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# The technological mediation of mathematics and its learning

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## Abstract

This paper examines the extent to which mathematical knowledge, and its related pedagogy, is inextricably linked to the tools – physical, virtual, cultural – in which it is expressed. Our goal is to focus on a few exemplars of computational tools, and to describe with some illustrative examples, how mathematical meanings are shaped by their use. We begin with an appraisal of the role of digital technologies, and our rationale for focusing on them. We present four categories of digital tool-use that distinguish their differing potential to shape mathematical cognition. The four categories are: i. dynamic and graphical tools, ii. tools that outsource processing power, iii. new representational infrastructures, and iv. the implications of high-bandwidth connectivity on the nature of mathematics activity. In conclusion, we draw out the implications of this analysis for mathematical epistemology and the mathematical meanings students develop. We also underline the central importance of design, both of the tools themselves and the activities in which they are embedded.

Keywords: digital technology; mathematics; dynamic tools; connectivity; design; representational infrastructure

## 1. Introduction

This paper addresses a central issue for mathematical development: to explore the extent to which mathematical knowledge is situated in the practices within which it was developed and the signs used in these situations. How far is mathematical knowledge, and its related pedagogy, inextricably linked to the tools – physical, virtual, cultural – in which it is expressed? Put another way, how are abstractions shaped by and expressed in the medium? To address these questions, our method in this paper is to focus on a few tools that exemplify particular tool-use, and to describe with some illustrative examples, how mathematical meanings – both pedagogical and epistemological – are shaped by their use.

The discourse of mathematics is inevitably expressed within a set of semiotic tools, so it is reasonable to conjecture that mathematical cognition evolves alongside the representational systems afforded by these tools (for related work on the shaping of representations, see Nunes, Schliemann & Carraher, 1993). The tools are cultural in the sense that they have evolved historically in response to the demands of mathematics itself, and, of course, the historical demands of the societies that gave rise to new mathematics. Modern mathematics in particular is intimately tied to algebraic expression – but this was not always so: consider, for example, the geometrical (and to us now, baroque) way that Newton expressed his laws of motion in *Principia* (diSessa, 2000).

Our interest in this paper will be on virtual tools and the computer will play a central role in what follows. There are two reasons for this. First, the relative novelty of digital technologies has offered us a chance to rethink the ways in which representations shape learning. In particular, it has fostered a sharper focus on trying to understand the role of tools more generally and how students' conceptions of mathematics are shaped, not only by the actions and attitudes of the teacher, but also by how far the students master what the French school of researchers term 'the process of instrumentation': the extent to which the learner is aware of the system, and is able to look *through* it as well as look *at* it (Artigue 2002).

This strand of work entails a more sensitive realisation that a fine balance is needed between the 'pragmatic' and 'theoretical' (or 'epistemic') roles of calculation, a point closely related to the dual nature of mathematics as both *tool* and *object* (Douady, 1991). The simple, and initially at least, widely-held assumption that technology could relieve the student of the need to calculate (in the broadest sense) and allow a sharper focus on structure and relationships, has given way to a more nuanced understanding that calculation and structure are intimately connected, and that an acute awareness of their relationship should guide the design of the technological artefacts intended for mathematical learning.

The second rationale for a focus on digital tools is their increasing ubiquity in mathematics classrooms together with their multi-faceted functionality. While any tool requires design and its integration into mathematical expression is worth close consideration (see, for example, Ruthven's analysis of the role of squared paper in mathematical pedagogy, Ruthven, 2009, in press), digital tools - by virtue of their infinite malleability - have encouraged researchers to consider not only how best to adapt tools to the learning of mathematics, but how to adapt the mathematics-to-be-learned in the light of new tool-rich possibilities. Thus design moves even more to centre-stage. Of course, this perspective leads to difficult questions of cultural legitimacy and what, in other contexts, one might call 'transfer'.

This last point needs a little elaboration. If our focus is on understanding how mathematical cognition evolves in tandem with the fluent use of digital tools embedded in learning situations, we will need at some point to ask whether and how such cognition generalises beyond the context in which it was developed. In this respect, we will borrow from Papert's analysis of how one can foster the development of a "Mathematical Way of Thinking" that goes beyond the teaching of specific content of mathematical topics. He asks:

> "Psychologists sometimes react by saying, 'Oh you mean the transfer problem". But I do not mean anything analogous to experiments on whether students who were taught algebra last year *automatically* learn geometry more easily than students who spent last year doing gymnastics. I am asking whether one can identify and teach (or foster the growth of) something *other* than algebra or geometry, which, once learned, will make it easy to learn algebra and geometry. No doubt, this other thing (let's call it the MWOT) can only be taught by using particular topics as vehicles. But the "transfer" experiment is profoundly changed if the question is whether one can *use* algebra as a *vehicle for deliberately teaching transferable general concepts and skills.* […] Yes, one can use algebra as a vehicle for initiating students into the mathematical way of thinking. But, to do so effectively one should first identify as far as possible components of the general intellectual skills one is trying to teach, and when this is done it will appear that algebra (in any traditional sense) is not a particularly good vehicle." (Papert, 1972, pp. 251).

Papert's focus on algebra, though pertinent given its hegemonic role as a modern medium of mathematical expression, should be seen as an instance of a more general insight that could equally reference geometry, number, statistics and calculus. Papert's position has not lost any of its force in the intervening three-and-a-half decades. It raises two major issues, each of which we will touch on below. First, and most obviously, it challenges us to conceptualise not only the design of pedagogic approaches and tools, but to understand more clearly *what kinds of knowledge* may be accessed through such tools. Second, it maps a research agenda to try to understand how mathematics can be expressed – and by implication, how mathematical knowledge such as MWOT can be developed.

