Thermodynamics in $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$ theory of gravity

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Abstract

First and second laws of black hole thermodynamics are examined at the apparent horizon of FRW spacetime in $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$ gravity, where R, $R_{\alpha\beta}$ and ϕ are the Ricci scalar, Ricci invariant and the scalar field respectively. In this modified theory, Friedmann equations are formulated for any spatial curvature. These equations can be presented into the form of first law of thermodynamics $T_h d\hat{S}_h + T_h d_i\hat{S}_h + WdV = dE$, where $d_i\hat{S}_h$ is an extra entropy term because of the non-equilibrium. The generalized second law of thermodynamics (GSLT) is expressed in an inclusive form where these results can be represented, in G.R., $f(R)$ and $f(R,\phi)$ gravities. Finally to check the validity of GSLT, we take some particular models and produce constraints of the parameters.

Keywords: Modified gravity; Dark energy theory; Thermodynamics.
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1 Introduction

The cosmological observations which leads to the recent accelerating expansion of the universe are weak lensing [1], large scale structure [2], cosmic microwave background (CMB) radiation [3, 4], Type Ia Supernovae [5] and baryon acoustic oscillations [6]. To explain the cosmic acceleration of the universe, two classical approaches are followed: first is to use General Relativity (GR) and introduce dark energy [7, 8]; and the second is modified theories of gravity, e.g., $f(R)$ gravity. The modified gravity theories have attain a great attention to study the current cosmic acceleration [3].

The first studies in black hole thermodynamics was done in 1970s, where physicists were thinking that there must be some connection among Einstein equations and thermodynamics because of the linkage between horizon area (geometric quantity) and entropy (thermodynamical quantity) of black hole. At that time, those thermodynamic studies were focused in the context of black hole in which the surface gravity (geometric quantity) is associated with its temperature (thermodynamical quantity) and the first law of thermodynamics (FLT) is satisfied by these quantities [9]. Using that the entropy is proportional to the horizon area of the BH and the first law of thermodynamics $\delta Q = TdS$, Jacobson [10] was surely capable to derive the Einstein equations in 1995. By his assumption the expression of FLT is valid for all Rindler causal horizons by every spacetime where T and δQ denote the Unruh temperature and energy flux which were watched by an accelerated observer within the horizon.

For radiation dominated (FRW) universe, Verlinde discovered that the Friedmann equation can be recomposed in the form like the Cardy-Verlinde formula [11]. In high-dimensional spacetime, this formula represents an entropy relation for a conformal field theory. It can be observed that radiation can be described by a conformal field theory. Therefore, thermodynamics of radiation in the universe has been formulated with the help of entropy formula can also be write in the form of friedmann equation, which describes the dynamics of spacetime. These two equations conflict one another [12], particularly when the Hubble entropy bound is saturated. Further, Verlinde discovered the relation between thermodynamics and Einstein equations. The discussion related to the relation between thermodynamics and the Einstein's equation was done in [13].

By the discovery of black hole thermodynamics, it was shown [14] that gravitation and thermodynamics are deeply connected. The Hawking temperature of apparent horizon is proportional to K_{sg} surface gravity, and horizon entropy $S = A/4G$ fulfil the first law of thermodynamics [9, 15]. The Bekenstein-Hawking entropy is defined as one quarter of the area of the BH's event horizon (which is measured in Planck units) [9, 15]. It was discovered that using an entropy formula related with the area of accelerated horizons and the first law of thermodynamics, the Einstein's field equations can be obtained [10]. After that, this relation was analysed, and it was found an equivalence between the Friedmann equations and FLT [16]. The relation between gravitation and thermodynamics has been vastly studied in the literature, and lastly, this topic has attracted a great interest in studying its properties in alternatives generalized theories of gravity [17]-[19].

Applying the first law of thermodynamics to the apparent horizon of FRW universe and assuming the geometric entropy given by a quarter of the apparent horizon area Cai derived the Friedmann equations which describes the dynamics of universe with any spatial curvature [20]. The relation between the Friedmann equations with the first law of thermodynamics for scalar-tensor gravity and $f(R)$ gravity is discussed by Akbar [21]. To satisfy the GSL constraints and conditions imposed on cosmological future horizon R_h, Hubble parameter H, and the temperature T, in a phantom-dominated universe are described in [22]. Akbar [23] has shown that the differential form of Friedmann equations of FRW universe filled with a viscous fluid can be rewritten as a similar form of the first law of thermodynamics at the apparent horizon of FRW universe. Bamba has studied the first and second laws of thermodynamics of the apparent horizon in f(R) gravity in the Palatini formalism [24]. He also explored both nonequilibrium and equilibrium descriptions of thermodynamics in f(R) gravity and conclude that equilibrium framework is more transparent than the non-equilibrium one. Further the laws of thermodynamics at the apparent horizon of FRW spacetime in $f(R,T)$ gravity are discussed by Sharif and Zubair [18] and found that the picture of equilibrium thermodynamics is not feasible in f(R, T) gravity, so the non-equilibrium treatment is used to study the laws of thermodynamics in both forms of the energy-momentum tensor of dark components. In [26] the laws of thermodynamics are studied by Wu for the generalized $f(R)$ gravity with curvaturematter coupling in spatially homogeneous, isotropic FRW universe whose results shows that the field equations of the generalized $f(R)$ gravity with curvaturematter coupling can be cast to the form of the first law of thermodynamics with the the entropy production terms and the GSL can be given by considering the FRW universe filled only with ordinary matter enclosed by the dynamical apparent horizon with the Hawking temperature.

In this paper, the horizon entropy is constructed from the first law of thermodynamics corresponding to the Friedmann equations in the context of $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$. We explore the generalized second law of thermodynamics (GSLT) and find out the necessary condition for its validity. The paper is organized as follows: In Sec. 2, we review $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$ gravity and formulate the field equations of FRW universe. Sec. 3 is devoted to study the first and second laws of thermodynamics. In Sec. 4, the validity of GSLT for different models are discussed. Finally in Sec. 5 we conclude our results. Throughout the paper we will use the metric signature $(-,+,+,+)$, $c=1$ and that the Ricci tensor $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$.

