BLOCK PARTITIONS: AN EXTENDED VIEW

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ABSTRACT. Given a sequence $S = (s_1, \ldots, s_m) \in [0,1]^m$, a block B of S is a subsequence $B = (s_i, s_{i+1}, \ldots, s_j)$. The size b of a block B is the sum of its elements. It is proved in [1] that for each positive integer n, there is a partition of S into n blocks $B_1, \ldots, B_n$ with $|b_i - b_j| \leq 1$ for every i, j. In this paper, we consider a generalization of the problem in higher dimensions.

1. INTRODUCTION

This paper is a follow-up to [1], which is about block partitions of sequences $S = (s_1, \ldots, s_m)$ of real numbers. A block B of S is either a sequence $B = (s_i, s_{i+1}, \ldots, s_j)$ where $i \leq j$ or the empty set. The size b of a block B is the sum of its elements. One of the main results of [1] says the following.

Theorem 1.1. *Given a sequence $S = (s_1, \ldots, s_m)$ of real numbers with $s_i \in [0,1]$ for all i, and an integer $n \in \mathbb{N}$, there is a partition of S into n blocks such that $|b_i - b_j| \leq 1$ for all i, j.*

The bound given here is best possible as shown by the example when all the $s_i = 1$ and n does not divide m.

Here, we rephrase or generalize the setting and the result in the following way. Define $a_i = \sum_{j=1}^{i} s_j$ (where $a_0 = 0$) and set $A = \{a_0, \ldots, a_m\}$. A partition of A into n blocks is the same as choosing indices $x_0 = 0 \leq x_1 \leq \cdots \leq x_n = m$ so that x_j is represented by $a_{x_j} \in A$, and then $b_j = a_{x_j} - a_{x_{j-1}}$ for all $i \in [n]$. Here $[n]$ denotes the set $\{1, \ldots, n\}$. For the more general setting, we consider closed sets $(A_i) = A_0, A_1, \ldots, A_n \subset \mathbb{R}$ satisfying the following conditions:

   (i) $A_0 = \{0\}$, $A_n = \{s\}$.
   (ii) $A_j \cap [a, a+1] \neq \emptyset$ for every $j \in [n-1]$ and for every $a \in \mathbb{R}$.

A *transversal* $T = (a_0, a_1, \ldots, a_n)$ of the system A_j is simply a selection of elements $a_j \in A_j$ for every $j = 0, 1, \ldots, n$. Given a transversal, we define $z_j = a_j - a_{j-1}$ for all $j \in [n]$. In this setting, z_j corresponds to the size b_j of the jth block.

Theorem 1.2. *Under the above conditions, there is a transversal $a_j \in A_j$ such that*
$$|z_i - z_j| \leq 1 \text{ for all } i, j \in [n].$$

The bound $|z_i - z_j| \leq 1$ is again best possible as shown by (essentially the same) example: $A_0 = \{0\}$, $A_i = \{0, \pm 1, \pm 2, \ldots\}$ for $i \in [n-1]$ and $A_n = \{s\}$, with an integer s not divisible by n. Without the closedness of the sets A_j, we would only have a transversal with $|z_i - z_j| \leq 1 + \varepsilon$ for each $\varepsilon > 0$, as shown by the following example: let $n \geq 3$, $A_j = (\infty, -1/2] \cup (1/2, \infty)$ for odd j, $A_j = (\infty, -1/2) \cup [1/2, \infty)$ for even j.

Theorem 1.2 could be easily deduced from Theorem 1.1, but we will present a new and shorter direct proof in the next section. Then we extend the new setting to higher dimensions.

Let B_d be the unit ball of a norm $\| \cdot \|$ in $\mathbb{R}^d$. Let $A_0, A_1, \ldots, A_n$ be a sequence of closed sets in $\mathbb{R}^d$. It is called a *grid-like sequence* if (i) $A_0 = \{0\}$ and $A_n = \{s\}$ for a fixed element $s \in \mathbb{R}^d$, and (ii) each of $A_1, A_2, \ldots, A_{n-1}$ intersect with all unit balls, or formally, $\forall i \in [n-1]$, $\forall a \in \mathbb{R}^d$:

$$(1) \qquad\qquad (a + B_d) \cap A_i \neq \emptyset.$$

Note that in the one dimensional case we required $A_i \cap [a, a+1] \neq \emptyset$ while the above condition would translate to $A_i \cap [a-1, a+1] \neq \emptyset$. So there is a factor of 2 in the new setting.

