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ON WELL-POSEDNESS OF SEMILINEAR STOCHASTIC EVOLUTION EQUATIONS ON L_p SPACES*

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Abstract. We establish well-posedness in the mild sense for a class of stochastic semilinear evolution equations on L_p spaces, driven by multiplicative Wiener noise, with a drift term given by a superposition operator that is assumed to be quasi-monotone and polynomially growing, but not necessarily continuous. In particular, we consider a notion of mild solution ensuring that the superposition operator applied to the solution is still function-valued but satisfies only minimal integrability conditions. The proofs rely on stochastic calculus in Banach spaces, monotonicity and convexity techniques, and weak compactness in L_1 spaces.

Key words. stochastic PDEs, monotone operators, convex analysis

AMS subject classifications. 60H15, 47H06, 46N30

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1. Introduction.

The purpose of this work is to prove well-posedness (existence, uniqueness, and continuous dependence of solutions on the initial datum) to stochastic evolution equations (SEEs) of the type

$$du(t) + Au(t)\,dt + f(u(t))\,dt = \eta u(t)\,dt + \sum_{k\in\mathbb{N}} b^k(t,u(t))\,dw^k(t), \qquad u(0) = u_0, \tag{1}$$

where $t \in [0,T]$, A is a linear m-accretive operator on $L_q(D)$, with D a bounded domain in $\mathbb{R}^n$ and $q \geq 2$, $f : \mathbb{R} \to \mathbb{R}$ is an increasing function of polynomial growth (without any continuity assumption), (w^k) are independent real standard Wiener processes, and (b^k) are (random, time-dependendent) maps from $L_q(D)$ to itself satisfying suitable integrability and Lipschitz continuity conditions. Precise assumptions on the notion of solution and on the data of the problem are given in section 2. In particular, we adopt three notions of solution that depend on the integrability properties of $f(u)$: strict mild and mild solutions are defined to be such that $f(u) \in L_1(0,T;L_q(D))$ almost surely and that $f(u) \in L_1(\Omega\times[0,T]\times D)$, respectively (here Ω stands for the underlying probability space); on the other hand, generalized solutions are defined as limits of strict mild solutions, so that, in general, $f(u)$ may not have any integrability. The first notion of solution is the simplest but also the most restrictive in terms of assumptions on the data of the problem. The second notion is the most natural if one wants $f(u)$ to be function-valued while satisfying minimal integrability conditions. The last notion, motivated by analogous constructions in the deterministic setting, apart from being the least demanding, is useful in several contexts, for instance, in the study of Kolmogorov operators and Markovian semigroups associated to stochastic PDEs (cf., e.g., [12]).

Our approach to the well-posedness problem is based, on the probabilistic side, on stochastic calculus for L_p-valued processes and, on the analytic side, on methods

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from the theory of (nonlinear) m-accretive operators and convex analysis. Some ideas developed here, concerning strict mild and generalized solutions, already appeared, in a more primitive form, in [23, 25] and, in a slightly different context, in [26].

The results are a step in the attempt to reduce the gap between the well-posedness theory for deterministic evolution equations of monotone type, which is essentially complete, and the one for stochastic equations, which is much less developed. In particular, if $\mathcal{A}$ is a (nonlinear, multivalued) m-accretive operator on a Banach space X and $g \in L_1(0,T;X)$, the equation

$$\frac{dy}{dt} + \mathcal{A}y \ni g, \qquad y(0) = y_0 \in X, \tag{2}$$

admits a unique solution $y \in C([0,T];X)$ that depends continuously on the initial datum and on the "forcing" term g. We recall that a graph (or, equivalently, a multivalued operator) $\mathcal{A} \subset X \times X$ is called accretive if, for every (x_1,y_1), $(x_2,y_2) \in \mathcal{A}$, there exists $z \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, z\rangle \geq 0$, where $J : X \to X'$ is the duality map. An accretive operator $\mathcal{A}$ is called m-accretive if $I + \mathcal{A}$ is surjective. Accretivity is thus a notion of monotonicity in Banach spaces: if X is a Hilbert space and $\mathcal{A}$ is single-valued (for simplicity), accretivity simply means that $\langle \mathcal{A}x_1 - \mathcal{A}x_2, x_1 - x_2\rangle \geq 0$ for all x_1, x_2 in the domain of $\mathcal{A}$. The X-valued continuous solution to (2) just mentioned is defined as the limit of solutions to suitably time-discretized equations, and the corresponding well-posedness theory was established in full generality by Crandall and Liggett [11]. If the Banach space X satisfies certain geometric assumptions, existence results for (2) can be obtained by (comparatively) simpler means, and they were known before the appearance of [11]. Let us consider, for instance, the case where X is uniformly smooth (a property that is fulfilled, e.g., by L_p spaces with $p \in \,]1,\infty[$): if y_0 belongs to the domain of $\mathcal{A}$ and the distributional derivative of g belongs to $L^1(0,T;X)$, then (2) admits a unique strong solution $y \in C([0,T];X)$, whose distributional derivative belongs to $L^\infty(0,T;X)$, that can be obtained as a limit of solutions $(y_\lambda)_{\lambda>0} \subset C^1([0,T];X)$ to the regularized equations

$$\frac{dy_\lambda}{dt} + \mathcal{A}_\lambda y_\lambda = g, \qquad y_\lambda(0) = y_0,$$

where $\mathcal{A}_\lambda := \frac{1}{\lambda}(I - (I + \lambda\mathcal{A})^{-1})$, $\lambda > 0$, the Yosida approximation of $\mathcal{A}$, is a Lipschitz continuous accretive operator on X. Since the map $(y_0, f) \mapsto y$ is continuous from $X \times L^1(0,T;X)$ to $C([0,T];X)$, a notion of solution to (2) as a limit of strong solutions to equations with more regular data is then inferred. A comprehensive treatment of these results, including historical remarks and bibliographical references, can be found in [1] (for the particular case of X being a Hilbert space, see also [8]).

The picture is completely different for equations of the type

$$du + \mathcal{A}u\,dt \ni \sum_{k\in\mathbb{N}} Bh_k\, w_k(t), \qquad u(0) = u_0 \in X, \tag{3}$$

where (h_k) is an orthonormal basis of a (separable) Hilbert space H, B is a linear operator from H to X satisfying suitable assumptions, and (w_k) are independent real standard Wiener processes. Regrettably, there is no *general* well-posedness theory for equation (3), even if X is a Hilbert space.[1] One of the reasons for this is that all

[1]The essentially different case where $\mathcal{A}$ is a maximal monotone operator from a Banach space V to its dual V', for which the gap between the deterministic and the stochastic theories is less wide, is not disussed here. The interested reader can refer to [1, Chapter 2], [20, 27, 28, 29], and references therein.

existing results of the deterministic theory depend in an essential way on g having finite variation, hence they cannot be extended to the case where the forcing term is stochastic. On the other hand, one can find in the literature (references will be given below) several well-posedness results for particular choices of $\mathcal{A}$, mostly for the semilinear case, i.e., assuming that $\mathcal{A} = A + F$, where A is a linear (unbounded) m-accretive operator and F is a nonlinear term. However, some results have recently been obtained also for the case where $\mathcal{A}$ is "fully" nonlinear, for instance, $\mathcal{A} = -\Delta\beta(\cdot)$ on H^{-1}, the so-called nonlinear diffusion or porous medium operator (see, e.g., [2, 3]).

From now on we shall focus our discussion on semilinear SEEs of type (3), with $\mathcal{A} = A+f$, where f is the superposition operator associated to a real-valued function, and X is an L_p space on a bounded domain of Euclidean space. The large existing literature on the subject (up-to-date references to which can be found, e.g., in [13]) deals mostly with the Hilbertian setting $p = 2$. For an extensive discussion of available results in such a setting we refer the interested reader to [27]. Semilinear equations on (non-Hilbert) L_p spaces are instead not widely studied. To our knowledge, the best existence results currently available are those in [21], where global well-posedness in the mild sense of (1) is obtained assuming that $-A$ generates an analytic semigroup and that f is polynomially bounded and locally Lipschitz continuous on $L_q(D)$ (not as function on $\mathbb{R}$!). Even though the condition on f is very restrictive, adapting ideas from [10], and considerably improving results thereof, well-posedness in spaces of continuous functions is obtained, allowing $f : \mathbb{R} \to \mathbb{R}$ to be a (monotone) polynomial, and assuming that the restriction of $-A$ to $C(\overline{D})$ generates a strongly continuous semigroup of contractions. The approach used in [21] relies on approximation of the coefficients and extension of local solutions, as well as on a reduction to a deterministic evolution equation with random coefficients, and on "sandwiching" $C(\overline{D})$ between UMD spaces defined as domains of fractional powers of $-A$. On the probabilistic side, the results of [21] rely in an essential way on the semigroup approach to SEE in UMD Banach spaces.

Even though a general theory for stochastic equations of monotone type remains currently unattainable, the well-posedness results obtained here complement the existing ones in the particular case of $\mathcal{A}$ as in the previous paragraph and $X = L_p$, with $p \neq 2$. In fact, our simplest existence results (see Theorems 2.3 and 2.7 below) assume only that $-A$ generates a continuous semigroup of contractions on L_p (for two different values of p) and that f is monotone and polynomially growing, without any continuity assumptions (thus we obviously cannot consider solvability in spaces of continuous functions). More sophisticated results (see Theorem 2.9 below) are obtained under the further assumption that the resolvent of A is sub-Markovian and a power thereof is hypercontractive. Such hypotheses are satisfied, for instance, by large classes of nondegenerate second-order elliptic differential operators. The approach we take follows the "classical" one described above for deterministic equations based on constructing solutions to regularized equations and then passing to the limit in an appropriate topology. The key difference is that the necessary a priori estimates on the approximating equations have to be obtained by other means. To this purpose, the essential tools are Itô's formula for the pth power of the L_p norm and techniques from convex analysis and the theory of nonlinear m-accretive operators.

The rest of the text is organized as follows. The main results are stated in section 2, and auxiliary results are collected in section 3. In sections 4, 5, and 6 we prove well-posedness in the strict mild, generalized, and mild senses, respectively.

We conclude this introductory section fixing some commonly used notation. All Lebesgue spaces on D will be denoted without explicit mention of the domain, e.g.,

$L_q := L_q(D)$. The mixed-norm spaces $L_p(0,T;L_q(D))$, $T \in \mathbb{R}_+$, will simply be denoted by $L_p(L_q)$. The domain and range of a map L will be denoted by $\mathsf{D}(L)$ and $\mathsf{R}(L)$, respectively. The standard notation $\mathscr{L}(E,F)$ will be used for the space of linear bounded operators between two Banach spaces E and F. We shall write $a \lesssim b$ to mean that a is less than or equal to b modulo a multiplicative constant, with subscripts to emphasize its dependence on specific quantities. Completely analogous meanings have the symbols $\gtrsim$ and $\eqsim$.

2. Main results. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, with $T > 0$ fixed, be a filtered probability space satisfying the "usual" conditions (see, e.g., [14]), and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. All stochastic elements will be defined on this stochastic basis, and any expression involving random quantities will be meant to hold $\mathbb{P}$-almost surely, unless otherwise stated. Given $p > 0$ and a separable Banach space X, we shall denote by $\mathbb{L}_p(X)$ the space of X-valued random variables ζ such that

$$\|\zeta\|_{\mathbb{L}_p(X)} := \big(\mathbb{E}\|\zeta\|_X^p\big)^{1/p} < \infty$$

and by $\mathbb{H}_p(X)$ the space of measurable[2] and adapted X-valued processes such that

$$\|u\|_{\mathbb{H}_p(X)} := \biggl(\mathbb{E}\sup_{t\leq T}\|u(t)\|_X^p\biggr)^{1/p} < \infty.$$

Both spaces are Banach spaces for $p \geq 1$ and quasi-Banach spaces for $0 < p < 1$.

To look for L_q-valued (mild) solutions to (1), it is clear that the linear operator A should be taken as the generator of a C_0-semigroup on L_q and that the maps (b^k) should satisfy suitable Lipschitz continuity assumptions. For compactness of notation, we set $B := (b^k)$, i.e., B denotes the whole sequence (b^k). Similarly, we shall write, just as notation, $W := (w^k)$, and

$$\int_0^t B(s)\,dW(s) := \sum_{k\in\mathbb{N}} \int_0^t b^k(s)\,dw^k(s).$$

A completely similar notation will be used for other series of stochastic integrals.

For later use, we introduce the following conditions, where $r > 0$, $s \geq 2$:

(A_s) A is a linear m-accretive operator on L_s.

($\mathrm{B}_{r,s}$) The maps $b^k : \Omega \times [0,T] \times L_s \to L_s$ are such that $b^k(\cdot,\cdot,x)$ is measurable and adapted for all $x \in L_s$ and all $k \in \mathbb{N}$, there exists a constant C_B such that

$$\big\|B(\omega,t,u) - B(\omega,t,v)\big\|_{L_s(\ell_2)} \leq C_B \big\|u-v\big\|_{L_s} \qquad \forall(\omega,t) \in \Omega \times [0,T],$$

and $B(\cdot,\cdot,0) \in \mathbb{L}_r(L_2(0,T;L_s(\ell_2)))$.

For simplicity, we shall often suppress explicit indication of the dependence on time of B. If A satisfies (A_s), the C_0-semigroup of contractions generated by $-A$ on L_s will be denoted by S. Should A satisfy (A_s) for different values of s, we shall not notationally distinguish among different (but consistent) realizations of A and S on different L_s spaces.

[2]Since we never need weak measurability, measurable will always mean strongly measurable.

We assume that the function $f : \mathbb{R} \to \mathbb{R}$ is increasing and that there exists $d \geq 1$ such that $|f(x)| \lesssim 1 + |x|^d$ for all $x \in \mathbb{R}$. In particular, f is *not* assumed to be continuous. The graph $f_0 \subset \mathbb{R} \times \mathbb{R}$ defined by

$$f_0(x) = \begin{cases} f(x), & x \in \mathbb{R} \setminus I, \\ [f(x-), f(x+)], & x \in I, \end{cases}$$

where I is the jump set of f, is maximal monotone and clearly extends the graph of f. Furthermore, the (multivalued) superposition operator f_1 associated to f_0, defined as

$$f_1 : u \longmapsto \bigl\{ v \in L_q : \; v \in f_0(u) \quad \text{a.e.} \bigr\}$$

on its obvious domain, is an m-accretive subset of $L_q \times L_q$ (see, e.g., [1, pp. 106–107]). We shall not notationally distinguish among the increasing function f, its maximal monotone extension f_0, and the associated superposition operator f_1: they will all be denoted simply by f.

Remark 2.1. Thanks to the linear term in u on the right-hand side of (1), nothing changes assuming that f (or A, or both) is quasi-monotone, i.e., that $f + \delta I$ is monotone for some $\delta > 0$.

We shall establish well-posedness of (1) in several classes of processes. The most natural, and most restrictive, notion of solution is the following. Here and in the following we denote the measure $\mathbb{P} \otimes dt \otimes dx$ on $\Omega \times [0,T] \times D$ by m (dt and dx being the Lebesgue measure on $[0,T]$ and on D, respectively).

DEFINITION 2.2. *Let u_0 be an L_q-valued $\mathcal{F}_0$-measurable random variable. A measurable adapted L_q-valued processes u is a* strict mild solution *to* (1) *if $u \in L_\infty(L_q)$, there exists an adapted L_q-valued process $g \in L_1(L_q)$, with $g \in f(u)$ m-a.e., and, for all $t \in [0,T]$, $S(t-\cdot)B(\cdot,u)$ is stochastically integrable and*

$$u(t) + \int_0^t S(t-s)\bigl(g(s) - \eta u(s)\bigr)\,ds = S(t)u_0 + \int_0^t S(t-s)B(s,u(s))\,dW(s). \tag{4}$$

Stochastic integrability of $G := (g^k)$ with respect to W here means, apart from the usual measurability conditions, $G \in L_2(0,T;L_q(\ell_2))$ $\mathbb{P}$-almost surely.[3]

Our first main result provides sufficient conditions for the well-posedness of (1) in $\mathbb{H}_p(L_q)$. The proof is given in section 4 below.

THEOREM 2.3. *Let $p > 0$ and $q \geq 2$ be such that*

$$p^* := \frac{p}{q}(2d + q - 2) > d.$$

Assume that

(a) $u_0 \in \mathbb{L}_{p^*}(L_{qd})$;

(b) *hypothesis* (A_s) *is satisfied for $s = q$ and $s = qd$;*

(c) *hypothesis* ($\mathrm{B}_{r,s}$) *is satisfied for $r = p$, $s = q$ and $r = p^*$, $s = qd$.*

Then there exists a unique strict mild solution $u \in \mathbb{H}_p(L_q)$ to (1)*. Moreover, u has continuous paths and the solution map $u_0 \mapsto u$ is Lipschitz continuous from $\mathbb{L}_p(L_q)$ to $\mathbb{H}_p(L_q)$.*

[3]This is not the most general definition of stochastic integrability (cf., e.g., [31]), but it suffices for our purposes.