2 $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$ gravity

Scalar tensor modified theories of gravity which are based on non-minimal coupling between matter and the geometry, have had very interesting applications in the thermodynamics context (See for instance []). Let us consider a very general theory based on a smooth arbitrary function $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$ on its arguments, where R, $R_{\alpha\beta}R^{\alpha\beta} \equiv Y$ and ϕ are the Ricci scalar, the Ricci invariant and the scalar field respectively within a scalar tensor context. The action of this modified theory reads [27],

$$S_m = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa^2}\left(f\left(R, R_{\alpha\beta}R^{\alpha\beta}, \phi\right) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}\right) + \mathcal{L}_m\right], \qquad (1)$$

where $\mathcal{L}_m$ and $\omega(\phi)$ are the matter Lagrangian density and a generic function of the scalar field ϕ respectively.

By varying the action (1) with respect to metric $g_{\mu\nu}$, the field equations obtained are:

$$\begin{aligned} & f_R R_{\mu\nu} - \frac{1}{2}\left(f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}\right)g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R + 2f_Y R^{\alpha}_{\mu}R_{\alpha\nu} \\ & -2[f_Y R^{\alpha}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta}g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} = \kappa^2 T^{(m)}_{\mu\nu}, \end{aligned} \qquad (2)$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, $f_R = \partial f/\partial R$, $f_Y = \partial f/\partial Y$ and $\kappa^2 \equiv 8\pi G$. The energy-momentum tensor for a perfect fluid is defined as

$$T^{(m)}_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu + p_m g_{\mu\nu}, \qquad (3)$$

where p_m, ρ_m and u_μ are the pressure, energy density and the four velocity of the fluid respectively. Hereafter, we will assume that the matter of the universe has zero pressure $p_m = 0$ (dust). An effective Einstein field equation from Eq. (2) can be written as (*error*)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_{eff}T^{(m)}_{\mu\nu} + T^{(d)}_{\mu\nu}, \tag{4}$$

where

$$G_{eff} = \frac{G}{\mathcal{F}}; \quad \mathcal{F} = f_R, \tag{5}$$

where G_{eff} is the effective gravitational matter and

$$T^{(d)}_{\mu\nu} = \frac{1}{\mathcal{F}}\Big[-\frac{1}{2}Rg_{\mu\nu}\mathcal{F} + \frac{1}{2}\left(f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}\right)g_{\mu\nu} + \mathcal{F}_{;\mu\nu} - g_{\mu\nu}\Box\mathcal{F} - 2f_Y \\ \times R^\alpha_\mu R_{\alpha\nu} + 2[f_Y R^\alpha_{(\mu}]_{;\nu)\alpha} - \Box[f_Y R_{\mu\nu}] - [f_Y R_{\alpha\beta}]^{;\alpha\beta}g_{\mu\nu} - \omega(\phi)\phi_{;\mu}\phi_{;\nu}\Big], \tag{6}$$

represents the energy-momentum tensor for dark components (*error*). The metric describing the FRW universe is

$$ds^2 = h_{\alpha\beta}dx^\alpha dx^\beta + \tilde{r}^2 d\Omega^2, \tag{7}$$

the 2-dimensional metric $h_{\alpha\beta} = diag\left(-1, \frac{a(t)^2}{1-kr^2}\right)$ with $(x^0, x^1) = (t, r)$, $a(t)$ is the scale factor and $k = \pm 1, 0$ is the spacial curvature. The second term is $\tilde{r} = a(t)r$ and $d\Omega^2 = d\theta^2 + \sin\theta^2 d\varphi^2$ is the 2-dimensional sphere with unit radius. The gravitational field equations for the metric (7) are given by

$$3\left(H^2 + \frac{k}{a^2}\right) = 8\pi G_{eff}\rho_m + \frac{1}{\mathcal{F}}\Big[\frac{1}{2}(R\mathcal{F} - f) - \frac{1}{2}\omega(\phi)\dot{\phi}^2 - 3H\partial_t\mathcal{F} - 6H \\ (2\dot{H} + 3H^2 + \frac{k}{a^2})\partial_t f_Y - f_Y\left(\dddot{H} + 4H\ddot{H} + 6\dot{H}H^2 - 2H^4 - \frac{4kH^2}{a^2}\right)\Big], \tag{8}$$

$$-\left(2\dot{H} + 3H^2 + \frac{k}{a^2}\right) = \frac{1}{\mathcal{F}}\Big[\frac{1}{2}(f - R\mathcal{F}) - \frac{1}{2}\omega(\phi)\dot{\phi}^2 + \partial_{tt}\mathcal{F} + 2H\partial_t\mathcal{F} \\ + \left(4\dot{H} + 6H^2 + \frac{2k}{a^2}\right)\partial_{tt}f_Y + 4H\left(\dot{H} + 3H^2 + \frac{2k}{a^2}\right)\partial_t f_Y + f_Y\Big(4\dddot{H} \\ +20H\ddot{H} + 10\dot{H}H^2 + 16\dot{H}^2 - 18H^4 - \frac{18k\dot{H}}{a^2} - \frac{20kH^2}{a^2} - \frac{18k^2}{a^4}\Big)\Big]. \tag{9}$$