Given a transversal T, we define again $z_i = a_i - a_{i-1}$ for $i \in [n]$ and set $Z = \text{conv}\{z_1, \ldots, z_n\}$. The goal is to find a transversal T such that

$$(2) \qquad\qquad D(T) = D\big(T, (A_i)\big) = \text{diam } Z = \max_{i,j} \|z_i - z_j\|$$

is as small as possible. Let

$$D_d^* = \sup_{(A_i)} \min_T D\big(T, (A_i)\big)$$

for grid-like sequences (A_i) and transversals T. Due to the closedness of each A_i, this minimum always exists. It is easy to see the following two propositions.

Proposition 1.3. $D_{d+1}^* \geq D_d^*$

Proof. If $A_0, A_1, ..., A_n \subset \mathbb{R}^d$ is a grid-like sequence, then $A_0, A_1 \times \mathbb{R}, A_2 \times \mathbb{R}...A_{n-1} \times \mathbb{R}, A_n \subset \mathbb{R}^{d+1}$ is also a grid-like sequence, and the n-dimensional projection of any transversal of the latter sequence is a transversal of the former sequence with at most the same diameter. $\square$

Proposition 1.4. $D_d^* \leq 4$, or in other words, there is always a transversal T with $D(T) \leq 4$.

Proof. Set $t = \frac{s}{n}$ and choose a point $a_i \in A_i$ from $it + B$ for $i \in [n-1]$, so $a_i = it + b_i$ with $b_i \in B$, for all $i = 0, \ldots, n$. Then $z_i = t + b_i - b_{i-1}$, and so $z_i - z_j = b_i - b_{i-1} - b_j + b_{j-1}$, clearly in $4B$. $\square$

One could hope that the better bound $D_d^* \leq 2$, which is valid for $d = 1$, also holds in higher dimensions. But this is not the case, at least with Euclidean norm:

Theorem 1.5.
$$D_2^* \geq 4\sqrt{2 - \sqrt{3}} \approx 2.071.$$

In more detail, for every $\varepsilon > 0$ and $n \geq n_0(\varepsilon)$, there exists a grid-like sequence $A_0, A_1, \ldots, A_n \subset \mathbb{R}^2$ such that for any transversal T, $D(T) \leq 4\sqrt{2 - \sqrt{3}}$.

We have a stronger bound $1 + \sqrt{2} \approx 2.414$, see Theorem 1.6 below, which may be sharp for $d = 2$ or maybe even for all d. Its proof is based on the same ideas as that of Theorem 1.5, but it is much longer and more complicated case analysis. Therefore, we prove Theorem 1.5 and give an informal description of the construction for Theorem 1.6, but omit the proof.

Theorem 1.6. $D_2^* \geq 1 + \sqrt{2} \approx 2.414$.

Apart from the trivial bound $D_d^* \leq 4$ given in Proposition 1.4, we cannot prove any upper bound,[1] not even in the case of the maximum norm $\| \cdot \|_\infty$. Note that the existence of a transversal T with "$D(T) \leq 2$ in the maximum norm" would imply the bound $D_d^* \leq 2\sqrt{2}$ (in the Euclidean norm). For some related problems and results we refer to [2].

Remark. We may assume without any loss that $s = 0$. Indeed, with the previous meaning of t, set $A_i^* = A_i - it$. Then the system A_i^* with $s^* = 0$ satisfies condition (1) and it is easy to check that for the transversals $a_i \in A_i$ and $a_i^* = a_i - it \in A_i^*$ one has the same $z_i - z_j = z_i^* - z_j^*$ and then the diameters of Z and Z^* coincide. Note that $0 = \frac{1}{n} \sum_{i=1}^n z_i^* \in Z^*$.

