrow down the scope of the research site to focus on particular aspects of the situation" and it forces one to "constantly compare and contrast what the data are saying with what the framework is saying."

On the one hand, these advantages have been recognised by researchers in the field of technology. Jones and Lagrange (2003) pointed out:

> *"there is a range of theoretical frameworks that appear to hold some promise when researching the use and impact of tools and technologies. Theories like embodied cognition and metaphor, cognitive gaps and transitions, situated abstraction, semiotic mediation, instrumentation… help to give relevant account of phenomena arising when students and teachers are using technology."*

On the other hand, the technology-rich classroom is a complex reality that necessitates observation and intervention from a wide range of perspectives. We argue, therefore, that it is dangerous prematurely to narrow down the scope of research, leading to a loss of recognition of this complexity. This loss is particularly disadvantageous when the aim of the researcher is to build an application of technology for use in ordinary classrooms. In this case, the framework is not just a means to collect data, but rather has to provide support for a continuous reworking of design, implementation, observation and adaptation. Thus in order to observe how the application works in the classroom, the researcher has to take into account a variety of 'ecological' conditions, or risk a lack of feasibility in the classroom. For example, analysing ten years of development of the E-slate project, Kynygos (2004) explains: "A reason (to take an ecological perspective) was our need for feedback from and interaction with people using E-slate in their daily routines, since we were keen to gain insight as early as possible".

Using a plurality of dimensions is a way to keep a focused view on teaching/learning phenomena and a scientific account of observations, while ensuring a sufficiently wide approach of classroom reality. More precisely, the hypothesis here is that in developing a technological application to teach and learn mathematics, it is possible to conceive a range of dimensions to make sense of the principles on which the development is based and then to anticipate consequences of its use.

How can dimensions be identified that serve to focus on a narrowed scope? Following Mewborne's (*ibid.*) theoretical frameworks may be helpful. Nonetheless, concerns rather than theories are taken here as starting points, since concerns help to achieve a narrowed scope, following which, a theory is needed to investigate the scope and to guide practical choices regarding software development and classroom implementation. A question that then arises is what theory should be used given that several compete: for instance, different theories pointed out by Jones and Lagrange (*ibid*.) address more or less a similar concern for the interaction between learners, technology and knowledge. Confronting and reconciling theories sensitive to the same concern is a work in itself (see Hoyles *et al*. 2004, on situated abstraction and instrumentation) that will not be undertaken here. Rather the choice of theories to underpin the concerns identified by the Casyopée team, was driven largely by familiarity.

Three dimensions will help in refocussing from each of the above concerns to practical choices regarding software development and the classroom implementation. We now present these dimensions along with two examples of classroom uses of Casyopée to illustrate implementation and how the dimensions helped in its examination.

Three dimensions

The anthropological dimension: transposing mathematicians' experimental activity

Introducing students into a "true mathematical activity", giving a significant part to experimentation and conjecture is assumed to be a way into meaningful algebra, and is especially topical, given the development of technology. Many educators, when thinking of a valid mathematical activity and of conjectures has in mind the practices of mathematicians. This reference is also especially present when dealing with technology, because tools and software proposed for classroom use often derive from instruments developed by mathematics research for its own needs, especially to make experimentation more productive. Thus mathematical research practices and tools represent a reference for enhanced classroom activity.

The concern in the Casyopée project for the conditions of a classroom experimental activity takes this reference into account and, in consequence, the first dimension will focus on the phenomenon of *didactic transposition* from research to classroom. The choice of a theoretical approach along this dimension will be the 'anthropological approach', which was initiated precisely by a conceptualisation of the didactic transposition.

Lagrange (2005a) explains:

> *The anthropological approach (Chevallard 1985, 1994, 1999) aims to give account of the conditions in which mathematical objects exist and live in institutions or more precisely how they are 'known and understood' as entities arising from practices. The word 'institution' has to be understood in a very broad sense as any social or cultural practice takes place within an institution. (We) will consider (a transposition) between scientific research institutions devoted to producing knowledge and didactic institutions devoted to apprenticeship…*
>
> *Other notions of the anthropological approach, especially useful to clarify the influence of technology on teaching/learning are the three components of practices in a institution: a type of task; the techniques used to solve this type of task and the 'theory' which is first the discourse used in order to explain and justify the techniques and then provides a structural basis for this discourse[2]…Chevallard (1999, p.231) explains that praxeologies (i.e. the above components of practices) are the matter of the transposition.*

The anthropological approach helps to identify the challenges that technology-aided experimental praxeologies in teaching/ learning have to meet: to ensure their legitimacy, they must be related to homologous practices in mathematical research and to be viable, they must be compatible with the constraints of the organization of the knowledge in teaching, that is dissimilar to mathematical research.

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[2] See Lagrange (2000, 2005a) for developments about the notion of praxeology when using technology and Monaghan (2005) for a discussion.

Let us study the similarities and differences of the experimental practices in research and teaching. In research, praxeologies are characterized by their consistency: mathematicians think of the objects (concepts, properties...) involved in their experimental practices according to the theory they want to build. They know the constraints that their conjectures must satisfy to be included into a deductive production. They try from the beginning to express the conjectures at the more general level, keeping in mind the theoretical apparatus they want to build.

In contrast, in teaching situations, theoretical objects do not come so easily out of experimental practices. As Joshua and Joshua (1987 p. 245) have noted:

> *The didactic mode (prevalent in the teaching of mathematics) is a rupture between a series of "activities" closed on themselves, and a further axiomatic presentation of a theoretical field that could, abstractedly, correspond.*

Even when pupils have an authentic experimental activity in a field of application, it is often difficult for them to establish a link with the theoretical entities modelled by the objects involved in the experimentation. The role of the proof is crucial since theoretical objects find their full relevance only when one goes beyond empirical validation.

What changes does technology bring? Can it cast experimentation as a means to enhance conceptualisation? The possibilities of experimentation are commonly advocated, as illustrated in the extract from the French curriculum for the beginning of algebra:

> *The computer enlarges considerably the possibilities of observation and manipulation; devoting to the computer a great number of calculations or a multitude of cases makes possible to observe and check empirically various properties.*

The idea that increasing the amount of data contributes in itself to the mathematical activity was justly criticized by Lakatos (quoted by Yerushalmy 1999 p. 80): '*If you believe that the longer the table the more conjectures it will suggest, you may waste your time compiling unnecessary data*'. To understand what is really at stake in the transposition of technology-aided experimental praxeologies, it is essential to take into account the practices since, as explained by Lagrange (2000), it is through these practices, where technical work plays a decisive role, that mathematical objects and the connections between them are constructed as a part of developing conceptual understanding. The difficulty is that mathematicians often admit devoting time to tasks of experimental nature, but the methods that made their conjectures possible and the role that these methods play in conceptualisation remain private. Indeed in the mathematical tradition, the way in which the results are conjectured is hidden beneath the deductive presentation of a theoretical structure. Concepts and properties are justified by the consistency of the structure rather than by the conditions of its development.

One branch of professional mathematics has broken with this tradition by expliciting how mathematical activity can take advantage of the computer. This branch —known as 'experimental mathematics'— stresses that using technology to make sense of empirical data and to conjecture and prove, requires to develop specific *methods*, and to build specific *computer environments*. Borwein (2005 p. 76) reports:

> *At CECM (Centre for Experimental and Constructive Mathematics) we are interested in developing methods for exploiting mathematical computation as a tool (1)in the development of mathematical intuition, (2)in hypotheses building, in the generation of symbolically assisted proofs, and (3)in the construction of a flexible computer environment in which researchers and research students can undertake such research.*

There is, however, much more at stake in terms of the transposition in experimental mathematics than simply using the computer to produce more data. With this transposition in view, the Casyopée project set out to be a contribution to the construction and study of viable and legitimate 'techno-experimental praxeologies'. Its work involved (1) thinking of techniques that could help students to benefit from the computer's possibilities of data production, (2) developing the use of the computer's processing capabilities to facilitate the search of conjectures and proof, and (3) thinking of an appropriate software environment.

The epistemological dimension: school algebra activities with technology.

In the Casyopée project, experimentation should be designed to help students access algebraic activity or to progress in this domain. This is a significant aspiration. Whereas basic numerical proficiency progresses - at least in 'developed' societies - a majority of citizens encounter difficulties at school in algebra and give up all practice after school. Thinking of a tool to enhance students' algebraic activity, highlights the concern about the relationship between mathematical knowledge and tools. This concern refers to the epistemology of algebra as a second dimension.

The choice for a theoretical approach in this dimension has been the "model for conceptualising algebraic activity" that Kieran (2006) introduced as a synthesis of a stream of research about school algebra. This model classifies school algebra activities into three categories: generational – becoming aware of a functional relationship and finding ways of expressing and exploring this relationship; transformational –changing the form of an expression in order to maintain equivalence; and global / meta-level – modelling, searching for structures and generalizing. The model was used for the conception of Casyopée as well as for the design of the associated classroom activities. Kieran (2004 p.23) suggests that there are two main frameworks – generalised arithmetic and functions – providing a "unique transversal thread to these three categories". The focus in Casyopée is on functions, because functions are objects and tools in many algebraic activities at upper secondary level.

This choice of functions had consequences on the objects that Casyopée handles:

a) letters identify the variable, values of the variable (*abscissas*), functions and parameters,
b) graphs and tabular representations of functions complement symbolic expressions,
c) equations correspond to the search for a value of the variable for a given value of the function or they can be about the equality of two functions, or the search for a value of a parameter. They can also be interpreted in the graphic register (intersection of curves).

Parameters are introduced in order to enrich the set of literals (beyond the variable and the functions names). They help the global / meta-level category of activity, because, in these activities, dependency has to be expressed at a sufficiently general level. For instance, (Lagrange 2005b p. 173) searching for a rectangle of maximum area on a triangle, the functional dependency between the area and the length of the rectangle includes, as parameters, the lengths of the sides of the triangle, giving a more general signification to the problem.

In problems about functions depending on parameters, the search for conjectures on graphic or numeric (tabular) representations has to be conducted more methodically (see the first example below) and empirical evidence is less convincing than in ‘ordinary’ one-variable functions, which provides an advantage to symbolic proof. In Casyopée, each parameter can exist symbolically or it can be instantiated and dynamically animated. This capability helps to explore and generate empirical evidence on numerical examples in parallel with the symbolic study of a generic case (see the example of the maximum area rectangle, (Lagrange *ibid* p. 175).

In its present state, Casyopée does not contribute to the category of generational activities, while software such as dynamic geometry would. That is why embedding dynamic geometry features into Casyopée, postponed until now, will be undertaken at a later stage, as the Casyopée teams would like to offer an 'all in one' software that offers support for all three categories of activities[3].

In the Casyopée project, transformational activities were considered significant. They are consistent with the objectives of the French curriculum:

> *Students should be able to recognize the form of an expression (sum, product, square, difference of two squares), to recognize various forms of an expression and to choose the most relevant form for a given work*

These objectives do not put at stake the technique of transformation itself, but rather the understanding of the multiplicity of equivalent forms and the role that a form can play in a proof. That is why computer symbolic calculation capabilities have been chosen in Casyopée to help students to easily obtain various forms for expressions of functions, their sub-expressions, and their values.

Proof is important to give sense to the transformational activities, keeping students away from unmotivated manipulation. Casyopée favours an approach to proving through meta-global activities. Casyopée also supports the building of a proof by offering students a set of elementary proofs (justifications) that a student can use to build his/her proof, justifying a property by a relevant form of the expression of the function. This set is designed to be an aid to students, rather than a constraint, by exploiting the “transformational knowledge” of computer symbolic calculation.

The third dimension: Designing a symbolic environment

After choosing computer symbolic calculation as the means to support students' transformational activities, it was decided to build Casyopée as a new environment rather than to use a standard symbolic application (like Derive or Maple). The team who started the Casyopée project, after trying to develop classroom uses of these applications found strong disadvantages.

A first disadvantage was the design of the interface. The power and openness of these applications imply a multiplicity of modes, menus, objects, and keywords, which is a cause of difficulties and erratic behaviours by students. Standard symbolic application’s main window is a 'history' of the calculation and definition. It provides no direct information about the present state of the application and the status of the objects and it is hard for a newcomer to

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[3] See the conclusion of this section.

have a proper representation of these. For instance, a very common difficulty for students arises when they have used a letter to name a value and then try to use the same letter as a formal parameter in a calculation. Actually, these applications are designed to be a powerful 'scratch paper' rather than a learning environment where students could develop methodical and reflective approaches to problem solving. The team felt the need for a working environment with a better balance between simplicity and power, where objects would have a clear status, and where their present state would be visible at the interface.

A second disadvantage with standard symbolic applications is that little care is taken for consistency with the curriculum. The consequence is that phenomena that students cannot understand (like complex values for 10th graders) constantly occur, complicating the task of the teacher. Inconsistencies between the way objects are handled in these applications and the way recommended by the curriculum also cause deep misunderstanding, for example when formal calculation simplifies x^2/x into x without warning or gives {0} as the set of solutions for *zeros(x^2/x, x).*

The following example shows inconsistencies that would cause no difficulty to mathematicians but led to serious problems among students.

