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Investigating the sources of variability in the dynamic response of built-up structures through a linear analytical model

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ABSTRACT

It is well established that the dynamic response of a number of nominally identical built-up structures are often different and the variability increases with increasing complexity of the structure. Furthermore, the effects of the different parameters, for example the variation in joint locations or the range of the Young's modulus, on the dynamic response of the system are not the same. In this paper, the effects of different material and geometric parameters on the variability of a vibration transfer function are compared using an analytical model of a simple linear built-up structure that consist of two plates connected by a single mount. Similar results can be obtained if multiple mounts are used. The scope of this paper is limited to a low and medium frequency range where usually deterministic models are used for vibrational analysis.

The effect of the mount position and also the global variation in the properties of the plate, such as modulus of elasticity or thickness, is higher on the variability of vibration transfer function than the effect of the mount properties. It is shown that the vibration transfer function between the plates is independent of the mount property if a stiff enough mount with a small mass is implemented. For a soft mount, there is a direct relationship between the mount impedance and the variation in the vibration transfer function. Furthermore, there are a range of mount stiffnesses between these two extreme cases at which the vibration transfer function is more sensitive to changes in the stiffness of the mount than when compared to a soft mount. It is found that the effect of variation in the mount damping and the mount mass on the variability is negligible. Similarly, the effect of the plate damping on the variability is not significant.

1. Introduction

Uncertainty and variability are inevitable in structural dynamics. There is always a level of variability in manufacturing as well as unavoidable uncertainty in defining the parameters of built-up structures. Increasing the complexity of a built-up structure will result in a higher level of variability in the dynamic response of those nominally identical structures.

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Automotive vehicles are an example of such complex structures with many different parts that possess a high level of variability in their noise and vibration Frequency Response Functions (FRF) (e.g. Refs. [1–5]).

Different methods are adopted to model the uncertainty and variability in structural dynamics [6–10]. Amongst different components, it is often suggested that bolted joints and fasteners are the most variable and uncertain as their properties vary for each individual joint and also over time. A comprehensive review on this uncertainty and the techniques used to model joint variability can be found in Ref. [11]. There have been also attempts to identify and model the variability in other components, for example, Scigliano et al. [12] assessed the variability of a car windscreen due to temperature variation numerically and verified it by experimental results, Resh [13] evaluated the variability in the dynamic characteristics of engine mounts and Donders et al. [14] used a Monte Carlo simulation to assess the effect of spot weld failure on the fundamental natural frequency of a vehicle body-in-white. Gärdhagen and Plunt [15] claimed that the variation in Eigen frequencies and large variation in modal damping ratio in a plate and an acoustic cavity can cause similar variations in FRF to that seen in vehicles. To have such similarity between experimental results and their simple model they used a high variation in modal damping ratio in a way that 70% of modes have a damping ratio in the range of 0.46–3.2%. This is a high level of variation in damping ratio even though it is the most uncertain parameter that needs to be estimated in structural dynamics [16]. Wood and Joachim [17] studied the variability of twelve nominally identical cars and concluded that the damping in the spot welds and joints is the main source of variability. However, these studies do not present a detailed comparison between the contributions made by different parameters of the system to the overall variability. Although they showed that FRF variability can be due to a variation of one of the system parameter, for example the joint properties, the same level of FRF variability can also be produced by a smaller change in another parameter of the system, for example the stiffness of one of the components.

The range over which one parameter can be expected to vary may be very different to that for another parameter. For example, the stiffness of rubber bushes can vary over a wide range while the variation in modulus of elasticity of a plate is very small. Furthermore, the effect of the variation of each parameter on the dynamic response of the structure may not be similar. This paper addresses the latter issue i.e. the effect of different parameters such as structure thickness, material properties, manufacturing tolerances, etc. on the variability of the dynamic response of a built-up structure. An assembly of two plates connected by a single mount at low and medium frequency consistent with interior automotive structure is considered here, where a simple built-up structure has been used in order to make it possible to obtain an analytical model that allows for the examination of the role of the different parameters in the overall variability. The model is linear and the effect of nonlinearity in mounts or other elements of a built-up structure is not considered in this study. A lumped parameter model for the mount is used as in practice the distributed mass of the mount usually affects the vibration transfer problem only at higher frequencies that are beyond the frequency of interest of this paper. The motivation behind this work is to understand the source of variability in noise and vibration FRFs in an automotive vehicle which are comprised of many structural parts and where some of these parts are connected with small mounting clips and joints, for example the attachment of plastic trim components to door panels via plastic push clips. Furthermore, the results can be implemented in methods that allow propagation of the variability from component level to the built-up structure (e.g. Ref. [18]). The frequency range of interest is limited to the low and medium frequency range. In this frequency range deterministic models are typically used for vibration analysis. The limitations of deterministic models at higher frequencies are discussed in Ref. [19] which provides an introduction to Statistical Energy Analysis.

The mathematical model and the derived approximate equations for various extreme cases are given in Section 2 of the paper. This is followed by a numerical example in Section 3 that has been used in this paper for comparison. In Section 4, the effects of different parameters on the vibration transfer function are examined and a sensitivity analysis is used to compare the effect of different parameters on the variability. Sensitivity analysis is widely used in structural design optimisation and its concept is well established [20–22]. Here, a normalised sensitivity function is used for comparison purposes.

2. Mathematical modelling

The mobility and impedance method can be employed to obtain vibration transfer functions of assemblies from a knowledge of the transfer functions of the individual sub-structures [23,24]. In the present work, the method is applied to two connected plates and mobility FRF is obtained. Such configurations are common in vehicles, for example the door trim is connected to the door by a series of clips. To simplify the analysis, only one mount is used in this study, but the same method can be applied to a more complex system (e.g. the representation cited in Ref. [25]). Since the focus of this study is on the low and medium frequency range, only bending modes of the plates are considered.

The schematic representation of the two connected plates is shown in Fig. 1, where the plates are considered simply supported on all four sides. Plate 1 is excited at point 1 and the response is obtained at point 4 on plate 2. The internal forces and velocities are shown at the mounting position on both plates and at the mount location. The points on the mount are referred to by the same number as their corresponding points on the plate with an additional dash.

In the following sub-sections, analytical formulation for out-of-plane mobility FRFs and impedances of a mount that is modelled as a mass-spring-damper is given. The mobility of the assembly as a function of mobility and impedance of its sub-structures is obtained. This is the model that the rest of the paper is based on. Approximate formulations are also given in the last sub-section here that provides an insight into the physics of the problem.

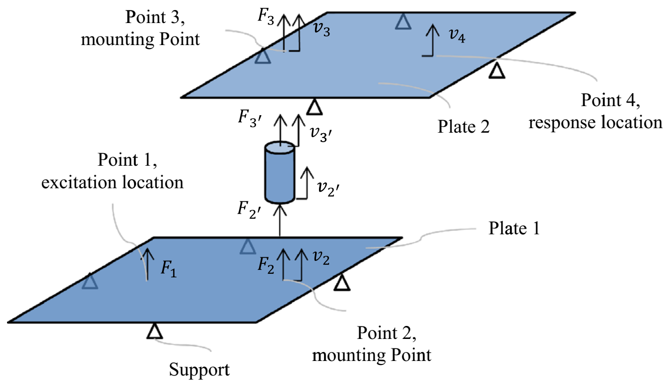

Fig. 1. Schematic view of two plates connected by a single mount.