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[1]Update: Endre Csóka recently claimed an unpublished upper bound $2\sqrt{2}$ using topology.

2. PROOF OF THEOREM 1.2

The idea is to find an $x \in \mathbb{R}$ such that the transversal we look for satisfies the condition

$$(3) \qquad a_i \in a_{i-1} + [x, x+1].$$

With this strategy the condition $|z_i - z_j| \leq 1$ is automatically guaranteed. The question is whether there is an $x \in \mathbb{R}$, which admits a transversal $a_0, \ldots, a_n$ satisfying condition (3) for every $i \in [n]$.

We analyze what happens when this strategy is followed. A *partial transversal* is just a selection of $a_i \in A_i$ for $i = 0, \ldots, h$, $h \in [n]$. We call it *x-good* if it satisfies condition (3) for every $i \in [h]$. As a first step, a_1 is to be chosen from the interval $J_1 = a_0 + [x, x+1] = [x, x+1]$, and $J_1 \cap A_1$ is nonempty because of condition (ii). Thus, a_0, a_1 is an x-good partial transversal for any $a_1 \in J_1 \cap A_1$.

It is easy to see that x-good transversals exist for every $x \in \mathbb{R}$ and $h \in [n-1]$. To construct such partial transversals we define J_i recursively as follows. Given J_i for some $i \in [n-1]$ and a fixed $x \in \mathbb{R}$, we let

$$J_{i+1} = (J_i \cap A_i) + [x, x+1] = \bigcup \{[a + x, a + x + 1] : a \in J_i \cap A_i\}.$$

A routine induction, based on condition (ii), shows that J_i is a closed interval of length at least one for every $i \in [n]$, and so J_i intersects A_i if $i \in [n-1]$. The intervals J_i of course depend on x, and we write $J_i(x) = [L_i(x), R_i(x)]$ to express this dependence. The definition of $J_i(x)$ implies that

$$(4) \qquad L_{i+1}(x) \;=\; x + \min\{a \in A_i : a \geq L_i(x)\},$$
$$(5) \qquad R_{i+1}(x) \;=\; x + 1 + \max\{a \in A_i : a \leq R_i(x)\}.$$

Note that both $L_i(x)$ and $R_i(x)$ are increasing functions of x satisfying $R_i(x) - L_i(x) \geq 1$. Also, $L_{i+1}(x) \geq x + L_i(x)$ implying via an easy induction that $L_i(x)$ and then $R_i(x)$ tends to infinity as $x \to \infty$. A similar argument shows that $\lim R_i(x) = \lim L_i(x) = -\infty$ as $x \to -\infty$. It is also clear that if $a_h \in J_h(x) \cap A_h$ for some $x \in \mathbb{R}$, then there exists an x-good partial transversal $a_0, a_1, \ldots, a_h$.

To complete the proof of Theorem 1.2 we only have to show that there is an $x \in \mathbb{R}$ such that $J_n(x) \cap A_n$ is non-empty, that is, $s \in J_n(x)$. Actually we prove more:

Lemma 2.1. *For every $h = 1, \ldots, n$ the intervals $J_h(x)$ ($x \in \mathbb{R}$) cover every point of $\mathbb{R}$: $\bigcup_{x \in \mathbb{R}} J_h(x) = \mathbb{R}$.*

Proof. First we claim that L_i is a left continuous function for every $i \in [n]$. This is evident for $i = 1$, so we assume that L_i is left continuous and proceed to prove that L_{i+1} is also left continuous. In view of (4) it amounts to checking that

$$f(x) = \min\{a \in A_i : a \geq L_i(x)\}$$

is a left continuous function of x. Let $x \in \mathbb{R}$ be arbitrary; we have to show that $x_j < x$, $x_j \to x$ implies $f(x_j) \to f(x)$. It is clear that if $[L_i(u), L_i(x)) \cap A_i = \emptyset$ for some $u < x$, then f is constant on the interval $[u, x]$ and $f(x_j) = f(x)$ for $x_j \geq u$.