Here, dots represents derivation with respect to the cosmic time t and $H = \dot{a}/a$ is the Hubble parameter. These equations can be rewritten as

$$3\left(H^2+\frac{k}{a^2}\right) = 8\pi G_{eff}(\rho_m+\rho_d), \tag{10}$$

$$-2\left(\dot{H}-\frac{k}{a^2}\right) = 8\pi G_{eff}\left(\rho_m+\rho_d+p_d\right), \tag{11}$$

where ρ_d and p_d are the energy density and pressure of dark components with $G = \mathcal{F}G_{eff}$, are given by

$$\begin{aligned}\rho_d &= \frac{1}{8\pi G}\left[\frac{1}{2}\left(R\mathcal{F}-f\right)-\frac{1}{2}\omega(\phi)\dot{\phi}^2-3H\partial_t\mathcal{F}-6H\left(2\dot{H}+3H^2+\frac{k}{a^2}\right)\right.\\ &\left.\times\partial_t f_Y - f_Y\left(\dddot{H}+4H\ddot{H}+6\dot{H}H^2-2H^4-\frac{4kH^2}{a^2}\right)\right],\end{aligned} \tag{12}$$

$$\begin{aligned}p_d &= \frac{1}{8\pi G}\left[\frac{1}{2}\left(f-R\mathcal{F}\right)-\frac{1}{2}\omega(\phi)\dot{\phi}^2+\partial_{tt}\mathcal{F}+2H\partial_t\mathcal{F}+\left(4\dot{H}+6H^2\right.\right.\\ &\left.+\frac{2k}{a^2}\right)\partial_{tt}f_Y+4H\left(\dot{H}+3H^2+\frac{2k}{a^2}\right)\partial_t f_Y+f_Y\left(4\dddot{H}+20H\ddot{H}\right.\\ &\left.\left.+10\dot{H}H^2+16\dot{H}^2-18H^4-\frac{18k\dot{H}}{a^2}-\frac{20kH^2}{a^2}-\frac{8k^2}{a^4}\right)\right].\end{aligned} \tag{13}$$

3 Generalized Thermodynamics laws

3.1 First Law of Thermodynamics

Here we analyse the validity of the first law of thermodynamics at the apparent horizon of FRW universe in $f(R,Y,\phi)$ gravity. The dynamical apparent horizon is derived by the relation $h^{\alpha\beta}\partial_\alpha\tilde{r}\partial_\beta\tilde{r} = 0$ from which we have the radius of apparent horizon $\tilde{r}_A = \left(H^2+\frac{k}{a^2}\right)^{-\frac{1}{2}}$. Taking time derivative of $\tilde{r}_A$ and using Eq. (11), we have

$$\mathcal{F}d\tilde{r}_A = 4\pi GH\tilde{r}_A^3(\hat{\rho_\nu}+\hat{p_\nu})dt, \tag{14}$$

where $\hat{\rho_\nu} = \hat{\rho_m}+\hat{\rho_d}$ and $\hat{p_\nu} = \hat{p_d}$. $d\tilde{r}_A$ represents the infinitesimal change in radius of the apparent horizon during a time interval dt. The temperature of apparent horizon is defined as $T_h = \frac{|K_{sg}|}{2\pi}$, where K_{sg} [20] is the surface gravity and is defined as $K_{sg} = -\frac{1}{\tilde{r}_A}(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A})$. In GR, the Bekenstein-Hawking defined the horizon entropy by the relation $S_h = A/4G$, where A is the area of the apparent horizon defined by $A = 4\pi\tilde{r}_A^2$ [9, 14, 15]. In the literature of modified theories of gravity , Wald [28] defined the horizon entropy with Noether charge. It can be obtained by varying the Lagrangian density of modified gravitational theories with respect to Riemann tensor. Wald entropy is defined as $\hat{S}_h = A/4G_{eff}$ [29], where G_{eff} is the effective gravitational coupling. The Wald entropy in $f(R, Y, \phi)$ gravity is defined as

$$\hat{S}_h = A\mathcal{F}/4G, \tag{15}$$

where $\mathcal{F} = f_R$. Differentiating Eq. (15) and using (14), we have

$$\frac{1}{2\pi\tilde{r}_A}d\hat{S}_h = 4\pi\tilde{r}_A^3\left(\hat{\rho}_\nu + \hat{p}_\nu\right)Hdt + \frac{\tilde{r}_A}{2G}d\mathcal{F}. \tag{16}$$

Multiplying both sides of above equation by $\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)$, we obtain

$$T_h d\hat{S}_h = -4\pi\tilde{r}_A^3\left(\hat{\rho}_\nu + \hat{p}_\nu\right)Hdt + 2\pi\tilde{r}_A^2\left(\hat{\rho}_\nu + \hat{p}_\nu\right)d\tilde{r}_A + \frac{\pi\tilde{r}_A^2 T_h}{G}d\mathcal{F}. \tag{17}$$

Now, we are defining the energy of the universe inside the apparent horizon. The Misner-Sharp energy defined in [30] is $E = \frac{\tilde{r}_A}{2G}$ and for $f(R, Y, \phi)$ gravity we can write it as [16, 31]

$$\hat{E} = \frac{\tilde{r}_A}{2G_{eff}}, \tag{18}$$

using volume $V = \frac{4}{3}\pi\tilde{r}_A^3$, we can write it as

$$\hat{E} = \frac{3V}{8\pi G_{eff}}\left(H^2 + \frac{k}{a^2}\right) = V\hat{\rho}_\nu, \tag{19}$$

which is the total energy inside the sphere of radius $\tilde{r}_A$. If we choose the effective gravitational coupling constant as positive in $f(R, Y, \phi)$ gravity then we have $G_{eff} = G/\mathcal{F} > 0$, from which we can conclude that $\hat{E} > 0$. From Eq's (10) and (19) we can write

$$d\hat{E} = 4\pi\tilde{r}_A^2\hat{\rho}_\nu d\tilde{r}_A - 4\pi\tilde{r}_A^3\left(\hat{\rho}_\nu + \hat{p}_\nu\right)Hdt + \frac{\tilde{r}_A}{2G}d\mathcal{F}. \tag{20}$$

Using Eq. (20) in (17), it follows that

$$T_h d\hat{S}_h = d\hat{E} - \hat{W} dV - \frac{(1 - 2\pi\tilde{r}_A T_h)\,\tilde{r}_A d\mathcal{F}}{2G}, \tag{21}$$

which involves the work density $\hat{W} = \frac{1}{2}(\hat{\rho}_\nu - \hat{p}_\nu)$ [32]. The above equation can be rewritten as

$$T_h d\hat{S}_h + T_h d_i\hat{S}_h = d\hat{E} - \hat{W} dV, \tag{22}$$

where

$$d_i\hat{S}_h = \frac{(1 - 2\pi\tilde{r}_A T_h)\,\tilde{r}_A d\mathcal{F}}{2G T_h} = \frac{\left(\hat{E} - \hat{S}_h T_h\right)}{T_h \mathcal{F}} d\mathcal{F}. \tag{23}$$

Comparing the $f(R,Y,\phi)$ gravity with GR, Lovelock gravity and Gauss-Bonnet gravity [33]-[35], we see an additional term $d_i\hat{S}_h$ in the first law of thermodynamics. We can call it the entropy production term which occurred due to the non-equilibrium behavior of $f(R,Y,\phi)$ gravity. From this result we have first law of thermodynamics for non-equilibrium behavior of $f(R)$ gravity [24] by setting $f(R,Y,\phi) = f(R)$. By choosing $f(R,Y,\phi) = R$, we can achieve the standard first law of thermodynamics in GR.

