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On a coefficient in trace formulas for Wiener–Hopf operators

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Dedicated to the memory of Yuri Safarov

Abstract. Let $a = a(\xi), \xi \in \mathbb{R}$, be a smooth function quickly decreasing at infinity. For the Wiener–Hopf operator $W(a)$ with the symbol a, and a smooth function $g\colon \mathbb{C} \to \mathbb{C}$, H. Widom in 1982 established the following trace formula:

$$\operatorname{tr}(g(W(a)) - W(g \circ a)) = \mathcal{B}(a; g),$$

where $\mathcal{B}(a; g)$ is given explicitly in terms of the functions a and g. The paper analyses the coefficient $\mathcal{B}(a; g)$ for a class of non-smooth functions g assuming that a is real-valued. A representative example of one such function is $g(t) = |t|^\gamma$ with some $\gamma \in (0, 1]$.

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1. Introduction

Let $a\colon \mathbb{R} \to \mathbb{C}$ be a function. On $\mathsf{L}^2(\mathbb{R}_+)$, $\mathbb{R}_+ = (0, \infty)$, define the Wiener–Hopf operator $W(a)$ with symbol a by

$$(W(a)u)(x) = \chi_+(x)\frac{1}{2\pi}\int e^{i(x-y)\xi} a(\xi)\chi_+(y)u(y)dyd\xi, \quad u \in \mathsf{L}^2(\mathbb{R}_+),$$

where χ_+ is the indicator of the half-line $\mathbb{R}_+$. If the limits are not specified, we always assume that the integration is taken over the entire line. We are interested in the operator

$$g(W(a)) - W(g \circ a), \tag{1.1}$$

with a suitable function $g\colon \mathbb{C} \to \mathbb{C}$. In [16], see also [18], H. Widom proved that this operator is trace class if

$$a \in \mathsf{L}^\infty(\mathbb{R}), \quad \iint \frac{|a(\xi_1) - a(\xi_2)|^2}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2 < \infty, \tag{1.2}$$

and established the following remarkable trace formula for the operator in (1.1). For any function $g\colon \mathbb{C} \to \mathbb{C}$ and any $s_1, s_2 \in \mathbb{C}$ denote

$$U(s_1, s_2; g) = \int_0^1 \frac{g((1-t)s_1 + ts_2) - [(1-t)g(s_1) + tg(s_2)]}{t(1-t)} dt, \tag{1.3}$$

and introduce

$$\mathcal{B}(a; g) = \frac{1}{8\pi^2} \iint \frac{U(a(\xi_1), a(\xi_2); g)}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2. \tag{1.4}$$

Both objects are well defined under the conditions of the next proposition:

Proposition 1.1. [*see* [16], *Theorem 1(a)*] *Suppose that* (1.2) *is satisfied, and let g be analytic on a neighbourhood of the closed convex hull of the function a. Then the operator* (1.1) *is trace class and*

$$\operatorname{tr}[g(W(a)) - W(g \circ a)] = \mathcal{B}(a; g). \tag{1.5}$$

If a is real-valued, then the analyticity assumptions on g can be replaced by some finite smoothness, see [16], Theorem 1(b). In paper [10] the assumptions on a and g are relaxed even further: the formula (1.5) is proved for real-valued a under the assumptions that the integral in (1.2) is finite and g belongs to the Besov class $B^2_{\infty,1}(\mathbb{R})$.

The quantity $\mathcal{B}(a; g)$ is an object that one encounters very often in the theory of Wiener–Hopf operators. It appears e.g. in [10], [11], [15], [16], [17], and [18] as an asymptotic coefficient in various trace formulas for truncated Wiener–Hopf and Toeplitz operators with smooth symbols. Moreover, the function $U(s_1, s_2; g)$ is present in a variety of trace formulas for the same operators with discontinuous symbols, see e.g. [1], [14], [12], [13], and references therein. Although the integral (1.3) is well defined for rather a wide class of functions g, the coefficient (1.4) itself has been considered so far for smooth functions g only. As observed in [16], if g is twice differentiable, we can integrate by parts in (1.3) to obtain that

$$U(s_1, s_2; g) = (s_1 - s_2)^2 \int_0^1 g''((1-t)s_1 + ts_2)(t \log t + (1-t)\log(1-t)) dt.$$

Thus, assuming that g'' is uniformly bounded, we obtain the estimate

$$|\mathcal{B}(a; g)| \le C \|g''\|_{\mathsf{L}^\infty} \iint \frac{|a(\xi_1) - a(\xi_2)|^2}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2,$$

with a universal constant $C > 0$, which guarantees the finiteness of $\mathcal{B}(a; g)$ under the condition (1.2). However, in applications one often needs non-smooth functions, see e.g. [3], [5], [6], [7], [8], and references therein. The main aim of this paper is to investigate the coefficient (1.4) for real-valued symbols a and non-smooth functions $g: \mathbb{R} \mapsto \mathbb{C}$, described in Condition 3.1 further on. A representative example of one such function is $g(t) = |t|^\gamma$ with some $\gamma \in (0, 1]$. Surprisingly, even finiteness of $\mathcal{B}(a; g)$ for such a function is far from trivial. The main result (see Theorem 3.2) is a bound on the coefficient $\mathcal{B}(a; g)$ that explicitly depends on the symbol a and function g. Formula (1.5) for non-smooth functions g is proved in [8].

Henceforth by C and c with or without indices we denote various positive constants whose precise value is of no importance. The value of constants may vary from line to line.

2. Smooth functions g

Before embarking on the formulation of the main theorem we provide some useful information on the smooth case. First we show how to extend formula (2.7) to $\mathsf{C}^{1,\varkappa}$-functions. Rewrite $U(s_1, s_2; g)$ in a different way introducing the integral

$$V(s_1, s_2; g) = \int_0^1 \frac{g((1-t)s_1 + ts_2) - g(s_2)}{1-t} dt. \tag{2.1}$$

This functional is well defined for any $\varkappa$-Hölder continuous function $g: \mathbb{C} \to \mathbb{C}$ with $\varkappa \in (0, 1]$, and

$$|V(s_1, s_2; g)| \le C_\varkappa [g]_\varkappa |s_1 - s_2|^\varkappa, \quad \text{for all } s_1, s_2 \in \mathbb{C}, \tag{2.2}$$

where we have denoted

$$[g]_\varkappa = \sup_{z,w \in \mathbb{C}, z \neq w} \frac{|g(z) - g(w)|}{|z - w|^\varkappa}.$$

If g is boundedly differentiable, then, integrating by parts once, we obtain

$$V(s_1, s_2; g) = (s_2 - s_1) \int_0^1 \log(1-t) g'((1-t)s_1 + ts_2) dt. \tag{2.3}$$

Due to the elementary formula

$$\begin{aligned} &\frac{g((1-t)s_1 + ts_2) - (1-t)g(s_1) - tg(s_2)}{t(1-t)} \\ &\quad = \frac{g((1-t)s_1 + ts_2) - g(s_1)}{t} + \frac{g((1-t)s_1 + ts_2) - g(s_2)}{1-t}, \end{aligned}$$

we have

$$U(s_1, s_2; g) = V(s_2, s_1; g) + V(s_1, s_2; g), \tag{2.4}$$

so that in combination with (2.3) we obtain

$$\begin{aligned} &U(s_1, s_2; g) \\ &\quad = (s_2 - s_1) \int_0^1 \log(1-t)[g'((1-t)s_2 + ts_1) - g'((1-t)s_1 + ts_2)]dt. \end{aligned} \tag{2.5}$$

Lemma 2.1. *Suppose that g' is $\varkappa$-Hölder continuous with some $\varkappa \in (0, 1]$. Then*

$$|\mathcal{B}(a; g)| \le C[g']_\varkappa \iint \frac{|a(\xi_1) - a(\xi_2)|^{1+\varkappa}}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2, \tag{2.6}$$

with a universal constant C.

Proof. Since

$$\begin{aligned} |g'((1-t)s_2 + ts_1) - g'((1-t)s_1 + ts_2)| &\le |1-2t|^\varkappa [g']_\varkappa |s_1 - s_2|^\varkappa, \\ &\le [g']_\varkappa |s_1 - s_2|^\varkappa, \quad \text{for all } t \in [0, 1], \end{aligned}$$

formula (2.5) gives

$$|U(s_1, s_2; g)| \le C[g']_\varkappa |s_1 - s_2|^{1+\varkappa}, \quad C = -\int_0^1 \log(1-t)dt.$$

This leads to the proclaimed bound. □

The double integral in (2.6) is the standard Gagliardo–Slobodetski seminorm of a in $\mathsf{W}^{s,p}(\mathbb{R})$ raised to power p, where $p = 1 + \varkappa$, and $s = (1 + \varkappa)^{-1}$, see e.g. [9].