Otherwise there is a sequence $a_j \in A_i$ such that

$$L_i(x_j) \leq f(x_j) = a_j < L_i(x).$$

Here $L_i(x_j) \to L_i(x)$ by the left continuity of L_i. Accordingly,

$$L_i(x) = \lim_{j \to \infty} a_j \in A_i$$

because A_i is closed. It follows that $f(x_j) \to L_i(x) = f(x)$. The left continuity of f at x is thus established.

Similarly, R_i is a right continuous function for every $i \in [n]$. To complete the proof of the lemma, consider any point $p \in \mathbb{R}$ and define

$$
\begin{aligned}
L^p = L_h^p &= \{x \in \mathbb{R} : L_h(x) \le p\}, \\
R_p = R_h^p &= \{x \in \mathbb{R} : R_h(x) \ge p\}.
\end{aligned}
$$

Here neither L^p nor R^p is empty since $\lim_{x \to -\infty} L_h(x) = -\infty$ and $\lim_{x \to \infty} R_h(x) = \infty$. The continuity and monotonicity properties of the functions L_h and R_h imply that both L^p and R^p are closed sets. Further, $L^p \cup R^p = \mathbb{R}$, as $x \notin L^p$ implies $L_h(x) > p$ and $x \notin R^p$ implies $R_h(x) < p$, and then $R_h(x) < p < L_h(x)$, which is impossible.

Now if two closed sets cover the connected space $\mathbb{R}$, then they have a point in common. So there is an $x_0 \in L^p \cap R^p$, and then $L_h(x_0) \le p \le R_h(x_0)$, so $p \in J_h(x_0)$. Thus every $p \in \mathbb{R}$ is contained in some $J_h(x)$. □

Remark. The proof of Theorem 1.1, which is an algorithm of complexity $O(nm^3)$ can be modified to give another (algorithmic) proof of Theorem 1.2. But the above proof can be turned into an approximation algorithm the following way. By the remark at the end of the introduction we may assume that $s = 0$. Note that for $x = -1$ no interval $J_h(x)$ contains a positive number, and similarly, for $x = 0$ no interval $J_h(x)$ contains a negative one. Using binary search, after k iteration, one finds an $x \in [-1, 0]$ that is within distance 2^{-k} of the solution.

3. Proof of Theorem 1.5

We begin with an informal description of the construction. We fix a large enough integer m. The grid-like sequence consists of $2m + 2$ sets $A_0, \ldots, A_{2m+1}$, where $A_0 = A_{2m+1} = \{0\}$. Recall that in this case $0 \in Z$. Each other A_i is the union of sets N_i, S_i, E_i, W_i (corresponding to North, South, East, and West) plus four corners $Q_i^{NE}, Q_i^{SE}, Q_i^{NW}, Q_i^{SW}$, see the figures below. The sets A_i are symmetric about the x and y axes. The sets $A_1, \ldots, A_m$ make up the first part of the construction. For $i \in \{m+1, \ldots, 2m\}$, A_i is the refection of A_{2m+1-i} about the line $x = y$. In particular, $A_m = A_{m+1}$.

The main characters in our construction are a square X and segments G_i of changing length that are either horizontal or vertical. The Minkowski sum $X + G_i$ is a hexagon shown on Figures 1 and 2. Its vertical and horizontal sides (or vertices) are drawn with heavy lines, its oblique sides with thin segments. The horizontal and vertical sides (or vertices) are extended to the sets N_i, S_i, E_i, W_i.

The main step of the proof is to show that an optimal tranversal $a_0, \ldots, a_{2m+1}$ has no point in the corners and that it does not visit the same region (N or S or E or W) twice. More precisely, if $a_i \in N_i$ and $a_j \in N_i$ for some $i < j$, then $a_h \in N_h$ for all $h \in \{i, \ldots, j\}$, and the same for the components of type S, E, W.