3.2 Generalized Second Law of Thermodynamics

In the circumstance of modified gravitational theories the GSLT has been discussed frequently [24, 25, 31, 36]. To check its validity in $f(R,Y,\phi)$ gravity, we have to prove the inequality [31]

$$\dot{\hat{S}}_h + d_i\dot{\hat{S}}_h + \dot{\hat{S}}_\nu \geq 0, \tag{24}$$

where $\hat{S}_h$, $d_i\dot{\hat{S}}_h = \partial_t(d_i\hat{S}_h)$ and $\hat{S}_\nu$ are horizon entropy, entropy due to all the matter inside the horizon and entropy due to energy sources inside the horizon. Now we continue with the Gibbs equation which includes the entropy of matter and energy fluid is given by [37]

$$T_\nu d\hat{S}_\nu = d(\hat{\rho}_\nu V) + \hat{\rho}_\nu dV, \tag{25}$$

where T_ν denote the temperature within the horizon. we are assuming here a relation between temperature within horizon and temperature of apparent horizon i.e., $T_\nu = bT_h$, where $0 < b < 1$ which guarantee the positivity of temperature and it is smaller than the horizon temperature. Substituting Eq's. (22) and (25) in Eq. (24), we obtain

$$\dot{\hat{S}}_{tot} = \dot{\hat{S}}_h + d_i\dot{\hat{S}}_h + \dot{\hat{S}}_\nu = \frac{2\pi\Sigma}{\tilde{r}_A bR} \geq 0, \qquad (26)$$

where

$$\Sigma = (1-b)\dot{\hat{\rho}}_\nu V + (1-\frac{b}{2})(\hat{\rho_\nu} + \hat{p_\nu})\dot{V},$$

which is the general condition to satisfy the GSLT in modified gravitational theories [31]. Using Eq's. (10) and (11), condition (24) is reduced to

$$\frac{2\pi\Xi}{Gb\left(H^2+\frac{k}{a^2}\right)\left(\dot{H}+2H^2+\frac{k}{a^2}\right)} \geq 0, \qquad (27)$$

where

$$\begin{aligned}\Xi &= (b-1)\partial_t f_R\left(H^2+\frac{k}{a^2}\right) + 2Hf_R(b-1)\left(\dot{H}-\frac{k}{a^2}\right) + (b-2)\\ &\quad \times f_R H\left(\dot{H}-\frac{k}{a^2}\right)^2\left(H^2+\frac{k}{a^2}\right)^{-1}. \qquad (28)\end{aligned}$$

To protect the GSLT the condition (26) is equivalent to $\Xi \geq 0$.

4 Validity of GSLT

We are using here some $f(R,\phi)$ models and the model constructed from $f(R,Y,\phi)$ gravity using power law method, and check the validity of $\dot{\hat{S}}_{tot} \geq 0$.

4.1 Model constructed from de-Sitter Universe

To explain the current cosmic era in cosmology the dS solutions are very importance. In dS universe scale factor, Hubble parameter, Ricci tensor and scalar field are defined as $a(t) = a_0 e^{H_0 t}$, $H = H_0$, $R = 12H_0^2$ and $\phi(t) \sim a(t)^\beta$, [38]. Substituting these terms we have constructed the model in our paper [40]

$$f(R,Y,\phi) = \alpha_1\alpha_2\alpha_3 e^{\alpha_1 R} e^{\alpha_2 Y}\phi^{\gamma_1} + \gamma_2\phi^{\gamma_3} + \gamma_4\phi^{\gamma_5},$$

where $\alpha_i's$ are constants of integration and

$$\gamma_1 = \frac{18\beta\alpha_1 H_0^2 - 108\beta\alpha_2 H_0^4 - 5 + 6\alpha_1 H_0^2 - 84\alpha_2 H_0^4}{6\left(H_0^2\alpha_1\beta - 6\beta\alpha_2 H_0^4\right)}$$

$$\gamma_2 = \omega_0\beta^2 H_0^2, \quad \gamma_3 = m+2, \quad \gamma_4 = -2\kappa^2\rho_0 a_0^3, \quad \gamma_5 = -\frac{3}{\beta}.$$

Introducing this model in (26) we have equation of the form

$$\hat{S}_{tot} = \frac{2\pi}{Gb}\Big[-12kH_0(b-1)\alpha_1^3\alpha_2\alpha_3 a_0^{\beta\gamma_1}e^{\alpha_1 R+\alpha_2 Y}\left(a_0^2H_0^2 + ke^{-2H_0t}\right)e^{-2H_0t} \times e^{\beta\gamma_1 H_0 t} - \frac{24}{a_0^2}kH_0(b-1)\left(a_0^2H_0^2 + ke^{-2H_0t}\right)\left(3a_0H_0^2e^{-3H_0t} + 2ke^{-4H_0t}\right)\alpha_1^2 \times\alpha_2^2\alpha_3 a_0^{\beta\gamma_1}e^{\alpha_1 R+\alpha_2 Y}e^{\beta\gamma_1 H_0 t} + \beta H_0(b-1)\alpha_1^2\alpha_2\alpha_3\gamma_1 a_0^{\beta+2}e^{\beta H_0 t}\left(a_0^2H_0^2 + k e^{-2H_0t}\right)e^{\alpha_1 R+\alpha_2 Y}a_0^{\beta(\gamma_1-1)}e^{\beta(\gamma_1-1)H_0t} - 2kH_0a_0^2(b-1)\alpha_1^2\alpha_2\alpha_3 e^{-2H_0t}a_0^{\beta\gamma_1} e^{\alpha_1 R+\alpha_2 Y}e^{\beta\gamma_1 H_0 t} + 2k^2H_0a_0^2\left(\frac{b}{2}-1\right)\alpha_1^2\alpha_2\alpha_3\left(a_0^2H_0^2 + ke^{-2H_0t}\right)^{-1}e^{-4H_0t} e^{\alpha_1 R+\alpha_2 Y}a_0^{\beta\gamma_1}e^{\beta\gamma_1 H_0 t}\Big]\left(a_0^2H_0^2 + ke^{-2H_0t}\right)^{-1}\left(2a_0^2H_0^2 + ke^{-2H_0t}\right)^{-1} \geq 0. \quad (29)$$

