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Morten O. Ravn

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ABSTRACT

This paper introduces a labor force participation choice into a labor market matching model embedded in a dynamic stochastic general equilibrium set-up with production and savings. The participation choice is modelled as a tradeoff between forgoing the expected benefits of being search active and engaging in costly labor market search. The model induces a symmetry in firms' and workers' search decision since both sides of the labor market vary search effort at the extensive margins. We show that this set-up is of considerable analytical convenience and that it gives rise to a linear relationship between labor market tightness and the marginal utility of consumption. We refer to the latter as the "consumption - tightness puzzle" because (a) it gives rise to a number of counterfactual implications, and (b) it is a robust implication of theory. Amongst the counterfactual implications are very low volatility of tightness, procyclical unemployment, and a positively sloped Beveridge curve. These implications all derive from procyclical variations in participation rates that follow from allowing for the extensive search margin.

Morten O. Ravn
Department of Economics
European University Institute
Via della Piazzuola 43
FI-50133 Florence
ITALY
and NBER
morten.ravn@iue.it

1 Introduction

This paper analyzes the properties of a Mortensen and Pissarides (1994), Pissarides (2000) style labor market matching model extended with a labor market participation choice embedded in a stochastic growth model. The bulk of modern business cycle theories assume instantaneous and costless matching of employers and workers, see e.g. Christiano and Eichenbaum (1992), Hansen (1985), or Prescott (1986). Labor market matching models instead realistically assume that it takes time and resources to match firms wishing to fill job vacancies with workers looking for jobs. This labor market matching process introduces frictional unemployment and it places the labor market in a central role in the transmission of shocks over time and across agents. Therefore, it is not surprising that this framework, which has proven *extremely* successful as a tool for understanding the long-run determinants of unemployment (see e.g. the review of Ljungqvist and Sargent, 2005), is receiving growing interest in the business cycle literature see e.g. Andolfatto (1996), Cheron and Langot (2004), den Haan, Ramey and Watson (2000), Gertler and Trigari (2005), Hall (2005), Merz (1995) and Shimer (2005).

In the Mortensen-Pissarides set-up, the labor market matching process is modelled on the basis of a matching function that relates the number of new job matches to the number of search active unmatched agents and to the number of job vacancies posted by firms. When deciding upon the number of job vacancies to post, firms consider the costs of setting aside resources to open a job vacancy relative to the expected benefits that a successful job match produces. Thus, on the part of firms, matching models allow for variations in the extensive search margin.

On the part of workers instead, most applications of matching / search theories in the business cycle literature assume that the labor market participation rate is constant. Therefore, variations in the extensive search margin occur only through changes in the net-hiring rate (the difference between the number of new job matches and the termination of existing job-worker relationships). This assumption might seem natural given that the labor market participation rate does not vary much over the business cycle. We argue that this latter argument is misleading on several grounds. First, consistently with the theory that we propose, US labor market participation rates display procyclical movements. Secondly, it is important to understand whether the relatively low volatility of the participation rate is consistent with economic theory. Third, in order to ask whether theory can account for the empirically observed moments of unemployment, the measurements of unemployment in the data and in theory need to be consistent and this requires one to introduce a participation choice. Finally, and this is a main contribution of this paper, we show that the matching model extended with a participation choice provides a series of strong predictions for indicators that are central in labor market matching models and for variables that are at the heart of business cycle research.

We assume that in order to participate in labor market activities, agents need to give up leisure which enables them to search for a job. In return, consistently with Flinn and Heckman (1983), search active agents face a potentially more favorable labor market outcome than non-participants. In particular, we assume that the matching probability of the former group of agents is higher than the matching probability of the latter group. Therefore, the participation choice is based upon the trade-off between giving up leisure to be search active vs. the expected (extra) benefits of being search active.

We are by no means the first to introduce a participation choice into models of labor market search and/or labor market matching. Burdett et al (1984) analyze and estimate a three state labor market search model with a participation choice, see also Bowlus (1997). Andolfatto and Gomme (1996) study a search model with a participation choice in order to analyze the effects of labor market policies. Following Pissarides (2000), a number of papers have analyzed matching models with a participation choice. Garibaldi and Wasmer (2005), Haefke and Reiter (2006), Pries and Rogerson (2004) and Yip (2003) all analyze dynamic search models with a participation choice in which shocks to the value of non-participation relative to participation generates flows in and out of the labor market. Each of these papers examine models without savings and assume incomplete markets. The current paper instead introduces production and savings. As argued by Hall (2006), savings and self-insurance are key when accounting for the search incentives of the unemployed. We assume complete markets since this gives rise to a much simpler framework than the more complicated incomplete markets settings. Furthermore, it appears that the complete markets setting emulates very well the main properties of the, perhaps more realistic, incomplete markets self-insurance model, see Hall (2006). Moreover, our analysis allows for risk aversion and we show that this is a key parameter. Similar complete markets settings have been analyzed by Veracierto (2003) and by Ravn (2005). Veracierto (2003) introduce a labor market participation choice into a Lucas-Alvarez type (island) search model with production and savings assuming complete markets. Ravn (2005) estimates a more complicated version of the model that is analyzed in the current paper. The main innovation of the current paper relative to Veracierto (2003) and Ravn (2005) is that we are able to derive a simple and robust relationship between labor market tightness and consumption that appears to have been overlooked in previous research.

The model that we study introduces a symmetry between firms' and workers' search activities since both sides of the labor market vary their search efforts at the extensive margin. This symmetry is shown to have important consequences and, surprisingly, turns out to be of considerable analytical convenience. When allowing for variations in the labor market participation rate, the first-order condition for households' search intensity along the extensive margin resembles the more familiar vacancy posting condition that derives from the firms' problem. In particular, variations in households' search intensity along the extensive margin equalize the marginal costs of search (the utility value of the loss of leisure) with the expected marginal benefit of labor market search which is the product of the probability that labor market search produces a match and the marginal benefit of being employed.

When this insight is combined with the assumption that wages are determined according to a (post-match) Nash bargain, it implies a linear relationship between labor market tightness and the marginal utility of consumption, a result that we refer to as the "*consumption - tightness puzzle*". This allows us to fully characterize the cyclical variations in labor market tightness on the basis of the cyclical variations in consumption. Therefore, a great advantage of our analysis is that we derive a simple testable relationship which does not depend upon the source of shocks to the economy nor on the persistence of these shocks.

We frame this relationship a puzzle for the following reasons. First, it implies very low volatility of the $vu-$ratio (or extreme volatility of consumption) since the standard deviation of the logarithm of the $vu-$ratio should equal the standard deviation of the logarithm of consumption times the curvature of the marginal utility of consumption. The latter is the coefficient of relative risk aversion (or the inverse of the intertemporal elasticity of substitution, the IES) and standard estimates of this parameter are small and values above 5 are usually claimed to be implausible. In contrast, in U.S. quarterly data, the standard deviation of the $vu-$ratio is around 20 times higher than the standard deviation of consumption at the business cycle frequencies. Thus, theory can account for maximum 25 percent of the observed volatility of the $vu-$ratio. Said differently, the model implies *procyclical* unemployment since vacancies are not only slightly more procyclical than realistic measures of the marginal utility of consumption, but also display much higher volatility. Alternatively, this latter insight insight can be formulated in terms of the slope of the Beveridge curve, which is positive in the model but negative in the data.

The intuition for why the matching model with an extensive search margin implies a positive correlation between unemployment and vacancies is straightforward. Consider a situation in which firms decide to post more vacancies. This increases households' payoff from labor market participation since, for a given unemployment rate, the probability that a job search results in a job-match increases. Therefore, there will be a positive correlation between vacancies and labor market participation. Moreover, since higher unemployment, all other things given, increases the returns from posting job vacancies, firms react by increasing job vacancies. This mechanism introduces a positive correlation between unemployment and vacancies unless the variations in labor market tightness are related to large (inversely signed) variations in the marginal utility of consumption and we argue that the latter is empirically implausible. This positive correlation between unemployment and vacancies also explains why the volatility of labor market tightness is low.

We show that these insights are robust. We study four extensions of the model. First we introduce an intensive search margin. We assume that agents can vary their search effort but that higher search effort is costly. We show that this extension leaves the consumption - tightness puzzle unaltered for plausible parametrizations of the search effort costs. Next, we introduce home-production. In this set-up the participation choice is a trade-off between forgoing the benefits of labor market search and giving up the resources generated by home-production. This implies a modification to the relationship between consumption (of market goods) and tightness but we argue that it might possibly worsen the consumption - tightness puzzle. The reason is that the implied relationship no longer depends on the intertemporal elasticity of substitution of consumption.

The final two extensions alter the assumptions on the matching framework. We first allow for "passive search", i.e. for the possibility that non-participants might be matched with a job vacancy despite not actively searching for a job. This set-up is potentially consistent with the fact that there are substantial flows from out-of-the-labor-force to employment. This extension does address the consumption- tightness puzzle because there is less incentive to become search active when non-participation also allows agents to find jobs. Nevertheless, for realistic assumptions regarding the size of flows into employment from unemployment and from non-participation, the consumption - tightness puzzle is approximately unchanged. In the second setting we assume that the matching technology is duration dependent. In particular, we assume that unemployed workers might be faced with either an efficient or an inefficient matching technology where the latter is associated with a smaller matching probability than the former. This set-up gives rise to a relationship between consumption and an altered version of tightness defined as the ratio of vacancies to the measure of search active workers faced with the inefficient matching function (who are on average longer term unemployed). In this case we leave it open whether duration dependence is important for the consumption - tightness puzzle because it is hard to match the implied measure of unemployment with official unemployment statistics.

A key aspect of the labor market matching model with an extensive search margin is that the participation rate should be procyclical (positively correlated with consumption). Such procyclical movements in the participation rate can actually be observed in U.S. data. In particular, the secular rise in the participation rate that has occurred in the U.S. over the last 60 years slowed down in each of the recessions dated by the NBER Business Cycle Dating Committee. Furthermore, we show that there exists a positive correlation between consumption and the participation rate at the business cycle frequencies. However, participation rates lag around a year after consumption and the elasticity of the participation rate to consumption is very low. We argue that future research need to look into the reasons for why labor market participation, although procyclical, vary little over the business cycle.

The remainder of the paper is structured as follows. Section 2 presents the basic model and derives the main result on the relationship between consumption and labor market tightness. Section 3 extends the basic set-up to include, in turn, an intensive search margin, homework, passive search, and duration dependent matching functions. Section 4 discusses the implications for variations in participation rates. Finally, Section 5 concludes and summarizes.

2 The Model

We study a stochastic optimal growth model combined with a labor market matching modeling of the labor market akin to Andolfatto (1996) and Merz (1995). We introduce a participation choice modelled as a trade-off between forgoing the opportunity of finding a job and the cost of giving up leisure in order to engage in labor market search activities. We show that introducing the extensive search margin (the participation choice) has fundamental implications.

2.1 Preferences and Technology

There is a measure one of households. Households consist of a continuum of agents and it is assumed that households pool the idiosyncratic labor market risk of their members. At any point in time a measure n_t of the household members are employed and earn labor income, a measure u_t are non-employed but search active, and a measure $(1 - n_t - u_t)$ are out of the labor force. Unemployment is measured by the second group of agents. Thus, consistently with the measurement of U.S. unemployment, we define unemployed agents as being characterized by (i) not being matched with an employer, but (ii) actively searching for a job.

Employed household members supply l_t hours of work and, as in the labor hoarding model of Burnside and Eichenbaum (1996), there is a fixed leisure cost $s \geq 0$ of engaging in labor market activities. Non-employed search active household members also face the fixed cost s of participating in labor market activities. Non-participants instead enjoy their entire time endowment as leisure.

The period utility function of a household member is given as:

$$u(c_{it}, e_{it}) = G(c_{it}) + H(e_{it}) \tag{1}$$

where c_{it} denotes consumption and e_{it} denotes leisure given labor market status $i = n, u, l$. We denote by $i = n$ that the household member is employed, by $i = u$ that the household member is unemployed, and by $i = l$ that the household member is not participating. The time-endowment is normalized to one unit. It follows that $e_{nt} = 1 - l_t - s$, $e_{ut} = 1 - s$, and $e_{lt} = 1$.