Relaxing the definition of solution, well-posedness for (1) can be proved for *any* $p > 0$ and $q \geq 2$. The following notion of solution derives from the definition of *solution faible* by Benilan and Brézis [4]. We first deal with equations with additive noise, i.e., of the type

$$du(t) + Au(t)\,dt + f(u(t))\,dt = \eta u(t)\,dt + B(t)\,dW(t), \qquad u(0) = u_0, \tag{5}$$

where the process B is measurable and adapted and belongs to $\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))$.

Definition 2.4. *Let $p > 0$ and $q \geq 2$. A process $u \in \mathbb{H}_p(L_q)$ is a* generalized solution *to* (5) *if there exist sequences $(u_{0n}) \subset \mathbb{L}_p(L_q)$, $(B_n) \subset \mathbb{L}_p(L_2(0,T;L_q(\ell_2)))$, and $(u_n) \subset \mathbb{H}_p(L_q)$ such that $u_{0n} \to u_0$ in $\mathbb{L}_p(L_q)$, $B_n \to B$ in $\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))$, and $u_n \to u$ in $\mathbb{H}_p(L_q)$ as $n \to \infty$, where u_n is the (unique) strict mild solution to*

$$du_n(t) + Au_n(t)\,dt + f(u_n(t))\,dt = \eta u_n(t)\,dt + B_n(t)\,dW(t), \qquad u_n(0) = u_{0n}.$$

Theorem 2.5. *Let $p > 0$ and $q \geq 2$. Assume that*
(a) *$u_0 \in \mathbb{L}_p(L_q)$;*
(b) *hypothesis* (A_s) *is satisfied for $s = q$ and $s = qd$;*
(c) *$B \in \mathbb{L}_p(L_2(0,T;L_q(\ell_2)))$.*
Then (5) *admits a unique generalized solution $u \in \mathbb{H}_p(L_q)$. Moreover, u has continuous paths and the solution map $u_0 \mapsto u$ is Lipschitz continuous from $\mathbb{L}_p(L_q)$ to $\mathbb{H}_p(L_q)$.*

In order to define generalized solutions to (1) we need some preparations. In particular, we (formally) introduce the map Γ on $\mathbb{H}_p(L_q)$ defined by $\Gamma : v \mapsto w$, where w is the unique generalized solution to

$$dw(t) + Aw(t)\,dt + f(w(t))\,dt = \eta w(t)\,dt + B(t,v(t))\,dW(t), \qquad u(0) = u_0,$$

if it exists. It is easy to see that, if $(\mathrm{B}_{p,q})$ holds, the domain of Γ is the whole $\mathbb{H}_p(L_q)$.

Definition 2.6. *A process $u \in \mathbb{H}_p(L_q)$ is a generalized solution to* (1) *if it is a fixed point of Γ in $\mathbb{H}_p(L_q)$.*

Theorem 2.7. *Let $p > 0$ and $q \geq 2$. Assume that*
(a) *$u_0 \in \mathbb{L}_p(L_q)$;*
(b) *hypothesis* (A_s) *is satisfied for $s = q$ and $s = qd$;*
(c) *hypothesis* $(\mathrm{B}_{p,q})$ *is satisfied.*
Then (1) *admits a unique generalized solution $u \in \mathbb{H}_p(L_q)$. Moreover, u has continuous paths and the solution map $u_0 \mapsto u$ is Lipschitz continuous from $\mathbb{L}_p(L_q)$ to $\mathbb{H}_p(L_q)$.*

The proofs of Theorems 2.5 and 2.7 are given in section 5 below. Note that if $u \in \mathbb{H}_p(L_q)$ is a generalized solution to (1), we cannot claim that $f(u)$ admits a selection g such that $\int_0^t S(t-s)g(s)\,ds$ in (4) is well defined, essentially because we do not have enough integrability for g.

Under additional assumptions we obtain existence of a (unique) solution u for which $f(u)$ admits a selection g satisfying the "minimal" integrability condition $g \in \mathbb{L}_1(L_1(L_1))$.

Definition 2.8. *Let u_0 be an L_q-valued $\mathcal{F}_0$-measurable random variable. A measurable adapted L_q-valued process $u \in L_\infty(L_q)$ is a* mild solution *to* (1) *if there exists $g \in \mathbb{L}_1(L_1(L_1))$, with $g \in f(u)$ m-a.e., and, for all $t \in [0,T]$, $S(t-\cdot)B(\cdot,u)$ is stochastically integrable and* (4) *is satisfied for all $t \in [0,T]$.*

The corresponding well-posedness result holds in a subset of $\mathbb{H}_p(L_q)$ defined in terms of the potential of f, for which we assume that $0 \in f(0)$. We need some definitions first: for $q \geq 2$, let ϕ_q be the homeomorphism of $\mathbb{R}$ defined by $\phi_q : x \mapsto x|x|^{q-2}$, and let $F : \mathbb{R} \to \mathbb{R}$ be the potential of f, i.e., a convex function such that $\partial F = f$, which we "normalize" so that $F(0) = 0$ (in particular $F \geq 0$). Similarly, setting $\tilde{f} := f \circ \phi_q^{-1}$, $\tilde{F}$ stands for the potential of $\tilde{f}$, subject to the same normalization, and $\tilde{F}^*$ for its Legendre–Fenchel conjugate. Finally, we set $\hat{F} := \tilde{F} \circ \phi_q$. A simple computation shows that the convex function $\hat{F}$ is the potential of the maximal monotone graph $x \mapsto f(x)\phi_q'(x)$.

THEOREM 2.9. *Let $p \geq q \geq 2$. Assume that*

(a) $u_0 \in \mathbb{L}_p(L_q)$*;*

(b) *hypothesis* (A_s) *is satisfied for* $s \in \{1, q, qd\}$*;*

(b′) *the resolvent* $R_\lambda := (I + \lambda A)^{-1}$*,* $\lambda > 0$*, is sub-Markovian and such that* $R_\lambda^\sigma(L_1) \subset L_q$ *for some* $\sigma \in \mathbb{N}$*;*

(c) *hypothesis* $(\mathrm{B}_{p,q})$ *is satisfied;*

(d) $0 \in f(0)$ *and* F *is even.*

Then (1) *admits a unique mild solution* $u \in \mathbb{H}_p(L_q)$ *such that* $\hat{F}(u)$*,* $\tilde{F}^*(g) \in \mathbb{L}_1(L_1(L_1))$*. Moreover,* u *has continuous paths and the solution map* $u_0 \mapsto u$ *is Lipschitz continuous from* $\mathbb{L}_p(L_q)$ *to* $\mathbb{H}_p(L_q)$*.*

The proof is given in section 6 below. It should be remarked that unconditional well-posedness in $\mathbb{H}_p(L_q)$, i.e., without any further conditions on u, remains an open problem.

Hypothesis (b) is satisfied by large classes of operators, for instance, all generators of sub-Markovian semigroups on $L_1(D)$ and of symmetric semigroups on $L_2(D)$. Their resolvent is obviously sub-Markovian. The "hypercontractivity" of the resolvent in hypothesis (b′) is satisfied, for example, by nondegenerate second-order elliptic operators, under very mild regularity assumptions on the coefficients, thanks to elliptic regularity results and Sobolev embedding theorems.

We conclude this section with a brief discussion on the relation among the three concepts of solutions: it is clear that, by definition, a strict mild solution is a generalized solution. The opposite implication is obviously false already in the deterministic setting. Again the definitions imply that a strict mild solution is a mild solution but not vice versa. On the other hand, the connection between mild and generalized solutions is subtler: we are going to show (see section 6) that the unique generalized solution is a mild solution and that, under the hypotheses of Theorem 2.9, the mild solution is unique. The proof of uniqueness of mild solutions is necessary because nothing forbids us to imagine that there are other ways of constructing mild solutions, without passing through generalized solutions as we do. It is, however, natural to ask whether a mild solution is a generalized solution. Again by looking at the deterministic situation, the answer is most likely negative. However, it is not difficult to adapt the proofs in section 6.2 to show that, under the assumptions of Theorem 2.9, a mild solution is a generalized solution and hence, a fortiori, unique.

3. Auxiliary results.

An essential tool will be the following Itô formula for the qth power of the L_q norm. For the proof (of a slightly more general case), which is based just on Itô's formula for real processes and Fubini's theorem, we refer to [19].

THEOREM 3.1. *Let u be an L_q-valued process, $q \geq 2$, such that*

$$u(t) = u_0 + \int_0^t b(s)\,ds + \int_0^t G(s)\,dW(s),$$

where

(a) $u_0 : \Omega \to L_q$ *is* $\mathcal{F}_0$*-measurable;*

(b) $b : \Omega \times [0,T] \to L_q$ *is measurable, adapted, and such that* $b \in L_1(0,T;L_q)$*;*

(c) $G = (g^k)$, *with* $g^k : \Omega \times [0,T] \to L_q$ *measurable and adapted for all* k, *satisfies* $G \in L_2(0,T;L_q(\ell_2))$.

Then

$$\begin{aligned}\|u(t)\|_{L_q} = \|u_0\|_{L_q} &+ \int_0^t \bigl\langle q|u(s)|^{q-2}u(s), b(s)\bigr\rangle\,ds\\ &+ \sum_k \int_0^t \langle q|u(s)|^{q-2}u(s), g^k(s)\rangle\,dw^k(s)\\ &+ \frac{1}{2}\int_0^t q(q-1)\bigl\langle |u(s)|^{q-2}, \|G\|^2_{\ell_2}\bigr\rangle\,ds.\end{aligned}$$

The map $\Phi_q := \|\cdot\|^q_{L_q}$ is continuously (Fréchet) differentiable from L_q to $\mathbb{R}$, with

$$\begin{aligned}\Phi_q' : L_q &\longrightarrow \mathscr{L}(L_q,\mathbb{R}) \simeq L_{q'}\\ v &\longmapsto q\bigl\langle |u|^{q-2}u, v\bigr\rangle,\end{aligned}$$

so that the sum of the second and third terms on the right-hand side of the above Itô formula can be concisely written as

$$\int_0^t \Phi_q'(s)b(s)\,ds + \int_0^t \Phi_q'(s)G(s)\,dW(s).$$

This equivalent rewriting will be frequently used. It will also be useful to note that, for any $u \in L_q$,

$$\Phi_q'(u) = q\|u\|^{q-2}_{L_q} J(u),$$

where J is the duality mapping of L_q, i.e., the map from L_q to its dual such that $\langle J(u), u\rangle = \|u\|^2_{L_q}$.

We proceed with a pathwise (i.e., valid $\mathbb{P}$-a.s.) estimate for the L_q norm of solutions to linear equations.

PROPOSITION 3.2. *Let* A *be a linear* m*-accretive operator on* L_q, $q \geq 2$, *and consider the unique mild solution* u *to the equation*

$$du(t) + Au(t) = b(t)\,dt + G(t)\,dW(t), \qquad u(0) = u_0,$$

where u_0, b, *and* G *satisfy the assumptions of Theorem* 3.1. *If* $u \in L_\infty(L_q)$, *then*

$$\begin{aligned}\|u(t)\|^q_{L_q} \leq \|u_0\|^q_{L_q} &+ \int_0^t \Phi_q'(u(s))b(s)\,ds + \int_0^t \Phi_q'(u(s))G(s)\,dW(s)\\ &+ \frac{1}{2}q(q-1)\int_0^t \|G(s)\|^2_{L_q(\ell_2)}\|u(s)\|^{q-2}_{L_q}\,ds.\end{aligned}$$

Proof. For any L_q-valued map h, we shall write, for $\varepsilon > 0$, $h^\varepsilon := (I+\varepsilon A)^{-1}h$. It is not difficult to verify that u^ε is the unique strong solution to

$$du^\varepsilon + Au^\varepsilon = b^\varepsilon\,dt + G^\varepsilon\,dW, \qquad u^\varepsilon(0) = u_0^\varepsilon$$

(cf., e.g., [24, Lemma 6]). Itô's formula then yields[4]

$$\begin{aligned}\big\|u^\varepsilon(t)\big\|_{L_q}^q + \int_0^t \Phi_q'(u^\varepsilon)Au^\varepsilon\,ds = \big\|u_0^\varepsilon\big\|_{L_q}^q + \int_0^t \Phi_q'(u^\varepsilon)b^\varepsilon\,ds + \int_0^t \Phi_q'(u^\varepsilon)G^\varepsilon\,dW\\ + \frac{1}{2}q(q-1)\int_0^t \big\langle |u^\varepsilon|^{q-2}, \|G^\varepsilon\|_{\ell_2}^2\big\rangle\,ds,\end{aligned}$$

where $\Phi_q'(u^\varepsilon)Au^\varepsilon = q\big\|u^\varepsilon\big\|_{L_q}^{q-2}\big\langle Au^\varepsilon, J(u^\varepsilon)\big\rangle \geq 0$ by accretivity of A on L_q, and $\big\|u_0^\varepsilon\big\|_{L_q} \leq \big\|u_0\big\|_{L_q}$ by contractivity of $(I+\varepsilon A)^{-1}$ on L_q. We are thus left with

$$\begin{aligned}\big\|u^\varepsilon(t)\big\|_{L_q}^q \leq \big\|u_0\big\|_{L_q}^q + \int_0^t \Phi_q'(u^\varepsilon)b^\varepsilon\,ds + \int_0^t \Phi_q'(u^\varepsilon)G^\varepsilon\,dW\\ + \frac{1}{2}q(q-1)\int_0^t \big\langle |u^\varepsilon|^{q-2}, \|G^\varepsilon\|_{\ell_2}^2\big\rangle\,ds.\end{aligned}$$

We are now going to pass to the limit as $\varepsilon \to 0$ in this inequality. One clearly has $\big\|u^\varepsilon(t)\big\|_{L_q} \to \big\|u(t)\big\|_{L_q}$ as $\varepsilon \to 0$ because $(I+\varepsilon A)^{-1}$ converges strongly to the identity in $\mathscr{L}(L_q)$ as $\varepsilon \to 0$. By the triangle inequality,

$$\begin{aligned}\sup_{t\leq T}\bigg|\int_0^t \Phi_q'(u^\varepsilon)b^\varepsilon\,ds - \int_0^t \Phi_q'(u)b\,ds\bigg| \leq \int_0^T \big|\big(\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big)b^\varepsilon\big|\,ds\\ + \int_0^T \big|\Phi_q'(u)(b^\varepsilon - b)\big|\,ds.\end{aligned}$$

The following reasoning is to be understood to hold for each fixed ω in a subset of Ω of full $\mathbb{P}$-measure. Since $u^\varepsilon(s) \to u(s)$ and $b^\varepsilon(s) \to b(s)$ in L_q, hence also in measure, for all $s \in [0,T]$, and Φ_q' is continuous, it follows that

$$\big|\big(\Phi_q'(u^\varepsilon(s)) - \Phi_q'(u(s))\big)b^\varepsilon(s)\big| \xrightarrow{\varepsilon\to 0} 0$$

in measure for all s. Moreover,

$$\begin{aligned}\big|\big(\Phi_q'(u^\varepsilon(s)) - \Phi_q'(u(s))\big)b^\varepsilon(s)\big| &\leq \big\|\Phi_q'(u^\varepsilon(s)) - \Phi_q'(u(s))\big\|_{L_{q'}}\big\|b(s)\big\|_{L_q}\\ &\lesssim \big\|u(s)\big\|_{L_q}^{q-1}\big\|b(s)\big\|_{L_q} \leq \big\|u\big\|_{L_\infty(L_q)}^{q-1}\big\|b(s)\big\|_{L_q}\end{aligned}$$

and $\big\|u\big\|_{L_\infty(L_q)}^{q-1}\big\|b\big\|_{L_q} \in L_1(0,T)$, which imply, by the dominated convergence theorem,

$$\int_0^T \big|\big(\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big)b^\varepsilon\big|\,ds \xrightarrow{\varepsilon\to 0} 0.$$

By a completely analogous argument one shows that $\int_0^T\big|\Phi_q'(u)(b^\varepsilon - b)\big|\,ds \xrightarrow{\varepsilon\to 0} 0$ and hence also that

$$\int_0^t \Phi_q'(u^\varepsilon)b^\varepsilon\,ds \xrightarrow{\varepsilon\to 0} \int_0^t \Phi_q'(u)b\,ds.$$