The GSLT constraint of dS $f(R,Y,\phi)$ depends on five parameters α_1, α_2, α_3, β and t. In this perspective, we are fixing two parameters and observe the feasible region by varying the possible ranges of other parameters. We are fixing α_1 and α_2 and show the results for $\hat{S}_{tot}$. Herein, we set the present day values of Hubble parameter and cosmographic parameters as $H_0 = 73.8$, $q = -0.81$, $j = 2.16$, $s = -0.22$ [39]. The feasible regions for all the possible cases for dS $f(R,Y,\phi)$ model are presented in Table 1.

Initially, we vary α_1 and α_2 to check the validity of $\hat{S}_{tot}$ for different values of α_3, β and t. If we set both α_1 and α_2 as positive then $\hat{S}_{tot}$ is valid for t, however β needs some particular ranges as ($\alpha_3 \geq 0$, $\beta \leq -0.78$) and ($\alpha_3 \leq 0$, $\beta \geq 0$). If $\alpha_1 < 0$ and $\alpha_2 > 0$, $\hat{S}_{tot}$ is valid for all values of t with ($\alpha_3 \geq 0$, $\beta \leq -0.78$) or ($\alpha_3 \leq 0$, $\beta \geq 0$). For ($\alpha_1 > 0$, $\alpha_2 < 0$), $\hat{S}_{tot}$ is valid for all values of α_3, β and t. For ($\alpha_1 > 0$, $\alpha_2 < 0$) and ($\alpha_1 < 0$, $\alpha_2 < 0$), $\hat{S}_{tot}$ is valid for all values of α_3, β and t. Evolution of GSLT constraint is shown here verses the parameters α_3, β and t by fixing α_1, α_2.

4.2 Model constructed from power Law method

Power solutions are very useful to discuss the different phases of cosmic evolution e.g., dark energy, matter and radiation dominated epochs. We are

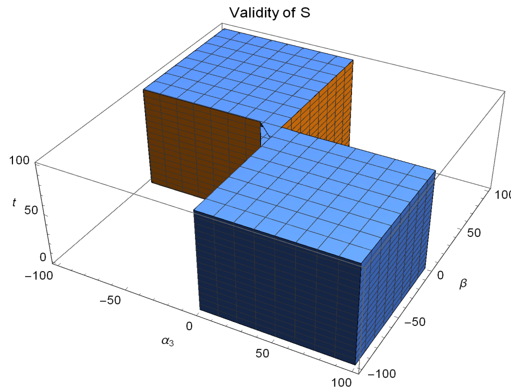

Figure 1: Variation of GSLT constraint for dS $f(R,Y,\phi)$ model with $\alpha_1 > 0$ and $\alpha_2 > 0$ and show the variation for all α_3 and β.

discussing here just one power law solution for $f(R,Y,\phi)$ gravity. For power law the scale factor is defined as [39, 44]

$$a(t) = a_0 t^n, \quad H(t) = \frac{n}{t}, \quad R = 6n(1-2n)t^{-2}, \tag{30}$$

where $n > 1$ shows the accelerating picture of the universe, $0 < n < 1$ leads to decelerated universe, ($n = \frac{2}{3}$) leads to dust dominated and ($n = \frac{1}{2}$) for radiation dominated. We have constructed the model $f(R,\phi)$ in our paper [40]. Here we are using this model to show the validity of GSLT

$$f(R,\phi) = \alpha_1\alpha_2\phi^{\gamma_1}R^{\gamma_2} + \gamma_3\phi^{\gamma_4} + \gamma_5\phi^{\gamma_6},$$

where $\alpha_i' s$ are constants of integration and

$$\begin{aligned}
\gamma_1 &= \frac{\alpha_1}{3n-1} + \frac{n-3}{n\beta} - \frac{2(3n-1)^2}{n^2\beta^2\alpha_1}, \quad \gamma_2 = \frac{n(n-3)\beta\alpha_1}{(3n-1)^2}, \\
\gamma_3 &= \omega_0\beta^2 n^2 a_0^{\frac{2}{n}}, \quad \gamma_4 = m + 2 - \frac{2}{n\beta}, \quad \gamma_5 = -2\kappa^2\rho_0 a_0^{3(1+w)}, \quad \gamma_6 = -\frac{3}{\beta}.
\end{aligned}$$

Substituting this model in (26) we get

$$\hat{S}_{tot} = \frac{2\pi}{Gb}\Bigg[H_0(b-1)\alpha_1\alpha_2\gamma_2(\gamma_2-1)a_0^{\beta\gamma_1}t^{n\beta\gamma_1}\left((j-q-2)\,H_0^2+\frac{k}{a_0^2t^{2n}}\right)$$
$$\times\left((1-q)H_0^2+\frac{k}{a_0^2t^{2n}}\right)^{\gamma_2-2}+\beta H_0\alpha_1\alpha_2\gamma_1\gamma_2\left((1-q)H_0^2+\frac{k}{a_0^2t^{2n}}\right)^{\gamma_2-1}$$
$$\times(b-1)a_0^{\beta\gamma_1}t^{n\beta\gamma_1}-2H_0(b-1)\alpha_1\alpha_2\gamma_2a_0^{\beta\gamma_1}t^{n\beta\gamma_1}\left((1-q)H_0^2+\frac{k}{a_0^2t^{2n}}\right)^{\gamma_2-1}$$
$$\times\left\{1+\frac{qH_0^2}{H_0^2+\frac{k}{a_0^2t^{2n}}}\right\}+2H_0\alpha_1\alpha_2\gamma_2a_0^{\beta\gamma_1}t^{n\beta\gamma_1}\left((1-q)H_0^2+\frac{k}{a_0^2t^{2n}}\right)^{\gamma_2-1}$$
$$\times\left(\frac{b}{2}-1\right)\left(1+\frac{qH_0^2}{H_0^2+\frac{k}{a_0^2t^{2n}}}\right)^2\Bigg]\left((1-q)H_0^2+\frac{k}{a_0^2t^{2n}}\right)^{-1}\geq 0. \qquad (31)$$

