Computations in the Derived Module Category

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I, Susanne Gollek, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

This thesis is centred around computations in the derived module category of finitely generated lattices over the integral group ring of a finite group G. Building upon the representability of the cohomology functor in the derived module category in dimensions greater than 0, we give a new characterisation of the cohomology of lattices in terms of their G-invariants, only having the syzygies of the trivial lattice to keep track of dimension. With the example of the dihedral group of order 6 we show that this characterisation significantly simplifies computations in cohomology. In particular, we determine the Bieberbach groups, that is, the fundamental groups of compact flat Riemannian manifolds, with dihedral holonomy group of order 6. Furthermore, we give an interpretation of the cup product in the derived module category and show that it arises naturally as the composition of morphisms. Inspired by the graded-commutativity of the cup product in singular cohomology we give a sufficient condition for the cohomology ring of a lattice to be graded-commutative in dimensions greater than 0.

For Leoni

Contents

1 Introduction

This thesis deals with computations in the derived module category of the category $\mathcal{F}(G)$ of finitely generated lattices over the integral group ring $\mathbb{Z}[G]$ of a finite group G. We are particularly interested in its application to the cohomology theory over these rings.

The theory of cohomology is fundamental to modern algebra. It is the dual version of homology and has its roots in the concept of a dual cell structure which H. Poincaré used in his proof of the Poincaré duality theorem. However its true importance was not realised until some 40 years after the development of homology. Although most of its ideas were around from as early as the 1890's the birth of homological algebra took place in the late 1930's with crystallisation of the notion of homology and cohomolgy of a topological space. The realisation that the same formalism can also be applied to other algebraic structures was primarily due to the work of S. Eilenberg until it reached its maturity in 1956 with the publication of Cartan and Eilenberg's book [5] and with the emergence of central notions of derived functors, and projective and injective modules. Today it extends to nearly every area in algebra.

In this thesis we will be working with the Eilenberg-MacLane definition of cohomology which is obtained by applying the $\mathrm{Hom}(\,\cdot\,, N)$ functor to a projective resolution $P_* \to M$ of a module M and taking the homology of the resulting chain complex, that is

$$H^n(M, N) = H_n(\mathrm{Hom}(P_*, N)).$$

With this definition $H^n(M, N)$ inherits the natural additive group structure from $\mathrm{Hom}(P, M)$. In his paper [17] Yoneda showed that alternatively we can define cohomology as module extensions which are naturally equipped with the product structure of concatenating extensions, the Yoneda product. This then allows us to introduce a product structure on cohomology, the cup product, and results in the notion of the cohomology ring $H^*(M, M)$ of a module. In the case of $H^*(\mathbb{Z}, \mathbb{Z})$, $\mathbb{Z}$ the trivial rank 1 module, coincides with the cup product of singular cohomology. In section 2.2 we will give an introduction to cohomology, its relation to module extensions and its algebraic structure.

The derived module category of a category of modules is the quotient category obtained by factoring through the subcategory of projective modules. In his PhD thesis [8] J. Humphreys showed that for a module over a ring R satisfying $\mathrm{Ext}^n(M, R) = 0$, for $n \geq 1$, the cohomology functor $H^n(M,\,\cdot\,)$ is co-representable in the derived module category with co-representing object the n-th syzygy of M, $\Omega_n(M)$, that is, for all $n \geq 1$ we have

$$H^n(M, N) = \mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N).$$

The idea of the derived module category and the co-representability of the cohomology functor has already been around for some years and was also observed for example by J.Carlson in the case of modules over the group ring kG of a finite group G with k a finite field of characteristic p, see [4], and has been formalised for certain classes of modules by F.E.A. Johnson in [10]. However Humphrey's result remains the most general. Furthermore, in his book [10] Johnson proved that under certain conditions the syzygy functors Ω_n and Ω_{-n} are adjoint in the derived module category, that is

$$\mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N) \simeq \mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-n}(N)).$$

This is mainly due to the fact that in these cases every homomorphism $f : M \to M'$ can be lifted to a chain transformation between two complete resolutions of M and M'.

Although the condition $\mathrm{Ext}^n(M, R) = 0$ fails for most rings it is true for many interesting ones, in particular the integral group ring $\mathbb{Z}[G]$ of a finite group G. As also mentioned by Carlson in [4] for group rings we obtain as a consequence of Frobenius reciprocity that

$$\Omega_k(M) \otimes \Omega_l(N) \simeq \Omega_{k+l}(M \otimes N),$$

where M and N are lattices and $M \otimes N$ is the tensor product over $\mathbb{Z}$ with diagonal G-action. This property will play an important role in our analysis of $\mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N)$ as it will allow us relate the syzygies of an arbitrary lattices to the ones of the trivial lattice. In section 2.3 we will give a brief introduction to syzygies and to the derived module category of $\mathcal{F}(G)$. We will prove two simple but important results, namely the adjointness formula

$$\mathrm{Hom}_{\mathcal{D}er}(R \otimes M, N) \simeq \mathrm{Hom}_{\mathcal{D}er}(M, R^* \otimes N)$$

and

$$\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, N) \simeq N^G / N\Sigma_G,$$

where N^G is the sub-lattices of invariant elements and Σ_G is the norm of G. These results, together with the compatibility of the tensor product and syzygies, will enable us to employ the representability of cohomology to express $H^n(M, N)$, $n \geq 1$, directly in terms of M and N only having the syzygies of the trivial lattice to keep track of the dimension. We will do this in section 2.4 and obtain the first result of this thesis

Theorem A: *Let M and N be lattices over a finite group G and let $n \geq 1$ then*

$$H^n(M, N) \simeq \mathcal{C}^n(M, N),$$

where $\mathcal{C}^n(M, N) = (\Omega_{-n}(\mathbb{Z}) \otimes M^* \otimes N)^G / (\Omega_{-n}(\mathbb{Z}) \otimes M^* \otimes N)\Sigma_G$.

The advantage of this formula is that it significantly reduces the amount of calculations necessary to determine cohomology as we will illustrate with the example of the dihedral group D_6 in chapter 3. First of all, since $\Omega_{-n}(\mathbb{Z}) = \Omega_{-1}(\mathbb{Z}) \otimes \Omega_{-n+1}(\mathbb{Z})$, it eliminates the need to determine a free resolution of $\mathbb{Z}$, and secondly since $H^n(M,N) \simeq H^n(\mathbb{Z}, M^* \otimes N)$ it allows us to calculate the cohomology of a lattice M in terms of the cohomology of the trivial lattice. Another advantage of this formula is that it also simplifies computations with cohomology such as calculating the induced map $f^* : H^n(M,N) \to H^n(M',N)$ of a homomorphism $f : M' \to M$. Again we will not have to work with resolutions and lift f to a chain transformation as f^* will simply map an element $[d \otimes m^* \otimes n] \in \mathcal{C}^n(M,N)$ to $[d \otimes (m^* \circ f) \otimes n] \in \mathcal{C}^n(M',N)$. In particular, the induced map $i^* : H^n(G,N) \to H^n(H, i^*(N))$ of a subgroup $i : H \hookrightarrow G$ is obtained as the projection

$$(\Omega_{-n}(\mathbb{Z}) \otimes N)^G / (\Omega_{-n}(\mathbb{Z}) \otimes N)\Sigma_G \to (\Omega_{-n}(\mathbb{Z}) \otimes N)^H / (\Omega_{-n}(\mathbb{Z}) \otimes N)\Sigma_H.$$

Recall that a Bieberbach group π is a fundamental group of a compact flat Riemannian manifold M and arises algebraically as a torsion-free extension of a finitely generated free abelian group N by a finite group G. In particular, we can regard π as a subgroup of the Euclidean group $E(n) = \mathbb{R}^n \rtimes O(n)$, the group of isometries of the n-dimensional Euclidean space, which acts freely and discontinuously on $\mathbb{R}^n$. Then the manifold M is given by the orbit space $\mathbb{R}^n/\pi$. Furthermore, the subgroup of pure translations in π is isomorphic to the free abelian group N and the holonomy group of M, the subgroup of $O(n)$ given by parallel translation along closed curves in M, is isomorphic to G. It is known that the torsion-free extensions of N by G correspond to those elements c in $H^2(G,N)$ for which $0 \neq i^*(c) \in H^2(C_p, i^*(N))$ for all cyclic subgroups $i : C_p \hookrightarrow G$ of prime order. Expressed in the above terminology this translates to the following

Theorem B: *The Bieberbach groups with holonomy group G are determined by those elements $c \in (\Omega_{-2}(\mathbb{Z}) \otimes N)^G / (\Omega_{-2}(\mathbb{Z}) \otimes N)\Sigma_G$ which are not in $(\Omega_{-2}(\mathbb{Z}) \otimes N)\Sigma_{C_p}$ for all cyclic subgroups $C_p \subseteq G$ of prime order.*

In chapter 3 we will use this result to determine the Bieberbach groups with holonomy group D_6.

The last section in chapter 2 is dedicated to the interpretation of the cup product in the derived module category and the analysis of the ring structure of $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M) = \sum_{n \geq 0} \mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), M)$. We will see that the adjointness of Ω_n and Ω_{-n} allows us to regard the composition of morphisms in $\mathrm{Hom}_{\mathcal{D}er}$ as a pairing

$$\bullet : \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), R) \otimes \mathrm{Hom}_{\mathcal{D}er}(\Omega_l(R), N) \to \mathrm{Hom}_{\mathcal{D}er}(\Omega_{k+l}(M), N)$$

and we will show that the cup product arises naturally as the composition $\bullet$, that is

Theorem C: *Let* $f \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), R)$ *and let* $h \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_l(M), R)$*. Let* $p_k^* : \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), R) \tilde{\to} H^k(M, R)$ *be the isomorphism giving the co-representabilty of cohomology then*

$$p_k^*(f) \cup p_l^*(h) = p_{k+l}^*(h \bullet f).$$

Notice that there also exists a natural product structure on the quotients $\mathcal{C}^n(M, N)$ given by the evaluation map

$$\begin{array}{rcl} \circ : \mathcal{C}^k(M, R) \otimes \mathcal{C}^l(R, N) & \to & \mathcal{C}^{k+l}(M, N) \\ [d_k \otimes m^* \otimes r'] \otimes [d_l \otimes r^* \otimes n] & \mapsto & [r^*(r')(d_k \otimes d_l \otimes m^* \otimes n)] \end{array}$$

and as a corollary of theorem A and theorem C we obtain

Corollary D: *The* $\bullet$*-composition in* $\mathrm{Hom}_{\mathcal{D}er}$ *corresponds to the evaluation map on* $\mathcal{C}$.

It is well known that the cup product in singular cohomolgy makes the cohomology ring $H^*(X, \mathbb{Z})$ a graded-commutative ring, that is $f \cup h = (-1)^{\deg(f)\deg(h)} h \cup f$. We obtain then quickly that the ring $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is graded-commutative for every lattice M which lies in some syzygy of the trivial lattice. However we will see that we can still improve on this and show

Theorem E: *Let* M *be lattice such that there exists a lattice* M' *and* $M \oplus M'$ *lies in the syzygy of some lattice of rank 1. Then the ring* $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ *is graded-commutative.*

In chapter 3 we will calculate the syzygies of the indecomposable D_6-lattices and use these results to give a list of lattices for which $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is graded-commutative.

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2 Cohomology and the derived module category

2.1 Preliminaries

In this section we will recall some linear algebra over the integral group ring of a finite group which will be important in subsequent sections. For a more detailed account see [2, 5]. Let G be a finite group. The integral group ring $\mathbb{Z}[G]$ of G is the set of all formal sums

$$\mathbb{Z}[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in \mathbb{Z} \right\},$$

where the sum is given by

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

and the product is given by

$$\sum_{g \in G} a_g g \cdot \sum_{h \in G} b_h h = \sum_{g \in G} \left(\sum_{h \in G} a_h b_{h^{-1}g} \right) g.$$

In $\mathbb{Z}[G]$ we distinguish a particular element $\Sigma_G = \sum_{g \in G} g$ called the norm of G.

By a G-lattice N we mean a right $\mathbb{Z}[G]$-module whose underlying abelian group is torsion free and finitely generated, that is, $N \simeq \mathbb{Z}^n$ as an abelian group and the action of G on N is given by a group representation

$$\rho_N : G \longrightarrow \mathrm{Gl}_n(\mathbb{Z}),$$

where $wg = \rho_N(g^{-1})w$, n is called the $\mathbb{Z}$-rank of N, $\mathrm{rk}_{\mathbb{Z}}(N) = n$. If there is no confusion about the group we simply refer to N as a lattice. We denote by $\mathcal{F}(G)$ the category whose objects are finitely generated G-lattices and whose morphisms are G-module homomorphisms.

A lattice N is called decomposable if there exist proper sub-lattices $0 \subsetneq M, M' \subsetneq N$ such that $N \simeq M \oplus M'$, N is called indecomposable if it is not decomposable. There are two indecomposable lattices which are of particular interest to us. The first one is the trivial lattices $\mathbb{Z}$ for which $\rho_{\mathbb{Z}}(g) = 1$ for all $g \in G$. The second one is the integral group ring $\mathbb{Z}[G]$, called the regular representation, for which a $\mathbb{Z}$-basis is given by the elements of G and the G-action is given by multiplication in G on the right. In each lattice we distinguish two particular sub-lattices. The first one is the set of G-invariant elements

$$N^G = \{ w \in N \mid wg = w \}$$

and the second one is the lattice

$$N\Sigma_G = \{w\Sigma_G \mid w \in N\}.$$

Notice that $N\Sigma_G \subset N^G$ since $\Sigma_G g = \Sigma_G$ for all $g \in G$.

The $\mathbb{Z}[G]$-dual of a lattice N is the lattice

$$N^* = \mathrm{Hom}_{\mathbb{Z}[G]}(N, \mathbb{Z}[G])$$

where the G-action is given by $\rho_{N^*}(g) = \rho_N(g^{-1})^t$. The $\mathbb{Z}$-dual of a lattice N is the lattice

$$N^\star = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$$

on which the G-action is given by $(fg)(w) = f(wg^{-1})$.

Let M and N be two lattices of rank m and n respectively, and with corresponding representations ρ_M and ρ_N. Then the tensor product $M \otimes N$ over $\mathbb{Z}$ is a lattice where the G-action is given by

$$(v \otimes w)g = vg \otimes wg.$$

In terms of group representations

$$\rho_{M\otimes N} : G \longrightarrow \mathrm{Gl}_{mn}(\mathbb{Z})$$

is given by $\rho_{M\otimes N}(g) = \rho_M(g) \otimes \rho_N(g)$ where the tensor product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined as $A \otimes B = (Ab_{ij})$. Furthermore the group of $\mathbb{Z}$-homomorphism $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ is a lattice on which G acts by $(fg)(v) = (f(vg^{-1}))g$ and $\mathrm{Hom}_{\mathbb{Z}}(M, N) \simeq M^\star \otimes N$. In particular $M^* \otimes M$ is the matrix ring $M_m(\mathbb{Z})$ on which G acts by conjugation, $Ag = \rho_M(g^{-1})A\rho_M(g)$.

Let $H \subset G$ be a subgroup and let $i : H \hookrightarrow G$ be the inclusion map. Then i induces maps

$$i^* : \mathcal{F}(G) \longrightarrow \mathcal{F}(H)$$

and

$$i_* : \mathcal{F}(H) \longrightarrow \mathcal{F}(G),$$

where i^* is given by restricting scalars to $\mathbb{Z}[H]$ and i_* is given by extending scalars to $\mathbb{Z}[G]$, in particular, $i_*(M) = M \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]$, where we consider $\mathbb{Z}[G]$ as a left $\mathbb{Z}[H]$-module and the G-action is given by $(v \otimes \alpha)g = v \otimes \alpha g$.

Lemma 2.1.1 *(Frobenius reciprocity)*

Let $H \subset G$ be a subgroup and let $i : H \hookrightarrow G$ be the inclusion map. Let M be an H-lattice and let N be a G-lattice then there exists an isomorphism

$$\psi : i_*(M) \otimes N \longrightarrow i_*(M \otimes i^*(N)),$$

where $\psi((v \otimes_{\mathbb{Z}[H]} g) \otimes w) = (v \otimes wg^{-1}) \otimes_{\mathbb{Z}[H]} g$ *and* $\psi^{-1}((v \otimes w) \otimes_{\mathbb{Z}[H]} g) = (v \otimes_{\mathbb{Z}[H]} g) \otimes wg$.

Corollary 2.1.2 *Let* M *be a* G*-lattice with* $\mathrm{rk}_{\mathbb{Z}}(M) = m$ *then* $M \otimes \mathbb{Z}[G]^k \simeq \mathbb{Z}[G]^{km}$.

Proof: Let i be the inclusion map of the trivial group into G then $\mathbb{Z}[G] = i_*(\mathbb{Z})$ and it follows from Forbenius reciprocity that

$$\mathbb{Z}[G]^k \otimes M = i_*(\mathbb{Z}^k) \otimes M \simeq i_*(\mathbb{Z}^k \otimes i^*(M)) = i_*(\mathbb{Z}^k \otimes \mathbb{Z}^m) = \mathbb{Z}[G]^{km}.$$

QED

Lemma 2.1.3 *(Eckmann-Shapiro lemma)*

Let $H \subset G$ *be a subgroup and let* $i : H \hookrightarrow G$ *be the inclusion map. If* M *is a* G*-lattice and* N *is an* H*-lattice then there exist isomorphisms*

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, i_*(N)) \simeq \mathrm{Hom}_{\mathbb{Z}[H]}(i^*(M), N)$$

and

$$\mathrm{Hom}_{\mathbb{Z}[G]}(i_*(N), M) \simeq \mathrm{Hom}_{\mathbb{Z}[H]}(N, i^*(M)).$$

Corollary 2.1.4 *Let* N *be a* G*-lattice then the* $\mathbb{Z}[G]$*-dual* N^* *and the* $\mathbb{Z}$*-dual* $N^\star$ *are isomorphic as* G*-lattices.*

Proof: Let i be the inclusion map of the trivial group into G. Then by the first isomorphism in the Eckmann-Shapiro lemma it follows that

$$\begin{aligned} N^* &= \mathrm{Hom}_{\mathbb{Z}[G]}(N, \mathbb{Z}[G]) = \mathrm{Hom}_{\mathbb{Z}[G]}(N, i_*(\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}(i^*(N), \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \\ &= N^\star. \end{aligned}$$

QED

Lemma 2.1.5 *Let $H \subset G$ be a subgroup and let $i : H \hookrightarrow G$ be the inclusion. Then $i_*(\mathbb{Z})$ is self-dual, that is $(i_*(\mathbb{Z}))^* \simeq i_*(\mathbb{Z})$.*

Proof: Let $\{x_1, \dots, x_k\}$ be representatives of $H\backslash G$ thus $G = Hx_1 \cup \dots \cup Hx_k$ and $Hx_i \cap Hx_j = \emptyset$ for $i \neq j$. Then $\{1 \otimes x_1, \dots, 1 \otimes x_k\}$ is a $\mathbb{Z}$-basis for $i_*(\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]$. Let $x \in G$ and let $x_{\sigma_x(i)} \in \{x_1, \dots, x_k\}$ be such that $x_i x^{-1} \in Hx_{\sigma_x(i)}$ thus $(1 \otimes x_i)x^{-1} = 1 \otimes x_{\sigma_x(i)}$. If $x_{i_1}x^{-1} \in Hx_0$ and $x_{i_2}x^{-1} \in Hx_0$ then $x_0 = hx_{i_1}x^{-1} = h'x_{i_2}x^{-1}$ and therefore $hx_{i_1} = h'x_{i_2}$, that is $x_{i_1} = x_{i_2}$. It follows that σ_x is a permutation on $\{1, \dots, k\}$ and therefore $\rho_{i_*(\mathbb{Z})}(x)$ is a permutation matrix, in particular

$$(\rho_{i_*(\mathbb{Z})}(x))_{kl} = \begin{cases} 1 & l = \sigma_x(k) \\ 0 & l \neq \sigma_x(k). \end{cases}$$

Since $x^{-1}x = 1$ it follows that $\sigma_{x^{-1}} = \sigma_x^{-1}$ so that

$$(\rho_{i_*(\mathbb{Z})}(x^{-1}))_{kl} = \begin{cases} 1 & l = \sigma_{x^{-1}}(k) \\ 0 & l \neq \sigma_{x^{-1}}(k) \end{cases} = (\rho_{i_*(\mathbb{Z})}(x))_{lk}$$

Thus $\rho_{i_*(\mathbb{Z})}(x) = \rho_{i_*(\mathbb{Z})}(x^{-1})^t$ and therefore $(i_*(\mathbb{Z}))^* = i_*(\mathbb{Z})$. QED

Proposition 2.1.6 *Let M, N and R be lattices. Then there exists an isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}[G]}(R \otimes M, N) \to \mathrm{Hom}_{\mathbb{Z}[G]}(R, M^* \otimes N).$$

Proof: Let $f \in \mathrm{Hom}_{\mathbb{Z}[G]}(R \otimes M, N)$ and define $\hat{f}(r)(m) = f(r \otimes m)$. Then $\hat{f} \in \mathrm{Hom}_{\mathbb{Z}}(R, M^* \otimes N)$. Furthermore

$$\begin{aligned} \hat{f}(zg)(w) &= f(zg \otimes w) = f((z \otimes wg^{-1})g) = f(z \otimes wg^{-1})g \\ &= (\hat{f}(z)(wg^{-1}))g = (\hat{f}(z)g)(w) \end{aligned}$$

thus $\hat{f} \in \mathrm{Hom}_{\mathbb{Z}[G]}(R, M^* \otimes N)$. Similarly let $\hat{f} \in \mathrm{Hom}_{\mathbb{Z}}(R, M^* \otimes N)$ and define $f(z \otimes w) = \hat{f}(z)(w)$. Then $f \in \mathrm{Hom}_{\mathbb{Z}}(R \otimes M, N)$. Furthermore

$$\begin{aligned} f((z \otimes w)g) &= f(zg \otimes wg) = \hat{f}(zg)(wg) = (\hat{f}(z)g)(wg) \\ &= \hat{f}(z)(wgg^{-1})g = \hat{f}(z)(w)g = f(z \otimes w)g. \end{aligned}$$

QED

2.2 Eilenberg-MacLane cohomology and Yoneda's theory of module extensions

In this section we recall the traditional Eilenberg-MacLane definition of cohomology, Yoneda's theory of module extensions and some basic properties. Furthermore, we recall the relationship of group cohomology with the compact flat Riemannian manifolds and singular cohomology. Although these definitions hold for any class of modules which admits enough projective modules we will restrict our attention to finitely generated lattices over the integral group ring of a finite group. We will here only give the proofs which convey the for us interesting properties of cohomology. For a more detailed account see [14, 15].

Definition 2.2.1 *Let M be a lattice. A free resolution $F_* \xrightarrow{\varepsilon} M$ of M is an exact sequence*

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where each F_i is free.

Lemma 2.2.2 *Every lattice admits a free resolution.*

Proof: Let $\{m_1, \ldots, m_k\}$ be a set of generators of M and let F_0 be the free module $F_0 = \mathbb{Z}[G]^k$ with standard generators $\{e_1, \ldots, e_k\}$. Then $\varepsilon : F_0 \to M$ with $\varepsilon(e_i) = m_i,\ i = 1, \ldots, k$, is a surjective homomorphism. Now choose a set of generators $\{d_1, \ldots, d_l\}$ for $\ker(\varepsilon)$ and let F_1 be the free module $F_1 = \mathbb{Z}[G]^l$ with standard generators $\{e_1, \ldots, e_l\}$. Then $p_1 : F_1 \to \ker(\varepsilon)$ with $p_1(e_i) = d_i$ is surjective homomorphism. Let $i_1 : \ker(\varepsilon) \to F_0$ be the inclusion and define $\partial_1 = p_1 \circ i_1$. Repeating this construction for $\ker(\partial_1)$ we obtain a free module F_2 and a homomorphism $\partial_2 : F_2 \to F_1$, and continuing this way we obtain inductively a sequence $\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$, where each F_i is free and $\mathrm{im}(\partial_{n+1}) = \ker(\partial_n)$ by definition. QED

Notice that a free resolution is by no means unique as we can always choose a different set of generators for M. However, the following proposition allows us to compare any two resolutions.