Let us now show that

$$M_\varepsilon(t) := \int_0^t \Phi_q'(u^\varepsilon)G^\varepsilon\,dW \xrightarrow{\varepsilon\to 0} M(t) := \int_0^t \Phi_q'(u)G\,dW$$

[4]From now on we shall occasionally omit the indication of the time parameter, if no confusion may arise, for notational compactness.

in probability. Recall that, for the sequence of continuous local martingales $(M_\varepsilon - M)$, one has $\sup_{t\leq T}|M_\varepsilon(t) - M(t)| \to 0$ in probability if and only if $[M_\varepsilon - M, M_\varepsilon - M](T) \to 0$ in probability (see, e.g., [18, Proposition 17.6]). We have

$$[M_\varepsilon - M, M_\varepsilon - M](T) = \int_0^T \big\|\Phi_q'(u^\varepsilon)G^\varepsilon - \Phi_q'(u)G\big\|_{\ell_2}^2\,ds,$$

where, by the triangle inequality,

$$\big\|\Phi_q'(u^\varepsilon)G^\varepsilon - \Phi_q'(u)G\big\|_{\ell_2} \leq \big\|\big(\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big)G^\varepsilon\big\|_{\ell_2} + \big\|\Phi_q'(u)(G^\varepsilon - G)\big\|_{\ell_2}.$$

By the identification $\mathscr{L}(L_q, \mathbb{R}) \simeq L_{q'}$, Minkowski's and Hölder's inequalities yield

$$\begin{aligned}\big\|\big(\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big)G^\varepsilon\big\|_{\ell_2} &= \bigg\|\int_D \big(\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big)G^\varepsilon\,dx\bigg\|_{\ell_2}\\ &\leq \int_D \big|\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big|\,\big\|G^\varepsilon\big\|_{\ell_2}\,dx\\ &\leq \big\|\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big\|_{L_{q'}}\big\|G^\varepsilon\big\|_{L_q(\ell_2)},\end{aligned}$$

as well as, similarly,

$$\big\|\Phi_q'(u)(G^\varepsilon - G)\big\|_{\ell_2} \leq \big\|\Phi_q'(u)\big\|_{L_{q'}}\big\|G^\varepsilon - G\big\|_{L_q(\ell_2)}.$$

Since $(I + \varepsilon A)^{-1}$ is a contraction on L_q, by a classical theorem of Marcinkiewicz and Zygmund (see, e.g., [15, p. 484]) it is also a contraction on $L_q(\ell_2)$, hence

$$\big\|\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big\|_{L_{q'}}\big\|G^\varepsilon\big\|_{L_q(\ell_2)} \leq \big\|\Phi_q'(u^\varepsilon) - \Phi_q'(u)\big\|_{L_{q'}}\big\|G\big\|_{L_q(\ell_2)} \xrightarrow{\varepsilon\to 0} 0$$

pointwise in the time variable, and $\|G^\varepsilon - G\|_{L_q(\ell_2)} \to 0$ because $(I + \varepsilon A)^{-1}$ converges in the strong operator topology to the identity in $\mathscr{L}(L_q)$. The above also yields

$$\big\|\Phi_q'(u^\varepsilon)G^\varepsilon - \Phi_q'(u)G\big\|_{\ell_2} \lesssim \|u\|_{L_q}^{q-1}\|G\|_{L_q(\ell_2)},$$

where, since $G \in L_2(0,T; L_q(\ell_2))$ and $u \in L_\infty(L_q)$,

$$\int_0^T \|u(s)\|_{L_q}^{2(q-1)}\|G(s)\|_{L_q(\ell_2)}^2\,ds \leq \|u\|_{L_\infty(L_q)}^{2(q-1)}\int_0^T \|G(s)\|_{L_q(\ell_2)}^2\,ds < \infty.$$

Therefore, by the dominated convergence theorem,

$$[M_\varepsilon - M, M_\varepsilon - M](T) \xrightarrow{\varepsilon\to 0} 0$$

in probability. Finally, by Hölder's inequality,

$$\begin{aligned}\int_0^t \big\langle |u(s)|^{q-2}, \|G^\varepsilon(s)\|_{\ell_2}^2\big\rangle\,ds &\leq \int_0^t \big\|G^\varepsilon(s)\big\|_{L_q(\ell_2)}^2\big\|u^\varepsilon(s)\big\|_{L_q}^{q-2}\,ds\\ &\leq \int_0^t \big\|G(s)\big\|_{L_q(\ell_2)}^2\big\|u(s)\big\|_{L_q}^{q-2}\,ds.\end{aligned}$$ ∎

We now establish a maximal inequality for stochastic convolutions that might be interesting in its own right (see Remark 3.4 below). We shall use the following notation, where S stands for the contraction semigroup on L_q generated by $-A$:

$$S \diamond G(t) := \int_0^t S(t-s)G(s)\,dW(s).$$

THEOREM 3.3. *Let $p > 0$ and $q \geq 2$. If G satisfies the hypothesis of Theorem* 3.1, *then the stochastic convolution $S \diamond G$ has (a modification with) continuous paths and*

$$\mathbb{E}\sup_{t\leq T}\big\|S \diamond G(t)\big\|_{L_q}^p \lesssim \mathbb{E}\bigg(\int_0^T \big\|G(s)\big\|_{L_q(\ell_2)}^2\,ds\bigg)^{p/2}.$$

Proof. We proceed in two steps, first assuming that G takes values in $\mathsf{D}(A)$, then removing this assumption.

Step 1. Let us assume for the moment that $G \in L_2(0,T;L_q(\ell_2))$ and $G \in D(A)$ almost everywhere. As in the proof of Proposition 3.2, it follows that $S \diamond G$ is the unique strong solution to

$$du(t) + Au(t)\,dt = G(t)\,dW(t), \qquad u(0) = 0.$$

Then Itô's formula yields

$$\big\|u(t)\big\|_{L_q}^q + \int_0^t \Phi_q'(u)Au\,ds = \int_0^t \Phi_q'(u)G\,dW + \frac{1}{2}q(q-1)\int_0^t \big\langle |u|^{q-2}, \|G\|_{\ell_2}^2\big\rangle\,ds.$$

Setting

$$v := \big\|u\big\|_{L_q}^q, \qquad \sigma := \Phi_q'(u)G,$$
$$b := \frac{1}{2}q(q-1)\big\langle |u|^{q-2}, \|G\|_{\ell_2}^2\big\rangle - \Phi_q'(u)Au,$$

we can write

$$v(t) = \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW(s).$$

Let $\alpha \geq 1$ be arbitrary but fixed. Then $\varphi : x \mapsto x^{2\alpha} \in C^2$ with

$$\varphi'(x) = 2\alpha x^{2\alpha-1}, \qquad \varphi''(x) = 2\alpha(2\alpha-1)x^{2(\alpha-1)}.$$

Therefore, by Itô's formula for real processes,

$$\begin{aligned}\big\|u(t)\big\|_{L_q}^{2\alpha q} = \varphi(v(t)) &= \int_0^t \Big(\varphi'(v(s))b(s) + \frac{1}{2}\varphi''(v(s))\|\sigma(s)\|_{\ell_2}^2\Big)\,ds\\ &\quad + \int_0^t \varphi'(v(s))\sigma(s)\,dW(s),\end{aligned}$$

where

$$\int_0^t \varphi'(v(s))b(s)\,ds = \int_0^t \big\|u(s)\big\|_{L_q}^{(2\alpha-1)q}\Big(\frac{1}{2}q(q-1)\big\langle |u|^{q-2}, \|G\|_{\ell_2}^2\big\rangle - \Phi_q'(u(s))Au(s)\Big)\,ds.$$

Accretivity of A implies

$$\Phi_q'(u)Au = q\|u\|_{L_q}^{q-2}\langle J(u), Au\rangle \geq 0;$$

hence, by Hölder's inequality and Young's inequality[5] in the form

$$xy \leq \varepsilon x^{\frac{\alpha q}{\alpha q-1}} + N(\varepsilon)y^{\alpha q} \qquad \forall x,y \geq 0,\ \varepsilon > 0,$$

[5]From now on, whenever we apply Young's inequality, we shall mostly state only the exponents used.

one has

$$\begin{aligned}\int_0^t \varphi'(v(s))b(s)\,ds &\lesssim \int_0^T \bigl\|u(s)\bigr\|_{L_q}^{2(\alpha q-1)} \bigl\|G(s)\bigr\|_{L_q(\ell_2)}^2\,ds\\ &\leq \bigl\|u\bigr\|_{L_\infty(L_q)}^{2(\alpha q-1)} \int_0^T \bigl\|G(s)\bigr\|_{L_q(\ell_2)}^2\,ds\\ &\leq \varepsilon\bigl\|u\bigr\|_{L_\infty(L_q)}^{2\alpha q} + N(\varepsilon)\biggl(\int_0^T \bigl\|G(s)\bigr\|_{L_q(\ell_2)}^2\,ds\biggr)^{\alpha q}.\end{aligned}$$

Similarly,

$$\int_0^t \varphi''(v(s))\|\sigma(s)\|_{\ell_2}^2\,ds \lesssim \int_0^t \bigl\|u(s)\bigr\|_{L_q}^{2(\alpha-1)q} \bigl\|\sigma(s)\bigr\|_{\ell_2}^2\,ds,$$

where, by a reasoning already used above,

$$\bigl\|g\bigr\|_{\ell_2} = \bigl\|\Phi_q'(u)G\bigr\|_{\ell_2} \leq \bigl\|\Phi_q'(u)\bigr\|_{L_{q'}} \bigl\|G\bigr\|_{L_q(\ell_2)} \lesssim \bigl\|u\bigr\|_{L_q}^{q-1} \bigl\|G\bigr\|_{L_q(\ell_2)},$$

hence, proceeding exactly as before,

$$\begin{aligned}\int_0^t \varphi''(v(s))\|\sigma(s)\|_{\ell_2}^2\,ds &\lesssim \int_0^t \bigl\|u(s)\bigr\|_{L_q}^{2(\alpha q-1)} \bigl\|G(s)\bigr\|_{L_q(\ell_2)}^2\,ds\\ &\leq \varepsilon\bigl\|u\bigr\|_{L_\infty(L_q)}^{2\alpha q} + N(\varepsilon)\biggl(\int_0^T \bigl\|G(s)\bigr\|_{L_q(\ell_2)}^2\,ds\biggr)^{\alpha q}.\end{aligned}$$

Finally, Davis' inequality yields

$$\mathbb{E}\sup_{t\leq T}\biggl|\int_0^t \varphi'(v(s))\sigma(s)\,dW(s)\biggr| \lesssim \mathbb{E}\biggl(\int_0^T \bigl\|\varphi'(v(s))\sigma(s)\bigr\|_{\ell_2}^2\,ds\biggr)^{1/2},$$

where

$$\bigl\|\varphi'(v)g\bigr\|_{\ell_2} \lesssim \bigl\|u\bigr\|_{L_q}^{2\alpha q-1} \bigl\|G\bigr\|_{L_q(\ell_2)},$$

which implies, by Young's inequality with exponents $2\alpha q/(2\alpha q-1)$ and $2\alpha q$,

$$\begin{aligned}\biggl(\int_0^T \bigl\|\varphi'(v(s))\sigma(s)\bigr\|_{\ell_2}^2\,ds\biggr)^{1/2} &\lesssim \bigl\|u\bigr\|_{L_\infty(L_q)}^{2\alpha q-1} \biggl(\int_0^T \bigl\|G(s)\bigr\|_{L_q(\ell_2)}^2\,ds\biggr)^{1/2}\\ &\leq \varepsilon\bigl\|u\bigr\|_{L_\infty(L_q)}^{2\alpha q} + N(\varepsilon)\biggl(\int_0^T \bigl\|G(s)\bigr\|_{L_q(\ell_2)}^2\,ds\biggr)^{\alpha q}.\end{aligned}$$

Taking ε small enough, the claim is proved in the case $p \geq 2q$. The case $0 < p < 2q$ follows by Lenglart's domination inequality (see [22]).

Step 2. Recall that $G^\varepsilon := (I+\varepsilon A)^{-1}G$ is $\mathsf{D}(A)$-valued and converges to G in $L_2(0,T;L_q(\ell_2))$ as $\varepsilon \to 0$, and $u^\varepsilon := S \diamond G^\varepsilon = (S \diamond G)^\varepsilon$ is the unique strong solution to

$$du^\varepsilon(t) + Au^\varepsilon(t)\,dt = G^\varepsilon(t)\,dW(t), \qquad u^\varepsilon(0) = 0.$$

It is elementary to show, by the previous step, that (u^ε) is a Cauchy sequence in $\mathbb{H}_p(L_q)$ and that its limit is a modification of $S \diamond G$. Since u^ε has continuous paths and the convergence in $\mathbb{H}_p(L_q)$ implies almost sure uniform convergence of paths (possibly along a subsequence), we conclude that u has a modification with continuous paths. ∎

Remark 3.4. The previous result is actually a special case of [32], who considered the case of X-valued stochastic convolutions, with X a 2-smooth Banach space. Our proof, although similar in spirit (the idea, as a matter of fact, goes back at least to [30]), is interesting in the sense that it does *not* use any infinite-dimensional calculus. In fact, as already mentioned, Itô's formula for the qth power of the L_q-norm reduces to nothing else but the one-dimensional Itô formula and Fubini's theorem (cf. the proof in [19]).

The estimate for stochastic convolutions just obtained allows one to prove well-posedness in the strict mild sense for equations with a Lipschitz continuous drift.

PROPOSITION 3.5. *Let $p > 0$ and $q \geq 2$. Assume that hypotheses* (A_q) *and* $(\mathrm{B}_{p,q})$ *are verified. If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $u_0 \in \mathbb{L}_p(L_q)$, then the equation*

$$du(t) + Au(t)\,dt + f(u(t))\,dt = B(t,u(t))\,dW(t), \qquad u(0) = u_0,$$

admits a unique strict mild solution $u \in \mathbb{H}_p(L_q)$ with continuous paths, and the solution map $u_0 \mapsto u$ is Lipschitz continuous from $\mathbb{L}_p(L_q)$ to $\mathbb{H}_p(L_q)$.

Proof. Since the proof proceeds by the classical fixed point argument, we omit some simple details. Consider the map, formally defined for the moment,

$$\Gamma : (u_0, u) \mapsto S(t)u_0 - \int_0^t S(t-s)f(u(s))\,ds + \int_0^t S(t-s)B(u(s))\,dW(s).$$

To prove existence and uniqueness, it suffices to show that $\Gamma(u_0, \cdot)$ is an everywhere defined contraction on $\mathbb{H}_p(L_q)$ for any $u_0 \in \mathbb{L}_p(L_q)$. One has, denoting the Lipschitz constant of f by C_f,

$$\begin{aligned}\sup_{t\leq T}&\Big\|\int_0^t S(t-s)\big(f(u(s)) - f(v(s))\big)\,ds\Big\|_{L_q} \\ &\leq \int_0^T \big\|f(u(s)) - f(v(s))\big\|_{L_q}\,ds \leq TC_f \sup_{t\leq T}\big\|u(t) - v(t)\big\|_{L_q},\end{aligned}$$

hence, writing $S * f(u)$ to denote the second term in the above definition of Γ,

$$\big\|S * (f(u) - f(v))\big\|_{\mathbb{H}_p(L_q)} \leq TC_f\big\|u - v\big\|_{\mathbb{H}_p(L_q)}.$$

Similarly, denoting the Lipschitz constant of B by C_B, it follows by Theorem 3.3 that

$$\begin{aligned}\mathbb{E}\sup_{t\leq T}&\Big\|\int_0^t S(t-s)\big(B(u(s)) - B(v(s))\big)\,dW(s)\Big\|^p_{L_q} \\ &\lesssim \mathbb{E}\Big(\int_0^T \big\|B(u(s) - B(v(s))\big\|^2_{L_q(\ell_2)}\,ds\Big)^{p/2} \\ &\leq T^{p/2}C_B^p\big\|u - v\big\|^p_{\mathbb{H}_p(L_q)},\end{aligned}$$

i.e., for a constant N independent of T,

$$\big\|S \diamond (B(u) - B(v))\big\|_{\mathbb{H}_p(L_q)} \leq NT^{1/2}\,C_B\big\|u - v\big\|_{\mathbb{H}_p(L_q)}.$$

Therefore, choosing T small enough, one finds a constant $c \in \,]0,1[$ such that

$$\big\|\Gamma(u_0, u) - \Gamma(u_0, v)\big\|_{\mathbb{H}_p(L_q)} \leq c\big\|u - v\big\|_{\mathbb{H}_p(L_q)}.$$

It is clear that $\Gamma(u_0, \mathbb{H}_p(L_q)) \subset \mathbb{H}_p(L_q)$. Recalling that the function

$$\begin{aligned} d : \mathbb{H}_p(L_q) \times \mathbb{H}_p(L_q) &\to \mathbb{R}_+ \\ (x, y) &\mapsto \big\|x - y\big\|^{1\wedge p}_{\mathbb{H}_p(L_q)} \end{aligned}$$

is a metric on $\mathbb{H}_p(L_q)$, Banach's contraction principle yields the existence of a unique fixed point of $\Gamma(u_0, \cdot)$ on the complete metric space $(\mathbb{H}_p(L_q), d)$, which is the unique strict mild solution we are looking for on the interval $[0, T]$. Writing $u = \Gamma(u_0, u)$, $v = \Gamma(v_0, v)$, the Lipschitz continuity of the solution map follows by $c < 1$ and

$$\begin{aligned} \big\|u - v\big\|_{\mathbb{H}_p(L_q)} &= \big\|\Gamma(u_0, u) - \Gamma(v_0, v)\big\|_{\mathbb{H}_p(L_q)} \\ &\le \big\|\Gamma(u_0, u) - \Gamma(u_0, v)\big\|_{\mathbb{H}_p(L_q)} + \big\|\Gamma(u_0, v) - \Gamma(v_0, v)\big\|_{\mathbb{H}_p(L_q)} \\ &\le c\big\|u - v\big\|_{\mathbb{H}_p(L_q)} + \big\|u_0 - v_0\big\|_{\mathbb{L}_p(L_q)}. \end{aligned}$$

By a classical patching argument, the smallness restriction on T can be removed. Continuity of paths follows by Theorem 3.3. ◻

Remark 3.6. Even though quite sophisticated well-posedness results exist for SEEs on L_q spaces with Lipschitz continuous coefficients (cf., e.g., [9, 31]), the previous simple proposition does not seem to follow from the existing literature. For instance, in *op. cit.* the semigroup S is assumed to be analytic (but not necessarily accretive), and (in [31]) solutions are sought in spaces strictly contained in $\mathbb{H}_p(L_q)$, and $p > 2$. It may indeed be possible to deduce the above well-posedness result from *op. cit.*, but it seems much easier to give a direct proof.