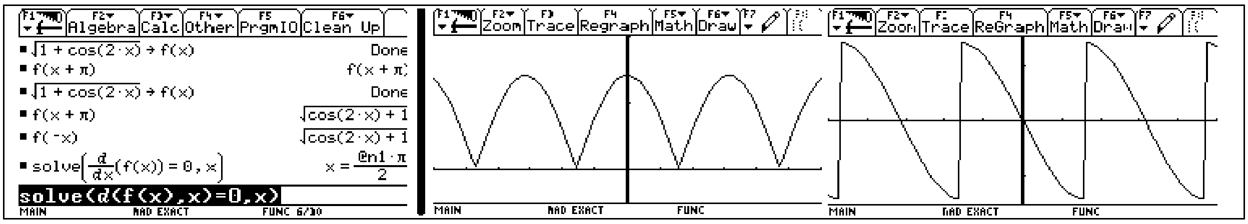

Figure 2.1: *Using the TI-92 to discuss a trigonometric function*

The task was to study the function **f(x)=** $\sqrt{1+\cos(2\cdot x)}$. It was set to 11th graders at the end of the year. The students used a TI-92[4] throughout the year within a research project (Lagrange 1999).

We expected that students would easily find periodicity and symmetry by observing the graph (figure 2.1, middle) and confirm their conjecture using the symbolic module (figure 2.1, left), then detect that the derivative is not defined at the points where the curve reaches the x-axis by observing that the curve has different non-zero gradients at these points. (S)he would conclude that the function has an 'ordinary maximum' (i.e. null derivative) every $k\pi$ and a 'special minimum' (i.e. no derivative) every $\pi/2 + k\pi$.

Observation showed that parity was not a problem for students. In contrast, they had difficulty in finding a period, because of the phenomenon on the screen (left): the curve seemed not to reach the **x** axis for $\pm 3\pi/2$.

Interpreting accurately the behaviour of the function at the points where the curve reaches the **x**-axis was not possible. Students persisted to think of an 'ordinary minimum', wondering why

[4] This Texas Instrument calculator (very similar to other calculators sold by this company: TI-89 ad Voyage 200) integrates symbolic calculation. The company claims that it has been designed for upper secondary level mathematics learning, but most of the remarks we address to standard symbolic application are relevant for this calculator (complexity, relationship to curriculum…)

repeated zooming in did not show a null gradient. They were reinforced in this idea by the false solution of the equation **`f'(x)=0`** by the symbolic module, giving zeros of the derivative for every **kπ/2**. Students thought this resolution was a reliable means to get extrema and had no reason to mistrust the result. Even the graph of the derivative (figure 2.1, right) was misleading because of the irrelevant line across the discontinuity. The reason for this behaviour was that the TI-92, like other symbolic systems, does not consider that functions are defined on a domain. Thus, solving the equation, it just looks for zeros of the numerators without considering possible zeros of the denominator.

These observations drew our attention towards the importance of a careful design that took into account the multiple constraints of classroom use: this is why *design* is a third dimension in the framework. Human Computer Interaction (HCI) researchers stressed for ten years that '*Laboratory-based usability studies are (only) part of the solution,'* and are best preceded by *"careful field studies"* to address question like: *'how technology can fit into users' actual social and material environments; the problems users have that technology can remedy; the applications that will promote creativity and enlightenment…'* (Nardi 1996).

As Yerushalmy (1999, p. 184) puts it, software design should give the learner control over experimentation by helping him/her to develop methods. It should also support the organization of the curriculum by being consistent, using "the same language of objects and actions that form the grid along which the curriculum is mapped".

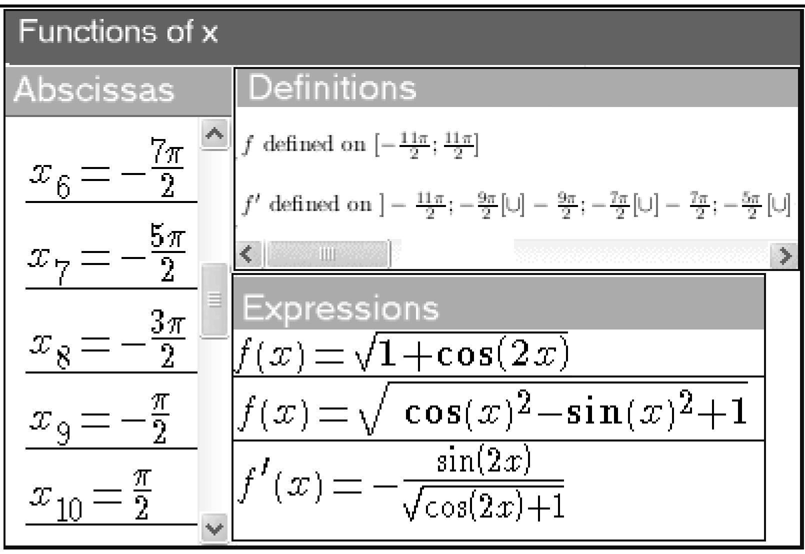

Figure 2.2 : *A main function window of Casyopée.*

A main function window identifies an independent variable (here x), "abscissas" that will be critical values of this variable (here zeros of the function which are also discontinuities of the derivative), the definition of the functions (union of intervals where they exist) and their expressions. Here the user created the function f by entering the definition and an expression. The symbolic kernel calculated an equivalent expression, the zeros and the derivative.

This is how the Casyopée team tried to take this dimension into account. First, the environment's interface displays windows that help to organize objects of different status (figure 2.2): values of the variable defining the intervals in which functions are defined and where properties can be proven or conjectured, functions with proven or conjectured properties, expressions with various algebraic equivalent definitions of the same function, equations. This organization is dynamic (as in a spreadsheet) by recalculation of the objects after instantiation of the parameters, and after modification of the functions. The history exists as a 'notebook', designed to be used as a basis for writing a report or a proof.

Objects are designed to be consistent with the usual repertory of secondary mathematics. For example, a basic choice was to define the functions on a domain (interval or union of intervals), rather than by just an expression as in symbolic systems. Casyopée evaluates the existence of the function on the domain and, on request, calculates the greatest domain. It is an example of assistance that the environment brings to various steps of the algebraic activity.

Finally, the properties of the functions – sign, variations and existence of zeros – are obtained as results of proofs. The elementary steps of proof (justifications) correspond to theorems familiar at secondary level and to properties of 'reference functions'. Casyopée can directly take for granted properties that, because of their simplicity, are not explicitly justified in usual practice.

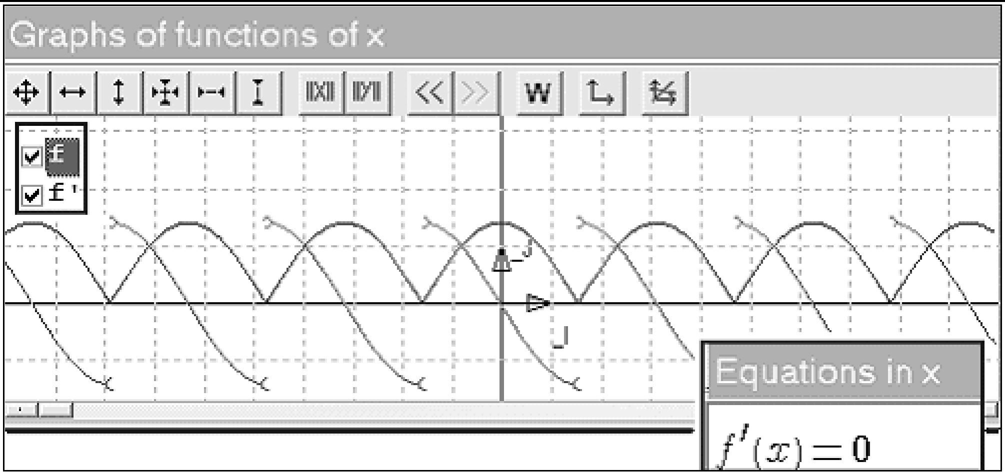

Figure 2.3: *Child windows*

From a "main function windows" a user can open five "child windows" : properties of functions involved in proof processes, exact and approximate values, graphs and equations. Each change in a main window is reflected in the child windows. Graphs and equations take the definition into account.

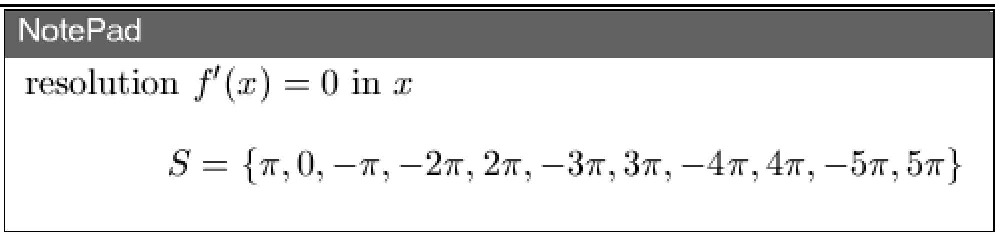

Figure 2.4: *The Notepad window*

Data resulting of actions (creation, computation, justification, equation solving) are automatically recorded in the Notepad (here solutions of the equation). Facilities for editing the Notepad and copying graphs are available.

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Classroom situations

This section presents two examples of classroom use of Casyopée. The first is a situation intended to help students progress towards a method for experimenting on algebraic expressions. It corresponds to our first concern (experimental activity) while also addressing the issue of a genuine algebraic activity especially at a global meta level. The second example is a situation where students used Casyopée to perform algebraic proof more effectively. The main concern is then the algebraic activity, especially *transformational* activity. These examples also provide insight into the influence of Casyopée's design on students' classroom activity, our third concern.

Towards a method for experimenting

You have to build a track for skateboarding. At one point the track is horizontal, and two meters farther it has to be horizontal again, but one meter higher. The goal is to find a function whose graph could be a track. The track has to be smooth. Try to make it as smooth as possible.

1° What axes can we choose? Which are the most interesting? Why?

2° What types of functions can we choose?

3° Use Casyopée to find functions whose graph could be a smooth track. Write a report on your work.

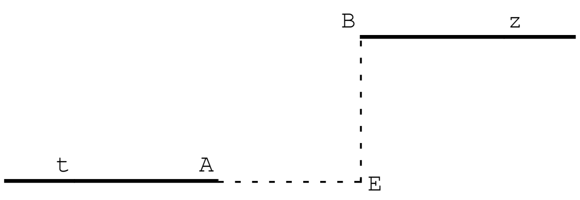

Figure 2.5 *The skateboarding track problem*.

The problem of figure 2.5 was proposed in a twelve-grade class, scientific stream, at the beginning of the year. The goal was consistent with the curriculum, that is to "motivate the study of functions by problem solving", especially problems whose solution uses the relationship between the properties of a function and its derivative. The session was two hours long in a computer room. The teacher had introduced the problem in a session before. Nineteen students were in this class. As often happens now in France even in the scientific stream, they had difficulties in algebraic manipulations and did not easily tackle problems by themselves but rather waited for the teacher's solution.

Mathematically, after choosing axes, a student has to look for a function *f* satisfying four conditions: $f(x_A) = y_A,\ f(x_B) = y_B \text{ and } f'(x_A) = f'(x_B) = 0$. He/she can think of cubic, piecewise quadratic and sine functions (figure 2.6). Depending on the type of function, the values of three or four parameters are to be found to satisfy the above conditions. Searching for these values can be done by animating parameters to adjust the curve and/or by algebraic calculations. In the team's *a priori* analysis, students would have not much trouble satisfying the first two conditions by animating parameters, but would less easily deal with conditions about the derivative. It was expected that the students could not obtain a solution just by animating parameters and thus the situation would bring to light algebraic conditions about the derivative necessary to get a 'smooth graph'. The use of Casyopée, was expected to help students to work by themselves in the session. Students were free to choose axes and a type of function (among those identified by collective discussion in the preceding session) to give them some sense of autonomy.

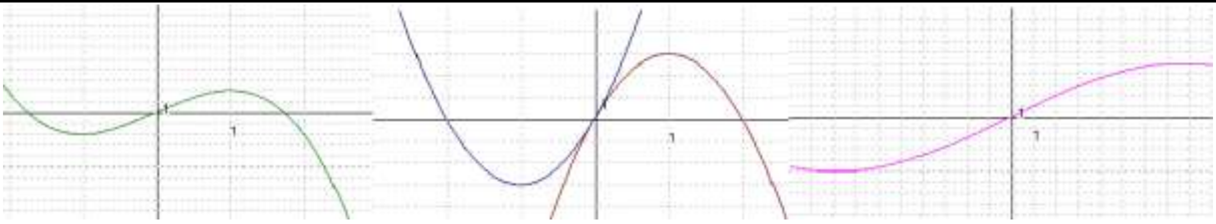

Figure 2.6: *Three types of function (cubic, piecewise quadratic and sine)*

<u>Report on the session</u>

Choosing axes

In the preceding session, students proposed axes with four different origins (A, B E and the middle of [AB]) and they usually searched for solutions in their own axes. After this session, classroom discussion made clear that setting the origin at the middle of [AB] helped to reach a solution more efficiently.

The team analysed this as an indication of the students' developing autonomy over decisions in their mathematical activity. This is a difference with another experiment of a similar problem (Artigue 2005, p.279) where students used the quite complex TI-92 calculator, and the teacher had to decide a common axis for all students, in order to engage students in a productive resolution.

Types of functions

Students found functions of different types. No type was chosen more frequently, which seems again to be an indication of students' developing autonomy. When a cubic function was chosen, the difficulty lay in the complexity of animating four parameters. The role of a parameter, like the 0-degree coefficient, is easily grasped, whereas the effect of changing other parameters confers little insight. The role of the parameters in a trigonometric function is more visible. Students nevertheless had difficulties in recognising the value $\pi/2$ after they found 1.6 by animating a parameter.