Proposition 2.2.3 *Let $\varphi : M \to M'$ be a homomorphism and let $F_* \xrightarrow{\varepsilon} M$ be a free resolution of M and $F'_* \xrightarrow{\varepsilon'} M'$ a free resolution of M'. Then there exists a chain transformation $\varphi_* : F_* \to F'_*$ which lifts φ. That is, φ_* is given by a family of homomorphisms $\varphi_k : F_k \to F'_k$, $k \geq 0$ such that $\varphi_k \circ \partial_{k+1} = \partial'_{k+1} \circ \varphi_{k+1}$ and $\varphi \circ \varepsilon = \varepsilon' \circ \varphi_0$.*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & F_2 & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow 0 \\
& \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & \\
\cdots \longrightarrow & F'_2 & \xrightarrow{\partial'_2} & F'_1 & \xrightarrow{\partial'_1} & F'_0 & \xrightarrow{\varepsilon'} & M' & \longrightarrow 0.
\end{array}$$

Furthermore, any two chain transformations φ_* and φ'_* lifting φ are chain homotopic, that is, there exist homomorphisms $s_k : F_k \to F'_{k+1}$ such that $\partial'_{k+1} \circ s_k + s_{k-1} \circ \partial_k = \varphi_k - \varphi'_k$, where $s_{-1} = 0$.

Proof: Let $0 \longrightarrow \mathrm{im}(\partial_1) \xrightarrow{i_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ be the first stage of the resolution of M, let $0 \longrightarrow \mathrm{im}(\partial'_1) \xrightarrow{i'_1} F'_0 \xrightarrow{\varepsilon'} M' \longrightarrow 0$ be the first stage of the resolution of M' and let $\varphi : M \to M'$ be a homomorphism. Since F_0 is free and ε' projective it follows that there exists a lift $\varphi_0 : F_0 \to F'_0$ of $\varphi \circ \varepsilon$, in particular $\varepsilon' \circ \varphi = \varphi \circ \varepsilon$. Furthermore, let $d \in \mathrm{im}(\partial_1)$ then $\varepsilon' \circ \varphi_0(d) = \varphi \circ \varepsilon(d) = 0$ thus φ_0 maps $\mathrm{im}(\partial_1)$ into $\mathrm{im}(\partial'_1)$ and we obtain a commutative diagram of exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{im}(\partial_1) & \xrightarrow{i_1} & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow 0 \\
& \downarrow{\scriptstyle \tilde{\varphi}_0} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & \\
0 \longrightarrow & \mathrm{im}(\partial'_1) & \xrightarrow{i'_1} & F'_0 & \xrightarrow{\varepsilon'} & M' & \longrightarrow 0,
\end{array}$$

where $\tilde{\varphi}_0 = \varphi_0|_{\mathrm{im}(\partial_1)}$. Repeating this construction for $\tilde{\varphi}_0$ then yields a homomorphism φ_1 and iterating this process gives a family of homomorphisms $\varphi_k : F_k \to F'_k$ which by definition satisfy $\varphi_k \circ \partial_{k+1} = \partial'_{k+1} \circ \varphi_{k+1}$.

Let φ'_* be another chain transformation lifting φ. Then $\varepsilon' \circ (\varphi_0 - \varphi'_0) = \varphi \circ \varepsilon - \varphi \circ \varepsilon = 0$ thus $\mathrm{im}(\varphi_0 - \varphi'_0) \subset \ker(\varepsilon') \simeq \mathrm{im}(\partial'_1)$ and hence $\varphi_0 - \varphi'_0$ defines a homomorphism $\tilde{s}_0 : F_0 \to \mathrm{im}(\partial'_1)$ such that $\varphi_0 - \varphi'_0 = i'_1 \circ \tilde{s}_0$. Again, since F_0 is free and $p'_1 : F'_1 \to \mathrm{im}(\partial'_1)$ projective we can lift $\tilde{s}_0$ to a homomorphism $s_0 : F_0 \to F'_1$ satisfying $p'_1 \circ s_0 = \tilde{s}_0$ and therefore $\partial'_1 \circ s_0 = i'_1 \circ p'_1 \circ s_0 = i'_1 \circ \tilde{s}_0 = \varphi_0 - \varphi'_0$.

Inductively, we obtain for $k \geq 1$

$$\begin{aligned}\partial'_k \circ (\varphi_k - \varphi'_k - s_{k-1} \circ \partial_k) &= (\varphi_{k-1} - \varphi'_{k-1}) \circ \partial_k - \partial'_k \circ s_{k-1} \circ \partial_k \\ &= (\varphi_{k-1} - \varphi'_{k-1}) \circ \partial_k - (\varphi_{k-1} - \varphi'_{k-1} - s_{k-2} \circ \partial_{k-1}) \circ \partial_k \\ &= 0,\end{aligned}$$

that is, $\mathrm{im}(\varphi_k - \varphi'_k - s_{k-1} \circ \partial_k) \subset \ker(\partial'_k) = \mathrm{im}(\partial'_{k+1})$ and, as before, $\varphi_k - \varphi'_k - s_{k-1} \circ \partial_k$ defines a homomorphism $\tilde{s}_k : F_k \to \mathrm{im}(\partial'_{k+1})$ with $\varphi_k - \varphi'_k - s_{k-1} \circ \partial_k = i'_{k+1} \circ \tilde{s}_k$. Again, since F_k is free and $p'_{k+1} : F'_{k+1} \to \mathrm{im}(\partial_{k+1})$ projective, $\tilde{s}_k$ lifts to a homomorphism $s_k : F_k \to F'_{k+1}$ satisfying $p'_{k+1} \circ s_k = \tilde{s}_k$ and therefore $\partial'_{k+1} \circ s_k = i'_{k+1} \circ p'_{k+1} \circ s_k = i'_{k+1} \circ \tilde{s}_k = \varphi_k - \varphi'_k - s_{k-1} \circ \partial_k$. It follows that $\varphi_k - \varphi'_k = s_{k-1} \circ \partial_k + \partial'_{k+1} \circ s_k$. QED

Notice that the existence of φ_* does not depend on the F_i''s being projective. Thus the proposition still holds if we replace $F'_* \xrightarrow{\varepsilon'} M'$ by an exact sequence ending in M'.

Let M and N be lattices and let

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0$$

be a truncated free resolution of M. Since the functor $\mathrm{Hom}(\,\cdot\,, N)$ is contravariant and left exact, applying it to $F_* \xrightarrow{\varepsilon} M$ yields a co-chain complex $\mathrm{Hom}(F_*, N)$

$$\cdots \longleftarrow \mathrm{Hom}(F_2, N) \xleftarrow{\partial_2^*} \mathrm{Hom}(F_1, N) \xleftarrow{\partial_1^*} \mathrm{Hom}(F_0, N),$$

that is, $\mathrm{im}(\partial_n^*) \subset \ker(\partial_{n+1}^*)$ where $\partial_i^*(f) = f \circ \partial_i$ for $f \in \mathrm{Hom}(F_{i-1}, N)$. An element in $\mathrm{im}(\partial_n^*)$ is called an n-coboundary and an element in $\ker(\partial_{n+1}^*)$ is called an n-cocycle. Two n-cocycles f and f' are called cohomologous if their difference is a coboundary, $f - f' = \partial_{n+1}(h)$.

Lemma 2.2.4 *Let $\varphi : M \to M'$ be a homomorphism, let $F_* \to M$ and $F'_* \to M'$ be free resolutions and let $\varphi_* : F_* \to F'_*$, $\varphi_* = (\varphi_k)_{k \geq 0}$ be the induced chain transformation. Then $f \circ \varphi_n$ is an n-cocycle (n-coboundary) for any n-cocycle (n-coboundary) $f : F'_n \to N$. In particular, if $M = M'$ and $\varphi = \mathrm{id}_M$ then φ lifts to chain transformations $\varphi_* : F_* \to F'_*$ and $\varphi'_* : F'_* \to F_*$ such that f and $f \circ (\varphi'_* \circ \varphi_*)$ are cohomologous for any n-cocyle $f : F_n \to N$.*

Proof: Let $f : F'_n \to N$ be an n-cocycle that is $\partial'^*_{n+1}(f) = f \circ \partial'_{n+1} = 0$. It then follows that $f \circ \varphi_n \circ \partial_{n+1} = f \circ \partial'_{n+1} \circ \varphi_{n+1} = 0$, thus $f \circ \varphi_n$ is an n-cocycle. Let $f : F'_n \to N$ be an n-coboundary, that is there exists a cochain $g : F_{n-1} \to N$ such that $f = \partial'^*_n(g) = g \circ \partial'_n$. It then follows that $f \circ \varphi_n = g \circ \partial'_n \circ \varphi_n = g \circ \varphi_{n-1} \circ \partial_n = \partial^*_n(g \circ \varphi_{n-1})$. Thus $f \circ \varphi_n$ is an n-coboundary.

Let $M = M'$ and $\varphi = \mathrm{id}_M$. Then $\mathrm{id}_* = (\mathrm{id}_{F_k})_{k\in\mathbb{N}}$ and $\varphi'_* \circ \varphi_* = (\varphi'_k \circ \varphi_k)_{k\in\mathbb{N}}$ are both chain transformations from $F_* \to M$ to itself and it follows from proposition 2.2.3 that there exist homomorphisms $s_k : F_k \to F_{k+1}$ such that $\partial_{k+1} \circ s_k + s_{k-1} \circ \partial_k = \mathrm{id}_k - (\varphi'_k \circ \varphi_k)$. Let $f : F_k \to N$ be a k-cocyle, then $f - f \circ (\varphi'_k \circ \varphi_k) = f \circ s_{k-1} \circ \partial_k = \partial^*_k(f \circ s_{k-1})$. Thus f and $f \circ (\varphi'_k \circ \varphi_k)$ are cohomologous. QED

Definition 2.2.5 *The n-th cohomology group of M with coefficients in N is defined as*

$$H^n(M,N) = H_n(\mathrm{Hom}(F_*,N)) = \begin{cases} \ker(\partial^*_1) = \mathrm{Hom}(M,N) & n = 0 \\ \ker(\partial^*_{n+1})/\mathrm{im}(\partial^*_n) & n > 0 \end{cases}$$

and an element $[f] \in H^n(M,N)$ is called the cohomology class of the cocycle $f \in \ker(\partial^*_{n+1})$. The cohomology group of M with coefficients in N is

$$H^*(M,N) = \sum_{n\geq 0} H^n(M,N).$$

The group structure of $H^n(M,N)$ is inherited from the group structure of $\mathrm{Hom}(F_n,N)$ *that is*

$$\begin{aligned} H^n(M,N) \times H^n(M,N) &\to H^n(M,N) \\ ([f],[h]) &\mapsto [f]+[h] = [f+h]. \end{aligned}$$

The group structure is indeed well defined as $f + h \in \ker(\partial^*_{n+1})$ whenever $f, h \in \ker(\partial^*_{n+1})$, and $[f] + [h]$ is independent of the choice of representatives as $(f + \partial^*_n(f')) + (h + \partial^*_n(h')) = f + h + \partial^*_n(f' + h') \in [f + h]$. Notice that the cohomology class $[f]$ of an n-cocycle $f : F_n \to N$ only depends on its value on $\mathrm{im}(\partial_{n+1}) \subset F_n$ since a general representative is of the form $f + g \circ \partial_n$ for $g : F_{n-1} \to N$. Thus the cohomology class [f] must be independent of the value on any point $x \in F_n$ with $\partial_n(x) \neq 0$. Furthermore, $\ker(\partial^*_1) = \mathrm{Hom}(M,N)$ since

$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ is exact, thus applying $\mathrm{Hom}(\,\cdot\,, N)$ yields the exact sequence $\mathrm{Hom}(F_1, N) \xleftarrow{\partial_1^*} \mathrm{Hom}(F_0, N) \xleftarrow{\varepsilon^*} \mathrm{Hom}(M, N) \longleftarrow 0$. It follows that $\ker(\partial_1^*) = \mathrm{im}(\varepsilon^*) \simeq \mathrm{Hom}(M, N)$. Thus

$$H^0(M, N) = \mathrm{Hom}(M, N).$$

A key characteristic of cohomology is its independence of the resolution for M since, by lemma 2.2.4, for any two resolutions $F_* \xrightarrow{\varepsilon} M$ and $F'_* \xrightarrow{\varepsilon'} M$ of M the identity map id_M induces an isomorphism on cohomology

$$\mathrm{id}_M^* : H_n(\mathrm{Hom}(F_*, N)) \xrightarrow{\sim} H_n(\mathrm{Hom}(F'_*, N)).$$

From a categorical point of view cohomology is a family of contra-variant functors $M \mapsto H^n(M, \cdot\,)$ from the category $\mathcal{F}(G)$ of finitely generated lattices to the category of abelian groups satisfying $H^n(M \oplus M', N) \simeq H^n(M, N) \oplus H^n(M', N)$ and the following properties.

Exactness: Let $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ be an exact sequence of lattices then there exists a long exact sequence

$$0 \longrightarrow H^0(M'', N) \xrightarrow{p^*} H^0(M, N) \xrightarrow{i^*} H^0(M', N) \xrightarrow{\delta_1} H^1(M'', N) \xrightarrow{p^*} \cdots$$

where δ is called the connecting homomorphism.

Homotopy: Two chain homotopic maps induce the same homomorphism on cohomology.

Similarly we can regard cohomology as a family of covariant functors $N \to H^n(\,\cdot\,, N)$ satisfying $H^n(M, N \oplus N') \simeq H^n(M, N) \oplus H^n(M, N')$ and the following property. Let $0 \longrightarrow N' \xrightarrow{i} N \xrightarrow{p} N'' \longrightarrow 0$ be an exact sequence of lattices then there exists a long exact sequence

$$0 \longrightarrow H^0(M, N') \xrightarrow{i_*} H^0(M, N) \xrightarrow{p_*} H^0(M, N'') \xrightarrow{\delta_1} H^1(M, N') \xrightarrow{i_*} \cdots$$

Lemma 2.2.6 *(Eckmann-Shapiro lemma)*

Let $H \subset G$ be a subgroup and let $i : H \hookrightarrow G$ be the inclusion. Let M be a G-lattice and let N be an H-lattices then there exist isomorphisms

$$H^n(M, i_*(N)) \simeq H^n(i^*(M), N)$$

and

$$H^n(i_*(N), M) \simeq H^n(N, i^*(M)).$$

Definition 2.2.7 *The cohomology of a group G with coefficients in N is defined as*

$$H^n(G, N) = H^n(\mathbb{Z}, N),$$

where $\mathbb{Z}$ is the trivial lattice. In particular, $H^0(G, N) = N^G$ is the sub-lattice of G-invariant elements of N.

Let $H \subset G$ be a subgroup, let $i : H \hookrightarrow G$ be the inclusion and let M be a G-lattice. By restricting scalars to $\mathbb{Z}[H]$ we can regard any free resolution $F_* \xrightarrow{\varepsilon} M$ of M over $\mathbb{Z}[G]$ as a free resolution $i^*(F)_* \xrightarrow{\varepsilon} i^*(M)$ of M over $\mathbb{Z}[H]$. In particular any coboundary (cocycle) over $\mathbb{Z}[G]$ is also a coboundary (cocycle) over $\mathbb{Z}[H]$. Thus i induces a homomorphism

$$i^* : H^n(G, N) \to H^n(H, i^*(N))$$

and by lemma 2.2.6 we can regard i^* as a homomorphism

$$i^* : H^n(G, N) \to H^n(\mathbb{Z} \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G], N)$$

induced by $\mathbb{Z} \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \to \mathbb{Z}$ where $1 \otimes g \mapsto 1$.

The group $H^2(G, N)$ has an interesting geometric interpretation. Let $E(n) = \mathbb{R}^n \rtimes O(n)$ be the group of isometries of $\mathbb{R}^n$ and let $\pi \subset E(n)$ be a subgroup. Let $t : E(n) \to \mathbb{R}^n$ and $r : E(n) \to O(n)$ be the projections. Then $\pi \cap \mathbb{R}^n$ is called the translational part of π and $r(\pi)$ is called the rotational part of π. The group π is called discontinuous if all its orbits in $\mathbb{R}^n$ are discrete and called irreducible if $t(\pi')$ spans $\mathbb{R}^n$ for all conjugates π' of π in $E(n)$. An irreducible and discontinuous subgroup of $E(n)$ is called a crystallographic group. Bieberbach's first theorem, [3], then states that the translational part $\pi \cap \mathbb{R}^n$ of a crystallographic group $\pi \subset E(n)$ is a free abelian group on n generators which are linearly independent translations. By a theorem of Auslander and Kuranishi, [1], an equivalent definition of a crystallographic group is the following. If π is given by a group extension of a finitely generated free abelian group N by a finite group G, that is, π occurs in an exact sequence

$$0 \longrightarrow N \longrightarrow \pi \longrightarrow G \longrightarrow 0 \qquad (2.1)$$

then π is a crystallographic group, where $N \simeq \pi \cap \mathbb{R}^n$ and $G \simeq r(\pi)$. It is well known that the orbit space $M = \mathbb{R}^n/\pi$ is a compact flat Riemannian manifold (a flat manifold for short) if and only if π is a torsion-free crystallographic group, [6, 7]. In particular, π is isomorphic to the fundamental group $\pi_1(M)$ of M. A torsion-free crystallographic group is called a Bieberbach group. Furthermore, any flat manifold M arises in this way and two Bieberbach groups π and π' are isomorphic if and only if there exists an affine diffeomorphism between $\mathbb{R}^n/\pi$ and $\mathbb{R}^n/\pi'$. Recall that the holonomy group of a manifold M is the subgroup in $O(n)$ given by parallel translation along closed curves in M. Furthermore a compact manifold is flat if and only if its holonomy group is finite, and every finite group is isomorphic to the holonomy group of a flat manifold M. A flat manifold with holonomy group isomorphic to a finite group G is called a G-manifold. In particular, if $\pi = \pi_1(M)$ then the holonomy group of M is isomorphic to $r(\pi)$. Thus the problem of classifying all G-manifolds is equivalent of determining all torsion-free extensions of a finitely generated free abelian group N by G. A detailed introduction to Bieberbach groups and compact flat Riemannian manifolds can be found in [6, 7, 16], a more general account of differential geometry can be found in [12].

It is well known that the group extensions of the above form (2.1), not necessarily torsion-free, are in one-to-one correspondence with $H^2(G, N)$, [14]. The following lemma due to P.A. Smith gives us some means of deciding which elements in $H^2(G, N)$ determine torsion-free extensions.

Lemma 2.2.8 *The elements $f \in H^2(G, N)$ determining torsion-free extensions are those for which $i^*(f) \neq 0$ in $H^2(C_p, i^*(N))$ for all subgroups $C_p \subset G$ of prime order p.*

A proof is given in [6, 7].

There exists another geometric meaning of the cohomology groups $H^n(G, \mathbb{Z})$ of a group G. Let X be a topological space. A group G is said to act on X if there exists a group homomorphism $G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the group of homeomorphism of X to itself. The group G is said to act properly if for all $x \in X$ there exists a neighbourhood U such that $Ug \cap U = \emptyset$ for all $1 \neq g \in G$. If a group G acts properly on a topological space X the singular complex $S(X)$ of X is a complex of free G-modules, see [14]. Furthermore, if the space X is acyclic, that is all higher homology groups vanish, $H_n(X) = 0,\ n \geq 1$, then the singular complex $S(X)$ is a free resolution of the trivial G-lattice $\mathbb{Z}$. If N is a trivial G-lattice then there exists an isomorphism between the cohomology groups of the quotient space X/G and those of G, that is,

$$H^n(X/G, N) \simeq H^n(G, N).$$

In other words if Y is aspherical, that is all higher homotopy groups vanish, with fundamental group G then its universal covering space is an acyclic space on which G acts properly. Thus the cohomology of Y is isomorphic to the cohomology of its fundamental group. For a detailed account of singular cohomology see [15]. The relationship to the cohomolgy of groups is explained in [7, 14, 16].

In singular cohomolgy there exists a well known product structure, namely the cup product, which is defined as follows. Let X be a topological space then the cup product is defined as

$$\begin{array}{rcl} \cup : H^k(X,\mathbb{Z}) \otimes H^l(X,\mathbb{Z}) & \longrightarrow & H^{k+l}(X,\mathbb{Z}) \\ f \otimes h & \mapsto & f \cup h, \end{array}$$

where $(f \cup h)[p_0, \ldots, p_{k+l}] = f[p_0, \ldots, p_k]\ h[p_k, \ldots, p_{k+l}]$ for a singular $(k+l)$-complex $[p_0, \ldots, p_{k+l}] \in S_{k+l}(X)$ with k-th front face $[p_0, \ldots, p_k] \in S_k(X)$ and l-th back face $[p_k, \ldots p_{k+l}]$. Furthermore, the cup product satisfies

$$f \cup h = (-1)^{kl} h \cup f,$$

Thus the cup product makes $H^*(X,\mathbb{Z}) = \sum_{k \geq 0} H^k(X,\mathbb{Z})$ into a graded-commutative ring. that

It follows that if X is an acyclic space on which G-acts properly then the isomorphism $H^n(X/G,\mathbb{Z}) \simeq H^n(G,\mathbb{Z})$ induces a product structure on $H^n(G,\mathbb{Z})$ making $H^*(G,\mathbb{Z})$ into a graded-commutative ring. We can extend this product structure to obtain a pairing

$$\cup : H^k(M,R) \otimes H^l(R,N) \longrightarrow H^{k+l}(M,N)$$

for any triple M, N, R of lattices. To do this we will need Yoneda's theory of module extensions which we introduce next.

Definition 2.2.9 *Let M and N be lattices, and let $n \geq 1$. An n-fold extension of M by N is an exact sequence*

$$\mathcal{E} : \quad 0 \longrightarrow N \xrightarrow{\nu_n} B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \xrightarrow{\nu_0} M \longrightarrow 0 .$$

The set of all n-fold extensions of M by N is denoted by $\mathcal{E}xt^n(M,N)$, where $\mathcal{E}xt^0(M,N) = \mathrm{Hom}(M,N)$, and the set of all extensions of M by N is denoted by $\mathcal{E}xt^*(M,N) = \sum_{k \geq 0} \mathcal{E}xt^k(M,N)$.

There is a natural pairing of extensions, called the Yoneda product,

$$\begin{array}{rcl}\circ : \mathcal{E}xt^k(M,R) \otimes \mathcal{E}xt^l(R,N) & \longrightarrow & \mathcal{E}xt^{k+l}(M,N) \\ \mathcal{E} \otimes \mathcal{E}' & \mapsto & \mathcal{E}' \circ \mathcal{E},\end{array}$$

where, if $\mathcal{E}$ is the extension

$$0 \longrightarrow R \xrightarrow{\nu_k} B_{k-1} \longrightarrow \cdots \longrightarrow B_0 \xrightarrow{\nu_0} M \longrightarrow 0$$

and $\mathcal{E}'$ is the extension

$$0 \longrightarrow N \xrightarrow{\nu'_l} B'_{l-1} \longrightarrow \cdots \longrightarrow B'_0 \xrightarrow{\nu'_0} R \longrightarrow 0,$$

then $\mathcal{E}' \circ \mathcal{E}$ is the extension

$$0 \longrightarrow N \xrightarrow{\nu'_l} \cdots \longrightarrow B'_0 \xrightarrow{\nu_k \circ \nu'_0} B_{k-1} \longrightarrow \cdots \xrightarrow{\nu_0} M \longrightarrow 0.$$

$$B'_0 \xrightarrow{\nu'_0} R \xrightarrow{\nu_k} B_{k-1}$$

This pairing allows us to regard an extension $\mathcal{E}$ as concatenated by short exact sequences

$$\mathcal{E}_k : \quad 0 \longrightarrow \mathrm{im}(\nu_k) \xrightarrow{j_k} B_{k-1} \xrightarrow{q_{k-1}} \mathrm{im}(\nu_{k-1}) \longrightarrow 0, \tag{2.2}$$

where $\nu_k = j_k \circ q_k$ for $1 < k < n$, $\nu_n = j_n$, $\nu_0 = q_0$, $\mathrm{im}(\nu_n) = N$ and $\mathrm{im}(\nu_0) = M$, and we write

$$\mathcal{E} = \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1.$$

Let $\mathcal{E} \in \mathcal{E}xt^1(M,N)$ and let $f : N \to N'$ be a homomorphism. Then there exists a commutative diagram of exact rows

$$\begin{array}{lccccccccc} \mathcal{E} : & 0 & \longrightarrow & N & \xrightarrow{j} & B & \xrightarrow{q} & M & \longrightarrow & 0 \\ & & & \downarrow f & & \downarrow s_f & & \downarrow \mathrm{id} & & \\ f_*\mathcal{E} : & 0 & \longrightarrow & N' & \xrightarrow{r_f} & \varinjlim(f,j) & \xrightarrow{t_f} & M & \longrightarrow & 0, \end{array} \tag{2.3}$$

where $\varinjlim(f,j) = (N' \oplus B)/\mathrm{im}(f,-j)$ is the push-out of f and j, $r_f(n') = [n',0]$, $s_f(b) = [0,b]$ and $t_f([n,b]) = q(b)$. Thus f defines a map

$$\begin{array}{rcl} f_* : \mathcal{E}xt^n(M,N) & \longrightarrow & \mathcal{E}xt^n(M,N') \\ \mathcal{E} & \mapsto & f_*\mathcal{E}, \end{array}$$

where $f_*\mathcal{E} = f_*\mathcal{E}_n \circ \cdots \circ \mathcal{E}_1$ for $\mathcal{E} = \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1$.