4. Strict mild solutions.

Consider the regularized equation

$$du_\lambda(t) + Au_\lambda(t)\,dt + f_\lambda(u_\lambda(t))\,dt = \eta u_\lambda(t)\,dt + B(t, u_\lambda(t))\,dW(t), \qquad u_\lambda(0) = u_0, \tag{6}$$

where $f_\lambda : \mathbb{R} \to \mathbb{R}$, $\lambda > 0$, is the Yosida approximation of f, defined as

$$f_\lambda := \frac{1}{\lambda}\big(I - (I + \lambda f)^{-1}\big).$$

As is well known (see, e.g., [1, pp. 99, 106–107]), f_λ is Lipschitz continuous on $\mathbb{R}$, as well as on L_q (when viewed as a superposition operator). Thanks to Proposition 3.5, well-posedness of (6) in the strict mild sense holds in $\mathbb{H}_p(L_q)$ for all $p > 0$ and $q \ge 2$.

We now proceed to considering (1). In this section we first show that a priori estimates on u_λ imply well-posedness of (1), then obtain such estimates (under additional assumptions on A and B), thus proving Theorem 2.3. Our argument depends on passing to the limit as $\lambda \to 0$ in the mild form of the regularized equation (6).

4.1. A priori estimates imply well-posedness.

We begin establishing sufficient conditions for (u_λ) to be a Cauchy sequence in $\mathbb{H}_p(L_q)$, whose limit is then a natural candidate as solution to (1).

It will be useful to consider the space $\mathbb{H}_p(X)$, with X a separable Banach space, endowed with the equivalent (quasi-)norm

$$\|u\|_{\mathbb{H}_{p,\alpha}(X)} := \Big(\mathbb{E}\sup_{t\le T}\big\|e^{-\alpha t}u(t)\big\|^p_X\Big)^{1/p}, \qquad \alpha \in \mathbb{R}_+,$$

and denoted by $\mathbb{H}_{p,\alpha}(X)$.

LEMMA 4.1. *Let $p > 0$, $q \geq 2$, $p^* := p(2d+q-2)/q$, and assume that hypotheses* (A_q) *and* $(\mathrm{B}_{p,q})$ *are satisfied. If the sequence (u_λ) is bounded in $\mathbb{H}_{p^*}(L_{2d+q-2})$, then (u_λ) is a Cauchy sequence in $\mathbb{H}_p(L_q)$.*

Proof. Let us define, for a constant parameter $\alpha > \eta$ to be chosen later, $v_\lambda(t) := e^{-\alpha t}u_\lambda(t)$ for all $t \geq 0$, so that

$$dv_\lambda(t) = -\alpha v_\lambda(t) + e^{-\alpha t}u_\lambda(t),$$

hence also, for $\mu > 0$,

$$\begin{aligned} d(v_\lambda - v_\mu) + \Big((\alpha-\eta)(v_\lambda - v_\mu) + A(v_\lambda - v_\mu) + e^{-\alpha t}\big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big)\Big)\,dt \\ = e^{-\alpha t}\big(B(u_\lambda) - B(u_\mu)\big)\,dW, \end{aligned} \tag{7}$$

in the (strict) mild sense, with initial condition $v_\lambda(0) - v_\mu(0) = 0$. Proposition 3.2 yields

$$\begin{aligned} &\big\|v_\lambda(t) - v_\mu(t)\big\|_{L_q}^q + q(\alpha-\eta)\int_0^t \|v_\lambda - v_\mu\|_{L_q}^q\,ds \\ &\qquad + \int_0^t e^{-\alpha s}\Phi_q'(v_\lambda - v_\mu)\big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big)\,ds \\ &\quad \leq \int_0^t e^{-\alpha s}\Phi_q'(v_\lambda - v_\mu)\big(B(u_\lambda) - B(u_\mu)\big)\,dW \\ &\qquad + \frac{1}{2}q(q-1)\int_0^t \big\|e^{-\alpha s}\big(B(u_\lambda) - B(u_\mu)\big)\big\|_{L_q(\ell_2)}^2 \big\|v_\lambda - v_\mu\big\|_{L_q}^{q-2}\,ds. \end{aligned} \tag{8}$$

We are going to estimate each term appearing in this inequality. Note that $\Phi_q'(cx) = c^{q-1}\Phi_q'(x)$ for all $c \in \mathbb{R}_+$ and $x \in L_q$, hence

$$\Phi_q'(v_\lambda - v_\mu)\big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big) = e^{-\alpha(q-1)s}\,\Phi_q'(u_\lambda - u_\mu)\big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big)$$

and

$$\Phi_q'(u_\lambda - u_\mu)\big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big) = q\int_D |u_\lambda - u_\mu|^{q-2}(u_\lambda - u_\mu)\big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big)\,dx,$$

where, setting $J_\lambda := (I + \lambda f)^{-1}$, $\lambda > 0$, and writing

$$\begin{aligned} u_\lambda - u_\mu &= u_\lambda - J_\lambda u_\lambda + J_\lambda u_\lambda - J_\mu u_\mu + J_\mu u_\mu - u_\mu \\ &= \lambda f_\lambda(u_\lambda) + J_\lambda u_\lambda - J_\mu u_\mu - \mu f_\mu(u_\mu), \end{aligned}$$

one has, by monotonicity of f and recalling that $f_\lambda = f \circ J_\lambda$,

$$\begin{aligned} \big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big)(u_\lambda - u_\mu) &\geq \big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big)\big(\lambda f_\lambda(u_\lambda) - \mu f_\mu(u_\mu)\big) \\ &\geq \lambda\big|f_\lambda(u_\lambda)\big|^2 + \mu\big|f_\mu(u_\mu)\big|^2 - (\lambda+\mu)\big|f_\lambda(u_\lambda)\big|\big|f_\mu(u_\mu)\big| \\ &\geq -\frac{\mu}{2}\big|f_\lambda(u_\lambda)\big|^2 - \frac{\lambda}{2}\big|f_\mu(u_\mu)\big|^2 \\ &\geq -\frac{1}{2}(\lambda+\mu)\big(\big|f_\lambda(u_\lambda)\big|^2 + \big|f_\mu(u_\mu)\big|^2\big). \end{aligned}$$

Moreover, since $|f_\lambda(x)| \leq |f(x)| \lesssim 1 + |x|^d$ for all $x \in \mathbb{R}$ and $|x-y|^{q-2} \lesssim \big(|x|+|y|\big)^{q-2}$ for all x, $y \in \mathbb{R}$ (the latter inequality holds because $q \geq 2$), one infers

$$
\begin{aligned}
&\big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big)(u_\lambda - u_\mu)|u_\lambda - u_\mu|^{q-2} \\
&\qquad \gtrsim -(\lambda+\mu)\big(1 + |u_\lambda|^{2d} + |u_\mu|^{2d}\big)|u_\lambda - u_\mu|^{q-2} \\
&\qquad \gtrsim -(\lambda+\mu)\Big(1 + \big(|u_\lambda| + |u_\mu|\big)^{2d}\Big)\big(|u_\lambda| + |u_\mu|\big)^{q-2} \\
&\qquad \gtrsim -(\lambda+\mu)\Big(1 + |u_\lambda|^{2d+q-2} + |u_\mu|^{2d+q-2}\Big),
\end{aligned}
$$

thus also

$$
\begin{aligned}
&\int_0^t e^{-\alpha s}\Phi_q'(v_\lambda - v_\mu)\big(f_\lambda(u_\lambda) - f_\mu(u_\mu)\big)\,ds \\
&\qquad \gtrsim -(\lambda+\mu)\int_0^t e^{-q\alpha s}\Big(1 + \big\|u_\lambda\big\|_{L_{2d+q-2}}^{2d+q-2} + \big\|u_\mu\big\|_{L_{2d+q-2}}^{2d+q-2}\Big)\,ds \\
&\qquad \gtrsim -(\lambda+\mu)\,\frac{1-e^{-q\alpha t}}{q\alpha}\Big(1 + \sup_{s\le t}\big\|u_\lambda(s)\big\|_{L_{2d+q-2}}^{2d+q-2} + \sup_{s\le t}\big\|u_\mu(s)\big\|_{L_{2d+q-2}}^{2d+q-2}\Big),
\end{aligned}
$$

which estimates the third term on the left-hand side of (8).

The Lipschitz continuity of B implies that the integrand in the last term on the right-hand side of (8) is estimated by

$$
C_B^2 e^{-q\alpha s}\,\big\|u_\lambda - u_\mu\big\|_{L_q}^q = C_B^2\,\big\|v_\lambda - v_\mu\big\|_{L_q}^q.
$$

In particular, collecting the second term on the right-hand side and the second term on the left-hand side of (8), we obtain

$$
\begin{aligned}
&\big\|v_\lambda(t) - v_\mu(t)\big\|_{L_q}^q + q\big(\alpha - \eta - C_B^2(q-1)/2\big)\int_0^t \|v_\lambda - v_\mu\|_{L_q}^q\,ds \\
&\qquad \lesssim (\lambda+\mu)\,\frac{1-e^{-q\alpha t}}{q\alpha}\Big(1 + \sup_{s\le t}\big\|u_\lambda(s)\big\|_{L_{2d+q-2}}^{2d+q-2} + \sup_{s\le t}\big\|u_\mu(s)\big\|_{L_{2d+q-2}}^{2d+q-2}\Big) \\
&\qquad\quad + \bigg|\int_0^t e^{-\alpha s}\Phi_q'(v_\lambda - v_\mu)\big(B(u_\lambda) - B(u_\mu)\big)\,dW\bigg|.
\end{aligned}
$$

For any $\alpha > 1/q$, raising both sides to the power p/q, taking supremum in time,[6] then expectation, one gets, setting $p^* := (2d+q-2)p/q$,

$$
\begin{aligned}
&\mathbb{E}\sup_{t\le T}\big\|v_\lambda(t) - v_\mu(t)\big\|_{L_q}^p \\
&\qquad + q^{p/q}\Big(\alpha - \eta - \frac{1}{2}C_B^2(q-1)\Big)^{p/q}\mathbb{E}\Big(\int_0^T \big\|v_\lambda - v_\mu\big\|_{L_q}^q\,ds\Big)^{p/q} \\
&\qquad \lesssim (\lambda+\mu)^{p/q}\,\Big(1 + \mathbb{E}\sup_{t\le T}\big\|u_\lambda(t)\big\|_{L_{2d+q-2}}^{p^*} + \mathbb{E}\sup_{t\le T}\big\|u_\mu(t)\big\|_{L_{2d+q-2}}^{p^*}\Big) \\
&\qquad\quad + \mathbb{E}\sup_{t\le T}\bigg|\int_0^t e^{-\alpha s}\Phi_q'(v_\lambda - v_\mu)\big(B(u_\lambda) - B(u_\mu)\big)\,dW\bigg|^{p/q},
\end{aligned}
\tag{9}
$$

where, by the Burkholder–Davis–Gundy inequality,

[6] Note that $A(t) + B(t) \le C(t)$ for all t, with A, B, C positive functions of t, implies $\sup_t A(t) \le \sup_t C(t)$ and $\sup_t B(t) \le \sup_t C(t)$, hence $\sup_t A(t) + \sup_t B(t) \le 2\sup_t C(t)$.

$$\mathbb{E}\sup_{t\le T}\biggl|\int_0^t e^{-\alpha s}\Phi_q'(v_\lambda-v_\mu)\bigl(B(u_\lambda)-B(u_\mu)\bigr)\,dW\biggr|^{p/q} \lesssim \mathbb{E}\biggl(\int_0^T \bigl\|e^{-\alpha s}\Phi_q'(v_\lambda-v_\mu)\bigl(B(u_\lambda)-B(u_\mu)\bigr)\bigr\|_{\ell_2}^2\,ds\biggr)^{\frac{p}{2q}}.$$

By an argument based on Minkowski's and Hölder's inequalities already used several times, and the Lipschitz continuity of B, one has

$$\begin{aligned}\int_0^T &\bigl\|e^{-\alpha s}\Phi_q'(v_\lambda-v_\mu)\bigl(B(u_\lambda)-B(u_\mu)\bigr)\bigr\|_{\ell_2}^2\,ds\\ &\le \int_0^T \bigl\|\Phi_q'(v_\lambda-v_\mu)\bigr\|_{L_{q'}}^2\bigl\|e^{-\alpha s}\bigl(B(u_\lambda)-B(u_\mu)\bigr)\bigr\|_{L_q(\ell_2)}^2\,ds\\ &\lesssim \int_0^T \bigl\|v_\lambda-v_\mu\bigr\|_{L_q}^{2(q-1)}\bigl\|e^{-\alpha s}\bigl(B(u_\lambda)-B(u_\mu)\bigr)\bigr\|_{L_q(\ell_2)}^2\,ds\\ &\le C_B^2\sup_{t\le T}\bigl\|v_\lambda-v_\mu\bigr\|_{L_q}^{2(q-1)}\int_0^T\bigl\|v_\lambda-v_\mu\bigr\|_{L_q}^2\,ds.\end{aligned}$$