If the central challenge of mathematical learning is to express mathematical abstraction, then we need to move beyond abstractions expressed only in traditional algebra. We have used the idea of *situated abstraction* as an orienting framework to describe and explore how interaction with semiotic tools shapes the development of mathematical meanings and in turn is shaped by the conceptions and social context of the students (see, for example, Hoyles & Noss, 1992; Noss & Hoyles, 1996). The distinction between conceiving abstractions as situated and the traditional view of abstraction that sidesteps the framing of representation tools, is both powerful and problematic. It is powerful because it seeks to legitimise forms of mathematical expression that are novel, and which may access precisely the alternatives to algebra that Papert sought. But it is problematic as it is easily misunderstood as a kind of pseudo-mathematics, falling short of traditional pedagogic practice, and too easily erecting a barrier rather than a doorway between situated and traditional abstraction.

A theoretical corpus of work relevant here is the analysis of 'instrumental genesis' that seeks to elaborate the mutual transformation of learner and artefact in the course of constructing knowledge with technological tools (Artigue, 2002; Trouche, 2005). Yet, as we have argued elsewhere (Hoyles, Noss and Kent, 2004), this instrumental genetic analysis leaves relatively unexplored the texture of the meanings evolved – the situated abstractions of mathematical ideas that are being developed and expressed, and how these abstractions are knitted together or 'webbed' (Noss & Hoyles, 1996) by the available tools and shaped by the interactions with these tools and with the social context.

This point is important because, although schemes of instrumented action provide an effective means for conceptualising tool-learner interaction, there remains a need to elaborate the *kinds* of mathematical knowledge that develop in such interactions. This knowledge, or at least its visible traces, may not look or sound like standard mathematical discourse. It is no coincidence that the idea of situated abstraction was born in the context of studying students' mathematical expression with computers, for example, by recording children expressing relationships, variants and invariants through a Logo program or a spreadsheet. It is in the nature of interactive, dynamic representations that digital systems afford – at least when designed thoughtfully – expression via tools that diverge from standard mathematics (recall Papert's point: standard expression may not be a particularly good vehicle for fostering what we are trying to teach!). We also recall Balacheff's argument (1993), when discussing the idea of 'computational transposition', that computer tools introduce a new model of knowledge related to the functioning of the machine and the interface designed for the software: i.e. the knowledge instantiated in a computer system is no longer *the same* knowledge. We seek here to present some elaboration of this idea.

In what follows, we present four categories that distinguish different ways that digital tools have the potential to shape mathematical cognition. We provide at least one illustrative example in each category. First, we will consider dynamic and graphical tools and ask how their use shapes mathematical activity and the kind of knowledge that is fostered by their use. Next, we consider how tools that outsource processing power from mind to machine can allow us to develop in more detail the didactical consequences of Artigue's epistemic/pragmatic distinction to which we referred above. Third, we will look more broadly at forms of new representational infrastructures, before finally considering the implications of the advent of high-bandwidth connectivity on the nature of mathematics activity and mathematical learning both within and across classrooms.

## 2. Dynamic and graphical tools

Digital technology can provide tools that are dynamic, graphical and interactive. Using these tools, learners can explore mathematical objects from different but interlinked perspectives, where the relationships that are key for mathematical understanding are highlighted, made more tangible and manipulable. The crucial point is that the semiotic mediation of the tools can support the process of mathematising by focussing the learner's attention on the things that matter: as Weir (1987) puts it, "the things that matter are the things you have commands to change." (p. 65). The computer screen affords the opportunity for teachers and students to make explicit that which is implicit, and draw attention to that which is often left unnoticed (Noss & Hoyles, 1996).

A more important point concerns the idea of expressing aspirations and ideas. We are accustomed to thinking of computers as precise, detached, accurate. We are less used to the idea of computers screens to express ideas, especially half-formed ones. In fact, with the advent of web 2.0, social networking, YouTube and so on, the conception of computers in popular culture is changing, and becoming more akin to the infrastructural role that, say, paper and pencil have historically played as a medium that is capable of supporting multiple modes of expression. But in education, and mathematical education particularly, this transformation has yet to become commonplace, and computers in formal educational settings are still largely associated with activities some way removed from sketching half-formed thoughts, or fostering creativity or inspiration.

By way of illustrating the point, we will give an example of how, using digital technologies, students can produce an accurate *sketch* of the solution to a problem. Here we use 'sketch' in a technical sense: it is accurate in that it meets the requirements of the problem situation but it is a sketch in that the necessary invariants of the mathematical structure of the problem are not formalised, see also Noss & Hoyles (1996). However the accuracy of the sketch means that by reflection on and manipulation of the sketch, the students can more easily come to notice what varies and what does not, and thus are more likely to become aware of what to focus on (Mason, 1996).

An example of this phenomenon is taken from Healy and Hoyles (2001). Here, two students are using a dynamic geometry system – *Cabri Geometry* in this case - to work on a task to construct a quadrilateral with the property that the *angle bisectors of two adjacent angles cross at right angles*. The students were asked that when they were convinced that they had constructed a quadrilateral that satisfied these initial conditions, they should seek to identify other properties of the quadrilateral that had of necessity to be satisfied.

Below, we reproduce part of a description of the pair successfully using the software to solve the challenge (for more detail see Healy & Hoyles, 2001). They exploited a mixture of creation and construction tools (this distinction is expressed by menu choices) to produce an accurate sketch of the quadrilateral required, explored it and through this exploration conjectured about the necessary geometrical relationships involved.