Formally we define the various components of A_i for $1 \le i \le m$ as

$$
\begin{aligned}
N(a, b) &= \{(x, y) : -a \le x \le a,\ y \ge b\}, \\
S(a, b) &= \{(x, y) : -a \le x \le a,\ y \le b\}, \\
E(c, d) &= \{(x, y) : x \ge c,\ -d \le y \le d\}, \\
W(c, d) &= \{(x, y) : x \le c,\ -d \le y \le d\},
\end{aligned}
$$

and

$$
\begin{aligned}
Q^{NW}(e, f) &= \{(x, y) : e \ge x,\ f \le y\}, \\
Q^{NE}(e, f) &= \{(x, y) : e \le x,\ f \le y\}, \\
Q^{SW}(e, f) &= \{(x, y) : e \ge x,\ f \ge y\}, \\
Q^{SE}(e, f) &= \{(x, y) : e \le x,\ f \ge y\}
\end{aligned}
$$
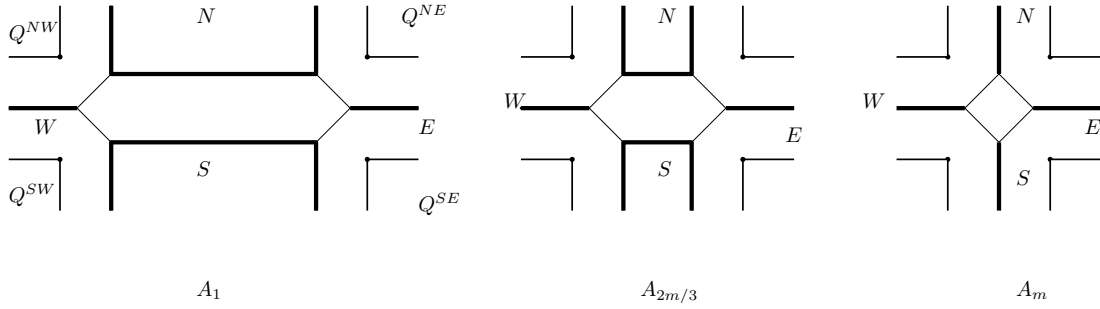
FIGURE 1. Some A_i from phase 1
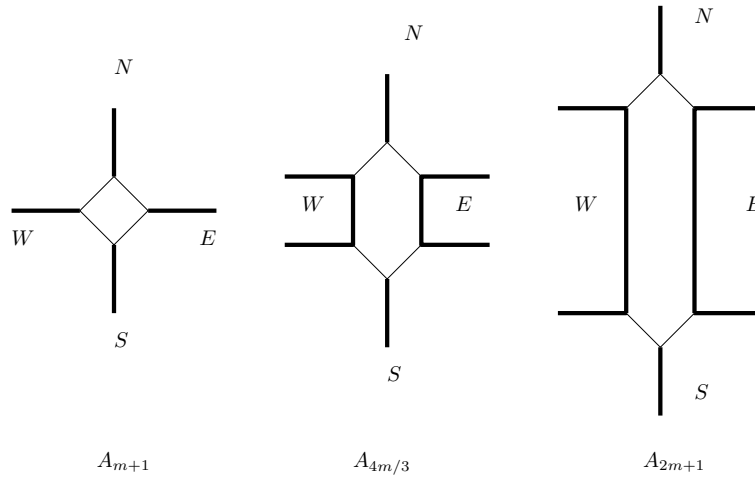
FIGURE 2. Some A_i from phase 2, corners suppressed

with appropriately chosen parameters a, b, c, d, e, f depending on i. We call $V(a, b) = N(a, b) \cup S(a, -b)$ and $H(c, d) = E(c, d) \cup W(-c, d)$ the vertical and horizontal parts, and the set

$$Q(e, f) = Q^{NW}(-e, f) \cup Q^{NE}(e, f) \cup Q^{SW}(-e, -f) \cup Q^{SE}(e, -f)$$

the union of the corners. Set first $A_1 = V_1 \cup H_1 \cup Q_1$, where

$$V_1 = V(3, 1), \quad H_1 = H(4, 0), \quad Q_1 = Q(4.5, 1.5).$$

Here the values 4.5 and 1.5 are chosen so that A_1, and later all other A_i satisfy condition (ii). Next, for $i = 2, \ldots, m$ we define $A_i = V_i \cup H_i \cup Q_i$, where