For the next theorem we rewrite the definition (1.4) of the coefficient $\mathcal{B}(a; g)$ as the principal value integral:

$$\mathcal{B}(a; g) = \lim_{\varepsilon \to 0} \mathcal{B}_\varepsilon(a; g), \quad \mathcal{B}_\varepsilon(a; g) = \frac{1}{8\pi^2} \iint\limits_{|\xi_1 - \xi_2| > \varepsilon} \frac{U(a(\xi_1), a(\xi_2); g)}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2. \tag{2.7}$$

Here and further we always assume that $\epsilon > 0$. In view of (2.4),

$$\mathcal{B}_\varepsilon(a; g) = \frac{1}{4\pi^2} \iint\limits_{|\xi_1 - \xi_2| > \varepsilon} \frac{V(a(\xi_1), a(\xi_2); g)}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2. \tag{2.8}$$

This representation can be transformed into a different formula for the coefficient $\mathcal{B}(a; g)$, known in the literature, see e.g. [17], Proposition 5.4 or [2], formula (1.5). For any $m \in \mathbb{R}$ and $n = 0, 1, 2 \dots$, denote

$$\|u\|_m^{(n)} = \max_{0\le k\le n} \sup_{\xi\in\mathbb{R}} (1 + |\xi|)^{m+k} |u^{(k)}(\xi)|.$$

Theorem 2.2. *Suppose that $g', g'' \in \mathsf{L}^\infty(\mathbb{R})$, and that $\|a\|_m^{(2)} < \infty$ with some $m \in (0, 1)$. Then the limit (2.7) exists and it is given by*

$$\mathcal{B}(a; g) = \frac{1}{4\pi^2} \int \lim_{\varepsilon\to 0} \int\limits_{|\xi_1-\xi_2|>\varepsilon} \frac{g(a(\xi_1)) - g(a(\xi_2))}{a(\xi_1) - a(\xi_2)} \frac{a'(\xi_1)}{\xi_1 - \xi_2} d\xi_1 d\xi_2. \tag{2.9}$$

Moreover,

$$|\mathcal{B}(a; g)| \le C[\|g'\|_{\mathsf{L}^\infty} \|a'\|_{m+1}^{(1)} + \|g''\|_{\mathsf{L}^\infty} (\|a'\|_{m+1}^{(0)})^2],$$

with a constant $C > 0$ independent of the functions a and g.

Before proving the above formula we point out some useful properties of the Hilbert transform

$$\tilde{u}(\xi) = \frac{1}{\pi} \lim_{\varepsilon\to 0} \int\limits_{|\eta-\xi|>\varepsilon} \frac{u(\eta)}{\eta - \xi} d\eta, \tag{2.10}$$

derived in [17], Lemmas 5.2 and 5.3.

Proposition 2.3. *Suppose that $\|u\|_m^{(1)} < \infty$ for some $m \in (0, 1)$. Then*

$$|\tilde{u}(\xi)| \le C\|u\|_m^{(1)} (1 + |\xi|)^{-m}.$$

If, in addition, $\|u\|_{m+1}^{(1)} < \infty$ and

$$\int u(\eta) d\eta = 0,$$

then

$$|\tilde{u}(\xi)| \le C\|u\|_{m+1}^{(1)} (1 + |\xi|)^{-m-1}.$$

The constants in the above inequalities do not depend on u.

Proof of Theorem 2.2. First we check that the integral on the right-hand side of (2.9) is finite. Observe that

$$\begin{aligned} u(\xi_1; \xi_2) := \frac{\partial}{\partial \xi_1} V(a(\xi_1), a(\xi_2); g) &= a'(\xi_1) \int_0^1 g'((1 - t)a(\xi_1) + ta(\xi_2)) dt \\ &= a'(\xi_1) \frac{g(a(\xi_1)) - g(a(\xi_2))}{a(\xi_1) - a(\xi_2)}, \end{aligned} \tag{2.11}$$

so that

$$\|u(\cdot;\xi_2)\|^{(1)}_{m+1} \le C(\|g'\|_{L^\infty}\|a'\|^{(1)}_{m+1} + \|g''\|_{L^\infty}(\|a'\|^{(0)}_{m+1})^2),$$

uniformly in $\xi_2 \in \mathbb{R}$. Moreover,

$$\int u(\xi_1;\xi_2)d\xi_1 = 0,$$

and consequently, by Proposition 2.3,

$$|\tilde{u}(\eta;\xi_2)| \le C(1+|\eta|)^{-m-1}(\|g'\|_{L^\infty}\|a'\|^{(1)}_{m+1} + \|g''\|_{L^\infty}(\|a'\|^{(0)}_{m+1})^2),$$

uniformly in $\xi_2 \in \mathbb{R}$, where $\tilde{u}(\eta;\xi_2)$ denotes the Hilbert transform of the function $u(\xi_1,\xi_2)$ in the variable ξ_1. Since

$$\mathcal{B}(a;g) = \frac{1}{4\pi}\int \tilde{u}(\xi_2,\xi_2)d\xi_2,$$

this leads to the required estimate.

Now we concentrate on the derivation of (2.9). To this end integrate (2.8) by parts:

$$\begin{aligned}4\pi^2\mathcal{B}_\varepsilon(a;g) &= \frac{1}{\varepsilon}\int [V(a(\xi_2+\varepsilon),a(\xi_2);g) + V(a(\xi_2-\varepsilon),a(\xi_2);g)]d\xi_2\\ &\quad + \int\int_{|\xi_1-\xi_2|>\varepsilon} \frac{1}{\xi_1-\xi_2}\frac{\partial}{\partial\xi_1}V(a(\xi_1),a(\xi_2);g)d\xi_1 d\xi_2.\end{aligned} \tag{2.12}$$

By (2.11), the double integral on the right-hand side of (2.12) coincides with the one in (2.9). To handle the first integral on the right-hand side of (2.12), note that by (2.2) with $\varkappa = 1$,

$$\begin{aligned}\frac{|V(a(\xi_2\pm\varepsilon),a(\xi_2);g)|}{\varepsilon} &\le C\|g'\|_{L^\infty}\frac{|a(\xi_2\pm\varepsilon)-a(\xi_2)|}{\varepsilon}\\ &\le C\|g'\|_{L^\infty}\|a\|^{(1)}_m(1+|\xi_2|)^{-m-1},\end{aligned}$$

uniformly in $\varepsilon \in (0,1]$, and that

$$\lim_{\varepsilon\to 0}\frac{V(a(\xi_2\pm\varepsilon),a(\xi_2);g)}{\varepsilon} = \pm a'(\xi_2)g'(a(\xi_2)) = \pm\frac{d}{d\xi_2}g(a(\xi_2)).$$

Clearly, the integral of the right-hand side equals zero. Thus by the Dominated Convergence Theorem the first term on the right-hand side of (2.12) tends to zero as $\varepsilon \to 0$, and the formula (2.9) is proved. □

3. Non-smooth functions

3.1. Main result. We concentrate on the very special non-smooth case, which is nonetheless interesting for applications. To distinguish from smooth functions, we change the notation from g to f and assume that f satisfies the following condition:

Condition 3.1. *For some integer $n \ge 1$, some $\gamma \in (0,1]$ and some $x_0 \in \mathbb{R}$, the function $f \in \mathsf{C}^n(\mathbb{R} \setminus \{x_0\}) \cap \mathsf{C}(\mathbb{R})$ satisfies the bound*

$$\|f\|_n = \max_{0 \le k \le n} \sup_{x \ne x_0} |f^{(k)}(x)||x - x_0|^{-\gamma+k} < \infty. \tag{3.1}$$

The constants in all subsequent estimates may depend on n, γ, but not on x_0. For a function f satisfying the above condition the bound holds:

$$|f^{(k)}(x)| \le \|f\|_n |x - x_0|^{\gamma-k}, k = 0, 1, \dots, n, \qquad x \ne x_0. \tag{3.2}$$

If $n \ge 1$, then the above condition implies that f is γ-Hölder continuous, and in particular,

$$|f(x_1) - f(x_2)| \le 2\|f\|_1 |x_1 - x_2|^\gamma, \quad \text{for all } x_1, x_2 \in \mathbb{R}. \tag{3.3}$$

For a function u denote

$$\mathsf{N}(u; \mathsf{l}^\delta(\mathsf{W}^{N,p})) = \left[\sum_{n \in \mathbb{Z}} \max_{0 \le k \le N} \left(\int_{(n,n+1)} |u^{(k)}(\xi)|^p d\xi\right)^{\frac{\delta}{p}}\right]^{\frac{1}{\delta}},$$

where $\delta \in (0, \infty]$, $p \in (0, \infty]$. Now we can state the main result.