The pair began the task by creating a quadrilateral ABCD consisting of 4 line segments arranged in no particular configuration. After labelling the four vertices, they added the angle bisectors of angles ABC and BCD, and used the angle-measuring tool to measure the angle where these two lines crossed. They then carefully dragged the vertices of the quadrilateral until this angle measured 90° (see Figure 1). Thus the constraints of the required quadrilateral were not constructed – i.e. the angle between the bisectors could be easily shifted from $90^o$ - but simply created "by eye". However, at the moment when the angle between the two bisectors in fact measured 90°, the pair noticed that BA was parallel to CD. There was no doubt in their minds, although they had, at that time, no validation of this hypothesis. Nonetheless they immediately conjectured, on the basis of this one example, that *whenever the two angle bisectors were at right angles, BA must be parallel to CD.* [1].

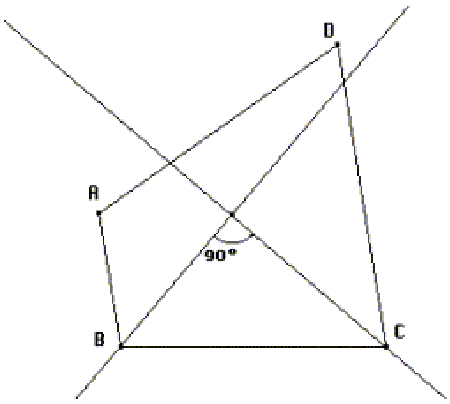


Figure 1: Sketching a quadrilateral with angle bisectors of two adjacent angles at right angles.

Such a conjecture can be designated as an abduction[2]. An abduction is characterised by noticing a local commonality, which depends on a recognition, or decision, about what counts as the same and different. This is subsequently generalised by identifying the constraints or structural relationships that appear to have given rise to the commonality (Radford, 2001): contrast with *de*duction that involves inferring what must follow from a set of structural constraints.

[1] Later, they went on to verify their conjectures in particular cases, explain why they must be true in an informal way, and finally wrote a deductive proof based on their experimentation.

[2] Arzarello, Micheletti, Olivero, & Robutti, 1998 also note how abduction is often used at the conjecturing stage with Cabri.

The important point is this: the key (correct) conjecture was triggered by reflection on an accurate sketch. During the process of dragging the sketch so that it corresponded by eye to the constraints of the problem, the students became aware that they should be keeping an eye open for possible relationships between the other elements. Without the dynamic aspect expressed through dragging, this would have been extremely difficult, as accuracy, as well as interactivity (through hand/eye coordination) is essential to the process of noticing such relationships. Notice too that this property of being dynamic is quite different from the sense of dynamic that characterises, say, animated diagrams. The key factor is the interplay between dynamic (while dragging) and static (stop when some relationship seems evident) and that this is crucially in the control of learners - so they can pause, reflect, go back and test in the light of feedback from the graphical image.

We conclude this section by noting another major function of the use of accurate sketches such as these in learning mathematics, which is to produce the *motivation* to hypothesise a theorem to account for the figures on the screen, prior to a conjecture and also subsequent to it[3]. This is a constant challenge in mathematics education: to motivate students to 'keep an eye on the general' in all that they do.

## 3. Outsourcing processing power

We would propose, alongside Jim Kaput in his seminal paper of 1992, that a fundamental property of digital technologies – one that distinguishes it from all other technologies – is its affordance in 'outsourcing' processing power from being the sole preserve of the human mind, to being capable of being undertaken by a machine.

Kaput's basic argument is that human history is entering a new phase, a virtual culture based on the externalisation of symbolic processing[4]. We will not elaborate the argument here (see Kaput, Hoyles & Noss, 2002). Instead, we will ask what kinds

[3] It is worth noting here that only did the tools afford mathematical learning but also a teacher is granted a way of appreciating the geometrical intuitions that the students had – and can model them again by positioning parts of the construction by eye with a group or students. This is another example of the crucial role of the computer as a window on mathematical meaning.

[4] Obvious exemplars of external processing are computer algebra systems.

of roles external symbolic processing might play in the generation and shaping of mathematical meaning.

There is little doubt that the outsourcing of processing power holds significant potential for the learning of mathematics. All too often, students become bogged down in procedures, lose touch with the problem they are tackling, make careless errors and lose motivation. In the case that calculation, *technique*, is required to achieve a numerical or algebraic result, then the devolution of processing power to the computer is unproblematic - and potentially renders all but a tiny part of conventional curricula redundant. However, if the goal is to achieve insight rather than answer – and such is typically the case in learning mathematics – then offloading technique may or may not be desirable. The difficulty resides in the recognition that, as we pointed out earlier, there exist facets of the technical alongside the conceptual that appear to be central to the process of semiotic construction. Thus, indiscriminate outsourcing of technical expertise from the learner to machine can make it more difficult still to foster in the learner the sense that mathematics is a coherent whole (Goldenberg 2000). Clearly, we need to exploit the massive processing power now at hand in ways that provide some glimpses of the structures that underlie calculations and manipulations. Put another way, students need to have some idea what produces the numbers or outcomes and at the same time gain some ownership of the process.

We have had some experience of how to deal with the problem of outsourcing in the context of the workplace, as part of the project, *Techno-mathematical Literacies in the Workplace*[5], in which we investigated the mathematical needs of employees in 'modern' workplaces, that is workplaces increasingly dependent on computer systems. In such workplaces, there tends to be a wide range of artefacts, many, if not most of which are mathematical, in the sense that mathematical relations drive their output. But this mathematics is largely invisible, outsourced to a computer system and hidden behind computer printouts, graphic displays, or dynamically-presented tables and figures.