The flow utility of a representative *household* is then given as:

$$u(c_t, e_t) = G(c_t) + n_t H(1 - l_t - s) + u_t H(1 - s) + (1 - n_t - u_t) H(1)$$

Here we have used the risk sharing principle which, due to separability of preferences, implies that each household member consumes the same amount of goods regardless of their labor market status.

The sub-utility functions G and H are assumed to be increasing and strictly concave. We restrict $G(c_t)$ to be of the form $G(c_t) = c_t^{1-\eta}/(1-\eta)$ for $\eta > 0$ and $\eta \neq 1$ or $G(c_t) = \ln c_t$. The parameter $1/\eta$ is the intertemporal elasticity of substitution (the IES from now on). Utility is assumed to be additively separable over time:

$$W_t = E_t \sum_{j=t}^{\infty} \beta^{j-t} u(c_{t+j}, e_{t+j})$$

where E_t denotes the expectations operator conditional on information available at date t, and $\beta < 1$ is the subjective discount factor.

Firms with vacancies and unemployed workers meet randomly in an anonymous matching market. Matches are formed according to the following matching function:

$$m_t = M(v_t, u_t) \tag{2}$$

where m_t is the measure of new matches between a measure of u_t unemployed workers and v_t vacant jobs in period t. The function M is assumed to be increasing and concave in each of its arguments, and to be homogeneous of degree one in vacancies and unemployment jointly. Given the constant returns assumption, we can express the matching function as:

$$m_t = u_t \varphi(\theta_t)$$

where $\theta_t = v_t/u_t$ is the ratio of vacancies to unemployment, and $\varphi(\theta_t) \equiv M(\theta_t, 1)$. Thus, the probability that a search active worker finds a job vacancy, $\gamma_t^h = m_t/u_t = \varphi(\theta_t)$, is an increasing function of θ_t while the probability that a job vacancy is matched with an unemployed worker, $\gamma_t^f = m_t/v_t = \varphi(\theta_t)/\theta_t$, is a decreasing function of θ_t. It follows that $\gamma_t^h/\gamma_t^f = \theta_t$. Hence, it is clear that the $vu-$ratio, aka labor market tightness, is a key variable since it determines the matching market prospects of firms and workers.

The matching technology above assumes that non-participants do not receive any job offers.[1] We later examine the consequences of allowing for "passive search" as well, i.e. assuming that job offers might arrive at non-participants.

Each period firms and employed households face an exogenously given probability that their match is terminated. This probability is given by $\sigma_t \in [0; 1]$. Thus, the transition equation for employment is given as:

$$n_{t+1} = (1 - \sigma_t) n_t + u_t \varphi(\theta_t) \tag{3}$$

We assume that the job-separation rate follows an autoregressive process:

$$\ln \sigma_{t+1} = (1 - \rho_\sigma) \ln \sigma + \rho_\sigma \ln \sigma_t + \varepsilon_{t+1}^\sigma \tag{4}$$

where $\rho_\sigma \in (-1; 1)$, $\sigma > 0$ denotes the unconditional mean of σ_t, ε_{t+1}^σ is assumed to be normally and independently distributed over time with mean 0 and variance v_σ.

Output is produced using inputs of labor (the product of employment and hours worked per employee), $n_t l_t$, capital, k_t, and is subject to stochastic productivity shocks, z_t. We assume that firms take capital rental rates, r_t, and the price of output (the numeraire) for given. As in Andolfatto (1996) we assume that firms have a number of different jobs that may either be filled, posted in the vacancy market, or dormant. If firms decide to post a vacancy it must pay a resource cost $\kappa > 0$ per vacancy per period. In equilibrium, firms determine the optimal number of vacancies by maximizing their profits taking into account the costs and benefits of vacancy postings. The firms are owned by the households and their profits are paid out to the households as dividends.

The production function is specified by:

$$y_t = f(k_t, n_t l_t, z_t) \tag{5}$$

which we assume satisfies the Inada conditions, is increasing and strictly concave in k_t and in $n_t l_t$, and homogeneous of degree one in $(k_t, n_t l_t)$. The process for productivity shocks is assumed to be stationary but possibly persistent:

$$\ln z_{t+1} = (1 - \rho_z) \ln \overline{z} + \rho_z \ln z_t + \varepsilon_{t+1}^z \tag{6}$$

where $\rho_z \in (-1; 1)$, $\overline{z} > 0$ denotes the unconditional mean of z, and ε_{t+1}^z is assumed to be normally and independently distributed over time with mean 0 and variance v_ε.

The capital stock evolves over time according to the standard neoclassical specification:

$$k_{t+1} = (1 - \delta) k_t + i_t \tag{7}$$

[1] Bowlus (1997) makes the same assumption in a search model as do Haefke and Reiter (2006).

where $\delta \in (0,1)$ denotes the depreciation rate, and i_t is gross investment.

The resource constraint of the economy is then given by:

$$y_t \geq c_t + i_t + \kappa v_t \tag{8}$$

We assume that wages are determined according to a standard Nash bargaining over the joint match surplus of a worker-job pair. We let ϑ denote the bargaining weight of the workers. We do not impose the Hosios (1990) condition since our results will hold regardless of this efficiency consideration.

We will now derive the implications of this model on the basis of the competitive search equilibrium. Given the recursive structure of the model, we remove time indices and use the notation x' to denote the next period value of the variable x.

2.2 The Households' Problem

The maximization problem of the representative household can be formulated on the basis of the following Bellman equation:

$$\begin{aligned} J(k,n) &= \max_{(c,k',u,n')} \{c^{1-\eta}/(1-\eta) + nH(1-l-s) + uH(1-s) \\ &\quad + (1-n-u)H(1) + \beta EJ(k',n')\} && (9) \\ c + k' &\leq (1-\delta+r)k + wnl + \pi && (10) \\ n' &= (1-\sigma)n + \gamma^h u && (11) \end{aligned}$$

$J(k,n)$ denotes the representative households' value function which depends on its holdings of capital and the share of the household members that are employed.[2] We use the notation Ex' to denote the expectation of x' conditional on all available current information (including the transition laws for the exogenous shocks and the aggregate state variables). Equation (10) is the budget constraint which states that total spending on consumption (c) and capital for the next period (k') cannot exceed the sum of the value of its remaining capital stock ($k - \delta k$), rental income from capital (rk), labor income (wnl), and the dividends received from its ownership of the firms (π).

Equation (11) is the households' employment transition function. It relates the share of household members that are employed next period (n') to this period's employment (n) corrected for *net* new employment. The latter is given by the number of new job-worker matches, $\gamma^h u$, less the separations of currently employed household members from their jobs, σn. Importantly, individual households take the matching probability, γ^h, for given.

The first-order conditions for c, k', u, and n', in that order, are given by:

$$\begin{aligned} c^{-\eta} &= \lambda_c && (12) \\ \lambda_c &= \beta E J_{k'}(k',n') && (13) \\ H(1) - H(1-s) &= \gamma^h \lambda_n && (14) \\ \lambda_n &= \beta E J_{n'}(k',n') && (15) \end{aligned}$$

[2] We simplify the notation slightly for presentational purposes. The state variables of the households include also the aggregate capital stock, aggregate employment, and the stochastic variables, z and σ.

and the envelope conditions are:

$$J_k(k,n) = \lambda_c(1-\delta+r) \tag{16}$$
$$J_n(k,n) = \lambda_c wl + (1-\sigma)\lambda_n + H(1-l-s) - H(1) \tag{17}$$

Combining the first-order conditions for u and n' implies that:

$$\gamma^h \beta E J_{n'}(k', n') = H(1) - H(1-s) \tag{18}$$

Equation (18) is key. The right hand side of this expression is the utility loss associated with a marginal change in the share of household members that are search active rather than non-participating. This utility loss comes from the fact that search active agents need to spend time on search activities that non-participants instead enjoy as leisure. The left hand side of the expression is the expectation of the change in the value of employment produced by a marginal change in the number of search active household members. This is given by the probability that a search active agent is matched with a vacancy, γ^h, times the expected marginal value of employment next period, $EJ_{n'}$, discounted at the rate of β.

Combining (18) with (17) gives us that:

$$\begin{aligned} &\frac{H(1)-H(1-s)}{\gamma^h} \\ &= \beta E\left\{w'l'c'^{-\eta} + (1-\sigma')\frac{H(1)-H(1-s)}{\gamma^{h\prime}} - (H(1) - H(1-l'-s))\right\} \end{aligned} \tag{19}$$

This is similar to the more familiar vacancy creation condition (which we derive below). It sets the "cost" of labor market search equal to the expected benefits. The latter consists of the sum of the (utility value of the) marginal increase in labor income and the future search costs savings less the utility value of the loss of leisure associated with working rather than enjoying the entire time-endowment as leisure.

2.3 The Firms' Problem

Bellman's equation for the firms' problem is given as:

$$Q(n) = \max_{k,v,n'}\left\{F(k,nl) - wnl - \kappa v - rk + \beta E\frac{u_c'}{u_c}Q(n')\right\} \tag{20}$$
$$n' = (1-\sigma)n + \gamma^f v \tag{21}$$

where $Q(n)$ is the value of a firm with n filled jobs. The objective function consists of the current profit flow, $\pi = F(k,nl) - wnl - \kappa v - rk$ plus the discounted expected future value. The maximization takes place subject to the job transition function which links the future number of filled jobs to the current stock of filled jobs plus net hiring where the latter is the difference between new hires, $\gamma^f v$, and exogenous terminations of current jobs, σn.

The first order conditions for this problem can be formulated as:

$$F_k = r \tag{22}$$
$$\frac{\kappa}{\gamma^f} = \beta E\frac{u_c'}{u_c}(Q_{n'}(n')) \tag{23}$$

and from the envelope condition it follows that:

$$Q_n(n) = F_n - wl + (1-\sigma)\beta E \frac{u_c'}{u_c}\left(Q_{n'}(n')\right) \tag{24}$$

Combining this with (23) gives us:

$$\frac{\kappa}{\gamma^f} = \beta E \frac{u_c'}{u_c}\left(F_{n'} - w'l' + (1-\sigma')\frac{\kappa}{\gamma^{f'}}\right) \tag{25}$$

Condition (22) equalizes the rental rate of capital with the marginal product of capital. Equation (23) is the condition for the optimal number of vacancy postings. The latter sets the vacancy posting cost, κ, equal to the expected discounted value of posting a vacancy which is given by the probability that a vacancy results in a new hire, γ^f, times the marginal value of filling a vacancy, $Q_{n'}(n')$, discounted by $\beta\frac{u_c'}{u_c}$. The value of filling a vacancy, in turn, is the sum of the marginal profit (the difference between the marginal product of a hire and the marginal wage cost) plus the expected future vacancy posting cost savings. Combining these expressions gives us the condition for vacancy postings given in (25).

2.4 Wages

Wages are determined by ex-post (after matching) Nash bargaining. This implies that employers and workers share the joint match surplus according to their bargaining power. Let $\vartheta \in (0;1)$ denote the firms' bargaining power and let S_n denote the joint match surplus. The match surplus is given as:

$$S_n = Q_n(n) + \frac{1}{c^{-\eta}} J_n(k,n)$$

and the surplus is divided so that:

$$\vartheta J_n(k,n) = c^{-\eta}(1-\vartheta) Q_n(n) \tag{26}$$

where Q_n and J_n were derived above. Evaluating condition (26) for the next period and taking expectations given today's information set, we have that:

$$\vartheta\beta E c^{\eta} J_{n'}(k',n') = (1-\vartheta)\beta E \frac{c'^{-\eta}}{c^{-\eta}} Q_{n'}(n')$$

This condition simplifies using the first order conditions from the households' and the firms' problems. In particular, we have that:

$$\begin{aligned}(1-\vartheta)\beta E \frac{c'^{-\eta}}{c^{-\eta}} Q_{n'} &= (1-\vartheta)\frac{\kappa}{\gamma^f} \\ \vartheta\beta E c^{\eta} J_{n'} &= \vartheta \frac{H(1) - H(1-s)}{\gamma^h} c^{\eta}\end{aligned}$$

Therefore the Nash bargaining outcome implies that:

$$(1-\vartheta)\frac{\kappa}{\gamma^f} = \vartheta\frac{H(1) - H(1-s)}{\gamma^h} c^{\eta} \tag{27}$$

which is the key relationship that we discuss below.