Theorem 3.2. *Suppose that the function $f : \mathbb{R} \to \mathbb{C}$ satisfies Condition 3.1 with $n = 2$, and some $\gamma \in (0,1]$, $x_0 \in \mathbb{R}$. Let a be a real-valued function such that $a \in \mathsf{W}^{N,p}_{\mathrm{loc}}(\mathbb{R})$ with some $p \in (1, \infty]$ and some N such that $N \ge \gamma^{-1} + p^{-1}$. Then the limit (2.7) exists and it satisfies the bound*

$$\begin{aligned} |\mathcal{B}(a; f)| \le C_\gamma \|f\|_1 \iint_{|\xi_1 - \xi_2| > 1} \frac{|a(\xi_1) - a(\xi_2)|^\gamma}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2 \\ + C_\gamma \|f\|_2 [\mathsf{N}(a'; \mathsf{l}^\gamma(\mathsf{W}^{N-1,p}))]^\gamma, \end{aligned} \tag{3.4}$$

where the constant C_γ is independent of the functions f, a, and the parameter x_0.

Note that the value of the right-hand side of (3.4) is preserved under the shift $a \to a + a_0$ with an arbitrary constant a_0. If we assume that $a - a_0 \in \mathsf{L}^\gamma(\mathbb{R})$ with some constant a_0, then the first integral in (3.4) can be estimated as follows:

$$\iint\limits_{|\xi_1-\xi_2|>1} \frac{|a(\xi_1) - a(\xi_2)|^\gamma}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2 \le 2 \iint\limits_{|\xi_1-\xi_2|>1} \frac{|a(\xi_1) - a_0|^\gamma}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2$$
$$\le 2 \int |a(\xi) - a_0|^\gamma d\xi.$$

3.2. Function f. Here we prove some elementary properties of the function f satisfying Condition 3.1 with $n = 2$.

Lemma 3.3. *If $\gamma \in (0, 1]$, then for any $t_1 \ne x_0, t_2 \ne x_0$, and any $\delta \in [0, 1]$, we have*

$$|f'(t_1) - f'(t_2)| \le 2 \mathbf{|} f \mathbf{|}_2 (\min_{j=1,2} |t_j - x_0|)^{\gamma-1-\delta} |t_1 - t_2|^\delta. \tag{3.5}$$

Proof. Suppose that either $t_1 > x_0, t_2 < x_0$, or $t_1 < x_0, t_2 > x_0$. According to (3.2), for any $\delta > 0$ we have

$$\begin{aligned} |f'(t_1)| &\le \mathbf{|} f \mathbf{|}_1 |t_1 - x_0|^{\gamma-1} \\ &= \mathbf{|} f \mathbf{|}_1 |t_1 - x_0|^{\gamma-1-\delta} |t_1 - x_0|^\delta \\ &\le \mathbf{|} f \mathbf{|}_1 (\min_{j=1,2} |t_j - x_0|)^{\gamma-1-\delta} |t_1 - t_2|^\delta, \end{aligned} \tag{3.6}$$

Estimating $f'(t_2)$ in the same way we get the claimed bound.

Suppose now that $t_2 \ge t_1 > x_0$ or $t_2 \le t_1 < x_0$. Then

$$|f'(t_1) - f'(t_2)| \le |f''(\theta)||t_1 - t_2|, \quad \text{with some } \theta \in (t_1, t_2),$$

and hence, by (3.2),

$$|f'(t_1) - f'(t_2)| \le \mathbf{|} f \mathbf{|}_2 |t_1 - x_0|^{\gamma-2} |t_1 - t_2|.$$

Together with (3.6), this gives

$$|f'(t_1) - f'(t_2)| \le \mathbf{|} f \mathbf{|}_2 2^{1-\delta} |t_1 - x_0|^{(\gamma-1)(1-\delta)} |t_1 - x_0|^{(\gamma-2)\delta} |t_1 - t_2|^\delta,$$

for any $\delta \in [0, 1]$. This leads to (3.5), as claimed.

The cases $t_1 > t_2 > x_0$ or $t_1 < t_2 < x_0$ are handled by exchanging the roles of t_1 and t_2. $\square$

3.3. Functional V. Let us derive some useful estimates for the functional V defined in (2.1). As before, we assume that $f : \mathbb{R} \mapsto \mathbb{C}$ in the definition (2.7) satisfies Condition 3.1 with some $\gamma \in (0, 1]$, $n = 2$ and $x_0 \in \mathbb{R}$.

First we make some straightforward observations. In view of (3.3) and (2.2),

$$|V(s_1, s_2; f)| \le C_\gamma \| f \|_1 |s_1 - s_2|^\gamma. \tag{3.7}$$

Furthermore, by definition (2.1) and by (3.3), for any $\mu \in (0, 1)$, we have

$$\begin{aligned} |V(s_1, s_2; f) - V(r_1, r_2; f)| \le C \| f \|_1 |\log \mu| (|s_1 - r_1|^\gamma + |s_2 - r_2|^\gamma) \\ + C \| f \|_1 \mu^\gamma (|s_1 - s_2|^\gamma + |r_1 - r_2|^\gamma)|, \end{aligned} \tag{3.8}$$

for any real s_1, r_1, s_2, r_2. This bound follows from (2.1) by splitting V into two integrals: over $(0, 1 - \mu)$ and over $(1 - \mu, 1)$.

Now introduce

$$Y(s_1, s_2; f) = \partial_{s_1} V(s_1, s_2; f) = \int_0^1 f'(s_1(1 - t) + s_2 t) dt, \tag{3.9}$$

$$X(s_1, s_2; f) = Y(s_1, s_2; f) - f'(s_1), \ s_1 \ne x_0. \tag{3.10}$$

Lemma 3.4. *Let f satisfy Condition 3.1 with $\gamma \in (0, 1]$, $n = 2$ and $x_0 \in \mathbb{R}$, and let $\delta \in [0, \gamma)$ be some number. Then for all real $s_1 \ne x_0$ and all real s_2,*

$$|X(s_1, s_2; f)| \le 4(\gamma - \delta)^{-1} \| f \|_2 |s_1 - s_2|^\delta |s_1 - x_0|^{\gamma - 1 - \delta}. \tag{3.11}$$

Proof. Represent X in the form

$$X(s_1, s_2; f) = \int_0^1 [f'((1 - t)s_1 + ts_2) - f'(s_1)] dt.$$

First suppose that either $s_1 > x_0, s_2 \ge x_0$, or $s_1 < x_0, s_2 \le x_0$. Then, by (3.5),

$$\begin{aligned} &|f'((1 - t)s_1 + ts_2) - f'(s_1)| \\ &\qquad \le 2 \| f \|_2 (1 - t)^{\gamma - 1 - \delta} t^\delta |s_1 - x_0|^{\gamma - 1 - \delta} |s_1 - s_2|^\delta, \end{aligned}$$

for any $\delta \in [0, 1]$. Consequently,

$$|X(s_1, s_2; f)| \le 2 \| f \|_2 |s_1 - s_2|^\delta |s_1 - x_0|^{\gamma - 1 - \delta} \int_0^1 (1 - t)^{\gamma - 1 - \delta} t^\delta dt.$$

The integral is bounded by $(\gamma - \delta)^{-1}$ for $\delta \in [0, \gamma)$, which leads to (3.11).