This implies

$$\begin{aligned}\mathbb{E}\sup_{t\le T}&\biggl|\int_0^t e^{-\alpha s}\Phi_q'(v_\lambda-v_\mu)\bigl(B(u_\lambda)-B(u_\mu)\bigr)\,dW\biggr|^{p/q}\\ &\lesssim C_B^{p/q}\mathbb{E}\sup_{t\le T}\bigl\|v_\lambda-v_\mu\bigr\|_{L_q}^{p(q-1)/q}\biggl(\int_0^T\bigl\|v_\lambda-v_\mu)\bigr\|_{L_q}^2\,ds\biggr)^{\frac{p}{2q}}\\ &\le \varepsilon C_B^{p/q}\mathbb{E}\sup_{t\le T}\bigl\|v_\lambda-v_\mu\bigr\|_{L_q}^p+N_1(\varepsilon)C_B^{p/q}\mathbb{E}\biggl(\int_0^T\bigl\|v_\lambda-v_\mu)\bigr\|_{L_q}^2\,ds\biggr)^{\frac{p}{2}}\\ &\le \varepsilon C_B^{p/q}\mathbb{E}\sup_{t\le T}\bigl\|v_\lambda-v_\mu\bigr\|_{L_q}^p+N_1(\varepsilon)C_B^{p/q}T^{1-\frac{2}{q}}\mathbb{E}\biggl(\int_0^T\bigl\|v_\lambda-v_\mu)\bigr\|_{L_q}^q\,ds\biggr)^{\frac{p}{q}},\end{aligned}$$

where we have used Young's inequality with exponents $q/(q-1)$ and q in the second-last step, and Hölder's inequality with exponents $q/2$ and $q/(q-2)$ in the last step (recall that $q\ge 2$). By (9), we conclude that there exist constants N_2, N_3, independent of λ, μ, and α, with N_2 also independent of ε, such that

$$\begin{aligned}\mathbb{E}\sup_{t\le T}&\bigl\|v_\lambda(t)-v_\mu(t)\bigr\|_{L_q}^p+q^{p/q}\Bigl(\alpha-\eta-\frac{1}{2}(q-1)C_B^2\Bigr)^{p/q}\mathbb{E}\biggl(\int_0^T\bigl\|v_\lambda-v_\mu\bigr\|_{L_q}^q\,ds\biggr)^{p/q}\\ &\le \varepsilon N_2\mathbb{E}\sup_{t\le T}\bigl\|v_\lambda(t)-v_\mu(t)\bigr\|_{L_q}^p+N_3\mathbb{E}\biggl(\int_0^T\bigl\|v_\lambda-v_\mu\bigr\|_{L_q}^q\,ds\biggr)^{p/q}\\ &\quad+(\lambda+\mu)^{p/q}\ \biggl(1+\mathbb{E}\sup_{t\le T}\bigl\|u_\lambda(t)\bigr\|_{L_{2d+q-2}}^{p^*}+\mathbb{E}\sup_{t\le T}\bigl\|u_\mu(t)\bigr\|_{L_{2d+q-2}}^{p^*}\biggr).\end{aligned}$$

It is immediately seen that, choosing ε small enough and α large enough, we are left with

$$\mathbb{E}\sup_{t\le T}\bigl\|v_\lambda(t)-v_\mu(t)\bigr\|_{L_q}^p\lesssim(\lambda+\mu)^{p/q}\ \biggl(1+\mathbb{E}\sup_{t\le T}\bigl\|u_\lambda(t)\bigr\|_{L_{2d+q-2}}^{p^*}+\mathbb{E}\sup_{t\le T}\bigl\|u_\mu(t)\bigr\|_{L_{2d+q-2}}^{p^*}\biggr),$$

which implies, by the boundedness of (u_λ) in $\mathbb{H}_{p^*}(L_{2d+q-2})$, that (u_λ) is a Cauchy sequence in $\mathbb{H}_{p,\alpha}(L_q)$, hence also in $\mathbb{H}_p(L_q)$ by equivalence of (quasi-)norms. ∎

The strong convergence of u_λ to a process $u \in \mathbb{H}_p(L_q)$ just established does not seem sufficient, unfortunately, to prove that u is a strict mild solution to (1). In fact, writing the regularized equation (6) in its integral form

$$
\begin{aligned}
(10)\quad u_\lambda(t) &+ \int_0^t S(t-s) f_\lambda(u_\lambda(s))\,ds \\
&= S(t)u_0 + \eta \int_0^t S(t-s) u_\lambda(s)\,ds + \int_0^t S(t-s) B(s, u_\lambda(s))\,dW(s),
\end{aligned}
$$

difficulties appear, as is natural to expect, when trying to pass to the limit in the integral on the left-hand side. We are going to show that boundedness assumptions on (u_λ) in a smaller space imply convergence of the term containing $f_\lambda(u_\lambda)$ in a suitable norm, which is turn yields well-posedness in the strict mild sense. First we state and prove a Lipschitz continuity result for the solution map $u_0 \mapsto u$ of strict mild solution, which immediately implies uniqueness.

Lemma 4.2. *Let u_1, u_2 be strict mild solutions in $\mathbb{H}_p(L_q)$ to (1) with initial conditions u_{01} and u_{02}, respectively. Then*

$$\|u_1 - u_2\|_{\mathbb{H}_p(L_q)} \lesssim \|u_{01} - u_{02}\|_{\mathbb{L}_p(L_q)}.$$

In particular, if (1) admits a strict mild solution $u \in \mathbb{H}_p(L_q)$, then it is unique and the solution map is Lipschitz continuous from $\mathbb{L}_p(L_q)$ to $\mathbb{H}_p(L_q)$.

Proof. We use again an argument based on Itô's formula and elementary inequalities. By definition of a strict mild solution, we have $f(u_1)$, $f(u_2) \in L_1(L_q)$. Therefore, from

$$
\begin{aligned}
&d(u_1 - u_2) + A(u_1 - u_2)\,dt + \big(f(u_1) - f(u_2)\big)\,dt \\
&\quad = \eta(u_1 - u_2)\,dt + \big(B(u_1) - B(u_2)\big)\,dW, \qquad u_1(0) - u_2(0) = u_{01} - u_{02},
\end{aligned}
$$

and Proposition 3.2, it follows that

$$
\begin{aligned}
&\big\|v_1(t) - v_2(t)\big\|_{L_q}^q + q(\alpha - \eta) \int_0^t \|v_1 - v_2\|_{L_q}^q\,ds \\
&\qquad \le \big\|u_{01} - u_{02}\big\|_{L_q}^q + M(t) \\
&\qquad\quad + \frac{1}{2} q(q-1) \int_0^t \big\|e^{-\alpha s}\big(B(u_1) - B(u_2)\big)\big\|_{L_q(\ell_2)}^2 \big\|v_1 - v_2\big\|_{L_q}^{q-2}\,ds,
\end{aligned}
$$

where $v_i := e^{-\alpha\cdot} u_i$, $i = 1, 2$, and

$$M(t) := \int_0^t e^{-\alpha s} \Phi_q'(v_1(s) - v_2(s))\big(B(u_1(s)) - B(u_2(s))\big)\,dW(s).$$

By the Lipschitz continuity of B,

$$\big\|e^{-\alpha s}\big(B(u_1) - B(u_2)\big)\big\|_{L_q(\ell_2)}^2 \big\|v_1 - v_2\big\|_{L_q}^{q-2} \le C_B^2 \big\|v_1 - v_2\big\|_{L_q}^q,$$

hence

$$
\begin{aligned}
&\big\|v_1(t) - v_2(t)\big\|_{L_q}^q + q\Big(\alpha - \eta - \frac{1}{2}(q-1)C_B^2\Big) \int_0^t \|v_1 - v_2\|_{L_q}^q\,ds \\
&\qquad \le \big\|u_{01} - u_{02}\big\|_{L_q}^q + M(t),
\end{aligned}
$$

and we choose α so that $\big(\alpha - \eta - (q-1)C_B^2/2\big) > 0$. Taking suprema in time, raising to the power p/q, taking expectation, and raising to the power $1/p$, we get

$$\big\|v_1 - v_2\big\|_{\mathbb{H}_p(L_q)} + q^{1/q}\Big(\alpha - \eta - \frac{1}{2}(q-1)C_B^2\Big)^{1/q}\big\|v_1 - v_2\big\|_{\mathbb{L}_p(L_q(L_q)))} \lesssim \big\|u_{01} - u_{02}\big\|_{\mathbb{L}_p(L_q)} + \big\|M_T^*\big\|_{\mathbb{L}_{p/q}}^{1/q},$$

where the implicit constant depends only on p and q, and $M_T^* := \sup_{t\le T}|M_t|$. The Burkholder–Davis–Gundy inequality yields

$$\begin{aligned}\big\|M_T^*\big\|_{\mathbb{L}_{p/q}}^{1/q} &\lesssim \big\|[M,M]_T^{1/2}\big\|_{\mathbb{L}_{p/q}}^{1/q} = \big\|[M,M]_T^{1/(2q)}\big\|_{\mathbb{L}_p}\\ &\lesssim \bigg\|\bigg(\int_0^T \big\|v_1 - v_2\big\|_{L_q}^{2(q-1)}\big\|e^{-\alpha s}(B(u_1) - B(u_2))\big\|_{L_q(\ell_2)}^2\bigg)^{\frac{1}{2q}}\bigg\|_{\mathbb{L}_p}\\ &\lesssim \big\|v_1 - v_2\big\|_{\mathbb{L}_p(L_{2q}(L_q))}\\ &\le \varepsilon\big\|v_1 - v_2\big\|_{\mathbb{H}_p(L_q)} + N(\varepsilon)\big\|v_1 - v_2\big\|_{\mathbb{L}_p(L_q(L_q))},\end{aligned}$$

where we have used the Lipschitz continuity of B and the inequality

$$\|\phi\|_{L_{2q}} \le \|\phi\|_{L_q}^{1/2}\|\phi\|_{L_\infty}^{1/2} \le \varepsilon\|\phi\|_{L_\infty} + N(\varepsilon)\|\phi\|_{L_q} \qquad \forall \phi \in L_q \cap L_\infty.$$

We are thus left with

$$\big\|v_1 - v_2\big\|_{\mathbb{H}_p(L_q)} + q^{1/q}\Big(\alpha - \eta - \frac{1}{2}(q-1)C_B^2\Big)^{1/q}\big\|v_1 - v_2\big\|_{\mathbb{L}_p(L_q(L_q)))} \lesssim \big\|u_{01} - u_{02}\big\|_{\mathbb{L}_p(L_q)} + \varepsilon\big\|v_1 - v_2\big\|_{\mathbb{H}_p(L_q)} + N(\varepsilon)\big\|v_1 - v_2\big\|_{\mathbb{L}_p(L_q(L_q))}.$$

Since the implicit constant is independent of α and ε, this implies, upon choosing α large enough and ε small enough, and recalling that the (quasi-)norms $\|\cdot\|_{\mathbb{H}_{p,\alpha}(L_q)}$ and $\|\cdot\|_{\mathbb{H}_p(L_q)}$ are equivalent,

$$\big\|u_1 - u_2\big\|_{\mathbb{H}_p(L_q)} \eqsim_T \big\|u_1 - u_2\big\|_{\mathbb{H}_{p,\alpha}(L_q)} = \big\|v_1 - v_2\big\|_{\mathbb{H}_p(L_q)} \lesssim \big\|u_{01} - u_{02}\big\|_{\mathbb{L}_p(L_q)}. \qquad \square$$

To prove uniqueness of solutions in $\mathbb{H}_p(L_q)$ we have used in a crucial way the condition $f(u) \in L_1(L_q)$, which allows one to apply Proposition 3.2 (i.e., to use Itô's formula). It is thus natural to look for conditions ensuring weak compactness of $f_\lambda(u_\lambda)$ in a functional space contained in $\mathbb{L}_0(L_1(L_q))$. This is the motivation for the following well-posedness result, conditional on boundedness of (u_λ) in a suitable norm.

PROPOSITION 4.3. *Let $p > 0$, $q \ge 2$ and $p^* := p(2d+q-2)/q > d$. If the sequence (u_λ) is bounded in $\mathbb{H}_{p^*}(L_{qd})$, then* (1) *admits a unique strict mild solution $u \in \mathbb{H}_p(L_q)$ with continuous paths, and $u_0 \mapsto u$ is Lipschitz continuous from $\mathbb{L}_p(L_q)$ to $\mathbb{H}_p(L_q)$.*

Proof. Since $d \ge 1$ implies $qd \ge 2d + q - 2$ and $L_{qd} \hookrightarrow L_{2d+q-2}$, it follows by Lemma 4.1 that u_λ converges strongly to $u \in \mathbb{H}_p(L_q)$ as $\lambda \to 0$. We are going to pass to the limit as $\lambda \to 0$ in a mild form of (6), i.e., in (10) above. The convergence

$$\int_0^t S(t-s)B(u_\lambda(s))\,dW(s) \xrightarrow{\lambda\to 0} \int_0^t S(t-s)B(u(s))\,dW(s)$$

in probability for all $t \le T$ follows by Theorem 3.3. In fact, by the contractivity of S and the Lipschitz continuity of B, one has

$$\mathbb{E}\Big\|\int_0^t S(t-r)\big(B(u_\lambda(r))-B(u(r))\big)\,dW(r)\Big\|_{L_q}^p \lesssim \mathbb{E}\biggl(\int_0^T \big\|u_\lambda(r)-u(r)\big\|_{L_q}^2\,dr\biggr)^{p/2},$$

where the last term tends to zero as $\lambda \to 0$ because $u_\lambda \to u$ in $\mathbb{H}_p(L_q)$ and $\mathbb{H}_p(L_q) \hookrightarrow \mathbb{L}_p(L_2(L_q))$.

We can now consider the term in (10) involving $f_\lambda(u_\lambda)$. It follows from $|f_\lambda| \leq |f|$ and $|f(x)| \lesssim 1+|x|^d$ for all $x \in \mathbb{R}$ that, for any $s > 1$,

$$\big\|f_\lambda(u_\lambda)\big\|_{\mathbb{L}_{p^*/d}(L_s(L_q))} \lesssim 1 + \big\|u_\lambda\big\|_{\mathbb{H}_{p^*}(L_{qd})}^d,$$

so that $f_\lambda(u_\lambda) = f(J_\lambda u_\lambda)$ is bounded, hence weakly compact, in the reflexive Banach space $E := \mathbb{L}_{p^*/d}(L_s(L_q))$ (recall that $p^*/d > 1$ by assumption). In particular, there exists $g \in E$ and a subsequence of λ, denoted by the same symbol, such that $f(J_\lambda u_\lambda) \to g$ weakly in E as $\lambda \to 0$. Since $J_\lambda u_\lambda \to u$ strongly in E as $\lambda \to 0$ and f, as an m-accretive operator on E, is also strongly-weakly closed thereon, we deduce that $g \in f(u)$ m-a.e. Since the linear operator

$$\phi \mapsto \int_0^\cdot S(\cdot - s)\phi(s)\,ds$$

is strongly (hence also weakly) continuous on E, we infer that

$$\int_0^\cdot S(\cdot - s)f_\lambda(u_\lambda(s))\,ds \xrightarrow{\lambda\to 0} \int_0^\cdot S(\cdot - s)g(s)\,ds$$

weakly in E and hence that

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\big(g(s) - \eta u(s)\big)\,ds + \int_0^t S(t-s)B(u(s))\,dW(s)$$

for almost all $t \in [0,T]$. However, since u admits a continuous L_q-valued modification, the identity must be satisfied for all $t \in [0,T]$. Existence is thus proved, and uniqueness as well as continuous dependence on the initial datum follow by the previous lemma. The mild solution u, being a strong limit in $\mathbb{H}_p(L_q)$ of (u_λ), inherits the path continuity of the latter. ∎

4.2. A priori estimates. As we have just seen, well-posedness in the strict mild sense in $\mathbb{H}_p(L_q)$ for (1) can be reduced to obtaining a priori estimates for (u_λ) in $\mathbb{H}_{p_1}(L_{q_1})$, with $p_1 > p$ and $q_1 > q$ suitably chosen.