Using these we can then derive the equilibrium wage bill per employee as:

$$wl = (1 - \vartheta) F_n + \vartheta c^\eta [H(1) - H(1 - l - s)]$$

which determines the wage as a weighted average of the marginal product of employment and the utility weighted leisure cost of working rather than enjoying the endowment as leisure.

2.5 The Consumption - Tightness Puzzle

We can now derive the key result which is summarized by the following proposition:

Proposition 1 *In the competitive search equilibrium, independently of the source of shocks to the economy, the vu-ratio is related to consumption through the following condition:*

$$\theta = \frac{\vartheta}{1 - \vartheta} \omega c^\eta \tag{28}$$

where ω is a constant given by $[H(1) - H(1 - s)] / \kappa$.

Proof. The result follows simply from re-arranging condition (27) using that $\gamma^h / \gamma^f = (m/u) / (m/v) = \theta$. ■

This equation summarizes in a simple way the central implications for variations in unemployment and vacancies in the labor market matching model with an endogenous participation choice. As we will show below, the relationship implies (a) low volatility of the vu-ratio, (b) a strong tendency for procyclical movements in unemployment and for (c) a positively sloped Beveridge curve. Before we show these results it is worth pointing out that the relationship between labor market tightness and consumption derived above does not depend on the stochastic processes for job separation shocks and technology shocks and neither does it depend on the absence of capital adjustment costs nor on the production technology.

Table 1 reports some selected moments of US aggregate output and labor market variables at the business cycle frequencies. We present moments of quarterly data for the sample period 1964-2004. In order to isolate the movements in the relevant variables at the business cycle frequencies, the data were detrended with either the Hodrick and Prescott (1997) filter or with the Baxter and King (1999) approximate band-pass filter.[3] We examine the properties of aggregate output, aggregate consumption, aggregate hours worked, aggregate unemployment, and vacancies all as ratios of the US civilian non-institutional population. The table also reports the moments of the $vu-$ratio. Consumption is measured as US private sector consumption of non-durables and services. Hours worked are aggregate hours worked in the non-farm part of the economy. Unemployment is the total number of unemployed persons as reported by the Bureau of Labor Statistics. Vacancies are measured on the

[3] As is standard in the business cycle literature, we use a value of 1600 for the smoothing parameter in the Hodrick-Prescott filter. For the Baxter-King filter we use an MA-length of 12 quarters and the cut-off frequencies are chosen as 6 quarter and 32 quarters, respectively.

basis of an index of "help wanted" advertisements.[4] The table reports the percentage standard deviations of these variables and some selected cross-correlations.

In the US, unemployment is strongly countercyclical and very volatile. At the business cycle frequencies, the standard deviation of unemployment is close to 11 percent per quarter, or more than 7 times higher than that of output (9 times that of consumption). The contemporaneous correlation between unemployment and output is close to -0.90. Vacancies are even more volatile than unemployment (its standard deviation is above 13 percent per quarter) and display very procyclical behavior at the business cycle frequencies (the correlation between vacancies and output is above 90 percent). The strong negative contemporaneous correlation between unemployment and vacancies that forms part of the classic Beveridge curve relationship then implies high volatility of the $vu-$ratio (its standard deviation is 16 times that of output, or around 20 times that of consumption) and a contemporaneous positive correlation with output in excess of 0.90.

Consistently with (28), the $vu-$ratio and consumption are positively correlated (the cross-correlation is approximately 80 percent in the US data). Figure 1 illustrates consumption plotted against the $vu-$ratio for the two detrending methods. The figure clearly visualizes the positive correlation between them. The R^2 measure of fit is as high as 60 percent and 62 percent for Hodrick-Prescott filtered data and Baxter-King filtered data, respectively.

However, for realistic degrees of the intertemporal elasticity of substitution, theory can account for only a small fraction of the observed volatility of the $vu-$ratio. Notice that (28) implies that regressing the (logarithm of the) $vu-$ratio on (the logarithm of) consumption should give an estimate of the inverse of the IES, or alternatively, that the standard deviation of the $vu-$ratio implied by the model is equal to the inverse of the IES times the standard deviation of consumption. The slopes of the regression lines in Figure 1 imply estimates of the inverse of the IES equal to 14.8 and 15.3, respectively, for the two panels of Figure 1. The ratio of the standard deviations instead imply values of η of 19.2 and 19.5 for Hodrick-Prescott filtered and Baxter-King filtered data, respectively. These estimates are far above the values of η normally considered realistic. Estimates by Eichenbaum, Hansen and Singleton (1988), Friend and Blume (1975), Neely, Roy and Whiteman (2001), and many others, indicate that realistic estimates of η are in the range of 0.5-3 (see Mehra and Prescott, 2003, for an extensive discussion). Said differently, for standard values of the IES, using the observed volatility of consumption, the model can account for only a small fraction (less than 25 percent) of the volatility of the $vu-$ratio.[5]

Another way of expressing these insights is in terms of the covariance implications. In particular, for realistic second moments of vacancies and consumption, the labor market matching model implies *pro-cyclical* unemployment. To see this, note that

[4] Table A.1 reports the definitions and sources of the data.

[5] The model can easily be extended to include productivity growth. In this case the condition in (28) is still valid but relates the $vu-$ratio to consumption relative to the level of productivity. Therefore one may wonder whether the calculations should not relate the level of the $vu-$ratio to detrended consumption. Following this strategy, however, implies even higher and more unrealistic estimates of η. For Baxter and King filtered consumption, forexample, the slope of the regression lines implies a value of η of 20 and the ratio of standard deviations of the $vu-$ratio and consumption gives a value of 40 for η.

(28) can be expressed as:

$$u = \frac{v}{c^{\eta}} \frac{1-\alpha}{\alpha\omega}$$

Taking logarithms gives us that:

$$cor\left(\widehat{u}, \widehat{c}\right) = \frac{cov\left(\widehat{c}, \widehat{v}\right)}{\left(var\left(\widehat{u}\right) var\left(\widehat{c}\right)\right)^{1/2}} - \eta \left(\frac{var\left(\widehat{c}\right)}{var\left(\widehat{u}\right)}\right)^{1/2}$$

where $\widehat{x}$ denotes $\ln\left(x_t\right)$.

Suppose the model would be able to reproduce the empirical estimates of the moments of the data that enter on the right hand side of this expression. In this case, using the estimates in Table 1, the cross-correlation between unemployment and consumption would equal approximately $0.99-\eta/10$.[6] Taking a value of η in the upper end of the empirically plausible estimates, $\eta = 3$, implies that $cor\left(\widehat{u}, \widehat{c}\right) = 0.69$. In US data instead, this correlation is -0.70, see Table 1. Therefore, even if the model could reproduce the correlation between vacancies and consumption and the variances of consumption, vacancies and unemployment, it would require very large and unrealistic values of η to account for the countercyclical movements in unemployment observed in the data.

In order to visualize the extent to which the actual and implied unemployment rates differ, Figure 2 plots the actual unemployment rate against the unemployment rate implied by the above relationship for $\eta = 1$ (a realistic value) and for $\eta = 10$ (an unrealistically high value) on the basis of HP-filtered US data. In both case, there is a strong *negative* association between the actual and implied unemployment rate

The intuition for the tendency for procyclical unemployment is straightforward. In this model, while employment is predetermined, unemployment is not a state variable since households can adjust the number of agents that are search active through variations in the participation rate. An increase in vacancies increases the expected payoff from labor market search since the probability of being matched with a vacancy rises. Therefore, the participation rate increases which leads to a tendency for procyclical unemployment. This effect is moderated only by the extent to which the underlying shock lowers the marginal utility of consumption (which lowers the payoff from search activities). The latter effect, however, is only quantitatively important when the curvature of the utility function is very large and we argue that plausible estimates of the IES imply moderate curvature. Therefore, fluctuations in vacancies tend to induce equally signed fluctuations in unemployment through variations in the participation rate. In other words, the Beveridge curve is - counterfactually - positively sloped when we allow for a participation choice.

In sum, once one allows agents to choose whether to be search active or not, the labor market matching model gives rise to a *consumption-tightness puzzle* in the sense that unrealistically high degrees of risk aversion (low degrees of intertemporal elasticity of substitution) are required to account for (a) the volatility of the $vu-$ratio, and (b) the countercyclical movements in unemployment (and a negatively sloped Beveridge curve).

[6] To get this expression express $cor\left(\widehat{u}, \widehat{c}\right)$ as $\left(var\left(\widehat{v}\right)/var\left(\widehat{u}\right)\right)^{1/2} cor\left(\left(\widehat{c}, \widehat{v}\right)\right) - \eta\left(var\left(\widehat{c}\right)/var\left(\widehat{u}\right)\right)^{1/2}$. Inserting the estimates on the basis of the HP-filtered data in Table 1 implies the formula in the text.

3 Extensions

We now examine a number of extensions of the basic model in order to gauge the robustness of the consumption-tightness puzzle highlighted in the previous section. As we will show, the qualitative features of the results above are robust.

3.1 Variable Search Effort

The first extension introduces variable search intensity into the above model. We assume that, for given levels of unemployment and vacancies, when more resources are spent on job search, more matches will be produced between unemployed workers and firms with vacant jobs. As in Merz (1995), higher search effort is assumed to give rise to a resource cost.[7] Allowing for variable search effort may therefore moderate the results above since the tendency for households to devote more resources to search activities when vacancies rise can also be achieved through variations in the intensive search margin.

With variable search effort, the matching technology is given as:

$$m_t = M\left(v_t, h_t u_t\right)$$

where h_t denotes search effort. We assume that the matching technology displays constant returns to $(v_t, h_t u_t)$ jointly. Thus, the probability that a search active worker finds a job vacancy, $m_t/u_t = h_t \phi\left(\theta_t/h_t\right)$, is an increasing function of θ_t and h_t.

We assume that higher search effort (along the intensive margin) gives rise to a resource cost, $d\left(h_t\right)$ per search active household member. The economy's resource constraint now reads:

$$y_t \geq c_t + i_t + \kappa v_t + u_t d\left(h_t\right)$$

where d is an increasing and convex function.

The first-order necessary conditions for the households' problem (see Appendix 1 for details) imply that optimal search effort, h, is determined such that:

$$c^{-\eta}\frac{\partial d\left(h\right)}{\partial h} = \beta\gamma^h E J_{n'}\left(k', n'\right) \tag{29}$$

where $\gamma^h = m/\left(uh\right)$.

This condition states that, in the optimum, marginal search costs equal the probability that a new match is formed times the marginal value of a match. Combining this equation with the households' first-order condition for the choice of u implies that:

$$H\left(1\right) - H\left(1-s\right) = c^{-\eta}\left[\psi\left(h\right) - 1\right] d\left(h\right) \tag{30}$$

where ψ is the elasticity of $d\left(h\right)$, $\psi\left(h\right) = \left(\partial d\left(h\right)/\partial h\right)\left(h/d\left(h\right)\right)$. Thus, if the elasticity of the search effort costs is constant, $c^{-\eta}$ and $d\left(h\right)$ will be perfectly negatively

[7] Search effort therefore has the interpretation of costs of filling in job applications, travelling to job interviews etc. Alternatively, one can assume that search efforts give rise to leisure costs, see eg. Andolfatto (1996). However, the latter modelling implies that search effort is constant in the optimum (see the next footnote) and is therefore less interesting for our purposes.

correlated. In other words, under these conditions search effort will be positively correlated with consumption no matter the source of shocks to the economy.[8]

After some algebra, we can show that when we allow for variable search efforts at both the intensive and the extensive margins, the following condition must hold:

$$\theta = \frac{\alpha}{1-\alpha}\omega c^{\eta}\frac{\psi(h)}{\psi(h)-1} \tag{31}$$

This expression differs from the one derived under assumption of constant search effort only by the term $\psi(h)/(\psi(h)-1)$ that appears on the left hand side. When $\psi(h)$ is constant, the model with variable search effort therefore delivers exactly the same predictions regarding the volatility of the $vu-$ratio and the cyclical features of unemployment as the model with constant search effort.

3.2 Homework

Next we consider an extension of the basic model with homework (see e.g. Benhabib, Rogerson, and Wright, 1991).[9] This extension modifies the trade-off between labor market search and non-participation since agents now have the opportunity of spending part of their time-endowment on home production.

Agents consume two types of goods: market goods (c_m) and goods produced in the home-sector (c_h). Both goods are produced using inputs of capital and labor and are subject to productivity shocks. Goods produced in the home-sector are used for consumption only.