The above constraint have five parameters α_1, α_2, n, β and t. We are checking the validity of $\hat{S}_{tot}$ for different values of n, β and t by fixing α_1, α_2. All possible cases of this model are discussed in Table 1.

Lets Start with $\alpha_1 > 0$ and check the viabe ranges of α_2, β and t. In this case we have three cases depending on the choice α_2, (i) $\alpha_2 < 0$, $n \geq 3$ with ($\beta \leq -35.8$, $t \geq 1$) or ($\beta \geq 2.81$, $t \geq 0.94$), (ii) $\alpha_2 = 0$, $n > 1$ & $\forall\ t$ with $\beta > 0$ or $\beta < 0$, (iii) $\alpha_2 > 0$ with ($n \geq 8.6$, $0 < \beta \leq 20$, $t \geq 0.8$) or ($n \geq 12.7$, $-20 \leq \beta < 0$, $t \geq 0.9$). Next we take $\alpha_1 < 0$ and again we have three cases based on the selection of α_2, i $\alpha_2 < 0$ with ($n \geq 8.6$, $-20 \leq \beta < 0$, $t \geq 0.8$) or ($n \geq 12.7$, $0 < \beta \leq 20$, $t \geq 0.9$), (ii) $\alpha_2 = 0$, $n > 1$ & $\forall\ t$ with $\beta > 0$ or $\beta < 0$, (iii) $\alpha_2 > 0$, $n \geq 3$ & $\forall\ t$ with ($\beta \leq -28.1$) or ($\beta \geq 35.7$). It can be noted that taking α_1 and α_2 with same sign $\hat{S}_{tot}$ is not valid for initial values of t and n and β is restricted to $\beta \leq 20$ & $\beta \geq -20$. To show the feasible regions of this model we show the plot of GSLT inequality.

4.3 $f(R,\phi)$ Models

1. $f(R,\phi) = \frac{R-2\Lambda(1-e^{B\phi\kappa^3R})}{\kappa^2}$
2. $f(R,\phi) = R\left(\frac{\omega_0\beta^2n^2a_0^{2/n}(mn\beta+2n\beta+6n-2)}{mn\beta+2n\beta-2}\right)\phi^{m+2-\frac{2}{n\beta}}$
3. $f(R,\phi) = R(1+\xi\kappa^2\phi^2)$
4. $f(R,\phi) = \phi(R+\alpha R^2)$

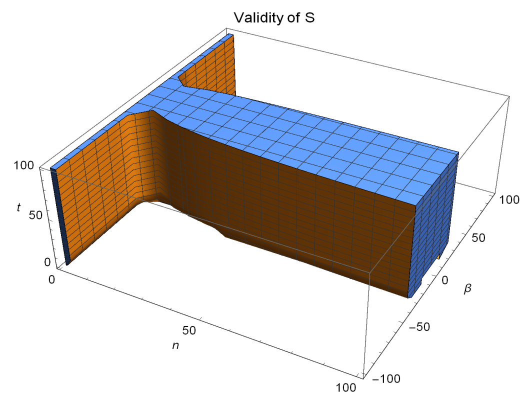

Figure 2: Plot of GSLT constraint for PL-$f(R,\phi)$ versus the parameters n, β and t with $\alpha_1 > 0$, $\alpha_2 = 5$.

4.3.1 Model-I

Myrzakulov et al. [41] has examined the spectral index and tensor-to-scalar ratio to describe the inflation in $f(R,\phi)$ theories and observed the results using the recent observational data. Here we are using the model

$$f(R,\phi) = \frac{R - 2\Lambda(1 - e^{B\phi\kappa^3 R})}{\kappa^2},$$

where κ^3 is introduced for dimensional reasons and b is a dimensionless number of order unity. Inserting the model in (26) the inequality becomes

$$\hat{S}_{tot} = \frac{2\pi}{Gb}\Bigg[2H_0(b-1)\Lambda B^2 a_0^{2\beta} t^{2n\beta}\kappa^4 R e^{Ba_0^\beta t^{n\beta}\kappa^3 R} + 2\Lambda B\kappa(b-1)\beta H_0 a_0^\beta t^{n\beta}$$
$$(1 + Ba_0^\beta t^{n\beta}\kappa^3 R)e^{Ba_0^\beta t^{n\beta}\kappa^3 R} - \frac{2}{\kappa^2}H_0(b-1)\{1 + 2\Lambda Ba_0^\beta t^{n\beta}\kappa^3 e^{Ba_0^\beta t^{n\beta}\kappa^3 R}\}$$
$$\left(1 + \frac{qH_0^2}{H_0^2 + \frac{k}{a_0^2 t^{2n}}}\right) + \frac{2H}{\kappa^2}\left(\frac{b}{2} - 1\right)\left\{1 + 2\Lambda Ba_0^\beta t^{n\beta}\kappa^3 e^{Ba_0^\beta t^{n\beta}\kappa^3 R}\right\}$$
$$\left(1 + \frac{qH_0^2}{H_0^2 + \frac{k}{a_0^2 t^{2n}}}\right)^2\Bigg]\left(H_0^2(1-q) + \frac{k}{a_0^2 t^{2n}}\right)^{-1}, \tag{32}$$

where $R = 6\left((1-q)H_0^2 + \frac{k}{a_0^2 t^{2n}}\right)$. Now we discuss the above constraint, which is depending on four parameters B, n, β and t. We find that $\hat{S}_{tot}$ is satisfied for two cases depending on the choice of B: (i) $B=0$ with $n>1$, $t\geq 0.96$ & $\forall\ \beta$, (ii) $B>0$ with $n>1$, $\beta\leq -0.6$ & $\forall\ t$ and $n\geq 2.5$, $\beta\geq 6$ & $t\geq 2.5$.