$$V_i = V(3 - (i-1)\delta, 1), \ H_i = H(4 - (i-1)\delta, 0), \ Q_i = Q(4.5 - (i-1)\delta, 1.5)$$

with $\delta = 3/(m-1)$. Note that A_m is made up of four halflines and the four corners. For $i = m+1, \ldots, 2m$ we let A_i be the reflected copy of A_{2m+1-i} about the line $x = y$. It is easy to check that $A_0, A_1, \ldots, A_{2m+1}$ is a grid-like sequence for $n = 2m + 1$.

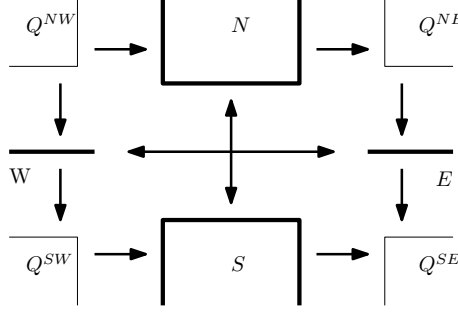**Claim 3.1.** *There is a transversal T with $D(T) \leq 4\sqrt{2 - \sqrt{3}}$.*

FIGURE 3. Forbidden jumps

Proof. Consider the transversal $T = \{a_0, \ldots, a_{2m+1}\}$ where

$$a_i = \left(\frac{m-i}{m-1} \left(2\sqrt{3} - 3 \right), 1 \right), \quad a_{m+i} = \left(1, \frac{i-1}{m-1} \left(2\sqrt{3} - 3 \right) \right)$$

for $i = 1, \ldots, m$. Then Z is an equilateral triangle whose vertices are

$$z_1 = \left(2\sqrt{3} - 3, 1 \right), \quad z_{m+1} = (1, -1), \quad z_{2m+1} = \left(-1, 3 - 2\sqrt{3} \right),$$

and diam $Z = 4\sqrt{2 - \sqrt{3}}$.   □

Fix an optimal transversal $T = \{a_0, \ldots, a_{2m+1}\}$, such a transversal exists by compactness. Write $z_i = (x_i, y_i)$. As $0 \in Z$, the above claim implies $||z_i|| \leq 4\sqrt{2 - \sqrt{3}}$.

We say that a_j *jumps* if a_{j-1} is in one type of component in A_{j-1} but a_j is in another type in A_j. Then z_j is called the corresponting *jump*. For instance a_j jumps if $a_{j-1} \in Q_{j-1}^{NW}$ but a_j is in N_j or in W_j. The important property is that $||z_j||$ is large when z_j is a jump. Therefore the structure of the sequence of jumps is rather restricted.

Assume by symmetry that $a_1 \in N_1$. Then $z_1 = a_1$ and $y_1 \geq 1$. Similarly, we may assume that $a_{2m} \in E_{2m}$. Then $z_n = -a_{n-1}$ and $x_n \leq -1$.

Fact 1. For every $i = 1, \ldots, 2m + 1$, $y_i \geq -1.1$ and $x_i \leq 1.1$. Proof: Otherwise the y component of $z_1 - z_i$ or the x-component of $z_i - z_n$ is larger than 2.1 and then diam $Z > 2.1$, which contradicts Claim 3.1 and the optimality of T.   □

This implies that the jumps $Q^{NW} \to N$, $N \to Q^{NE}$, $Q^{SW} \to S$, $S \to Q^{SE}$ and $W \to E$ are forbidden, and so are the jumps $Q^{NW} \to W$, $W \to Q^{SW}$, $Q^{NE} \to E$, $E \to Q^{SE}$ and $N \to S$, see Figure 3. In particular, Q^{SE} and Q^{NW} is never visited by T, because all other components T could jump from Q^{NW} into or from which T could jump into Q^{SE} are too far away. Moreover, as indicated on Figure 3, there cannot be $E \to W$ or $S \to N$ jumps either. Indeed, suppose that there is a jump from some E_i to W_{i+1}. To get back to the point $a_{2m} \in E_{2m}$ there must be another jump to E_j for some $j > i + 1$. But then $x_{i+1} \leq -2$ and $x_j \geq 1 - \delta$, yielding $||z_j - z_{i+1}|| \geq 3 - \delta$, a contradiction. A similar argument applies to an $S \to W$ jump.