Thus a key utility of the artefact seems to be precisely that the mathematics is safely outsourced to the technology or to an expert team (see for example, Kent & Noss, 2000). But we found that judgement of implications of the output could not simply be left to the machine, but rather demanded some *mathematical* interpretation.

[5] Grant number: L139-25-0119 (Economic and Social Research Council, UK).

Thus there was a need for employees to appreciate this ‘concealed’ knowledge, at least at some level. In addition, we found that the mathematical knowledge was shaped by the artefacts and systems within the workplace and the justifications had to adhere to the discourse of the workplace. We identified what we called *technomathematical literacies;* ‘technomathematical’ to express the idea that the mathematics is expressed with and through the artefacts and ‘literacies’, to underline the idea that making meaning out of computational artefacts requires interpretation and familiarity considered as a cultural form (for details of the research and the evidence from which the following example is based, see Hoyles et al, 2007).

The symbolic artefacts on which we centred our attention in this research were intended as catalysts for communication between different layers of the workforce — such as between the manufacturing shop floor, middle and senior managers, and process/systems engineers. Middle-level employees were key in this communicative task, but were often at a loss as to how to exploit the artefacts to facilitate their interactions, to explain where and why the outputs had arisen. In other words, the artefacts generally failed in their intended function as *boundary objects,* that is affording the communication of meanings across communities (see, for example, Bowker & Star, 1999). From our observations of workplaces, it was evident that for the artefacts to serve as boundary objects, some grasp of the mathematical underpinnings needed to be communicated, and this we set out to undertake in the second phase of our study. Since it was clear that a deep and detailed mathematical appreciation was neither necessary nor feasible, we set out to design for ‘layered learning’ (see Kahn *et al.*, 2006), that is, to construct a pedagogical and technical approach that allowed our learners (shop floor employees for example) to drill down to an ‘appropriate layer of detail’- to ‘get the idea’ or glimpse the relevant structure.

The example we will briefly outline is derived from our case study in a car factory, where it was evident that one artefact the SPC[6] chart was supposed to serve as a boundary object. This time-series graph was generated by the workers on the production line to monitor a wide range of processes (see Figure 2 for an example). The chart is intended as a means to share information between shop floor and management as to how any given process was performing The workers enter figures

[6] SPC means statistical process control SPC a set of techniques widely used in workplaces as part of process improvement activities (see for example, Oakland, 2003),

on the chart, and plot the graphical elements by hand: these charts are then handed over to the process engineers, who undertake a series of complex calculations to produce measures of efficiency (shown in the bottom right corner of Fig 2), which become the subject of discussion at team meetings.

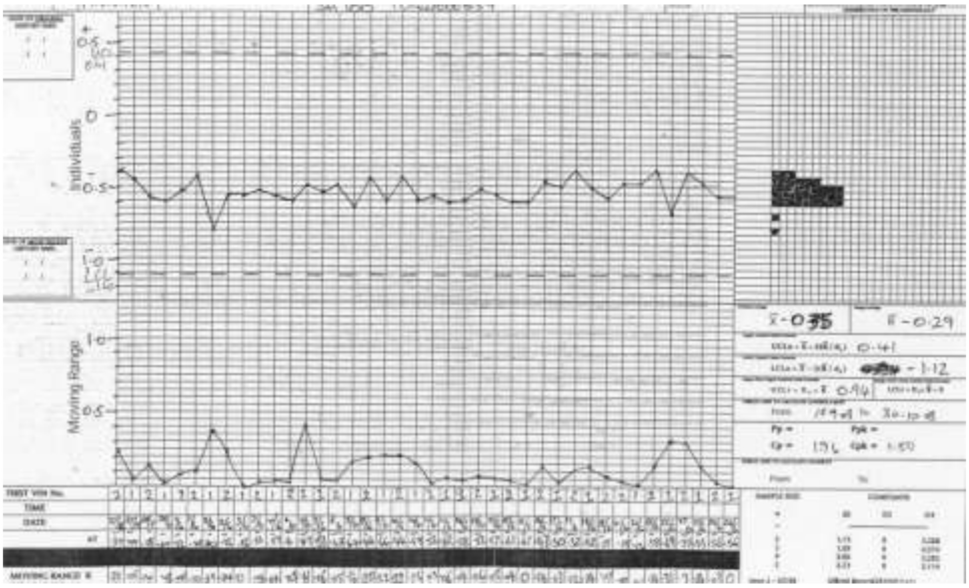


Figure 2. An example of an SPC chart, an intended boundary object

Our ethnography derived some understanding of how the charts were used, what they were intended to do, and the kinds of technomathematical knowledge necessary for their effective interpretation. In the pedagogic phase of our work we enhanced the charts electronically: in fact, this became a general methodological gambit and we coined the term *“technologically enhanced boundary object, or TEBO,* to describe the designed artefact. The idea was straightforward: to open up some of the layers of mathematical structure hidden in the artefact, sometimes by opening black-boxed calculations to reveal key variables, and in other cases (as in this example), by outsourcing to our TEBO some of what the employees previously had to understand.