4.3.2 Model-II

We have derived $f(\phi)$ from Klein-Gordon equation by inserting $\omega(\phi)=\omega_0\phi^m$ and $\phi=a(t)^\beta$ given in [27] and compose the model of the form $f(R,\phi)=Rf(\phi)$ is given by

$$f(R,\phi)=R\left(\frac{\omega_0\beta^2n^2a_0^{2/n}(mn\beta+2n\beta+6n-2)}{mn\beta+2n\beta-2}\right)\phi^{m+2-\frac{2}{n\beta}},$$

where ω_0 and a_0 are constants. Introducing this model in (26) we find the constraint

$$\begin{aligned}\hat{S}_{tot}&=\frac{2\pi}{Gb}\Bigg[\beta^2H_0(b-1)\omega_0n\left(mn\beta+2n\beta+6n-2\right)a_0^{\beta(m+2)}t^{mn\beta+2n\beta-2}\\&-2H_0(b-1)\frac{\omega_0\beta^2n^2a_0^{2/n}\left(mn\beta+2n\beta+6n-2\right)}{mn\beta+2n\beta-2}a_0^{\beta\left(m+2-\frac{2}{n\beta}\right)}t^{mn\beta+2n\beta-2}\\&\left\{1+qH_0^2(H_0^2+\frac{k}{a_0^2}t^{2n})^{-1}\right\}+2\left(\frac{b}{2}-1\right)H_0a_0^{\beta\left(m+2-\frac{2}{n\beta}\right)}t^{mn\beta+2n\beta-2}\\&\frac{\omega_0\beta^2n^2a_0^{2/n}\left(mn\beta+2n\beta+6n-2\right)}{mn\beta+2n\beta-2}\left(1+\frac{qH_0^2}{H_0^2+\frac{k}{a_0^2t^{2n}}}\right)^2\Bigg]\\&\left(H_0^2(1-q)+\frac{k}{a_0^2t^{2n}}\right)^{-1}.\end{aligned}\tag{33}$$

One can see that the inequality of this model is depending on four parameters β, m, n and t. We will discuss the viability of $\hat{S}_{tot}$ for different values of β and m by fixing n. For $n>1$ and $\forall\ t$ we have two cases: $m\geq 2$ with $\beta\leq -1.5$ and $m\leq -3.2$ with $\beta\geq 5$. We show the plot of GSLT constraint verses the parameters m, β and t by fixing $n>1$.

4.3.3 Model-III

Now we are presenting the model which is applied to describe the cosmological perturbations for non-minimally coupled scalar field dark energy in both

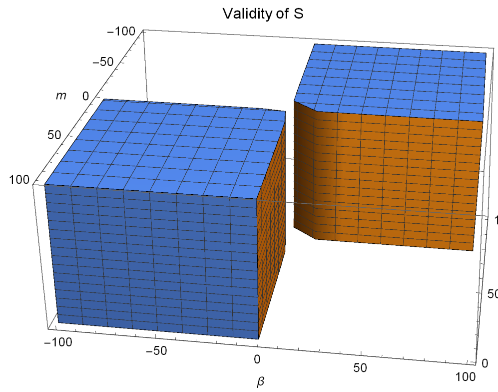

Figure 3: Variation of GSLT constraint for Model-II versus the parameters m, β and t with $n = 1.1$.

metric and Palatini formalisms [42].

$$f(R, \phi) = R(1 + \xi\kappa^2\phi^2),$$

where ξ is the coupling constant. Using this model in (26) we have

$$\hat{S}_{tot} = \frac{2\pi}{Gb}\Bigg[2\beta H_0\xi\kappa^2(b-1)a_0^{2\beta}t^{2n\beta} - 2H_0(b-1)\left(1+\xi\kappa^2 a_0^{2\beta}t^{2n\beta}\right)\times \left\{1 + qH_0^2\left(H_0^2 + \frac{k}{a_0^2}t^{2n}\right)^{-1}\right\} + 2\left(\frac{b}{2}-1\right)H_0\left(1+\xi\kappa^2 a_0^{2\beta}t^{2n\beta}\right) \left(1 + \frac{qH_0^2}{H_0^2 + \frac{k}{a_0^2 t^{2n}}}\right)^2\Bigg]\left(H_0^2(1-q) + \frac{k}{a_0^2 t^{2n}}\right)^{-1}.$$

Here we have four parameters n, ξ, β and t. By fixing n we will find the values of ξ and β for which $\hat{S}_{tot}$ is satisfied. For $n > 1$ it is valid for $\beta \leq -3.5$ with ($\forall\ \xi$, $t \geq 4$) and for $\beta \geq 0.15$ with ($\xi \leq 0$, $t \geq 1$).

4.3.4 Model-IV

To reproduce inflation, a very familiar gravitational action is defined as [43]

$$f(R, \phi) = \phi(R + \alpha R^2),$$

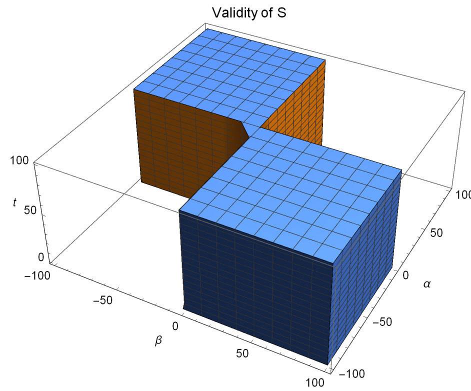

Figure 4: validity regions of GSLT constraint for Model-IV verses the parameters α and β with $n = 1.1$.