Per capita hours supplied to the home-sector are given as

$$\mu = n\mu_n + u\mu_u + (1-n-u)\,\mu_l$$

where μ_n denotes hours worked at home of an employed worker, μ_u hours worked at home of an unemployed household member, and μ_l hours worked at home of a non-participant. The home-production resource constraint is given as:

$$c_h \leq g\left((1-x)\,k, \mu, z^h\right) \tag{32}$$

where c_h is the consumption of home-goods, x denotes the fraction of the aggregate capital stock that is used for production in the market sector. z^h are temporary productivity shocks to the home-production technology which we assume are generated by a first-order autoregressive process with innovations that are possibly correlated

[8] Merz (1995) also finds procyclical search intensity in a standard labor market matching model without the participation choice. Her result is derived on the basis of the impulse responses in a numerically solved version of the model. For the Andolfatto (1996) specification of leisure costs of search effort we would assume that $d(h) = 0$ and that $x_u = (1-h-s)$. This implies, however, that optimal search effort is constant since the first-order condition for h can be expressed as: $-\partial H(1-h-s)/\partial h = [H(1) - H(1-h-s)]/h$ which involves only h and constants. Therefore the optimal h is constant.

[9] Garibaldi and Wasmer (2005) also introduce homework into a matching framework with a participation choice. Cooley and Quadrini (1999) include homework in a matching framework with limited asset market participation.

with the innovations to z. We assume that g is increasing and concave in $(1-x)k$ and in μ, and that it is homogeneous of degree one in $((1-x)k, \mu)$ jointly.

The period utility function is given as:

$$u(c, e_i) = c^{1-\eta}/(1-\eta) + H(e_i)$$

where $e_n = 1 - s - l - \mu_n$, $e_u = 1 - s - \mu_u$, and $e_l = 1 - \mu_l$. c is an aggregate of the consumption of the two goods:

$$c = C(c_m, c_h)$$

We assume that C is increasing, concave, and homogeneous of degree 1. Finally, the resource constraint for the market sector now reads:

$$c_m + k' + \kappa v \leq f(xk, nl, z) + (1-\delta)k \tag{33}$$

The households' problem can now be expressed as choosing sequences of consumption, capital stocks, hours worked in the home sector, the share of search active agents, and the division of capital between sectors to solve:

$$\begin{aligned} J(k,n) &= \max_{(c,n',k',u,\mu_i,x)} \{c^{1-\eta}/(1-\eta) + nH_w(1-l-s-\mu_n) + uH_u(1-s-\mu_u) \\ &\quad + (1-n-u)H_n(1-\mu_l) + \beta EJ(k',n')\} \end{aligned} \tag{34}$$

subject to the constraints:

$$\begin{aligned} c_m + k' &\leq (1-\delta+rx)k + wnl + \pi \\ n' &= (1-\sigma)n + \gamma^h u \\ c_h &\leq g\left((1-x)k, \mu, z^h\right) \end{aligned}$$

The first-order conditions are described in detail in Appendix 2. A key implication follows from the first-order condition for hours devoted to homework which is given as:

$$\frac{\partial H(x_i)}{\partial \mu_i} = \frac{\partial C/\partial c_h}{\partial C/\partial c_m}\frac{\partial g}{\partial \mu}\beta EJ_{k'}(k',n')$$

Notice that the right hand side of this expression does not depend on the labor market status. Therefore, under the condition that H is strictly concave, it follows that:

$$\mu_l = \mu_u + s = \mu_n + l + s \tag{35}$$

In other words, leisure does not depend on labor market status. Thus, agents that are non-participants compensate for their lack of hours devoted to market activities by working s more hours at home than agents that are search active, and $s+l$ hours more at home than agents that are employed. This reflects risk sharing: the marginal disutility of work is equalized across agents (that differ by their labor market status).

We follow Gomme, Kydland and Rupert (2001) and assume that the consumption aggregator, C, is given by a Cobb-Douglas function:

$$c = c_m^{\xi} c_h^{1-\xi},\ \xi \in (0;1) \tag{36}$$

and that the home-good production function is given by a Cobb-Douglas production function:

$$g\left((1-x)k, \mu, z^h\right) = z^h\left((1-x)k\right)^{\tau}\mu^{1-\tau},\ \tau \in (0;1) \tag{37}$$

We can now derive the following result:

Proposition 2 *In the homework economy with a Cobb-Douglas consumption aggregator and Cobb-Douglas home-production function, the $vu-$ratio and consumption of market goods are related as:*

$$\theta = \frac{c_m}{\mu} \frac{\vartheta}{1-\vartheta} \omega_h \tag{38}$$

where $\omega_h = \frac{1-\xi}{\xi} (1-\tau) \frac{s}{\kappa}$.

Proof. Using that $\mu_l = \mu_u + s = \mu_n + l + s$, the first-order conditions for the optimal choices of u, μ_i, c_m, and c_h and the outcome of the wage bargaining implies that:

$$\frac{\partial C}{\partial c_h} (\mu_l - \mu_u) \frac{\partial g}{\partial \mu} = \frac{1-\vartheta}{\vartheta} \kappa \frac{\partial C}{\partial c_m} \theta$$

where equation (35) implies that $\mu_l - \mu_u = s$. Using (37) we have that $\frac{\partial g\left((1-x)k, \mu, z^h\right)}{\partial \mu} = (1-\tau_h) c_h / \mu$ and from (36) it follows that $\frac{\partial c}{\partial c_h} / \frac{\partial c}{\partial c_m} = (1-\xi)/\xi \, (c_m/c_h)$. Inserting these gives (38) ■

This relationship differs from (28) in two ways. First, it no longer involves the risk aversion parameter, η. Secondly, the relationship involves also the number of hours supplied to the home-sector, r. Potentially the latter aspect might help addressing the consumption-tightness puzzle. In particular, a negative covariance between consumption of market goods and hours supplied to the home sector induces volatility in the $vu-$ratio. Most models with home production do indeed imply strongly countercyclical movements in hours supplied in the home sector (see e.g. Gomme, Kydland and Rupert, 2001).

Quantitatively, however, even a substantial negative covariance between consumption of market goods and homework hours is unlikely to help much in explaining the gap between the observed volatility of the $vu-$ratio and that implied by the growth model with labor market frictions and a participation choice and the model therefore still has a strong tendency for procyclical movements in unemployment and for a positive contemporaneous correlation between unemployment and vacancies. To see this, consider the following calculation. The standard deviation of Hodrick-Prescott filtered (per capita) hours worked in the market sector is around 1.75 percent per quarter in the U.S. (see Table 1). The volatility of hours worked in the home sector is unlikely to be higher than this. Thus, even if consumption of market goods and hours worked in the home sector were perfectly negatively correlated, the implied standard deviation of the $vu-$ratio, would be no higher than 2.59 percent, around 10 times lower than the standard deviation of the vu-ratio in the U.S. data.[10] For the same reasons, the model with homework implies a positive correlation between unemployment and vacancies and procyclical unemployment.

Therefore we conclude that the introduction of homework does not impact on the consumption - labor market tightness puzzle; On the contrary it may even worsen.

[10] To get this number, assuming a Cobb-Douglas matching technology, it follows from (38) that the standard deviation of the logarithm of the $vu-$ratio is given as $\left(\sigma_c^2 + \sigma_r^2 - 2cov(c,r)\right)^{1/2}$. Assuming that $cov(c,r) = -\sigma_c \sigma_r$, and using the values for σ_c and σ_r from Table 1 for the Hodrick-Prescott filter data gives the number in the text.

3.3 Passive Search

The matching technology analyzed so far assumes that non-participants do not receive any job offers. However, in U.S. data there are substantial flows from out-of-the-labor-force directly into employment, see e.g. Davis, Faberman and Haltiwanger (2006) for a recent review. For this reason, Andolfatto and Gomme (1996), Pries and Rogerson (2004), and Yip (2003), for example, assume that wage offers might be received by both unemployed workers and by non-participants.[11]

We now extend the model of Section 2 by allowing for "passive search". We assume that the aggregate matching function is given as:

$$m_t = M^u\left(v_t, u_t\right) + M^l\left(v_t, 1 - n_t - u_t\right)$$

We will assume that $m_t^u/u_t \geq m_t^l/\left(1 - n_t - u_t\right)$ in order to be consistent with the observation that the matching frequency of unemployed workers is much higher than the matching frequency of non-participants. This assumption also squares well with Flinn and Heckman's (1983) finding that unemployment helps facilitate job search relative to non-participation.

The households' problem is given as:

$$\begin{aligned}
J\left(k, n\right) &= \max_{\left(c, k', u, n'\right)} \left\{c^{1-\eta}/\left(1-\eta\right) + nH\left(1 - l - s\right) + uH\left(1 - s\right)\right. \\
&\qquad \left. + \left(1 - n - u\right)H\left(1\right) + \beta EJ\left(k', n'\right)\right\} && (39) \\
c + k' &\leq \left(1 - \delta + r\right)k + wnl + \pi && (40) \\
n' &= \left(1 - \sigma\right)n + \gamma_1^h u + \gamma_2^h\left(1 - n - u\right) && (41)
\end{aligned}$$

where (41) now takes into account that non-participants as well as unemployed household members might become matched. In this equation we define $\gamma_1^h = m^u/u$ and $\gamma_2^h = m^l/\left(1 - n - u\right)$.

The firms' problem is unchanged (apart from the change in the probability that a vacancy is filled). Going through the same steps as in Section 2 gives us the following condition:

$$\theta \frac{u + \left(1 - n\right)\frac{m^u}{m}}{1 - n - u} = \frac{\vartheta}{1 - \vartheta}\omega c^\eta \qquad (42)$$

which is identical to (28) apart from the ratio $\frac{u + (1-n)\frac{m^u}{m}}{1-n-u}$ that appears on the left hand side. Notice that this ratio is equal to 1 when $m^u/m = 1$ as we assumed in Section 2. When m^u/m approaches 0 instead, this ratio becomes equal to $u/\left(1 - n - u\right)$. In this case, the left hand side of (42) becomes equal to ratio of vacancies to non-participation.

According to Fallick and Fleischman (2004), the mean flow from non-employment into employment is approximately equal to 4.6 millions in the United States and the mean flow from unemployment accounts for around 1.8 millions of these new job findings. Thus, m^u/m is approximately equal to 40 percent. Assuming that this ratio is constant, we can then compute the left hand side of (42). Figure 3 illustrates the

[11] In principle, time-aggregation might account for the recorded flows from out-of-the labor force to employment even if non-participants have to become search active to find a job match.

implies standard deviation of (HP-filtered values of) the left hand side of (42) for alternative values of $\frac{m^u}{m}$. When m^u/m approaches 0 the implied percentage standard deviation of the left hand side is 13.2 which is 40 percent lower than the standard deviation of tightness itself. However, when we set m^u/m equal to its mean US value, the implied standard deviation of the left hand side of (42) is 21.2 which is only marginally lower than the volatility of tightness. Therefore, while allowing for "passive search" helps addressing the consumption-tightness puzzle, quantitatively this feature does not appear to matter much because the left hand side of (42) is insensitive to $\frac{m^u}{m}$ unless this ratio becomes very small.

Therefore, we conclude that allowing for passive search is inessential for the results.

3.4 Duration Dependent Matching Functions

The previous Section analyzed a setting in which search active agents and non-participants face heterogenous matching functions. An alternative modelling of this feature is that unmatched agents differ in their labor market prospects even when search active. In particular, recently unemployed agents may face more efficient matching functions than longer term unemployed workers or out-of-the-labor-force. We now analyze such a setting.

We assume that there are two types of unemployed workers that differ in their prospects of being matched with vacancies, "short-term unemployed" and "long-term unemployed". Long-term unemployed workers face a less efficient matching technology than the short-term unemployed and this group of agents may choose to become non-participants. As in Section 2, we assume that only search active agents receive job offers.

The labor market flow dynamics are as follows. Every period a fraction σ of the currently employed worker-job matches are terminated and a measure M new matches are formed. Workers that experience a termination of their matches, enter into short-term unemployment.[12] A short term unemployed household member may either remain short term unemployed, become matched with a vacancy, or experience a transition to long-term unemployment. We assume that the latter event occurs with probability $\mu \in [0;1]$. New matches are formed between vacant jobs and search active unmatched agents but the number of matches depends now on both labor market tightness and on the structure of unemployment.