Animating parameters versus considering algebraic conditions

We observed very varied student behaviour. Some students persisted in randomly animating parameters, although it did not lead them to a solution. Most students organised the animation, giving constant values to some parameters and animating others. The remaining students discovered by themselves that writing algebraic conditions helped to decrease the number of parameters. Nearly all solutions were found by reflective animation. Seven students found solutions by themselves. The others could not reach a solution alone in the two hours, but most did personal and productive work that they could reuse after a collective synthesis to write a solution.

It seemed that Casyopée helped to make a relatively complex research situation 'live' in a class with students likely to be passive in normal lessons, with a clear integration into the curriculum[5]. Moreover, decreasing the number of parameters appeared as a generic method to solve this type of problem.

Building an algebraic proof

This is an example of a session for 11th grade vocational students in electronics. As mentioned earlier, proving is a way to give meaning to transformational activity. Proof, however, is thought irksome and irrelevant by many students, because in ordinary cases, conjectures can be validated through a graph or a table. Students experience difficulties calculating algebraic transformations, but also in organising and writing proof. The hypothesis for this session was that studying a function with a parameter could bring them towards a symbolic proof, and that Casyopée could help them not only by performing algebraic calculations, but also by providing for the means to build a proof.

In the vocational part students were learning about band-pass filters. These electronic devices attenuate all signals below a given frequency and all signals above another given frequency. Students considered a practical device made of resistors and capacities of given values R and C using an oscilloscope to observe for a given input tension V_{in}, the evolution of the output tension V_{out} against the frequency. They also calculated the transfer function, that is the absolute value of the complex quotient V_{out}/V_{in}, which depends on a parameter *T*, product of the device's resistance and capacity (Figure 2.7). They characterized the filter as band-pass because the limits of this function are zero for frequencies approaching zero and infinity.

In the mathematics classroom they had to go further in the study of transfer functions. Using Casyopée, after entering the function, students could perform algebraic transformations by means of the "Compute" menu (Fig. 2.8) and elementary proofs by way of the "Justify" menu. (Fig. 2.9) These actions have a result in the functions windows (new functions or expressions, properties and so on) and also produce information in the 'Notepad' (Fig. 2.10). A standard study begins by using the " Compute" menu to obtain the derivative. Then it is to get the factorised form and finally to select a sub-expression informative of the sign of the derivative by way of the same menu. Then the proof consists in the justification of the sign of this sub-expression and of the derivative. Note that Casyopée, as in ordinary practice 'admits' that a 'visibly positive' factor (for instance a square) does not change the sign of a product. The study is classically achieved by justifying the variation of the function by way of the derivative's sign.

[5] In France, the 12th grade is the 'Terminale', i.e. a class preparing to the baccalaureate and the curriculum's pressure cannot be ignored.

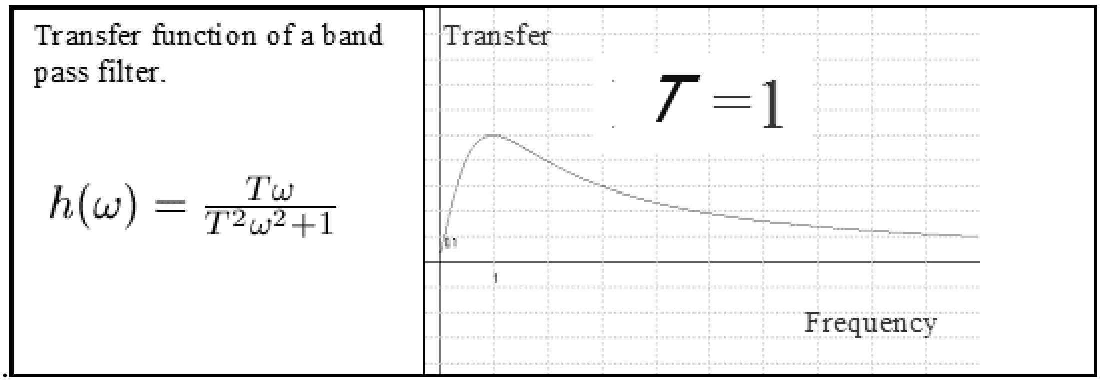

Figure 2.7 : *The band pass filter.*

Transfer function and curve for a value of the resistor and capacitor

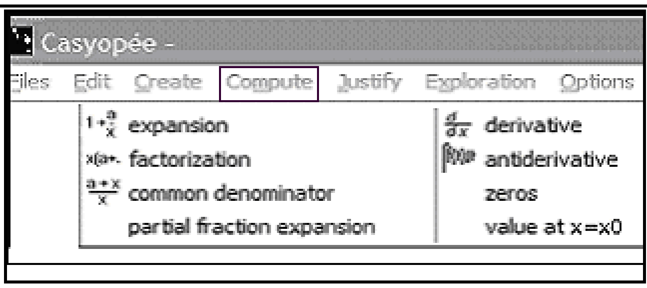

Fig 2.8: *The 'Compute' menu*

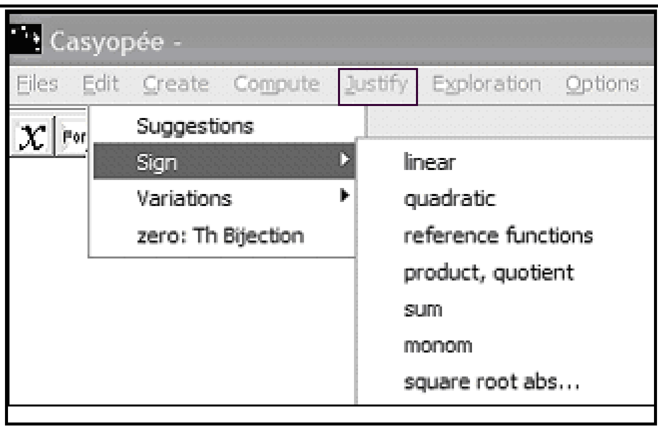

Fig 2.9: *The 'Justify' menu*

NotePad

Derivative of $h(x)$: $\frac{T}{(T^2x^2+1)} - \frac{2T^3x^2}{(T^2x^2+1)^2}$

Factorization of $h'(x)$: $-\frac{T(Tx-1)(Tx+1)}{(T^2x^2+1)^2}$

Function defined on $[0;\infty[$: $h'0(x) = Tx - 1$

Sign: linear $h'0(x)$
negative upon $]0;1/T[$; positive upon $]1/T;\infty[$

Sign: product, quotient $h'(x)$
opposite sign of $h'0(x)$
positive upon $]0;1/T[$; negative upon $]1/T;\infty[$

Variation: signs of derivative known $h(x)$
increasing upon $]0;1/T[$; decreasing upon $]1/T;\infty[$

Fig 2.10: *Indications given by Casyopée in the NotePad after (1) calculating the derivative, (2) factoring this derivative, (3) selecting a sub-expression, (4)justifying by 'sign: linear' (5) justifying by 'product-quotient' (6) justifying by 'signs of the Derivative'*

Students were familiar with the study of function, but here it was presented as a new task because of the parameter *T*. The 'electronics' context contributed, however, in providing sense to this 'generalized' study. Students were asked to do this study and then to use the information given in Casyopée's notepad to write a solution. For instance, students were expected to motivate their choice of a factored form of the derivative by a comment such as: *the factorisation is the form that permits to study the sign of the function.*

The team analysed the record of the students' actions and their written productions with the aim of evaluating how Casyopée helped them find and write proofs. Nearly all students correctly did the first part of the study, using the 'Compute' menu; most justified correctly the derivative's sign and half of them interpreted correctly the solution relatively to the nature of the filter. Globally, this is satisfying, because even students who did not totally succeed had a consistent approach to developing a proof.

Differences were observed in the written productions that students completed from Casyopée's notepad. Half of them merely added informative subtitles to the successive steps of proof while the other half produced results that were of better quality, especially when compared with their usual written work. Some productions were very personal, detailing the steps they had gone through in a narrative way, or, conversely, synthesising the proof by reorganising and rewriting the notebook. So there is some justification for acknowledging Casyopée's potential as an aid to writing a proof.

Summary of Section 2

This Section set out to explain how the development of a digital tool and associated classroom situations could usefully start from a selection of concerns, and is summarised in the table below. Aiming to make the algebraic representation more learnable brought about three concerns (third row) that defined three dimensions. In each dimension, a theoretical approach (fourth row) brought central ideas and concepts. The five central rows of the table show how basic choices made in the Casyopée project are related to each dimension and the two last lines put the two described classroom situations into relation with the three dimensions above.

The three dimensions are based on distinctive approaches of teaching learning and software development. Each of them helps to focus on specific aspects of the project, informing basic choices. Nearly all rows corresponding to basic choices have more than one empty cell, showing that choices in software development cannot be informed by a single dimension.

Intersecting the two examples of classroom activity with the dimensions has helped to evaluate how the work with Casyopée contributed to situations taking into account the three concerns. There is an obvious gap as generational activities were not involved in the present state of Casyopée. Casyopée works on functions given by algebraic representations and till now provides no enactive representation for non-algebraic functional exploration. One consequence has been limitations in the students' experimental activity, especially when modelling phenomena: except when the phenomenon is directly described by a curve, like in the skateboarding problem, students cannot build and try models by themselves.

This summary gives pointers to additional work: to develop features allowing students to work with enactive non-algebraic representations of phenomena and to pass fluidly between algebraic and non-algebraic representations[6]. Thus we argue that an analysis around dimensions provides a basis for future work in design, implementation, observation, and adaptation.

[6] This objective will be pursued inside the European project ReMath (Representing Mathematics with Digital Media http://remath.cti.gr/).

<table>
<tr><td></td><td colspan="3">Dimensions</td></tr>
<tr><td>Concerns</td><td>Classroom experimental activity</td><td>Students' algebraic activity</td><td>Software environment for classroom use</td></tr>
<tr><td>Ideas,

concepts</td><td><u>Anthropological approach</u>

Transposition

Techno-Experimental praxeologies</td><td><u>Epistemology</u>

Categories of algebraic activity</td><td>Design

Users' social and material environment

Problems that technology can remedy</td></tr>
<tr><td colspan="4"><u>Basic choices in the Casyopée project</u></td></tr>
<tr><td>Symbolic calculation</td><td>Relationship with mathematicians' tools</td><td>Transformational activity</td><td>Difficulties with standard symbolic systems</td></tr>
<tr><td>Proof</td><td>Methods for proving</td><td>Transformational and global meta level activities</td><td>Help to search conjectures and write proofs</td></tr>
<tr><td>Interface</td><td>Methods for experimenting</td><td></td><td>Students' control over experimentation</td></tr>
<tr><td>Objects</td><td></td><td></td><td>Consistency with the curriculum</td></tr>
<tr><td>Dynamic parameters</td><td></td><td>Global meta level activity</td><td></td></tr>
<tr><td colspan="4"><u>Examples of classroom situation</u></td></tr>
<tr><td>Smooth track</td><td>Methods for experimenting</td><td>Global meta level activity</td><td rowspan="2">Organisation of objects and actions at the interface</td></tr>
<tr><td>Filter transfer function</td><td>Help to find and write proofs</td><td>Transformational activity</td></tr>
</table>

Table 2.1: *Summary of dimensions, concerns, choices and classroom situations*

3. The WebLabs project: developing an alternative infrastructure

WebLabs was a three-year project (http://www.lkl.ac.uk/kscope/weblabs/) whose overarching aim was to create an alternative infrastructure with which students (age 13-15 years) in 6 different European countries could construct, share, comment on and evaluate representations of their evolving mathematical and scientific ideas[7]. There were two main focal points of our design effort. First, to construct a set of tools and activities, based on *ToonTalk* – a programming language in the style of a videogame - that allowed students to address various knowledge domains in mathematics in ways that resonated with the activities of an experimental lab: to do 'experiments', test conjectures, look for counterexamples, and share their evolving ideas. We designed and built a series of toolsets of working models of mathematical objects and relationships for students to think about and manipulate, which were transparent in the sense that it was easy to look not only at what the models *did,* but how they *worked*, through expressing them in a programming language rather than with standard numerical or algebraic notation. By making their thoughts 'visible' in the form of working models or programs, we hoped to leverage students' intuitions and add to them a formalism that would become generative in developing their understandings. Alongside this development we designed sequences of activities to explore several mathematical domains using the toolkits.

The second focus of the project involved the construction of *WebReports*, a web-based system that included simple mechanisms for uploading and downloading models to form a basis of collaboration, co-construction and comment. *WebReports* also afforded us a window on the ways students could share their models of evolving knowledge at a distance, what they felt was important to discuss, to change and manipulate, providing a way to assess how the new representational structures influenced the trajectories of student thinking (for a discussion of the role of expressive tools, see Noss & Hoyles, 2006).

Thus the twin objectives of the work were iteratively to design, develop and evaluate tools both for *constructing* and *sharing* evolving knowledge of mathematical relationships. The key idea was that learners could not only discuss, conjecture with and comment upon each others' ideas, but they could inspect and edit each others' *working models* of ideas, computer programs – rather than in algebraic notation - that instantiated the state of their current knowledge.