We say that a pair of jumps is *opposite* if either (a) one is $N \to E$ or $W \to S$ and the other is $E \to N$ or $S \to W$, or (b) one is $N \to W$ or $E \to S$ and the other is $W \to N$ or $S \to E$.

Fact 2. There cannot be an opposite pair of jumps if m is large enough. Proof: If z_i and z_j are opposite jumps, then $|x_j - x_i| \geq 2 - 2\delta$ and $|y_j - y_i| \geq 2 - 2\delta$. Thus $||z_j - z_i|| > 2.8$ if δ is small enough, a contradiction.   □

Lemma 3.2. *There is a single jump and it goes from N_{j-1} to E_j for some $j = 2, \ldots, 2m$.*
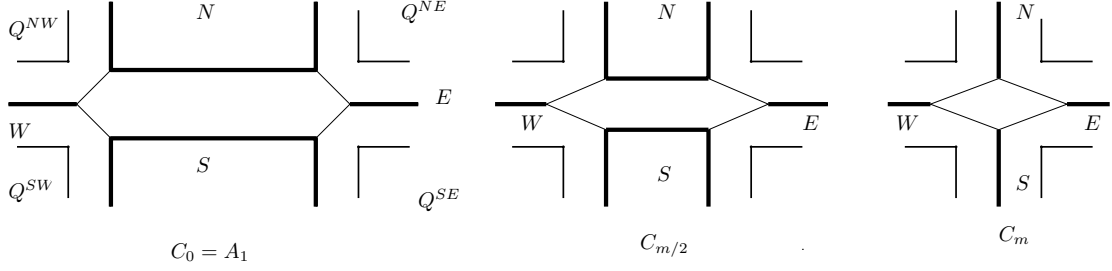
FIGURE 4. $C_0, C_{m/2}, C_m$, the rhombus R is drawn with thin segments

Proof. There must be a jump since $a_1 \in N_1$ and $a_{2m} \in E_{2m}$. Assume that the first jump is z_j, that is $a_i \in N_i$ for $i = 1, \ldots, j-1$ but $a_j \notin N_j$. As we have seen, a_j is either in E_j or in W_j, Q_j^{SW} and Q_j^{SE} too far away.

Suppose first that $a_j \in W_j$, we will see that it leads to a contradiction. Note that z_j is fairly large: $x_j \leq -1 + \delta$ and $y_j \leq -1 + \delta$. Also, there must be a further jump, say z_k meaning that $a_i \in W_i$ for $i = j, \ldots, k-1$ but $a_k \notin W_k$. So a_k is in S_k or in N_k since Q_k^{NE} is too far away if m is large enough. Now $a_{2m} \in E_{2m}$ which cannot be reached from a_k without creating an opposite pair: along the way there is an $S \to E$ jump or a $N \to E$ one. A contradiction.

We can conclude that the first jump goes from N_{j-1} to E_j.

If there is a further jump, then the next jump, z_k say, must go from E_{k-1} to Q_k^{NE} or S_k or N_k, as Q_k^{SW} is too far away. Here N_k is excluded as then z_j, z_k are opposite. If $a_k \in Q_k^{NE}$, then $||z_k - z_j|| > |y_k - y_j| \geq 2.5 - \delta$. Assume finally that $a_k \in S_k$. Then, again, $a_{2m} \in E_{2m}$ cannot be reached from $a_k \in S_k$ without using an opposite pair: along the way there is an $S \to E$ jump or an $W \to N$ one. $\square$