Now suppose that either $s_1 > x_0, s_2 < x_0$, or $s_1 < x_0, s_2 > x_0$. According to (3.5),

$$\begin{aligned}
&|f'((1-t)s_1 + ts_2) - f'(s_1)| \\
&\quad \le 2^{1-\delta} \mathbf{|} f \mathbf{|}_2 |(1-t)s_1 + ts_2 - x_0|^{\gamma-1-\delta} t^\delta |s_1 - s_2|^\delta \\
&\quad = 2^{1-\delta} \mathbf{|} f \mathbf{|}_2 |s_1 - s_2|^{\gamma-1-\delta} \left| t - \frac{s_1 - x_0}{s_1 - s_2} \right|^{\gamma-1-\delta} t^\delta |s_1 - s_2|^\delta, \quad t \ne \frac{s_1 - x_0}{s_1 - s_2}.
\end{aligned}$$

Since $\gamma \le 1$ and $|s_1 - s_2| > |s_1 - x_0|$, we estimate

$$|s_1 - s_2|^{\gamma-1-\delta} < |s_1 - x_0|^{\gamma-1-\delta}.$$

Furthermore,

$$\int_0^1 |t - z|^{\gamma-1-\delta} dt \le \frac{2}{\gamma - \delta}$$

uniformly in $z \in [0, 1]$. This implies (3.11). □

4. Two lemmas on integrals of polynomials

In this section we prepare two elementary results involving real-valued polynomial functions a.

For a closed interval $I \subset \mathbb{R}$ we denote by $|I|$ its length (the Lebesgue measure). For a smooth function a on I we denote by $\|a\|_{\mathsf{L}^p}$ its L^p-norm on the interval I.

Lemma 4.1. *Let $I \in \mathbb{R}$ be a closed interval, and let a be a real-valued polynomial. Suppose that I contains at least $N - 1$ distinct critical points of the function a, with some $N = 1, 2, \dots$. Let $p \in [1, \infty]$ be arbitrary. Then for any $\gamma \in (0, 1]$ and any two points $\eta_1, \eta_2 \in I$ the bound holds*

$$\big||a(\eta_1)|^\gamma - |a(\eta_2)|^\gamma\big| \le \|a^{(N)}\|_{\mathsf{L}^p}^\gamma |I|^{\gamma(N - \frac{1}{p})}. \tag{4.1}$$

If I contains exactly $N - 1$ distinct critical points of a, then the total variation $\operatorname{Var}[|a|^\gamma; I]$ *of the function $|a|^\gamma$ on the interval I satisfies the bound*

$$\operatorname{Var}[|a|^\gamma; I] \le (N + 1)^2 \|a^{(N)}\|_{\mathsf{L}^p}^\gamma |I|^{\gamma(N - \frac{1}{p})}. \tag{4.2}$$

Proof. Assume without loss of generality that $\|a^{(N)}\|_{\mathsf{L}^p} \le 1$. Since the interval I contains at least $N-1$ distinct zeros of a', by an elementary argument, the interval I also contains at least $N-2$ distinct zeros of a'', $N-3$ distinct zeros of a''', and eventually, at least one point ξ_0, such that $a^{(N-1)}(\xi_0) = 0$. This means that

$$|a^{(N-1)}(\xi)| \le \int_{\xi_0}^{\xi} |a^{(N)}(\eta)|d\eta \le \|a^{(N)}\|_{\mathsf{L}^p}|I|^{1-\frac{1}{p}} \le |I|^{1-\frac{1}{p}}, \quad \text{for all } \xi \in I.$$

From this bound we obtain consecutively that

$$|a^{(N-2)}(\xi)| \le |I|^{2-\frac{1}{p}}, \quad |a^{(N-3)}(\xi)| \le |I|^{3-\frac{1}{p}},$$

and, in general,

$$|a^{(k)}(\xi)| \le |I|^{N-k-\frac{1}{p}}, \quad k = 1, 2, \dots, N-1.$$

In particular, $|a'(\xi)| \le |I|^{N-1-\frac{1}{p}}$, so that for any $\eta_1, \eta_2 \in I$ we have

$$a(\eta_1) - a(\eta_2) = w(\eta_1, \eta_2), \quad |w(\eta_1, \eta_2)| \le |I|^{N-1-\frac{1}{p}}|\eta_1 - \eta_2| \le |I|^{N-\frac{1}{p}}.$$

Thus

$$||a(\eta_1)|^\gamma - |a(\eta_2)|^\gamma| \le |w(\eta_1, \eta_2)|^\gamma \le |I|^{\gamma(N-\frac{1}{p})},$$

as claimed.

In order to prove (4.2), note that the polynomial a has at most N distinct roots on I, and hence there are at most $N+1$ intervals where the polynomial a is sign-definite. Using the additivity of total variation, it suffices to prove that on each of these intervals the total variation does not exceed $(N+1)\|a^{(N)}\|_{\mathsf{L}^p}^\gamma |I|^{\gamma(N-\frac{1}{p})}$. Assume for simplicity that $a(\xi) \ge 0$ for all $\xi \in I$. Partition I into intervals $\{I_j\}$ on which the function a is monotone. Thus by (4.1),

$$\mathrm{Var}[|a|^\gamma; I_j] \le \|a^{(N)}\|_{\mathsf{L}^p}^\gamma |I|^{\gamma(N-\frac{1}{p})}.$$

As the number of intervals I_j does not exceed N, we immediately obtain the required bound. □

Lemma 4.2. *Let $I \in \mathbb{R}$ be a closed interval, such that $|I| \le r$ with some number $r > 0$, and let a be a real-valued polynomial. Let $\gamma \in (0, 1]$, $p \in [1, \infty]$ and $N \ge \gamma^{-1} + p^{-1}$. Then the total variation* $\mathrm{Var}[|a|^\gamma; I]$ *of the function $|a|^\gamma$ on the interval I satisfies the bound*

$$\mathrm{Var}[|a|^\gamma; I] \le C_\gamma (N+1)^2 \|a'\|_{\mathsf{W}^{N-1,p}}^\gamma |I|^{(1-p^{-1})\gamma}, \tag{4.3}$$

and hence,

$$\int_I |a'(\xi)||a(\xi)|^{\gamma-1} d\xi \le C_\gamma (N+1)^2 \|a'\|_{\mathsf{W}^{N-1,p}}^\gamma |I|^{(1-p^{-1})\gamma}, \tag{4.4}$$

with a constant $C_\gamma = C_\gamma(r)$ independent of a and N.

Proof. Let $I_k, k = 1, 2, K$, be non-empty closed intervals with disjoint interiors such that $I = \bigcup_k I_k$, and satisfying the following requirements:

- each I_k, $k = 1, 2, \dots, K-1$, contains exactly $N-1$ critical points of a,
- the interval I_K contains no more than $N-1$ critical points of a.

By (4.2), for any $k = 1, 2, \dots, K-1$ we have

$$\begin{aligned} \operatorname{Var}[|a|^\gamma; I_k] &\le (N+1)^2 \|a^{(N)}\|_{\mathsf{L}^p}^\gamma |I_k|^{\gamma(N-\frac{1}{p})} \\ &\le r^{\gamma(N-\frac{1}{p})-1} (N+1)^2 \|a^{(N)}\|_{\mathsf{L}^1}^\gamma |I_k|, \end{aligned} \tag{4.5}$$

where we have used that $\gamma(N - p^{-1}) \ge 1$. Furthermore, by (4.2) again,

$$\begin{aligned} \operatorname{Var}[|a|^\gamma; I_K] &\le (L+1)^2 \|a^{(L)}\|_{\mathsf{L}^p}^\gamma |I|^{(L-\frac{1}{p})\gamma} \\ &\le r^{(L-1)\gamma} (N+1)^2 \|a'\|_{\mathsf{W}^{N-1,p}}^\gamma |I|^{(1-\frac{1}{p})\gamma}, \end{aligned} \tag{4.6}$$

where $L-1 \le N-1$ is the number of critical points on I_K. By the additivity, the inequalities (4.5) and (4.6) lead to (4.3). The left-hand side of (4.3) coincides with that of (4.4) (up to a positive multiplicative constant). This completes the proof. $\square$

5. Proof of Theorem 3.2

We begin the proof of Theorem 3.2 with estimating $\mathcal{B}_1(a; f)$, which will produce the integral term on the right-hand side of (3.4). The function f is assumed to satisfy Condition 3.1. As before, all constants in the estimates below are independent of the symbol a, function f, parameter x_0, but may depend on $\gamma \in (0, 1]$ and other relevant parameters unless otherwise stated.