An objective of the WebLabs project was to explore the extent to which some of the apparent complexity and difficulties of mathematical and scientific ideas is due to the symbols and language used to express them. Such a hypothesis has a strong design implication, namely to develop a system and an activity structure in which students could express their ideas in novel ways, *without* sacrificing what makes the ideas powerful and rigorous. Our theoretical framework is based on two distinct and interrelated themes. The first is *Constructionism***,** an 'orienting framework' (Cobb *et al.*, 2003) suggested by Papert in the late nineteen-eighties as a

[7] We acknowledge the support of Grant IST 2001-3220 of the Information Society Technologies Programme of the European Commission. We also acknowledge the contribution of all the *WebLabs* team (from participating countries, Portugal, Bulgaria, Sweden, Cyprus, Italy as well as UK), and notably the UK researchers, Y. Mor and G. Simpson. See http://www.weblabs.eu.com.

pedagogical counterpart of *constructivism.* The idea is that students learn by building with appropriate tools, virtual 'external' realities that mirror their developing mathematical or scientific meanings, and by sharing this public or semi-public entity with a community. The second orienting idea that guided our design decisions was to exploit the benefits for learning mathematics and science of collaborative interaction, by including (mainly asynchronous) discussion and evaluation at a distance as part of the programme of activities, as well as face-to-face interchange.

To support the work of WebLabs, we designed sequenced activities in five knowledge domains: *Sequences, Infinity, Collisions, Lunar Lander* and *Models, Systems and Randomness*:

a. In *Sequences* students construct and analyse number sequences, which after common introductory activities focussed either on the Fibonacci sequence or on explorations of sequences that converge and diverge.
b. In *Infinity,* students explore the cardinality of infinite sets and the relationships between different infinite sets.
c. In *Collisions,* students build models of objects colliding in 1-dimension, iteratively test them against reality and refine their models to cover more cases of collision.
d. In *Lunar Lander*, students control the motion of virtual objects, record data and plot the resulting position-time and velocity-time graphs, thereby investigating acceleration and the relationships between different representations of motion.
e. In *Models, Systems and Randomness* students build computational models that represent and explore various real-world phenomena, and investigate the concept of randomness, and how it could be understood and used.

In each domain, we expended considerable effort in iteratively designing these sequences of activities; starting from distinguishing the core epistemological ideas of the domain, predicting potential obstacles and then building tools that would assist exploration and problem solving. The activity sequences were also designed to fit into, complement or extend the present mathematics curriculum for secondary school students. We are unable here to deal with more than a small fraction of these activities: the interested reader may wish to consult www.lkl.ac.uk/kscope/weblabs/. In what follows, we distinguish between *design* outcomes – a main focus in this chapter – and then report on general *learning* outcomes, which we illustrate with specific examples of what some students achieved with our system.

The methodological approach

We provide a brief overview of our methodological approach. In order to research both technological design and learning, our methodology fits the paradigm of the *iterative design experiment* - theory-based interventions that aim for specific learning goals alongside the development of theoretical frameworks for learning, in general and within a particular knowledge domain (diSessa & Cobb, 2004). We have sought to discover how different aspects of learning are supported and mediated by the toolsets and the activity systems we designed. Our approach was iterative, in the sense that initial evaluations of the research team, collaborating teachers and partners in other countries fed into subsequent phases of design. Technical development, assessment of engagement with the core ideas proceeded in tandem and informed further design cycles.

This methodology was challenging. It inevitably drew on inter-disciplinary expertise as well as requiring systematic evaluations of learning based on prior research, while remaining open to the potential of the new tools. Our evaluation of learning was almost entirely qualitative, largely because of the small numbers of students (a maximum of two classrooms in each country) and their diversity in terms of language and prior attainment. We stress, therefore, that this research aimed to provide proof of concept – to describe the conditions (technical, cultural, pedagogical) in which learning took place, and the specific kinds of domain-specific learning that could occur. In what follows we give examples of the design outcomes of the project; we then briefly describe our pedagogical approach and activity structures, illustrate them by one example of an extended interaction; finally, we summarise the learning outcomes of WebLabs and suggest its limitations.

Design outcomes

We chose for our modelling environment, *ToonTalk*, a concurrent constraint-based programming language, in which the source code consists of actions of animated cartoon-like characters. ToonTalk is an object-oriented language so that tools can be attached to the back of any object to give it functionality and reused, inspected, or combined. The basic idea of ToonTalk is that it provides a rich programming environment capable of supporting the construction of, for example, games, simulations and animations: it is a general-purpose computation engine with an interface that is concrete and playful. We will give a flavour of what is involved in what follows: for the moment, we should simply note that there is no textual editing involved in ToonTalk programming, that it involves manipulating animated characters rather than, say, static icons (more information can be gained at www.toontalk.com).

From a design point of view, a programming environment provided a necessary but far from sufficient basis for our system. By analogy with, say, Lego bricks, construction of complex structures is substantially facilitated if one is provided with ready-made working parts of modules that can be combined into larger more complex structures, but which can also be broken apart to their elements to see how each works. This idea of *layering* turned out to be an important design criterion, and as the project developed, we sought to understand how the different *layers of interaction* that characterised student engagement with the system and its related activities, could engender learning at different structural layers of knowledge – what students came to express and know about specific pieces of knowledge.

Before such layers could be developed, we faced the challenge of tuning the substrate on which they could be built; that is, to enhance ToonTalk in many different ways so that the representational infrastructure could support innovative expression. We begin, therefore, by pointing to three examples of how ToonTalk was tuned to provide learners with the right kinds of functionalities required for their activities.

Extending to very large numbers

Standard ToonTalk only supported the standard computer programming size for integer numbers. Yet as our activities developed, it became natural to encourage learners to imagine what would happen if their robots continued to run forever, generating larger and larger integers. Accordingly, we devised a means by which programs could produce *very* large numbers, supporting integers of *any* size within memory limitations. At one level, of course,

in Maasz, J, & Schloeglmann, W. (eds) *New Mathematics Educations Research and Practice*
Sense Publishers ISBN 9077874747, (2006) pp 278- 300

very big integers behave just like small ones: the laws of combination are the same, checking whether a number is divisible by, say, 3 involves the same algorithm and so on. But there are possibilities that open up with very large numbers that generate a sense of surprise – an unexpected pattern in the final digits of 100! for example – and a sense of engagement that accrues from being able to – literally – hold in one's hand and integer that has tens of thousands of digits.

There are dangers too. We intended that such activities would be part of a transitional set of activities during which it became logical to ask what would happen if the number of digits (or the number itself) actually became infinite. Of course, gaining a sense of what happens "at infinity" could easily be seen as being at odds with what happens when the integers are 'merely large'. Plenty of scope, here, for what mathematics educators could label as "misconceptions"! But there is also a sense in which very large numbers almost demand questions about infinitely large ones: if one has a sense of a number taking longer and longer to write, then it becomes acceptable to ask whether one could write one just a little bit longer, or a lot longer.

Writing, of course, has its own limitations. It is easy to imagine that 100, 1000, 10000 is a sequence that could go on for a *very* long time. After a few terms, it seems rather cumbersome to apply pen to paper. But supposing one could go for a walk along a number, looking at patterns or, looking *for* patterns? See Figure 3.1 for a possible view of the situation, which involves looking at – actually walking along –100!.

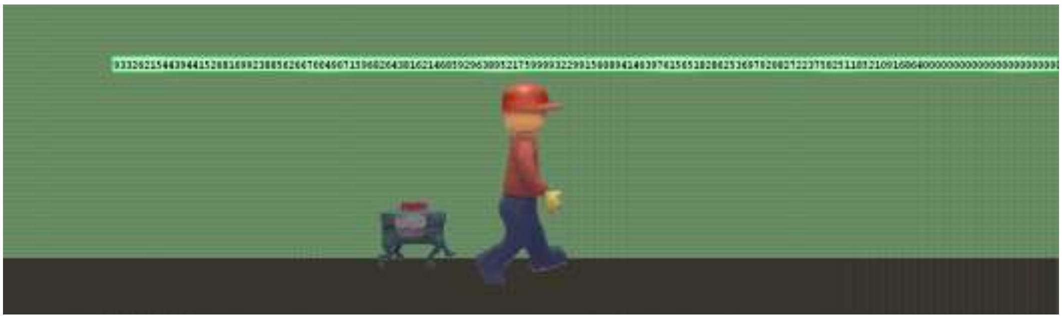

Figure 3.1: *The result of a process that computes 100!. The programmer can literally "walk along" the length of the number to get a sense of its size.*

It is worth asking what kind of affordances this (relatively small) change in representational form might make? Consider a simple example. Any teacher of mathematics knows that students routinely confuse squaring with doubling. 3 squared is 9; 3 * 2 is 6. There is not much difference! More seriously, there is no real sense of what squaring does (as a function), particularly when examples are routinely confined to small numbers less than 10. Now consider programming a ToonTalk "robot" to produce 1,000,000,000 squared. Laying the result out as in Figure 3.1 will soon reveal how much "longer" it is than the same number doubled. "How much longer?" becomes a sensible question, and one that generalises to cubes and so on. Merely being able to walk up and down numbers, and get a sense of their size makes – potentially at least – a huge difference to the kinds of questions it is natural to ask, and the sorts of knowledge that are likely to be developed.

One last point. While we remarked that walking along a number is rather a different way to think of it compared to writing it down, we might ask how else we could "view" such an object. In ToonTalk it is possible to zoom out and look at any object from above, in a helicopter. This revealed a surprising (to students) fact about a very large number (in this case 10000!): namely that there was a large number of zeros at the end, a fact that would have been rather time-consuming to reveal if one was confined to walking!

Infinite Decimal representation

Part of our evolving set of activities involved students interacting with rational numbers. For example, in our work on infinite sequences and series, we engaged students with the sum of sequences like 1, ½, ¼, $^1/_8$, ... and 1, ½, $^1/_3$, ¼, ... In such a scenario, there are several difficulties with the conventional representation. The first is evident with the use of ellipsis to denote "and so on". Not all students see that, for example, 0.1428571... as an infinite decimal, preferring instead to seeing 1 as the "last" digit. Indeed, the fact that it takes an infinite number of digits to represent a tangible entity like $^1/_7$ is a paradoxical situation for many students – the difference between a number and its (various) representations is far from obvious! So a second difficulty – more serious than the first – is that it is, in conventional representations, *impossible* to write down an equation like $^1/_7$ = 0.1428571 without some convention peculiar to the representational infrastructure (such as judicious placing of dots either at the end, or above some of the digits).

Our challenge, therefore, was to eliminate rounding errors. We achieved this by the implementation of exact rational arithmetic in ToonTalk. In ToonTalk, it really is the case that there is an exact decimal expansion of a rational number, and moreover, that this is recognised by the system ($^1/_7$ = 0.1428571... is "true").

But how to represent the "..." to the right of the decimal expansion? Clearly this is a serious design challenge: no truncation should return 'true', yet there *is* a decimal expansion of $^1/_7$ that is *exactly* equal to it. We remark in passing that we met this situation many times in our iterative design process: solving one problem of representation threw up a new problem.

Our solution was to invent the idea of *shrinking digits*. Digits are displayed in gradually decreasing size until they reach the size of a pixel. In this way the idea that an infinite number of digits follow the decimal point is conveyed visually. By using the ToonTalk ‘pumping’ tool for increasing the size of an object, a student can view more and more of the digits that initially were too small to see. This process can take place indefinitely: there is a theoretical size limit based on the memory of the computer, although there is nothing to stop the process being transferred to a second computer when the memory is full! Figure 3.2 provides an illustration of a decimal representation of the rational number $^5/_{49}$.

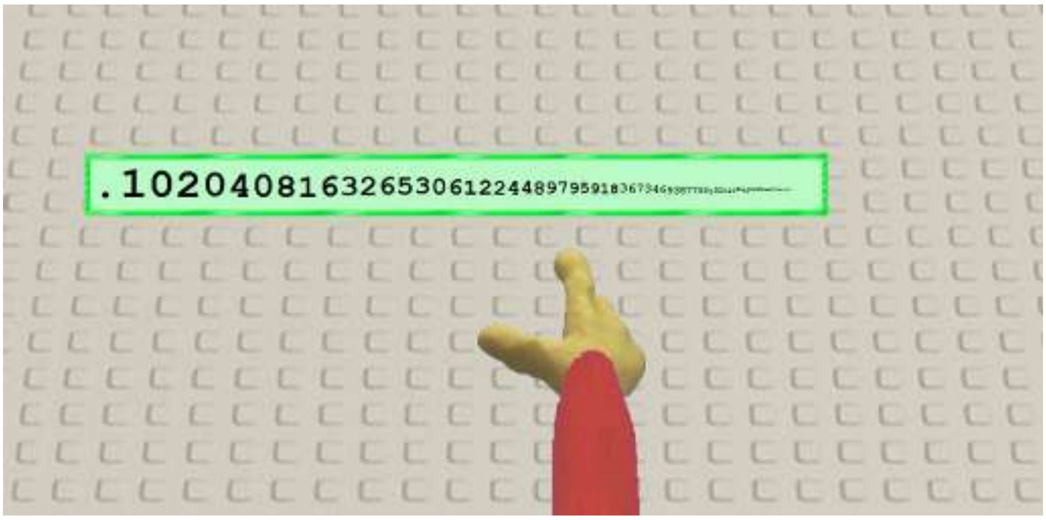

Figure 3.2: *An example of the new shrinking digit display, showing the result of dividing 5 by* 49.

Once shrinking digits were implemented, we could incorporate in activities discussions of the equivalence of different representations of the same number. Figures 3.3 shows how ToonTalk tests for equality and inequality by showing a balance between any two numbers, regardless of how they are represented (as rationals, infinite decimals or mixed fractions).