Let z_j be the single jump along the way. Note that $x_j \geq 1 - \delta$ and $y_j \leq -1 + \delta$. This leads to a simple minimisation problem. Given points $z_1, z_j, z_n \in \mathbb{R}^2$, with $y_1 \geq 1$, $x_j \geq 1 - \delta$, $y_j \leq -1 + \delta$ and $x_n \leq -1$, find the minimum diameter of the triangle formed by these points. As δ tends to 0, the unique solution to this problem converges to the equilateral triangle specified in the proof of Claim 3.1. Thus, $D(T) > 4\sqrt{2 - \sqrt{3}} - \varepsilon$ for $m \geq m_0(\varepsilon)$. This completes the proof of Theorem 1.5 for odd values of n. For even values of n some straightforward modifications are needed, which are left to the reader.

4. SKETCH OF THE PROOF OF THEOREM 1.6

In this construction m is a large integer again, and the grid-like sequence of sets is $C_{-1}, C_0, \ldots,$ C_{6m+1}, C_{6m+2}, where $C_{-1} = C_{6m+2} = \{(0,0)\}$. Further $C_0 = A_1, C_{6m+1} = A_{2m}$, where A_1, A_{2m} come from the previous construction to guarantee that $|y_0| \geq 1$ and $|x_{6m+1}| \geq 1$ with the previous notation $z_i = a_i - a_{i-1} = (x_i, y_i)$.

The main characters are a rhombus R (instead of the square X) and the same segments G_i. The length of the shorter diagonal of the rhombus is $\sqrt{2}$, its sides are of length $\sqrt{2 + \sqrt{2}}$, and its centre is the origin. But this time rotated copies of R are needed, so let $R(\alpha)$ stand for the rotated copy of R; rotated by angle α in clockwise direction around its centre.

The first m sets $C_1, \ldots, C_m$ come from the Minkowski sum of R and the previous horizontal G_i: This sum is a hexagon again and the components N_i, S_i, E_i, W_i are just extended from the horizontal and
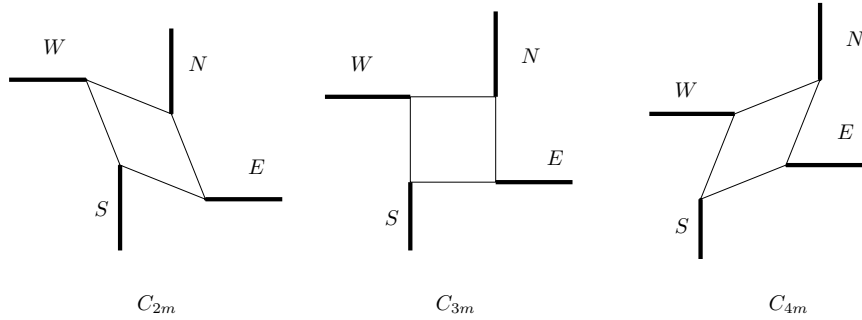
FIGURE 5. Deformation of the rhombus from C_{2m} to C_{4m}, corners suppressed

vertical sides of this hexagon the same way as in Theorem 1.5. The corners are at the same distance from the horizontal and vertical components as before. See Figure 4. The last $C_{5m+1} \ldots, C_{6m}$ sets come from the Minkowski sum of $R(\pi/2)$ and the corresponding vertical segments G_i analogously.

Note that N_m, S_m, E_m, W_m are halflines, and they remain halflines in all C_i with $i \in \{m, \ldots, 5m\}$. The sets $C_{m+1}, \ldots, C_{2m}$ come from gradually rotated copies of R. Then the rhombus $R(\pi/4)$ that defines C_{2m} is gradually deformed to a square (of side legth $\sqrt{2}$) in C_{3m}, which is further deformed to the rhombus $R(-\pi/4)$ in C_{4m}, see Figure 5. Then $R(-\pi/4)$ rotates back to $R(-\pi/2) = R(\pi/2)$ in C_{5m}.

The proof that this construction gives $D(T) > 1 + \sqrt{2} - \varepsilon$ is based on ideas similar to those used in Theorem 1.5: First one shows that an optimal transversal T has no point in the corners, and second, that T does not visit the same type component N, S, E, W twice. We omit the details.

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