Similarly, let $h : M' \to M$ be a homomorphism. Then there exists a commutative diagram of exact rows

$$\begin{array}{ccccccccccc} \mathcal{E}h^* : & 0 & \longrightarrow & N & \xrightarrow{u_h} & \varprojlim(h,q) & \xrightarrow{w_h} & M' & \longrightarrow & 0 \\ & & & \downarrow \mathrm{id} & & \downarrow v_h & & \downarrow h & & \\ \mathcal{E} : & 0 & \longrightarrow & N & \xrightarrow{j} & B & \xrightarrow{q} & M & \longrightarrow & 0, \end{array} \tag{2.4}$$

where $\varprojlim(h,q) = \{(m',b) \mid h(m') = q(b)\}$ is the pull-back of h and q, $u_h(n) = (j(n), 0)$, $v_h(m', b) = b$ and $w_h(m', b) = m'$. Thus h defines a map

$$\begin{aligned} h^* : \mathcal{E}xt^n(M,N) &\longrightarrow \mathcal{E}xt^n(M,N') \\ \mathcal{E} &\mapsto \mathcal{E}h^*, \end{aligned}$$

where $\mathcal{E}h^* = \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1 h^*$ for $\mathcal{E} = \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1$.

Next we define an additive structure on $\mathcal{E}xt^n(M,N)$. Let $\mathcal{E} \in \mathcal{E}xt^n(M,N)$

$$\mathcal{E} : \quad 0 \longrightarrow N \xrightarrow{\nu_n} B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \xrightarrow{\nu_0} M \longrightarrow 0$$

and $\mathcal{E}' \in \mathcal{E}xt^n(M', N')$

$$\mathcal{E}' : \quad 0 \longrightarrow N' \xrightarrow{\nu'_n} B'_{n-1} \longrightarrow \cdots \longrightarrow B'_0 \xrightarrow{\nu'_0} M' \longrightarrow 0$$

be two n-fold extensions. Then the direct sum of $\mathcal{E}$ and $\mathcal{E}'$ is the following n-fold extension in $\mathcal{E}xt^n(M \oplus M', N \oplus N')$

$$\mathcal{E} \oplus \mathcal{E}' : 0 \longrightarrow N \oplus N' \xrightarrow{\nu_n \oplus \nu'_n} B_{n-1} \oplus B'_{n-1} \longrightarrow \cdots \longrightarrow B_0 \oplus B'_0 \xrightarrow{\nu_0 \oplus \nu'_0} M \oplus M' \longrightarrow 0\,.$$

Let

$$\begin{aligned} \triangle_M : M &\to M \oplus M \\ m &\mapsto (m,m) \end{aligned}$$

be the diagonal map on M, and let

$$\begin{aligned} \nabla_N : N \oplus N &\to N \\ (n_1, n_2) &\mapsto n_1 + n_2 \end{aligned}$$

be the addition in N. Then the Baer sum of extensions is defined as

$$\begin{aligned} \mathcal{E}xt^n(M,N) \times \mathcal{E}xt^n(M,N) &\longrightarrow \mathcal{E}xt^n(M,N) \\ (\mathcal{E}, \mathcal{E}') &\mapsto \mathcal{E} + \mathcal{E}' = (\nabla_N)_*(\mathcal{E} \oplus \mathcal{E}')(\triangle_M)^* \end{aligned}$$

Finally we introduce an equivalence relation on $\mathcal{E}xt^n(M,N)$ which is compatible with the Yoneda product and the Baer sum. This will allow us to identify cohomology with module extensions and to introduce a product structure on cohomology which generalises the above introduced cup product.

We call two 1-fold extensions $\mathcal{E}, \mathcal{E}' \in \mathcal{E}xt^1(M,N)$ equivalent, $\mathcal{E} \equiv \mathcal{E}'$, if there exists a homomorphism $\varphi : B \to B'$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{j} & B & \xrightarrow{q} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N & \xrightarrow{j'} & B' & \xrightarrow{q'} & M & \longrightarrow & 0. \end{array}$$

The set of equivalence classes is denoted by $\mathrm{Ext}^1(M,N)$.

Let $\mathcal{E}, \mathcal{E}' \in \mathcal{E}xt^2(M,N)$ be two 2-fold extensions of M by N where $\mathcal{E} = \mathcal{E}_2 \circ \mathcal{E}_1$, $\mathcal{E}' = \mathcal{E}_2' \circ \mathcal{E}_1'$, $\mathcal{E}_1' \in \mathcal{E}xt^1(N,R)$ and $\mathcal{E}_2' \in \mathcal{E}xt^1(R,M)$. Then $\mathcal{E}$ and $\mathcal{E}'$ are called equivalent, $\mathcal{E} \equiv \mathcal{E}'$, if either $\mathcal{E}_i \equiv \mathcal{E}_i'$, $i = 1,2$ or there exists a lattice S such that $\mathcal{E} = \mathcal{E}_1'\pi^* \circ \iota_*\mathcal{E}_2'$, where $\pi : R \oplus S \to R$ is the projection and $\iota : R \to R \oplus S$ is the inclusion.

Let $\mathcal{E} = \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1$ and $\mathcal{E}' = \mathcal{E}_n' \circ \cdots \circ \mathcal{E}_1'$ be two n-fold extensions of M by N. Then $\mathcal{E}$ and $\mathcal{E}'$ are called equivalent, $\mathcal{E} \equiv \mathcal{E}'$, if $\mathcal{E}$ can be transformed into $\mathcal{E}'$ by either replacing a 1-fold extension $\mathcal{E}_i$ by an equivalent 1-fold extension, or by replacing a 2-fold extension $\mathcal{E}_{i+1} \circ \mathcal{E}_i$ by an equivalent 2-fold extension.

Definition 2.2.10 *Let M and N be lattices then*

$$\mathrm{Ext}^n(M,N) = \begin{cases} \mathrm{Hom}(M,N) & n = 0 \\ \mathcal{E}xt^n(M,N)/\equiv & n \geq 1 \end{cases}$$

and

$$\mathrm{Ext}^*(M,N) = \sum_{k \geq 0} \mathrm{Ext}^k(M,N).$$

Definition 2.2.11 *A morphism $(\alpha, \beta) : \mathcal{E} \to \mathcal{E}'$ of extensions is a family of homomorphisms $(\alpha, \ldots, \beta)$ such that the following diagram commutes*

$$\begin{array}{cccccccccccccc} \mathcal{E}: & 0 & \longrightarrow & N & \longrightarrow & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \alpha} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\ \mathcal{E}': & 0 & \longrightarrow & N' & \longrightarrow & B_{n-1}' & \longrightarrow & \cdots & \longrightarrow & B_0' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

Lemma 2.2.12 *Let $\mathcal{E}, \mathcal{E}' \in \mathcal{E}xt^n(M,N)$ and let $(\alpha, \beta) : \mathcal{E} \to \mathcal{E}'$ be a morphism of extensions then $\alpha_* \mathcal{E} \equiv \mathcal{E}' \beta^*$.*

A proof is given in [14], proposition 5.1 pp 84.

Let $\mathcal{E}_1, \mathcal{E}_1' \in \mathcal{E}xt^n(M,R)$ and let $\mathcal{E}_2, \mathcal{E}_2' \in \mathcal{E}xt^m(R,N)$ such that $\mathcal{E}_i \equiv \mathcal{E}_i'$. Then it follows immediately from the definition that $\mathcal{E}_2 \circ \mathcal{E}_1 \equiv \mathcal{E}_2' \circ \mathcal{E}_1'$. Thus the above defined Yoneda product extends to the quotients.

Definition 2.2.13 *The Yoneda product of extensions is defined as the following pairing*

$$\begin{aligned} \circ : \mathrm{Ext}^n(M,R) \otimes \mathrm{Ext}^m(R,N) &\longrightarrow \mathrm{Ext}^{n+m}(M,N) \\ [\mathcal{E}] \otimes [\mathcal{E}'] &\mapsto [\mathcal{E}'] \circ [\mathcal{E}] = [\mathcal{E}' \circ \mathcal{E}]. \end{aligned}$$

Let $\mathcal{E}_1, \mathcal{E}_1' \in \mathcal{E}xt^n(M,N)$ and $\mathcal{E}_2, \mathcal{E}_2' \in \mathcal{E}xt^n(M',N')$ such that $\mathcal{E}_i \equiv \mathcal{E}_i'$ for $i = 1,2$ then $\mathcal{E}_1 \oplus \mathcal{E}_2 \equiv \mathcal{E}_1' \oplus \mathcal{E}_2'$. Thus the direct sum, and therefore the Baer sum, is well defined on Ext^n .

Definition 2.2.14 *The Baer sum of extensions is defined as*

$$\begin{aligned} + : \mathrm{Ext}^n(M,N) \times \mathrm{Ext}^n(M,N) &\to \mathrm{Ext}^n(M,N) \\ (\mathcal{E}, \mathcal{E}') &\mapsto \mathcal{E} + \mathcal{E}' = (\nabla_N)_*(\mathcal{E} \oplus \mathcal{E}')(\triangle_M)^* \end{aligned}$$

Lemma 2.2.15 *Let $f, f' \in \mathrm{Hom}(N,N')$, $h, h' \in \mathrm{Hom}(M',M)$ and let $\mathcal{E} \in \mathrm{Ext}^n(M,N)$ then*

1) $(f + f')_* \mathcal{E} = f_* \mathcal{E} + f'_* \mathcal{E}$

2) $\mathcal{E}(h + h')^* = \mathcal{E} h^* + \mathcal{E} h'^*$

Proof: 1) Let $\mathcal{E} = \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1$ where $\mathcal{E}_k$ is of the form (2.2). Then

$$(f + f')_* \mathcal{E} = ((f + f')_* \mathcal{E}_n) \circ \cdots \circ \mathcal{E}_1$$

and

$$f_*\mathcal{E} + f'_*\mathcal{E} = (\nabla_{N'})_*(f_*\mathcal{E}_n \oplus f'_*\mathcal{E}_n) \circ (\mathcal{E}_{n-1} \oplus \mathcal{E}_{n-1}) \circ \cdots \circ (\mathcal{E}_1 \oplus \mathcal{E}_1)(\triangle_M)^*.$$

From the definition of the push-out it follows that there is a commutative diagram of exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \xrightarrow{r} & \varinjlim(\nabla_{N'}, r_f \oplus r_{f'}) & \xrightarrow{t} & \operatorname{im}(\nu_{n-1}) \oplus \operatorname{im}(\nu_{n-1}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \nabla_{N'}} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & N' \oplus N' & \xrightarrow{r_f \oplus r_{f'}} & \varinjlim(f, \nu_n) \oplus \varinjlim(f', \nu_n) & \xrightarrow{t_f \oplus t_{f'}} & \operatorname{im}(\nu_{n-1}) \oplus \operatorname{im}(\nu_{n-1}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle f \oplus f'} & & \uparrow{\scriptstyle s_f \oplus s_{f'}} & & \uparrow{\scriptstyle \triangle_{\operatorname{im}(\nu_{n-1})}} & & \\
0 & \longrightarrow & N & \xrightarrow{\nu_n} & B_{n-1} & \xrightarrow{q_{n-1}} & \operatorname{im}(\nu_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f+f'} & & \downarrow{\scriptstyle s_{f+f'}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & N' & \xrightarrow{r_{f+f'}} & \varinjlim(f + f', \nu_n) & \xrightarrow{t_{f+f'}} & \operatorname{im}(\nu_{n-1}) & \longrightarrow & 0
\end{array}$$

where the top row is $(\nabla_{N'})_*(f_*\mathcal{E}_n \oplus f'_*\mathcal{E}_n)$ and the bottom row is $(f + f')_*\mathcal{E}_n$ Furthermore, since $r \circ (f + f') = r \circ \nabla_{N'} \circ (f \oplus f') = s \circ (r_f \oplus r_{f'}) \circ (f \oplus f') = s \circ (s_f \oplus s_{f'}) \circ \nu_n$ it follows from the universal property of the push-out that there exists a homomorphism $\alpha : \varinjlim(f + f', \nu_n) \to \varinjlim(\nabla_{N'}, r_f \oplus r_{f'})$ such that

$$\alpha \circ r_{f+f'} = r$$

and $\alpha \circ s_{f+f'} = s \circ (s_f \oplus s_{f'})$. In particular, $\alpha([n', x]_{f+f'}) = r(n') + (s \circ (s_f \oplus s_{f'}))(x)$ for $[n', x]_{f+f'} \in \varinjlim(f + f', \nu_n),\ n' \in N'$ and $x \in B_{n-1}$ so that

$$\begin{aligned}
(t \circ \alpha)([n', x]_{f+f'}) &= t(r(n') + (s \circ (s_f \oplus s_{f'}))(x)) = (t \circ s \circ (s_f \oplus s_{f'}))(x) \\
&= (t_f \oplus t_{f'}) \circ (s_f \oplus s_{f'})(x) = \triangle_{\operatorname{im}(\nu_{n-1})} \circ q_{n-1}(x) \\
&= \triangle_{\operatorname{im}(\nu_{n-1})} \circ t_{f+f'}([n', x]_{f+f'}).
\end{aligned}$$

Thus we obtain a commutative diagram of exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \xrightarrow{r_{f+f'}} & \varinjlim(f + f', \nu_n) & \xrightarrow{t_{f+f'}} & \operatorname{im}(\nu_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \triangle_{\operatorname{im}(\nu_{n-1})}} & & \\
0 & \longrightarrow & N' & \xrightarrow{r} & \varinjlim(\nabla_{N'}, r_f \oplus r_{f'}) & \xrightarrow{t} & \operatorname{im}(\nu_{n-1}) \oplus \operatorname{im}(\nu_{n-1}) & \longrightarrow & 0
\end{array}$$

That is a morphism $(\mathrm{id}, \triangle_{\operatorname{im}(\nu_{n-1})})$ such that

$$(\mathrm{id}, \triangle_{\operatorname{im}(\nu_{n-1})})((f + f')_*\mathcal{E}_n) = (\nabla_{N'})_*(f_*\mathcal{E}_n \oplus f'_*\mathcal{E}_n)$$

and the claim follows for $n = 1$ from lemma 2.2.12.

For $n \geq 2$ let $\mathcal{E}_1$ be the short exact sequence $0 \longrightarrow R \xrightarrow{j} B_1 \xrightarrow{q} M \longrightarrow 0$ and let $\mathcal{E}_2$ be the short exact sequence $0 \longrightarrow N \xrightarrow{i} B_2 \xrightarrow{p} R \longrightarrow 0$. Then $(\mathcal{E}_2 \oplus \mathcal{E}_2)(\triangle_R)^* \circ \mathcal{E}_1$ is the exact sequence

$$0 \longrightarrow N \oplus N \xrightarrow{u_R} \varprojlim(\triangle_R, p \oplus p) \xrightarrow{j \circ w_R} B_1 \xrightarrow{q} M \longrightarrow 0$$

and $(\mathcal{E}_2 \oplus \mathcal{E}_2) \circ (\mathcal{E}_1 \oplus \mathcal{E}_1)(\triangle_M)^*$ is the exact sequence

$$0 \longrightarrow N \oplus N \xrightarrow{i \oplus i} B_2 \oplus B_2 \xrightarrow{u_M \circ (p \oplus p)} \varprojlim(\triangle_M, q \oplus q) \xrightarrow{w_M} M \longrightarrow 0$$

Since $\triangle_M \circ q = (q \oplus q) \circ \triangle_M$ it follows from the universal property of the pull-back that there exists a homomorphism $\beta : B_1 \to \varprojlim(\triangle_M, q \oplus q)$ where $\beta(x) = ((x, x), q(x))$ such that $v_M \circ \beta = \triangle_M$ and $w_M \circ \beta = q$ where $v_M : \varprojlim(\triangle_M, q \oplus q) \to B_1 \oplus B_1$ is the projection. Furthermore, $(\beta \circ j)(r) = ((r, r), 0) = (u_M \circ \triangle_R)(r)$ so that we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & N \oplus N & \xrightarrow{u_R} & \varprojlim(\triangle_R, p \oplus p) & \xrightarrow{j \circ w_R} & B_1 & \xrightarrow{q} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle v_R} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & N \oplus N & \xrightarrow{i \oplus i} & B_2 \oplus B_2 & \xrightarrow{u_M \circ (p \oplus p)} & \varprojlim(\triangle_M, q \oplus q) & \xrightarrow{w_M} & M & \longrightarrow & 0
\end{array}$$

that is a morphism $(\mathrm{id}, \mathrm{id})$ from $(\mathcal{E}_2 \oplus \mathcal{E}_2)(\triangle_R)^* \circ \mathcal{E}_1$ to $(\mathcal{E}_2 \oplus \mathcal{E}_2) \circ (\mathcal{E}_1 \oplus \mathcal{E}_1)(\triangle_M)^*$. By lemma 2.2.12. it follows that $(\mathcal{E}_2 \oplus \mathcal{E}_2)(\triangle_R)^* \circ \mathcal{E}_1 \equiv (\mathcal{E}_2 \oplus \mathcal{E}_2) \circ (\mathcal{E}_1 \oplus \mathcal{E}_1)$. It now follows inductively that the above morphism yields the equivalence

$$\begin{aligned}
(f + f')_* \mathcal{E} &= (f + f')_* \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1 \\
&\equiv (\nabla_{N'})_* (f_* \mathcal{E}_n \oplus f'_* \mathcal{E}_n) \triangle^*_{\mathrm{im}(\nu_{n-1})} \circ \cdots \circ \mathcal{E}_1 \\
&\equiv (\nabla_N)_* (f_* \mathcal{E}_n \oplus f'_* \mathcal{E}_n) \circ (\mathcal{E}_{n-1} \oplus \mathcal{E}_{n-1}) \circ \cdots \circ (\mathcal{E}_1 \oplus \mathcal{E}_1)(\triangle_M)^* \\
&= f_* \mathcal{E} + f'_* \mathcal{E}.
\end{aligned}$$

Similarly we can prove 2) by employing the universal property of the pull-back we obtain the equivalences

$$(\nabla_{\mathrm{im}(\nu_1)})_* (\mathcal{E}_1 h^* \oplus \mathcal{E}_1 h'^*)(\triangle_{M'})^* \equiv \mathcal{E}_1 (h + h')^*$$

and

$$\mathcal{E}_2 \circ (\nabla_R)_* (\mathcal{E}_1 \oplus \mathcal{E}_1) \equiv (\nabla_N)_* (\mathcal{E}_2 \oplus \mathcal{E}_2) \circ (\mathcal{E}_1 \oplus \mathcal{E}_1)$$

and inductively we obtain

$$\mathcal{E} h^* \oplus \mathcal{E} h'^* \equiv \mathcal{E}(h + h')^*.$$

QED

Theorem 2.2.16 *Let M and N be lattices then $(\mathrm{Ext}^n(M,N),+)$ is an abelian group. The zero element is the equivalence class of the split extension and the inverse of an extension $\mathcal{E}$ is given by $(-\mathrm{id}_N)_*\mathcal{E}$. Furthermore,*

$$\mathcal{E}\circ(\mathcal{E}_1+\mathcal{E}_2)=\mathcal{E}\circ\mathcal{E}_1+\mathcal{E}\circ\mathcal{E}_2$$

and

$$(\mathcal{E}_1+\mathcal{E}_2)\circ\mathcal{E}'=\mathcal{E}_1\circ\mathcal{E}'+\mathcal{E}_2\circ\mathcal{E}'$$

for all $\mathcal{E}\in\mathrm{Ext}^m(N,R)$ $\mathcal{E}'\in\mathrm{Ext}^{m'}(R,M)$ and $\mathcal{E}_1,\mathcal{E}_2\in\mathrm{Ext}^n(M,N)$.

In particular, $\mathrm{Ext}^*(M,M)$ equipped with Yoneda product and Baer sum is an associative graded ring with unit $\mathrm{id}_M : M\to M$.

A proof is given in [14], theorem 5.3 pp 85.

We are now ready to show the equivalence of cohomology and module extensions. This relationship was first studied by N. Yoneda in his paper [17]. To do this let

$$0\longrightarrow N\xrightarrow{\nu_n}B_{n-1}\longrightarrow\cdots\longrightarrow B_0\xrightarrow{\nu_0}M\longrightarrow 0$$

be a representative of $\mathcal{E}\in\mathrm{Ext}^n(M,N)$, let $F_*\xrightarrow{\varepsilon}M$ be a free resolution of M and let $\iota=(\iota_k)_{k\le n}$ be a chain transformation lifting id_M, where we denote $\iota_n=\iota_n(\mathcal{E})$. That is, we have a commutative diagram of exact rows

$$\begin{array}{ccccccccccc}
\cdots\longrightarrow & F_n & \xrightarrow{\partial_n} & F_{n-1} & \longrightarrow\cdots\longrightarrow & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \iota_n(\mathcal{E})} & & \downarrow{\scriptstyle \iota_{n-1}} & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle \mathrm{id}} & \\
0\longrightarrow & N & \xrightarrow{\nu_n} & B_{n-1} & \longrightarrow\cdots\longrightarrow & B_0 & \xrightarrow{\nu_0} & M & \longrightarrow 0.
\end{array}$$

Theorem 2.2.17 *Let M and N be lattices. Then for all $n\ge 0$ there exists a group isomorphism*

$$\mathcal{Y}:\mathrm{Ext}^n(M,N)\ \xrightarrow{\sim}\ H^n(M,N),$$

which for $n\ge 1$ is given by $\mathcal{Y}(\mathcal{E})=[\iota_n(\mathcal{E})]$.

Proof: For $n = 0$ we already have by definition

$$H^0(M, N) = \mathrm{Hom}(M, N) = \mathrm{Ext}^0(M, N).$$

Let $n \geq 1$, let $F_* \xrightarrow{\varepsilon} M$ be a free resolution of M and let

$$\mathcal{E}: \quad 0 \longrightarrow N \xrightarrow{\nu_n} B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \xrightarrow{\nu_0} M \longrightarrow 0$$

be a representative of an extension in $\mathrm{Ext}^n(M, N)$. Let $\iota_n(\mathcal{E}) : F_n \to N$ be a lift of id_M. Then, since $\iota_n(\mathcal{E})$ commutes with the boundary operators and $\nu_{n+1} = 0$, it follows that $\partial^*_{n+1}(\iota_n(\mathcal{E})) = \iota_n(\mathcal{E}) \circ \partial_{n+1} = 0$. Thus $\iota_n(\mathcal{E})$ is a cocycle.

Let $\iota'_n(\mathcal{E})$ be another lift of id_M. Then, by proposition 2.2.3 there are homomorphisms, $s_n : F_n \to 0$ and $s_{n-1} : F_{n-1} \to N$ such that $\iota_n(\mathcal{E}) - \iota'_n(\mathcal{E}) = s_{n-1} \circ \partial_n + 0 = \partial^*_n(s_{n-1})$. Thus $\iota_n(\mathcal{E})$ and $\iota'_n(\mathcal{E})$ are cohomologous.

Let $\mathcal{E}' \in \mathcal{E}xt^n(M, N)$ such that $\mathcal{E} \equiv \mathcal{E}'$. It is sufficient to consider a morphism $(\mathrm{id}_N, \mathrm{id}_M)$ from $\mathcal{E}$ to $\mathcal{E}'$. It then follows that $\iota_n(\mathcal{E}') = \mathrm{id}_N \circ \iota_n(\mathcal{E})$. Thus $\mathcal{E}$ and $\mathcal{E}'$ define the same cohomology class and we obtain a well-defined homomorphism $\mathrm{Ext}^n(M, N) \to H^n(M, N)$.