Fig 3.3: *a) The fraction 54/49 is the same regardless of whether it is displayed as a proper fraction or an infinite decimal expansion.*

Fig 3.3: *b) If both sides are multiplied by 49 the results are exactly 54.*

Fig 3.3: *c) If the decimal is approximated then it is not equal to the original fraction.*

Fig 3.3: *d) If the original fraction and the approximation are multiplied by 49 the approximated decimal expansion no longer becomes exactly 54.*

The WebReport System

We now turn to the collaborative dimension of the work. As we explained above, we designed a web-based collaboration system, called *WebReports*. The primary aim of this system was to allow learners to reflect on each others' work by sharing working models of their ideas. To help the students navigate the system, it was organised around the different knowledge domains each with a repository of tools, and online guidance as to how to use the system in general as well as hints to support both teacher and students in their exploration of the knowledge domain. Figure 3.4 shows the front page of the system.

Fig 3.4: *Front page of the Webreport system showing the topics available, the 10 most recent reports and the link to the repository of tools*

WebReports could include formatted text, comments, diagrams and multi-media objects, and most importantly – ToonTalk models. These models were embedded in the report as images, which linked to the actual code object. When clicked, they automatically opened in the reader's ToonTalk environment – which could be in another classroom or another country. The reader could then manipulate the object, modify it, and respond with a comment that might include her own model.

Interaction between groups and individuals was promoted by a layered commenting facility, inspired by Knowledge Forum and the work of Scardamalia and Bereiter (2006). We wanted, like them, to build a community of learners of mathematics who would increasingly take control of monitoring their own learning, sharing and building on ideas and raising counter examples to refute conjectures. Thus each report ended with a selection of comment options, which included "Can you explain?", "What if...", "I have a conjecture..", and "This doesn't work because…" as well as a box to insert a new custom comment type or an unclassified comment. Commenting on someone else's report provided the same functionality as posting a report – a *wysiwyg* editor and the facility to include images, embedded ToonTalk objects and external links. Comments could also be posted as replies to other comments so that threads of discussion could be created (in much the same way as in internet newsgroups) and monitored.

The idea was that after discussion of a phenomenon in a group at one site, the group would publish a report of their collective observations, models, conjectures and conclusions. The key idea was that they would focus on the *process* of reasoning (the construction and then running of ToonTalk programs) and then illustrate this with outcomes that might be, for example, sequences of numbers, or spreadsheet graphs, that could then become the subject of discussion and further experimentation. Finally when a task sequence was completed, we planned that groups would publish a concluding report devised after extensive within-site negotiation to achieve a consensus and through this report they would share conclusions with remote peers. Thus in order maximally to exploit the collaborative dimension, we developed a common frame for activities that evolved iteratively – these focused on intra- and inter-classroom collaboration (see Figure 3.5). For some background to the theoretical rationale for such an approach, see Hoyles *et al.* (1992).

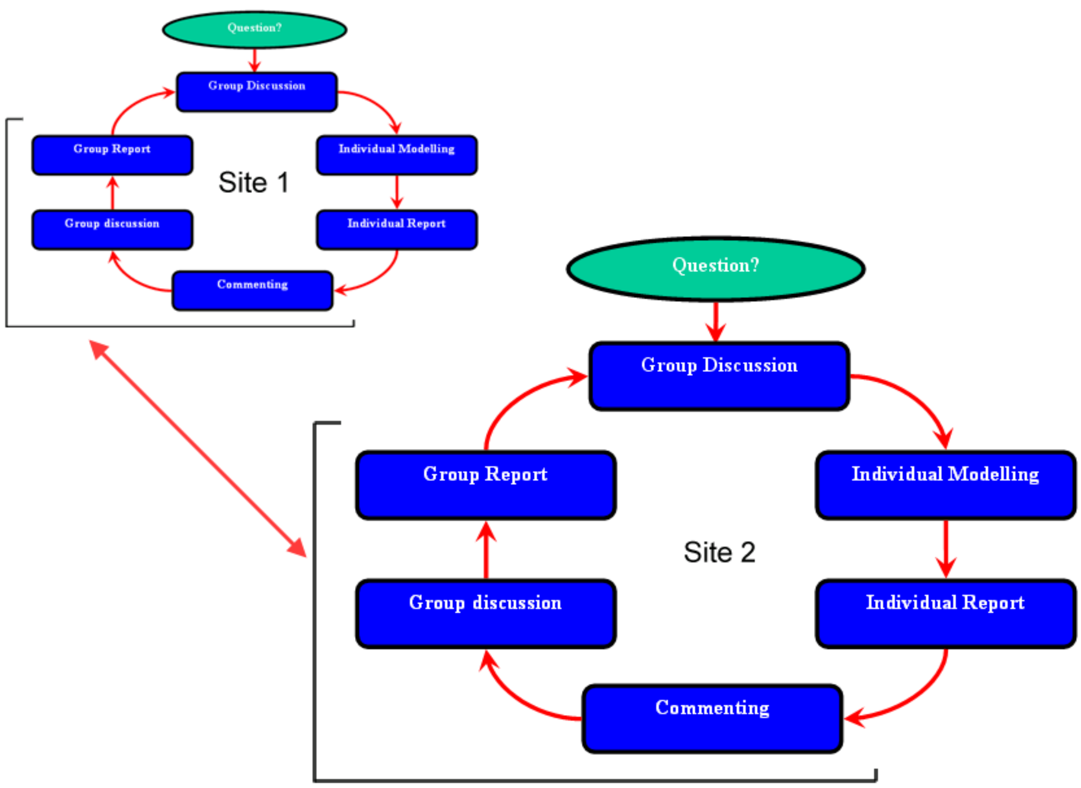

Figure 3.5: *The common frame for WebLabs activities*

So the innovative component of WebReports is that in addition to students talking to each other about what they *think*, they can discuss what they have *done*: the models they upload with their discussion become objects to argue about, modify, reconstruct and build upon. Building on the success we achieved with students sharing each others' ToonTalk models, we found a way to pipe data from running programs directly into Excel, so that students could easily generate and upload graphical representations where appropriate: see Simpson *et al.* (2005; 2006) for detailed examples of this functionality and the way students appropriated it.

Our focus so far has been on the design of tools for construction and collaboration and a general framework into which they were inserted. We now turn to describe the activity sequences we iteratively devised after experimentation with students and the pedagogical approach we planned to adopt.

Pedagogy and activity

Having set up our general framework, we designed, again iteratively, sequences of activities in each knowledge domain that sought to exploit the representational system we had designed. Each activity sequence had explicit overarching learning aims as well as aims for each of its component tasks, each taking into account the mathematical background of the students and the curriculum they would have followed. Each sequence also focussed on the tools to be used, the need to encourage prediction and reflection, and our intention to capitalise on collaborative exchange, both face-to-face and at a distance, as relevant to the knowledge domain. Rather than describe the process in general, we present part of one activity sequence with respect to one knowledge domain, *sequences, cardinality and infinity*, which is particularly relevant to a discussion of an alternative infrastructure for algebra. Finally, we provide an illustrative example of the implementation of one activity.

Sequences, cardinality and infinity

In many countries pattern recognition and generalisation are considered fundamental to mathematical thinking, and a fruitful pathway into algebraic thinking. Yet at the same time, a number of researchers have pointed to the difficulties students encounter in shifting from pattern spotting to structural understanding (Stacey 1989; Lee and Wheeler 1987; MacGregor and Stacey 1992; Arzarello 1991; Hoyles & Noss, 1996).

A set of activities was designed for students to investigate number sequences with the main aim being for them to learn to reason and argue about the structure of number sequences. Students started by modelling the most basic sequence: the natural numbers. However, the way we encouraged them to model in ToonTalk afforded easy generalisation to any arithmetic sequence, and later to any iterative sequence, developing a shared language for describing their sequences that formed the basis for mathematical discussion. How was the sequence generated? Were different generating rules mathematically equivalent? Could different sequences be generated by the same programs? We tried to produce situations that generated surprises and we then formulated two different directions – one pointing toward Fibonacci sequences, and the other to an exploration of convergence and divergence – which were both tried with students in different classrooms. Figure 3.6 outlines the structure of the activities in the number sequences domain.

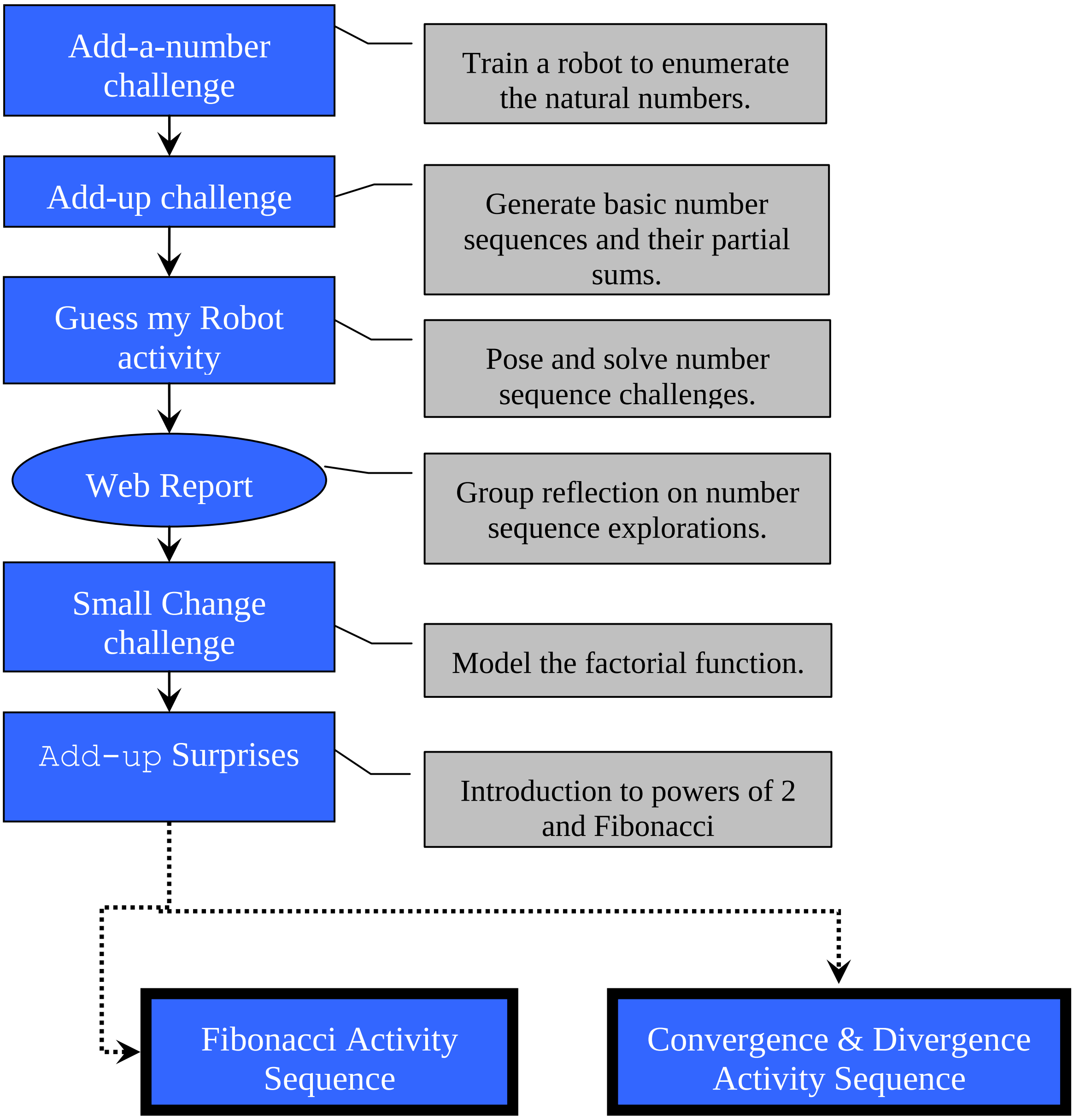

Figure 3.6: *the number sequence activities.*

From the point of view of this chapter, our focus is on how the representational infrastructure shaped – and was shaped by – our intention to highlight the collaborative dimension. We now illustrate this by reference to one episode of student interaction during one activity, the *Guess my Robot* activity.

An episode from Guess my Robot

The key players in the story were Rita, a 12-year old girl in a school near Lisbon, and Nasko, a 12-year-old boy in Sofia[8]. The Sofia group consisted of 6 boys and girls, aged 11-12, working with *WebLabs* researchers. They had been working with *ToonTalk* for several months, approximately once a week for a couple of hours. The second group was from a village south of Lisbon. Paula, a teacher and researcher in the *WebLabs* team, worked with a school group there (aged 12-13) during the first project year. Researchers in both groups acted as teachers, guiding the students through the mathematical ideas and activities as well as through the programming skills. At the same time, the researchers facilitated collaborative interaction, by pointing children to interesting and relevant peer Webreports and helping them to add a few words in English to their own reports.

The activity was based on the well-known “Guess my rule” game, which has been used in many classrooms over many years to provoke children to discuss and compare the formulation of rules. In its classical form, it was used as an introduction to functions and to formal algebraic notation. As Carraher and Earnest (2003) have recently reported, even children in younger grades enjoy participating in this game, and can be drawn into discussions of an algebraic nature through playing it.