Formally, we assume that the aggregate number of matches is given as:

$$\begin{aligned} M(v, u_1, u_2) &= m_1(v, u_1) + m_2(v, u_2) \\ m_1(v, u) &> m_2(v, u) \text{ for } \forall v, u > 0 \end{aligned}$$

where u_1 denotes the measure of short-term unemployed workers, and u_2 the measure of long-term unemployed.

The employment transition equation is now given as:

$$n' = (1 - \sigma) n + m_1 + m_2 \tag{43}$$

[12]Strictly speaking, the use of the 'long-term' unemployment and 'short-term' unemployment is misleading since the transition from the latter group to the latter group occurs independently of the duration of unemployment. However, on average, the latter group will have experienced shorter unemployment spells than the former.

and the transition equation for short term unemployment is given as:

$$u_1' = (1-\phi)\,u_1 + \sigma n - m_1 \tag{44}$$

where ϕ is the probability that a currently short-term unemployed worker becomes long-term unemployed.

Bellman's equation for the households' problem is given as:

$$\begin{aligned} J(k,n,u_1) &= \max_{(c,k',u)} \{c^{1-\eta}/(1-\eta) + nH(1-l-s) + (u_1+u_2)\,H(1-s) \\ &\quad + (1-n-u_1-u_2)\,H(1) + \beta EJ(k',n',u_1)\} \end{aligned} \tag{45}$$

where we note that short-term unemployment is now a state variable. The Bellman equation is maximized subject to the constraints:

$$c + k' \leq (1-\delta+r)\,k + wnl + \pi \tag{46}$$

$$n' = (1-\sigma)\,n + \gamma_1^h u_1 + \gamma_2^h u_2 \tag{47}$$

$$u_1' = (1-\phi)\,u_1 + \sigma n - \gamma_1^h u_1 \tag{48}$$

γ_1^h denotes the probability that a short-term unemployed search active household member is matched with a vacancy and γ_2^h is the equivalent probability for a long-term unemployed worker.

In this model, the participation choice is relevant for long-term unemployed household members; Under mild conditions on γ_1^h relative to γ_2^h, household members are better off searching as long as they are faced with the more efficient matching technology.

The first order condition for the optimal choice of u_2 is given by:

$$\gamma_2^h \beta EJ_{n'}(k',n') = H(1) - H(1-s) \tag{49}$$

This condition is equivalent to condition (18) derived in Section 2 apart from the definition of the matching market prospect. The marginal value of employment, however, now takes into account the multiple matching functions. It is given as:

$$\begin{aligned} J_n(k,n,u_1) &= c^{-\eta}wl - (H(1) - H(1-l-s)) \\ &\quad + (1-\sigma)\,\beta EJ_{n'}(k',n',u_1') + \sigma\beta EJ_{u_1'}(k',n',u_1') \end{aligned}$$

This determines the marginal value of a job as sum of the utility value of the labor income, the expected marginal value of being employed the next period times the probability that the match survives (discounted one period), the expected marginal value of short-term unemployment times the probability that the match is terminated, less the utility value of the loss of leisure of working rather than enjoying the time endowment as leisure. The marginal value of short term unemployment, in turn is given as:

$$\begin{aligned} J_{u_1}(k,n,u_1) &= \gamma_1^h \beta EJ_{n'}(k',n',u_1') \\ &\quad + \left[(1-\phi) - \gamma_1^h\right] \beta EJ_{u_1'}(k',n',u_1') - (H(1) - H(1-s)) \end{aligned}$$

A short term unemployed worker finds a job match with probability γ_1^h which gives her the value $\beta E J_{n'}(k', n', u_1')$; With probability $\left[(1-\phi)-\gamma_1^h\right]$ a currently short-term unemployed worker is still unemployed next period giving her a value $\beta E J_{u_1'}(k', n', u_1')$; Finally, being search active rather than non-participating gives rise to a utility loss $(H(1)-H(1-s))$ due to the search effort that must be exerted.

The firms' problem is now given as:

$$Q(n) = \max_{k,v}\left\{F(k, nl) - wnl - \kappa v - rk + \beta E\frac{u_c'}{u_c}Q(n')\right\} \tag{50}$$

subject to:

$$n' = (1-\sigma)n + \left(\gamma_1^f + \gamma_2^f\right)v \tag{51}$$

Notice that we assume that firms cannot target any of the two matching markets individually. The first-order condition for the choice of capital is identical to the model of Section 2. The vacancy posting condition, however, is now given by:

$$\kappa = \left(\gamma_1^f + \gamma_2^f\right)\beta E\frac{u_c'}{u_c}(Q_{n'}) \tag{52}$$

where:

$$Q_n = F_n - wl + (1-\sigma)\beta E\frac{u_c'}{u_c}(Q_{n'})$$

It is important to notice that the relevant first-order condition for households' search efforts at the extensive margin involves the probability that long-term unemployed household members find a job match while firms' first-order condition for vacancy postings involve the probability of meeting any unmatched search active worker. If possible, firms would prefer target vacancies at the matching market that yields the highest possible probability of a match with an unemployed worker. This possibility is, however, ruled out by assumption and this creates the wedge between the relevant matching market first-order conditions.

Wages are again determined by an ex-post Nash bargain. Following the same steps as in the previous models gives us that in equilibrium:

$$\theta\frac{u}{u_2} = \frac{\vartheta}{1-\vartheta}c^{\eta}\frac{H(1)-H(1-s)}{\kappa} \tag{53}$$

where $\theta = v/(u_1+u_2)$.

This relation is similar to the one derived in the previous sub-section (equation (42)) since it implies a modification to the appropriate measure of tightness. In the current setting, the "consumption-tightness puzzle" involves the ratio of vacancies to "long-term" unemployment rather than the standard definition of tightness that enter equation (28).

If u_2 is literally interpreted as "long-term" unemployment, this model leads to an even bigger consumption - tightness puzzle than the model we analyzed in Section 2. The reason for this is that longer term unemployment is even more volatile than overall unemployment. In Table 1 we report, for example, the moments of vacancies to unemployment above 15 weeks of duration. The standard deviation of this ratio is around 35 percent per quarter which is 59 percent higher than the standard deviation of tightness itself.

However, this calculation might be misleading for two reasons. First, u_2 does not directly measure long-term unemployment although the unemployment duration of agents faced with the inefficient matching technology will on average be longer than the mean duration of agents faced with the more efficient matching technology. Secondly, the measurement of the duration of unemployment applied by the Bureau of Labor Statistics defines the duration of unemployment as the length of "in-progress spell of joblessness". The duration of unemployment of a search active agent faced with the inefficient matching technology who was previously out of the labor force will therefore be measured by the duration of the *current* job search rather than the length of time since the last job match.

For these reasons, the measurement of u_2 on the basis of longer term unemployment might be misleading. In essence, u_2 denotes the measure of agents that despite being faced with a potentially quite inefficient matching technology still find it worthwhile to be search active. It is not clear to match this measure up with the data and we therefore leave it open whether duration dependence of the matching market prospects is important for accounting for the consumption - tightness puzzle.

4 Discussion

The analysis above has illustrated the robustness of the relationship between the marginal utility of consumption and labor market tightness that we derived in Section 2. We now want to discuss some wider aspects of the result and its implications.

The low volatility of labor market tightness and procyclical movements in unemployment derive from the variations in labor market participation. In the set-up that we study, households optimally choose to increase labor market participation in response to increases in labor market tightness. It is this mechanism that implies low volatility of labor market tightness in equilibrium.[13]

Hence, it is clear that variations in the participation rate are key and that the introduction of an extensive search margin leads to a strong tendency for procyclical variations in labor market participation. Figure 4 illustrates the US labor market participation rate from 1947 onwards. The figure clearly illustrates the secular increase in the U.S. participation rate. It rose from around 58 percent in the late 1940's to approximately 67 percent by the 2000's, an increase that is dominated by an increase in the employment rate (from 56 percent to 64 percent).

Figure 4 also illustrates (with shaded areas) the recessions of the US economy according to the NBER business cycle dating committee. The figure indicates that the secular increase in the participation rate predominantly took place during periods of high activity. In particular, the secular increase in the participation rate either slowed down or was reversed during each of the recessions. Thus, consistently with the theory, there appears to be some cyclical features of the movements in the participation rate.

[13] Notice, however, that the response of unemployment to vacancies may lead to high volatility of vacancies itself.

To examine this further, the last rows of the two panels of Table 1 report the moments of HP-filtered and BK-filtered participation rates. The participation rate is procyclical but displays low volatility at the business cycle frequencies irrespective of the detrending method. In particular, relative to trend, the standard deviation of the participation rate is around one fourth of the standard deviation of consumption at the business cycle frequencies and the cross-correlation between these variables is just below 30 percent.

Figure 5 illustrates in the top panel the HP-filtered US data for consumption and the participation rate. This illustrates quite clearly that the participation rate is much smoother than consumption at the business cycle frequencies. The figure also hints that there might be a phase-shift between consumption and participation rate. In particular, with the exception of the late 1970's, the participation rate appears to lag the fluctuations in consumption. The lower panel illustrates the cross-correlation function between consumption and leads and lags of the participation rate. The results indicate that the participation rate lags around 4 quarters after consumption. Moreover, with a 4 quarter lag the cross-correlation is as high as 65 percent.

Nevertheless, despite this high correlation, the elasticity of the participation rate to consumption is still estimated to be low. Using the estimates of Table 1, the elasticity of the participation rate with respect to consumption is around 17 percent (with a four quarter lag). Thus, even large cyclical fluctuations in consumption are associated with small variations in participation rates.

Similarly, Figure 6 illustrates the relationship between the $vu-$ratio and labor market participation rates. The top panel shows the deviations from (Hodrick-Prescott) trends of the $vu-$ratio and of the participation rate. Given the large difference in their volatility, the $vu-$ratio is plotted against the left axis and the participation rate against the right axis. Consistently with the model that we have analyzed these two variables are clearly *positively* related. The lower panel shows the cross-correlation function at leads and lags. As above, there is a substantial positive correlation and it occurs with a lag (but slightly shorter than above). At a 2 quarter lag (of the participation rate), the cross-correlation is close to 70 percent.

This suggests that perhaps the findings of this paper are related to costs of entering and exiting the labor force. Such costs might explain why the participation rate moves little in response to variations in the benefits of job search and why the participation rate appears to lag behind output and consumption over the business cycle. Such costs also appear realistic for some parts of the agents that compose the out-of-the-labor-force group. Young people under education or workers that have to move geographically in order to search for a job, for example, might find it costly to change their labor market status. On the other hand, using a limited information approach, Ravn (2005) estimates such costs to be *very large* in order to account for the labor market movements over the business cycle which casts doubt on this aspect being the sole explanation for the findings of this paper.

An alternative assumption adopted by e.g. Garibaldi and Wasmer (2005) and Haefke and Reiter (2006) is that agents differ in their evaluation of leisure (or in their productivity in homework). This implies that the group of non-participants will be heterogenous with respect to how close their valuation of non-participation is to their valuation of labor market search. In particular, those agents that value leisure highly may find it optimal to remain out of the labor force even for large increases in the value of search. Through this mechanism, heterogeneity in the valuation of leisure (or in home-productivity) can limit the tendency for procyclical movements in labor force participation that we have derived in this paper. Nonetheless, this set-up appears to be in contradiction with the large observed flows of agents from non-participation into employment (and into unemployment) that we discussed in Section 3.3 unless there are large idiosyncratic shocks to preferences or home-productivity. We find such idiosyncratic preferences shocks hard to interpret.[14]

Therefore we find it more promising to explore in more details the type of settings that we studied in Sections 3.3 and 3.4. Here duration dependence and passive search jointly imply that (i) there is less incentive to join the labor force in response to increases in the value of search, and (ii) some of the non-participants might find it unprofitable actively to search for a job not because they value leisure highly but because they face little prospect of finding a job through active search.

5 Summary and Conclusions

An important line of research in business cycle theory has studied the effects of matching frictions in the labor market. This is an important development in business cycle theory since fluctuations in labor are key for understanding the business cycle, see e.g. Kydland (1995). The matching frictions assumed in Mortensen-Pissarides set-up places the labor market in a central role in the propagation of shocks over time and across agents.