2.1. An individual plate mobility

The equation of motion for a thin plate using Kirchhoff–Love plate theory can be solved to obtain the out-of-plane displacement of a simply supported plate [26]. For a thin plate that is subject to a harmonic point force $F_e \cos(\omega t)$ at an arbitrary location (x_e, y_e), the mobility function can be obtained for response at an arbitrary location (x_r, y_r) using the following equation,

$$Y_{re}=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{4}{\rho hab}\Phi_{repq}\Omega_{pq} \tag{1}$$

where

$$\Phi_{repq}=\sin\frac{p\pi x_r}{a}\sin\frac{q\pi y_r}{b}\sin\frac{p\pi x_e}{a}\sin\frac{q\pi y_e}{b} \tag{2}$$

and

$$\Omega_{pq}=\frac{j\omega}{\omega_{pq}^2-\omega^2+2j\zeta_{pq}\omega_{pq}\omega} \tag{3}$$

where ρ is the plate density, h is the thickness, a and b are plate dimensions, ζ_{pq} is the modal damping ratio and ω_{pq} is the natural frequency for mode (p, q) of the plate, obtained from the following equation for a simply supported plate, where D is flexural rigidity of the plate, $D = Eh^3/\left(12\left(1-\nu^2\right)\right)$;

$$\omega_{pq}=\pi^2\left[\left(\frac{p}{a}\right)^2+\left(\frac{q}{b}\right)^2\right]\left(\frac{D}{\rho h}\right)^{1/2}, \text{ for } p, q=1, 2,\ldots \tag{4}$$

A viscous damping model is used here but similar results can be obtained if structural damping was used.

2.2. The mount impedance

The mount can be considered as a combination of two masses, a spring and a damper as shown in Fig. 2. The mass of the mount is divided equally between two ends of the mount and the damping is assumed of viscous type.

The impedance matrix of the mount is given by [24]:

$$\mathbf{Z}_m=\begin{bmatrix} c+j\omega m/2+k/j\omega & -(c+k/j\omega) \\ -(c+k/j\omega) & c+j\omega m/2+k/j\omega \end{bmatrix}. \tag{5}$$

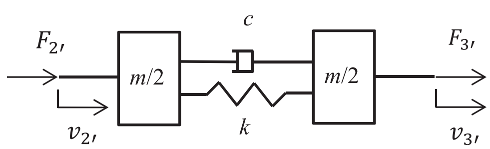

Fig. 2. A lumped parameter model of the solid mount that connects the two plates.

2.3. Vibration transfer function

The velocity at point 2 can be found from the mobility formulation for plate 1,

$$v_2 = Y_{22}F_2 + Y_{21}F_1 \tag{6}$$

where Y_{ij} is the ith and jth element of a mobility matrix for an individual part. The dynamics of the mount can be obtained from an impedance approach, $\mathbf{f}_m = \mathbf{Z}_m \mathbf{v}_m$ where,

$$\mathbf{f}_m = \begin{bmatrix} F_{2'} \\ F_{3'} \end{bmatrix}, \mathbf{Z}_m = \begin{bmatrix} Z_{2'2'} & Z_{2'3'} \\ Z_{3'2'} & Z_{3'3'} \end{bmatrix}, \mathbf{v}_m = \begin{bmatrix} v_{2'} \\ v_{3'} \end{bmatrix}. \tag{7}$$

For the receiver plate, the velocity of the mounting point can be obtained from the following mobility equation, assuming no external force is applied to the second plate,

$$v_3 = Y_{33}F_3. \tag{8}$$

The continuity and the equilibrium conditions require that $v_{n'} = v_n$ and $F_{n'} = -F_n$ for $n = \{2,3\}$. By substitution, the velocity of the plates at the mounting position can be obtained as a function of mount impedances and internal forces,

$$\begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = -\mathbf{Z}_m^{-1} \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}. \tag{9}$$

Also, the velocity of the plates at the mounting point can be obtained from Eq. (6) and Eq. (8),

$$\mathbf{v}_{pm} = \mathbf{Y}_{pm}\mathbf{f}_{pm} + \mathbf{Y}_{pe}\mathbf{f}_e \tag{10}$$

where

$$\mathbf{v}_{pm} = \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}, \mathbf{f}_{pm} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}, \mathbf{f}_e = F_1, \mathbf{Y}_{pm} = \begin{bmatrix} Y_{22} & 0 \\ 0 & Y_{33} \end{bmatrix}, \mathbf{Y}_{pe} = \begin{bmatrix} Y_{21} \\ 0 \end{bmatrix}. \tag{11}$$

It should be noted that Eq. (10) is not the mobility formulation for the whole system but it is a combination of the mobilities of the individual sub-structures. By substituting velocities at the mount position into Eq. (10) the following equation can be obtained for the internal forces $\mathbf{f}_{pm}$,

$$\mathbf{f}_{pm} = \left(-\mathbf{Y}_{pm} - \mathbf{Z}_m^{-1}\right)^{-1}\mathbf{Y}_{pe}\mathbf{f}_e. \tag{12}$$

The mobility approach can be employed again to obtain the response at point 4, $v_4 = Y_{43}F_3$ and the transfer mobility between point 4 and 1 can be obtained by substituting the internal force F_3 from Eq. (12) into this equation,

$$Y_{41}^b(\omega) = \frac{v_4}{F_1} = -\frac{Y_{43}Z_{23}Y_{21}}{Z_{22}Z_{33}Y_{22}Y_{33} - Z_{23}Z_{32}Y_{22}Y_{33} + Z_{22}Y_{22} + Z_{33}Y_{33} + 1}. \tag{13}$$

The superscript ^{b} is used in this paper for the transfer functions of the assembly, e.g. Y_{41}^b and those without any superscript are used for the transfer functions of the sub-structures, e.g. Y_{22} (point mobility of the plate one at point 2). The above equation shows the relationship between the dynamic properties of the individual components and the vibration transfer function of the built-up structure which is a complex interaction of different component's properties.

2.3.1. Approximate equations for vibration transfer function

One of the main approaches to vibration control is to isolate the source of the vibration from its receiver. A mount with a low impedance can be used for this purpose [27,28]. On the other hand, a mount with high impedance that can be considered as a rigid link is used in many applications to limit the deformation under loading. This vibration transfer problem has been studied previously and approximate equations that govern the system response have been obtained, for example refer to Refs. [27,29,30]. Similar approximate equations for these two extreme cases are obtained here in order to provide an insight into the physics of the problem, as they show how the natural frequencies of the built-up structure change by varying the mount impedance. The rigid mount is considered first.