Our version of the game was somewhat different. The idea was that a student set out a challenge, in the form of a sequence produced by a robot (a ToonTalk computer program) he or she has built, and posted it on the *Webreport* system in the form of the first few terms. The challenge was for the second player to produce a robot that resulted in *the same* sequence. Responders had to build a robot that would produce this sequence, and in doing so work out an underlying rule for its generation. The new element in our variant of the game was that “rules” had to be encoded as programs: one responded to a challenge sequence by posting a program that produced “the same” sequence. Managing to reproduce someone else's sequence by training a robot was the way to show that a learner had grasped *how* the sequence might have been originally generated. As one girl said:

“*So, like, the robot is my proof that I got it?*”

Rita found the 'guess my robot' activity, and decided to pose her own challenge. The sequence she posted was *2, 16, 72, 296, 1192 …* (see Figure 3.7).

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[8] This episode is based on a description that is due to appear in Mor, Hoyles, Simpson & Noss (submitted).

in Maasz, J, & Schloeglmann, W. (eds) *New Mathematics Educations Research and Practice*
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Figure 3.7: *Rita's "Guess My Robot" challenge. The five numbers in the boxes were produced by Rita's program (a ToonTalk robot).*

A few days after she had posted her Webreport, the Sofia *WebLabs* group held a session, and some of the students tried to solve Rita's challenge. Nasko posted his response. He had built a robot that produced Rita's five terms, but the robot turned out to be different from Rita's. Nasko also realised that the same robot could be used to generate other sequences by changing its initial inputs. So, he posed a two-part challenge back at Rita:

- Could she use *his* robot to generate a new sequence of five terms?
- Could she use *her* robot to generate the same sequence?

We remark in passing, how questions of uniqueness and existence arose as part of the collaborative exchange, apparently naturally – if one *sees* a set of numbers being generated by two robots, it is natural to ask whether each robot is doing the same thing (and conversely, if two robots are each generating different sequences, whether they must necessarily be different).

After a few days, Rita came to her next session to find comments on her page – and from children on the other side of Europe! She immediately clicked on the *ToonTalk* robots in the responses, and watched them step through the process of rule-generation. She was totally surprised: Nasko and Ivan had solved her challenge, but their robots seemed completely different from hers. We will suppress details of the evolving story. Here we will focus on just one 'ending', which involved Rita's response to Nasko. She worked out what inputs Nasko must have given his robot, and showed that *her* robot could in fact generate the same output as his. Her response nicely captured the way the structure of the programming system shaped her thinking.

Later we asked Rita again: "How did you know that the two robots generated the same sequence"? The next day, Rita surprised us. She had generated two robots, one was hers and the other Nasko's. Then she has made a new robot that subtracted one stream of outputs from the other and had watched the robots create a stream of zeros. She had generated thousands of zeros in this way and was convinced that this was a 'proof' of her conjecture that the sequences were the same.

Well, not quite a proof, at least not in any conventional sense, but one that was generated by the tools available and became acceptable in the community of students engaged with the activity. If Rita had found that there were 6000 terms that were equal to zero, she was as likely to wonder if the 6001st would be zero, as to assume that *all* terms to infinity would be zero. The point is not that Rita had constructed a proof for her community of the equality of the robots (incidentally, she has implicitly defined two robots as equal if they generate the same output for ever). She had developed a tacit situated abstraction: "For any two corresponding actions of the two original robots, my robot will produce a zero." From a pedagogical point of view, the construction of large but finite streams of zeros *raises the question* of equality in a natural way, which could provide the basis for a conventional proof in some future pedagogical scenario.

We end this section by highlighting some issues illustrated by this episode. First, we point to the way in which the responses made by the children was shaped by the models and by their medium of communication. This was particularly visible in Rita's unexpected (by us) construction of a new robot to generate the differences. In fact several times in this activity sequence when their mathematical argument was challenged, students initiated programming a model to support it or reject it. They displayed the confidence to reflect on their own errors either individually or collectively and were able to compare or accept algorithms and use counterexamples to refute theories, a remarkable focus on the processes by which the sequences were generated, and in rather stark contrast to usual interactions with algebra.

Second, we recognised the substantive possibility opened up by the imperatives of asynchronous discourse. The formality required for articulation in Webreports was shaped by the need to communicate unambiguously. We also note the power of sharing models and ideas in a dynamic medium when embedded in this game like situation. This undoubtedly provided a strong affective component, to respond to challenges, to build on them or rebut them and then finally to decide if there were any equivalences in the responses provided. This method of interacting at a distance was generative in developing similar activities in other domains, and we referred to them collectively as *Guess my X* activities. Guess my X seemed to attract more sustained participation than, for example, the development of group webreports, by providing a good balance of competition and consensus; the need to negotiate the criteria for assessing equivalence.

Learning Outcomes

While the focus of this chapter is on the design of an expressive alternative infrastructure of constructing and sharing, we turn – very cursorily in the form of a brief summary - to report some general learning outcomes of the project, based on an evaluation methodology that was essentially qualitative, although supported by some quantitative data.

Developing a rigorous language: Students developed a model-based language and symbols to express ideas rigorously that served at least in the limited domains under investigation as an alternative to algebra. Judged by the criteria of the quality of WebReport interactions, and the nature of classroom discussion, we saw an emergence of structural reasoning based on models, and an improved awareness of mathematical ways of thinking, including fundamental ideas such as generalisation, invariance, consistency and proof.

Layered learning: Students worked at different layers: running ready-made models/programs; inspecting programs and changing parameters; modifying programs and programming from scratch. The depth at which the students interacted with the system influenced the quality of their engagement with the topic.

Motivation and engagement leading to enhanced responsibility for learning: Having built models around a long-term motivational challenge, learners became committed to them, and were willing to argue about the correctness of the models posted by others. They did not always change their mind when confronted with conflicting models and arguments but nonetheless engaged in extended argumentation, and took responsibility for their own and their group's learning, quite unlike normal classroom interactions. Students' awareness of their audience was a strong motivational factor, provoking them to invest great efforts in articulating and illustrating their arguments.

Interpreting and comparing representations: We found students better able to make connections between different representations through constructing and sharing them, by identifying with the structure of the representational system. The facility to share comments and hypotheses together with working models was a valuable tool for critiquing and sharing representations of mathematical and scientific phenomena.

Collaboration and formalisation: Asynchronous communication through the WebReports encouraged formalisation, since the normal contextual cues were absent from the interaction. But for collaboration to be effective, we needed to design carefully for it. We identified the need for the distributed group to sign up to a joint enterprise, which could be the development of a shared product, but could also be engagement in a game or series of challenges, such as in Guess my Robot, which was extended to a more general category 'Guess my X'. We also recognized the importance of a group facilitator to ensure sustained interchange.

Effective inter-site collaboration: When inter-site collaboration was successful it tended not to be about group knowledge building but evolving products or cumulative challenges. Final group products as originally conceived, were rarely achieved, largely due to pragmatic reasons of language and curricular organisation and the fact that the production of an inter-site group product tended to be counter to traditional school culture in terms of the long-term engagement necessary and the requirement to pursue collaborative rather than individual goals.

Time to develop fluency with tools: Substantial time and effort was necessary before students could become fluent enough with the programming tools and with the WebReports system to engage with the tools and to express their ideas to each other. Nevertheless, particularly with the students of 13 or 14 years, we found a surprising readiness to learn and become fluent with the interface and the tools.

Summary of Section 3

In this section, we have outlined what becomes possible with a radically different infrastructure from the standard algebra which was developed with static media. Nevertheless, such an innovation brings with it new and often unforeseen difficulties. From the research point of view, the most challenging element has been to explore the sustainability of an online community. This is a far from trivial enterprise. We saw from the example above that students were beginning to develop autonomy and to be able to manage their own learning, listen to, challenge and learn from others from diverse backgrounds, as well as manage multiple technologies. Yet sustained interactions of this nature were rather rare and for the most part, only happened with appropriate facilitation from a teacher or researcher: thus students did not necessarily engage with the distributed community spontaneously. We should not be surprised: the teacher's role does not disappear in this new scenario, although it certainly changes. In fact, there is a research agenda here. This should include studying the mentoring role, which becomes necessary to balance the trend towards student self-managed mathematical work and the need for guidance and instruction, the kinds of support that foster collaborative engagement and, perhaps most challenging of all, the extent to which *some* of the 'functionality' of the teacher might, with the necessary computational support, be devolved to the system.

4. Comparing and contrasting Casyopée and WebLabs: a contribution to convergence in Mathematics Education

The ambition of this book is to combat the fragmentation of knowledge in Mathematics Education arising from the wealth of research undertaken in many different countries and situations. Although the authors of this chapter share many ideas and conceptualisations, the differing approaches of Casyopée and WebLabs could be interpreted as instances of this fragmentation arising from the different research cultures and contexts in which the two projects were developed. Casyopée was much influenced by the work around the classroom implementation of computer symbolic computation and associated theoretical reflection in terms of instrumentation and praxeologies whose genesis was analysed by Artigue (2002) and more recently by Monaghan (2005). WebLabs was derived from constructionism[9] and more than two decades of research in this tradition from which notions such as situated abstraction and webbing had been derived (Noss & Hoyles 1996).

There are, however, points of convergence. The first point was articulated in section one: most students do not have access to interesting and complex mathematical problems because the traditional school setting does not allow them to master the symbolism necessary to express solutions. A second point is that both projects consider that creating new representational infrastructures for mathematical expression is essential for the future. A third point is that both projects emphasised the process of design on the basis of carefully selected criteria, both focussed carefully on key mathematical objects and relationships (related, perhaps, to what Schweiger – this volume – calls 'fundamental ideas') and both adopted iterative trialling to adapt designs on the basis of feedback from trials with students.

From this starting point, this last section will try in more depth to compare and contrast the two projects with the aim of drawing common trends linked to technology use in mathematics

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[9] For Harel & Papert (1991) “constructionism shares constructivism's connotation of learning as "building knowledge structures" (and) then adds the idea that this happens especially effectively when learners are engaged in construction for a “public” audience".

teaching and learning and more generally to mathematics education. Before this, we develop the idea of a plurality of dimensions as a tool to analyse educational uses of technology.

A meta-study of publications about digital technology in mathematics

The idea of a plurality of dimensions as a tool to analyse educational use of technology came, at the end of the nineties, from a contract that researchers in France had with the Ministry of Research to do a meta-study of publications in the educational use of digital technology. The Ministry wanted to know what were really the efficient uses of technology for teaching and learning. It initially appeared that it was an impossible task, since from reading all the literature about ICT and Education, it appeared that although researchers found a substantial amount of interesting potential uses of technology, this contrasted with what they knew of the poor classroom integration of technology (Lagrange et al. 2003).

Thus the researchers found it more useful to search for reasons for this discrepancy. Assuming that the classroom situation is complex and that technology introduces even more complexity, the hypothesis was that much research and innovation failed to take this complexity into account because it tended to restrict its focus to only a few dimensions. The first step of the methodology was to take a broad view of all publications (nearly 800) we could access from the years 1994 to 1998 in order to identify dimensions of analyses from questions or concerns that authors put forward to justify an innovation or a research study. Then, a set of 79 research papers was selected on the basis of representativity and quality. This set was analysed statistically to specify how these dimensions were taken into account by research and to identify trends. Finally, the analysis focused on ten papers representing these trends.

The first step produced six dimensions. The second step led to a classification into three groups: two dimensions were widely considered, two had limited consideration and two were 'embryonic' in a sense that will be explained later.

Widely considered dimensions

A first dimension (epistemological and semiotic) considered the influences of ICT on the mathematical knowledge taught and on the way mathematical objects could be represented and manipulated. Most papers considered this influence, generally seeing advantages to new meanings and new ways of representing mathematics that technology fostered.

The second dimension dealt with cognition: many papers offered a cognitive framework within which to explain how the student might learn with ICT, referring to general mathematics education frameworks or to more specifically technology-oriented theorizations, using a wealth of concepts. Jones and Lagrange (*ibid*.) pointed out some of these and stressed that further work needed to be done to understand their connections and specificities.

Dimensions given limited consideration

The meta-study had prepared two dimensions that were thought important while reading literature in the first step. The first is the situational dimension. 'Situational' refers to the work of Brousseau (1997) but, in our meaning, this dimension was not necessarily linked to a specific didactic theorization. We meant that a learning situation had an 'economy', that is a specific organization of the many different components intervening in the classroom and that technology brings changes and specificities in this economy. For instance, technological tools have a deep impact on the 'didactical contract', which is a continually evolving agreement between teacher and students about this organization. Thus one would have anticipated a need for 'situational' analysis given the wealth of new situations provided by the literature. Surprisingly, very few research papers were identified that took such an approach.

We also prepared a dimension of analysis of the role of the teacher, because it appeared to us that benefits or disadvantages of technology reported or assumed in the literature could not be explained without considering the many aspects of the teacher's classroom preparation and management. Again, very few papers investigated research questions related to this aspect.

"Embryonic" dimensions

We designated as "embryonic" the two remaining dimensions, because they were not explicitly mentioned but concerns and analyses that could be interpreted in these frames were found in some research papers.

Instrumental dimension

The instrumental approach (Lagrange 1999, Artigue 2003, Trouche 2005) takes a tool first as an artifact. For instance a scientific calculator is at a material level just plastic and silicon. A human being has to elaborate an instrument from this artifact.

The following ideas are important in this dimension :

- the instrument is built during human activity;
- this activity is dependent on features of the artifact: precisely, its constraints and potentialities;
- it has two components, the first one –instrumentalization-- is directed toward the artifact, when the human being creates uses of the tool for himself: the second --instrumentation-- is directed towards the human himself when he builds understanding of the tool's operation[10] .