To define an inverse regard ∂_n as the composition $i_n \circ p_n$, where $p_n : F_n \to \mathrm{im}(\partial_n)$ is the projection and $i_n : \mathrm{im}(\partial_n) \to F_{n-1}$ is the inclusion. Consider the n-fold extension

$$\mathcal{E}: \quad 0 \longrightarrow \mathrm{im}(\partial_n) \xrightarrow{i_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Since any cocycle $f : F_n \to N$ vanishes on $\ker(\partial_n) = \ker(p_n)$ it can be written uniquely in the form $f = f' \circ p_n$ for some $f' : \mathrm{im}(\partial_n) \to N$, and we can construct an extension $\mathcal{E}(f) = f'_*\mathcal{E} \in \mathrm{Ext}^n(M, N)$. Let $f = g \circ \partial_n$ be a coboundary. Then $f' \circ p_n = g \circ \partial_n = g \circ i_n \circ p_n$. Since p_n is surjective it follows that $f' = g \circ i_n$ and $f'_*\mathcal{E} = (g \circ i_n)_*\mathcal{E} = g_* i_{n*}\mathcal{E}$. But by proposition 1.7, chapter 3 in [14] $(i_n)_*\mathcal{E} \equiv 0$. Thus $f'_*\mathcal{E} \equiv 0$ and $f \mapsto \mathcal{E}(f)$ gives a well defined map $H^n(M, N) \to \mathrm{Ext}^n(M, N)$.

Let $\iota_n(\mathcal{E})$ be the cocycle given by $\mathcal{E}$ then $\iota_n(\mathcal{E}) = \iota_n(\mathcal{E})' \circ p_n$. Consider the induced extension $(\iota_n(\mathcal{E})')_*\mathcal{E}$. Then we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{r} & \varinjlim(\iota_n(\mathcal{E})', i_n) & \xrightarrow{t} & \mathrm{im}(\partial_{n-1}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \iota_n(\mathcal{E})'} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathrm{im}(\partial_n) & \xrightarrow{i_n} & F_{n-1} & \xrightarrow{p_{n-1}} & \mathrm{im}(\partial_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_n(\mathcal{E})'} & & \downarrow{\scriptstyle \iota_{n-1}} & & \downarrow{\scriptstyle \iota_{n-2}} & & \\
0 & \longrightarrow & N & \xrightarrow{\nu_n} & B_{n-1} & \xrightarrow{q_{n-1}} & \mathrm{im}(\nu_{n-1}) & \longrightarrow & 0
\end{array}$$

Since $\nu_n \circ \iota_n(\mathcal{E})' = \iota_{n-1} \circ i_n$ it follows by the universal property of the push-out that there exists a homomorphism $\alpha : \varinjlim(\iota_n(\mathcal{E})', i_n) \to B_{n-1}$ such that $s \circ \alpha =$

ι_{n-1}, $r \circ \alpha = \nu_n$ and $\alpha([v,x]) = \nu_n(v) + \iota_{n-1}(x)$. Furthermore

$$\begin{aligned}(q_{n-1} \circ \alpha)[v,x] &= q_{n-1}(\nu_n(v) + \iota_{n-1}(x)) = (\iota_{n-2} \circ p_{n-1})(x) \\ &= (\iota_{n-2} \circ t)([v,x]).\end{aligned}$$

Thus we obtain $(\iota_n(\mathcal{E})')_*\mathcal{E} \equiv \mathcal{E}$. Finally, by lemma 2.2.15, it follows that $\mathcal{E}(f+h) = \mathcal{E}(f)+\mathcal{E}(h)$ so that $H^n(M,N) \simeq \mathrm{Ext}^n(M,N)$ is indeed a group isomorphism.QED

The isomorphism in theorem (2.2.17) now enables us to introduce the following product structure on $H^*(M,N)$

Definition 2.2.18 *The cup product in cohomology is defined as*

$$\begin{aligned}\cup : H^k(M,R) \otimes H^l(R,N) &\rightarrow H^{k+l}(M,N) \\ f \otimes h &\mapsto f \cup h = \mathcal{Y}(\mathcal{Y}^{-1}(h) \circ \mathcal{Y}^{-1}(f)).\end{aligned}$$

In particular, $H^*(M,M)$ equipped with the cup product is an associative graded ring with unit id_M.

2.3 Stable modules, syzygies and the derived module category

A key characteristic of cohomology is its independence of the free resolution over which it is computed. Considering two free resolutions $F_* \xrightarrow{\varepsilon} M$ and $F'_* \xrightarrow{\varepsilon'} M$ of a lattice M, Schanuel's lemma, see [2], states that $\mathrm{im}(\partial_n)$ and $\mathrm{im}(\partial'_n)$ are stably equivalent, that is, there exists free modules F and F' such that $\mathrm{im}(\partial_n) \oplus F \simeq \mathrm{im}(\partial'_n) \oplus F'$. Now since the cohomology class of an n-cocycle $f : F_n \rightarrow N$ only depends on its value on $\mathrm{im}(\partial_{n+1}) \subset F_n$ it is only natural to try and express cohomology in terms of the stable class $\Omega_n(M)$ of $\mathrm{im}(\partial_n)$. To do this we will need to recall some theory of stable modules, syzygies and the derived module category which we are going to do here. For a detailed account we refer the reader to [10].

Definition 2.3.1 *Two lattices M and M' are called stably equivalent, $M \sim M'$, if there exist free modules F and F' such that*

$$M \oplus F \simeq M' \oplus F'.$$

The stable class $[M]$ *of a lattice* M *is the corresponding equivalence class*

$$[M] = \{\ M' \in \mathcal{F}(G) \mid M' \sim M\}.$$

$[M]$ *is also called a stable lattice.* M *is called minimal provided* $\mathrm{rk}_{\mathbb{Z}}(M) \leq \mathrm{rk}_{\mathbb{Z}}(M')$ *for all* $M' \in [M]$*.*

The direct sum of stable lattices is defined as $[M] \oplus [N] = [M \oplus N]$ and the tensor product of stable lattices is defined as $[M] \otimes [N] = [M \otimes N]$.

Lemma 2.3.2 *(Schanuel's lemma)*

Let

$$0 \longrightarrow N \xrightarrow{\ i\ } F \xrightarrow{\ p\ } M \longrightarrow 0$$

and

$$0 \longrightarrow N' \xrightarrow{\ i'\ } F' \xrightarrow{\ p'\ } M' \longrightarrow 0$$

be two short exact sequences where F *and* F' *are free. Then* $N \sim N'$ *if and only if* $M \sim M'$*.*

Proof: Let $M \sim M'$ thus there exist free modules E and E' and an isomorphism $\psi : M' \oplus E' \tilde{\rightarrow} M \oplus E$. Let $X = \varprojlim(\psi \circ (p' \oplus \mathrm{id}_{E'}), p \oplus \mathrm{id}_E)$ be the pull back of $\psi \circ (p' \oplus \mathrm{id}_{E'})$ and $p \oplus \mathrm{id}_E$ then we obtain a commutative diagram of exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N & \xrightarrow{\ \mathrm{id}\ } & N & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & N' & \longrightarrow & X & \longrightarrow & F \oplus E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle p \oplus \mathrm{id}} & & \\
0 & \longrightarrow & N' & \xrightarrow{\ i'\ } & F' \oplus E' & \xrightarrow{\ \psi\circ(p'\oplus\mathrm{id})\ } & M \oplus E & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $F \oplus E$ and $F' \oplus E'$ are free the top row and the left column split and it follows that

$$N' \oplus F \oplus E \simeq X \simeq N \oplus F' \oplus E'$$

and therefore $N \sim N'$.

Similarly, if $N \sim N'$ then there exist free modules E and E, and a homomorphism $\psi : N \oplus E \to N' \oplus E'$. Let $X = \varinjlim(i \oplus \mathrm{id}_E, (i' \oplus \mathrm{id}_{E'}) \circ \psi)$ be the push-out of $i \oplus \mathrm{id}_{E'}$ and $(i' \oplus \mathrm{id}_{E'}) \circ \psi$ so that we obtain a commutative diagram of exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N \oplus E & \xrightarrow{(i' \oplus \mathrm{id}_{E'}) \circ \psi} & F' \oplus E' & \xrightarrow{p'} & M' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i \oplus \mathrm{id}_E} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & F \oplus E & \longrightarrow & X & \longrightarrow & M' & \longrightarrow & 0. \\
 & & \downarrow{\scriptstyle p} & & \downarrow & & & & \\
 & & M & \xrightarrow{\mathrm{id}} & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since a projective module is injective relative to $\mathcal{F}(G)$ it follows that a short exact sequence with a free module on the left splits and we obtain

$$M \oplus F' \oplus E' \simeq X \simeq M' \oplus F \oplus E$$

and therefore $M \sim M'$. QED

Definition 2.3.3 *A complete free resolution* $F_* \xrightarrow{\partial_0} F^*$ *of a lattice* M *is an exact sequence*

$$\cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \longrightarrow \cdots,$$

of free modules such that $\partial_0 = \mu \circ \varepsilon$, *where* $\varepsilon : F_0 \longrightarrow M$ *is surjective and* $\mu : M \longrightarrow F_{-1}$ *is injective.*

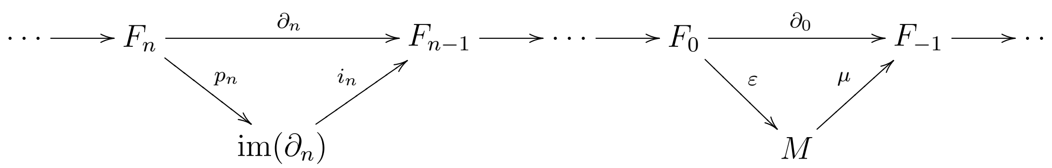
Lemma 2.3.4 *Every lattice admits a complete free resolution.*

Proof: Let M be a lattice. By lemma 2.2.2 there exists a free resolution $F_* \xrightarrow{\varepsilon} M$ which gives the left hand side of a complete free resolution. To obtain the right hand side let $F_{-1} = M \otimes \mathbb{Z}[G]$ then by corollary 2.1.2 it follows that F_{-1} is free. Let $\nu : M \to F_{-1}$ be the map $\mu(m) = m \otimes \Sigma_G$. Then $\mu(mg) = mg \otimes \Sigma_G = (m \otimes \Sigma_G)g$ and $\mu(m) = 0$ if and only if $m = 0$. Thus μ is an injective homomorphism and we can define $\partial_0 = \mu \circ \varepsilon$. Now let p_{-1} be the projection $p_{-1} : F_{-1} \to F_{-1}/\mathrm{im}(\mu)$. Then, similar to M, we obtain a free module $F_{-2} = F_{-1}/\mathrm{im}(\mu) \otimes \mathbb{Z}[G]$ and an inclusion $i_{-1} : F_{-1}/\mathrm{im}(\mu) \to F_{-2}$ and we can define $\partial_{-1} = i_{-1} \circ p_{-1}$. Iterating this construction then yields the remaining right hand side. QED

As for module extensions, we can regard a complete free resolution as being concatenated by short exact sequences

$$0 \longrightarrow \mathrm{im}(\partial_{n+1}) \xrightarrow{i_{n+1}} F_n \xrightarrow{p_n} \mathrm{im}(\partial_n) \longrightarrow 0,$$

called the n-th stage of the resolution, where $p_0 = \varepsilon$, $\mathrm{im}(\varepsilon) = M$ and $i_0 = \mu$. Notice, that we can regard any complete free resolution $F_* \longrightarrow F^*$ of a lattice M as a complete free resolution $F'_* \longrightarrow F'^*$ of $\mathrm{im}(\partial_n)$



simply by re-indexing $F'_k = F_{k+n}$, $\varepsilon' = p_n$ and $\mu' = i_n$ and $\partial'_k = \partial_{k+n}$. This method is called dimension shifting.

Let $F_* \longrightarrow F^*$ and $F'_* \longrightarrow F'^*$ be two complete free resolutions of M. It then follows from Schanuel's lemma that

$$\mathrm{im}(\partial_n) \sim \mathrm{im}(\partial'_n)$$

for all $n \in \mathbb{Z}$. Thus the stable class of $\mathrm{im}(\partial_n)$ is independent of the choice of resolution and only depends on the stable class of M.

Definition 2.3.5 *Let $F_* \xrightarrow{\partial_0} F^*$ be a complete free resolution of a lattice M.*

The n-th syzygy $\Omega_n(M)$ of M is defined as the stable class of $\mathrm{im}(\partial_n)$*, that is,*

$$\Omega_n(M) = \{R \mid \mathrm{im}(\partial_n) \sim R\}.$$

Proposition 2.3.6 *Let M and N be lattices then*

1) $\Omega_n(M) = \Omega_n(N)$ *if and only if* $M \sim N$,

2) $\Omega_n(\Omega_m(M)) = \Omega_{n+m}(M)$ *and*

3) $\Omega_n(M) \otimes \Omega_m(N) = \Omega_{n+m}(M \otimes N)$.

Proof: 1) follows inductively from the definition of Ω_n and Schanuel's lemma and 2) follows immediately by dimension shifting. To prove 3) Let $F_* \longrightarrow F^*$ be a complete free resolution of M. It then follows by corollary 2.1.2 that

$$\cdots \longrightarrow F_1 \otimes N \xrightarrow{\partial_1 \otimes \mathrm{id}} F_0 \otimes N \xrightarrow{\varepsilon \otimes \mathrm{id}} M \otimes N \xrightarrow{\mu \otimes \mathrm{id}} F_{-1} \otimes N \xrightarrow{\partial_{-1} \otimes \mathrm{id}} F_{-2} \otimes N \longrightarrow \cdots$$

is a complete free resolution of $M \otimes N$. Thus $\Omega_n(M \otimes N) = \Omega_n(M) \otimes N$. Similarly, by tensoring a complete free resolution of N with M we also obtain a free resolution of $M \otimes N$, and in this case we obtain $\Omega_m(M \otimes N) = M \otimes \Omega_m(N)$. Together with 2) it then follows that

$$\Omega_{n+m}(M \otimes N) = \Omega_n(M \otimes \Omega_m(N)) = \Omega_n(M) \otimes \Omega_m(N).$$

QED

Definition 2.3.7 *Let M and N be lattices. Two homomorphisms $f, h : M \to N$ are called equivalent, $f \approx h$, if $f - h$ factors through a free module, that is, if there exist a free module F and homomorphisms $\alpha : F \to N$ and $\beta : M \to F$ such that $f - h = \alpha \circ \beta$.*

Lemma 2.3.8 *The relation $\approx$ is additive and compatible with composition, that is,*

1) Let $f, f'h, h' \in \mathrm{Hom}(M, N)$ such that $f \approx f'$ and $h \approx h'$ then $f + h \approx f' + h'$, and

2) Let $f, f' \in \mathrm{Hom}(M, R)$ and $h, h' \in \mathrm{Hom}(R, N)$ such that $f \approx f'$ and $h \approx h'$ then $h \circ f \approx h' \circ f$.

Proof: 1) Let $f, f'h, h' \in \mathrm{Hom}(M,N)$ such that there exist free modules F and F' and homomorphisms $\alpha : F \to N$, $\beta : M \to F$, $\alpha' : F' \to N$ and $\beta' : M \to F'$ such that $f - f' = \alpha \circ \beta$ and $h - h' = \alpha' \circ \beta'$ then

$$(f+h)-(f'+h') = (f-f')+(h-h') = \alpha\circ\beta+\alpha'\circ\beta' = (\nabla_N\circ(\alpha\oplus\alpha'))\circ((\beta\oplus\beta')\circ\triangle_M)$$

where $\triangle_M : M \to M \oplus M$ is the diagonal map, $\triangle_M(m) = (m,m)$, and $\nabla_N : N \oplus N \to N$ is the sum, $\nabla_N(n,n') = n + n'$. In particular, $\nabla_N \circ (\alpha \oplus \alpha') : F \oplus F' \to N$ and $(\beta \oplus \beta') \circ \triangle_M : M \to F \oplus F'$. Thus $f + h \approx f' + h'$.

2) Let $f, f' \in \mathrm{Hom}(M,R)$ and $h, h' \in \mathrm{Hom}(R,N)$ such that there exist free modules F and F' and homomorphisms $\alpha : F \to R$, $\beta : M \to F$, $\alpha' : F' \to N$ and $\beta' : R \to F'$ such that $f-f' = \alpha\circ\beta$ and $h-h' = \alpha'\circ\beta'$. Then $(h-h')\circ f = \alpha'\circ\beta'\circ f$ and $f' = f - \alpha \circ \beta$ so that

$$\begin{aligned}(h \circ f) - (h' \circ f') &= (h \circ f) - (h' \circ (f - \alpha \circ \beta)) = (h - h') \circ f + h' \circ \alpha \circ \beta \\ &= \alpha' \circ \beta' \circ f + h' \circ \alpha \circ \beta.\end{aligned}$$

Since $\alpha' \circ \beta' \circ f$ and $h' \circ \alpha \circ \beta$ factor through a free module it follows by 1) that $\alpha'\circ\beta'\circ f+h'\circ\alpha\circ\beta$ factors through a free module and therefore $h\circ f \approx h'\circ f'$.QED

It now follows that the category $\mathcal{F}(G)$ of finitely generated lattices over a finite group G has the following quotient category.

Definition 2.3.9 *The derived module category $\mathcal{D}er = \mathcal{D}er(\mathcal{F}(G))$ of $\mathcal{F}(G)$ is the one whose objects are the same as for $\mathcal{F}(G)$, and for any two lattices M and N the group of morphisms is given by the quotient*

$$\mathrm{Hom}_{\mathcal{D}er}(M,N) = \mathrm{Hom}(M,N)/\approx .$$

Alternatively we can define $\mathcal{D}er(\mathcal{F}(G))$ as the quotient category of $\mathcal{F}(G)$ by factoring through the sub-category of free modules, that is, $\mathcal{D}er(\mathcal{F}(G))$ is the category of stable classes in $\mathcal{F}(G)$. A proof that these definitions are equivalent is given in [11].

Lemma 2.3.10 *Let N be a G-lattice and let $\mathbb{Z}$ be the trivial lattice then there exists an isomorphism*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z},N) &\simeq N^G/N\Sigma_G \\ f &\mapsto f(1).\end{aligned}$$

Proof: Let $\mathrm{Hom}_0(\mathbb{Z}, N) = \{f \in \mathrm{Hom}(\mathbb{Z}, N) \mid f \approx 0\}$ then for $f \in \mathrm{Hom}_0(\mathbb{Z}, N)$ there exist a free module $F \simeq \mathbb{Z}[G]^k$ and homomorphisms $\alpha \in \mathrm{Hom}(\mathbb{Z}[G]^k, N)$ and $\beta \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}[G]^k)$ such that $f = \alpha \circ \beta$. In terms of the standard basis $(e_1, \ldots, e_k)$ of $\mathbb{Z}[G]^k$ we can regard α as a k-tuple $\alpha = (n_1, \ldots, n_k)$, $n_i \in N$, where $\alpha(e_i) = n_i$ and β as the vector $\beta(1) = (a_1, \ldots, a_k)^t \Sigma_G$, $a_k \in \mathbb{Z}$. We obtain $f(1) = n\Sigma_G$, $n = \sum_{i=1}^k a_i n_i$ and therefore $\mathrm{Hom}_0(\mathbb{Z}, N) \simeq N\Sigma_G$.

Since $\mathrm{Hom}(\mathbb{Z}, N) \simeq N^G$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, N) = \mathrm{Hom}(\mathbb{Z}, N)/\mathrm{Hom}_0(\mathbb{Z}, N)$ it follows that $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, N) = N^G/N\Sigma_G$. QED

Lemma 2.3.11 *(adjointness formula)*

Let M, N and R be lattices then there exists an isomorphism

$$\mathrm{Hom}_{\mathcal{D}er}(R \otimes M, N) \simeq \mathrm{Hom}_{\mathcal{D}er}(R, M^* \otimes N).$$

Proof: By proposition 2.1.6 there exists an isomorphism $\mathrm{Hom}(R \otimes M, N) \simeq \mathrm{Hom}(R, M^* \otimes N)$ where $f \in \mathrm{Hom}(R \otimes M, N)$ maps to $\hat{f} \in \mathrm{Hom}(R, M^* \otimes N)$ and $\hat{f}(r)(m) = f(r \otimes m)$. Thus it remains to show that $f \approx 0$ if and only if $\hat{f} \approx 0$.

Assume $f \approx 0$, that is, there exist a free module F and homomorphisms $\alpha : F \to N$ and $\beta : R \otimes M \to F$ such that $f = \alpha \circ \beta$. Then

$$\hat{f}(r)(m) = f(r \otimes m) = (\alpha \circ \beta)(r \otimes m) = \alpha(\hat{\beta}(r)(m)) = ((\alpha_* \circ \hat{\beta})(r))(m)$$

where $\hat{\beta} : R \to M^* \otimes F$ is the image of β in $\mathrm{Hom}(R, M^* \otimes F)$ and $\alpha_* : M^* \otimes F \to M^* \otimes N$ maps a homomorphism $h : M \to F$ to $\alpha_*(h) = \alpha \circ h$. Thus since $M^* \otimes F$ is free by corollary 2.1.2 it follows that $\hat{f} \approx 0$.

Similarly, assume $\hat{f} \approx 0$, thus there exist a free module F and homomorphisms $\hat{\alpha} : F \to M^* \otimes N$ and $\beta : R \to F$ such that $\hat{f} = \hat{\alpha} \circ \beta$. Then

$$f(r \otimes m) = \hat{f}(r)(m) = ((\hat{\alpha} \circ \beta)(r))(m) = (\hat{\alpha}(\beta(r))(m) = (\alpha \circ (\beta \otimes \mathrm{id}))(r \otimes m)$$

where $\beta \otimes \mathrm{id} : R \otimes M \to F \otimes M$ and $\alpha : F \otimes M \to N$ is the pre-image of $\hat{\alpha}$ in $\mathrm{Hom}(F \otimes M, N)$. Since $F \otimes M$ is free it follows that $f \approx 0$. QED

The formula $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, N) \simeq N^G/N\Sigma_G$, lemma 2.3.10, and the adjointness formula lemma, 2.3.11, together with the results of the next section will enable us to express the cohomology groups $H^n(M, N)$ directly in terms of M and N only having the syzygies $\Omega_{-n}(\mathbb{Z})$ of the trivial module to keep track of dimension. In chapter 3 we will use this expression to calculate the cohomology of D_6-lattices and will see that this significantly minimises the involved computation.

2.4 Co-representability of the cohomology functor

In this section we first recall that for the category $\mathcal{F}(G)$ of finitely generated lattices the cohomology functor $M \mapsto H^n(M, \cdot)$, $n \geq 1$, is co-representable in the derived module category $\mathcal{D}er(\mathcal{F}(G))$ with co-representing object $\Omega_n(M)$, the n-th syzygy of M. That is, $H^n(M, N) \simeq \mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N)$. The co-representability of cohomology was first shown by J. Humphreys in his PhD-thesis [8]. In particular he showed that a necessary and sufficient condition for the above isomorphism to hold for a R-module M is $\mathrm{Ext}^n(M, R) = 0$. In his book [10], F. Johnson investigates this subject and shows that the syzygy functors Ω_n and Ω_{-n} are adjoint, that is, $\mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N) \simeq \mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-n}(N))$. This, together with the results of the previous section, will give the first result of this thesis, that is, for $n \geq 1$ we obtain

$$H^n(M, N) \simeq \mathcal{C}^n(M, N),$$

where $\mathcal{C}^n(M, N) = (\Omega_{-n}(\mathbb{Z}) \otimes M^* \otimes N)^G / (\Omega_{-n}(\mathbb{Z}) \otimes M^* \otimes N)\Sigma_G$. This will allow us to significantly reduce the calculations involved in determining cohomology as we will illustrate with the example of the dihedral group D_6 in chapter 3. This expression also simplifies operations on cohomology such as calculating induced homomorphisms. We will illustrate this on the example of the induced homomorphism of a subgroup $H \subset G$ and express lemma 2.2.8 in terms of the above quotient $\mathcal{C}^n(M, N)$. In chapter 3 we will see on the example of D_6 that this allows us to quickly determine the Bieberbach groups with holonomy group D_6.

Theorem 2.4.1 *(Co-representability of cohomology)*

For $n \geq 1$ the cohomology functor $M \mapsto H^n(M, \cdot\,)$ on $\mathcal{F}(G)$ is co-representable in the derived module category $\mathcal{D}er(\mathcal{F}(G))$ with co-representing object $\Omega_n(M)$. That is, let M and N be lattices and let $n \geq 1$. Then there exists an isomorphism

$$p_n^* : \mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N) \xrightarrow{\sim} H^n(M, N),$$

where p_n^* is induced by the projection $p_n : F_n \to \mathrm{im}(\partial_n)$ of some free resolution of M.