In the SPC training provided by the factory, we had observed trainees engaging in physical experiments catalysed by a version of a “shove ha’penny” game in order to generate sample process data[7]. By a set of various improvements in

[7] Shove ha’penny is a British game that used to be played in pubs, in which coins are pushed or flicked up a graduated horizontal board, and bets cast as to where they will land.

process, such as using a ruler to simulate pushing by hand to systematise measurement, and plotting the outcome on an SPC chart, the trainees were encouraged to see how the process could become more tightly controlled

With the TEBO we developed, shown in Figure 3, the employees could generate trials of 50 'flicks' in a simulated shove ha'penny game and the TEBO plotted where the coin stopped each time on the chart. Employees could therefore generate large data sets, watch the time series and the histogram of the data grow simultaneously, and thus observe the key ideas more readily: notice trends over time, aggregate statistical patterns, see how they emerged from individual trials and how they were constrained within certain limits in situations of random variation. Our study indicated that our design was largely successful in enabling the mathematical underpinnings of the SPC charts to, at least to some extent, be revealed while maintaining a link with the practice; to 'reduce the magic' as described by one worker.

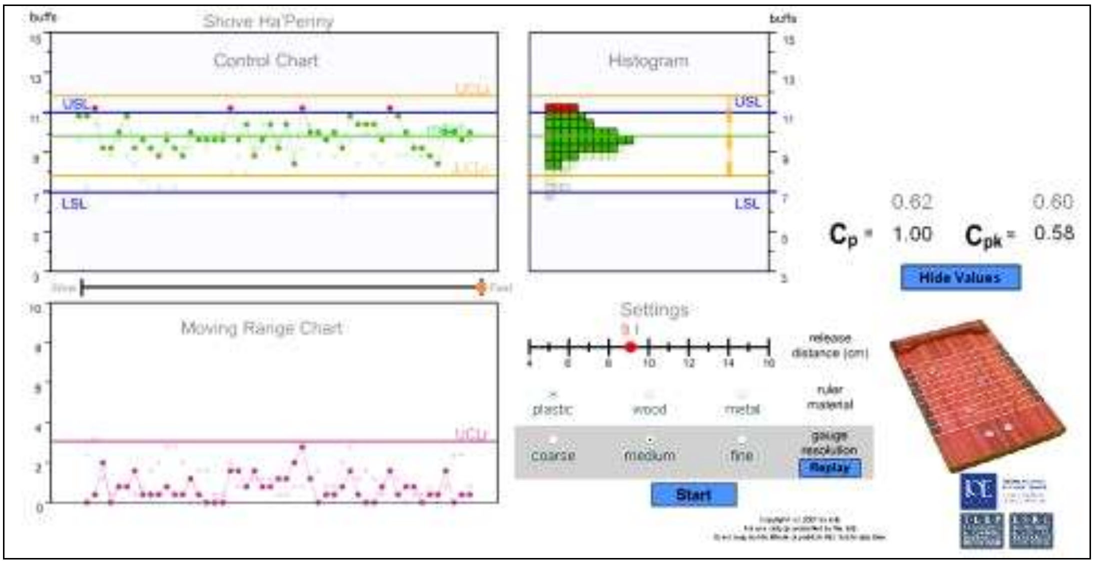


Figure 3. The TEBO: automating the processes underlying the construction of an SPC chart

Our research indicates an important, and in the context of this paper, ironic point about outsourcing (both social and technical) of processing power. The calculations that powered our TEBO were, of course, outsourced to the machine so became invisible. Yet for effective communication, we took careful design decisions so that some of the *processes underlying this outsourcing* – which parameters were crucial, how the different representations contributed to the calculated results – became *more* visible; and, as we pointed out above, we designed in *layers* that allowed different grain-sizes of interaction with the key mathematical ideas. We conjecture that much the same could be true of the classroom: in order to benefit from the pedagogic gain of outsourcing calculation to, say, the calculator, some attention must be given to providing glimpses of the process in the interests of learning and debugging. Opening access to some layers of the system while achieving an optimal grain size is a matter of careful and expert iterative design.

## 4. New semiotic tools and representational infrastructures

We begin with an example drawn from Seymour Papert (2006). He invites us to join him in a thought experiment at an undefined time when the Roman numeral system was in use. We are to imagine that the restricted number of experts versed in doing multiplication suddenly became insufficient for the needs of their society, and that mathematics educators were asked to remedy the situation. Naturally, they adopted a range of carefully designed teaching experiments and their efforts were rewarded: more people than before were able to multiply. But 'something else did this far more effectively: the invention of Arabic arithmetic turned an esoteric skill into one of "the basics".' (*ibid.,* p. 582).

It was Kaput who coined the term 'representational infrastructure' to refer to the kind of cultural tool epitomized by the Arabic numeral system (his work in this regard and its implications for mathematics learning is summarised in Hoyles & Noss, 2008). One characteristic of such a representational system is that it is taken-for-granted: this ubiquity and invisibility are critical facets of tool systems that become infrastructural. A key point here is that students of mathematics learning need to be aware not only of *how* mathematics is learned but also *what* is learned and the language in which this is expressed. Multiplication, like Newton's laws, or elementary calculus, is *learnable,* precisely *because* we have Arabic numerals, the machinery of simple equations and Leibniz's calculus notation respectively. What is to be learned depends on the representational forms with which it is expressed, shaping and sometimes defining what can be considered as learnable. Thus we would argue that those who study mathematical cognition ignore semiotic mediation at their peril!

We give two examples. The first is derived from the WebLabs project[8], in which we employed *ToonTalk* as a programming 'language' for children to build models of mathematical phenomena[9]. Our aim was to design tasks that would, we thought, be relatively unrealistic for 13/14 year-old students with only conventional infrastructures for expression. Or, to put it another way, to see if we could design new representations that would make relatively unlearnable mathematics more learnable for these students. For example, we designed tasks on infinite sequences and series, and engaged students with the sum of sequences like 1, ½, ¼, $^1/_8$, ... and 1, ½, $^1/_3$, ¼, ... .