For a rigid mount, the stiffness of the mount and its impedance will be infinite which can be modelled by a mount with a very high stiffness. The natural frequency of the mount with free boundary conditions is $\omega_m = \sqrt{k/(m/2)}$. For a finite frequency range $\omega \ll \omega_m$, the mass impedance term in Eq. (5) for the lumped parameter model can be neglected compared to the stiffness impedance term, $|mj\omega/2| \ll |k/j\omega|$. As a result, for mounts at frequencies further below the mounts natural frequency, the cross impedance terms and point impedance terms can be considered equal,

$$\omega \ll \omega_m \Rightarrow |Z_{22}| = |Z_{33}| \approx |Z_{23}| = |Z_{32}|. \tag{14}$$

For a mount with a high natural frequency, the first two terms in the denominator of Eq. (13) can be neglected comparing with the rest of the denominator providing,

$$|kmY_{22}Y_{33}| \ll \left|\frac{k}{j\omega}\left(Y_{22}+Y_{33}\right)\right|. \tag{15}$$

Langley [31] obtained equations for envelopes of the spatially averaged mobility of a plate. A simpler equation for low and medium frequency where the modal overlap factor is not very high can be obtained by approximating Eq. (3) at its natural frequencies, max $(\Omega_{pq}) \approx 1/2\zeta_{pq}\omega_{pq}$. The maximum of the modal contribution, Eq. (2) is unity and assuming that the response is dominated by the resonant mode at a specific frequency the maximum of the point mobility can be obtained as max $(Y_{rr}) = 2/(m_i\zeta_{pq_i}\omega)$ where $m_i(i = 1,2)$ is the mass of the plate. Substituting this into Eq. (15) results in $m \ll (m_1\zeta_{pq_1}+m_2\zeta_{pq_2})/2$ for the mass of the mount.

Similarly, $\left|jk/\omega\left(Y_{22}+Y_{33}\right)\right| \gg 1$ allows neglecting 1 in the denominator of Eq. (13). The minimum of Y_{22} and Y_{33} can be obtained from the definition given in Ref. [31] which sets the limit for stiffness to be considered as rigid link,

$$k \gg \frac{\omega}{\min(Y_{22})+\min\left(Y_{33}\right)}. \tag{16}$$

Considering these assumptions, Eq. (13) can be reduced to the following approximate equation for a rigid mount,

$$Y_{41}^{b} \approx -\frac{Y_{43}Y_{21}}{Y_{22}+Y_{33}}. \tag{17}$$

For a rigid mount, the vibration transfer function is independent of the mount properties and the peaks of the transfer function of the system should occur at the minimums of the denominator of Eq. (17), $Y_{22}+Y_{33}$. Since Y_{22} and Y_{33} are complex functions and the phase changes before and after each mode of the system for well separated modes, there would be a peak in Y_{41}^b between each pair of natural frequencies of two individual plates.

For a soft mount, the stiffness of the mount is very low and as a result the amplitude of impedance terms are small, so neglecting the terms in the denominator of Eq. (13) compared to one and the vibration transfer function Y_{41}^b can be approximated to,

$$Y_{41}^{b} \approx -Y_{43}Z_{23}Y_{21}. \tag{18}$$

This is valid when $\left|jk/\omega\left(Y_{22}+Y_{33}\right)\right| \ll 1$, which by considering max (Y_{rr}) as given above implies that,

$$k \ll \frac{\omega^2\zeta_{pq_1}m_1\zeta_{pq_2}m_2}{2(\zeta_{pq_1}m_1+\zeta_{pq_2}m_2)}. \tag{19}$$

The natural frequencies of the built-up structure coincide with those of two separate plates in this case. It can be seen that there is a direct relation between the vibration transfer function Y_{41}^b and the impedance of the mount and a change in the impedance of the mount can affect the vibration transfer function. This is in contrast to the case of a rigid mount where the impedance of the mount does not affect the transfer mobility. Thus, to isolate two systems the impedance of the mount should be low. However, if the aim is to minimise the effect of the variability due to variation in mount properties a rigid mount must be used.

3. A numerical example

A numerical example of the configuration that is shown in Fig. 1 is considered here that is used in the following sections of the paper as a physical representation of the model. As the focus of the study is on the variation in noise and vibration at low and medium frequency range, the frequency range of the interest is limited to that below 600 Hz.

The dimensions of the two plates are chosen to have well separated modes at lower frequencies and a higher modal overlap factor at the end of the frequency range of interest. For the purpose of this study, the first plate is chosen to be 500 mm by 300 mm and the second plate has a dimension of 600 by 400 mm in x and y directions respectively so that the natural frequencies of two plates do not coincide. Both plates have a thickness of 1.5 mm and are made of steel. Two levels of modal damping ratio are considered in this paper, a low modal damping ratio of ζ_{pq}=0.01 and a high level of ζ_{pq}=0.03. These damping levels are commonly used for modelling the dynamic response of body-in-white and trimmed body assemblies respectively in the automotive industry. The mount damping ratio is considered equal to 1% with a mass of 1 g, which is the mass of the small clips used to connect vehicle trim to its body. The same numbering convention has been used as in Fig. 1 where coordinates (measured from the left corner) of plate 1 for points 1 and 2 and of left corner of plate 2 for points 3 and 4 are given in Table 1.

The transfer function mobility between point 1 in plate one and point 4 on plate 2, Y_{41}^b, is obtained for two values of mount stiffness; a soft mount k = 1 kN/m and a stiff mount k = 5000 kN/m (both relative to the stiffness of the steel plates) and are shown in Fig. 3.

As can be seen, the approximate formulas obtained in the previous section represent the system very well at the extreme cases. The soft mount acts as a vibration isolator and reduces the transmitted vibration significantly compared to the rigid mount. Furthermore, the resonant frequencies of the dynamic response when the soft mount is used coincide with the

Table 1
The position of the excitation, mounting and measurement position on plates.

Point	x (mm)	y (mm)	Point	x (mm)	y (mm)
1	177	71	3	104	69
2	353	140	4	520	346

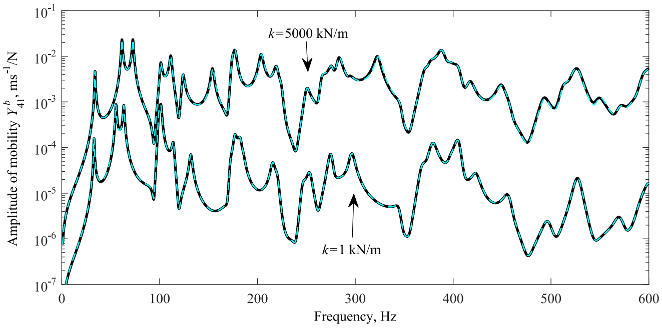

Fig. 3. Amplitude of mobility Y^b_{41} for two plates with modal damping ratio of ζ_{pq}=0.01 connected by a mount with two different stiffness of k = 1 kN/m and k = 5000 kN/m. Solid thick line (—): exact solution. Dashed thin line (- - -): approximate solution.

resonant frequencies of the individual plates, while for the case of a rigid mount, they lie between modes of the individual plates as discussed in the previous section.

4. Effect of different parameters on the vibration transfer function

By changing a parameter of an individual component of a built-up structure, not only the dynamic response of that component varies but also the dynamic response of the assembly changes. As the variation propagates in the structure from component level to the assembly, its effect would not be the same. In this section, the effect of the mount properties, i.e. its stiffness, damping and mass, on the vibration transfer function is evaluated as are mounting position and plate properties.