The instrument is therefore a mixture of features of the artifact and a mental construct of the user. The process of elaboration is what Rabardel names 'instrumental genesis'. In the case of tools to do mathematics, a student learns mathematics while instrumenting the tool. That is why we speak of interwoven mathematical and instrumental genesis, which means that mathematical understanding will be dependent on features of the tool and that schemes of use of the tool will be dependent on mathematical knowledge. This notion complements the idea of situated abstraction, in which the tool shapes the evolving conceptions of learning while, at

[10] This understanding is not just declarative knowledge. That is why authors referring to the instrumental approach generally use the notion of scheme in Vergnaud (1985)'s acceptation: "A scheme is an invariant organization of activity for a given class of situations. It has an intention and a goal and constitutes a functional dynamic entity". In this chapter I do not want to enter into a detailed conceptualization of the mental activity in instrumented situations: scheme can be taken as a mental construct pre-organizing the subject's activity.

the same time, being shaped by learners *in use*. In Hoyles, Noss and Kent (2004), we address the complementarity between the theory of instrumental genesis and the ideas of situated abstraction. In that paper, we suggest the importance of this complementarity as follows:

> This is what the notion of situated abstraction seeks to address, by providing a means to describe and validate an activity from a mathematical vantage point but without *necessarily* mapping it onto standard mathematical discourse. The notion is particularly pertinent in computational environments, since the process of instrumental genesis involving the new representational infrastructure supported by the computer will tend to produce individual understandings and ways of working that are divergent from standard mathematics. (*ibid* p. 314).

We give an illustrative example. When students graph functions in a computer environment (or with a graphic calculator), they are faced with the fact that a function graph depends on parameters of the 'graphing window' and they have to develop specific 'framing schemes', typically interweaving knowledge in mathematics and on the calculator. This is far from a spontaneous and immediate process.

Anthropological dimension

This dimension was introduced in section 2. Here the notion of praxeology will be explained in more detail. Analyzing the transposition of mathematicians' experimental activity into education helped in section 2 to make clear that knowledge cannot be seen independently of institutions. Bosch et al. (2004, p. 4) explain: "The process of didactic transposition highlights the institutional relativity of knowledge and situates didactic problems at an institutional level, beyond individual characteristics of the institutions' subjects". Section 2 pointed out that 'institution' has to be taken in a very broad sense: a school system in a country is an institution, but a branch of this system is also one. Then, mathematical activity can be modeled as a human institutionally-situated activity among others. In a given institution, among many problems or questions, some are recognized as a 'type of task' and 'techniques' are identified as specific ways to do these tasks. Tasks and techniques together make up the practical component of "know-how"; the praxeologies (*praxis* + *logos*) integrate into a theoretical component.

Techniques have a central role in this model. They cannot be seen just as 'skills'. Certainly, they sometimes mean routines, especially when the purpose is to perform a sub-task in a problem, but they also imply reasoning about mathematical entities, especially during their creation and when questioning their consistency and their domain of validity. As Artigue (2002) pointed out, techniques have both a pragmatic and an epistemic value. The pragmatic value is related to the technique's usefulness and efficiency. It is directed towards tasks. The epistemic value is the light that the technique sheds on properties of mathematical objects. It is directed towards the theoretical component[11].

The following example presents a problem to show how considering this level of the techniques helps us to understand the impact of technology on classroom mathematical

[11] This block has two levels. One is 'technological' —in the etymological acceptation of 'discourse about techniques'—and the other theoretical. Considering the 'technological' level in the context of use of 'technology' —in the ordinary acceptation— is useful because as the example below will show, classroom conceptualization of mathematical objects and properties is aimed through a 'discussion about the techniques'.

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practices. The following problem was taken from a 10th grade textbook and is representative of a type of task existing at this level: reducing an expression with radicals. The task was to prove the equality:

$$\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}=1$$

It is motivating because the expression on the left side is quite complicated and on the right side it is quite simple. Without technology, students should rewrite each term on the left side without surds in the denominator and there is a technique for that. In the textbook, it is written like this.

If a denominator is $a+\sqrt{b}$ *then multiply numerator and denominator by* $a-\sqrt{b}$

If a denominator is $a-\sqrt{b}$ *then multiply numerator and denominator by* $a+\sqrt{b}$.

The textbook also provides for a number of exercises for training, numerical or more theoretical.

Write with an integer denominator $\frac{1}{\sqrt{5}-1}$ *and* $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

Show the equality $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x-y}}=\frac{\sqrt{x-y}}{\sqrt{x}+\sqrt{y}}$

With this technique, the problem can be solved, albeit with careful and accurate manipulation. A question is why teachers ask students to learn and practice this technique, to train in its application and to use it to solve problems. A first reason is pragmatic. The technique helps a learner to obtain canonical expressions that are easier to handle in calculations. If this technique would have only this pragmatic role, teaching would be too much oriented towards skills and training. But this technique has also an epistemic role relative to more theoretical knowledge. When practising the technique, a student has to reflect on the structure of an expression to consider, for instance, the denominator and its structure. He (she) has to use properties of equivalent quotients and of the square of surds. He (she) has to also to consider algebraic facts like the factorisation of the difference of two squares. Questions like "does this technique work for every expression?" can begin to develop a student's appreciation of the structure of sets of these expressions.

When students use technology the problems and the exercises become (almost) trivial. Even an ordinary numeric calculator (Fig. 4.1 top) computes the sum into 1. It is a numerical approximation but to students, it is a strong indication that the equality is true. A symbolic calculator also simplifies the sum into 1. It also transforms the expressions of the exercises just in the form a teacher would expect. The equality with x and y is not directly proved, but the proof can be done by a simple transformation (see Fig. 4.1 bottom).

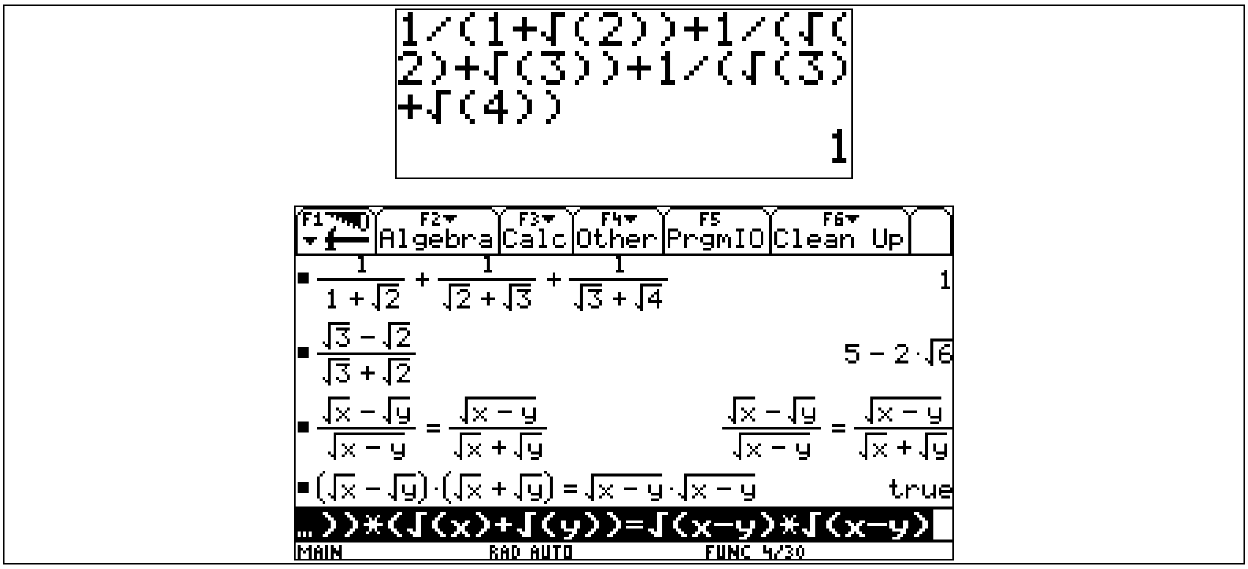

Fig 4.1: *Contrasting a numerical and symbolic calculater*

This means that a valuable praxeology has been destroyed by the use of technology because calculators do computations that could be done before only at the cost of a laborious but potentially epistemic technique. Such a phenomenon can be easily explained: technology was developed precisely to help people perform smoothly algorithmic techniques. As valuable praxeologies disappear, teaching has to create new praxeologies in which techniques performed using technologies retain an epistemic value. New types of tasks have to be thought of and evaluated by considering the possible techniques to solve them and their epistemic value.

Lagrange (2005a) showed that this is a realistic challenge, not least because of the variety of new tasks and techniques and of their epistemic potentialities that technology, especially symbolic calculation, brings about: CAS, for example, aided pattern discovery, problems and techniques to access generalization and the management of expression by way of symbolic calculation.

New dimensions

The meta-study was undertaken in 2000 and was derived from a corpus of papers published up to 1998. Now, some years later, two other dimensions should be added. One is collaboration. We are seeing rapid developments in the ways that it is possible to interact and collaborate through technological devices and many papers now stress the dimension of interactive and collaborative learning, especially when dealing with web-based applications: as we saw, WebLabs attempted to exploit these technological possibilities with some success. A wealth of new questions opens, especially about the contribution to mathematics learning of different levels and modalities of interactivity and collaboration, and about the potential of virtual communities, and how they might be fostered and sustained.

The other new dimension is design. There is growing awareness that, as educators, we cannot simply orchestrate software applications that industry or computer science creates. Yerushalmy (*ibid*.) developed artefacts to experiment with classroom use of technology as a part of her research activity and she reflected on principles that could orientate design. She sees a discrepancy between encouraging evidence about the impact of various specific software capabilities and discouraging evidence about work with educational software that does not always act as the idea generator it was designed to be. She asks designers for more work, not just having good ideas, but also to realize and articulate their perhaps unconscious decisions and turn them into conscious design considerations. Design also needs to be iterative: to have clear aims and criteria but also to be flexible enough so as to be adaptive to and support student learning requirements during mathematical activities.

Towards a synthesis

The dimensions were conceived so as to encompass the varied aspects of the educational use of digital technologies in an analytic approach. To move closer towards a synthesis, it is useful to consider what dimensions have in common, and to group them around centres of interest, as shown in table 4.1.

The epistemological and semiotic, cognitive and instrumental dimensions are principally concerned with new ways of representing and manipulating objects supported by digital technologies. The epistemological and semiotic dimensions privilege the relationship to mathematical knowledge, while the cognitive dimension focuses on the learner. The instrumental dimension takes into account the user's operative knowledge related to the representations and manipulations.

The 'situational' and 'design' dimensions share a common interest for the 'economy' of the learning situation in the sense described earlier, recognizing that technological tools shape mathematical activity and trying to predict as much as possible the ways they do so. The notion of *scenario of use* should encompass both the design dimension in the narrow sense of software development and implementation in teaching/learning.

Sensitivity to contexts of learning brings together the anthropological, 'teacher' and 'collaboration′ dimensions. These dimensions recognize the complexity of human thinking and learning, emphasizing the social aspects of these activities where technology is seen as providing cultural artifacts supplementing language and written expression. While the anthropological dimension considers institutions and the transposition of practices between these, the teacher dimension takes into account the process of mediation in teaching/learning practices and its necessary adaptation to new artifacts. Through software and networks, digital technologies can afford a means to develop collaboration in social activities, creating new contexts and dramatically changing existing ones.

Dimensions	*Common centre of interest*
Epistemological & semiotic	Influence of digital representations on conceptualisations
Cognitive	
Instrumental	
Situations	Influence of tools on teaching/learning situations
Design	
Anthropological	Sensitivity of technology use to contexts of learning
Teacher	
Collaboration	

Table 4.1: *Grouping Dimensions*

Comparing and contrasting the approaches in Casyopée and WebLabs

As indicated in the introduction to this section, approaches to technology use in the research contexts where Casyopée and WebLabs were developed seem, at first sight, rather distinct. Reflection on the use of computer symbolic computation, which is the context of Casyopée, comes from experiments of the introduction of technology into educational settings that can tolerate from some adaptation, but are not supposed to change fundamentally. In contrast, the central orientation of WebLabs was a design experiment to test out conditions for more or less radical change.

In this concluding section we will see how a comparison between the two projects helps to draw common trends linked to technology use in mathematics teaching/learning and more generally to mathematics education. The above centres of interest, or groups of dimensions, will help to organize the comparison.

The influence of digital representations on conceptualisations

In both projects, the influence of digital representations on conceptualisations is seen from an epistemological point of view. Both projects share a common concern for students' access to formal thinking and to formal objects, and recognise that formal representations should be learnt as part of a culture of empowerment. They also share a common motivation: to use technology to provide students with representational infrastructures to help them make sense of mathematical ideas and over time to take responsibility for their activity using these infrastructures.

Starting from this common assumption and motivation, Casyopée’s and WebLabs’ position towards standard mathematical notation differ. Casyopée keeps to this notation as one which proved powerful for mathematicians for centuries. It postulates that technology in the form of symbolic calculation offers a means for more fruitful problem solving with the more or less standard algebraic notation and thus could contribute to make this notation more accessible and learnable.

WebLabs, on the other hand, chooses to explore the extent to which some of the apparent complexity and difficulties of mathematical and scientific ideas is due to symbols and language that were the only ones available in the pre-computer age. Thus, technology is seen as a means to broaden mathematical knowledge, either by opening access to problems for Casyopée and by new linkages with mathematical content for WebLabs.