Lemma 5.1. *Assume that f is as specified above. Then*

$$|\mathcal{B}_1(a; f)| \le C_\gamma {\vert\kern-0.25ex\vert\kern-0.25ex\vert} f {\vert\kern-0.25ex\vert\kern-0.25ex\vert}_1 \iint\limits_{|\xi_1-\xi_2|>1} \frac{|a(\xi_1) - a(\xi_2)|^\gamma}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2.$$

Proof. The required bound immediately follows from (2.8) and (3.7). $\square$

The remaining part of the coefficient $\mathcal{B}(a; f)$ is studied with the help of a suitable partition of unity on $\mathbb{R}$. For a function $\zeta \in \mathsf{C}_0^\infty(\mathbb{R})$ and numbers $R > 0$, $\varepsilon \in (0, R)$, define

$$\mathcal{D}_{\varepsilon,R}(a; \zeta, f) = \frac{1}{4\pi^2} \iint\limits_{\varepsilon<|\xi_1-\xi_2|<R} \zeta(\xi_1) \frac{V(a(\xi_1), a(\xi_2); f)}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2. \tag{5.1}$$

In all the subsequent bounds the constants are independent of the cut-off ζ, and of the parameters ε, R.

Theorem 5.2. *Let* $\zeta \in \mathsf{C}_0^1(-1,1)$*, and let* $a \in \mathsf{W}^{N,p}(-2,2)$ *with some* $p \in (1,\infty]$ *and* $N \ge \gamma^{-1} + p^{-1}$*. Then*

$$|\mathcal{D}_{\varepsilon,R}(a;\zeta,f)| \le C_{\gamma,\delta} \|\zeta\|_{\mathsf{C}^1} {\mathopen{|\!|\!|}} f {\mathclose{|\!|\!|}}_2 R^{(1-\frac{1}{p})\delta} A_{N,p}(a)^\gamma, \tag{5.2}$$

for any $\delta \in (0,\gamma)$*, uniformly in* $R \in (0,1]$ *and* $\varepsilon \in (0,R]$*. Here*

$$A_{N,p}(a) = \|a'\|_{\mathsf{W}^{N-1,p}}, \tag{5.3}$$

where the norm is taken on the interval $(-2,2)$*.*

Furthermore, the limit of $\mathcal{D}_{\varepsilon,R}(a;\zeta,f)$ *as* $\varepsilon \to 0$*, exists.*

Note the following straightforward estimate:

$$|a(\xi_1) - a(\xi_2)| \le A_{1,p}(a)|\xi_1 - \xi_2|^{1-\frac{1}{p}}, \quad \xi_1, \xi_2 \in (-2,2). \tag{5.4}$$

Integrating (5.1) by parts we get

$$\mathcal{D}_{\varepsilon,R}(a;\zeta,f) = \mathcal{D}^{(1)}_{\varepsilon,R}(a;\zeta,f) + \mathcal{D}^{(2)}_{\varepsilon,R}(a;\zeta,f) + \mathcal{D}^{(3)}_{\varepsilon}(a;\zeta,f) - \mathcal{D}^{(3)}_{R}(a;\zeta,f) \tag{5.5}$$

with

$$\mathcal{D}^{(1)}_{\varepsilon,R}(a;\zeta,f) = \frac{1}{4\pi^2} \iint\limits_{\varepsilon<|\xi_1-\xi_2|<R} \frac{\zeta(\xi_1)}{\xi_1-\xi_2} \frac{\partial}{\partial \xi_1} V(a(\xi_1),a(\xi_2);f)\,d\xi_1 d\xi_2,$$

$$\mathcal{D}^{(2)}_{\varepsilon,R}(a;\zeta,f) = \frac{1}{4\pi^2} \iint\limits_{\varepsilon<|\xi_1-\xi_2|<R} \frac{V(a(\xi_1),a(\xi_2);f)}{\xi_1-\xi_2} \frac{\partial}{\partial \xi_1} \zeta(\xi_1)\,d\xi_1 d\xi_2,$$

and

$$\begin{aligned} \mathcal{D}^{(3)}_{\varepsilon}(a;\zeta,f) = \frac{1}{4\pi^2\varepsilon} \int & [\zeta(\xi+\varepsilon) V(a(\xi+\varepsilon),a(\xi);f) \\ & + \zeta(\xi-\varepsilon) V(a(\xi-\varepsilon),a(\xi);f)]\,d\xi, \end{aligned} \tag{5.6}$$

Below we estimate each term separately.

Lemma 5.3. *Suppose that* $\zeta \in \mathsf{C}_0^\infty(-1,1)$ *and that* $a \in \mathsf{W}^{1,p}(-2,2)$*,* $p \in (1,\infty]$*. Then*

$$|\mathcal{D}^{(2)}_{\varepsilon,R}(a;\zeta,f)| \le C_\gamma {\mathopen{|\!|\!|}} f {\mathclose{|\!|\!|}}_1 \max|\zeta'| R^{(1-\frac{1}{p})\gamma} A_{1,p}(a)^\gamma, \tag{5.7}$$

uniformly in $R \in (0,1]$ *and* $\varepsilon \in (0,R]$*.*

Proof. By (3.7) and (5.4) we have

$$|V(a(\xi_1), a(\xi_2); f)| \le C {\bf |} f {\bf |}_1 A_{1,p}(a)^\gamma |\xi_1 - \xi_2|^{(1-\frac{1}{p})\gamma},$$

so that (5.7) follows immediately. □

For the next group of results we need to assume that a is a real-valued polynomial.

Lemma 5.4. *Suppose that* $\zeta \in \mathsf{C}_0^\infty(-1,1)$, *and that* a *is a real-valued polynomial. Then*

$$|\mathcal{D}_\varepsilon^{(3)}(a; \zeta, f)| \le C_{\gamma,\delta} \|\zeta\|_{\mathsf{C}^1} {\bf |} f {\bf |}_2 \varepsilon^{(1-\frac{1}{p})\delta} A_{N,p}(a)^\gamma,$$

for any $\delta \in [0, \gamma)$, $p \in [1, \infty]$ *and any* $N \ge \gamma^{-1} + p^{-1}$, *uniformly in* $\varepsilon \in (0, 1]$. *The constant* $C_{\gamma,\delta}$ *may depend on the parameter* N.

Proof. Without loss of generality assume that $\|\zeta\|_{\mathsf{C}^1} = 1$. Represent

$$\begin{aligned}
&\int \zeta(\xi \pm \varepsilon) V(a(\xi \pm \varepsilon), a(\xi); f) d\xi \\
&\quad = \pm \int\int_0^\varepsilon [\zeta'(\xi \pm \nu) V(a(\xi \pm \nu), a(\xi); f) \\
&\qquad\qquad + \zeta(\xi \pm \nu) a'(\xi \pm \nu) Y(a(\xi \pm \nu), a(\xi); f)] d\nu d\xi \\
&\quad = \pm \int\int_0^\varepsilon [\zeta'(\xi) V(a(\xi), a(\xi \mp \nu); f) \\
&\qquad\qquad + \zeta(\xi) a'(\xi) Y(a(\xi), a(\xi \mp \nu); f)] d\nu d\xi,
\end{aligned}$$

see (3.9) for the definition of the function Y. Let us simplify the formula for $\mathcal{D}_\varepsilon^{(3)}$, introducing the integrals

$$S_1^{(\pm)} = \frac{1}{\varepsilon} \int \zeta'(\xi) \int_0^\varepsilon V(a(\xi), a(\xi \mp \nu); f) d\nu d\xi,$$

$$S_2^{(\pm)} = \frac{1}{\varepsilon} \int \zeta(\xi) \int_0^\varepsilon a'(\xi) X(a(\xi), a(\xi \mp \nu); f) d\nu d\xi,$$

see (3.10) for the definition of X. Therefore

$$4\pi^2 \mathcal{D}_\varepsilon^{(3)}(a; \zeta, f) = S_1^{(+)} - S_1^{(-)} + S_2^{(+)} - S_2^{(-)}.$$

By (3.7) and (5.4),

$$\begin{aligned}|S_1^{(\pm)}| &\le \frac{C}{\varepsilon} \| f \|_1 \int |\zeta'(\xi)| \int_0^\varepsilon |a(\xi) - a(\xi - \nu)|^\gamma d\nu d\xi \\ &\le \frac{C}{\varepsilon} \| f \|_1 A_{1,p}(a)^\gamma \int_0^\varepsilon \nu^{(1-\frac{1}{p})\gamma} d\nu \\ &\le C \| f \|_1 A_{1,p}(a)^\gamma \varepsilon^{(1-\frac{1}{p})\gamma}.\end{aligned}$$