The sensitivity analysis allows one to investigate the change in the system response due to variation in its parameters [6]. If p is one of the parameters of a system function called Y, a change in system parameter, Δp, causes a change in system function, ΔY. The *sensitivity function* can be defined as a function that relates the variation in system function ΔY, to the change in the parameter Δp with a nominal value of p_0, $\Delta Y = \tilde{S}(p_0)\Delta p$. If the function is normalised it makes it possible to compare the sensitivity of the system to different parameters, which in the limit becomes,

$$S_p^Y = \frac{\partial Y/Y}{\partial p/p}. \tag{20}$$

This is a normalised measure of the change in a function (e.g. mobility function) due to a change in a parameter (e.g. thickness). It should be noted that the vibration transfer function is a complex function (magnitude and phase) which results in a complex sensitivity function, where in this paper the magnitude of the sensitivity functions given by Eq. (20) is only considered. The analytical results of sensitivity are verified by using the finite difference method which is not shown in the paper for brevity.

4.1. The effect of mount properties

Three parameters of stiffness, damping ratio and mass are defining the properties of the mount. Their effects on the vibration transfer function are considered here first.

4.1.1. The effect of the mount stiffness

The effect of variation in the mount stiffness on the mobility was demonstrated in Section 2.3.1 for two extreme cases, i.e. an isolator mount and a rigid link. The amplitude of mobility Y^b_{41} is plotted for 10 different values of mount stiffness spaced logarithmically in the range of 1 kN/m to 10000 kN/m in Fig. 4. The solid line in the figures is the locus of the peaks of three lowest modes as the stiffness of the mount increases. While resonant frequencies are changed by increasing the mount stiffness, the anti-resonance frequency is almost constant and only its amplitude changes. Increasing the mount stiffness (from low values) increases the amplitude of the response. It can also be seen that by increasing the stiffness there is a reduction in the spacing between FRF curves. In order to develop an improved understanding of the effect of the mount stiffness on the variability the sensitivity function can be used.

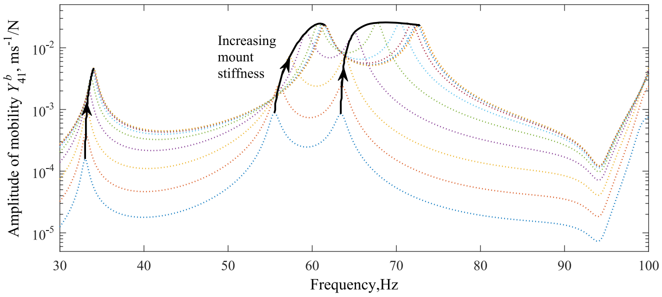

Fig. 4. The effect of the mount stiffness on the amplitude of mobility Y^b_{41} for the first three modes of the two connected plates of the numerical example with a modal damping ratio of ζ_{pq}=0.01, connected by a mount with m = 0.001 kg and ζ = 0.01. Mount stiffness is logarithmically spaced in the range of 1 kN/m to 10000 kN/m. Dashed lines (- - -): mobility Y^b_{41}. Solid lines (—): locus of peaks.

By substituting the mount impedance terms in Eq. (13) and differentiating according to the definition of Eq. (20), the sensitivity function of transfer mobility $Y^b_{41}$with respect to the mount stiffness, $S_k^{Y^b_{41}}$, can be obtained as a direct function of the mount stiffness.

The contours of the amplitude of sensitivity $S_k^{Y^b_{41}}$ as a function of the mount stiffness and frequency are shown in Fig. 5 for the numerical example presented in Section 3, with two levels of modal damping ratio: ζ_{pq}=0.01 as a low damping ratio and ζ_{pq}=0.03 as a high damping ratio of the plates.

For a relatively low mount stiffness, it was previously shown that the transfer mobility is in direct relationship with the mount impedance Z_{23}, implying a unity sensitivity which is visible in Fig. 5(a) and (b) for a low mount stiffness. The dashed line is the limit of low stiffness given by Eq. (19). For a stiff mount as given in Eq. (17), the transfer mobility Y^b_{41} is independent of the mount impedance which results in a sensitivity of zero to the mount stiffness. The solid line in the figure is the high stiffness limit given by Eq. (16) above, in which the mount can be considered rigid. This line is very close to a straight line and it does not change considerably with frequency.

The effect of the mount stiffness on the variability of the vibration transfer function is higher when the property of a mount changes from a soft mount to a stiff mount and the resonance peaks shift in frequency. This is the range at which peaks can be seen in the sensitivity contour between dashed and solid lines and is called the "transition zone" here.

Increasing the modal damping of the plates results in an increase in modal overlap and a reduction in resonance amplitude of the mobility. Sensitivity to the stiffness as well as the width of the transition zone is reduced by increasing the modal damping. At high frequencies, where modal overlap factor is high, the transition zone disappears and the sensitivity changes from almost unity for a mount with a low stiffness to zero for a mount with a high stiffness.

Due to the shift of resonance frequencies, the peaks of the sensitivity curve coincide only with the resonance frequencies of the system. At peak values, the sensitivity is of the order 5, this value being used as a comparison later in the paper. Note that the amplitude of the sensitivity is not directly correlated to the amplitude of the resonance.

4.1.2. Effect of the mount damping

To investigate the effect of the mount damping on the vibration transfer function, the sensitivity function with respect to

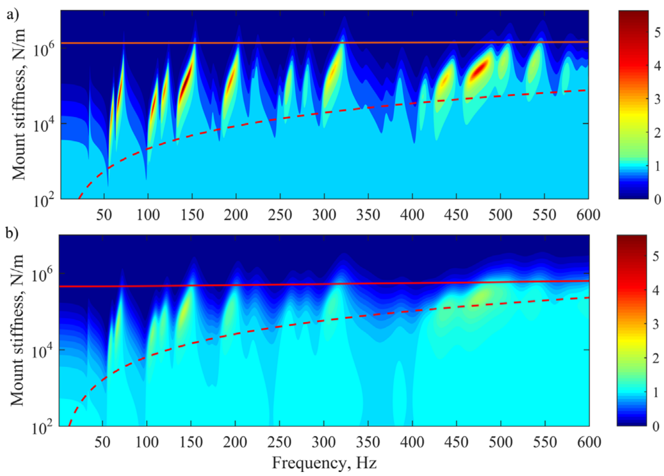

Fig. 5. Contour plots for the amplitude of the sensitivity of the transfer mobility with respect to the mount stiffness $S_k^{Y^b_{14}}$ as a function of frequency and mount stiffness for the model given in the numerical example of Section 3 and for two different values of plates modal damping, a) ζ_{pq}=0.01, b) ζ_{pq}=0.03. Solid line (—): stiffness limit for a stiff mount (Eq. (16)). Dashed line (- - -): stiffness limit for a soft mount (Eq. (19)).