This paper has studied the effects of introducing a labor market participation choice. Surprisingly, we have shown that the introduction of a labor market participation choice is of considerable analytical convenience since it allows us to derive a very simple testable relationship between labor market tightness and consumption. Moreover, this relationship is robust to various extensions of the baseline model that we proposed. The advantage of this result is that it involves only observable variables and that gives rise to a relationship that does not depend on the properties of the stochastic processes of exogenous variables.

A standard intuition from such labor market matching models is that unemployment, consistently with the data, behaves countercyclically as job matches increases in good times when firms increase their investment in job hiring activities. This paper has shown, however, that once one introduces an endogenous labor market participation choice, there is a strong tendency for procyclical behavior of unemployment. The reason is that labor market non-participants have an incentive to enter the labor market, i.e. become search active, when labor market prospects improve. These procyclical movements in labor market participation rates imply low volatility of the ratio of vacancies to unemployment and a positive slope of the Beveridge curve. Evidently, in U.S. data, although participation rates do move procyclically, the elasticity

[14] A similar mechanism can be introduced into the set-up studied in the present paper by allowing for shocks to the marginal rate of substitution between leisure and consumption. Denoting such a taste shock for ζ_t, the equivalent of equation (28) becomes $\theta = \frac{\vartheta}{1-\vartheta}\omega\zeta c^{\eta}$. A large variance of the taste shock may therefore break the link between consumption and tightness.

of the participation rate is very low.

Understanding better why this is the case is an important issue for further research. There are various avenues open for addressing this. One possibility is to introduce costs of entering and exiting the labor force. Another possibility is to introduce some of the aspects that we examined in Section 3. It is possible that when these features are joined and possibly combined with other extensions (such as habit persistence, non-separable preferences, incomplete markets) that this will yield a solution to the consumption - tightness puzzle. We will examine this in future research.

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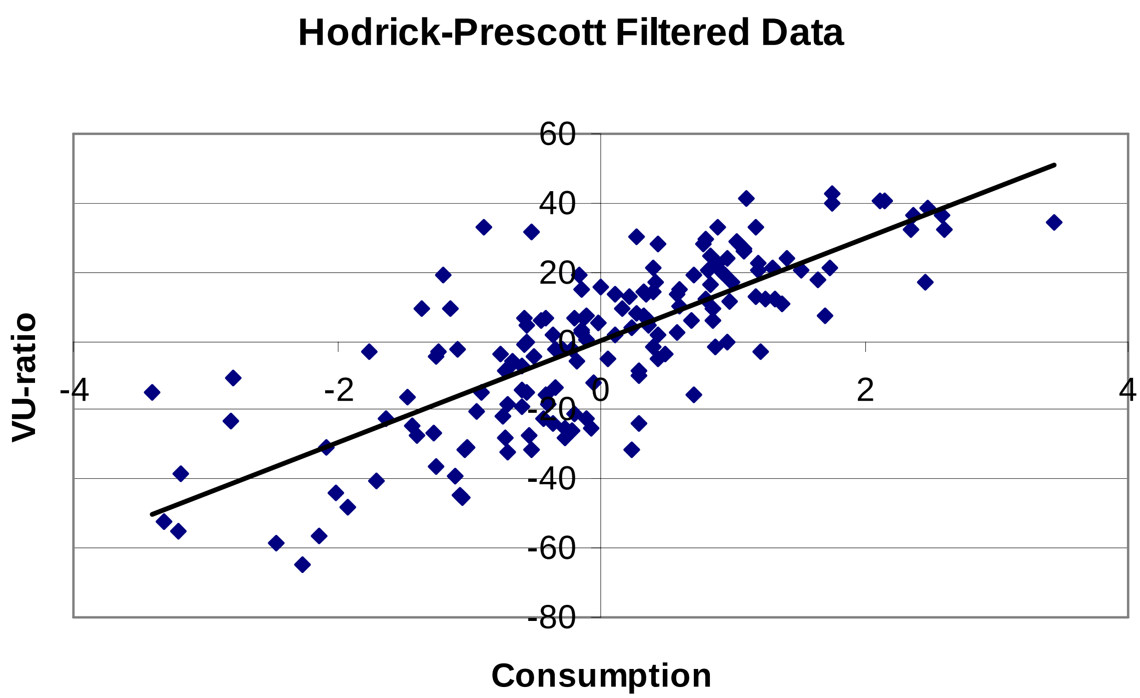
Table A.1: Definitions and Sources of Data (sample period: 1964 Q.1 - 2004 Q.1

Name	Definition	Source
Civilian non-institutional pop.	Civilian non-institutional population 16 years of age and above	Economagic, Fed. of St. Louis
Output	Gross Domestic Product in chained year 2000 prices divided by civilian non-institutional population	Economagic, Fed. of St. Louis
Consumption	Personal Consumption Expenditure in chained year 2000 prices divided by civilian non-institutional population	Economagic, Fed. of St. Louis
Total hours	Non-farm hours divided by civilian non-institutional population	DRI database
Unemployment	Total unemployment divided by civilian non-institutional population	Economagic, Fed. of St. Louis
Vacancies	Index of help wanted advertising in newspapers divided by civilian non-institutional population	Economagic, Fed. of St. Louis
VU-ratio	Vacancies divided by unemployment	-
Unemployment > 15 weeks	Civilians unemployed 15 weeks and above divided by civilian non-institutional population	Economagic, Fed. of St. Louis
Participation rate	Sum of civilian employment and unemployment divided by civilian non-institutional population	Economagic, Fed. of St. Louis

Table 1. US Business Cycle Statistics, 1964-2004			
	HP-filtered Data		
Variable	Standard deviation (percent)	Correlations with output	Correlations with consumption
Output	1.56	1	-
Consumption	1.23	0.87	-
Total hours	1.75	0.91	-
Unemployment	10.81	-0.87	-0.71
Vacancies	13.18	0.91	0.81
$vu-$ratio	23.66	0.90	0.77
Ratio of vacancies to unempl>15 weeks	34.71	0.85	0.68
Participation rate	0.31	0.45	0.27
	BK-filtered Data		
Variable	Standard deviation	Correlations with output	Correlations with consumption
Output	1.51	1	-
Consumption	1.22	0.89	-
Total Hours	1.77	0.88	-
Unemployment	10.97	-0.89	-0.73
Vacancies	13.15	0.93	0.81
$vu-$ratio	23.85	0.92	0.78
Ratio of vacancies to unempl.>15 weeks	35.52	0.87	0.70
Participation rate	0.28	0.48	0.29

Figure 1: The Consumption - VU-ratio Relationship



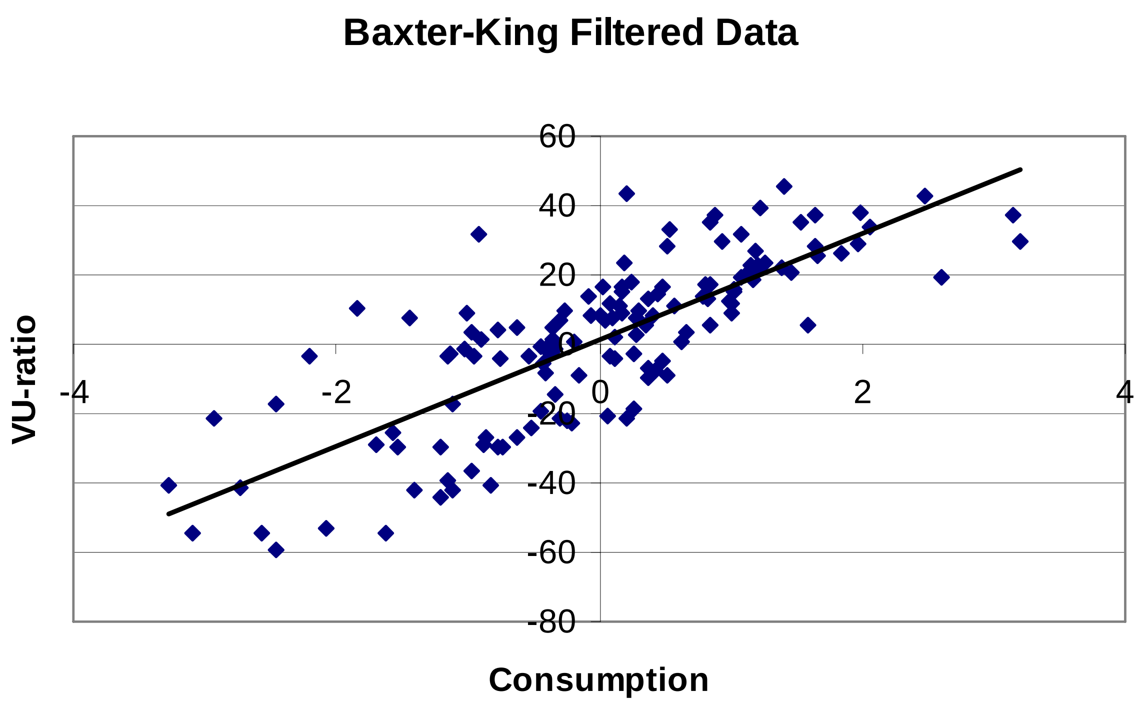
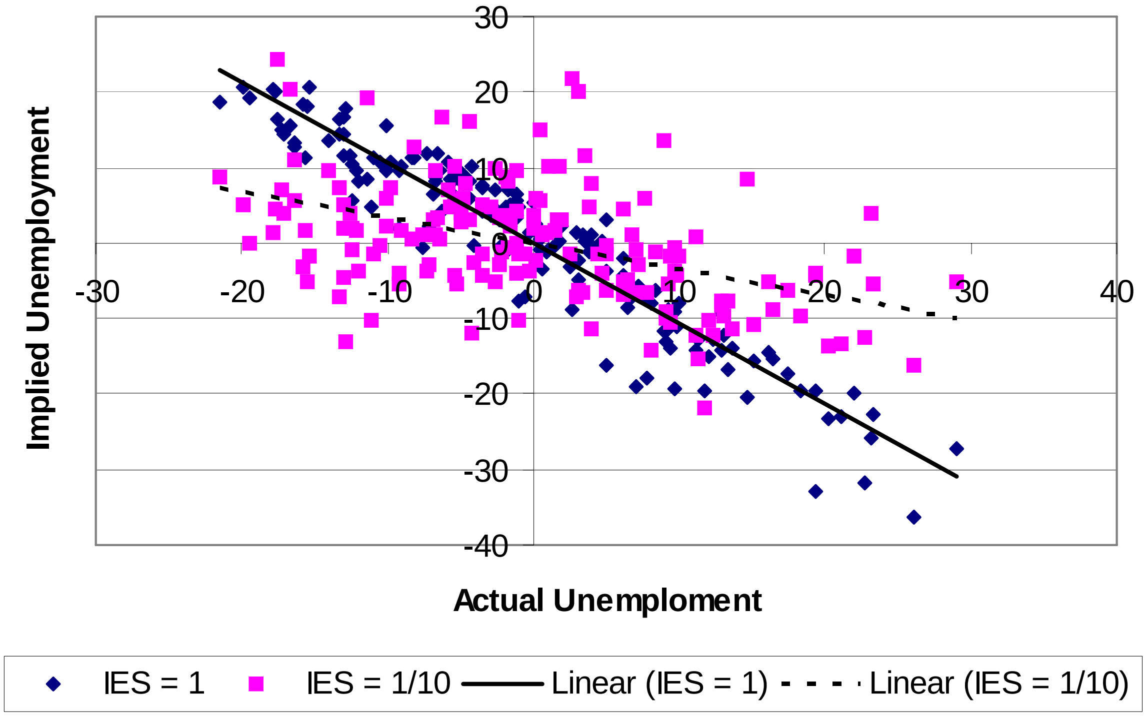
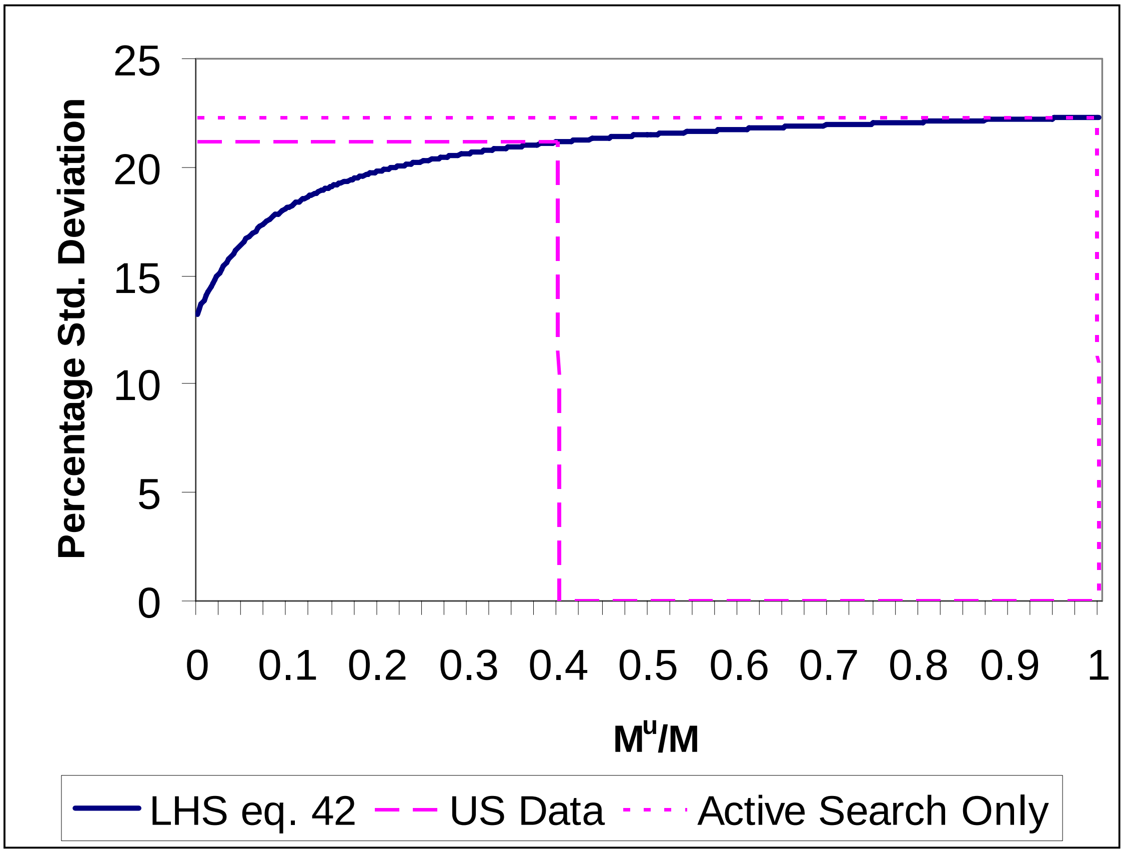