where α is a constant with suitable dimensions. Introducing this model in (26) we have inequality of the form

$$\hat{S}_{tot} = \frac{2\pi}{Gb}\Bigg[12\alpha H_0(b-1)a_0^{\beta}t^{n\beta}\left\{(j-q-2)H_0^2+\frac{k}{a_0^2t^{2n}}\right\}+\beta H_0(b-1) \times a_0^{\beta}t^{n\beta}\left\{1+12\alpha\left((1-q)H_0^2+\frac{k}{a_0^2t^{2n}}\right)\right\}-2H_0(b-1)a_0^{\beta}t^{n\beta}\Bigg\{1+12\alpha \times\left((1-q)H_0^2+\frac{k}{a_0^2t^{2n}}\right)\Bigg\}\left\{1+qH_0^2\left(H_0^2+\frac{k}{a_0^2t^{2n}}\right)^{-1}\right\}+2\left(\frac{b}{2}-1\right)\times H_0a_0^{\beta}t^{n\beta}\left\{1+12\alpha\left((1-q)H_0^2+\frac{k}{a_0^2t^{2n}}\right)\right\}\left(1+\frac{qH_0^2}{H_0^2+\frac{k}{a_0^2t^{2n}}}\right)^2\Bigg] \left(H_0^2(1-q)+\frac{k}{a_0^2t^{2n}}\right)^{-1}. \quad (34)$$

Here we intend to discuss the validity of $\hat{S}_{tot}$ and constrain the parameters like n, α, β and t. For $n > 1$ we have two cases depending on the choice of α: (i) $\alpha < 0$ with $(\beta \geq 0, \forall\ t)$ (ii) $\alpha \geq 0$ with $(\beta \leq -0.25,\ t \geq 1)$. We are presenting here the evolution of GSLT to show some viable regions in this case.

Models	Variations of parameters	Validity of S
de-Sitter Model $f(R,Y,\phi)$	$\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 < 0$, $\alpha_2 > 0$	$\forall\, t$ with ($\alpha_3 \geq 0$, $\beta \leq -0.78$) or ($\alpha_3 \leq 0$, $\beta \geq 0$)
	$\alpha_1 > 0$, $\alpha_2 < 0$ and $\alpha_1 < 0$, $\alpha_2 < 0$	$\forall\, \alpha_3$, β & t
Power Law Model $f(R,\phi)$	$\alpha_1 > 0$ with $\alpha_2 < 0$	$n \geq 3$ with ($\beta \leq -35.8$, $t \geq 1$) or ($\beta \geq 2.81$, $t \geq 0.94$)
	$\alpha_2 = 0$	$n > 1$, $\beta > 0$ or $\beta < 0$, $\forall\, t$
	$\alpha_2 > 0$	($n \geq 8.6$, $0 < \beta \leq 20$, $t \geq 0.8$) or ($n \geq 12.7$, $-20 \leq \beta < 0$, $t \geq 0.9$)
	$\alpha_1 < 0$ with $\alpha_2 < 0$	($n \geq 8.6$, $-20 \leq \beta < 0$, $t \geq 0.8$) or ($n \geq 12.7$, $0 < \beta \leq 20$, $t \geq 0.9$)
	$\alpha_2 = 0$	$n > 1$, $\beta > 0$ or $\beta < 0$, $\forall\, t$
	$\alpha_2 > 0$	$n \geq 3$ & $\forall\, t$ with ($\beta \leq -28.1$) or ($\beta \geq 35.7$)
Model-I	$B = 0$	$n > 1$, $\forall\, \beta$, $t \geq 0.96$
	$B > 0$	($n > 1$ with $\beta \leq -0.6$, $\forall\, t$) or ($n \geq 2.5$ with $\beta \geq 6$, $t \geq 2.5$)
	$B < 0$	not valid
Model-II	$n > 1$	$\forall\, t$ with ($m \geq 2$, $\beta \leq -1.5$) or ($m \leq -3.2$, $\beta \geq 5$)
Model-III	$n > 1$	($\forall\, \xi$, $\beta \leq -3.5$ & $t \geq 4$) or ($\xi \leq 0$, $\beta \geq 0.15$ & $t \geq 1$)
Model-IV	$n > 1$	($\beta \geq 0$, $\alpha < 0$ & $\forall\, t$) or ($\beta \leq -0.25$, $\alpha \geq 0$ & $t \geq 1$)

Table 1: Validity regions of S for different models.

5 Conclusions

To analyse the accelerated cosmic expansion and to estimate the universe destiny, scalar tensor theories of gravity are very effective. $f(R, R_{\mu\nu}R^{\mu\nu}, \phi)$ is one of more general modified gravity, which involves the contraction of Ricci tensors $Y = R_{\mu\nu}R^{\mu\nu}$ and scalar field ϕ. In this paper, the thermodynamic study is implemented to the more general modified theory $f(R, R_{\mu\nu}R^{\mu\nu}, \phi)$ which can be considered as an extended form of $f(R)$ and $f(R, \phi)$ gravity. However the laws of thermodynamics for $f(R)$ and $f(R, \phi)$ and other modified theories have been established but there is a remarkable difference between results of this theory and other modifications. Further we have consider some models to check the viability of GSLT. The general dS case $f(R, Y, \phi)$ the GSLT constraint is depending on five parameters α_1, α_2, α_3, β and t. In this perspective, we are fixing α_1 and α_2 and show the viable region by varying the other parameters. In power law case $f(R, \phi)$ by varying α_1, α_2 we have examined the feasible constraints on β, n and t. Next we have considered four models of $f(R, Y, \phi)$ gravity independent of Y, which are of the form $f(R, \phi)$, $Rf(\phi)$, $\phi f(R)$. Model-I is depending on four parameters B, n, β and t, we have checked the validity of $\hat{S}_{tot}$ by varying B. Model-II is a function of four parameters β, m, n and t, by fixing n we will discuss the viability of $\hat{S}_{tot}$ for different values of β, m and t. In model-III the constraint is depending on four parameters n, ξ, β and t. By fixing $n > 1$ we examined the possible regions for the other parameters. Next in model-IV we have four parameters n, α, β and t. For $n > 1$ we have find the feasible constraints on other parameters.

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