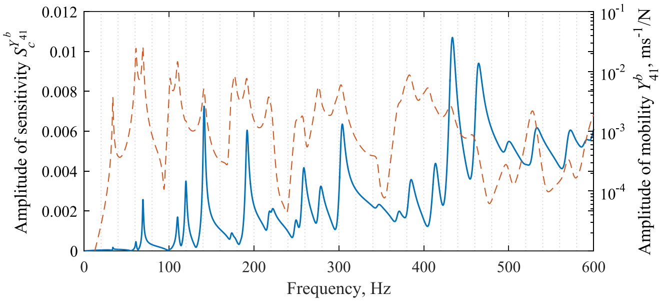

Fig. 6. Amplitude of sensitivity $S_c^{Y^b_{41}}$ for a mount with stiffness of k = 100 kN/m and a mount damping ratio of ζ = 1%. Solid line (—): Amplitude of sensitivity $S_c^{Y^b_{14}}$. Dashed line (- - -): Amplitude of mobility Y^b_{41}.

the mount damping can be obtained in a similar way, $S_c^{Y^b_{41}}=(\partial Y^b_{41}/Y^b_{41})/(\partial c/c)$. By substituting Y^b_{41} and mount impedances in this equation, and comparing it with the sensitivity of the mobility with respect to the mount stiffness, the following equation can be obtained for the sensitivity with respect to the damping coefficient,

$$S_c^{Y^b_{41}}=\frac{2j\zeta\omega}{\omega_m}S_k^{Y^b_{41}}. \tag{21}$$

Although increasing the damping level will increase its effectiveness, the sensitivity of mobility $Y^b_{41}$with respect to damping will be much smaller than the sensitivity to the stiffness since it is multiplied by the mount damping ratio and the ratio between the frequency and natural frequency of the free-free mount, ω/ω_m, which will be very small number in practice for the low frequency range.

There is a widely-held opinion that the variability and uncertainty in damping ratio is much higher than the variation in other parameters of the system (e.g. refer to Ref. [16]). As a result, damping is mentioned as the main source of variability in the vibration transfer functions in some literature (e.g. Ref. [17]). However, it can be anticipated from Eq. (21) that even higher variation in the damping ratio of the mount would not contribute to the overall variability when compared with the effect of the stiffness of the mount. Even if a mount is used as an isolator, the internal resonances (free-free natural frequencies) of the mount will be high enough to ensure effective isolation [27]. For an isolating mount, the higher damping ratio causes an increase in the transferred vibration at high frequencies.

Peaks in sensitivity become distinctive at the transition zone due to the shift in natural frequencies. For such a mount with a stiffness of 100 kN/m, the sensitivity of mobility Y^b_{14} with respect to the mount damping is shown in Fig. 6. As can be seen the sensitivity is lower than 0.012 and is much smaller than the sensitivity to the mount stiffness (which was of the order 5). The effect of modal damping of the plates is addressed in Section 4.2.3.

4.1.3. Effect of the mount mass

It is shown in Section 2.3.1 that the mass of the mount can be neglected if the inequality of $m \ll (m_1\zeta_{pq_1}+m_2\zeta_{pq_2})/2$ holds. In practice, the mount mass should at least be smaller than one tenth of the right hand expression in the above inequality. For the numerical example introduced in Section 3 with a modal damping ratio for plates of ζ_{pq}=0.01, the average mass of two plates will be 2.29 kg which implies a limit of 0.022 kg for the mass of the mount. To demonstrate the effect of the mount mass on the mobility FRF, three different masses of 0.001 kg, 0.01 kg and 0.1 kg are chosen and the corresponding mobility FRF are shown in Fig. 7. The mobility curves for two smaller mount mass, 0.001 kg and 0.01 kg which are below the limit of 0.022 kg, are very similar implying a limited effect on variability.

To have a better understanding and to compare the effect of the mount mass on the vibration transfer function in a systematic way, the sensitivity analysis can be used. By following the same procedure, the sensitivity with respect to the

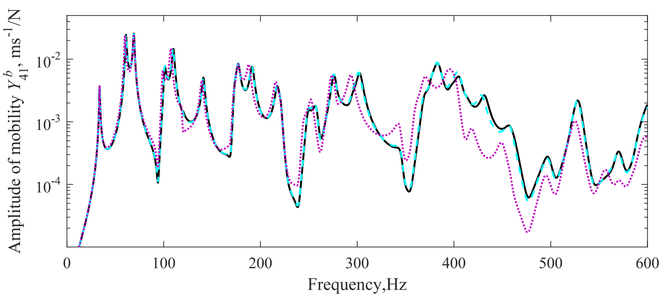

Fig. 7. The effect of the mount mass on the amplitude of mobility Y^b_{41} for two plates connected by a mount with stiffness of 100 kN/m, mount damping ratio ζ=0. 01 and three different masses of the mount. Solid line (—): m=0. 001 kg. Dashed line (- - -): m=0. 01 kg. Dotted line (….): m=0. 1 kg.

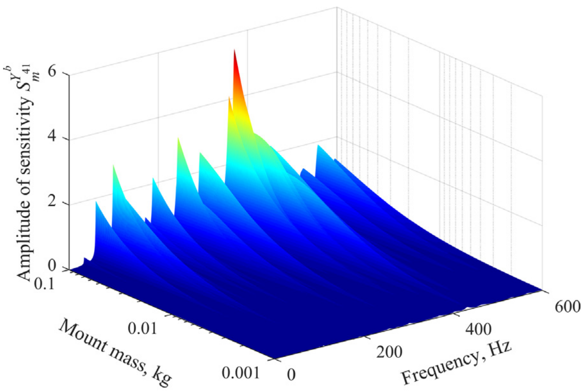

Fig. 8. Amplitude of sensitivity of mobility Y_{41}^b with respect to the mount mass as a function of frequency and mount mass for k=100kN/m, ζ=0. 01 and ζ_{pq}=0. 01.

mass of the isolator can be obtained. The amplitude of the sensitivity of mobility Y_{41}^b with respect to the mount mass, $S_m^{Y_{41}^b}$, is shown in Fig. 8 as a function of mount mass and frequency for a stiffness of the mount equal to 100 kN/m. The modal damping ζ_{pq} for two plates is 0.01. The higher value for modal damping would reduce the amplitude of peaks and have similar effect to that shown in Section 4.1.1 for the sensitivity with respect to the mount stiffness. For masses smaller in value than the threshold of 0.022 kg, the sensitivity is very small. Increasing the mass will increase the sensitivity but in practice as the mount mass will be low, the sensitivity is low in general when compared to the sensitivity with respect to the stiffness (with a maximum amplitude of approximately 0.1 for a mount mass of 1 g and about 0.8 for a mount mass of 0.01 kg). The peaks of the sensitivity curve coincide with the resonance frequencies of mobility Y_{41}^b similar to the two previous cases; the sensitivity with respect to the mount damping and the mount stiffness.

4.2. Effect of variation in the parameters of the plate

Referring to Eq. (13), any change in the parameters of the plate will have an effect on the assembly through the point mobilities at the mount positions (Y_{22} or Y_{33}) and the mobilities between excitation or measurement locations and the mount positions (Y_{21} or Y_{43} respectively) for plate one or plate two.