To estimate $S_2^{(\pm)}$ use (3.11) with $\delta \in [0, \gamma)$ and (5.4) again:

$$\begin{aligned}|X(a(\xi), a(\xi \mp \nu); f)| &\le C \| f \|_2 |a(\xi) - a(\xi \mp \nu)|^\delta |a(\xi) - x_0|^{\gamma-1-\delta} \\ &\le C \| f \|_2 A_{1,p}(a)^\delta |\nu|^{(1-\frac{1}{p})\delta} |a(\xi) - x_0|^{\gamma-1-\delta}.\end{aligned}$$

Therefore

$$|S_2^{(\pm)}| \le \frac{C}{\varepsilon} A_{1,p}(a)^\delta \| f \|_2 \int_0^\varepsilon \nu^{(1-\frac{1}{p})\delta} d\nu \int_{-1}^1 |a'(\xi)||a(\xi) - x_0|^{\gamma-1-\delta} d\xi.$$

By virtue of (4.4),

$$|S_2^{(\pm)}| \le C A_{1,p}(a)^\delta \| f \|_2 \, \varepsilon^{(1-\frac{1}{p})\delta} A_{N,p}(a)^{\gamma-\delta}.$$

Since $A_{1,p}(a) \le A_{N,p}(a)$, the required bound follows. □

Lemma 5.5. *Suppose that* $\zeta \in \mathsf{C}_0^\infty(-1, 1)$ *and that* a *is a real-valued polynomial. Then*

$$|\mathcal{D}_{\varepsilon,R}^{(1)}(a; \zeta, f)| \le C_{\gamma,\delta} \| f \|_2 \max |\zeta| R^{(1-\frac{1}{p})\delta} A_{N,p}(a)^\gamma, \tag{5.8}$$

for any $\delta \in [0, \gamma)$, $p \in (1, \infty]$ *and any* $N \ge \gamma^{-1} + p^{-1}$, *uniformly in* $R \in (0, 1]$ *and* $\varepsilon \in (0, R]$.

Proof. Without loss of generality assume that $\max |\zeta| = 1$. Since

$$\iint\limits_{\varepsilon<|\xi_1-\xi_2|<R} \frac{1}{\xi_1 - \xi_2} \zeta(\xi_1) a'(\xi_1) g'(a(\xi_1)) d\xi_1 d\xi_2 = 0,$$

the integral $\mathcal{D}_{\varepsilon,R}^{(1)}$ can be rewritten as

$$\iint\limits_{\varepsilon<|\xi_1-\xi_2|<R} \frac{1}{\xi_1 - \xi_2} \zeta(\xi_1) a'(\xi_1) X(a(\xi_1), a(\xi_2); f) d\xi_1 d\xi_2,$$

see (3.10) for the definition of the function X.

By virtue of (3.11) and (5.4), for any $\delta \in [0, \gamma)$ the integrand is bounded from above by

$$
\begin{aligned}
&C_{\gamma,\delta} \| f \|_2 \frac{|a(\xi_1) - a(\xi_2)|^\delta}{|\xi_1 - \xi_2|} |a'(\xi_1)||a(\xi_1) - x_0|^{\gamma-1-\delta} \\
&\qquad \le C_{\gamma,\delta} \| f \|_2 A_{1,p}(a)^\delta |\xi_1 - \xi_2|^{(1-p^{-1})\delta - 1} |a'(\xi_1)||a(\xi_1) - x_0|^{\gamma-1-\delta},
\end{aligned}
$$

for all ξ_1 where $a(\xi_1) \ne x_0$. Assuming that $\delta > 0$, and using (4.4), we obtain that

$$
|\mathcal{D}^{(1)}_{\varepsilon,R}(a; \zeta, f)| \le C_{\gamma,\delta} \| f \|_2 R^{(1-\frac{1}{p})\delta} A_{1,p}(a)^\delta A_{N,p}(a)^{\gamma-\delta}.
$$

As $A_{1,p} \le A_{N,p}$, the bound (5.8) follows. □

Proof of Theorem 5.2. Collecting the bounds established in Lemmas 5.3–5.5, and using the representation (5.5), we arrive at the bound (5.2) for a polynomial a.

For an arbitrary function $a \in \mathsf{W}^{N,p}(-2, 2)$, $p \in (1, \infty]$, and a number $q \le p$, $1 < q < \infty$, find a polynomial $\tilde{a} = \tilde{a}_\varepsilon$, such that

$$
\|a - \tilde{a}\|_{\mathsf{W}^{N,q}} < A_{1,p}(a) R^{\gamma^{-1}} \varepsilon^{4\gamma^{-1}}. \tag{5.9}
$$

This implies that

$$
A_{N,q}(\tilde{a}) \le A_{N,q}(a) + A_{1,p}(a) \le A_{N,p}(a)(4^{\frac{1}{q}-\frac{1}{p}} + 1). \tag{5.10}
$$

For subsequent calculations we assume without loss of generality that $\| f \|_2 = 1$ and $\|\zeta\|_{\mathsf{C}^1} = 1$. In view of (3.8), for any $\mu \in (0, 1)$ we have

$$
\begin{aligned}
&|V(a(\xi_1), a(\xi_2); f) - V(\tilde{a}(\xi_1), \tilde{a}(\xi_2); f)| \\
&\quad \le C|\log \mu|(|a(\xi_1) - \tilde{a}(\xi_1)|^\gamma + |a(\xi_2) - \tilde{a}(\xi_2)|^\gamma) \\
&\qquad + C\mu^\gamma (A_{1,q}(\tilde{a})^\gamma |\xi_1 - \xi_2|^{(1-\frac{1}{q})\gamma} + A_{1,p}(a)^\gamma |\xi_1 - \xi_2|^{(1-\frac{1}{p})\gamma}),
\end{aligned}
$$

where we have also used (5.4). Consequently,

$$
\begin{aligned}
|\mathcal{D}_{\varepsilon,R}(a; \zeta, f) - \mathcal{D}_{\varepsilon,R}(\tilde{a}; \zeta, f)| &\le \frac{C}{\varepsilon^2}[|\log \mu| \|a - \tilde{a}\|^\gamma_{\mathsf{L}^q} + \mu^\gamma A_{1,p}(a)^\gamma R] \\
&\le \frac{C}{\varepsilon^2} R A_{1,p}(a)^\gamma (|\log \mu| \varepsilon^4 + \mu^\gamma),
\end{aligned}
$$

where we have used (5.9). Take $\mu = \varepsilon^{3\gamma^{-1}}$, so that

$$
|\mathcal{D}_{\varepsilon,R}(a; \zeta, f) - \mathcal{D}_{\varepsilon,R}(\tilde{a}; \zeta, f)| \le C R A_{1,p}(a)^\gamma \varepsilon. \tag{5.11}
$$

Let $\tilde{\delta}$ be given by

$$\tilde{\delta} = \delta \, \frac{1 - p^{-1}}{1 - q^{-1}},$$

where $\delta \in (0, \gamma)$. By picking a suitable q one ensures that $\tilde{\delta} < \gamma$ as well. Now use Theorem 5.2 for the polynomial $\tilde{a}$ with the parameter $\tilde{\delta}$ instead of δ, remembering (5.10):

$$|\mathcal{D}_{\varepsilon,R}(\tilde{a}; \zeta, f)| \le C R^{(1-\frac{1}{q})\tilde{\delta}} A_{N,q}(\tilde{a})^{\gamma} \le C R^{(1-\frac{1}{p})\delta} A_{N,p}(a)^{\gamma}.$$

Combining this bound with (5.11) we obtain (5.2).

