Pach's selection theorem does not admit a topological extension

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Abstract

Let $U_1, \dots, U_{d+1}$ be n-element sets in $\mathbb{R}^d$ and let $\langle u_1, \dots, u_{d+1} \rangle$ denote the convex hull of points $u_i \in U_i$ (for all i) which is a simplex. Pach's selection theorem is about such simplices. It says that there are sets $Z_1 \subset U_1, \dots, Z_{d+1} \subset U_{d+1}$ and a point $u \in \mathbb{R}^d$ such that each $|Z_i| \ge c_1(d)n$ and $u \in \langle z_1, \dots, z_{d+1} \rangle$ for every choice of $z_1 \in Z_1, \dots, z_{d+1} \in Z_{d+1}$. Here we show that this theorem does not admit a topological extension with linear size sets Z_i. Further we prove a topological extension where each $|Z_i|$ is of order $(\log n)^{1/d}$.

1 Introduction

Let $U_1, \dots, U_{d+1}$ be n-element sets in $\mathbb{R}^d$, and let $\langle u_1, \dots, u_{d+1} \rangle$ denote the convex hull of points $u_i \in U_i$, $i = 1, \dots, d+1$. Pach's selection theorem, that we like to call a *homogeneous selection theorem* is about convex hulls of this type. It says the following.

Theorem 1.1 (Pach [9])**.** *Under the above conditions there are sets $Z_1 \subset U_1, \dots, Z_{d+1} \subset U_{d+1}$ and a point $u \in \mathbb{R}^d$ such that each $|Z_i| \ge c_1(d)n$ and $u \in \langle z_1, \dots, z_{d+1} \rangle$ for every choice of $z_1 \in Z_1, \dots, z_{d+1} \in Z_{d+1}$ where $c_1(d) > 0$ is a constant depending on d only.*

This result was proved by Bárány, Füredi and Lovász [2] for $d = 2$ and by Pach [9] for general d. Here we show that this theorem does not admit a topological extension when the size of the Z_i is linear in n but does admit one when the sizes are of order $(\log n)^{1/d}$. The formulation of this topological extension is the following.

Set $N = (d+1)n$ and consider the $(N-1)$-dimensional simplex Δ_{N-1} and a partition of its vertex set of $d+1$ sets $V_1, \dots, V_{d+1}$ each of size n. Trivially, there is an affine map $f : \Delta_{N-1} \to \mathbb{R}^d$ that is a bijection between V_i and U_i for each i. In this setting the homogeneous selection theorem says that there are subsets $Z_i \subset V_i$ such that $|Z_i| \ge c_1(d)n$ and

$$\bigcap_{z_1 \in Z_1, \dots, z_{d+1} \in Z_{d+1}} f(\langle z_1, \dots, z_{d+1} \rangle) \neq \emptyset.$$

Assume now that f is not affine but only continuous. Viewing each V_i as a 0-dimensional complex, consider the join

$$(\Delta_{n-1}^{(0)})^{*(d+1)} = V_1 * \dots * V_{d+1} = \{\sigma \subset \bigcup_{i=1}^{d+1} V_i : |\sigma \cap V_i| \le 1 \text{ for all } 1 \le i \le d+1\},$$

which is a subcomplex of the d-skeleton of Δ_{N-1}. For a mapping $f : (\Delta_{n-1}^{(0)})^{*(d+1)} \to \mathbb{R}^d$, let $\tau(f)$ denote the maximal m such that there exist m-element subsets $Z_1 \subset V_1, \ldots, Z_{d+1} \subset V_{d+1}$ that satisfy

$$\bigcap_{z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}} f(\langle z_1, \ldots, z_{d+1} \rangle) \neq \emptyset.$$

Define the *topological Pach number* $\tau(d,n)$ to be the minimum of $\tau(f)$ as f ranges over all continuous maps from $(\Delta_{n-1}^{(0)})^{*(d+1)}$ to $\mathbb{R}^d$. Our main result is the following:

Theorem 1.2. *For $d \geq 1$ there exists a constant $c_2(d)$ such that $\tau(d,n) \leq c_2(d) n^{1/d}$ for all n. Further, $c_2(d) = O(d)$ for all large enough n.*

For a lower bound on $\tau(d,n)$ we only have the following:

Theorem 1.3. *For $d \geq 1$ there exists a constant $c_3(d) > 0$ such that $\tau(d,n) \geq c_3(d)(\log n)^{1/d}$ for all n.*

The paper is organized as follows. In Section 2 we state Theorem 2.1 that describes a connection between $\tau(d,n)$ and the expansion of the bipartite graph of the atoms vs. coatoms in a graded lattice of rank $d+1$. This result is then used to prove Theorem 1.2. The proof of Theorem 2.1 is given in Section 3. In Section 4 we prove Theorem 1.3 by using results of Gromov [6] and Erdős [5].