To have an understanding of how changes in the plate response causes variation in the assembly, the sensitivity function can be used with regard to parameter of the plate p,

$$S_p^{Y_{41}^b}=S_{Y_{22}}^{Y_{41}^b}S_p^{Y_{22}}+S_{Y_{21}}^{Y_{41}^b}S_p^{Y_{21}}. \tag{22}$$

Using the definition given in Eq. (20) the sensitivity of the mobility Y_{41}^b with respect to the mobility of the plates can be obtained,

$$S_{Y_{22}}^{Y_{41}^b}=-\frac{(Z_{22}Z_{33}Y_{33}-Z_{23}Z_{32}Y_{33}+Z_{22})Y_{22}}{Z_{22}Z_{33}Y_{22}Y_{33}-Z_{23}Z_{32}Y_{22}Y_{33}+Z_{22}Y_{22}+Z_{33}Y_{33}+1}, \tag{23}$$

$$S_{Y_{21}}^{Y_{41}^b}=1. \tag{24}$$

Mobility Y_{41}^b has a direct relation with mobility Y_{21} (Y_{43} in case of plate two) which results in a sensitivity of unity as given in Eq. (24). This is not the case for point mobility for which the sensitivity is a complex function of different properties of the system, Eq. (23). For a soft mount, mobility Y_{22} does not appear in the mobility Y_{41}^b equation and it is expected that the sensitivity function $S_{Y_{22}}^{Y_{41}^b}$ approaches zero for such a mount. An example of how a change in the mount stiffness can affect the sensitivity with respect to the plate parameters is shown in Section 4.2.1.

4.2.1. Effect of the location of the mount

It is well known that the location of the response point has a great effect on the FRFs of the structure [16]. Manufacturing tolerances always exist which can result in a variation in mount position. Furthermore in many practical cases, adjustment in the position of the connectors are often used to compensate for inaccuracies in manufacturing. This will cause variation in the response of individual components resulting in a variation in the vibration transfer function of the assembly.

The variation in mobility Y_{41}^b due to the changing the mount position on plate 1, (x_2, y_2), is shown in Fig. 9. The changes in mobility are due to the change in modal contribution of the individual modes. Referring back to Eq. (1) to Eq. (3), the variation in mount position only changes the value of Φ_{repq} given in Eq. (2), i.e. it only changes the amplitude of the response FRF close to the resonances, but not its frequency content. However, as the anti-resonances are due to interaction of two adjacent modes, the changes in modal amplitude can change the frequency of anti-resonances. For the assembly, the frequencies of the peaks and troughs also change as they are due to interaction in modes of two plates. The level of variability

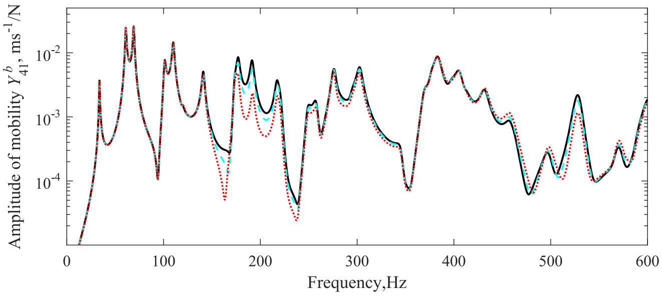

Fig. 9. The effect of changing the mount location on the amplitude of mobility Y^b_{41} for two plates of the numerical example with modal damping ζ_{pq}=0. 01, connected by a mount with stiffness of 100 kN/m, damping ratio ζ=0. 01 and mass of 0.001 kg. Solid line (—): mount at its original position. Dashed line (- - -): Δx_2=2 mm and Δy_2=2 mm. Dotted line (….): Δx_2=5 mm and Δy_2=5 mm.

in the built-up structure is of a similar level to that of the individual plates.

Eq. (22) to Eq. (24) can be used to obtain the sensitivity $S^{Y^b_{41}}_{x_2}$ by obtaining the sensitivities of the mobility functions of plate one, Y_{22} and Y_{21}, with respect to x_2. The measurement and excitation points coordinates appear in the equation for mobility as trigonometric functions in Eq. (2) which are periodic and the same value can be obtained for different locations. Hence, half of the plate length $a/2$ is used as the normalising parameter for location x_2,

$$S^{Y_{re}}_{x_2} = \frac{\partial Y_{re}/Y_{re}}{\partial x_2/(a/2)}. \tag{25}$$

The amplitude of sensitivity $S^{Y^b_{41}}_{x_2}$ is shown in Fig. 10 for the same case study with a low modal damping of ζ_{pq}=0.01 as a function of the mount stiffness and frequency. Increasing modal damping has a similar effect to that shown in Section 4.1.1 for the mount stiffness. The contribution of mobility Y_{21} to the sensitivity function is constant and there is always a direct relationship between Y_{21} and Y^b_{41} according to Eq. (24). The peaks of sensitivity $S^{Y^b_{41}}_{x_2}$ that are not changing by increasing the mount stiffness can be attributed to mobility Y_{21}, while ones that are appearing by increasing the mount stiffness are due to variation in mobility Y_{22}. This is because for a soft mount (low mount stiffness), the point mobility Y_{22} does not contribute to the mobility Y^b_{41} (Eq. (18)). Thus the structures with soft mounts are less sensitive in general to the position of the mount when compared to those with rigid mounts.

Peaks of sensitivity correspond with both resonance and anti-resonances of mobility Y^b_{41} curve. For example, the maximum of the sensitivity which occurs at a frequency of 240 Hz corresponds to an anti-resonance of the mobility. However, the amplitude of sensitivity is still very high for some of the resonance frequencies of the system, e.g. order of magnitudes of 40. Therefore, for general automotive noise, vibration and harshness (NVH) interests, attention should be focused on the tolerances of the joint position in particular.

The sensitivity to the stiffness which is obtained in Section 4.1.1 shows that a 1% change in position is equivalent to about 10% change in stiffness. The existence of manufacturing tolerance alongside the high sensitivity to the mount location makes the position one of the main factors that contributes to the variation in the vibration transfer function.

4.2.2. Effect of the modulus of the elasticity of the plates

The variation in modulus of elasticity of the plate affects the natural frequencies of the plate, which in turn cause a

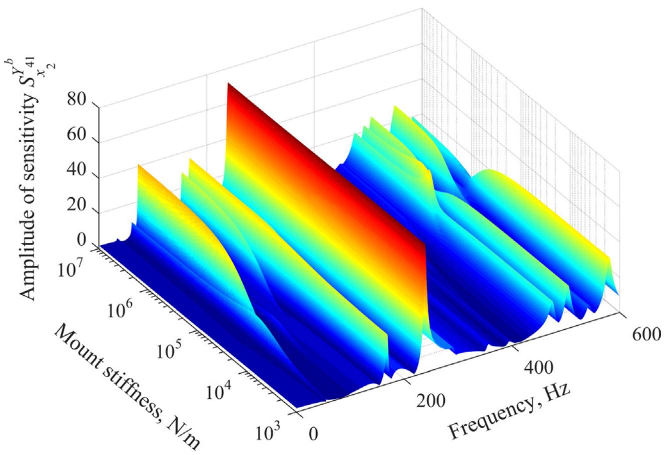

Fig. 10. Amplitude of sensitivity of mobility Y^b_{41} with respect to the location x_2 of the mount on plate one as a function of frequency and mount stiffness for m=0. 001 kg, ζ=0. 01 and plates of the numerical example with ζ_{pq}=0. 01.