Finally, the existence of the limit

$$\lim_{\varepsilon \to 0} \mathcal{D}_{\varepsilon,R}(a; \zeta, f)$$

follows from the fact that the right-hand side of (5.2) tends to zero as $R \to 0$, $\varepsilon \to 0$. $\square$

Proof of Theorem 3.2. Let $\zeta_k \in \mathsf{C}_0^\infty(\mathbb{R})$, $k \in \mathbb{Z}$, be a family of functions constituting a partition of unity subordinate to the covering of the real axis by intervals $(k-1, k+1)$, $k \in \mathbb{Z}$. We may assume that the norms $\|\zeta_k\|_{\mathsf{C}^1}$ are bounded uniformly in $k \in \mathbb{Z}$. Represent $\mathcal{B}_\varepsilon(a; f)$ as

$$\mathcal{B}_\varepsilon(a; f) = \mathcal{B}_1(a; f) + \sum_{k \in \mathbb{Z}} \mathcal{D}_{\varepsilon,1}(a; \zeta_k, f).$$

The first term on the right-hand side is estimated by Lemma 5.1. Due to the bound (5.2) the second term is bounded by $C \,\|\!| f \|\!|_2 \mathsf{N}(a'; \mathsf{l}^\gamma(\mathsf{W}^{N-1,p}))^\gamma$. Furthermore, since the N-(quasi)-norm is finite, the sum has a limit as $\varepsilon \to 0$. This completes the proof. $\square$

6. A special case

In the previous Section, in the proof of Theorem 3.2, we use the covering of the real axis by intervals $(k-1, k+1)$, $k \in \mathbb{Z}$ that obviously all have length 2. Now we derive an estimate for $\mathcal{B}(a; f)$ using a covering by intervals whose size is sensitive to the rate of change of the function a. Let us describe in more precise terms the conditions on a. Let $\tau: \mathbb{R} \to \mathbb{R}$ be a positive function satisfying the condition

$$|\tau(\xi) - \tau(\eta)| \le \nu |\xi - \eta|, \quad \text{for all } \xi, \eta \in \mathbb{R}, \tag{6.1}$$

with some $\nu \in (0,1)$. It is straightforward to check that

$$(1+\nu)^{-1} \le \frac{\tau(\xi)}{\tau(\eta)} \le (1-\nu)^{-1}, \quad \text{for all } \eta \in J(\xi) = (\xi - \tau(\xi), \xi + \tau(\xi)). \tag{6.2}$$

We call τ *the scale function*. Let $v: \mathbb{R} \to \mathbb{R}$ be another continuous positive function such that

$$C_1 \le \frac{v(\eta)}{v(\xi)} \le C_2, \quad \text{for all } \eta \in J(\xi), \tag{6.3}$$

with some positive constants C_1, C_2 independent of ξ and η. We call v *the amplitude function*. Since $\nu < 1$, one can construct a covering of $\mathbb{R}$ by open intervals $J(\xi_j)$ centered at some points $\xi_j, j \in \mathbb{Z}$, which satisfies *the finite intersection property,* i.e. the number of intersecting intervals is bounded from above by a constant depending only on the parameter ν, see [4], Chapter 1. Moreover, there exists a partition of unity $\phi_j \in \mathsf{C}_0^\infty(\mathbb{R})$ subordinate to the above covering such that

$$|\phi_j^{(k)}(\xi)| \le C_k \tau(\xi)^{-k}, \quad k = 0, 1, \dots, \tag{6.4}$$

with some constants C_k independent of $j \in \mathbb{Z}$.

It is convenient for us to use a covering with finite intersection property, constructed with the help of the function $\tau/2$ instead of τ itself. Let

$$I_j = \Big(\eta_j - \frac{\tau_j}{2}, \eta_j + \frac{\tau_j}{2}\Big), \quad \tau_j = \tau(\eta_j), j \in \mathbb{Z},$$

be intervals forming such a covering, and let $\phi_j \in \mathsf{C}_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, be a subordinate partition of unity satisfying (6.4).

Consider a symbol $a \in \mathsf{C}^N(\mathbb{R})$, satisfying the bounds

$$|a(\xi) - a_0| \le C v(\xi), \quad |a^{(k)}(\xi)| \le C_k \tau(\xi)^{-k} v(\xi), \quad k = 1, 2, \dots, N, \tag{6.5}$$

with some functions τ and v described above, and with some constant a_0.

In all the bounds below the constants are independent of the functions f, τ, and v, but may depend on the parameter ν and the constants in (6.3) and (6.5).

Theorem 6.1. *Suppose that f satisfies Condition 3.1 with $n = 2$ and $\gamma \in (0,1]$. Let τ, v, a satisfy (6.1), (6.3), and let a satisfy (6.5) with some $N \ge \gamma^{-1}$. Then*

$$|\mathcal{B}(a; f)| \le C_\gamma {|\!|\!|} f {|\!|\!|}_2 \int \frac{v(\xi)^\gamma}{\tau(\xi)} d\xi. \tag{6.6}$$

A similar bound holds also for functions f with higher smoothness.

Theorem 6.2. *Suppose that $f:\mathbb{C}\to\mathbb{C}$ is a function such that f' is $\varkappa$-Hölder continuous with some $\varkappa\in(0,1]$. Let τ, v, a satisfy (6.1), (6.3), and let a satisfy (6.5) with $N=1$. Then*

$$|\mathcal{B}(a;f)| \le C_\varkappa \|f'\|_{\mathrm{C}^{0,\varkappa}} \int \frac{v(\xi)^{1+\varkappa}}{\tau(\xi)} d\xi. \tag{6.7}$$

First we give a detailed proof of Theorem 6.1.

Represent $\mathcal{B}(a;f)$ as follows:

$$\mathcal{B}_\varepsilon(a;f) = \sum_{j\in\mathbb{Z}} \mathcal{D}_{\varepsilon,\infty}(a;\phi_j,f), \tag{6.8}$$

see (5.1) for the definition of $\mathcal{D}_{\varepsilon,R}(\cdots)$. Split each summand into two components:

$$\mathcal{D}_{\varepsilon,\infty}(a;\phi_j,f) = \mathcal{D}_{\varepsilon,R_j}(a;\phi_j,f) + \mathcal{D}_{R_j,\infty}(a;\phi_j,f), \quad R_j = \frac{\tau_j}{2}.$$

Lemma 6.3. *Suppose that the scaling function τ satisfies (6.1) with some $\nu\in(0,1)$. If f satisfies the conditions of Theorem 6.1, then*

$$\sum_{j\in\mathbb{Z}} |\mathcal{D}_{R_j,\infty}(a;\phi_j,f)| \le C \,{\left|\!\left|\!\left| f \right|\!\right|\!\right|}_1 \int \frac{|a(\xi)-a_0|^\gamma}{\tau(\xi)} d\xi. \tag{6.9}$$

Proof. Assume without loss of generality that ${\left|\!\left|\!\left| f \right|\!\right|\!\right|}_1 = 1$ and $a_0=0$. For all $\xi_1\in I_j$ we get from (6.1) that

$$\tau(\xi_1) \le \frac{2+\nu}{2}\tau_j.$$

Thus for all ξ_2 such that $|\xi_1-\xi_2|>\tau_j/2$ we have

$$\tau(\xi_1) \le (2+\nu)|\xi_1-\xi_2|,$$

$$\tau(\xi_2) \le \nu|\xi_1-\xi_2| + \tau(\xi_1) \le 2(\nu+1)|\xi_1-\xi_2|,$$

which leads to

$$c_\nu(\tau(\xi_1)+\tau(\xi_2)) \le |\xi_1-\xi_2|, c_\nu = \frac{1}{4(1+\nu)}.$$

Therefore

$$|\mathcal{D}_{R_j,\infty}(a;\phi_j,f)| \le \iint\limits_{|\xi_1-\xi_2|>c_\nu(\tau(\xi_1)+\tau(\xi_2))} \phi_j(\xi_1) \frac{|V(a(\xi_1),a(\xi_2);f)|}{|\xi_1-\xi_2|^2} d\xi_1 d\xi_2.$$

By (3.7), the right-hand side does not exceed

$$C \iint\limits_{|\xi_1-\xi_2|>c_\nu\tau(\xi_1)} \phi_j(\xi_1)\frac{|a(\xi_1)|^\gamma}{|\xi_1-\xi_2|^2}d\xi_1 d\xi_2 + C\iint\limits_{|\xi_1-\xi_2|>c_\nu\tau(\xi_2)} \phi_j(\xi_1)\frac{|a(\xi_2)|^\gamma}{|\xi_1-\xi_2|^2}d\xi_1 d\xi_2.$$

Thus the sum over j is bounded from above by

$$\begin{aligned} C \iint\limits_{|\xi_1-\xi_2|>c_\nu\tau(\xi_1)} & \frac{|a(\xi_1)|^\gamma}{|\xi_1-\xi_2|^2}d\xi_1 d\xi_2 + C\iint\limits_{|\xi_1-\xi_2|>c_\nu\tau(\xi_2)} \frac{|a(\xi_2)|^\gamma}{|\xi_1-\xi_2|^2}d\xi_1 d\xi_2 \\ & \le C'\int \frac{|a(\xi)|^\gamma}{\tau(\xi)}d\xi, \end{aligned}$$

as claimed. □

Lemma 6.4. *Let a satisfy (6.5) with some functions $\tau = \tau(\xi)$ and $v = v(\xi)$ satisfying (6.1) and (6.3). Suppose also that $N \ge \gamma^{-1}$ and $R \le R_j$. Then for any $\delta \in [0,\gamma)$ the bound holds*

$$|\mathcal{D}_{\varepsilon,R}(a;\phi_j,f)| \le C_\delta \| f \|_2 R^\delta \int_{I_j} \frac{v(\xi)^\gamma}{\tau(\xi)^{1+\delta}}d\xi, \tag{6.10}$$

uniformly in $\varepsilon \in (0,R]$.