Both authors are concerned with proof. Casyopée focuses on formal proof, principally for epistemological reasons: formal proof is what gives sense to transformational activity. In the WebLabs example, proof is grounded on, first an emphasis on making explicit the processes of reasoning, and second on students deriving situated abstractions rather than on explicit (algebraic) formal reasoning. The motivation for proving is in the social relationship when students want to compare different models. In WebLabs, the technology helps to create situations where students feel the need for a proof and find informal approaches to it. Proof in WebLabs is a way to convince each other of the validity of a property. In Casyopée, technology is an aid, principally when students have to write a proof: here proof is more a written text conforming to institutional standards. As the literature on proof abundantly reports, these two aspects of proof are complementary.

The instrumental dimension is important in both examples, although not developed here: the difficulty of instrumenting standard Computer Algebra Systems has been a strong motivation to design a new environment (Lagrange 2005b).

The influence of tools on teaching/learning situations

Both projects take design as a very central dimension. Sections 2 and 3 showed how the design of an application has a deep impact on the way teachers and students can use it and what they learn from its use. The authors of this chapter consider the design of new environments as a crucial dimension of their work as mathematics educators, by opening windows on all elements of the teaching/learning process: situational, cognitive and didactical. A further common orientation is iterative design, starting from initial reflection of a research team and taking advantage of collaborating teachers and partners' feedback during subsequent phases of design. Designers of computer environments for learning should be aware that, in many aspects, the impact of new software on classroom practices can never be totally anticipated. This is particularly the case in environments providing for new representational infrastructure, because, as we remarked in section 3, solving one problem of representation often throws up new unexpected problems. There is evidence that trying small-scale implementations and studying the effect on teachers and students in successive iterations is a way to take account of epistemological relevance and classroom complexity.[12]

Casyopée and WebLabs are mathematical educational applications developed on an underlying "general purpose" platform. In Casyopée, this platform was a computer symbolic kernel and in WebLabs a programming language. In both projects a first task was to "tune" the platform to provide learners with just the right functionalities. In WebLabs the platform is seen as a first level of a layered design, with layers that structure not only the tool but also student engagement with the system and finally the knowledge that he (she) develops. Casyopée, in fact, might also be analysed as a layered tool. It would help students to situate

[12] Iterative design in Weblabs is explained in section 3 above. For Casyopée, see Lagrange (2005b p. 173).

the idea of symbolic computation, of general forms of expressions and of symbolic rules inside strategies of exploration and proof. Tuning the layers and organising them consistently with the knowledge at stake are important principles.

Transparency is another important principal of design. In the WebLabs project, this implies "that it is easy to look not only at what the models *do,* but how they *work*". In Casyopée , the most important design principle, consistent with Yerushalmy's idea of tools supporting the curriculum, is that teachers and students should easily recognise objects manipulated in the environment by referring to standard mathematical objects. Thus, Casyopée and WebLabs do not privilege transparency in the same sense. In Casyopée transparency is 'external': it is in the relationship that users perceive between the functioning the environment and the ordinary mathematics. By contrast, WebLabs' transparency is 'internal' referring to the representation itself rather than to standard mathematical representations. For WebLabs, curriculum support is linked to the appropriate design of activity sequences using the new tools to achieve the learning aims. Both understandings of transparency seem important when dealing with formal representations, although it is not so clear how they can be reconciled.

Contexts of use

Regarding the contexts of use, the legitimacy of technology in the mathematics classroom is an important issue for Casyopée. Many authors stress the idea of experimental approaches for pedagogical reasons loosely referring to mathematicians' practices. The idea underlying the development of Casyopée is that, to develop viable implementations of this idea, it is necessary to discuss how mathematical practices can be transposed to students. The link between experimentation and conceptualisation appears critical. Technology can help to make this link if it promotes methods and tools for conjecturing and proving.

WebLabs focuses on collaboration between students: distance is taken as an opportunity to allow students to share their models of evolving knowledge, discuss, change and manipulate them. Computer programs are seen as formal objects, by which students operationalise their ideas and edit each others', instantiating the state of their current knowledge and at the same time beginning to appreciate the need for a shared formal language. While WebLabs does not consider the idea of transposition from professional to classroom mathematics, and Casyopée does not involve collaboration, it is interesting to note that distance collaboration is now a growing dimension of mathematicians' work and thus this might form an interesting basis for reconciliation. The idea of alternative systems of representation could also be considered from the perspective of the intermediate systems mathematicians use when investigating a new problem (especially with computers) before having recourse to the standard notation.

Finally, we should consider the relationship between theoretical frameworks and the dimensions. Research on the use of technology in mathematics education exploits many frameworks to help to interpret many aspects of a complex reality. This does make it difficult for researchers to communicate their goals and findings cumulatively. The relationship of dimensions to theoretical frameworks is, therefore, not uniform. In the epistemological, semiotic and cognitive dimensions, a researcher can choose among frameworks that reflect the research emphasis on these dimensions, while in others the choice is limited.

This chapter presents, through a comparison of two projects, the considerable potential in seeking out convergences among existing frameworks using structuring dimensions as a tool.

It also points to both practical and theoretical research around those dimensions that are, until now, under-researched.

References

Artigue, M. (2005), The integration of symbolic calculation into secondary education: some lessons from didactical engineering. In Guin D. & Trouche L., *The integration of symbolic calculators.* Ch. 9. Dordrecht: Kluwer

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning* 7(3), 254-274.

Arzarello, F. (1991). Pre-algebraic Problem Solving. Paper presented at a seminar on problem-solving. Viana do Castelo, Portugal.

Bell A. (1996). Problem solving approaches to algebra: Two aspects. Bednarz, N., Kieran, C. and Lee, L. (Eds) *Approaches to Algebra: Perspectives for Research and Teaching*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Borwein, J.,(2005),The experimental mathematician: the pleasure of discovery and the role of proof, *International Journal of Computers for Mathematical Learning* 10.2: 75–108.

Bosch, M., Chevallard, Y. & Gascòn, J. 2005 - Science or magic? The use of models and theories in didactics of mathematics. *Proceedings of CERME 4, European research in mathematics education - cerme4.crm.es*

Brousseau, 1997, *Theory of Didactic Situations*, Kluwer.

Carraher D. & Earnest D. (2003) Guess my rule revisited. Proceedings of the 27th. conference of the International Group for the Psychology of Mathematics Education. Honolulu.

Chevallard, Y. (1985). *La transposition didactique*, Editions La Pensée Sauvage, Grenoble.

Chevallard, Y. (1994). Les processus de transposition didactique et leur théorisation, in Arsac, Martinand, Tiberghien (eds.) *La Transposition didactique à l'épreuve des faits,* Editions La Pensée Sauvage, Grenoble.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques,* 19, 221-266

Cobb, P., Confrey J., diSessa, A., Lehrer R. and Schauble, L. (2003) Design Experiments in Educational Research. *Educational Researcher*, Vol. 32, No. 1, pp. 9–13, 2003

diSessa, AA, & Cobb, P. (2004). Ontological Innovation and the Role of Theory in Design Experiments. *Journal of the Learning Sciences*, 13(1), 77-103
Educational Studies in Mathematics, 20, 147-164.

Harel, I. and Papert, S., 1991, *Constructionism*. Norwood, NJ: Ablex

Hoyles, C., Healy, L. & Pozzi, S. (1992). Interdependence and autonomy: Aspects of groupwork with computers. In Mandel, H., De Corte, E., Bennett S.N. and Friedrich H.F (eds.), Learning and Instruction, European Research in International Context. Vol. 2, pp. 239-257.

Hoyles C. & Noss R. (1996) The visibility of meanings: modelling the mathematics of banking. *International Journal of Computers for Mathematical Learning* 1,1, 3-31.

Hoyles C., Noss R. & Kent, P. (2004) On the Integration of Digital Technologies into Mathematics Classrooms. *International Journal of Computers for Mathematical Learning*, 9, 3, 309-326.

Hoyles, C. Morgan and G. Woodhouse (eds.), *Rethinking the Mathematics Curriculum,* pp. 22-28. London: Falmer Press.

Hoyles, C., Noss, R. and Kent, P. (2004), On the Integration of Digital Technologies into Mathematics Classrooms, *International Journal of Computers for Mathematical Learning*, Volume 9.3, pp. 309 – 326

Jones and Lagrange (2003) Tools And Technologies IN Mathematical Didactics: Research Findings AND Future Directions. In Mariotti (chief ed.), CERME3 Proceedings, ermeweb.free.fr

Joshua, S., Joshua, M.A. (1988). Les fonctions didactiques de l'expérimental dans l'enseignement scientifique. *Recherches en Didactique des Mathématiques*, vol. 9.1, 5-27.

Kieran (2006) Research on learning and teaching Algebra. In Angel Gutirrez and Paolo Boero (eds.), *Handbook of Research on the Psychology of Mathematics Education, Sense Publishers, Rotterdam/Taipe*, pp. 11-40

Kieran, C. (2004). The Core of Algebra: Reflections on its Main Activities, in Stacey et al. (eds.), *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study*. Springer 2004

Kynigos, C. (2004). « A 'Black-and-White Box' approach to user empowerment with component computing », *Interactive Learning Environments*. Vol. 12, Nos. 1–2, pp. 27–71

Lagrange , J.B., (1999) Techniques and concepts in pre-calculus using CAS: a two year classroom experiment with the TI-92 *International Journal for Computer Algebra in Mathematics Education. 1999 6.2*

Lagrange, J.B. (2005b) Curriculum, classroom practices and tool design in the learning of functions through technology-aided experimental approaches. *International Journal of Computers for Mathematical Learning*. 10.2, 143–189.

Lagrange, J.B., (2005a). Using symbolic calculators to study mathematics. The case of tasks and techniques. Chapter 5 in Guin D., Trouche L. (eds.), *The integration of symbolic calculators*. Kluwer

Lagrange, J.B., 2000. « L'intégration d'instruments informatiques dans l'enseignement : une approche par les techniques » *Educational Studies in mathematics*. 43.1

Lagrange, J.-B., Artigue, M., Laborde, C. and Trouche, L.: 2003, Technology and Mathematics Education: a Multidimensional Study of the Evolution of Research and Innovation, in A. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick and F.K.S. Leung (eds.), *Second International Handbook of Mathematics Education*, Kluwer Academic Publishers, Dordrecht, pp. 239-271.

Lee, L., & Wheeler, D. H. (1987). Algebraic Thinking in High School Students: Their Conceptions of Generalisation and Justification. Montreal: Concordia University.

MacGregor, M., & Stacey, K. (1992). Seeing a Pattern and Writing a Rule. Proceedings of the Sixteenth International Conference for the Psychology of Mathematics Education, pp. 181-188. New Hampshire: Program Committee of the 16th PME Conference.

Mewborne (2005), Framing Our Work. In G. M. Lloyd, M. R. Wilson, J. L. M. Wilkins, & S. L. Behm (Eds.), *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Retrieved October 31, 2005, from http://convention2.allacademic.com/index.php?cmd=pmena_guest

Monaghan, J. D. (2005). Computer algebra, instrumentation and the anthropological approach. *CAME 2005 - The Fourth CAME Symposium.* http://www.lonklab.ac.uk/came/events/CAME4

Mor, Y., Hoyles, C., Simpson G. & Noss R. (in press) Designing to see and share structure in number sequences. *International Journal for Technology in Mathematics Education*

Nardi, B. A. (1996). Introduction to Context and Consciousness: Activity Theory and Human-Computer Interactions, Cambridge MA: MIT press

Noss R. & Hoyles C. (1996) *Windows on Mathematical Meanings: Learning Cultures and Computers.* Dordrecht: Kluwer.

Noss R. & Hoyles, C. (2006) Exploring Mathematics through Construction and Collaboration. In K.R. Sawyer (Ed) *Cambridge handbook of the Learning Sciences.* Cambridge: CUP.

Noss, R. (1998) New Numeracies for a technological culture. *For the Learning of Mathematics*, 18, 2, 2-12. 1998

Papert, S., and Harel, I. (1991) Situating Constructionism. In Harel & Papert (eds.) *Constructionism*. Norwood, NJ: Ablex

Scardamalia, M. & Bereiter, C. (2006) Knowledge Building: Theory, Pedagogy and Technology. In: Sawyer, K. (Ed) Cambridge Handbook of the Learning Sciences. pp. 97-118.Cambridge UK: C.U.P

Simpson, G., Hoyles, C. & Noss, R. (2005) Designing a programming-based approach for modelling scientific phenomena. *Journal of Computer Assisted Learning.* 21, pp. 143–158.

Simpson, G., Hoyles, C. & Noss, R. (2006) Exploring the mathematics of motion through construction and collaboration. *Journal of Computer Assisted Learning.* 22, pp.1-23

Stacey, K. (1989). Finding and Using Patterns in Linear Generalising Problems.

Trouche, L., 2005, An instrumental approach to mathematics learning. Chapter 6 in Guin D., Trouche L. (eds.), *The integration of symbolic calculators.* Kluwer.

Vergnaud, G., (1985) Concepts et schèmes dans une théorie opératoire de la représentation. *Psychologie Française* , 30, 3/4, pp. 245-251.

Yerushalmy, M. (1999). Making exploration visible: on software design and school algebra curriculum, *International Journal of Computers for Mathematical Learning*, vol 4, 169-189.