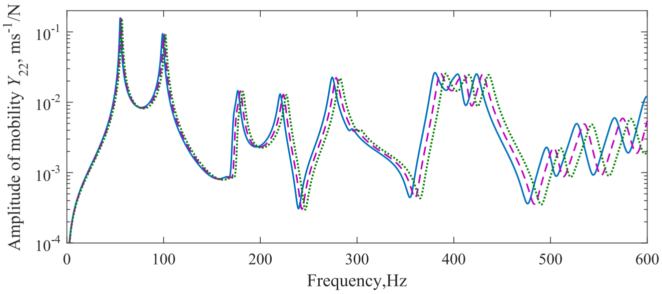

Fig. 11. Amplitude of mobility Y_{21} for plate 1 of the numerical example with modal damping ζ_{pq}=0. 01, connected by a mount with stiffness of 100 kN/m, damping ratio ζ=0. 01 and mass of 0.001 kg for three values of modulus of elasticity. Solid line (—): original parameters ΔE_1=0. Dashed line (- - -): ΔE_1=3%. Dotted line (….): ΔE_1=6%.

variation in the mobility of individual plates and hence mobility Y_{41}^b of the assembly. The effect of increasing the modulus of elasticity on mobility of plate one, Y_{22}, is shown in Fig. 11. Referring to Eq. (4), it can be seen that the natural frequencies of the plate have a direct relationship with its flexural rigidity D, which is a function of root square of the modulus of elasticity. As a result, an increase in Young's modulus causes a shift in the frequency of the mobility curve as can be seen in Fig. 11. If the change in modulus of elasticity is ΔE then the shift in frequency will be equal to $\Delta\omega = (\sqrt{\mathrm{E} + \Delta E}/\sqrt{E} - 1)\omega$.

The change in amplitude of mobility Y_{41}^b for an increase in modulus of elasticity of plate one is shown in Fig. 12. The shift in frequency is visible again here but in contrast to the mobility of the individual plate (Fig. 11), the change in the mobility is not only a shift in frequency but also a change of the shape of the FRF as it is the result of the interaction between the modes of two individual plates described by Eq. (13).

The sensitivity function with respect to modulus of elasticity can be obtained by following the same procedure which makes it possible to compare the role of the modulus of elasticity with another parameter. The amplitude of the sensitivity of mobility Y_{41}^b with respect to the modulus of elasticity of plate one,$S_{E_1}^{Y_{41}^b}$, is shown in Fig. 13 for the numerical example. As previously discussed, at low stiffness the point mobility Y_{22} does not contribute to the variability of the vibration transfer function, which justifies the changes due to stiffness of the mount in the graph.

Peaks of sensitivity correspond with both resonances and anti-resonances of the mobility curve. The variation in modulus of elasticity affects almost all resonances and anti-resonances while the variation of position of the mount causes a high sensitivity only at some of the resonances and anti-resonances. This can be due to the fact that the sensitivity with location depends on the proximity of the location to nodes or anti-nodes of the mode shapes while the variation in the modulus of elasticity affects all the natural frequencies of the plate.

The amplitude of the sensitivity is similar to that found for sensitivity with respect to the mount position (order of magnitudes of 40). It is reported in the literature that the elasticity modulus can be varied by less than 1% [15], which is not a very large number but when considered alongside high sensitivity of vibration transfer function to the modulus of elasticity, it can be a significant source of variability in built-up structures.

4.2.3. Effect of modal damping of the plate

The amplitude of sensitivity of the transfer mobility with respect to modal damping of plate one $S_{\zeta_{pq_1}}^{Y_{41}^b}$ for two different values of modal damping is shown in Fig. 14(a) and the amplitudes of corresponding mobility Y_{41}^b are shown in Fig. 14(b). By increasing the modal damping, the average value of the sensitivity function increases but the peaks of sensitivity have similar values in both cases. This is in contrast to all previous cases where increasing the modal damping cause a reduction in peak amplitude of the sensitivity function in a similar way to the mobility. At high frequencies where modal overlap

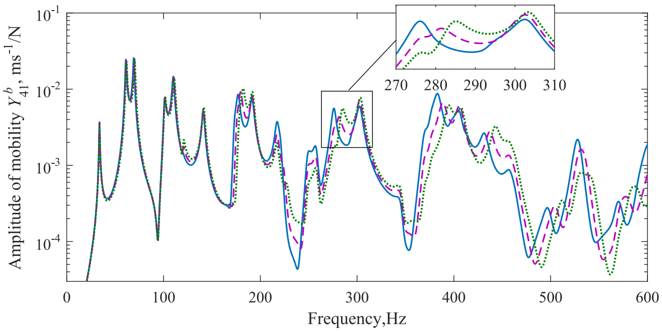

Fig. 12. Amplitude of mobility Y_{41}^b of two connected plate of the numerical example with modal damping ζ_{pq}=0. 01, connected by a mount with stiffness of 100 kN/m, damping ratio ζ=0. 01 and mass of 0.001 kg for three values of modulus of elasticity of plate 1. Solid line (—): original parameter ΔE_1=0. Dashed line (- - -): ΔE_1=3%. Dotted line (……): ΔE_1=6%.

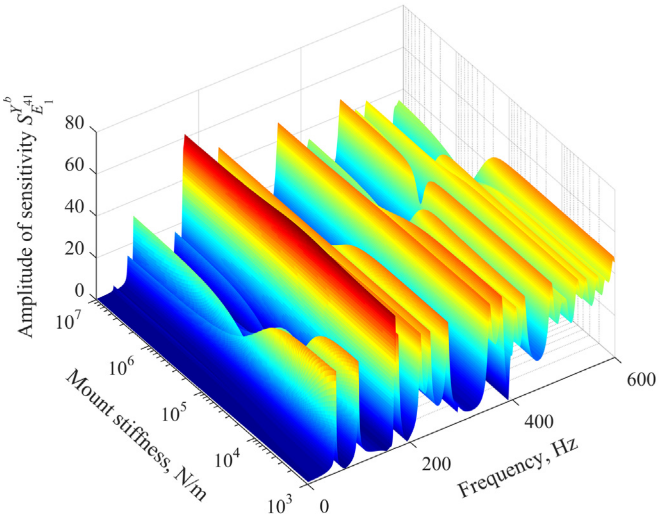

Fig. 13. Amplitude of sensitivity of mobility Y^b_{41} with respect to the modulus of elasticity of plate one as a function of frequency and mount stiffness for $m = 0.001$ kg, $\zeta = 0.01$ and plates of the numerical example with ζ_{pq}=0.01.