Proof. Without loss of generality assume $\| f\|_2 = 1$. Let

$$\tilde a(\eta) = a(\eta_j + R_j\eta),\ \tilde\phi_j(\eta) = \zeta_j(\eta_j + R_j\eta).$$

Thus by (6.4), $\|\tilde\phi_j\|_{\mathsf{C}^1} \le C$, $\operatorname{supp}\tilde\phi_j \subset (-1,1)$ uniformly in j, and in view of (6.3) and (6.5),

$$|\tilde a^{(n)}(\eta)| \le C_n v(\eta_j), \quad \text{for all } \eta \text{ such that } |\eta| \le 2,$$

for all $n = 1,\dots,N$, so that $A_{N,\infty}(\tilde a) \le C v(\eta_j)$, see (5.3) for the definition. Thus by Theorem 5.2 with $p=\infty$, and arbitrary $\delta\in[0,\gamma)$,

$$|\mathcal{D}_{\varepsilon,R_j}(a;\phi_j,f)| = |\mathcal{D}_{\varepsilon R_j^{-1},RR_j^{-1}}(\tilde a,\tilde\phi_j,f)| \le C(RR_j^{-1})^\delta v(\eta_j)^\gamma.$$

The right-hand side is trivially estimated by

$$CR^\delta\int_{I_j} v(\eta_j)^\gamma\tau_j^{-1-\delta}d\xi.$$

By virtue of (6.1) and (6.3), this is bounded by the right-hand side of (6.10). This completes the proof. □

Corollary 6.5. *Suppose that* $\tau_{\inf} = \inf \tau(\xi) > 0$, *and that* $R \le \tau_{\inf}/2$. *Then for any* $\delta \in [0, \gamma)$ *the bound holds*:

$$|\mathcal{D}_{\varepsilon,R}(a; 1, f)| \le C_\delta \| f \|_2 R^\delta \int \frac{v(\xi)^\gamma}{\tau(\xi)^{1+\delta}} d\xi, \tag{6.11}$$

uniformly in $\varepsilon \in (0, R]$.

Proof of Corollary 6.5 *and Theorem* 6.1. Since the covering $\{I_j\}$ possesses the finite intersection property, the bound (6.11) follows from the bound (6.10) by summing over all j's.

Using the bound (6.10) with $R = R_j$ we obtain that

$$|\mathcal{D}_{\varepsilon,R_j}(a; \phi_j, f)| \le C \| f \|_2 \int_{I_j} \frac{v(\xi)^\gamma}{\tau(\xi)} d\xi,$$

uniformly in $\varepsilon \in (0, 1]$. In view of the finite intersection property of the covering $\{I_j\}$, we get

$$\sum_{j \in \mathbb{Z}} |\mathcal{D}_{\varepsilon,R_j}(a; \phi_j, f)| \le C \| f \|_2 \int \frac{v(\xi)^\gamma}{\tau(\xi)} d\xi.$$

In view of the representation (6.8) this bound together with (6.9) lead to (6.6). □

For Theorem 6.2 we give only a sketch of the proof. The details are either the same as in the preceding proof, or they can be easily filled in.

Sketch of the proof of Theorem 6.2. By (2.6), the proof reduces to estimating the integral

$$\iint \frac{|a(\xi_1) - a(\xi_2)|^{1+\varkappa}}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2.$$

As in Lemma 6.3 one can show that

$$\sum_j \iint_{|\xi_1 - \xi_2| > R_j} \phi_j(\xi_1) \frac{|a(\xi_1) - a(\xi_2)|^{1+\varkappa}}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2 \le C \int \frac{|a(\xi) - a_0|^{1+\varkappa}}{\tau(\xi)} d\xi$$

Furthermore, if $|\xi_1 - \xi_2| < R_j$, $\xi_1 \in I_j$, then by (6.5), (6.1), and (6.3),

$$|a(\xi_1) - a(\xi_2)| \le C v(\eta_j)\tau_j^{-1}|\xi_1 - \xi_2|,$$

so that

$$\begin{aligned}
&\iint\limits_{|\xi_1-\xi_2|<R_j} \phi_j(\xi_1)\frac{|a(\xi_1) - a(\xi_2)|^{1+\varkappa}}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2 \\
&\le C v(\eta_j)^{1+\varkappa}\tau_j^{-1-\varkappa} \int_{I_j}\int_{|\xi_1-\xi_2|<R_j} |\xi_1 - \xi_2|^{\varkappa-1} d\xi_2 d\xi_1 \\
&\le C v(\eta_j)^{1+\varkappa} \\
&\le C \int_{I_j} v(\eta_j)^{1+\varkappa}\tau_j^{-1} d\xi.
\end{aligned}$$

Now, as in the proof of Lemma 6.4, the last integral is bounded by $\int v^{1+\varkappa}\tau^{-1} d\xi$. This completes the proof of Theorem 6.2. □

We illustrate the usefulness of the bound (6.6) with the example of the symbol

$$a(\xi) = a_T(\xi) = \frac{1}{1 + \exp\frac{\xi^2-\mu}{T}}, \tag{6.12}$$

where $T \in (0, T_0]$, $T_0 > 0$ and $\mu \in \mathbb{R}$ are some parameters. This symbol is nothing but the Fermi symbol for non-interacting Fermions at positive temperature T and chemical potential μ, see e.g. [7]. We are interested in the small T behaviour, whereas the value μ is kept fixed. Assume for simplicity that $\mu = 1$. It is clear that in a neighbourhood of the points $\xi = \pm 1$ the derivatives of a grow as $T \to 0$. It is straightforward to check that

$$|a^{(n)}(\xi)| \le C_n a(\xi)(1 - a(\xi))(1 + |\xi|)^n T^{-n}, \quad n = 1, 2, \dots, \tag{6.13}$$

and

$$a(\xi)(1 - a(\xi)) \le \exp\Big(-\frac{|\xi^2 - 1|}{T}\Big), \quad \xi \in \mathbb{R}. \tag{6.14}$$

Thus Theorem 3.2 with any $p \in (1, \infty]$ leads to the estimate

$$|\mathcal{B}(a; f)| \le C \, ❙ f ❙_1 + \widetilde{C} \, ❙ f ❙_2 T^{-N\gamma+\frac{\gamma}{p}}. \tag{6.15}$$

The right-hand side is greater than CT^{-1}, since $N \ge \gamma^{-1} + p^{-1}$.

Let us now estimate $\mathcal{B}(a_T; f)$ in a different way, by applying Theorem 6.1. Since

$$(1+|\xi|)^n T^{-n} \exp\Big(-\frac{|\xi^2-1|}{2T}\Big) \le C_n(||\xi|-1|+T)^{-n}, \quad C_n = C_n(T_0),$$

in view of (6.13) and (6.14), we have

$$|a^{(n)}(\xi)| \le C_n(||\xi|-1|+T)^{-n} \exp\Big(-\frac{|\xi^2-1|}{2T}\Big), \quad n = 1, 2, \ldots.$$

This shows that a satisfies (6.5) with

$$a_0 = 0, \quad \tau(\xi) = \frac{1}{2}(||\xi|-1|+T), \quad v(\xi) = v_\beta(\xi) = (1+|\xi|)^{-\beta},$$

with an arbitrary $\beta > 0$. Consequently, by Theorem 6.1,

$$\begin{aligned} |\mathcal{B}(a; f)| &\le C \| f \|_2 \int (||\xi|-1|+T)^{-1}(1+|\xi|)^{-\beta\gamma} d\xi \\ &\le C \| f \|_2 (|\log T| + 1). \end{aligned}$$

This bound is clearly sharper than (6.15), and its precision (as $T \to 0$) is confirmed by the asymptotic formula for $\mathcal{B}(a_T; f)$, $T \to 0$, announced in [7] and proved in [8].

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