Figure 2. Actual vs. Implied Unemployment



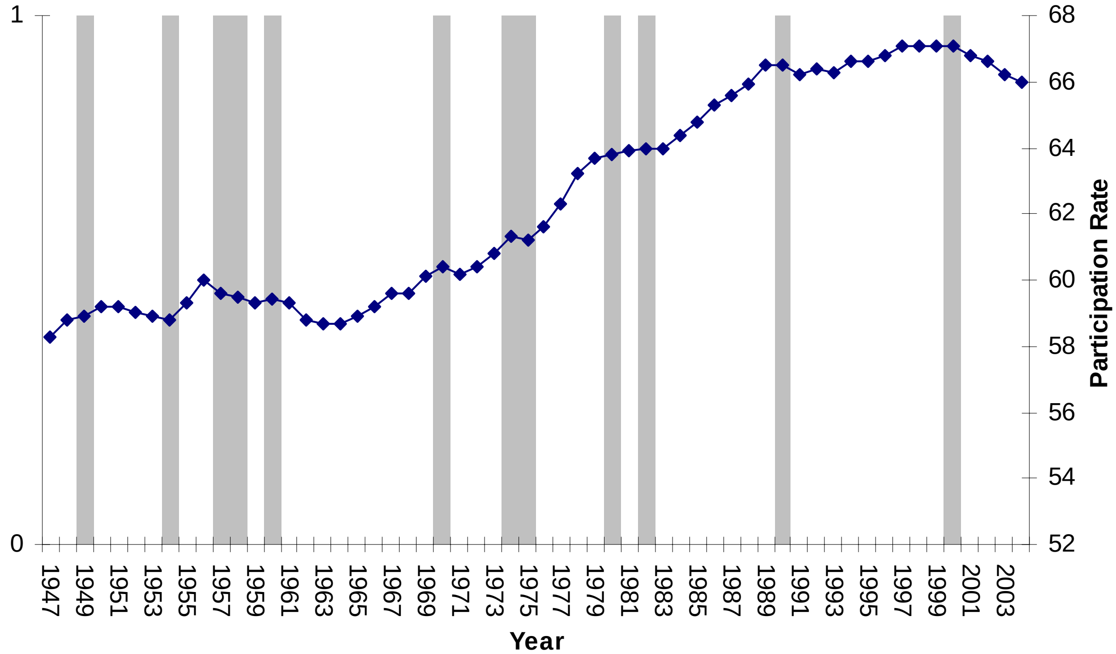
This figure illustrates the actual US unemployment level with the unemployment level implied by Theorem 1. The diamonds (squares) illustrate the relationship when assuming that $\eta = 1$ ($\eta = 10$). The linear regression lines show that there is a negative relationship between the actual and predicted unemployment levels.

Figure 3. Passive Search Model



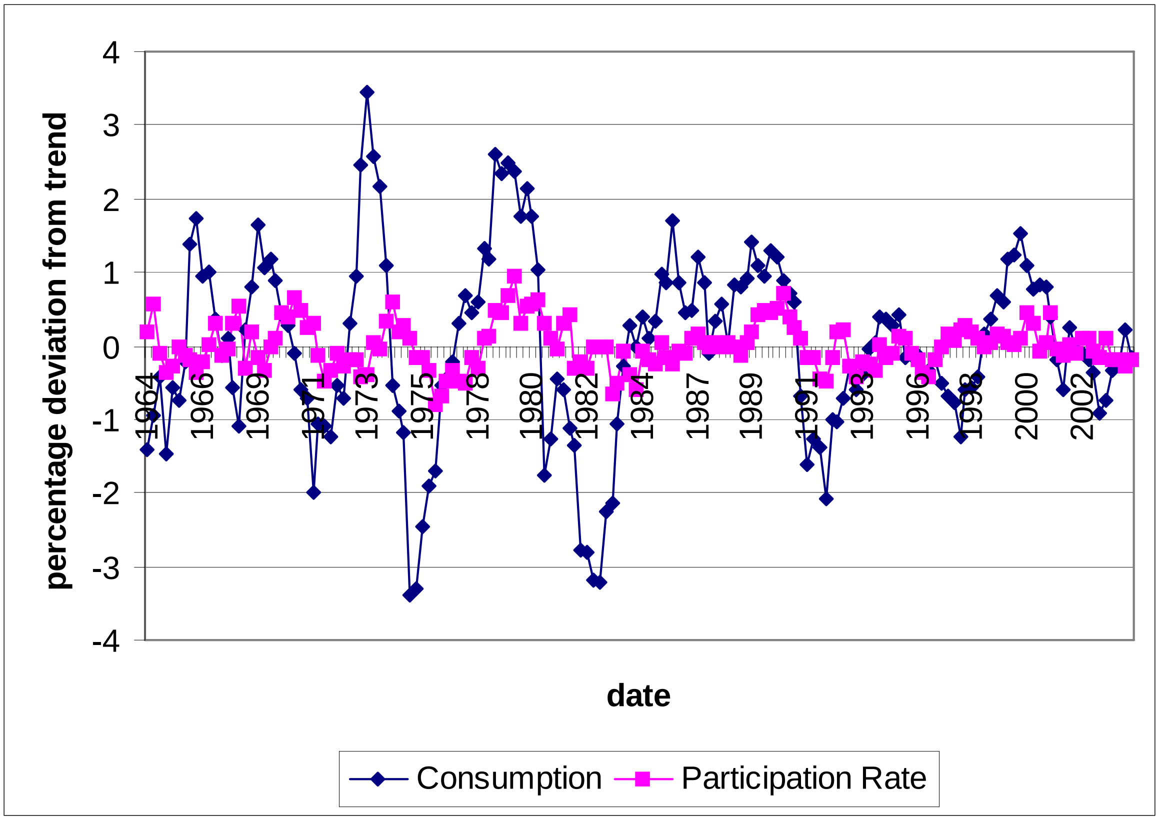
Note: The full drawn line illustrates the percentage standard deviation of HP-filtered values of the left hand side of equation (42) for alternative values of the share of total matches due to "passive search" (M^u/M). The mean value of M^u/M in US data is around 40 percent. The model of Section 2 of the paper assumes that M^u/M is equal to 1.

Figure 4: US Participation Rate



Note: The graph illustrates the civilian non-institutional labor force as a share of the civilian non-institutional population of age 16 and above. The shaded areas are recessions as defined by the NBER dating committee.

Figure 5: Consumption and Participation Rate



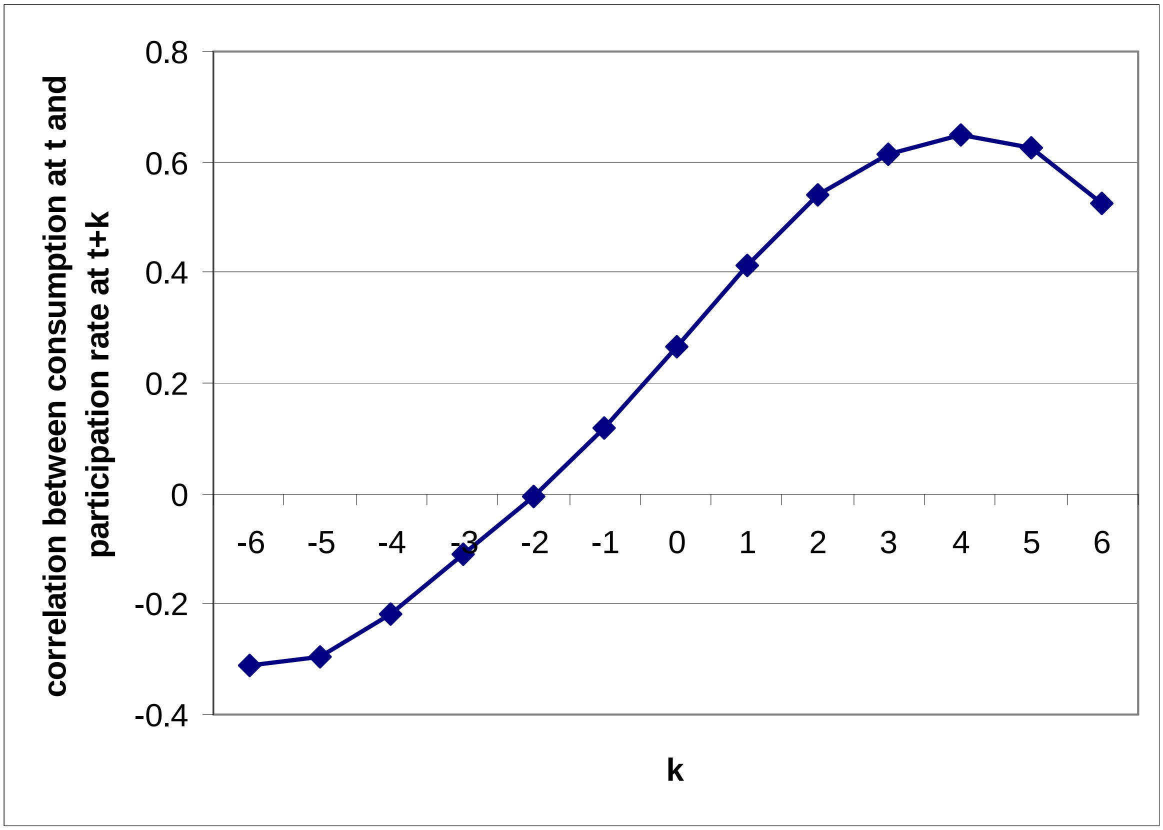
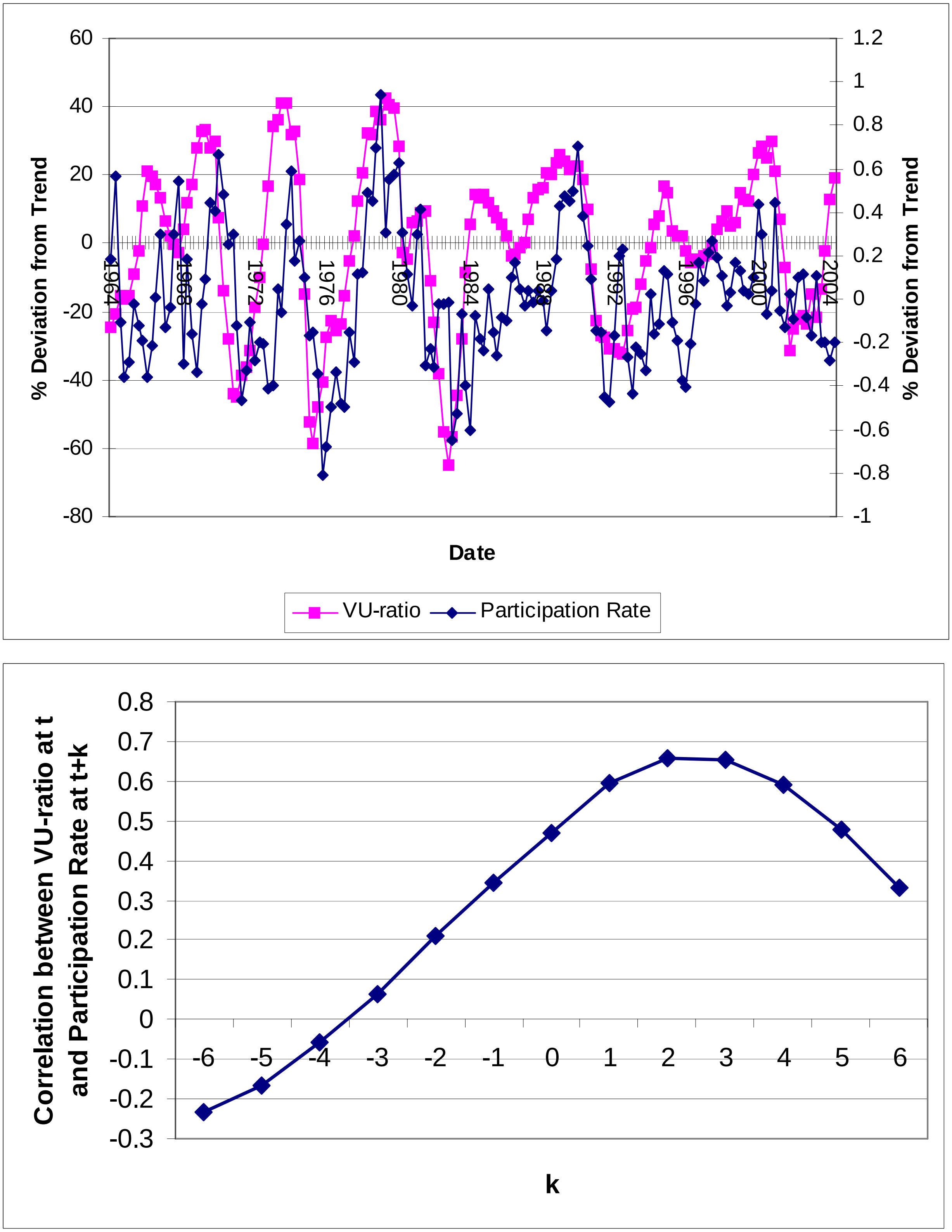