PROPOSITION 4.4. *Let $p > 0$ and $q \geq 2$. If $u_0 \in \mathbb{L}_p(L_q)$ is $\mathcal{F}_0$-measurable and hypotheses* (A_q), $(\mathrm{B}_{p,q})$ *are satisfied, then there exists a constant N, independent of λ, such that*

$$\mathbb{E}\sup_{t\leq T}\big\|u_\lambda(t)\big\|_{L_q}^p \leq N\Big(1+\mathbb{E}\big\|u_0\big\|_{L_q}^p\Big).$$

Proof. The proof uses arguments analogous to ones already seen, hence we omit some detail. As in previous proofs, we begin observing that the regularized equation (6) admits a unique L_q-valued solution u_λ, and, setting $v_\lambda(t) := e^{-\alpha t}u_\lambda(t)$ for all $t \geq 0$, with $\alpha > \eta$ a constant to be fixed later, Proposition 3.2 implies

$$\begin{aligned}\big\|v_\lambda(t)\big\|_{L_q}^q + q(\alpha-\eta)\int_0^t \|v_\lambda\|_{L_q}^q\,ds + \int_0^t e^{-\alpha s}\Phi_q'(v_\lambda)f_\lambda(u_\lambda)\,ds \\ \leq \int_0^t e^{-\alpha s}\Phi_q'(v_\lambda)B(u_\lambda)\,dW \\ + \frac{1}{2}q(q-1)\int_0^t \big\|e^{-\alpha s}B(u_\lambda)\big\|_{L_q(\ell_2)}^2\big\|v_\lambda\big\|_{L_q}^{q-2}\,ds.\end{aligned}$$

We shall denote the stochastic integral in the previous inequality by M. By the homogeneity of order $q-1$ of Φ_q', the monotonicity of f_λ, the inequality $|f_\lambda| \leq |f|$, the identity $\|\Phi_q'(x)\|_{L_{q'}} = q\|x\|_{L_q}^{q-1}$, and the elementary inequality $a^{q-1} \leq 1 + a^q$ for all $a \geq 0$, we have

$$\begin{aligned}e^{-\alpha s}\Phi_q'(v_\lambda)f_\lambda(u_\lambda) &= e^{-q\alpha s}\Phi_q'(u_\lambda)\big(f_\lambda(u_\lambda) - f_\lambda(0) + f_\lambda(0)\big) \\ &\geq e^{-q\alpha s}\Phi_q'(u_\lambda)f_\lambda(0) \geq -e^{-q\alpha s}\Phi_q'(u_\lambda)|f(0)| \\ &\geq -qe^{-q\alpha s}|f(0)|\|u_\lambda\|_{L_q}^{q-1} \\ &\geq -qe^{-q\alpha s}|f(0)| - qe^{-q\alpha s}|f(0)|\|u_\lambda\|_{L_q}^q,\end{aligned}$$

hence

$$\int_0^t e^{-\alpha s}\Phi_q'(v_\lambda)f_\lambda(u_\lambda)\,ds \geq -\frac{|f(0)|}{\alpha}(1-e^{-q\alpha t}) - q|f(0)|\int_0^t \big\|v_\lambda\big\|_{L_q}^q\,ds.$$

Similarly, by the triangle inequality and Lipschitz continuity of B,

$$\big\|e^{-\alpha s}B(u_\lambda)\big\|_{L_q(\ell_2)}^2 \leq 2C_B^2\big\|v_\lambda\big\|_{L_q}^2 + 2e^{-2\alpha s}\big\|B(0)\big\|_{L_q(\ell_2)}^2,$$

hence, thanks to the elementary inequality $a^{q-2} \leq 1 + a^q$, $a \geq 0$,

$$\begin{aligned}\frac{1}{2}q(q-1)\int_0^t \big\|e^{-\alpha s}B(u_\lambda)\big\|_{L_q(\ell_2)}^2\big\|v_\lambda\big\|_{L_q}^{q-2}\,ds \\ \leq q(q-1)C_B^2\int_0^t \big(1 + e^{-2\alpha s}\big\|B(0)\big\|_{L_q(\ell_2)}^2\big)\big\|v_\lambda\big\|_{L_q}^q\,ds \\ + \frac{q(q-1)}{2\alpha}\big\|B(0)\big\|_{L_q(\ell_2)}^2(1-e^{-2\alpha t}).\end{aligned}$$

We can thus write

$$\begin{aligned}\big\|v_\lambda(t)\big\|_{L_q}^q + q\big(\alpha - \eta - |f(0)| - (q-1)C_B^2\big)\int_0^t \|v_\lambda\|_{L_q}^q\,ds \\ \leq \frac{|f(0)|}{\alpha} + \frac{q(q-1)}{2\alpha}\big\|B(0)\big\|_{L_q(\ell_2)}^2 + M_T^*,\end{aligned}$$

and we choose the constant α larger than $\eta + |f(0)| + (q-1)C_B^2$. Raising to the power p/q, taking suprema, then expectation, and taking the power $1/p$, we are left with

$$\big\|v_\lambda\big\|_{\mathbb{H}_p(L_q)} + N_1\big\|v_\lambda\big\|_{\mathbb{L}_p(L_q(L_q))} \lesssim N_2 + \big\|M_T^*\big\|_{\mathbb{L}_{p/q}}^{1/q},$$

where

$$\begin{aligned}N_1 &:= q^{1/q}\Big(\alpha - \eta - |f(0)| - (q-1)C_B^2\Big)^{1/q}, \\ N_2 &:= \Big(\frac{|f(0)|}{\alpha} + \frac{q(q-1)}{2\alpha}\big\|B(0)\big\|_{L_q(\ell_2)}^2\Big)^{1/q},\end{aligned}$$

and, by an argument based on the Burkholder–Davis–Gundy inequality and norm interpolation, as in the proof of Lemma 4.2,

$$\begin{aligned}\big\|M_T^*\big\|_{\mathbb{L}_{p/q}}^{1/q} &\lesssim \big\|[M,M]_T^{1/2}\big\|_{\mathbb{L}_{p/q}}^{1/q} = \big\|[M,M]_T^{\frac{1}{2q}}\big\|_{\mathbb{L}_p}\\ &\lesssim \varepsilon\big\|v_\lambda\big\|_{\mathbb{H}_p(L_q)} + N_3(\varepsilon)\big\|v_\lambda\big\|_{\mathbb{L}_p(L_q(L_q))}.\end{aligned}$$

Collecting estimates yields

$$\begin{aligned}&\big\|v_\lambda\big\|_{\mathbb{H}_p(L_q)} + N_1(\alpha)\big\|v_\lambda\big\|_{\mathbb{L}_p(L_q(L_q))}\\ &\qquad\le N_4\Big(N_2 + \varepsilon\big\|v_\lambda\big\|_{\mathbb{H}_p(L_q)} + N_3(\varepsilon)\big\|v_\lambda\big\|_{\mathbb{L}_p(L_q(L_q))}\Big)\end{aligned}$$

for a constant N_4 independent of α and ε. The proof is completed choosing first ε small enough, then α large enough, and appealing to the equivalence of the norms of $\mathbb{H}_p(L_q)$ and $\mathbb{H}_{p,\alpha}(L_q)$. □

4.3. Proof of Theorem 2.3. Since $u_0 \in \mathbb{L}_{p^*}(L_{qd})$ and hypotheses (A_{qd}), $(\mathrm{B}_{p^*,qd})$ are satisfied, the regularized equation (6) admits a unique L_{qd}-valued (strict) mild solution u_λ for all $\lambda > 0$. By Proposition 4.4 the sequence (u_λ) is bounded in $\mathbb{H}_{p^*}(L_{qd})$; hence Proposition 4.3 allows us to conclude that (1) admits a unique strict mild solution $u \in \mathbb{H}_p(L_q)$ and that the solution map $u_0 \mapsto u$ is Lipschitz continuous from $\mathbb{L}_p(L_q)$ to $\mathbb{H}_p(L_q)$.

5. Generalized solutions. In this section we prove Theorems 2.5 and 2.7. The main tool is the Lipschitz continuity of the map $(u_0, B) \mapsto u$ established in the next lemma. For reasons of notational compactness, we set $L_{r,\alpha}(0,T;X) := L_r([0,T],\mu;X)$, where μ is the measure on $[0,T]$ with density $t \mapsto e^{-r\alpha t}$.

LEMMA 5.1. *Assume that $p > 0$ and $q \ge 2$. Let u_1, $u_2 \in \mathbb{H}_p(L_q)$ be strict mild solutions to*

$$du_1 + Au_1\,dt + f(u_1)\,dt = \eta u_1\,dt + B_1\,dW, \qquad u_1(0) = u_{01},$$

and

$$du_2 + Au_2\,dt + f(u_2)\,dt = \eta u_2\,dt + B_2\,dW, \qquad u_2(0) = u_{02},$$

respectively, where B_1, $B_2 \in \mathbb{L}_p(L_2(0,T;L_q(\ell_2)))$ and u_{01}, $u_{02} \in \mathbb{L}_p(L_q)$. Then, for any $\alpha > \eta$,

$$\big\|u_1 - u_2\big\|_{\mathbb{H}_{p,\alpha}(L_q)} \lesssim \big\|u_{01} - u_{02}\big\|_{\mathbb{L}_p(L_q)} + \big\|B_1 - B_2\big\|_{\mathbb{L}_p(L_{2,\alpha}(0,T;L_q(\ell_2)))}. \tag{11}$$

In particular,

$$\big\|u_1 - u_2\big\|_{\mathbb{H}_p(L_q)} \lesssim \big\|u_{01} - u_{02}\big\|_{\mathbb{L}_p(L_q)} + \big\|B_1 - B_2\big\|_{\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))}. \tag{12}$$

Moreover, the same estimates hold for generalized solutions.

Proof. The proof uses again arguments analogous to those used in the proof of Lemma 4.2, therefore some detail will be omitted.

Setting $v_i(t) := e^{-\alpha t}u_i(t)$, $i = 1,2$, for all $t \ge 0$, with $\alpha > \eta$, it follows by Proposition 3.2 and monotonicity of f,

$$\begin{aligned}\big\|v_1(t)-v_2(t)\big\|_{L_q}^q + q(\alpha-\eta)\int_0^t \|v_1-v_2\|_{L_q}^q\,ds \\ \le \big\|u_{01}-u_{02}\big\|_{L_q}^q + \int_0^t e^{-\alpha s}\Phi_q'(v_1-v_2)\big(B_1-B_2\big)\,dW \\ + \frac{1}{2}q(q-1)\int_0^t \big\|e^{-\alpha s}\big(B_1-B_2\big)\big\|_{L_q(\ell_2)}^2 \big\|v_1-v_2\big\|_{L_q}^{q-2}\,ds,\end{aligned}$$

where, by Young's inequality with exponents $q/(q-2)$ and $q/2$, the last term is estimated by

$$\begin{aligned}\sup_{s\le t}\big\|v_1(s)-v_2(s)\big\|_{L_q}^{q-2}\int_0^t e^{-2\alpha s}\big\|B_1-B_2\big\|_{L_q(\ell_2)}^2\,ds \\ \le \varepsilon \sup_{s\le t}\big\|v_1(s)-v_2(s)\big\|_{L_q}^q + N(\varepsilon)\biggl(\int_0^t e^{-2\alpha s}\big\|B_1-B_2\big\|_{L_q(\ell_2)}^2\,ds\biggr)^{q/2}.\end{aligned}$$

Choosing ε smaller than one, hence, as before, raising to the power p/q, taking suprema in time, then expectation, and finally power $1/p$, we get

$$\begin{aligned}\big\|v_1-v_2\big\|_{\mathbb{H}_p(L_q)} + (\alpha-\eta)^{1/q}\big\|v_1-v_2\big\|_{\mathbb{L}_p(L_q(L_q))} \\ \lesssim \big\|u_{01}-u_{02}\big\|_{\mathbb{L}_p(L_q)} + \big\|B_1-B_2\big\|_{\mathbb{L}_p(L_{2,\alpha}(0,T;L_q(\ell_2)))} + \big\|M_T^*\big\|_{\mathbb{L}_{p/q}}^{1/q},\end{aligned}$$

where M denotes the stochastic integral with respect to W in the first inequality above. Applying the Burkholder–Davis–Gundy inequality, elementary estimates, and Young's inequality with exponents $q/(q-1)$ and q, we have

$$\begin{aligned}\big\|M_T^*\big\|_{\mathbb{L}_{p/q}}^{1/q} &\lesssim \big\|[M,M]_T^{\frac{1}{2q}}\big\|_{\mathbb{L}_p} \\ &= \biggl\|\biggl(\int_0^T \big\|v_1-v_2\big\|_{L_q}^{2(q-1)}\big\|B_1-B_2\big\|_{L_q(\ell_2)}^2 e^{-2\alpha s}\,ds\biggr)^{\frac{1}{2q}}\biggr\|_{\mathbb{L}_p} \\ &\lesssim \varepsilon\big\|v_1-v_2\big\|_{\mathbb{H}_p(L_q)} + N(\varepsilon)\big\|B_1-B_2\big\|_{\mathbb{L}_p(L_{2,\alpha}(0,T;L_q(\ell_2)))}.\end{aligned}$$

Choosing ε suitably small, the last two inequalities yield

$$\big\|v_1-v_2\big\|_{\mathbb{H}_p(L_q)} \lesssim \big\|u_{01}-u_{02}\big\|_{\mathbb{L}_p(L_q)} + \big\|B_1-B_2\big\|_{\mathbb{L}_p(L_{2,\alpha}(0,T;L_q(\ell_2)))},$$

which, recalling the equivalence of the norms in $\mathbb{H}_{p,\alpha}(L_q)$, $\alpha \ge 0$, establishes the claim because

$$\begin{aligned}\big\|u_1-u_2\big\|_{\mathbb{H}_{p,\alpha}(L_q)} &= \big\|v_1-v_2\big\|_{\mathbb{H}_p(L_q)}, \\ \big\|u_1-u_2\big\|_{\mathbb{L}_p(L_{q,\alpha}(L_q))} &= \big\|v_1-v_2\big\|_{\mathbb{L}_p(L_q(L_q))}.\end{aligned}$$

It is easily seen that estimates (11) and (12) are stable with respect to passage to the limit, hence they remain true for generalized solutions. ∎

Proof of Theorem 2.5. Let $p_1 \ge p$ be such that

$$p_1^* := \frac{p_1}{q}(2d+q-2) > d.$$

Note that $d \geq 1$ implies $p_1^* \geq p_1 \geq p$, hence $\mathbb{L}_{p_1^*}(L_{qd})$ is dense in $\mathbb{L}_p(L_q)$, so that there exists a sequence

$$(u_{0n})_{n\in\mathbb{N}} \subset \mathbb{L}_{p_1^*}(L_{qd})$$

such that $u_{0n} \to u_0$ in $\mathbb{L}_p(L_q)$ as $n \to \infty$. Similarly, there also exists a sequence

$$(B_n)_{n\in\mathbb{N}} \subset \mathbb{L}_{p_1^*}(L_2(0,T;L_{qd}(\ell_2)))$$

such that $B_n \to B$ in $\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))$ as $n \to \infty$. Then, by Theorem 2.3, for each $n \in \mathbb{N}$ the equation

$$du_n(t) + Au_n(t)\,dt + f(u_n(t))\,dt = \eta u(t)\,dt + B_n(t)\,dW(t), \qquad u_n(0) = u_{0n},$$

admits a unique strict mild solution $u_n \in \mathbb{H}_{p_1}(L_q) \hookrightarrow \mathbb{H}_p(L_q)$, and the previous lemma yields

$$\|u_n - u_m\|_{\mathbb{H}_p(L_q)} \lesssim \|u_{0n} - u_{0m}\|_{\mathbb{L}_p(L_q)} + \|B_n - B_m\|_{\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))},$$

hence (u_n) is a Cauchy sequence in $\mathbb{H}_p(L_q)$. This implies that its strong limit $u \in \mathbb{H}_p(L_q)$ is a generalized solution to (5). Uniqueness and Lipschitz dependence on the initial datum follow immediately by (11). ∎

Proof of Theorem 2.7. Let $w_1, w_2 \in \mathbb{H}_p(L_q)$ and consider the equation, for $i = 1, 2$,

$$du_i(t) + Au_i(t)\,dt + f(u_i(t))\,dt = \eta u_i(t)\,dt + B(w_i(t))\,dW(t), \qquad u_i(0) = u_{0i}.$$

The assumptions on B immediately imply that $B(w) \in \mathbb{L}_p(L_2(0,T;L_q(\ell_2)))$ for all $w \in \mathbb{H}_p(L_q)$, hence the previous equation admits a unique generalized solution $u_i \in \mathbb{H}_p(L_q)$ by Theorem 2.5. In particular, the domain of the map Γ is the whole $\mathbb{H}_p(L_q)$ and its image is contained in $\mathbb{H}_p(L_q)$. We are now going to show that Γ is a contraction in $\mathbb{H}_{p,\alpha}(L_q)$ for small T. In fact, inequality (11) yields

$$\|u_1 - u_2\|_{\mathbb{H}_{p,\alpha}(L_q)} \lesssim \|u_{01} - u_{02}\|_{\mathbb{L}_p(L_q)} + \|B(w_1) - B(w_2)\|_{\mathbb{L}_p(L_{2,\alpha}(0,T;L_q(\ell_2)))},$$

where, by the Lipschitz continuity of B,

$$\begin{aligned}
&\|B(w_1) - B(w_2)\|^p_{\mathbb{L}_p(L_{2,\alpha}(0,T;L_q(\ell_2)))}\\
&\qquad = \mathbb{E}\biggl(\int_0^T e^{-2\alpha s}\|B(w_1) - B(w_2)\|^2_{L_q(\ell_2)}\,ds\biggr)^{p/2}\\
&\qquad \leq C_B^p\,\mathbb{E}\biggl(\int_0^T e^{-2\alpha s}\|w_1 - w_2\|^2_{L_q}\,ds\biggr)^{p/2}\\
&\qquad \leq C_B^p T^{p/2}\|w_1 - w_2\|^p_{\mathbb{H}_{p,\alpha}(L_q)}.
\end{aligned}$$

This implies that Γ is a contraction on $\mathbb{H}_{p,\alpha}(L_q)$ for T small enough, hence that a unique generalized solution exists that depends Lipschitz continuously on the initial datum. By a classical patching procedure, the result can be extended to arbitrary finite T. ∎

6. Mild solutions. In this section we prove Theorem 2.9. The proof is split into two parts: first we prove existence, showing that one can pass to the limit in the term of (10) containing $f_\lambda(u_\lambda)$ in the weak topology of $\mathbb{L}_1(L_1(L_1))$. Then we prove uniqueness, as a consequence of continuous dependence on the initial datum, via an extension of Proposition 3.2. We proceed this way because, as will be apparent soon, the symmetry condition on F and the "regularizing" assumptions on A are needed only to prove uniqueness.