2 Finite Lattices and Topological Pach Numbers

Let L be a finite graded lattice with a rank function $\mathrm{rk}(\cdot)$. Let $\widehat{0}$ and $\widehat{1}$ be the minimal and maximal elements of L. Assume that $\mathrm{rk}(\widehat{1}) = d+1$ and let

$$A = \{x \in L : \mathrm{rk}(x) = 1\} \quad , \quad C = \{x \in L : \mathrm{rk}(x) = d\}$$

be respectively the sets of atoms and coatoms of L. For $x \in L$ let

$$A_x = \{a \in A : a \leq x\} \quad , \quad C_x = \{c \in C : x \leq c\}.$$

For a set of atoms $Z \subset A$ let $\Gamma(Z) = \cup_{z \in Z} C_z$. Let G_L denote the bipartite graph on the vertex set $A \cup C$ with edges (a,c) iff $a \leq c$, for $a \in A$ and $c \in C$.

The main ingredient of the proof of Theorem 1.2 is the following connection between $\tau(d,n)$ and the expansion of G_L.

Theorem 2.1. *Let L be a graded lattice of rank $d+1$ such that $|A| \geq n(d+1)$. Then $m = \tau(d,n)$ satisfies*

$$\min_{Z \subset A, |Z| = m} |\Gamma(Z)| \leq \frac{d}{d+1} \big(\max_{a \in A} |C_a| + |C| \big).$$

The proof of Theorem 2.1 is deferred to Section 3.

Proof of Theorem 1.2: Let $q \geq 2d$ be a prime power and let $\mathbb{F}_q$ be the finite field of order q. Let $L = L(d+1,q)$ denote the lattice of linear subspaces of $\mathbb{F}_q^{d+1}$, ordered by containment. The sets of atoms and coatoms of L satisfy $|A| = |C| = N_d = \frac{q^{d+1}-1}{q-1}$ and $|C_a| = N_{d-1} = \frac{q^d-1}{q-1}$ for all $a \in A$. Two distinct coatoms, that is, two distinct d-dimensional subspaces intersect in a $(d-1)$-dimensional subspace whose size is $N_{d-2} = \frac{q^{d-1}-1}{q-1}$. For a given $Z \subset A$ the sets C_z form an N_{d-1}-uniform hypergraph on vertex set $\Gamma(Z)$ with $|Z|$ edges, and any two edges intersect in a set of size N_{d-2}. In this case a result of Corrádi [3] (see also exercise 13.13 in [7] and Theorem 2.3(ii) in [1]) implies that

$$|\Gamma(Z)| \geq \frac{|Z|N_{d-1}^2}{N_{d-1} + (|Z|-1)N_{d-2}} = \frac{|Z|N_{d-1}^2}{q^{d-1} + |Z|N_{d-2}} \geq N_d - \frac{N_d^{1+\frac{1}{d}}}{|Z|}. \tag{1}$$

The last inequality here follows from a routine computation using the values of N_k. Let $n = \lfloor \frac{|A|}{d+1} \rfloor$. Applying Theorem 2.1 together with (1) it follows that $m = \tau(d,n)$ satisfies

$$\begin{aligned} N_d - \frac{N_d^{1+\frac{1}{d}}}{m} &\leq \min_{Z \subset A, |Z| = m} |\Gamma(Z)| \\ &\leq \frac{d}{d+1}\big(\max_{a \in A} |C_a| + |C|\big) \\ &= \frac{d}{d+1}(N_{d-1} + N_d). \end{aligned} \tag{2}$$

Rearranging (2) and using $q \geq 2d$ and $n+1 \geq \frac{|A|}{d+1} = \frac{N_d}{d+1}$ we obtain

$$\begin{aligned} m &\leq \frac{(d+1)N_d^{1+\frac{1}{d}}}{N_d - dN_{d-1}} \leq 2(d+1)N_d^{\frac{1}{d}} \\ &\leq 2(d+1)\big((d+1)(n+1)\big)^{\frac{1}{d}}. \end{aligned}$$