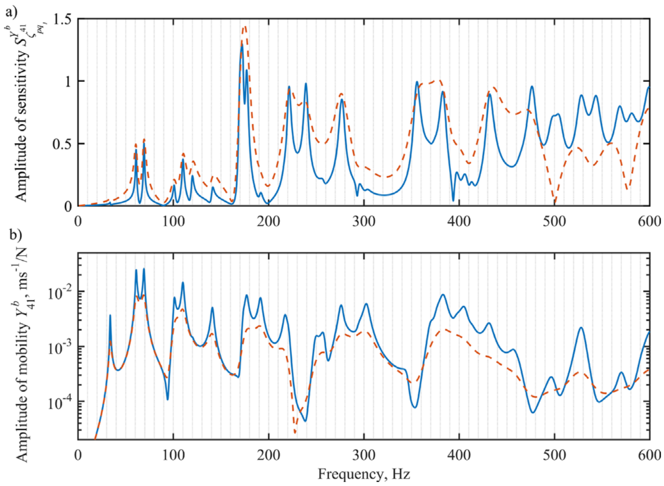

Fig. 14. Amplitude of sensitivity $S^{Y^b_{41}}_{\zeta_{pq1}}$ and amplitude of mobility Y^b_{41} for plates of the numerical example connected by a mount with stiffness of k=100 kN/m, ζ=0. 01 and m=0. 001 kg for two different values of modal damping. Solid line (—): ζ_{pq}=0. 01. Dashed line (- - -): ζ_{pq}=0. 03.

factor is above unity, an increase in modal damping has decreased the peak value of the sensitivity. As the modal overlap factor increases at high frequency, the plate mobility approaches the mobility of an infinite plate which is independent of damping and justifies the decrease in the amplitude of sensitivity with respect to the modal damping at high frequencies. The peaks of the sensitivity coincide with the resonance and anti-resonance frequencies of the assembly. The maximum amplitude of the sensitivity is about 1.5, which is much smaller than the sensitivity to other parameters of the plate such as modulus of elasticity or stiffness and position of the mount. Therefore, the effect of modal damping of the main structural components on the variability of the vibration transfer function in complex built-up structures should be relatively small.

4.2.4. Effect of the other parameters of the plate

The thickness, density and Poisson's ratio of the plates all have a similar effect on the mobility of the plate such that variation of these parameters will change the flexural stiffness of the plate. The density and thickness of the plates also appear in Eq. (1) explicitly, which results in a direct inverse relation between them and the mobility of the plates. Thus it is expected that they have a similar effect to the modulus of elasticity on the vibration transfer function. The sensitivity analysis allows a comparison to be made of their effect with that of the modulus of elasticity in a quantitative fashion. It should be noted that if a function is in form of a polynomial of a parameter, the sensitivity of that function with respect to the parameter will be constant and equal to the power of that polynomial, i.e.

$$f(x)=x^n \rightarrow S_x^f = \left(\frac{df}{dx}\right)\left(\frac{x}{f}\right) = nx^{n-1}\left(\frac{x}{x^n}\right)=n. \tag{26}$$

Differentiating the plate mobility with respect to its thickness results in,

$$\frac{\partial Y_{re}}{\partial h} = \left(\frac{\partial Y_{re}}{\partial h}\right)_{\omega_{pq}} + \left(\frac{\partial Y_{re}}{\partial \omega_{pq}}\right)_h \frac{\partial \omega_{pq}}{\partial h}. \tag{27}$$

This results in the sensitivity function in the form of,

$$S_h^{Y_{re}} = -1 + \frac{\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \Phi_{repq}\left(\Omega_{pq,\omega_{pq}}\omega_{pq}\right)}{\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \Phi_{repq}\Omega_{pq}} S_h^{\omega_{pq}}. \tag{28}$$

By comparison with sensitivity $S_E^{Y_{re}}$, the above equation can be rewritten as,

$$S_h^{Y_{re}} = -1 + S_E^{Y_{re}} \frac{S_h^{\omega_{pq}}}{S_E^{\omega_{pq}}} \tag{29}$$

and by substituting the values of $S_h^{\omega_{pq}}$ and $S_E^{\omega_{pq}}$,

$$S_h^{Y_{re}} = -1 + 2S_E^{Y_{re}}. \tag{30}$$

As it can be seen in the above equation, the sensitivity of plate mobility with respect to thickness $S_h^{Y_{re}}$ has a direct relation with sensitivity $S_E^{Y_{re}}$, which is due to their similar effect on the flexural rigidity and this causes a shift in the natural frequencies, but the shape of the individual plate mobility Y_{re} would not change as discussed in Section 4.2.2. There is also an offset of -1 in Eq. (30). This is due to the inverse relation between mobility Y_{re} and thickness, which changes the amplitude of the mobility but not the resonant frequencies of the response.

To compare the effect of these parameters, the sensitivity of the natural frequency of the plate with respect to them is given in Table 2. It can be seen that the highest sensitivity between these parameters belongs to the thickness $S_h^{\omega_{pq}}$ (as the bending stiffness of a plate is proportional to the thickness raised to the third power), while the lowest is sensitivity with respect to Poisson's ratio.

The dimensions of the plate a and b appear in both of Φ_{repq} and ω_{pq} when obtaining the mobility of the plate Y_{re}. The sensitivity of mobility with respect to the plate's dimension would comprise of functions similar to both $S_E^{Y_{re}}$ and $S_{x_2}^{Y_{re}}$. The numerical value of the sensitivity depends on the dimensions and the location of the mount and hence it is not provided here, but an interested reader can follow the same procedure to obtain that.

5. Conclusions

A simplified model has been used to assess the effect of different parameters, such as stiffness and damping, on the variability of the dynamic response. The mobility between two points on two connected plates with a single connecting mount was modelled and an exact equation for the vibration transfer function obtained, together with approximate equations appropriate for the cases of a soft and stiff mount. The effects of variation in different parameters on the vibration transfer function were investigated and the sensitivity function was used to compare them quantitatively.

It was found that, generally, the mount parameters have a smaller effect on the variability of vibration transfer function compared to the parameters of the plates. For a stiff enough mount that can be considered as a rigid connection, the variations in its parameters have no effect on the dynamic response of the assembly. For a soft mount that can be considered as an isolator, there is a direct relationship between the mount properties and the dynamic response. The vibration transfer function becomes more sensitive to the stiffness of the mount when its behaviour is changed from a soft mount to a stiff mount. In this range of stiffness, the resonance frequencies of the system are changing as well as the amplitude of the response. It was also shown that the effect of mount damping on the variation is negligible. For a more complex system, the mount mass should be compared with the point mobilities at the mount connecting point.

Any change in the position of the mount has a great effect on the variability of the dynamic response. Although the change in

Table 2
Sensitivity of the natural frequency of a single plate with respect to its parameter.

$S_E^{\omega_{pq}}$	$S_h^{\omega_{pq}}$	$S_\rho^{\omega_{pq}}$	$S_\nu^{\omega_{pq}}$
$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{\nu^2}{1-\nu^2}$

location of the mount on the individual plate only changes the amplitude of the response for certain modes due to variation in modal contribution, when its effect propagates to the assembly, it changes both the resonance frequencies and the FRF. When compared to the stiffness of the mount, the dynamic response is about an order of magnitude more sensitive to the location of mount.

Variation in the modulus of elasticity, density, thickness or Poisson's ratio will cause a change in the flexural rigidity, which appears mainly as a frequency shift in FRF of the plate when considered individually. Their effect on the dynamic response of the assembly does not appear as mere shift in frequency and it can cause a change in the shape of the FRF depending on the mode involved in the response. The sensitivity to these parameters of the plate is higher than the sensitivity to the location of the mount. Modal damping of the plate has a much lesser effect on the overall variability when compared to the other parameters, but as the uncertainty about its value is higher and it can be more variable, it may contribute to the overall variability of the dynamic response.

An increase in plate's damping flattens the peaks of the sensitivity function with respect to both plate's and mount's parameters. The only exception is the sensitivity to the plate's damping itself for which its average at low frequencies increases by increasing plate's damping while peaks of sensitivity function only slightly increase.

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