Proof: Let $F_* \xrightarrow{\varepsilon} M$ be a free resolution of M and consider the exact sequence

$$F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{p_n} \mathrm{im}(\partial_n) \longrightarrow 0\,.$$

Since $\operatorname{Hom}(\,\cdot\,, N)$ is a left exact and contravariant functor we obtain an exact sequence

$$0 \longrightarrow \operatorname{Hom}(\operatorname{im}(\partial_n), N) \xrightarrow{p_n^*} \operatorname{Hom}(F_n, N) \xrightarrow{\partial_{n+1}^*} \operatorname{Hom}(F_{n+1}, N).$$

Thus p_n^* induces an isomorphism $p_n^* : \operatorname{Hom}(\operatorname{im}(\partial_n), N) \to \operatorname{im}(p_n^*) = \ker(\partial_{n+1}^*)$. In particular, since $\partial_n^* = p_n^* \circ i_n^*$, where $i_n : \operatorname{im}(\partial_n) \to F_{n-1}$ is the inclusion, p_n^* induces an isomorphism

$$p_n^* : \operatorname{Hom}(\operatorname{im}(\partial_n), N)/\operatorname{im}(i_n^*) \xrightarrow{\sim} H^n(M, N).$$

Let $f \in \operatorname{Hom}(\operatorname{im}(\partial_n), N)$ such that there exist a free module F and homomorphisms $\alpha : F \to N$, $\beta : \operatorname{im}(\partial_n) \to F$ with $f = \alpha \circ \beta$, and let $X = \varinjlim(i_n, \beta)$. Thus there is a commutative diagram of exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im}(\partial_n) & \xrightarrow{i_n} & F_{n-1} & \xrightarrow{p_{n-1}} & \operatorname{im}(\partial_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle=} & & \\
0 & \longrightarrow & F & \xrightarrow{j} & X & \xrightarrow{p} & \operatorname{im}(\partial_{n-1}) & \longrightarrow & 0. \\
 & & \downarrow{\scriptstyle\alpha} & & & & & & \\
 & & N & & & & & &
\end{array}$$

Since for a finite group G any extension of a module by $\mathbb{Z}[G]$ splits it follows that the bottom sequence splits. Thus there exists a homomorphism $\mu : X \to F$ such that $\mu \circ j = \operatorname{id}_F$ and therefore $f = \alpha \circ \mu \circ \phi \circ i_n = i_n^*(\alpha \circ \mu \circ \phi) \in \operatorname{im}(i_n^*)$.

On the other hand, if $f \in \operatorname{im}(i_n^*)$ then there is a homomorphism $g : F_{n-1} \to N$ such that $f = g \circ i_n$. Thus $f \approx 0$ and we obtain

$$\operatorname{im}(i_n^*) = \{f \in \operatorname{Hom}(\operatorname{im}(\partial_n), N) \mid f \approx 0\}.$$

It follows that p_n^* is the isomorphism

$$p_n^* : \operatorname{Hom}_{\mathcal{D}er}(\Omega_n(M), N) \xrightarrow{\sim} H^n(M, N).$$

QED

Let

$$\operatorname{Hom}_{\mathcal{D}er}(\Omega_*(M), N) = \sum_{k \geq 0} \operatorname{Hom}_{\mathcal{D}er}(\Omega_k(M), N)$$

then there exists a surjective level preserving homomorphism

$$P : H^*(M, N) \longrightarrow \operatorname{Hom}_{\mathcal{D}er}(\Omega_*(M), N),$$

where for $n \geq 1$ $P|_{H^n(M,N)} = (p_n^*)^{-1}$ is the isomorphism in theorem 2.4.1, and $P|_{H^0(M,N)} : \operatorname{Hom}(M, N) \to \operatorname{Hom}_{\mathcal{D}er}(M, N)$ is the projection.

Theorem 2.4.2 *(adjointness formula)*

Let M and N be lattices and let $k \geq 0$. Then there exists an isomorphism

$$\psi_k : \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-k}(N)).$$

In particular, for any homomorphism f the family $\{\psi_k(f)\}_{k\in\mathbb{Z}}$ is a chain transformation between a complete resolution of $\Omega_k(M)$ and a complete resolution of N, and $\{\psi_k(f)\}_{k\in\mathbb{Z}}$ is unique in the derived module category.

Proof: Since $\Omega_k(\Omega_l(M)) = \Omega_{k+l}(M)$ by proposition 2.3.6 it is sufficient to prove the theorem for $k = 1$. By theorem 28.5 pp.118 in [10] it follows that any projective module in $\mathcal{F}(G)$ is injective relative to $\mathcal{F}(G)$. Let

$$\mathcal{E}_1(M): \quad 0 \longrightarrow \mathrm{im}(\partial_1^M) \xrightarrow{i_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be the 1st stage of a complete free resolution of M and let

$$\mathcal{E}_{-1}(N): \quad 0 \longrightarrow N \xrightarrow{\nu} E_{-1} \xrightarrow{q_{-1}} \mathrm{im}(\partial_{-1}^N) \longrightarrow 0$$

be the -1st stage of a complete free resolution of N. Let $f : \mathrm{im}(\partial_1^M) \to N$ be a representative of a morphism in $\mathrm{Hom}_{\mathcal{D}er}(\Omega_1(M), N)$. Since E_{-1} is free we can extend $\nu \circ f$ to a homomorphism $\hat{f} : F_0 \to E_{-1}$ such that $\hat{f} \circ i_1 = \nu \circ f$, and since $q_{-1} \circ \hat{f} \circ i_1 = q_{-1} \circ \nu \circ f = 0$, $\hat{f}$ maps $\mathrm{im}(i_1)$ to $\ker(q_{-1}) = \mathrm{im}(\nu)$ and therefore induces a homomorphism $\psi_1(f) : M \to \mathrm{im}(\partial_{-1}^N)$. Thus we obtain a commutative diagram of exact sequences

$$\begin{array}{lccccccccc}
\mathcal{E}_1(M): & 0 & \longrightarrow & \mathrm{im}(\partial_1^M) & \xrightarrow{i_1} & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \psi_1(\hat{f})} & & \\
\mathcal{E}_{-1}(N): & 0 & \longrightarrow & N & \xrightarrow{\nu} & E_{-1} & \xrightarrow{q_{-1}} & \mathrm{im}(\partial_{-1}^N) & \longrightarrow & 0.
\end{array}$$

Now assume $f \approx 0$, that is, there exist a free module F and homomorphisms $\alpha : F \to N$ and $\beta : \mathrm{im}(\partial_1^M) \to F$ such that $f = \alpha \circ \beta$. Consider the pushout $\beta_*\mathcal{E}_1(M)$ and construct the $\hat{\alpha}$ and $\psi_1(\alpha)$ such that we obtain a commutative diagram of exact rows

$$\begin{array}{lccccccccc}
\mathcal{E}_1(M): & 0 & \longrightarrow & \mathrm{im}(\partial_1^M) & \xrightarrow{i_1} & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle s_\beta} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\
\beta_*\mathcal{E}_1(M): & 0 & \longrightarrow & F & \xrightarrow{r_\beta} & \varinjlim(i_1, \beta) & \xrightarrow{t_\beta} & M & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \hat{\alpha}} & & \downarrow{\scriptstyle \psi_1(\hat{\alpha})} & & \\
\mathcal{E}_{-1}(N): & 0 & \longrightarrow & N & \xrightarrow{\nu} & E_{-1} & \xrightarrow{q_{-1}} & \mathrm{im}(\partial_{-1}^N) & \longrightarrow & 0.
\end{array}$$

Thus, since $(\hat{\alpha} \circ s_\beta) \circ i_1 = \hat{\alpha} \circ r_\beta \circ \beta = \nu \circ (\alpha \circ \beta) = \nu \circ f$, we can choose $\hat{f} = \hat{\alpha} \circ s_\beta$ and $\psi_1(f) = \psi_1(\alpha)$. Furthermore, since F is free the middle exact sequence splits thus there exists an homomorphism $\tau : M \to \varinjlim(i_1, \beta)$ such that $\tau \circ t_\beta = \mathrm{id}_M$. It follows that $\psi_1(f) = q_{-1} \circ (\hat{\alpha} \circ \tau) \approx 0$ and therefore

$$\begin{aligned} \psi_1 : \mathrm{Hom}_{\mathcal{D}er}(\Omega_1(M), N) &\to \mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-1}(N)) \\ [f] &\mapsto [\psi_1(f)] \end{aligned}$$

is a well-defined homomorphism.

Let $f : M \to \mathrm{im}(\partial^N_{-1})$ be a representative of a morphism in $\mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-1}(N))$. Since F_0 is free the composition $f \circ \varepsilon$ lifts to a homomorphism $\tilde{f}$. Furthermore, since $q_{-1} \circ \tilde{f} \circ i_1 = f \circ \varepsilon \circ i_1 = 0$ it follows that $\tilde{f}$ restricts to a homomorphism $\psi_{-1}(f) = \tilde{f}|_{\mathrm{im}(\partial^M_1)} : \mathrm{im}(\partial^M_1) \to N$ and we obtain a commutative diagram

$$\begin{array}{llcccccccc}
\mathcal{E}_1(M): & 0 & \longrightarrow & \mathrm{im}(\partial^M_1) & \xrightarrow{i_1} & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \psi_{-1}(f)} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} & & \\
\mathcal{E}_{-1}(N): & 0 & \longrightarrow & N & \xrightarrow{\nu} & E_{-1} & \xrightarrow{q_{-1}} & \mathrm{im}(\partial^N_{-1}) & \longrightarrow & 0.
\end{array}$$

Now assume $f \approx 0$, that is, there exist a free module F and homomorphisms $\alpha : F \to \mathrm{im}(\partial^N_{-1})$ and $\beta : M \to F$ such that $f = \alpha \circ \beta$. Consider the pullback $\mathcal{E}_{-1}(N)\alpha^*$ and construct $\tilde{\beta}$ and $\psi_{-1}(\beta)$. Then we obtain a commutative diagram with exact rows

$$\begin{array}{llcccccccc}
\mathcal{E}_1(M): & 0 & \longrightarrow & \mathrm{im}(\partial^M_1) & \xrightarrow{i_1} & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & & & {\scriptstyle \psi_{-1}(\beta)}\downarrow & & \downarrow{\scriptstyle \tilde{\beta}} & & \downarrow{\scriptstyle \beta} & & \\
\mathcal{E}_{-1}(N)\alpha^*: & 0 & \longrightarrow & N & \xrightarrow{u_\alpha} & \varprojlim(\alpha, q_{-1}) & \xrightarrow{w_\alpha} & F & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle v_\alpha} & & \downarrow{\scriptstyle \alpha} & & \\
\mathcal{E}_{-1}(N): & 0 & \longrightarrow & N & \xrightarrow{\nu} & E_{-1} & \xrightarrow{q_{-1}} & \mathrm{im}(\partial^N_{-1}) & \longrightarrow & 0.
\end{array}$$

Since $q_{-1} \circ (v_\alpha \circ \tilde{\beta}) = (\alpha \circ \beta) \circ \varepsilon = f \circ \varepsilon$ we can choose $\tilde{f} = v_\alpha \circ \tilde{\beta}$ and $\psi_{-1}(f) = \psi_{-1}(\beta)$, and since F is free the middle exact sequence splits and there exists a homomorphism $\kappa : \varprojlim(\alpha, q_{-1}) \to N$ such that $\kappa \circ u_\alpha = \mathrm{id}_N$. It follows that $\psi_{-1}(f) = \kappa \circ (\tilde{\beta} \circ i_1) \approx 0$ and therefore

$$\begin{aligned} \psi_{-1} : \mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-1}(N)) &\longrightarrow \mathrm{Hom}_{\mathcal{D}er}(\Omega_1(M), N) \\ [f] &\mapsto \psi_{-1}([f]) = [\psi_{-1}(\tilde{f})] \end{aligned}$$

is a well defined homomorphism. Furthermore $\psi_{-1} \circ \psi_1 = \mathrm{id}_{\mathrm{Hom}_{\mathcal{D}er}(\Omega_1(M), N)}$ and $\psi_1 \circ \psi_{-1} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-1}(N))}$ by definition. QED

Corollary 2.4.3 *In the category of finitely generated lattices over the integral group ring of a finite group the cohomology functor* $N \mapsto H^n(\ \cdot, N),\ n \geq 1$ *is representable on the derived module category with representing object* $\Omega_{-n}(N)$. *That is there exists an isomorphism*

$$H^n(M, N) \simeq \mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-n}(N)).$$

Proof: The proof follows immediately from the co-representability theorem 2.4.1 and the adjointness formula theorem 2.4.2. QED

The representability of cohomology together with the results of the previous section now enable us to express the cohomology groups $H^n(M, N)$ directly in terms of M and N only having $\Omega_{-n}(\mathbb{Z})$, the syzygies of the trivial lattice, to keep track of the dimension. To see this, consider the following

Definition 2.4.4 *Let* M *and* N *be lattices over a finite group* G *and define*

$$\mathcal{C}^n(M, N) = (\mathrm{im}(\partial_{-n}) \otimes M^* \otimes N)^G / (\mathrm{im}(\partial_{-n}) \otimes M^* \otimes N)\Sigma_G.$$

Lemma 2.4.5 *The quotient* $\mathcal{C}^n(M, N)$ *only depends on the stable classes of* M, N *and* $\mathrm{im}(\partial_{-n})$.

Proof: It suffices to show that $(M \otimes N)^G/(M \otimes N)\Sigma_G \simeq (M' \otimes N)^G/(M' \otimes N)\Sigma_G$ if $M' \sim M$. Let F be free module then $F^G \simeq F\Sigma_G$ so that $F^G/F\Sigma_G \simeq 0$. Furthermore $(M \oplus N)^G/(M \oplus N)\Sigma_G = (M^G/M\Sigma_G) \oplus (N^G/N\Sigma_G)$. Thus if there exists free modules E and E' such that $M \oplus E \simeq M' \oplus E'$ it follows that

$$\begin{aligned}
((M \otimes N)^G/(M \otimes N)\Sigma_G) &\simeq ((M \otimes N)^G/(M \otimes N)\Sigma_G) \oplus ((E \otimes N))^G/(E \otimes N)\Sigma_G) \\
&\simeq ((M \oplus E) \otimes N)^G/((M \oplus E) \otimes N)\Sigma_G \\
&\simeq ((M' \oplus E') \otimes N)^G/((M' \oplus E') \otimes N)\Sigma_G \\
&\simeq ((M' \otimes N)^G/(M' \otimes N)\Sigma_G)
\end{aligned}$$

QED

Theorem 2.4.6 *Let M and N be lattices. Then for $n \geq 1$ there exists a group isomorphism*

$$\Psi_n : H^n(M,N) \;\simeq\; \mathcal{C}^n(M,N),$$

where $\Psi_n(f)(m) = \psi_n(f')(m)$, $f = p_n^*(f')$, p_n^* is the isomorphism in theorem 2.4.1 and ψ_n is the isomorphism in theorem 2.4.2.

Proof: From the representability of cohomology, corollary 2.4.3, and proposition 2.3.6 it follows that $H^n(M,N) \simeq \mathrm{Hom}_{\mathcal{D}er}(M, \Omega_{-n}(\mathbb{Z}) \otimes N)$, and from the adjointness formula lemma 2.3.11 and lemma 2.3.10 it follows that $H^n(M,N) \simeq \mathcal{C}^n(M,N)$. QED

To see how Ψ_n maps a cohomology class $[f] \in H^n(M,N)$ to an element in $\mathcal{C}^n(M,N)$ let $f = f' \circ p_n$ where $[f'] \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N)$ and let $F_* \xrightarrow{\partial_0} F^*$ be a complete free resolution of the trivial lattice $\mathbb{Z}$. Then for any lattice M a complete free resolutions is given by $F_* \otimes M \xrightarrow{\partial_0 \otimes \mathrm{id}_M} F^* \otimes M$. Thus we can choose f' to be a homomorphism $f' : \mathrm{im}(\partial_n) \otimes M \to \mathbb{Z} \otimes N$ and calculate $\psi_n(f')$ as the following chain transformation

$$\begin{array}{ccccccccc}
\mathrm{im}(\partial_n) \otimes M & \xrightarrow{i_n \otimes \mathrm{id}_M} & F_{n-1} \otimes M & \longrightarrow & \cdots & \longrightarrow & F_0 \otimes M & \xrightarrow{\varepsilon \otimes \mathrm{id}_M} & \mathbb{Z} \otimes M \\
\downarrow{\scriptstyle f'} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \psi_n(f')} \\
\mathbb{Z} \otimes N & \xrightarrow{\mu \otimes \mathrm{id}_N} & F_{-1} \otimes N & \longrightarrow & \cdots & \longrightarrow & F_{-n} \otimes N & \xrightarrow{p_{-n} \otimes \mathrm{id}_N} & \mathrm{im}(\partial_{-n}) \otimes N.
\end{array}$$

It follows that

$$\Psi_n(f) = \hat{\psi}_n(f')(1)$$

where $\hat{\psi}_n(f')(1)(m) = \psi_n(f')(1 \otimes m)$.

Next we will analyse how the induced homomorphisms on cohomology relate to the homomorphisms on the quotients $\mathcal{C}^n(M,N)$. Let $\varphi : M' \to M$ be a homomorphism. Then φ induces a homomorphism

$$\begin{aligned} \varphi^* : \mathcal{C}(M,N) &\;\to\; \mathcal{C}(M',N) \\ [d \otimes m^* \otimes n] &\;\mapsto\; [d \otimes (m^* \circ \varphi) \otimes n]. \end{aligned}$$

Similarly, let $\xi : N \to N'$ be a homomorphism. Then ξ induces a homomorphism

$$\begin{aligned} \xi_* : \mathcal{C}^n(M,N) &\;\to\; \mathcal{C}^n(M',N) \\ [d \otimes m^* \otimes n] &\;\mapsto\; [d \otimes m^* \otimes \xi(n)]. \end{aligned}$$

Proposition 2.4.7 *The isomorphism* $\Psi_n : H^n(M,N) \simeq \mathcal{C}^n(M,N)$ *commutes with induced homomorphisms. That is, if* $\varphi : M' \to M$ *and* $\xi : N \to N'$ *are homomorphisms then* $\varphi^* \circ \Psi_n = \Psi_n \circ \varphi^*$ *and* $\xi_* \circ \Psi_n = \Psi_n \circ \xi_*$.

Proof: Let $\varphi : M' \to M$ be a homomorphism and let $F_* \xrightarrow{\partial_0} F^*$ be a complete free resolution of the trivial lattice $\mathbb{Z}$. Then a complete free resolution of M is given by $F_* \otimes M \xrightarrow{\partial_0 \otimes \mathrm{id}_M} F^* \otimes M$ and a complete free resolution of M' is given by $F_* \otimes M' \xrightarrow{\partial_0 \otimes \mathrm{id}_{M'}} F^* \otimes M'$. Then for the chain transformation induced by φ we obtain $\varphi_* = (\mathrm{id}_{F_k} \otimes \varphi)_{k \in \mathbb{Z}}$. Let $f' : \mathrm{im}(\partial_n) \otimes M \to \mathbb{Z} \otimes N$ be a representative of a morphism $f' \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N)$. Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{im}(\partial_n) \otimes M' & \xrightarrow{i_n \otimes \mathrm{id}_{M'}} & F_{n-1} \otimes M' & \longrightarrow & \cdots & \longrightarrow & F_0 \otimes M' & \xrightarrow{\varepsilon \otimes \mathrm{id}_{M'}} & \mathbb{Z} \otimes M' \\
\downarrow{\scriptstyle \mathrm{id}\otimes\varphi} & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi} & & & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi} & & \downarrow{\scriptstyle \varphi} \\
\mathrm{im}(\partial_n) \otimes M & \xrightarrow{i_n \otimes \mathrm{id}_M} & F_{n-1} \otimes M & \longrightarrow & \cdots & \longrightarrow & F_0 \otimes M & \xrightarrow{\varepsilon \otimes \mathrm{id}_M} & \mathbb{Z} \otimes M \\
\downarrow{\scriptstyle f} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \psi_n(f)} \\
\mathbb{Z} \otimes N & \xrightarrow{\mu \otimes \mathrm{id}_N} & F_{-1} \otimes N & \longrightarrow & \cdots & \longrightarrow & F_{-n} \otimes N & \xrightarrow{p_{-n} \otimes \mathrm{id}_N} & \mathrm{im}(\partial_{-n}) \otimes N
\end{array}$$

and it follows that $\varphi^*(\psi_n(f))) = \psi_n(\varphi^*(f))$. In particular, for $f = f' \circ p_n$

$$\begin{aligned}
(\Psi_n \circ \varphi^*)(f)(m) &= (\psi_n \circ \varphi^*)(f')(1 \otimes m) = (\varphi^* \circ \psi_n)(f')(1 \otimes m) \\
&= (\varphi_* \circ \Psi_n)(f)(m).
\end{aligned}$$

Similarly, for $\xi : N \to N'$ we obtain $\xi_*(\psi_n(f))) = \psi_n(\xi_*(f)))$ and therefore $\xi_* \circ \hat{\Psi}_n = \hat{\Psi}_n \circ \xi_*$ QED

A special case of an induced homomorphism is given by a subgroup $H \subset G$. Let $i : H \hookrightarrow G$ be the inclusion map then i induces a homomorphism $i^* : H^n(G,N) \to H^n(H, i^*(N))$. In section 2.2 we saw that the Bieberbach groups with holonomy D_6, that is torsion free extensions of a free abelian group N by G, are determined by those elements $c \in H^2(G,N)$ for which $0 \neq i^*(c) \in H^2(C_p, i^*(N))$ for all cyclic subgroups $C_p \subset G$ of prime order. In terms of the quotient $\mathcal{C}$ this corresponds to

Theorem 2.4.8 *The Bieberbach groups with holonomy group* G *are determined by those elements* $c \in (\Omega_{-2}(\mathbb{Z}) \otimes N)^G / (\Omega_{-2}(\mathbb{Z}) \otimes N)\Sigma_G$ *which are not in* $(\Omega_{-2}(\mathbb{Z}) \otimes N)\Sigma_{C_p}$ *for all cyclic subgroups* $C_p \subset G$.

Proof: We will show that for an arbitrary subgroup $H \subset G$, with inclusion map $i : H \hookrightarrow G$, the induced homomorphism $i^* : H^n(G, N) \to H^n(H, i^*(N))$ corresponds to the projection

$$i^* : (\Omega_{-n}(\mathbb{Z}) \otimes N)^G / (\Omega_{-n}(\mathbb{Z}) \otimes N)\Sigma_G \longrightarrow (\Omega_{-n}(\mathbb{Z}) \otimes N)^H / (\Omega_{-n}(\mathbb{Z}) \otimes N)\Sigma_H.$$

Then the claim follows immediately by considering cyclic subgroups of G and $n = 2$. First Notice that for every G-lattice M we have $M^G \subset M^H$ and $M\Sigma_G \subset M\Sigma_H$ since $\Sigma_G = \sum_i x_i \Sigma_H$ where the x_i's are representatives of the cosets in G/H. Thus the above projection is well-defined.

Let $\mathcal{D}(G)$ denote the derived module category of $\mathcal{F}(G)$, let $\mathcal{D}(H)$ denote the derived module category of $\mathcal{F}(H)$. The induced map $i^* : \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, N) \to \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, N)$ is defined in the obvious way by restricting scalars. Then, since $i^*(\mathbb{Z}[G]) \simeq \mathbb{Z}[H]^{[G:H]}$, it follows that any homomorphism which factors through a free module over $\mathbb{Z}[G]$ also factors through a free module over $\mathbb{Z}[H]$ so that

$$i^* : \mathrm{Hom}_{\mathcal{D}(G)}(\mathbb{Z}, N) \to \mathrm{Hom}_{\mathcal{D}(H)}(\mathbb{Z}, N)$$

is well-defined. Thus, by lemma 2.3.10, we obtain a homomorphism

$$\begin{aligned} i^* : N^G / N\Sigma_G &\longrightarrow N^H / N\Sigma_H \\ [v]_G &\mapsto [v]_H. \end{aligned}$$

In particular $i^*([v]_G) = 0$ if and only if $v \in N\Sigma_H$. Now the induced map on cohomology $i^* : H^n(G, N) \to H^n(H, i^*(N))$ is given by restricting scalars and therefore corresponds to the projection

$$i^* : (\Omega_{-n}(\mathbb{Z}) \otimes N)^G / (\Omega_{-n}(\mathbb{Z}) \otimes N)\Sigma_G \longrightarrow (\Omega_{-n}(\mathbb{Z}) \otimes N)^H / (\Omega_{-n}(\mathbb{Z}) \otimes N)\Sigma_H.$$

QED

In section 3.3 we will use this result to determine the Bieberbach groups with holonomy group D_6.

2.5 Composition in the derived module category and the cup product

In section 2.2 we saw that there exists a product structure on cohomology, namely the cup product, which is induced by the Yoneda product on module extension and extends the cup product in singular cohomology. In this section we show that in terms of the derived module category the cup product arises naturally as the composition of morphisms in $\mathrm{Hom}_{\mathcal{D}er}$. Furthermore we give a sufficient condition on M for $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ to be a graded-commutative ring.