By Bertrand's postulate, for any large enough integer n (specifically, for any $n \geq \frac{(2d)^{d+1}-1}{(2d-1)(d+1)}$) one can find $q \geq 2d$ for which $\lfloor \frac{N_d}{d+1} \rfloor \leq 2^d n$. Plugging into the above upper bound on $m = \tau(d,n)$ for such n, the resulted constant $c_2(d)$ just multiplies by 2 so still $c_2(d) = O(d)$. For $n \leq \frac{(2d)^{d+1}-1}{(2d-1)(d+1)} := c_2(d)$, trivially $m \leq n \leq c_2(d)$.

□

3 Continuous Maps of Finite Lattices

In this section we prove Theorem 2.1. We first recall some definitions. Let $\overline{L} = L - \{\hat{0}, \hat{1}\}$. The order complex $\Delta(\overline{L})$ is a simplicial complex on the vertex set $\overline{L}$ whose p-dimensional simplices are increasing chains $x_0 < \cdots < x_p$ in $\overline{L}$. For a subset $\sigma \subset L$ let $\vee\sigma = \vee_{x \in \sigma} x$. Let $\mathcal{A}(L)$ be the simplicial complex on the vertex set A whose simplices are all $\sigma \subset A$ such that $\vee\sigma < \hat{1}$. For $x \in \overline{L}$ let $\overline{L}_{\leq x} = \{y \in \overline{L} : y \leq x\}$. The main ingredient in the proof of Theorem 2.1 is the following result.

Proposition 3.1. *There exists a continuous map $f : \mathcal{A}(L) \to \mathbb{R}^d$ such that for any $u \in \mathbb{R}^d$*

$$|\{c \in C : u \in f(\langle A_c \rangle)\}| \leq d \max_{a \in A} |C_a|. \tag{3}$$

We first note the following:

Claim 3.2. *There exists a continuous map $g : \mathcal{A}(L) \to \Delta(\overline{L})$ such that for all $x \in \overline{L}$*

$$g(\langle A_x \rangle) \subset \Delta(\overline{L}_{\leq x}).$$

Proof. We define g inductively on the k-skeleton $\mathcal{A}(L)^{(k)}$. On the vertices $a \in A$ of $\mathcal{A}(L)$ let $g(a) = a$. Let $k > 0$ and suppose g has been defined on $\mathcal{A}(L)^{(k-1)}$. Let $\sigma = \langle a_0, \ldots, a_k \rangle \in \mathcal{A}(L)^{(k)}$ and let $y = \vee\sigma < \widehat{1}$. For $0 \leq i \leq k$ let $\sigma_i = \langle a_0, \ldots, a_{i-1}, \widehat{a_i}, a_{i+1}, \ldots, a_k \rangle$ be the i-th face of σ. Let $y_i = \vee\sigma_i$. Then g is defined on σ_i and by induction hypothesis

$$g(\sigma_i) \subset \Delta(\overline{L}_{\leq y_i}) \subset \Delta(\overline{L}_{\leq y}).$$

Being a cone, $\Delta(\overline{L}_{\leq y})$ is contractible and hence g can be continuously extended from $\partial\sigma$ to the whole of σ so that $g(\sigma) \subset \Delta(\overline{L}_{\leq y})$. □

Proof of Proposition 3.1: By a general position argument we choose a mapping $e : \overline{L} \to \mathbb{R}^d$ with the following property: For any pairwise disjoint $S_1, \ldots, S_{d+1} \subset \overline{L}$, if $|S_i| \leq d$ for all $1 \leq i \leq d+1$, then $\bigcap_{i=1}^{d+1} \mathrm{aff}\,\big(e(S_i)\big) = \emptyset$, which implies of course that

$$\bigcap_{i=1}^{d+1} \mathrm{relint}\,\mathrm{conv}\,\big(e(S_i)\big) = \emptyset. \tag{4}$$

Extend e by linearity to the whole of $\Delta(\overline{L})$ and let $f = e \circ g : \mathcal{A}(L) \to \mathbb{R}^d$. We claim that the map f satisfies (3). Let $u \in \mathbb{R}^d$ and let

$$T = \{\tau \in \Delta(\overline{L}) : u \in \mathrm{relint}\, e(\langle \tau \rangle)\}.$$