In such a scenario, there are several difficulties with the conventional representation. The first is evident with the use of ellipsis to denote "and so on". Not all students see that, for example, 0.1428571... is an infinite decimal, preferring instead to see the 1 on the right as the "last" digit. Indeed, the fact that it takes an infinite number of digits to represent a tangible entity like $^1/_7$ is a paradoxical situation for many students – the difference between a number and its (various) representations is far from obvious. So a second difficulty – more serious than the first – is that it is, in conventional representations, *impossible* to write down an equation like $^1/_7$ = 0.1428571 without some convention peculiar to the representational infrastructure (such as judicious placing of dots either at the end, or above some of the digits).

To design our new representation we had, therefore, to eliminate rounding errors. We achieved this by the implementation of exact rational arithmetic in ToonTalk. In ToonTalk, it really is the case that there is an exact decimal expansion of a rational number, and moreover, that this is recognised by the system ($^1/_7$ = 0.1428571... is "true").

But how to represent the "..." to the right of the decimal expansion? Clearly this is a serious design challenge: no truncation should return 'true', yet there *is* a decimal expansion of $^1/_7$ that is *exactly* equal to it. We remark in passing that we met

[8] Grant IST 2001-3220 of the Information Society Technologies Programme of the European Commission.

[9] We have put quotation marks around the word 'language' to underline that ToonTalk is far from a standard representational infrastructure for programming. Instead of the standard lines of code, Toontalk is a programming system in which programs are instantiated as 'robots', trained what to do by – literally – being shown by the user's avatar, present in the form of its (your) hand.

this situation many times in our iterative design process: solving one problem of representation threw up a new problem.

Our solution was to invent the idea of *shrinking digits*. Digits are displayed in gradually decreasing size until they reach the size of a pixel. In this way the idea that an infinite number of digits follow the decimal point is conveyed visually. By using the ToonTalk ‘pumping’ tool for increasing the size of an object, a student can view more and more of the digits that initially were too small to see. This process can take place indefinitely: there is a theoretical size limit based on the memory of the computer, although there is nothing to stop the process being transferred to a second computer when the memory is full! Figure 4 provides an illustration of a decimal representation of the rational number $^{5}/_{49}$.

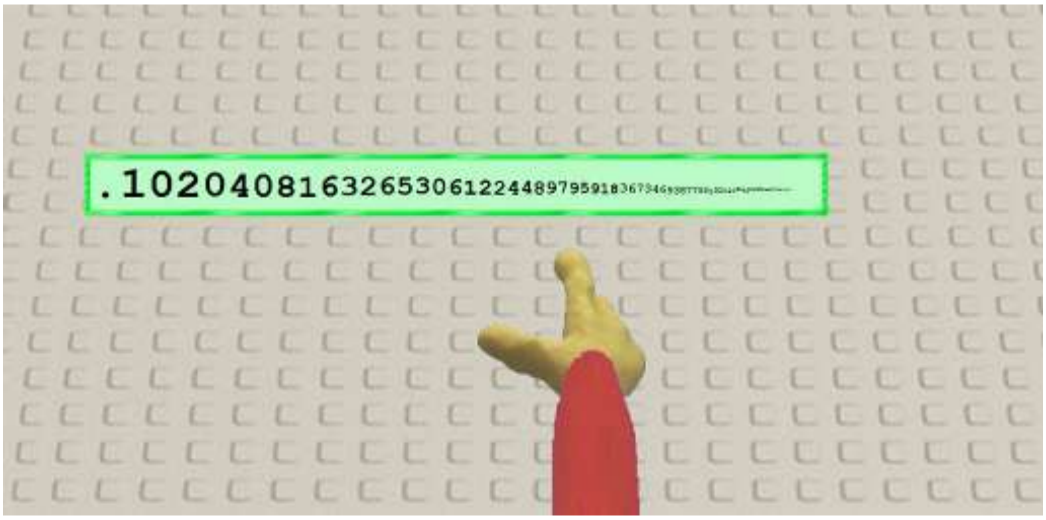


Figure 2. An example of the new shrinking digit display, showing the result of dividing 5 by 49. (You can move ‘your hand’ to the right, hover over the tiny digits and then pump them up to a size large enough to read)

Dividing the infinite shrinking digit representing $^{6}/_{7}$ by $^{2}/_{7}$ really does return the exact value 3.

Our evidence as to the extent the new representational infrastructure enhanced the mathematical meanings developed by students when compared with the meanings developed, in paper and pencil, is mixed. We were unable to undertake any large scale trials due to constraints of technology access and time – inevitable in such experimental situations  - but we did have existence theorems: instances of students engaging with and undertaking tasks that would, we think, have been impossible using traditional paper and pencil infrastructures (see Mor, Noss, Kahn, Hoyles & Simpson, 2006).

# 5. Connectivity and shared mathematics

Connectivity continues to change the landscape of human-human and human-computer interaction. To what extent is this shift reflected in the mathematical meanings learners develop? There is no lack of potential: indeed Roschelle, Penuel, & Abrahamson (2004) have argued that the connectivity made possible by computational media constitutes a profoundly important set of affordances, ranking alongside the 'representational-simulation affordances' of computational media as described in the previous section. Given that this connectivity has only recently been implemented and access is still an issue in many schools, there is rather limited research at the time of writing this paper to test this conjecture or to identify in any systematic way the implications of enhanced connectivity on mathematical development. We draw from the work of the panel on connectivity that was brought together by Study Group of the International Congress of Mathematics Instruction, ICMI 17 (see Hivon et al, in press. While noting the technological challenge of creating the appropriate means to share knowledge between students and teacher, the authors also pointed to its potential for mathematical learning.