The proofs in this section rely on some results of convex analysis whose proofs can be found, for instance, in [17, Chapter E].

6.1. Existence. We shall use the following weak convergence criterion (see [7, Theorem 18]).

LEMMA 6.1. *Let $(Y, \mathcal{A}, \mu)$ be a finite measure space. Assume that γ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $\mathsf{D}(\gamma) = \mathbb{R}$ and $0 \in \gamma(0)$. If the sequences of functions (z_n), $(y_n) \subset L_0(Y, \mathcal{A}, \mu)$ indexed by $n \in \mathbb{N}$ are such that $\lim_{n\to\infty} y_n = y$ μ-a.e., $z_n \in \gamma(y_n)$ μ-a.e. for all n, and there exists a constant N such that*

$$\int_Y z_n y_n \, d\mu < N \qquad \forall n \in \mathbb{N},$$

then there exist $z \in L_1(Y, \mu)$ and a subsequence $(n_k)_k$ such that $z_{n_k} \to z$ weakly in $L_1(Y, \mu)$ as $k \to \infty$ and $z \in \gamma(y)$ μ-a.e.

Sketch of proof. The weak compactness in $L_1(Y, \mu)$ of (z_n) is a consequence of $z_n y_n = G(y_n) + G^*(z_n)$, where G is a convex function with $G(0) = 0$ such that γ is the subdifferential of G and G^* is the conjugate of G. In fact, $\mathsf{D}(\gamma) = \mathbb{R}$ implies that G^* is superlinear at infinity, which in turn implies, by the criterion of de la Vallée-Poussin (see, e.g., [5, Theorem 4.5.9]), that (z_n) is uniformly integrable in $L_1(Y, \mu)$; hence, by the Dunford–Pettis theorem, it is relatively weakly compact thereon (see, e.g., [5, Corollary 4.7.19]). The fact that $z \in \gamma(y)$ μ-a.e. requires a further (short) argument based on monotonicity (see [7] for details). ∎

Let us also introduce some further notation. For $q \geq 2$, define the homeomorphism ϕ_q of $\mathbb{R}$ and the maximal monotone graph $\tilde{f}$ in $\mathbb{R} \times \mathbb{R}$ as

$$\phi_q : x \mapsto x|x|^{q-2} = |x|^{q-1} \operatorname{sgn} x, \qquad \tilde{f} := f \circ \phi_q^{-1}.$$

Since $0 \in \tilde{f}(0)$, there exists a convex function $\tilde{F} : \mathbb{R} \to \mathbb{R}$ with $\tilde{F}(0) = 0$ such that $\partial \tilde{F} = \tilde{f}$, where ∂ stands for the subdifferential (in the sense of convex analysis). As usual, we shall denote the convex conjugate of $\tilde{F}$ by $\tilde{F}^*$. We recall that $\tilde{F}^*$ is convex and superlinear at infinity, because $\tilde{f}$ is finite on the whole real line.

In the next statement g stands for the process defined in Definition 2.8.

PROPOSITION 6.2. *Let $p \geq q \geq 2$ and $0 \in f(0)$. Assume that*

(a) *$u_0 \in \mathbb{L}_p(L_q)$;*

(b) *hypothesis (A_s) is satisfied for $s \in \{1, q, qd\}$;*

(c) *hypothesis $(\mathrm{B}_{p,q})$ is satisfied.*

Then there exists a mild solution $u \in \mathbb{H}_p(L_q)$ to (1). Moreover, u has continuous paths and satisfies $\hat{F}(u)$, $\tilde{F}^*(g) \in \mathbb{L}_1(L_1(L_1))$.

Proof. We proceed in several steps.

Step 1. We begin showing that the generalized solution to (1), which exists and is unique thanks to Theorem 2.7, can be approximated by strict mild solutions to suitable equations. Let u be the generalized solution to (1) and $\delta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\|u - u_n\|_{\mathbb{H}_p(L_q)} < \delta$ for all $n > n_0$, where u_n is the unique generalized solution to

$$du_n + Au_n\,dt + f(u_n)\,dt = \eta u_n\,dt + B(u_{n-1})\,dW, \qquad u_n(0) = u_0.$$

In turn, for any $n > n_0$, there exists $\nu = \nu(n)$ and $u_{0\nu}$, $\big[B(u_{n-1})\big]_\nu$ such that

$$\|u_n - u_n^\nu\|_{\mathbb{H}_p(L_q)} \lesssim \|u_0 - u_{0\nu}\|_{\mathbb{L}_p(L_q)} + \big\|[B(u_{n-1})]_\nu - B(u_{n-1})\big\|_{\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))} < \delta,$$

where u_n^ν is the unique strict mild solution to

$$du_n^\nu + Au_n^\nu\,dt + f(u_n^\nu)\,dt = \eta u_n^\nu\,dt + \big[B(u_{n-1})\big]_\nu\,dW, \qquad u_n^\nu(0) = u_{0\nu}.$$

In particular, by the triangle inequality, one can construct a sequence of strict mild solutions $\bar{u}_n := u_n^{\nu(n)}$ such that $\|u - \bar{u}_n\|_{\mathbb{H}_p(L_q)}$ is less than (a constant times) δ, i.e., being δ arbitrary, the sequence $\bar{u}_n := u_n^{\nu(n)}$ converges to u in $\mathbb{H}_p(L_q)$.

Step 2. We are now going to show that there exists a constant N, independent of n, such that

$$\mathbb{E}\int_0^T \Phi_q'(\bar{u}_n(s))\bar{g}_n(s)\,ds < N,$$

where $\bar{g}_n \in f(\bar{u}_n)$ m-a.e. is the selection of $f(\bar{u}_n)$ appearing in the definition of strict mild solution. Let $\bar{B}_n := [B(u_{n-1})]_{\nu(n)}$ and $v_n(t) := e^{-\alpha t}\bar{u}_n(t)$ for all $t \in [0,T]$, with $\alpha > \eta$ a constant. Proposition 3.2 and obvious estimates yield

$$\begin{aligned}\big\|v_n\big\|^q_{L_\infty(0,T;L_q)} + q(\alpha-\eta)\int_0^T \|v_n(s)\|^q_{L_q}\,ds + \int_0^T e^{-\alpha s}\Phi_q'(v_n(s))\bar{g}_n(s)\,ds \\ \lesssim \big\|u_{0n}\big\|^q_{L_q} + \sup_{t\le T}\bigg|\int_0^t e^{-\alpha s}\Phi_q'(v_n(s))\bar{B}_n(s)\,dW\bigg| \\ + \frac{1}{2}q(q-1)\int_0^T \big\|e^{-\alpha s}\bar{B}_n(s)\big\|^2_{L_q(\ell_2)}\big\|v_n(s)\big\|^{q-2}_{L_q}\,ds.\end{aligned}$$

As before, let us denote the stochastic integral on the right-hand side by M. Appealing to Young's inequality with exponents $q/(q-2)$ and $q/2$, the last term can be estimated by

$$\varepsilon\big\|v_n\big\|^q_{L_\infty(0,T;L_q)} + N(\varepsilon)\big\|\bar{B}_n\big\|^q_{L_2(0,T;L_q(\ell_2))},$$

so that, choosing ε small enough, raising to the power p/q, and taking expectation, we obtain

$$\begin{aligned}\big\|v_n\big\|^p_{\mathbb{H}_p(L_q)} + \mathbb{E}\bigg(\int_0^T e^{-\alpha s}\Phi_q'(v_n(s))\bar{g}_n(s)\,ds\bigg)^{p/q} \\ \lesssim \big\|u_{0n}\big\|^p_{\mathbb{L}_p(L_q)} + \mathbb{E}(M_T^*)^{p/q} + \big\|\bar{B}_n\big\|^p_{\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))}.\end{aligned}$$

By an argument already used before, based on the Burkholder–Davis–Gundy inequality and Young's inequality with exponents $q/(q-1)$ and q, we have

$$\mathbb{E}(M_T^*)^{p/q} \lesssim \varepsilon\big\|v_n\big\|^p_{\mathbb{H}_p(L_q)} + N(\varepsilon)\big\|\bar{B}_n\big\|^p_{\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))},$$

thus also, choosing ε small enough,

$$\mathbb{E}\biggl(\int_0^T e^{-\alpha s}\Phi_q'(v_n(s))\bar{g}_n(s)\,ds\biggr)^{p/q} \lesssim \|u_{0n}\|^p_{\mathbb{L}_p(L_q)} + \|\bar{B}_n\|^p_{\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))},$$

where the first term on the right-hand side is bounded because, by definition of a generalized solution, u_{0n} converges to u_0 in $\mathbb{L}_p(L_q)$. Moreover, denoting the norm of $\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))$ by $\|\cdot\|$ for simplicity,

$$\|\bar{B}_n - B(u_{n-1})\| = \|[B(u_{n-1})]_{\nu(n)} - B(u_{n-1})\| < \delta,$$

hence $\|\bar{B}_n\| < \|B(u_{n-1})\| + \delta$, where, by the Lipschitz continuity of B,

$$\begin{aligned}\|B(u_{n-1})\| &\le \|B(u_{n-1}) - B(0)\| + \|B(0)\|\\ &\lesssim \|u_{n-1}\|_{\mathbb{H}_p(L_q)} + \|B(0)\|.\end{aligned}$$

Since $u_n \to u$ in $\mathbb{H}_p(L_q)$ as $n \to \infty$, it follows that $\|u_n\|_{\mathbb{H}_p(L_q)}$ is bounded, which in turn implies that $\|\bar{B}_n\|$ is also bounded. We conclude that there exists a constant N, independent of n, such that

$$\mathbb{E}\biggl(\int_0^T e^{-\alpha s}\Phi_q'(v_n(s))\bar{g}_n(s)\,ds\biggr)^{p/q} < N.$$

Since $p/q \ge 1$, it follows by Jensen's inequality that

$$\mathbb{E}\int_0^T e^{-\alpha s}\Phi_q'(v_n(s))\bar{g}_n(s)\,ds < N$$

(where N might differ from the previous one). The proof is finished observing that $e^{-\alpha s}\Phi_q'(v_n(s)) = e^{-q\alpha s}\Phi_q'(\bar{u}_n(s))$ and $e^{-q\alpha s} \ge e^{-q\alpha T}$ for all $s \in [0,T]$.

Step 3. We are going to show that the uniform estimate of the previous step implies that the generalized solution u is also a mild solution to (1). The conclusion of the previous step can equivalently be written as

$$\int_\Xi \phi_q(\bar{u}_n)\bar{g}_n\,dm < N,$$

where $\Xi := \Omega \times [0,T] \times D$, and N is a constant independent of n. Since ϕ_q is a homeomorphism of $\mathbb{R}$, setting $v_n := \phi_q(\bar{u}_n)$ and recalling the definition $\tilde{f} := f \circ \phi_q^{-1}$, we have (see, e.g., [6, p. II.12] about the associativity of composition of graphs)

$$\bar{g}_n \in f(\bar{u}_n) = f \circ \phi_q^{-1}\bigl(\phi_q(\bar{u}_n)\bigr) = \tilde{f}(v_n);$$

hence the previous estimate can be written as

$$\int_\Xi v_n\bar{g}_n\,dm < N,$$

where $\bar{g}_n \in \tilde{f}(v_n)$ m-a.e. Since, by continuity of ϕ_q, $v_n = \phi_q(u_n) \to \phi_q(u) =: v$ m-a.e. as $n \to \infty$ and $\mathsf{D}\bigl(f \circ \phi_q^{-1}\bigr) = \mathbb{R}$, Lemma 6.1 implies that there exists $g \in L_1(m)$ and a subsequence (n') of (n) such that $\bar{g}_{n'} \to g$ weakly in $L_1(m)$ as $n' \to \infty$, and

$$g \in f \circ \phi_q^{-1}(v) = f \circ \phi_q^{-1}(\phi_q(u)) = f(u) \quad m\text{-a.e.}$$

Since $-A$ generates a C_0-semigroup of contractions on $L_1(D)$ by assumption, one obtains

$$\int_0^t S(t-s)\tilde{g}_n(s)\,ds \to \int_0^t S(t-s)g(s)\,ds$$

weakly in $\mathbb{L}_1(L_1(L_1))$ as $\lambda \to 0$, by a reasoning completely analogous to that used in the last part of the proof of Proposition 4.3. Similarly, convergence of the stochastic convolutions follows as in the proof just mentioned because

$$\left\|\bar{B}_n - B(u)\right\|_{\mathbb{L}_p(L_2(0,T;L_q(\ell_2)))} \xrightarrow{n\to\infty} 0.$$

Step 4. It remains to prove that $\hat{F}(u)$, $\tilde{F}^*(g) \in \mathbb{L}_1(L_1(L_1))$. In fact, we have $\bar{g}_n v_n = \tilde{F}(v_n) + \tilde{F}^*(\tilde{g}_n)$ because $g_n \in \tilde{f}(v_n) = \partial\tilde{F}(v_n)$; hence, by the previous step, there exists a constant N, independent of n, such that

$$\int_\Xi \tilde{F}(v_n)\,dm < N, \qquad \int_\Xi \tilde{F}^*(g_n)\,dm < N.$$

The convexity of $\tilde{F}$ and $\tilde{F}^*$ implies the weak lower semicontinuity in $L_1(m)$ of

$$\phi \mapsto \int_\Xi \tilde{F}(\phi)\,dm, \qquad \phi \mapsto \int_\Xi \tilde{F}^*(\phi)\,dm,$$

hence $\tilde{F}^*(g) \in L_1(m)$ because $\bar{g}_n \to g$ weakly in $L_1(m)$. Moreover, $v_n \to v$ in m-measure and

$$\|v_n\|_{L_1(m)} = \|\bar{u}_n\|_{L_{q-1}(m)} \to \|u_n\|_{L_{q-1}(m)} = \|v\|_{L_1(m)},$$

by virtue of the strong convergence $\bar{u}_n \to u$ in $\mathbb{H}_p(L_q)$ and the embedding $\mathbb{H}_p(L_q) \hookrightarrow L_{q-1}(m)$. We thus have $v_n \to v$ strongly in $L_1(m)$, hence, similarly as above, $\hat{F}(u) = \tilde{F}(v) \in L_1(m)$. ∎

Remark 6.3. If $p < q$, the above proof does not work because Jensen's inequality reverses. However, a weaker integrability result can still be obtained. Namely, again by Jensen's inequality, we have

$$\int_\Xi \left|f_n\phi_q(u_n)\right|^{p/q} d\mu < N,$$

uniformly over n, where the constant N depends also on the Lebesgue measure of D. Setting $x^{\langle a\rangle} := |x|^a \operatorname{sgn} x$ for all $x \in \mathbb{R}$ and $a > 0$, taking into account that $0 \in f(0)$, the previous estimate can equivalently be written as

$$\int_\Xi f_n^{\langle p/q\rangle}\phi_q^{\langle p/q\rangle}(u_n)\,d\mu < N.$$

For any $a > 0$ the function $x \mapsto x^{\langle a\rangle}$ is a homeomorphism of $\mathbb{R}$, hence the function $\psi_{p,q} : x \mapsto \phi_q^{\langle p/q\rangle}(x)$ is also a homeomorphism of $\mathbb{R}$. We clearly have

$$f_n^{\langle p/q\rangle} \in f^{\langle p/q\rangle} \circ \psi_{p,q}^{-1}\big(\psi_{p,q}(u_n)\big)$$

μ-a.e., hence there exists $z \in L_1(\mu)$ such that $f_{n_k}^{\langle p/q\rangle} \to z$ weakly in $L_1(\mu)$ along a subsequence (n_k), with $z \in f^{\langle p/q\rangle} \circ \psi_{p,q}^{-1}\big(\psi_{p,q}(u)\big) = f^{\langle p/q\rangle}(u)$. We thus have, for a generalized solution, that $\zeta \in f(u)$ is only in $L_{p/q}(\mu)$, rather than in $L_1(\mu)$. This in particular implies that it does not seem possible any longer to claim that u is a mild solution, even in a very weak sense, as the semigroup S is not defined in $L_q(D)$ spaces with $0 < q < 1$.