Choose a maximal pairwise disjoint subfamily $T' \subset T$. It follows by (4) that $|T'| \leq d$. For each $\tau' \in T'$ choose an atom $a(\tau') \in A$ such that

$$a(\tau') \leq \min \tau'. \tag{5}$$

Now let $c \in C$ such that $u \in f(\langle A_c \rangle)$. Then there exists a $b \in g(\langle A_c \rangle) \subset \Delta(\overline{L}_{\leq c})$ such that $u = e(b)$. Let $\tau \in T$ such that $b \in \mathrm{relint}\langle \tau \rangle$. Then

$$\tau \in \Delta(\overline{L}_{\leq c}). \tag{6}$$

By maximality of T' there exists a simplex $\tau' \in T'$ and a vertex $x \in \tau' \cap \tau$. It follows by (5) and (6) that $a(\tau') \leq x \leq c$, i.e. $c \in C_{a(\tau')}$. Therefore

$$|\{c \in C : u \in f(\langle A_c \rangle)\}| \leq \sum_{\tau' \in T'} |C_{a(\tau')}| \leq d \max_{a \in A} |C_a|.$$

□

Proof of Theorem 2.1: Let L be a lattice of rank $d+1$ whose set of atoms A satisfies $|A| \geq (d+1)n$. Let $V_1, \ldots, V_{d+1}$ be disjoint n-subsets of A. By Proposition 3.1 there exists a continuous map $f : \mathcal{A}(L) \to \mathbb{R}^d$ such that for any $u \in \mathbb{R}^d$

$$|\{c \in C : u \in f(\langle A_c \rangle)\}| \leq d \max_{a \in A} |C_a|.$$

Let $m = \tau(d,n)$. Then there exist $Z_1 \subset V_1, \ldots, Z_{d+1} \subset V_{d+1}$ and a $u \in \mathbb{R}^d$ such that $|Z_i| \geq m$ for all $1 \leq i \leq d+1$ and

$$u \in \bigcap_{z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}} f(\langle z_1, \ldots, z_{d+1} \rangle).$$

Write

$$C(Z_1, \ldots, Z_{d+1}) = \bigcap_{i=1}^{d+1} \{c \in C : A_c \cap Z_i \neq \emptyset\}.$$

If $c \in C(Z_1, \ldots, Z_{d+1})$ then there exist $z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}$ such that $z_i \leq c$ for all i and hence $u \in f(\langle z_1, \ldots, z_{d+1} \rangle) \subset f(\langle A_c \rangle)$. Hence by Proposition 3.1

$$|C(Z_1, \ldots, Z_{d+1})| \leq d \max_{a \in A} |C_a|. \tag{7}$$

On the other hand

$$\begin{aligned} |C(Z_1, \ldots, Z_{d+1})| &= |C - \bigcup_{i=1}^{d+1} (C - \Gamma(Z_i))| \\ &\geq |C| - \sum_{i=1}^{d+1} (|C| - |\Gamma(Z_i)|) = \sum_{i=1}^{d+1} |\Gamma(Z_i)| - d|C| \\ &\geq (d+1) \min_{Z \subset A, |Z| = m} |\Gamma(Z)| - d|C|. \end{aligned} \tag{8}$$

Theorem 2.1 now follows from (7) and (8).

□

4 The Lower Bound

Theorem 1.3 is a direct consequence of Gromov's topological overlap Theorem [6] combined with a result of Erdős on complete $(d+1)$-partite subhypergraphs in $(d+1)$-uniform dense hypergraphs [5]. We first recall these results. Let X be a finite d-dimensional pure simplicial complex. For $k \geq 0$, let $X^{(k)}$ denote the k-dimensional skeleton of X and let $X(k)$ be the family of k-dimensional faces of X, $f_k(X) = |X(k)|$. Define a positive weight function $w = w_X$ on the simplices of X as follows. For $\sigma \in X(k)$, let $c(\sigma) = |\{\eta \in X(d) : \sigma \subset \eta\}|$ and let