From this and other sources, we distinguish two areas where we consider connectivity has considerable potential for enhancing the teaching and learning of mathematics. First, for connectivity within and between classrooms, an individual's communication can be changed into an object in a shared workspace, and thus become available for collective reflection and manipulation by the originator of the communication - but also by others. Second, the very need for remote communication of mathematical ideas – either synchronous or asynchronous – provides a motivation to produce explicit formal expression of mathematical ideas. We now look at each of these scenarios in turn.

## *i. Objects for reflection and manipulation in a shared classroom space*

There are technologies where each student in a class can build a particular case or part of a mathematical object, and these different instances can be brought together in a common workspace. Students can therefore view their own production and that of their peers and all responses can become an object of collective reflection and can be manipulated accordingly. This affordance appears to have – so far from mainly anecdotal evidence - a marked impact on mathematical learning. As Trouche and Hivon argue (in the case of a class of students working with *TI Navigator*): "Each student becomes detached from his/her production as a distance is created between student and the expression of his/her creation and this distance seemed to improve collective reflection on practice. The student becomes involved in the class activity in a different way as the tool maintains this distance between a student and the results proposed to the class and to the teacher". (Trouche & Hivon, in press). This type of connectivity might have considerable impact on the potential of dynamic and graphical tools for the development of mathematical meanings as set out in section 2 in this paper, since the sketch is now available to all for collective consideration.

While this observation refers to the effect of connectivity on teaching and learning, there are epistemological possibilities as well. Consider, for example, viewing a family of objects in the shared space, with each object belonging, say, to a single student. The group as a whole can view *the family* as a new mathematical object with its own parameters. This potential for the study of hitherto inaccessible mathematical objects and relationships is a largely untapped, but nonetheless tantalising, prospect (see for example Hegedus & Penuel, 2008). One set of studies that deals with this epistemological dimension has been reported in a series of papers by Wilensky and his colleagues. They report on studies that have added synchronous connectivity to the agent-based system *NetLogo*, so the students in a class can all become engaged in a participatory simulation rather than simply a modelling activity (see, for example, Wilensky & Stroup, 1999, Wilensky, & Reisman (2006). These studies have pointed to a range of benefits for learning, not least that it introduced a shared experience of a complex system: "There are very few opportunities, in the classroom or in life, for students collectively to witness the same complex system unfolding. Focal attention to such a system is hard to achieve outside of the virtual and, even when achieved, if the viewing does not connect the micro-level behaviour to the macro-level outcomes, then only the appearance is shared, not the mechanisms of action" (Wilensky, in press).

### *ii. Designing to share objects at a distance*

Turning to the issue of sharing at a distance, we have undertaken two projects that both set out to exploit intersite connectivity (as well as face-to-face collaboration) to promote synchronous and asynchronous sharing, discussion and co-development of mathematical ideas. The overarching objective of both studies was to foster appreciation of the structures and processes *underlying* a set of mathematical ideas through carefully designed collaborative activities. The first project, the *Playground project* sought to design systems in which children aged between 4 and 8 years, could design, build and share simple video games. (see for example, Hoyles, Noss, & Adamson, 2002)

As part of the study we noted an interesting shift when children moved from face-to-face collaboration to collaborating across remote sites. This shift was characterised by a move from socially derived rules to govern the games in the former scenario to system rules (computational expressions) in the latter. This shift seemed to be a result of the necessity to formalise in the absence of all the normal richness of interaction that characterises face-to-face collaboration, where the narrative of the game was fore grounded and rules frequently only tacitly agreed. At a distance such tacit agreements were not available, and the narrative had to be translated into a form that the computer could accept (for elaboration, see Noss, R., Hoyles, C., Gurtner, J-L., Adamson, R. & Lowe, S, 2002).

The absence of face-to-face collaboration does not in any sense guarantee the shift towards formalisation. That it arose at all, undoubtedly owes much to the activity structures, relationships between children, and of course, the presence of the researchers. Nevertheless, it is interesting to speculate whether, by a more focused and prolonged emphasis on remote collaboration with suitably designed computational systems, new kinds of formalised discourse might be engendered in a wider range of learning environments.

In a later project, *WebLabs*, (described earlier) (www.lkl.ac.uk/kscope/weblabs), we attempted to scaffold interactions at a distance by devising a web-based system, *WebReports*, that allowed students to post their ideas—*and their working models* (using the ToonTalk programming system used in the project) — so that students working in other classrooms could download the models, run and interpret them, reflect on them before sending comments and possibly amended models (see for example, Simpson, Hoyles, & Noss, 2005). This work built on the importance for learning of externalising cognitive processes and sharing these externalised representations: for example, Scardamalia & Bereiter had argued that an electronic and networked discussion board would foster conversations between students and thus would “contribute to the development of a “knowledge building community” (Scardamalia & Bereiter, 1996). Our key idea was that learners could not only discuss, conjecture with and comment upon each others' ideas, but they could also inspect and edit each others' working models of ideas, the computer programs – so that the processes underlying the outcomes were made visible at least to some extent. Again, the idea of appropriate layers of visibility was crucial in the design. This proposed functionality of collaborative knowledge construction is, at least so far, one of the most promising avenues we perceive of connectivity: the possibility of building mathematical understandings in shared remote space, in settings that transcend that of a single classroom[10].