The adjointness formula, theorem 2.4.2, allows us to regard the composition of morphisms in $\mathcal{D}er(\mathcal{F}(G))$ as the following pairing.

Definition 2.5.1 *Let M, N and R be lattices and let $\bullet$ be the following homomorphism*

$$\begin{aligned} \bullet : \mathrm{Hom}_{\mathcal{D}}(\Omega_k(M), R) \otimes \mathrm{Hom}_{\mathcal{D}}(\Omega_l(R), N) &\to \mathrm{Hom}_{\mathcal{D}}(\Omega_{k+l}(M), N) \\ f \otimes h &\mapsto h \bullet f = h \circ \psi_{-l}(f). \end{aligned}$$

Proposition 2.5.2 *Let M be a lattice. Then*

$$\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M) = \sum_{k \geq 0} \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), M)$$

equipped with the $\bullet$-composition is a graded associative ring with unit id_M. If M' is another lattices such that $M' \in \Omega_{n_0}(M)$ for some $n_0 \in \mathbb{Z}$. Then there exists a ring isomorphism

$$\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M) \simeq \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M'), M').$$

Proof: For any morphism $f \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_l(M), M)$ the family $\psi_k(f)_{k \in \mathbb{Z}}$ is a chain transformation between complete free resolutions of $\Omega_l(M)$ and M it follows that $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is a graded associative ring. Furthermore, it follows immediately from definition 2.5.1 that $f \bullet \mathrm{id}_M = f \circ \psi_{-k}(\mathrm{id}) = \psi_{-k}(\psi_k(f) \circ \mathrm{id}) = f$ and $\mathrm{id}_M \bullet f = f$.

Now let $M' \in \Omega_{n_0}(M)$ for some $n_0 \in \mathbb{Z}$. It then follows from lemma 2.3.6 that $\Omega_k(M') = \Omega_{k+n_0}(M)$ and from theorem 2.4.2 that

$$\mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M'), M') = \mathrm{Hom}_{\mathcal{D}er}(\Omega_{k+n_0}(M), \Omega_{n_0}(M)) \simeq \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), M)$$

for all $k \in \mathbb{N}$. QED

Theorem 2.5.3 *The cup product in cohomology corresponds to the* $\bullet$*-composition in the derived module category. That is, for all* $k, l \geq 1$, $f \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), R)$ *and* $h \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_l(R), N)$ *we have*

$$p_k^*(f) \cup p_l^*(h) = p_{k+l}^*(h \bullet f).$$

Proof: By theorem 2.2.17 and definition 2.2.18 of the cup product it is sufficient to show that for $k, l \geq 1$ we have $\mathrm{Ext}^{k+l}(h \bullet f) = \mathrm{Ext}^l(h) \circ \mathrm{Ext}^k(f)$, where

$$\mathrm{Ext}^n = \mathcal{Y}^{-1} \circ p_n^* : \mathrm{Hom}_{\mathcal{D}er}(\Omega_n(M), N) \xrightarrow{\sim} \mathrm{Ext}^n(M, N)$$

is the composition of the isomorphisms p_n^*, theorem 2.4.1, and $\mathcal{Y}^{-1}$, theorem 2.2.17. That is, $\mathrm{Ext}^n(f) = f_*\mathcal{E}_n(M)$ where $\mathcal{E}_n(M)$ is given by a truncated free resolution

$$\mathcal{E}_n(M): \quad 0 \longrightarrow \mathrm{im}(\partial_n^M) \xrightarrow{i_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Let

$$0 \longrightarrow \mathrm{im}(\partial_{k+l}^M) \xrightarrow{i_{k+l}} F_{k+l-1} \xrightarrow{p_{k+l-1}} \mathrm{im}(\partial_{k+l-1}^M) \longrightarrow 0$$

be the $(k+l)$-th stage of a free resolution $F_* \xrightarrow{\varepsilon} M$ of M, and let

$$0 \longrightarrow \mathrm{im}(\partial_l^R) \xrightarrow{j_l} E_{l-1} \xrightarrow{q_{l-1}} \mathrm{im}(\partial_{l-1}^R) \longrightarrow 0$$

be the l-th stage of a free resolution $E_* \xrightarrow{\epsilon} R$ of R. Let $f : \mathrm{im}(\partial_k^M) \to R$ and $h : \mathrm{im}(\partial_l^R) \to N$ be representatives of morphisms in $\mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), R)$ and $\mathrm{Hom}_{\mathcal{D}er}(\Omega_l(R), N)$ respectively. Let ψ_n be the isomorphism in the adjointness formula, theorem 2.4.2, and consider $\psi_{-l}(f) : \mathrm{im}(\partial_{k+l}^M) \to \mathrm{im}(\partial_l^R)$ and the push-outs $\varinjlim(h, j_l)$ and $\varinjlim(h \bullet f, i_{k+l})$. Thus we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{im}(\partial_{k+l}^M) & \xrightarrow{i_{k+l}} & F_{k+l-1} \\ \downarrow{\scriptstyle \psi_{-l}(f)} & & \downarrow{\scriptstyle \widetilde{\psi_{-l+1}(f)}} \\ \mathrm{im}(\partial_l^R) & \xrightarrow{j_l} & E_{l-1} \end{array}$$

and

$$\begin{array}{ccc} \mathrm{im}(\partial_l^R) & \xrightarrow{j_l} & E_{l-1} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle s_h} \\ N & \xrightarrow{r_h} & \varinjlim(h, j_l). \end{array}$$

It follows that $s_h \circ \widetilde{\psi_{-l+1}(f)} \circ i_{k+l} = r_h \circ (h \circ \psi_{-l}(f)) = r_h \circ (h \bullet f)$. Thus by the universal property of the push-out there exists a homomorphism

$$\alpha : \varinjlim(h \bullet f, i_{k+l}) \to \varinjlim(h, j_l)$$

such that the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{im}(\partial^M_{k+l}) & \xrightarrow{i_{k+l}} & F_{k+l-1} & & \\
\downarrow{\scriptstyle h\bullet f} & & \downarrow{\scriptstyle s_{h\bullet f}} & \searrow{\scriptstyle s_h\circ\widetilde{\psi_{-l+1}(f)}} & \\
N & \xrightarrow{r_{h\bullet f}} & \varinjlim(h \bullet f, i_{k+l}) & & \\
& \searrow{\scriptstyle r_h} & & \searrow{\scriptstyle \alpha} & \\
& & & & \varinjlim(h, j_l).
\end{array}$$

In particular, α maps an element $[n,p]_{h\bullet f} \in \varinjlim(h \bullet f, i_{k+l})$ to $\alpha([n,p]_{h\bullet f}) = r_h(n) + (s_h \circ \widetilde{\psi_{-l+1}(f)})(p) = [n, \widetilde{\psi_{-l+1}(f)}(p)]_h \in \varinjlim(h, j_l)$. Now consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{im}(\partial^M_{k+l}) & \xrightarrow{i_{k+l}} & F_{k+l-1} & \xrightarrow{p_{k+l-1}} & \mathrm{im}(\partial^M_{k+l-1}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle h\bullet f} & & \downarrow{\scriptstyle s_{h\bullet f}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & N & \xrightarrow{r_{h\bullet f}} & \varinjlim(h \bullet f, i_{k+l}) & \xrightarrow{t_{h\bullet f}} & \mathrm{im}(\partial^M_{k+l-1}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \psi_{-l+1}(f)} & & \\
0 & \longrightarrow & N & \xrightarrow{r_h} & \varinjlim(h, j_l) & \xrightarrow{t_h} & \mathrm{im}(\partial^R_{l-1}) & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle s_h} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathrm{im}(\partial^R_l) & \xrightarrow{j_l} & E_{l-1} & \xrightarrow{q_{l-1}} & \mathrm{im}(\partial^R_{l-1}) & \longrightarrow & 0.
\end{array}$$

Then the diagram consisting of the top two rows and the diagram consisting of the bottom two rows commute by definition of the push-out. The diagram consisting of the middle two rows commutes since $\alpha \circ r_{h\bullet f} = r_h$ and

$$\begin{aligned}
(t_h \circ \alpha)[n,p]_{h\bullet f} &= t_h[n, \widetilde{\psi_{-l+1}(f)}(p)]_h \\
&= (q_{l-1} \circ \widetilde{\psi_{-l+1}(f)})(p) \\
&= (\psi_{-l+1}(f) \circ p_{k+l-1})(p) \\
&= (\psi_{-l+1}(f) \circ t_{h\bullet f})[n,p]_{h\bullet f}.
\end{aligned}$$

Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{r_{h\bullet f}} & \varinjlim(h\bullet f, i_{k+l}) & \xrightarrow{i_{k+l-2}\circ t_{h\bullet f}} & F_{k+l-2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \widetilde{\psi_{-l+1}(f)}} & & \\
0 & \longrightarrow & N & \xrightarrow{r_h} & \varinjlim(h, j_l) & \xrightarrow{j_{l-1}\circ t_h} & E_{l-2} & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_k & \xrightarrow{\partial_k} & F_{k-1} & \longrightarrow \cdots \longrightarrow & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle s_f} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & E_0 & \xrightarrow{r_f\circ\epsilon} & \varinjlim(f, i_k) & \xrightarrow{i_{k-1}\circ t_f} \cdots \longrightarrow & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0,
\end{array}$$

where the top sequence is a representative of $\mathrm{Ext}^{k+l}(h \bullet f)$ and the bottom sequence is a representative of $\mathrm{Ext}^l(h) \circ \mathrm{Ext}^k(f)$. Therefore there exists a morphism $(\mathrm{id}_N, \mathrm{id}_M)$ such that

$$(\mathrm{id}_N, \mathrm{id}_M)(\mathrm{Ext}(h \bullet f)) = \mathrm{Ext}(h) \circ \mathrm{Ext}(f)$$

and by lemma 2.2.12 it follows that $\mathrm{Ext}^{k+l}(h \bullet f) = \mathrm{Ext}^l(h) \circ \mathrm{Ext}^k(f)$. QED

Now let

$$\begin{array}{rcl} \circ : R^* \otimes R & \to & \mathbb{Z} \\ r^* \otimes r & \mapsto & r^*(r) \end{array}$$

be the evaluation map on R. Let M and N be two further lattices. Choose $\mathbb{Z}$-basis $(a_1, \ldots, a_m), (b_1, \ldots, b_n)$ and $(e_1, \ldots, e_r)$ for M, N and R respectively, and choose the corresponding dual basis $(a^1, \ldots, a^m), (e^1, \ldots, e^r)$ as $\mathbb{Z}$-basis for M^* and R^*. Let ρ_R, ρ_M and ρ_N be the corresponding group representations. Recall that we can regard an element in $M^* \otimes R$ as an $(r \times m)$-matrix A on which G-acts by $Ag = \rho_R(g^{-1})A\rho_M(g)$. Then the evaluation map on R induces a pairing

$$\begin{array}{rcl} \circ : (M^* \otimes R) \otimes (R^* \otimes N) & \to & (M^* \otimes N) \\ A \otimes B & \mapsto & B \circ A \end{array}$$

which is given by matrix multiplication, where $A = \sum_{ij} \alpha^i_j (a^j \otimes e_i), B = \sum_{kl} \beta^k_l (e^l \otimes b_k)$ and $B \circ A = \sum_{ijkl} \alpha^i_j \beta^k_l (a^j \otimes e^l(e_i) \otimes b_k) = \sum_{ijk} \alpha^i_j \beta^k_i (a^j \otimes b_k)$. Furthermore if $A \in (M^* \otimes R)^G$ and $B \in (R^* \otimes N)^G$, that is, $\rho_R(g^{-1})A\rho_M(g) = A$ and $\rho_N(g^{-1})B\rho_R(g) = B$, then

$$\rho_N(g^{-1})BA\rho_M(g) = \rho_N(g^{-1})B\rho_R(g)\rho_R(g^{-1})A\rho_M(g) = BA.$$

Thus $BA \in (M^* \otimes N)^G$. Also, if $A \in (M^* \otimes R)\Sigma_G$, that is, there exists an element $A' \in (M^* \otimes R)$ such that $A = \sum_{g \in G} \rho_R(g^{-1})A'\rho_M(g)$ and $B \in (R^* \otimes N)^G$ then

$$BA = \sum_{g \in G} B\rho_R(g^{-1})A'\rho_M(g) = \sum_{g \in G} \rho_N(g^{-1})BA'\rho_M(g),$$

that is $BA \in (M^* \otimes N)\Sigma_G$. Thus the evaluation map on R extends to a well-defined pairing

$$\begin{aligned} \circ : \mathcal{C}^k(M,R) \otimes \mathcal{C}^l(R,N) &\rightarrow \mathcal{C}^{k+l}(M,N) \\ [d_k \otimes m^* \otimes r'] \otimes [d_l \otimes r^* \otimes n] &\mapsto [r^*(r')(d_k \otimes d_l \otimes m^* \otimes n)]. \end{aligned}$$

Corollary 2.5.4 *The $\bullet$-composition in $\mathrm{Hom}_{\mathcal{D}er}$ corresponds to evaluation map $\mathcal{C}$, that is, if $f \in H^k(M,R)$ and $h \in H^l(R,N)$ then*

$$\Psi_{k+l}(f \cup h) = \Psi_l(h) \circ \Psi_k(f),$$

where Ψ_k is the isomorphism in theorem 2.4.6.

Proof: Let $f \in H^k(M,R)$ and $h \in H^l(R,N)$ then $f = p_k^*(f')$ and $h = p_l^*(h')$ where $f' \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M),R)$ and $h' \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_l(R),N)$. Let $f'' = \psi_k(f') \in \mathrm{Hom}_{\mathcal{D}er}(M,\Omega_{-k}(R))$ and $h'' = \psi_l(h') \in \mathrm{Hom}_{\mathcal{D}er}(R,\Omega_{-l}(N))$. Then $h' \bullet f' = h' \circ \psi_{-l}(f') = \psi_{-(k+l)}(\psi_k(h'') \circ f'')$ and it follows from theorem 2.5.3 that

$$\begin{aligned} \Psi_{k+l}(f \cup h) &= \Psi_{k+l}(p_k^*(f') \cup p_l^*(h')) = \Psi_{k+l}(p_{k+l}^*(h' \bullet f')) \\ &= \psi_{k+l}(h' \bullet f') = \psi_k(h'') \circ f''. \end{aligned}$$

As representatives for f'' and h'' choose $f'' : M \rightarrow \mathrm{im}(\partial_{-k}) \otimes R$ and $h'' : R \rightarrow \mathrm{im}(\partial_{-l}) \otimes N$ where $\mathrm{im}(\partial_n) \in \Omega_n(\mathbb{Z})$, $n = -k, -l$. Choose $\mathbb{Z}$-basis $(a_i), (b_i), (e_i), (c_i)$ and (d_i) for $M, N, R, \mathrm{im}(\partial_{-k})$ and $\mathrm{im}(\partial_{-l})$ respectively, and choose the corresponding dual basis $(a^j), (e^j)$ as $\mathbb{Z}$-basis for M^* and R^*. Let $\Psi_k(f) = \sum_{ijr} f_j^{ir}[c_i \otimes a^j \otimes e_r]$ and $\Psi_l(h) = \sum_{stu} h_t^{su}[d_s \otimes e^t \otimes b_u]$ then

$$\Psi_l(h) \circ \Psi_k(f)(a_\mu) = \sum_{irstu} f_\mu^{ir} h_r^{su}[c_i \otimes d_s \otimes b_u],$$

$f''(a_\mu) = \sum_{ir} f_\mu^{ir}[c_i \otimes e_r]$ and $h''(e_\nu) = \sum_{su} h_\nu^{su}[d_s \otimes b_u]$. As we saw earlier we can calculate $\psi_k(h'') = \mathrm{id} \otimes h''$ where id is the identity on $\mathrm{im}(\partial_{-k})$ thus we obtain

$$\begin{aligned} (\psi_k(h'') \circ f'')(a_\mu) &= (\mathrm{id} \otimes h'')(f(a_\mu)) = \textstyle\sum_{ir} f_\mu^{ir}(\mathrm{id} \otimes h'')[c_i \otimes e_r] \\ &= \textstyle\sum_{ir} f_\mu^{ir}[c_i \otimes h''(e_r)] = \sum_{irsu} f_\mu^{ir} h_r^{su}[c_i \otimes d_s \otimes b_u]. \end{aligned}$$

It follows that $\Psi_{k+l}(f \cup h) = \psi_k(h'') \circ f'' = \Psi_l(h) \circ \Psi_k(f)$. QED

We know that $H^*(G,\mathbb{Z})$ equipped with the cup product is a graded-commutative ring. Thus it follows that $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(\mathbb{Z}),\mathbb{Z})$ equipped with the $\bullet$-compostion is graded-commuative, where the graded-commutativity in dimension 0 is induced by the projection $H^0(G,\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z},\mathbb{Z})$. It then follows immediately from proposition 2.5.2 that $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is graded-commuative for any lattice M which lies in some syzygy, $\Omega_n(\mathbb{Z})$, of the the trivial lattice. However we can still improve on this.

From theorem 2.3.11 and lemma 2.3.6 it follows that we can regard the composition in $\mathrm{Hom}_{\mathcal{D}er}$ as a pairing

$$\bullet : \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(\mathbb{Z}), M^*\otimes R)\otimes\mathrm{Hom}_{\mathcal{D}er}(\Omega_l(\mathbb{Z}), R^*\otimes N) \to \mathrm{Hom}_{\mathcal{D}er}(\Omega_{k+l}(\mathbb{Z}), M^*\otimes N)$$

which we will now investigate in more detail for $M = N = R$.

Let $f \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_k(M), M)$, $h \in \mathrm{Hom}_{\mathcal{D}er}(\Omega_l(M), M)$, and let

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

be a free resolution of $\mathbb{Z}$. Tensoring with M then yields a free resolution of M

$$\cdots \longrightarrow F_1 \otimes M \xrightarrow{\partial_1\otimes\mathrm{id}} F_0 \otimes M \xrightarrow{\varepsilon\otimes\mathrm{id}} \mathbb{Z}\otimes M$$

and we can choose as representatives for f and h homomorphisms $f : \mathrm{im}(\partial_k)\otimes M \to M$ and $h : \mathrm{im}(\partial_l)\otimes M \to M$. As a free resolution for $\mathrm{im}(\partial_k)\otimes M$ choose

$$\cdots \longrightarrow F_1 \otimes \mathrm{im}(\partial_k)\otimes M \xrightarrow{\partial_1\otimes\mathrm{id}} F_0 \otimes \mathrm{im}(\partial_k)\otimes M \xrightarrow{\varepsilon\otimes\mathrm{id}} \mathbb{Z}\otimes \mathrm{im}(\partial_k)\otimes M\ .$$

Then $\psi_{-l}(f)$ can be obtained as the lift of f between these two resolution which gives $\psi_{-l}(f) = \mathrm{id}\otimes f$, that is,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{im}(\partial_l)\otimes\mathrm{im}(\partial_k)\otimes M & \longrightarrow \cdots \xrightarrow{\varepsilon\otimes\mathrm{id}} & \mathbb{Z}\otimes\mathrm{im}(\partial_k)\otimes M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \mathrm{im}(\partial_l)\otimes M & \longrightarrow \cdots \xrightarrow{\varepsilon\otimes\mathrm{id}} & \mathbb{Z}\otimes M & \longrightarrow & 0.
\end{array}$$

Thus for $d' \in \mathrm{im}(\partial_l)$, $d \in \mathrm{im}(\partial_k)$ and $m \in M$ we obtain

$$\begin{aligned}
(h \bullet f)(d'\otimes d\otimes m) &= (h\circ(\mathrm{id}\otimes f))(d'\otimes d\otimes m) \\
&= h(d'\otimes f(d\otimes m))
\end{aligned}$$

and the corresponding composition on $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(\mathbb{Z}), M^*\otimes M)$ is given by

$$\begin{aligned}
(\widehat{h\bullet f})(d'\otimes d)(m) &= (h\bullet f)(d'\otimes d\otimes m) = h(d'\otimes f(d\otimes m)) = h(d'\otimes \hat{f}(d)(m)) \\
&= \hat{h}(d')(\hat{f}(d)(m)) = (\hat{h}(d')\hat{f}(d))(m)
\end{aligned}$$

Similarly we can calculate $\widehat{f \bullet h}$ as

$$(\widehat{f \bullet h})(d \otimes d')(m) = (\hat{f}(d)\hat{h}(d'))(m)$$

Then, since $d' \otimes d = (-1)^{kl} d \otimes d'$, it follows that

$$(\widehat{f \bullet h})(d' \otimes d)(m) = (-1)^{kl}(\widehat{f \bullet h})(d \otimes d')(m) = (-1)^{kl}(\widehat{h \bullet f})(d' \otimes d)(m)$$

if and only if

$$\hat{f}(d)\hat{h}(d') = \hat{h}(d')\hat{f}(d)$$

in $M^* \otimes M$. Since $M^* \otimes M$ is the matrix ring $M_m(\mathbb{Z})$, $m = \mathrm{rk}_{\mathbb{Z}}(M)$, it follows that $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is graded-commuative if $\mathrm{rk}_{\mathbb{Z}}(M) = 1$.

Theorem 2.5.5 *Let $\mathcal{Z} \in \mathcal{F}(\mathbb{Z}[G])$ be a lattice with $\mathrm{rk}_{\mathbb{Z}}(\mathcal{Z}) = 1$. Let $M \in \mathcal{F}(\mathbb{Z}[G])$ be a lattice such that there exists an $M' \in \mathcal{F}(\mathbb{Z}[G])$ and an $n_0 \in \mathbb{Z}$ with $M \oplus M' \in \Omega_{n_0}(\mathcal{Z})$. Then $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is a graded-commutative ring.*

Proof: First assume that $M \in \Omega_{n_0}(\mathcal{Z})$ for some $n_0 \in \mathbb{Z}$. Then it follows by proposition 2.5.2 that

$$\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M) \simeq \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(\mathcal{Z}), \mathcal{Z}) \simeq \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(\mathbb{Z}), \mathcal{Z}^* \otimes \mathcal{Z}).$$

Thus, since $\mathrm{rk}_{\mathbb{Z}}(\mathcal{Z}) = 1$ it follows that $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is graded commutative.

Now assume there exists a lattice M' such that $M \oplus M' \in \Omega_{n_0}(\mathcal{Z})$ for some $n_0 \in \mathbb{Z}$. Then $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M \oplus M'), M \oplus M')$ is graded commutative and since

$$\begin{aligned}
& \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M \oplus M'), M \oplus M') \\
\simeq\ & \mathrm{Hom}_{\mathcal{D}er}(\Omega_*, (M \oplus M')^* \otimes (M \oplus M')) \\
=\ & \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(\mathbb{Z}), M^* \otimes M) \oplus \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(\mathbb{Z}), M'^* \otimes M) \\
& \oplus \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(\mathbb{Z}), M^* \otimes M') \oplus \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(\mathbb{Z}), M'^* \otimes M') \\
=\ & \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M) \oplus \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M'), M) \\
& \oplus \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M') \oplus \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M'), M')
\end{aligned}$$

it follows that

$$\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M) \hookrightarrow \mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M \oplus M'), M \oplus M')$$

is a subring and therefore itself graded-commutative. QED

3 The dihedral group D_6

In this chapter we will use the results of chapter 2 to calculate the syzygies and cohomologies of lattices over the dihedral group, D_6, of order 6. Furthermore we will determine the Bieberbach groups with holonomy group D_6 and the lattices M for which the ring $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is graded-commutative. The majority of the work will be to determine M^*, M^{D_6}, $M\Sigma_{D_6}$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, M)$ for the indecomposable lattices, and the indecomposable components of the tensor products over $\mathbb{Z}$., which we will do in section 3.1. Having done this we will see in section 3.2 that most of the results then follow immediately.

Throughout this chapter we will work with the following presentation of D_6

$$D_6 =< x, y \mid y^2 = x^3,\ xyx^{-1} = y > .$$

3.1 Indecomposable D_6-lattices

The following is a complete list of the indecomposable lattices over D_6 given in terms of a $\mathbb{Z}$-basis and representation $\rho_M : D_6 \to \mathrm{Gl}_m(\mathbb{Z}),\ m = \mathrm{rk}_{\mathbb{Z}}(M)$. A detailed computation is given in [13].