6.2. Uniqueness. The aim of this subsection is to prove continuous dependence on the initial datum (from which uniqueness follows immediately) for mild solutions to (1), *without* assuming that $g \in L_1(L_q)$. We need to assume, however, the same integrability conditions on the solution that are established in the proof of Proposition 6.2, as well as positivity and regularizing properties for the resolvent of A and a symmetry condition on F.

The key is the following estimate for the difference of two mild solutions to (1), whose proof is inspired by an analogous result, in a different setting, in [2].

LEMMA 6.4. *Under the hypotheses of Theorem* 2.9*, assume that* u_i*,* $i = 1, 2$*, satisfies*

$$u_i(t) + \int_0^t S(t-s)\big(g_i(s) - \eta u_i(s)\big)\,ds = S(t)u_{0i} + \int_0^t S(t-s)B(u_i(s))\,dW(s)$$

for all $t \in [0,T]$*, where* $g_i \in \mathbb{L}_1(L_1(L_1))$*,* $g_i \in f(u_i)$ *m-a.e., and* $\hat{F}(u_i)$*,* $\tilde{F}^*(g_i) \in \mathbb{L}_1(L_1(L_1))$*. Then, setting* $v_i(t) := e^{-\alpha t}u_i(t)$*,* $t \in [0,T]$*, for* $\alpha \geq 0$ *constant, one has*

$$\begin{aligned}
&\big\|v_1(t) - v_2(t)\big\|_{L_q}^q + q(\alpha-\eta)\int_0^t \big\|v_1(s) - v_2(s)\big\|_{L_q}^q\,ds \\
&\qquad + \int_0^t e^{-\alpha s}\Phi_q'(v_1(s) - v_2(s))(g_1(s) - g_2(s))\,ds \\
&\qquad \leq \big\|u_{01} - u_{02}\big\|_{L_q}^q \\
&\qquad\quad + \int_0^t e^{-\alpha s}\Phi_q'(v_1(s) - v_2(s))\big(B(u_1(s)) - B(u_2(s))\big)\,dW(s) \\
&\qquad\quad + \frac{1}{2}q(q-1)\int_0^t \big\|e^{-\alpha s}\big(B(u_1(s)) - B(u_2(s))\big)\big\|_{L_q(\ell_2)}^2 \big\|v_1(s) - v_2(s)\big\|_{L_q}^{q-2}\,ds
\end{aligned}$$

for all $t \in [0,T]$.

Proof. Given $\sigma \in \mathbb{N}$ such that $(I + \varepsilon A)^{-\sigma}$ maps $L_1(D)$ to $L_q(D)$, set $h_i^\varepsilon := (I+\varepsilon A)^{-\sigma}h$ for all $h \in \{u_i, u_{0i}, g_i, v_i\}$, and $B_i^\varepsilon := (I+\varepsilon A)^{-\sigma}B(u_i)$. Then $g_i \in L_1(L_q)$ and v_i^ε is the unique L_q-valued strong solution to

$$dv_i^\varepsilon + Av_i^\varepsilon\,dt + (\alpha-\eta)v_i^\varepsilon\,dt + e^{-\alpha t}g_i^\varepsilon\,dt = e^{-\alpha t}B_i^\varepsilon\,dW, \qquad u_i^\varepsilon(0) = u_{0i}^\varepsilon.$$

Itô's formula then yields

$$\begin{aligned}
&\big\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\big\|_{L_q}^q + q(\alpha-\eta)\int_0^t \big\|v_1^\varepsilon(s) - v_2^\varepsilon(s)\big\|_{L_q}^q\,ds \\
&\qquad + \int_0^t e^{-\alpha s}\Phi_q'(v_1^\varepsilon(s) - v_2^\varepsilon(s))(g_1^\varepsilon(s) - g_2^\varepsilon(s))\,ds \\
&\qquad \leq \big\|u_{01}^\varepsilon - u_{02}^\varepsilon\big\|_{L_q}^q \\
&\qquad\quad + \int_0^t e^{-\alpha s}\Phi_q'(v_1^\varepsilon(s) - v_2^\varepsilon(s))\big(B_1^\varepsilon(s) - B_2^\varepsilon(s)\big)\,dW(s) \\
&\qquad\quad + \frac{1}{2}q(q-1)\int_0^t \big\|e^{-\alpha s}\big(B_1^\varepsilon(s) - B_2^\varepsilon(s)\big)\big\|_{L_q(\ell_2)}^2 \big\|v_1^\varepsilon(s) - v_2^\varepsilon(s)\big\|_{L_q}^{q-2}\,ds.
\end{aligned} \tag{13}$$

We are now going to pass to the limit as $\varepsilon \to 0$ in the above inequality. Since $(I + \varepsilon A)^{-\sigma}$ converges to the identity in $\mathscr{L}(L_q)$ in the strong operator topology, it immediately follows that

$$\begin{aligned}\big\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\big\|_{L_q} &\xrightarrow{\varepsilon\to 0} \big\|v_1(t) - v_2(t)\big\|_{L_q} \qquad \forall t \in [0,T],\\ \big\|u_{01}^\varepsilon - u_{02}^\varepsilon\big\|_{L_q} &\xrightarrow{\varepsilon\to 0} \big\|u_{01} - u_{02}\big\|_{L_q}.\end{aligned}$$

Since $(I+\varepsilon A)^{-\sigma}$ is contracting in L_q, $\|v_1^\varepsilon - v_2^\varepsilon\|_{L_q} \leq \|v_1\|_{L_q} + \|v_2\|_{L_q}$ pointwise; hence, by Fubini's theorem, $v \in \mathbb{H}_p(L_q)$, and the dominated convergence theorem,

$$\int_0^t \big\|v^\varepsilon(s)\big\|_{L_q}^q \, ds \xrightarrow{\varepsilon\to 0} \int_0^t \big\|v(s)\big\|_{L_q}^q \, ds \qquad \forall t \leq T.$$

The dominated convergence theorem also immediately shows that

$$\begin{aligned}&\int_0^t \big\|e^{-\alpha s}\big(B_1^\varepsilon(s) - B_2^\varepsilon(s)\big)\big\|_{L_q(\ell_2)}^2 \big\|v_1^\varepsilon(s) - v_2^\varepsilon(s)\big\|_{L_q}^{q-2} \, ds\\ &\qquad \xrightarrow{\varepsilon\to 0} \int_0^t \big\|e^{-\alpha s}\big(B(u_1(s)) - B(u_2(s))\big)\big\|_{L_q(\ell_2)}^2 \big\|v_1(s) - v_2(s)\big\|_{L_q}^{q-2} \, ds.\end{aligned}$$

Let us now consider the last term on the left-hand side of (13). Recalling the definition of the homeomorphism $\phi_q : x \mapsto x|x|^{q-2}$, we have

$$\begin{aligned}&\int_0^t e^{-\alpha s}\Phi_q'(v_1^\varepsilon(s) - v_2^\varepsilon(s))(g_1^\varepsilon(s) - g_2^\varepsilon(s)) \, ds\\ &\qquad = q\int_0^t e^{-q\alpha s}\Phi_q'\big(u_1^\varepsilon(s) - u_2^\varepsilon(s)\big)\big(g_1^\varepsilon(s) - g_2^\varepsilon(s)\big) \, ds\\ &\qquad \eqsim \int_0^t \int_D \big(g_1^\varepsilon(s) - g_2^\varepsilon(s)\big)\phi_q\big(u_1^\varepsilon(s) - u_2^\varepsilon(s)\big) \, dx \, ds.\end{aligned}$$

The properties of $(I+\varepsilon A)^{-\sigma}$ imply easily that $g_i^\varepsilon \to g_i$ in $\mathbb{L}_1(L_1(L_1))$, hence in m-measure, as $\varepsilon \to 0$. Similarly, since $u_i^\varepsilon \to u_i$ in m-measure and ϕ_q is continuous, $\phi_q(u_1^\varepsilon - u_2^\varepsilon) \to \phi_q(u_1 - u_2)$ in m-measure. In particular,

$$\big(g_1^\varepsilon - g_2^\varepsilon\big)\phi_q\big(u_1^\varepsilon - u_2^\varepsilon\big) \xrightarrow{\varepsilon\to 0} (g_1 - g_2)\phi_q(u_1 - u_2)$$

in m-measure. We are going to show that this convergence takes place in $\mathbb{L}_1(L_1(L_1))$. To this purpose, it suffices to show, by Vitali's theorem, that the sequence on the left-hand side is uniformly integrable. Let $\delta \in \,]0,1/2]$ be arbitrary but fixed. By Young's inequality with conjugate functions $\tilde{F}$ and $\tilde{F}^*$ and the definition $\hat{F} := \tilde{F} \circ \phi_q$,

$$\begin{aligned}\big|\big(g_1^\varepsilon - g_2^\varepsilon\big)\phi_q\big(u_1^\varepsilon - u_2^\varepsilon\big)\big| &\eqsim \big|\delta(g_1^\varepsilon - g_2^\varepsilon)\big| \, \big|\phi_q\big(\delta(u_1^\varepsilon - u_2^\varepsilon)\big)\big|\\ &\leq \hat{F}\big(\big|\delta(u_1^\varepsilon - u_2^\varepsilon)\big|\big) + \tilde{F}^*\big(\big|\delta(g_1^\varepsilon - g_2^\varepsilon)\big|\big)\\ &= \hat{F}\big(\delta(u_1^\varepsilon - u_2^\varepsilon)\big) + \tilde{F}^*\big(\delta(g_1^\varepsilon - g_2^\varepsilon)\big).\end{aligned}$$

In the last step we have used that $\hat{F}$ and $\tilde{F}^*$ are even: in fact, since F is even and $F(0) = 0$, we infer that f is odd, $\tilde{f}$ is odd, and hence $\tilde{F}$, $\tilde{F}^*$, and $\hat{F}$ are even with $\tilde{F}(0) = \tilde{F}^*(0) = \hat{F}(0) = 0$. Then it follows that $\tilde{F}^*$ and $\hat{F}$ are increasing on $\mathbb{R}_+$ (this can also be seen by $\partial\tilde{F}^* = \tilde{f}^{-1} = \phi_q \circ f^{-1} \geq 0$ and $\partial\hat{F} = f\phi_q' \geq 0$ on $\mathbb{R}_+$). Therefore $\hat{F}(cx) = \hat{F}(c|x|) \leq \hat{F}(|x|) = \hat{F}(x)$ for all $x \in \mathbb{R}$ and $c \in [0,1]$, and the same holds for $\tilde{F}^*$. In particular,

$$\begin{aligned}\hat{F}\big(\delta(u_1^\varepsilon - u_2^\varepsilon)\big) &= \hat{F}\Big(\frac{1}{2}(2\delta u_1^\varepsilon) + \frac{1}{2}(-2\delta u_2^\varepsilon)\Big)\\ &\leq \frac{1}{2}\hat{F}\big(2\delta u_1^\varepsilon\big) + \frac{1}{2}\hat{F}\big(2\delta u_2^\varepsilon\big)\\ &\leq \frac{1}{2}\big(\hat{F}(u_1^\varepsilon) + \hat{F}(u_2^\varepsilon)\big),\end{aligned}$$

and, completely analogously,

$$\tilde{F}^*\big(\delta(g_1^\varepsilon - g_2^\varepsilon)\big) \leq \frac{1}{2}\big(\tilde{F}^*(g_1^\varepsilon) + \tilde{F}^*(g_2^\varepsilon)\big),$$

thus also

$$\big|(g_1^\varepsilon - g_2^\varepsilon)\phi_q\big(u_1^\varepsilon - u_2^\varepsilon\big)\big| \lesssim \hat{F}(u_1^\varepsilon) + \hat{F}(u_2^\varepsilon) + \tilde{F}^*(g_1^\varepsilon) + \tilde{F}^*(g_2^\varepsilon).$$

Let us now observe that, by Jensen's inequality for positive operators (see, e.g., [16]),

$$\begin{aligned}\tilde{F}^*(g_i^\varepsilon) &= \tilde{F}^*\big((I+\varepsilon A)^{-\sigma} g_i\big) \leq (I+\varepsilon A)^{-\sigma}\tilde{F}^*(g_i),\\ \hat{F}^*(u_i^\varepsilon) &= \hat{F}\big((I+\varepsilon A)^{-\sigma} u_i\big) \leq (I+\varepsilon A)^{-\sigma}\hat{F}(u_i).\end{aligned}$$

But since $\hat{F}(u_i)$, $\tilde{F}^*(g_i) \in \mathbb{L}_1(L_1(L_1))$ by assumption, hence $(I+\varepsilon A)^{-\sigma}\hat{F}(u_i) \to \hat{F}(u_i)$ and $(I+\varepsilon A)^{-\sigma}\tilde{F}^*(g_i) \to \tilde{F}^*(g_i)$ as $\varepsilon \to 0$ in $\mathbb{L}_1 L_1 L_1$, it follows that the sequence $\big|(g_1^\varepsilon - g_2^\varepsilon)\phi_q(u_1^\varepsilon - u_2^\varepsilon)\big|$ is dominated by a convergent sequence of $\mathbb{L}_1(L_1(L_1))$, which is a fortiori uniformly integrable. Then $(g_1^\varepsilon - g_2^\varepsilon)\phi_q(u_1^\varepsilon - u_2^\varepsilon)$ is also uniformly integrable, because a (positive) sequence dominated by a uniformly integrable sequence is itself uniformly integrable. We have thus proved that the last term on the left-hand side of (13) converges in probability for all $t \in [0,T]$ to

$$\int_0^t e^{-\alpha s}\Phi_q'(v_1(s) - v_2(s))(g_1(s) - g_2(s))\,ds.$$

It remains only to consider the stochastic integral on the right-hand side of (13), which converges to

$$\int_0^t e^{-\alpha s}\Phi_q'(v_1(s) - v_2(s))\big(B(u_1(s)) - B(u_2(s))\big)\,dW(s)$$

in probability for all $t \in [0,T]$ as $\varepsilon \to 0$. The proof is based on an argument entirely analogous to the one already used in the proof of Proposition 3.2 and is hence omitted. ∎

We can now prove uniqueness of mild solution to (1) and their continuous dependence on the initial datum.

PROPOSITION 6.5. *Under the hypotheses of Theorem* 2.9, *assume that* $u \in \mathbb{H}_p(L_q)$ *is a mild solution to* (1). *Then* u *is the unique mild solution such that* $\hat{F}(u) + \tilde{F}^*(g) \in \mathbb{L}_1(L_1(L_1))$. *Moreover, the solution map* $u_0 \mapsto u$ *is Lipschitz continuous from* $\mathbb{L}_p(L_q)$ *to* $\mathbb{H}_p(L_q)$.

Proof. Let u_1, u_2 be as in the previous lemma, with u_{01}, $u_{0,2} \in \mathbb{L}_p(L_q)$. Then

$$\begin{aligned}\big\|v_1(t) - v_2(t)\big\|_{L_q}^q &+ q(\alpha - \eta)\int_0^t \big\|v_1 - v_2\big\|_{L_q}^q\,ds + \int_0^t e^{-\alpha s}\Phi_q'(v_1 - v_2)(g_1 - g_2)\,ds\\ &\leq \big\|u_{01} - u_{02}\big\|_{L_q}^q + \int_0^t e^{-\alpha s}\Phi_q'(v_1 - v_2)\big(B(u_1) - B(u_2)\big)\,dW\\ &\quad + \frac{1}{2}q(q-1)\int_0^t \big\|e^{-\alpha s}\big(B(u_1) - B(u_2)\big)\big\|_{L_q(\ell_2)}^2 \big\|v_1 - v_2\big\|_{L_q}^{q-2}\,ds,\end{aligned}$$

where

$$\Phi'_q(v_1 - v_2)\big(g_1 - g_2\big) = qe^{-(q-1)\alpha\cdot}\big\langle g_1 - g_2, \phi_q(u_1 - u_2)\big\rangle \geq 0.$$

We are now in the condition to use exactly the same proof of Lemma 4.2, arriving at

$$\big\|u_1 - u_2\big\|_{\mathbb{H}_p(L_q)} \lesssim \big\|u_{01} - u_{02}\big\|_{\mathbb{L}_p(L_q)},$$

which proves that $u_0 \mapsto u$ is Lipschitz continuous from $\mathbb{L}_p(L_q)$ to $\mathbb{H}_p(L_q)$, and hence, as an immediate consequence, that the solution is unique. □

Remark 6.6. It is clear by the previous proof that we do *not* have well-posedness in the space $\mathbb{H}_p(L_q)$, as our uniqueness result holds only under additional assumptions on the solution itself. The problem of unconditional uniqueness in $\mathbb{H}_p(L_q)$ remains therefore open.

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