To sum up the outcomes of these two projects, (see also Noss & Hoyles, 2006), we note that where we did achieve success, engagement tended to derive from the sense of audience we created and the need to make arguments explicit when removed from the presence of others. This led to some interesting discussion threads about deep mathematical topics – it is not commonplace to have students routinely chatting about mathematics! Nevertheless, there were considerable challenges concerning the need to take account of the mediation of tools operating at two levels, first in the construction process and second in the communication infrastructure: both influenced the development of mathematical meanings. The teacher had to cope with these twin challenges in orchestrating optimal student-student and student-teacher interaction in relation to the knowledge at stake.

For interaction at a distance to lead to developing mathematical meanings, there needs to be more investigation of the kinds of support required to foster longer communication turns by each contributor. It appears evident that a necessary – but far from sufficient – condition for connectivity to foster learning, is for interaction to be extended and productive: off-task interaction is unlikely to lead to mathematical

[10] It is worth noting that had this project been a few years later, we could have employed one of the many ‘social networking’ sites to achieve much the same effect at a fraction of the time and effort).

development. Some researchers have suggested that simple statistics on thread lengths in threaded discussion systems indicate that communication does not usually continue long enough to get much beyond chatting (Stahl, 2001, p. 179). Thus a particular requirement suggests the need to support interactions so that communication is stimulated and maintained over time as well as space. We had some success in our work by contriving competitive challenges that stimulated a game-like discourse. Other possible strategies include pointing to conflicting arguments from others in the group that have to be resolved. This strategy can be used by a teacher but, we now think, more effectively supported by the technological system itself, Although it is outside the scope of this paper, we note that it is this realisation that has stimulated our latest research, MiGen[11], in which we seek to introduce various supports from the computational system (see Noss, Hoyles, Geraniou, Gutiérrez-Santos, Mavrikis & Pearce, under review).

# 6. Conclusions

This paper has raised issues concerning the ways that mathematical meanings are shaped by the symbolic tools in use, and the representational infrastructures that hold them together to express mathematics and to communicate and share mathematical ideas. We have distinguished different ways that tools can shape mathematical cognition: these require future investigation to establish if they do reliably enhance learning.

We began with the idea of dynamic and graphical tools, and our example involved ‘sketching’, as a way for students to consider and choose for themselves on what it is important to focus. This is a key obstacle in learning mathematics: ironically enough, given that the search for variants and invariants is, perhaps, *the* crucial mathematical activity. And ironic too, in that sketching – which does not, at least in our example, involve rigorous expression of mathematical ideas, but rather getting a sense of the possible relationships involved – and only subsequently employing the computer in its most obvious role, as a mechanism for expressing rigour.

We then considered the implications of the outsourcing of processing power to the technology, and chose as our example, our research intervention in a car

[11] Funded under the ESRC/EPSRC, TLRP-TEL programme Grant reference RES-139-25-0381.

manufacturing plant. Our example indicates that outsourcing is not unproblematic. It does not remove the necessity to understand at some level, and it neither removes the necessity for pedagogic design, nor the need to make visible some of the processes underlying the outsourced mathematics. While the devolution of mathematical technique to the machine is a superb advance for mathematics as a discipline, it nonetheless presents a major challenge for learning scientists who must decide, first what needs to be maintained as visible – the parameters and variables, relationships and techniques that contribute towards 'epistemic' development – and second, how to present these key factors in a layered learning sequence.

In considering the question of representational infrastructure, we noted that there were sufficient indications that many commonly encountered obstacles to understanding mathematics lay in the chosen representational infrastructure, rather than any in the complexity of the idea itself. Put another way, we might conjecture that Bruner's often-quoted aphorism could be rephrased as: *any mathematical idea is learnable and teachable, provided we find the right representational infrastructure within which to express it.* We would prefer not to be taken literally: but we do think that research is beginning to point to instances of how technology can be utilised to realise this aim.

Finally, we considered the question of connectivity, and gave two ways in which it may have implications for mathematical development; in the possibility of bringing students' constructions together as objects for reflection and manipulation in a shared space, and in the need for explicit formal expression of mathematical ideas when they are to be shared at a distance. This area of research is in its infancy: it is, after all, much harder to think of ways that connectivity could revolutionise mathematics than almost any other domain. One reason is that the balance between information in the form of facts, and concepts is titled strongly on the former. Nevertheless, there are signs that there may yet be the beginnings of, not just a pedagogical transformation but also an epistemological one, catalysed by connectivity.

We conclude by noting that there are two key unifying ideas in this paper. The first is design, the obvious but often overlooked fact that technology *per se* is unlikely to influence mathematical development in any significant ways, it is how it is designed to support learning and how it is embedded in activities designed with specific learning objectives. The second is the importance of tools (tools that express the mathematics and tools that that connect the learners) in shaping and enhancing the meanings developed by the participants articulated as situated abstractions in each case.

The research challenges are considerable, not least because of the rapid advance of the technology that might render the categories described in this paper inadequate. For example, we have barely had a chance to consider the implications of multi-touch screens or mobile handheld devices on learning; yet these too hold the promise of pedagogic potential and also will shape both what is learned as well as how it is learned. There are many such advances in the pipeline. But just in case we are accused of technocentrism, we reiterate that none of these developments will happen without more design research to tease out the ways the tools shape mathematics and its learning, and reciprocally, an understanding of how individuals and communities can shape the evolving technology.

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