I. The trivial rank 1 representation $\mathbb{Z}$

$\rho_{\mathbb{Z}}(x) = 1$ and $\rho_{\mathbb{Z}}(y) = 1$

II. The non-trivial rank 1 representation $\mathbb{Z}^t$

$\rho_{\mathbb{Z}^t}(x) = 1$ and $\rho_{\mathbb{Z}^t}(y) = -1$

III. The integral group ring $\mathbb{Z}[C_2]$

Let $C_2 = \{1, t \mid t^2 = 1\}$ be the cyclic group of order 2. Let x act trivially and let y act by multiplication with t. Then the corresponding representation of D_6 in terms of the $\mathbb{Z}$-basis $\{1, t\}$ is given by

$$\rho_{\mathbb{Z}[C_2]}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_{\mathbb{Z}[C_2]}(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

IV. The Eisenstein integers $\mathcal{R} = \mathbb{Z}[\omega]$

Let $\mathbb{Z}[\omega] = \mathbb{Z}[x]/x^2+x+1, \quad \omega = \frac{1}{2}(-1+i\sqrt{3})$. Let x act by multiplication with ω and let y act by complex conjugation. Then the corresponding representation of D_6 in terms of the $\mathbb{Z}$-basis $\{e_1, e_2\},\ e_1 = -\omega,\ e_2 = 1 + \omega$, is given by

$$\rho_{\mathcal{R}}(x) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \rho_{\mathcal{R}}(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

V. The submodule $\mathcal{P} = (1-\omega)\mathcal{R} \subset \mathcal{R}$

The representation of D_6 in terms of the $\mathbb{Z}$-basis $\{f_1, f_2\}$, $f_1 = -2-\omega$, $f_2 = 1-\omega$ is given by

$$\rho_{\mathcal{P}}(x) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \rho_{\mathcal{P}}(y) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

VI. $\mathrm{ind}_{\mathrm{C}_2}^{\mathrm{D}_6}(\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}[\mathrm{C}_2]} \mathbb{Z}[\mathrm{D}_6]$

Let $\mathbb{Z}$ be the trivial rank 1 C_2-lattice and regard $\mathbb{Z}[D_6]$ as a left C_2-lattices in the obvious way. Then the corresponding representation of D_6 in terms of the $\mathbb{Z}$-basis $\{1\otimes 1,\ 1\otimes x,\ 1\otimes x^2\}$ is given by

$$\rho_{\mathrm{ind}_{\mathrm{C}_2}^{\mathrm{D}_6}(\mathbb{Z})}(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \rho_{\mathrm{ind}_{\mathrm{C}_2}^{\mathrm{D}_6}(\mathbb{Z})}(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

VII. $\mathrm{ind}_{\mathrm{C}_2}^{\mathrm{D}_6}(\mathbb{Z}^{\mathrm{t}}) = \mathbb{Z}^{\mathrm{t}} \otimes_{\mathbb{Z}[\mathrm{C}_2]} \mathbb{Z}[\mathrm{D}_6]$,

Let $\mathbb{Z}^t$ be the non-trivial rank 1 C_2-lattice, that is $1t = -1$. Then the corresponding representation of D_6 in terms of the $\mathbb{Z}$-basis $\{1\otimes 1,\ 1\otimes x,\ 1\otimes x^2\}$ is given by

$$\rho_{\mathrm{ind}_{\mathrm{C}_2}^{\mathrm{D}_6}(\mathbb{Z}^{\mathrm{t}})}(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \rho_{\mathrm{ind}_{\mathrm{C}_2}^{\mathrm{D}_6}(\mathbb{Z}^{\mathrm{t}})}(y) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Consider the representation $\rho = \rho_{\mathcal{R}} : D_6 \to \mathrm{Gl}_2(\mathbb{Z})$ with

$$\rho(x) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},\ \rho(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the image of the induced ring homomorphism $\rho : \mathbb{Z}[D_6] \to \mathrm{M}_2(\mathbb{Z})$ is given by

$$\mathrm{im}(\rho_*) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a-b+c-d \equiv 0 \bmod 3 \right\}$$

and isomorphic to $\mathcal{R} \oplus \mathcal{P}$, where the above $\mathbb{Z}$-basis $\{e_1, e_2\}$ of $\mathcal{R}$ maps into $\mathrm{im}(\rho)$ as $e_1 \mapsto \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_2 \mapsto \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$, and the above $\mathbb{Z}$-basis $\{f_1, f_2\}$ of

$\mathcal{P}$ maps into $\mathrm{im}(\rho)$ as $f_1 \mapsto \begin{pmatrix} 0 & -2 \\ 0 & -1 \end{pmatrix}$ and $f_2 \mapsto \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$. Furthermore, the kernel

$$\ker(\rho) = \mathrm{span}_{\mathbb{Z}}(1 + x + x^2, y + xy + x^2y)$$

is isomorphic to $\mathbb{Z}[C_2]$. Thus we obtain a split short exact sequence

$$0 \longrightarrow \mathbb{Z}[C_2] \xrightarrow{i} \mathbb{Z}[D_6] \xrightarrow{\rho} \mathcal{R} \oplus \mathcal{P} \longrightarrow 0. \tag{3.5}$$

VIII. Let $Y_0 = \rho^{-1}(R)$.

A $\mathbb{Z}$-basis of Y_0 is given by $\{a_1, a_2, a_3, a_4\}$, $a_1 = -1+y+xy, a_2 = 1+x-y, a_3 = 1+x+x^2, a_4 = y+xy+x^2y$, with respect to which the corresponding representation of D_6 is given by

$$\rho_{Y_0}(x) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \rho_{Y_0}(y) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

IX. Let $Y_1 = \rho_{-1}(\mathcal{P})$.

A $\mathbb{Z}$-basis of Y_1 is given by $\{b_1, b_2, b_3, b_4\}$, $b_1 = x - y, b_2 = 1 - xy, b_3 = 1 + x + x^2, b_4 = y + xy + x^2y$, with respect to which the corresponding representation of D_6 is given by

$$\rho_{Y_1}(x) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad \rho_{Y_1}(y) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

X. The regular representation $\mathbb{Z}[D_6]$

The regular representation in terms of the $\mathbb{Z}$-basis $\{1, x, x^2, y, xy, yx\}$ is given by

$$\rho_{\mathbb{Z}[D_6]}(x) = \begin{pmatrix} \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \end{pmatrix} \rho_{\mathbb{Z}[D_6]}(y) = \begin{pmatrix} 0 & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & 0 \end{pmatrix}.$$

For the remainder of this chapter we will consider a D_6-lattices as being given in terms of these basis and representations.

Proposition 3.1.1 *For the indecomposable D_6-lattices we have*

N	N^*	N^{D_6}	$N\Sigma_{D_6}$	$\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, N)$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$6\mathbb{Z}$	$\mathbb{Z}_6$
$\mathbb{Z}^t$	$\mathbb{Z}^t$	0	0	0
$\mathbb{Z}[C_2]$	$\mathbb{Z}[C_2]$	$\mathbb{Z}$	$3\mathbb{Z}$	$\mathbb{Z}_3$
$\mathcal{R}$	$\mathcal{P}$	0	0	0
$\mathcal{P}$	$\mathcal{R}$	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}_2$
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	0	0	0
Y_0	Y_0	$\mathbb{Z}$	$\mathbb{Z}$	0
Y_1	Y_1	$\mathbb{Z}$	$3\mathbb{Z}$	$\mathbb{Z}_3$
$\mathbb{Z}[D_6]$	$\mathbb{Z}[D_6]$	$\mathbb{Z}$	$\mathbb{Z}$	0

Proof:
For $\mathbb{Z}[D_6]$ we know that $\mathbb{Z}[D_6]^* = \mathbb{Z}[D_6]$, $\mathbb{Z}[D_6]^{D_6} = \mathbb{Z}[D_6]\Sigma_{D_6} = \mathbb{Z}\Sigma_{D_6} \simeq \mathbb{Z}$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, \mathbb{Z}[D_6]) = 0$.

1) The dual lattices N^*:

It is immediate that $\rho_N(g^{-1})^t = \rho_N(g)$ for all $g \in D_6$ for $N = \mathbb{Z}, \mathbb{Z}^t, \mathbb{Z}[C_2], \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$ and $\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$, thus in these case N is self-dual, $N \simeq N^*$.

Let $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$P^{-1}\rho_{\mathcal{R}}(g^{-1})^t P = \rho_{\mathcal{P}}(g)$$

for all $g \in D_6$. Thus $\mathcal{R}^* = \mathcal{P}$ and $\mathcal{P}^* = \mathcal{R}$.

Let

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

Then $P^{-1} = \begin{pmatrix} -2 & -1 & 0 & 1 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and

$$P^{-1}\rho_{Y_0}(g^{-1})^t P = \rho_{Y_0}(g)$$

for all $g \in D_6$. Thus $Y_0^* \simeq Y_0$, and since the dual of an indecomposable lattice is also indecomposable and of the same rank it follows that $Y_1^* \simeq Y_1$.

2) The invariant lattices N^{D_6}:

For $N = \mathbb{Z}$ we have $\mathbb{Z}^{D_6} = \mathbb{Z}$ since D_6 acts trivially on $\mathbb{Z}$. For $N = \mathbb{Z}^t$ we have $(\mathbb{Z}^t)^{D_6} = 0$ since $ny = -n = n$ if and only if $n = 0$.

For the remaining lattices write an element $n \in N$ as a $\mathbb{Z}$-vector $n = (n_1, \ldots, n_r)^t$ and consider the D_6-action in terms of the corresponding group representation.

For $N = \mathbb{Z}[C_2]$

$$\rho_{\mathbb{Z}[C_2]}(x)n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

and

$$\rho_{\mathbb{Z}[C_2]}(y)n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} n_2 \\ n_1 \end{pmatrix}.$$

Thus

$$\mathbb{Z}[C_2]^{D_6} = \{(a,a)^t \in \mathbb{Z}^2\} \simeq \mathbb{Z}.$$

For $N = \mathcal{R}$ and $N = \mathcal{P}$

$$\rho(x)n = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -n_2 \\ n_1 - n_2 \end{pmatrix}.$$

Thus

$$\mathcal{R}^{D_6} = 0 \quad \text{and} \quad \mathcal{P}^{D_6} = 0.$$

For $N = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$

$$\rho_{\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})}(x)n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} n_2 \\ n_3 \\ n_1 \end{pmatrix}$$

and

$$\rho_{\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})}(y)n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_3 \\ n_2 \end{pmatrix}.$$

Thus

$$(\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}))^{D_6} = \{(a,a,a)^t \in \mathbb{Z}^3\} \simeq \mathbb{Z}.$$

For $N = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$

$$\rho_{\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)}(x)n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} n_2 \\ n_3 \\ n_1 \end{pmatrix}$$

and

$$\rho_{\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)}(y)n = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} -n_1 \\ -n_3 \\ -n_2 \end{pmatrix}.$$

Thus

$$(\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t))^{D_6} = 0.$$

For $N = Y_0$

$$\rho_{Y_0}(x)n = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} -n_2 \\ n_1 - n_2 \\ -n_1 + n_2 + n_3 \\ n_1 + n_4 \end{pmatrix}$$

and

$$\rho_{Y_0}(y)n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} n_2 \\ n_1 \\ n_4 \\ n_3 \end{pmatrix}$$

It follows that

$$Y_0^{D_6} = \{(0,0,a,a)^t \in \mathbb{Z}^4\} \simeq \mathbb{Z}.$$

For $N = Y_1$

$$\rho_{Y_1}(x)n = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} -n_2 \\ n_1 - n_2 \\ n_2 + n_3 \\ -n_2 + n_4 \end{pmatrix}$$

and

$$\rho_{Y_1}(y)n = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} -n_2 \\ -n_1 \\ n_4 \\ n_3 \end{pmatrix}.$$

Thus

$$Y_1^{D_6} = \{(0,0,a,a)^t \in \mathbb{Z}^4\} \simeq \mathbb{Z}.$$

3) The sublattices $N\Sigma_{D_6}$ and the quotients $N^{D_6}/N\Sigma_{D_6} = \mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, N)$:

Since $N\Sigma_{D_6} \subset N^{D_6}$ it follows from the preceding part that $N\Sigma_{D_6} = 0$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, N) = 0$ for $N = \mathbb{Z}^t, \mathcal{R}, \mathcal{P}$ and $\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$. For the remaining lattices we need to calculate $N\Sigma_{D_6} = \rho_N(\Sigma_{D_6})N$.

For $N = \mathbb{Z}$ we have $\rho_{\mathbb{Z}}(\Sigma_{D_6}) = 6$ thus $N\Sigma_{D_6} \simeq 6\mathbb{Z}$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}_6$.

For $N = \mathbb{Z}[C_2]$ we have

$$\rho_{\mathbb{Z}[C_2]}(\Sigma_{D_6}) = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

thus $N\Sigma_{D_6} \simeq 3\mathbb{Z}$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, \mathbb{Z}[C_2]) = \mathbb{Z}_3$.

For $N = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$ we have

$$\rho_{\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})}(\Sigma_{D_6}) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

thus $N\Sigma_{D_6} \simeq 2\mathbb{Z}$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})) = \mathbb{Z}_2$.

For $N = Y_0$ we have

$$\rho_{Y_0}(\Sigma_{D_6}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \end{pmatrix}$$

thus $N\Sigma_{D_6} \simeq \mathbb{Z}$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, Y_0) = 0$.

For $N = Y_1$ we have

$$\rho_{Y_1}(\Sigma_{D_6}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

thus $N\Sigma_{D_6} \simeq 3\mathbb{Z}$ and $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, Y_1) = \mathbb{Z}_3$. QED

The following is a complete list of the tensor products $M \otimes N$ of the indecomposable lattices.

Proposition 3.1.2 *The indecomposable components of a tensor product $M \otimes N$ where M and N are indecomposable are given as follows:*

We have $\mathbb{Z} \otimes N \simeq N$ and $\mathbb{Z}[D_6] \otimes N \simeq \mathbb{Z}[D_6]^{\mathrm{rk}_{\mathbb{Z}}(N)}$ for all lattices N. For the remaining tensor products we obtain

I. For $M = \mathbb{Z}^t$ we obtain

$$\begin{aligned} \mathbb{Z}^t \otimes \mathbb{Z}^t &\simeq \mathbb{Z}, & \mathbb{Z}^t \otimes \mathcal{P} &\simeq \mathcal{R}, & \mathbb{Z}^t \otimes Y_0 &\simeq Y_1, \\ \mathbb{Z}^t \otimes \mathbb{Z}[C_2] &\simeq \mathbb{Z}[C_2], & \mathbb{Z}^t \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t), & \mathbb{Z}^t \otimes Y_1 &\simeq Y_0 \\ \mathbb{Z}^t \otimes \mathcal{R} &\simeq \mathcal{P}, & \mathbb{Z}^t \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) &\simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}), \end{aligned}$$

II. For $M = \mathbb{Z}[C_2]$ we obtain

$$\begin{aligned} \mathbb{Z}[C_2] \otimes \mathbb{Z}[C_2] &\simeq \mathbb{Z}[C_2] \oplus \mathbb{Z}[C_2], & \mathbb{Z}[C_2] \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) &\simeq \mathbb{Z}[D_6] \\ \mathbb{Z}[C_2] \otimes \mathcal{R} &\simeq \mathcal{R} \oplus \mathcal{P}, & \mathbb{Z}[C_2] \otimes Y_0 &\simeq Y_0 \oplus Y_1, \\ \mathbb{Z}[C_2] \otimes \mathcal{P} &\simeq \mathcal{R} \oplus \mathcal{P}, & \mathbb{Z}[C_2] \otimes Y_1 &\simeq Y_0 \oplus Y_1, \\ \mathbb{Z}[C_2] \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq \mathbb{Z}[D_6] \end{aligned}$$

III. For $M = \mathcal{R}$ we obtain

$$\begin{aligned} \mathcal{R} \otimes \mathcal{R} &\simeq Y_0, & \mathcal{R} \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq \mathbb{Z}[D_6], & \mathcal{R} \otimes Y_0 &\simeq \mathcal{P} \oplus \mathbb{Z}[D_6], \\ \mathcal{R} \otimes \mathcal{P} &\simeq Y_1, & \mathcal{R} \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) &\simeq \mathbb{Z}[D_6], & \mathcal{R} \otimes Y_1 &\simeq \mathcal{R} \oplus \mathbb{Z}[D_6], \end{aligned}$$

IV. For $M = \mathcal{P}$ we obtain

$$\begin{aligned} \mathcal{P} \otimes \mathcal{P} &\simeq Y_0, & \mathcal{P} \otimes Y_0 &\simeq \mathcal{R} \oplus \mathbb{Z}[D_6], \\ \mathcal{P} \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq \mathbb{Z}[D_6], & \mathcal{P} \otimes Y_1 &\simeq \mathcal{P} \oplus \mathbb{Z}[D_6], \\ \mathcal{P} \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) &\simeq \mathbb{Z}[D_6], \end{aligned}$$

V. For $M = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$ *we obtain*

$$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \oplus \mathbb{Z}[D_6], \qquad \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \otimes Y_0 \simeq \mathbb{Z}[D_6]^2,$$

$$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \oplus \mathbb{Z}[D_6], \qquad \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \otimes Y_1 \simeq \mathbb{Z}[D_6]^2,$$

VI. For $M = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$ *we obtain*

$$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \oplus \mathbb{Z}[D_6], \qquad \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \otimes Y_0 \simeq \mathbb{Z}[D_6]^2,$$

$$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \otimes Y_1 \simeq \mathbb{Z}[D_6]^2,$$

VII. For $M = Y_0$ *we obtain*

$$Y_0 \otimes Y_0 \simeq Y_1 \oplus \mathbb{Z}[D_6], \quad Y_0 \otimes Y_1 \simeq Y_0 \oplus \mathbb{Z}[D_6]$$

VIII. For $M = Y_1$ *we obtain*

$$Y_1 \otimes Y_1 \simeq Y_1 \oplus \mathbb{Z}[D_6]$$

Proof: For $\mathbb{Z} \otimes N$ the claim is immediate and for $\mathbb{Z}[D_6] \otimes N$ the claim follows from lemma 2.1.2.

I. Let $M = \mathbb{Z}^t$. Then $\rho_{\mathbb{Z}^t \otimes N}(x) = \rho_N(x)$ and $\rho_{\mathbb{Z}^t \otimes N}(y) = -\rho_N(y)$ and we obtain

- for $N = \mathbb{Z}^t$: $\rho_{\mathbb{Z}^t \otimes \mathbb{Z}^t}(g) = \rho_{\mathbb{Z}^t}(g)\rho_{\mathbb{Z}^t}(g) = \rho_{\mathbb{Z}}(g)$ for all $g \in D_6$ and it follows that $\mathbb{Z}^t \otimes \mathbb{Z}^t \simeq \mathbb{Z}$.

- for $N = \mathbb{Z}[C_2]$: Let $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then $P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$P^{-1}\rho_{\mathbb{Z}^t \otimes \mathbb{Z}[C_2]}(g)P = \rho_{\mathbb{Z}[C_2]}(g)$$

and it follows that $\mathbb{Z}^t \otimes \mathbb{Z}[C_2] \simeq \mathbb{Z}[C_2]$.

- for $N = \mathcal{R}, \mathcal{P}$: $\rho_{\mathbb{Z}^t \otimes \mathcal{R}}(g) = \rho_{\mathcal{P}}(g)$ and $\rho_{\mathbb{Z}^t \otimes \mathcal{P}}(g) = \rho_{\mathcal{R}}(g)$ for all $g \in D_6$ and it follows that $\mathbb{Z}^t \otimes \mathcal{R} \simeq \mathcal{P}$ and $\mathbb{Z}^t \otimes \mathcal{P} \simeq \mathcal{R}$.

- for $N = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}), \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$: It follows immediately from Frobenius reciprocity, lemma 2.1.1, that

$$\begin{aligned}\mathbb{Z}^t \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq \mathbb{Z}^t \otimes (\mathbb{Z} \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6]) \simeq \mathbb{Z}^t \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)\end{aligned}$$

and

$$\begin{aligned}\mathbb{Z}^t \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) &\simeq \mathbb{Z}^t \otimes (\mathbb{Z}^t \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6]) \simeq \mathbb{Z} \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}).\end{aligned}$$

- for $N = Y_0, Y_1$: Since $Y_0 \simeq \mathcal{R} \otimes \mathcal{R} \simeq \mathcal{P} \otimes \mathcal{P}$ and $Y_1 \simeq \mathcal{R} \otimes \mathcal{P}$, see below, it follows that

$$\mathbb{Z}^t \otimes Y_0 \simeq \mathbb{Z}^t \otimes \mathcal{R} \otimes \mathcal{R} \simeq \mathcal{P} \otimes \mathcal{R} \simeq Y_1$$

and similarly

$$\mathbb{Z}^t \otimes Y_1 \simeq \mathbb{Z}^t \otimes \mathcal{R} \otimes \mathcal{P} \simeq \mathcal{P} \otimes \mathcal{P} \simeq Y_0.$$

II. Let $M = \mathbb{Z}[C_2]$. For a lattice N we can write the representation of D_6 corresponding to $\mathbb{Z}[C_2] \otimes N$ as

$$\rho_{\mathbb{Z}[C_2]\otimes N}(x) = \begin{pmatrix} \rho_N(x) & 0 \\ 0 & \rho_N(x) \end{pmatrix},\ \rho_{\mathbb{Z}[C_2]\otimes N}(y) = \begin{pmatrix} 0 & \rho_N(y) \\ \rho_N(y) & 0 \end{pmatrix}.$$

- for $N = \mathbb{Z}[C_2]$: Let

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

then $P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $P^{-1}\rho_{\mathbb{Z}[C_2]\otimes\mathbb{Z}[C_2]}(g)P = \rho_{\mathbb{Z}[C_2]\oplus\mathbb{Z}[C_2]}$, and it follows that $\mathbb{Z}[C_2] \otimes \mathbb{Z}[C_2] \simeq \mathbb{Z}[C_2] \oplus \mathbb{Z}[C_2]$.

- for $N = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}), \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$: It follows immediately from Frobenius reciprocity, lemma 2.1.1, that

$$\begin{aligned}\mathbb{Z}[C_2] \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq \mathbb{Z}[C_2] \otimes (\mathbb{Z} \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6]) \simeq \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathbb{Z}[D_6]\end{aligned}$$

similarly

$$\begin{aligned}\mathbb{Z}[C_2] \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) &\simeq \mathbb{Z}[C_2] \otimes (\mathbb{Z}^t \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6]) \simeq \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathbb{Z}[D_6]\end{aligned}$$

- for $N = \mathcal{R}, \mathcal{P}$: Let

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

then $P^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$ and we obtain $P^{-1}\rho_{\mathbb{Z}[C_2]\otimes\mathcal{R}}(g)P = \rho_{\mathcal{R}\oplus\mathcal{P}}(g)$ and $P^{-1}\rho_{\mathbb{Z}[C_2]\otimes\mathcal{P}}(g)P = \rho_{\mathcal{P}\oplus\mathcal{R}}(g)$ for all $g \in D_6$. It follows that

$$\mathcal{R} \otimes \mathbb{Z}[C_2] \simeq \mathcal{P} \otimes \mathbb{Z}[C_2] \simeq \mathcal{R} \oplus \mathcal{P}.$$

- for $N = Y_0, Y_1$: Since $Y_0 \simeq \mathcal{R} \otimes \mathcal{R} \simeq \mathcal{P} \otimes \mathcal{P}$ and $Y_1 \simeq \mathcal{R} \otimes \mathcal{P}$, see below, it follows that

$$\mathbb{Z}[C_2] \otimes Y_0 \simeq \mathbb{Z}[C_2] \otimes \mathcal{R} \otimes \mathcal{R} \simeq (\mathcal{R} \oplus \mathcal{P}) \otimes \mathcal{R} \simeq Y_0 \oplus Y_1$$

and similarly

$$\mathbb{Z}[C_2] \otimes Y_1 \simeq \mathbb{Z}[C_2] \otimes \mathcal{R} \otimes \mathcal{P} \simeq (\mathcal{R} \oplus \mathcal{P}) \otimes \mathcal{P} \simeq Y_0 \oplus Y_1.$$

III. and IV. Let $M = \mathcal{R}, \mathcal{P}$. For a lattice N we can write the representation of D_6 corresponding to $\mathcal{R} \otimes N$ as

$$\rho_{\mathcal{R}\otimes N}(x) = \begin{pmatrix} 0 & -\rho_N(x) \\ \rho_N(x) & -\rho_N(x) \end{pmatrix}, \ \rho_{\mathcal{R}\otimes N}(y) = \begin{pmatrix} 0 & \rho_N(y) \\ \rho_N(y) & 0 \end{pmatrix}$$

and the representation of D_6 corresponding to $\mathcal{P} \otimes N$ as

$$\rho_{\mathcal{P}\otimes N}(x) = \begin{pmatrix} 0 & -\rho_N(x) \\ \rho_N(x) & -\rho_N(x) \end{pmatrix}, \ \rho_{\mathcal{P}\otimes N}(y) = \begin{pmatrix} 0 & -\rho_N(y) \\ -\rho_N(y) & 0 \end{pmatrix}.$$

- For $\mathcal{R} \otimes \mathcal{R}$ and $\mathcal{P} \otimes \mathcal{P}$ let

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

then $P^{-1} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$ and we obtain $P^{-1}\rho_{\mathcal{R}\otimes\mathcal{R}}(g)P = \rho_{Y_0}(g)$ and $P^{-1}\rho_{\mathcal{P}\otimes\mathcal{P}}(g)P = \rho_{Y_0}(g)$ for $g \in D_6$. It follows that

$$\mathcal{R} \otimes \mathcal{R} \simeq \mathcal{P} \otimes \mathcal{P} \simeq Y_0.$$

- For $\mathcal{R}\otimes\mathcal{P}$ let

$$P=\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

then $P^{-1}=\begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ and we obtain $P^{-1}\rho_{\mathcal{R}\otimes\mathcal{P}}(g)P=\rho_{Y_1}(g)$ for $g\in D_6$. It follows

$$\mathcal{R}\otimes\mathcal{P}\simeq Y_1.$$

- for $\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}),\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$: As $\mathbb{Z}[C_2]$-modules $\mathcal{R}$ and $\mathcal{P}$ are isomorphic to $\mathbb{Z}[C_2]$. Thus it follows immediately from Frobenius reciprocity, lemma 2.1.1, that

$$\begin{aligned}\mathcal{R}\otimes\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq \mathcal{R}\otimes(\mathbb{Z}\otimes_{\mathbb{Z}[C_2]}\mathbb{Z}[D_6])\simeq\mathbb{Z}[C_2]\otimes_{\mathbb{Z}[C_2]}\mathbb{Z}[D_6]\\ &\simeq \mathbb{Z}[D_6]\end{aligned}$$

and similarly $\mathcal{R}\otimes\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)\simeq\mathcal{P}\otimes\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})\simeq\mathcal{P}\otimes\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)\simeq\mathbb{Z}[D_6]$.