$$w(\sigma) = \frac{c(\sigma)}{\binom{d+1}{k+1} f_d(X)}.$$

Let $C^k(X)$ denote the space of $\mathbb{F}_2$-valued k-cochains of X with the coboundary map $d_k : C^k(X) \to C^{k+1}(X)$. As usual, the space of k-coboundaries is denoted by $d_{k-1}\big(C^{k-1}(X)\big) = B^k(X)$. For $\phi \in C^k(X)$, let $[\phi]$ denote the image of ϕ in $C^k(X)/B^k(X)$. Let

$$\|\phi\| = \sum_{\sigma \in X(k) : \phi(\sigma) \neq 0} w(\sigma)$$

and

$$\|[\phi]\| = \min\{\|\phi + d_{k-1}\psi\| : \psi \in C^{k-1}(X)\}.$$

The *k-th coboundary expansion constant* of X is

$$h_k(X) = \min\left\{\frac{\|d_k\phi\|}{\|[\phi]\|} : \phi \in C^k(X) - B^k(X)\right\}.$$

Note that $h_k(X) = 0$ iff $\tilde{H}^k(X;\mathbb{F}_2) \neq 0$. One may regard $h_k(X)$ as a sort of distance between X and the family of complexes Y that satisfy $\tilde{H}^k(Y;\mathbb{F}_2) \neq 0$. Gromov's celebrated topological overlap result is the following:

Theorem 4.1 (Gromov [6])**.** *For any integer $d \geq 0$ and any $\epsilon > 0$ there exists a $\delta = \delta(d,\epsilon) > 0$ such that if $h_k(X) \geq \epsilon$ for all $0 \leq k \leq d-1$, then for any continuous map $f : X \to \mathbb{R}^d$ there exists a point $u \in \mathbb{R}^d$ such that*

$$|\{\sigma \in X(d) : u \in f(\sigma)\}| \geq \delta f_d(X).$$

We next describe a result of Erdős that generalizes the well known Erdős-Stone and Kővári-Sós-Turán theorems from graphs to hypergraphs.

Theorem 4.2 (Erdős [5])**.** *For any d and $c' > 0$ there exists a constant $c = c(d,c') > 0$ such that for any $(d+1)$-uniform hypergraph $\mathcal{F}$ on N-element set V with at least $c'N^{d+1}$ hyperedges, there exists an $m \geq c(\log N)^{1/d}$ and disjoint m-element sets $Z_1,\ldots,Z_{d+1} \subset V$ such that $\{z_1,\ldots,z_{d+1}\} \in \mathcal{F}$ for all $z_1 \in Z_1,\ldots,z_{d+1} \in Z_{d+1}$.*

Proof of Theorem 1.3: Recall that $V_1,\ldots,V_{d+1}$ are disjoint n-element sets and let $V = V_1 \cup \cdots \cup V_{d+1}$, $|V| = N = (d+1)n$. Let $X = V_1 * \ldots * V_{d+1}$ and let $f : X \to \mathbb{R}^d$ be a continuous map. It was shown by Gromov [6] (see also [4, 8]) that the expansion constants $h_i(X)$ are uniformly bounded away from zero. Concretely, it follows from Theorem 3.3 in [8] that $h_i(X) \geq \epsilon = 2^{-d}$ for $0 \leq i \leq d-1$. Let $\delta = \delta(d,2^{-d})$. Then by Theorem 4.1 there exists a $u \in \mathbb{R}^d$ and a family $\mathcal{F} \subset X(d)$ of cardinality

$$|\mathcal{F}| \geq \delta f_d(X) = \delta n^{d+1} = \delta(d+1)^{-(d+1)}N^{d+1}$$

such that $u \in f(\sigma)$ for all $\sigma \in \mathcal{F}$. Writing $c' = \delta(d+1)^{-(d+1)}$ and $c_3(d) = c(d,c')$, it follows from Theorem 4.2 that there exists an $m \geq c_3(d)(\log N)^{1/d} \geq c_3(d)(\log n)^{1/d}$ and disjoint m-sets $Z_1,\cdots,,Z_{d+1} \subset V$ such that $u \in f(\langle z_1,\ldots,z_{d+1}\rangle)$ for all $z_1 \in Z_1,\ldots,z_{d+1} \in Z_{d+1}$. Clearly, there exist a permutation π on $\{1,\ldots,d+1\}$ such that $Z_{\pi(i)} \subset V_i$ for all $1 \leq i \leq d+1$.

□

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