Figure 6: Labor Market Tightness and Participation Rate



The top panel illustrates percentage deviations from an HP-trend of the VU-ratio (left scale) and of labor market participation (right scale)

Appendix 1. Derivation of the results for the model with variable search effort

In this model, the households' problem is given as:

$$
\begin{aligned}
J(k,n) &= \max_{(c,k',u,n',h)} \Big[c^{1-\eta}/(1-\eta) + nH(1-l-s) + uH(1-s) \\
&\quad + (1-n-u)H(1) + \beta EJ(k',n') \Big]
\end{aligned}
$$

subject to:

$$
\begin{aligned}
c + k' &\leq (1-\delta+r)k + wnl + \pi - ud(h) \\
n' &= (1-\sigma)n + \gamma^h uh
\end{aligned}
$$

We let λ_c denote the multiplier on the first constraint and λ_n the multiplier on the second constraint. The first-order condition are given by:

$$
\begin{aligned}
c &: c^{-\eta} = \lambda_c \\
k' &: \lambda_c = \beta EJ_{k'}(k',n') \\
h &: \lambda_c \frac{\partial d(h)}{\partial h} = \lambda_n \gamma^h \\
k &: J_k(k,n) = \lambda_c(1-\delta+r) \\
u &: H(1) - H(1-s) = \gamma^h \lambda_n h - \gamma d(h)\lambda_c \\
n' &: \lambda_n = \beta EJ_{n'}(k',n') \\
n &: J_n(k,n) = \lambda_c wl + (1-\sigma)\lambda_n + H(1-l-s) - H(1)
\end{aligned}
$$

Combining the first-order conditions for h and n' gives us that:

$$
c^{-\eta}\frac{\partial d(h)}{\partial h} = \beta\gamma^h EJ_{n'}(k',n')
$$

which is equation (29) in the text.

Next, combining the conditions for u, c, and h implies that:

$$
\begin{aligned}
H(1) - H(1-s) &= \gamma^h \lambda_n h + d(h)c^{-\eta} \Rightarrow \\
H(1) - H(1-s) &= c^{-\eta}\gamma d(h)[\psi(h) - 1]
\end{aligned}
$$

which is equation (30) in the text. The firms' problem is unchanged relative to the basic model. The Nash wage bargaining therefore implies that:

$$
\vartheta J_n = c^{-\eta}(1-\vartheta)Q_n
$$

where:

$$
\begin{aligned}
J_n(k,n) &= \lambda_c wl + H(1-l-s) - H(1) + (1-\sigma)\beta EJ_{n'}(k',n') \\
Q_n(n) &= F_n - wl + (1-\sigma)\beta E\frac{u_c'}{u_c}(Q_{n'}(n'))
\end{aligned}
$$

and from the envelope conditions, we have that:

$$\begin{aligned} \beta E \frac{u_c'}{u_c} \left(Q_{n'} \left(n' \right) \right) &= \frac{\kappa}{\gamma^f} \\ \beta E J_{n'} \left(k', n' \right) &= \lambda_n \end{aligned}$$

From the first-order condition for h we have that we can express the latter as:

$$\beta E J_{n'} \left(k', n' \right) = \frac{H(1) - H(1-s) + \gamma d(h) c^{-\eta}}{\gamma^h h}$$

Therefore, it follows that:

$$\vartheta \frac{H(1) - H(1-s) + \gamma d(h) c^{-\eta}}{\gamma^h h} = (1-\vartheta) c^{-\eta} \frac{\kappa}{\gamma^f}$$

which can be re-arranged to gives us that:

$$\theta = \frac{\vartheta}{1-\vartheta} c^{\eta} \kappa \left[H(1) - H(1-s) \right] \left[\frac{\psi(h)}{\psi(h) - 1} \right]$$

which corresponds to equation (31).

Appendix 2: Homework

The firms' problem is again unchanged so we concentrate on the households' problem. It can be formulated as:

$$\begin{aligned} J(k,n) = \max_{(c,n',k',u,r,x)} & \left[c^{1-\eta} / (1-\eta) + nH(1-l-s-\mu_w) + uH(1-s-\mu_u) \right. \\ & \left. + (1-n-u) H(1-\mu_n) + \beta E J(k', n') \right] \end{aligned}$$

subject to:

$$\begin{aligned} c_m + k' &\leq (1-\delta+rx) k + wnl + \pi \\ n' &= (1-\sigma) n + \gamma^h u \\ c_h &= g\left((1-x) k, n\mu_n + u\mu_u + (1-n-u)\mu_l, z^h \right) \end{aligned}$$

We denote the multipliers on these restrictions (in that order) by λ_1, λ_2 and λ_3.

The first-order conditions (and envelope conditions) are given as:

$$
\begin{aligned}
c_m &: \; c^{-\eta}\frac{\partial C}{\partial c_m} = \lambda_1 \\
c_h &: \; c^{-\eta}\frac{\partial C}{\partial c_h} = \lambda_3 \\
k' &: \; \lambda_1 = \beta E J_{k'}\left(k', n'\right) \\
u &: \; \lambda_2 \gamma^h + \lambda_3 \frac{\partial g}{\partial \mu}\left(\mu_u - \mu_l\right) = H\left(1 - \mu_l\right) - H\left(1 - s - \mu_u\right) \\
\mu_n &: \; nH'\left(1 - l - s - \mu_n\right) = \lambda_3 \frac{\partial g}{\partial \mu} n \\
\mu_u &: \; uH'\left(1 - s - \mu_u\right) = \lambda_3 \frac{\partial g}{\partial \mu} u \\
\mu_l &: \; \left(1 - n - u\right) H'\left(1 - \mu_l\right) = \lambda_3 \frac{\partial g}{\partial \mu}\left(1 - n - u\right) \\
x &: \; \lambda_1 r k = \lambda_3 \frac{\partial g}{\partial k_h} k \\
n' &: \; \lambda_2 = \beta E J_{n'}\left(k', n'\right) \\
n &: \; J_n = H\left(1 - l - s - \mu_n\right) - H\left(1 - \mu_l\right) + \lambda_1 w l + \lambda_2\left(1 - \sigma\right) + \lambda_3 \frac{\partial g}{\partial \mu}\left(\mu_n - \mu_u\right) \\
k &: \; J_k = \lambda_1\left(1 - \delta + r x\right) + \lambda_3 \frac{\partial g}{\partial k} x
\end{aligned}
$$

The first-order conditions for μ_n, μ_u and μ_l immediately imply that:

$$\mu_l = \mu_u + s = \mu_n + l + s$$

since:

$$H'\left(1 - l - s - \mu_n\right) = H'\left(1 - s - \mu_u\right) = H'\left(1 - \mu_l\right)$$

Turn now to the wage-bargaining. We have that:

$$\vartheta J_n\left(k, n\right) = \lambda_1\left(1 - \vartheta\right) Q_n\left(n\right)$$

where:

$$
\begin{aligned}
J_n\left(k, n\right) &= \lambda_1 w l + \lambda_3 \frac{\partial g}{\partial \mu}\left(\mu_n - \mu_u\right) + \left(1 - \sigma\right)\beta E J_{n'}\left(k', n'\right) \\
Q_n\left(n\right) &= F_n - w l + \left(1 - \sigma\right)\beta E \frac{u_c'}{u_c}\left(Q_{n'}\left(n'\right)\right)
\end{aligned}
$$

and the envelope conditions imply that:

$$
\begin{aligned}
\beta E \frac{u_c'}{u_c}\left(Q_{n'}\left(n'\right)\right) &= \frac{\kappa}{\gamma^f} \\
\beta E J_{n'}\left(k', n'\right) &= \frac{\lambda_3 \frac{\partial g}{\partial \mu}\left(\mu_l - \mu_u\right)}{\gamma^h}
\end{aligned}
$$

Therefore, we get that:

$$
\begin{aligned}
\vartheta\beta E\frac{1}{\lambda_1}J_{n'}\left(k',n'\right) &= \left(1-\vartheta\right)\beta E\frac{\lambda_1'}{\lambda_1}Q_{n'}\left(n'\right)\Rightarrow \\
\vartheta\frac{1}{\lambda_1}\frac{\lambda_3\frac{\partial g}{\partial\mu}\left(\mu_l-\mu_u\right)}{\gamma^h} &= \left(1-\vartheta\right)\frac{\kappa}{\gamma^f}\Rightarrow \\
\frac{\partial C}{\partial c_h}\left(\mu_l-\mu_u\right)\frac{\partial g}{\partial r} &= \frac{1-\vartheta}{\vartheta}\kappa\frac{\partial C}{\partial c_m}\theta
\end{aligned}
$$

We now use the Cobb-Douglas assumptions and the result from above that $r_l - r_u = s$ which allows us to express this condition as:

$$
\left(1-\xi\right)\frac{C}{c_h}s\left(1-\tau\right)\frac{c_h}{\mu}=\frac{1-\vartheta}{\vartheta}\kappa\frac{C}{c_m}\theta\xi
$$

which can be re-arranged to give us equation (38) :

$$
\theta=\frac{c_m}{\mu}\frac{\vartheta}{1-\vartheta}\frac{1-\xi}{\xi}\left(1-\tau\right)\frac{s}{\kappa}
$$