- for $\mathcal{R}\otimes Y_0$ let

$$P=\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$P^{-1}=\begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $P^{-1}\rho_{\mathcal{R}\otimes Y_0}P\simeq\rho_{\mathcal{P}\oplus\mathbb{Z}[D_6]}(g)$ for all $g\in D_6$, and it follows that

$$\mathcal{R}\otimes Y_0\simeq\mathcal{P}\oplus\mathbb{Z}[D_6].$$

- for $\mathcal{P} \otimes Y_0$ let

$$P = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

and $P^{-1}\rho_{\mathcal{P}\otimes Y_0}P \simeq \rho_{\mathcal{R}\oplus\mathbb{Z}[D_6]}(g)$ for all $g \in D_6$, and it follows that

$$\mathcal{P} \otimes Y_0 \simeq \mathcal{R} \oplus \mathbb{Z}[D_6].$$

- for Y_1: Since $Y_1 \simeq \mathcal{R} \otimes \mathcal{P}$ it follows that

$$\mathcal{R} \otimes Y_1 \simeq \mathcal{R} \otimes \mathcal{R} \otimes \mathcal{P} \simeq Y_0 \otimes \mathcal{P} \simeq \mathcal{R} \oplus \mathbb{Z}[D_6]$$

and similarly $\mathcal{P} \otimes Y_1 \simeq \mathcal{P} \oplus \mathbb{Z}[D_6]$.

V. and VI. Let $M = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}), \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$.

- for $N = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}), \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$. As $\mathbb{Z}[C_2]$-lattices $\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}[C_2]$ and $\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \simeq \mathbb{Z}^t \oplus \mathbb{Z}[C_2]$. Thus it follows from Frobenius reciprocity, lemma 2.1.1, that

$$\begin{aligned} \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq ((\mathbb{Z} \oplus \mathbb{Z}[C_2]) \otimes \mathbb{Z}) \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \oplus \mathbb{Z}[D_6] \\ \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) &\simeq ((\mathbb{Z}^t \oplus \mathbb{Z}[C_2]) \otimes \mathbb{Z}) \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \oplus \mathbb{Z}[D_6] \\ \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) &\simeq ((\mathbb{Z}^t \oplus \mathbb{Z}[C_2]) \otimes \mathbb{Z}^t) \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \oplus \mathbb{Z}[D_6]. \end{aligned}$$

- for $N = Y_0, Y_1$: As $\mathbb{Z}[C_2]$-lattices $Y_0 \simeq Y_1 \simeq \mathbb{Z}[C_2]^2$. Thus it follows from Frobenius reciprocity for $i = 0, 1$ that

$$\begin{aligned}(\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}) \otimes Y_i) &\simeq (\mathbb{Z} \otimes \mathbb{Z}[C_2]^2) \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathbb{Z}[D_6]^2 \\ (\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \otimes Y_i) &\simeq (\mathbb{Z}^t \otimes \mathbb{Z}[C_2]^2) \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \\ &\simeq \mathbb{Z}[C_2]^2 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[D_6] \simeq \mathbb{Z}[D_6]^2\end{aligned}$$

VII. and VIII. For $M = Y_0, Y_1$ and $N = Y_0, Y_1$ it follows

$$\begin{aligned}Y_0 \otimes Y_0 &\simeq Y_0 \otimes (\mathcal{R} \otimes \mathcal{R}) \simeq (\mathcal{P} \oplus \mathbb{Z}[D_6]) \otimes \mathcal{R} \simeq \mathcal{P} \otimes \mathcal{R} \oplus \mathbb{Z}[D_6]^2 \\ &\simeq Y_1 \oplus \mathbb{Z}[D_6]^2 \\ Y_1 \otimes Y_1 &\simeq (\mathcal{R} \otimes \mathcal{P}) \otimes (\mathcal{R} \otimes \mathcal{P}) \simeq (\mathcal{R} \otimes \mathcal{R}) \otimes (\mathcal{P} \otimes \mathcal{P}) \simeq Y_0 \otimes Y_0 \\ &\simeq Y_1 \oplus \mathbb{Z}[D_6]^2 \\ Y_0 \otimes Y_1 &\simeq Y_0 \otimes (\mathcal{R} \otimes \mathcal{P}) \simeq (\mathcal{P} \oplus \mathbb{Z}[D_6]) \otimes \mathcal{P} \simeq \mathcal{P} \otimes \mathcal{P} \oplus \mathbb{Z}[D_6]^2 \\ &\simeq Y_0 \oplus \mathbb{Z}[D_6]^2\end{aligned}$$

QED

3.2 Syzygies and cohomologies of D_6-lattices

In this section we will use the results of chapter 2 and section 3.1 to calculate the syzygies and cohomologies of D_6-lattices. We will see that all that is required are some initial calculations for the trivial lattice $\mathbb{Z}$ and that the rest will follow immediately.

Theorem 3.2.1 *The following is a complete list of the minimal representatives of $\Omega_n(N)$ for an indecomposable D_6-lattices N*

N	$\Omega_1(N)$	$\Omega_2(N)$	$\Omega_3(N)$	$\Omega_4(N)$
$\mathbb{Z}$	$\mathcal{P} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$Y_0 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathcal{R} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$Y_1 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$
$\mathbb{Z}^t$	$\mathcal{R} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$Y_1 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathcal{P} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$Y_0 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$
$\mathbb{Z}[C_2]$	$\mathcal{R} \oplus \mathcal{P}$	$Y_0 \oplus Y_1$	$\mathcal{R} \oplus \mathcal{P}$	$Y_0 \oplus Y_1$

N	$\Omega_1(N)$	$\Omega_2(N)$	$\Omega_3(N)$	$\Omega_4(N)$
$\mathcal{R}$	Y_1	$\mathcal{P}$	Y_0	$\mathcal{R}$
$\mathcal{P}$	Y_0	$\mathcal{R}$	Y_1	$\mathcal{P}$
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$
Y_0	$\mathcal{R}$	Y_1	$\mathcal{P}$	Y_0
Y_1	$\mathcal{P}$	Y_0	$\mathcal{R}$	Y_1

In general, we have $\Omega_n(N) = \Omega_{n+4}(N)$ for $n \neq -4$, $\Omega_{-4}(N) = \Omega_4(N)$ and $\Omega_n(\mathbb{Z}[D_6])$ is the class of free lattices over $\mathbb{Z}[D_6]$ for all $n \in \mathbb{Z}$. For an arbitrary lattice $N = N_1 \oplus \cdots \oplus N_k$ we have $\Omega_n(N) = \Omega_n(N_1) \oplus \cdots \oplus \Omega_n(N_k)$.

Proof: Since D_6 has a periodic free resolution of period 4, see [9] chapter 7, it follows that $\Omega_n(\mathbb{Z}) = \Omega_{n+4}(\mathbb{Z})$ for $n \neq -4$, $\Omega_{-4}(\mathbb{Z}) = \Omega_4(\mathbb{Z})$ and since $\Omega_n(N) = \Omega_n(\mathbb{Z}) \otimes N$ the same holds for any lattice N.

Let $\varepsilon : \mathbb{Z}[D_6] \to \mathbb{Z}$ be the augmentation map. Then $\Omega_1(\mathbb{Z})$ is the stable class of the augmentation ideal $\ker(\varepsilon)$. It is well known that for any group G the elements $1 - g \in \mathbb{Z}[G]$, $1 \neq g$ form a $\mathbb{Z}$-basis of the augmentation ideal. Thus for $G = D_6$ a $\mathbb{Z}$-basis of the augmentation ideal is given by $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5\}$ where

$$\hat{e}_1 = 1 - x,\ \hat{e}_2 = 1 - x^2,\ \hat{e}_3 = 1 - y,\ \hat{e}_4 = 1 - xy,\ \hat{e}_5 = 1 - x^2y.$$

Let

$$\begin{aligned} e_1 &= -\hat{e}_1 + \hat{e}_3 - \hat{e}_5 = -1 + x - y + x^2y \\ e_2 &= \hat{e}_2 - \hat{e}_3 + \hat{e}_4 = 1 - x^2 + y - xy \\ e_3 &= -\hat{e}_1 + \hat{e}_4 = x - xy \\ e_4 &= -\hat{e}_2 + \hat{e}_3 = x^2 - y \\ e_5 &= \hat{e}_5 = 1 - x^2y \end{aligned}$$

then $\{e_1, e_2, e_3, e_4, e_5\}$ is also a $\mathbb{Z}$-basis of $\ker(\varepsilon)$ (since $\hat{e}_1 = e_2 - e_3 + e_5,\ \hat{e}_2 = e_1 + e_2 - e_3 + e_4,\ \hat{e}_3 = e_1 + e_2 - e_3 + e_4 + e_5,\ \hat{e}_4 = e_4,\ \hat{e}_5 = e_2 + e_5$) on which D_6 acts via

$$\rho_{\ker(\varepsilon)}(x) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \rho_{\ker(\varepsilon)}(y) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

It follows that

$$\rho_{\ker(\varepsilon)} = \begin{pmatrix} \rho_{\mathcal{P}} & 0 \\ 0 & \rho_{\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)} \end{pmatrix}$$

thus a minimal representative of $\Omega_1(\mathbb{Z})$ is

$$\mathcal{P} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t).$$

To obtain the remaining syzygies we use lemma 2.3.6, $\Omega_n(\mathbb{Z}) \otimes \Omega_1(\mathbb{Z}) = \Omega_{n+1}(\mathbb{Z})$ and $\Omega_n(\mathbb{Z}) \otimes N = \Omega_n(N)$, and the calculations of the tensor products in proposition 3.1.2. For example $\Omega_2(\mathbb{Z}) = \Omega_1(\mathbb{Z}) \otimes \Omega_1(\mathbb{Z})$ and

$$\begin{aligned} & (\mathcal{P} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)) \otimes (\mathcal{P} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)) \\ \simeq\ & (P \otimes P) \oplus (P \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t))^2 \oplus (\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \otimes \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)) \\ \simeq\ & Y_0 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t) \oplus \mathbb{Z}[D_6]^2. \end{aligned}$$

Thus a minimal representative of $\Omega_2(\mathbb{Z})$ is given by $Y_0 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$. QED

Corollary 3.2.2 *For the following lattices M the ring $\mathrm{Hom}_{\mathcal{D}er}(\Omega_*(M), M)$ is graded-commutative*

$$\begin{aligned} M \ =\ & \mathbb{Z},\ \mathbb{Z}^t,\ \mathcal{R},\ \mathcal{P},\ \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}),\ \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t),\ Y_0,\ Y_1, \\ & \mathcal{R} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}),\ \mathcal{R} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t),\ \mathcal{P} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}),\ \mathcal{P} \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t), \\ & Y_0 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}),\ Y_0 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t),\ Y_1 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}),\ Y_1 \oplus \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t), \end{aligned}$$

Proof: The claim follows immediately from theorem 2.5.5 and theorem 3.2.1. QED

Theorem 3.2.3 *The following is a complete list of the cohomology groups $H^n(M, N)$ of indecomposable D_6-lattices.*

I. For $M = \mathbb{Z}$ the cohomology groups $H^n(\mathbb{Z}, N) = H^n(D_6, M)$ are

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$
$\mathbb{Z}^t$	0	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0
$\mathbb{Z}[C_2]$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
$\mathcal{R}$	0	0	0	$\mathbb{Z}_3$	0
$\mathcal{P}$	0	$\mathbb{Z}_3$	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
Y_0	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	0
Y_1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_3$
$\mathbb{Z}[D_6]$	$\mathbb{Z}$	0	0	0	0

II. For $M = \mathbb{Z}^t$ the cohomology groups are

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0
$\mathbb{Z}^t$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$
$\mathbb{Z}[C_2]$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
$\mathcal{R}$	0	$\mathbb{Z}_3$	0	0	0
$\mathcal{P}$	0	0	0	$\mathbb{Z}_3$	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
Y_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_3$
Y_1	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	0
$\mathbb{Z}[D_6]$	$\mathbb{Z}$	0	0	0	0

III. For $M = \mathbb{Z}[C_2]$ the cohomology groups are

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
$\mathbb{Z}^t$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
$\mathbb{Z}[C_2]$	$\mathbb{Z}^2$	0	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$
$\mathcal{R}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
$\mathcal{P}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}$	0	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathbb{Z}$	0	0	0	0
Y_0	$\mathbb{Z}^2$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
Y_1	$\mathbb{Z}^2$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
$\mathbb{Z}[D_6]$	$\mathbb{Z}^2$	0	0	0	0

IV. For $M = \mathcal{R}$ the cohomology groups are

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	0	0
$\mathbb{Z}^t$	0	0	0	$\mathbb{Z}_3$	0
$\mathbb{Z}[C_2]$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
$\mathcal{R}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_3$
$\mathcal{P}$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}$	0	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathbb{Z}$	0	0	0	0
Y_0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	0
Y_1	$\mathbb{Z}$	$\mathbb{Z}_3$	0	0	0
$\mathbb{Z}[D_6]$	$\mathbb{Z}^2$	0	0	0	0

V. For $M = \mathcal{P}$ the cohomology groups are

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	0	0	0	$\mathbb{Z}_3$	0
$\mathbb{Z}^t$	0	$\mathbb{Z}_3$	0	0	0
$\mathbb{Z}[C_2]$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
$\mathcal{R}$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	0
$\mathcal{P}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_3$
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}$	0	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathbb{Z}$	0	0	0	0
Y_0	$\mathbb{Z}$	$\mathbb{Z}_3$	0	0	0
Y_1	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	0
$\mathbb{Z}[D_6]$	$\mathbb{Z}^2$	0	0	0	0

VI. For $M = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$ the cohomology groups are

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$\mathbb{Z}^t$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
$\mathbb{Z}[C_2]$	$\mathbb{Z}$	0	0	0	0
$\mathcal{R}$	$\mathbb{Z}$	0	0	0	0
$\mathcal{P}$	$\mathbb{Z}$	0	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
Y_0	$\mathbb{Z}^2$	0	0	0	0
Y_1	$\mathbb{Z}^2$	0	0	0	0
$\mathbb{Z}[D_6]$	$\mathbb{Z}^3$	0	0	0	0

VII. For $M = \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$ *the cohomology groups are*

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
$\mathbb{Z}^t$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$\mathbb{Z}[C_2]$	$\mathbb{Z}$	0	0	0	0
$\mathcal{R}$	$\mathbb{Z}$	0	0	0	0
$\mathcal{P}$	$\mathbb{Z}$	0	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
Y_0	$\mathbb{Z}^2$	0	0	0	0
Y_1	$\mathbb{Z}^2$	0	0	0	0
$\mathbb{Z}[D_6]$	$\mathbb{Z}^3$	0	0	0	0

VIII. For $M = Y_0$ *the cohomology groups are*

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	0
$\mathbb{Z}^t$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_3$
$\mathbb{Z}[C_2]$	$\mathbb{Z}^2$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
$\mathcal{R}$	$\mathbb{Z}$	$\mathbb{Z}_3$	0	0	0
$\mathcal{P}$	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}^2$	0	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathbb{Z}^2$	0	0	0	0
Y_0	$\mathbb{Z}^2$	0	0	0	$\mathbb{Z}_3$
Y_1	$\mathbb{Z}^2$	0	$\mathbb{Z}_3$	0	0
$\mathbb{Z}[D_6]$	$\mathbb{Z}^4$	0	0	0	0

IX. For $M = Y_1$ the cohomology groups are

N	$H^0(M,N)$	$H^1(M,N)$	$H^2(M,N)$	$H^3(M,N)$	$H^4(M,N)$
$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_3$
$\mathbb{Z}^t$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	0
$\mathbb{Z}[C_2]$	$\mathbb{Z}^2$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$
$\mathcal{R}$	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	0
$\mathcal{P}$	$\mathbb{Z}$	$\mathbb{Z}_3$	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}^2$	0	0	0	0
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z}^t)$	$\mathbb{Z}^2$	0	0	0	0
Y_0	$\mathbb{Z}^2$	0	0	0	$\mathbb{Z}_3$
Y_1	$\mathbb{Z}^2$	0	$\mathbb{Z}_3$	0	0
$\mathbb{Z}[D_6]$	$\mathbb{Z}^4$	0	0	0	0

and $H^n(M,N) = H^{n+4}(M,N)$ for $n \geq 1$. For a decomposable lattice $M = M_1 \oplus \cdots \oplus M_k$, we have

$$H^n(M,N) = \bigoplus_{i=1}^{k} H^n(M_i,N),$$

and for a decomposable lattice $N = N_1 \oplus \cdots \oplus N_k$, we have

$$H^n(M,N) = \bigoplus_{i=1}^{k} H^n(M,N_i).$$

Proof: For $n = 0$ it follows from proposition 2.1.6 that $H^0(M,N) = \mathrm{Hom}(M,N) \simeq \mathrm{Hom}(\mathbb{Z}, M^* \otimes N) \simeq (M^* \otimes N)^{D_6}$. And the claim follows from proposition 3.1.1 and proposition 3.1.2.

For $n \geq 1$ the claim follows from theorem 2.4.6,

$$H^n(M,N) \simeq (\mathrm{im}(\partial_{-n}) \otimes M^* \otimes N)^G / (\mathrm{im}(\partial_{-n}) \otimes M^* \otimes N)\Sigma_G,$$

the calculations for $\Omega_n(\mathbb{Z})$, theorem 3.2.1, and proposition 3.1.1 and proposition 3.1.2. QED

3.3 Bieberbach groups with holonomy group D_6

As mentioned in section 2.2 a Bieberbach group π with holonomy group D_6 is given by a torsion free group extensions

$$0 \longrightarrow N \longrightarrow \pi \longrightarrow D_6 \longrightarrow 0,$$

where N is a D_6-lattice. In theorem 2.4.8 we proved that these extensions correspond to elements $c \in \mathcal{C}^2(\mathbb{Z}, N) \simeq H^2(D_6, N)$ which are not in $(\Omega_{-2}(\mathbb{Z}) \otimes N)\Sigma_{C_p}$ for all cyclic subgroups $C_p \subset D_6$ of prime order. The cyclic subgroup of D_6 are $C_2 = \{1, y\}$ and $C_3 = \{1, x, x^2\}$. In the proof of proposition 3.1.1 we calculated the representatives of $\mathrm{Hom}_{\mathcal{D}er}(\mathbb{Z}, N)$ in N^G which will give us the representatives of $H^2(D_6, N)$ in $(\Omega_{-2}(\mathbb{Z}) \otimes N)^{D_6}$. Thus, to see which one of them determines a torsion free extension we need to check if they lie in $(\Omega_{-2}(\mathbb{Z}) \otimes N)\Sigma_{C_p},\ p = 2, 3$.

In theorem 3.2.3 we showed that the indecomposable lattices N for which $H^2(D_6, N) \neq 0$ are $N = \mathbb{Z}, \mathbb{Z}^t, \mathbb{Z}[C_2], \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$ and Y_0, and for these lattices $H^2(D_6, N)$ and $\Omega_{-2}(\mathbb{Z}) \otimes N$ are of the form

N	$H^2(D_6, N)$	$\Omega_{-2}(\mathbb{Z}) \otimes N$
$\mathbb{Z}$	$\mathbb{Z}_2$	$Y_0 \oplus \mathrm{ind}_{C_2}(\mathbb{Z})$
$\mathbb{Z}^t$	$\mathbb{Z}_3$	$Y_1 \oplus \mathrm{ind}_{C_2}(\mathbb{Z}^t)$
$\mathbb{Z}[C_2]$	$\mathbb{Z}_3$	$Y_0 \oplus Y_1 \oplus \mathbb{Z}[D_6]$
$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$	$\mathbb{Z}_2$	$\mathrm{ind}_{C_2}(\mathbb{Z}) \oplus \mathbb{Z}[D_6]$
Y_0	$\mathbb{Z}_3$	$Y_1 \oplus \mathbb{Z}[D_6]$

where the claim for $\Omega_{-2}(\mathbb{Z}) \otimes N$ follows from theorem 3.2.1 and proposition 3.1.2. Comparing with the proof of proposition 3.1.1 we see that the representatives of $H^2(D_6, N)$ only come from Y_1 or $\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$ and are of the form $a(0, 0, 1, 1)^t,\ a = 0, 1, 2$, and $b(1, 1, 1)^t,\ b = 0, 1$, respectively. Thus we only need to determine $Y_1\Sigma_{C_p}$ and $\mathrm{ind}_{C_2}(\mathbb{Z})\Sigma_p,\ p = 2, 3$.

For $Y_1\Sigma_{C_2}$ we obtain

$$\rho_{Y_1}(\Sigma_{C_2}) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

so that

$$Y_1\Sigma_{C_2} = \left\{ c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \mid c_1, c_2 \in \mathbb{Z} \right\}$$

and all representatives $a(0,0,1,1)^t,\ a=0,1,2$ lie in $Y_1\Sigma_{C_2}$.

For $Y_1\Sigma_{C_3}$ we obtain

$$\rho_{Y_1}(\Sigma_{C_3}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -1 & -1 & 0 & 3 \end{pmatrix}$$

so that

$$Y_1\Sigma_{C_3} = \left\{ c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + 3c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 3c_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mid c_1, c_2, c_3 \in \mathbb{Z} \right\}$$

and a representative $a(0,0,1,1)^t,\ a=0,1,2$ lies in $Y_1\Sigma_{C_3}$ if and only if $a=0$.

For $\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})\Sigma_{C_2}$ we obtain

$$\rho_{\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})}(\Sigma_{C_2}) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

so that

$$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})\Sigma_{C_3} = \left\{ 2c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mid c_1, c_2 \in \mathbb{Z} \right\}$$

and a representative $b(1,1,1)^t,\ b=0,1,$ lies in $\mathrm{ind}_{C_2}(\mathbb{Z})\Sigma_2$ if and only if $b=0$.

For $\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})\Sigma_{C_3}$ we obtain

$$\rho_{\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})}(\Sigma_{C_3}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

so that

$$\mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})\Sigma_{C_3} = \left\{ c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{Z} \right\}$$

and all representatives $b(1,1,1)^t,\ b=0,1,$ lie in $\mathrm{ind}_{C_2}(\mathbb{Z})\Sigma_3$.

We obtain

Theorem 3.3.1 *Let N be a D_6-lattice. Then there exists a torsion-free extension of N by D_6 if and only if $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \subseteq H^2(D_6, N)$, and any element $(a, b) \in \mathbb{Z}_2 \oplus \mathbb{Z}_3 \subseteq H^2(D_6, N)$ with $a \neq 0$ and $b \neq 0$ determines a torsion-free extensions. In particular, N must satisfy $N_1 \oplus N_2 \subset N$ where $N_1 = \mathbb{Z}, \mathrm{ind}_{C_2}^{D_6}(\mathbb{Z})$ and $N_2 = \mathbb{Z}^t, \mathbb{Z}[C_2], Y_0$.*

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